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REGULARITY OF THE SOLUTIONS FOR ELLIPTIC PROBLEMS ON NONSMOOTH DOMAINS IN $\mathbb{R}^3$

PART I: COUNTABLY NORMED SPACES ON POLYHEDRAL DOMAINS

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This is the first of a series of three devoted to the regularity of solution of elliptic problems on nonsmooth domains in $\mathbf{R}^3$. The present paper introduces various weighted spaces and countably weighted spaces in neighborhood of edges and vertices of polyhedral domains.

These well-defined spaces are the foundations for comprehensive study of the regularity theory of elliptic problem with piecewise analytic date in $\mathbf{R}^3$.

REGULARITY OF THE SOLUTIONS FOR ELLIPTIC PROBLEMS ON NONSMOOTH DOMAINS IN $\mathbb{R}^3$

Part I: Countably Normed Spaces On Polyhedral Domains

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Abstract

This is the first of a series of three devoted to the regularity of solution of elliptic problems on nonsmooth domains in $\mathbb{R}^3$. The present paper introduces various weighted spaces and countably weighted spaces in neighbourhood of edges and vertices of polyhedral domains, and it concentrates on exploring the structure of these spaces such as the imbeddings of weighted Sobolev spaces, the relation between weighted Sobolev spaces and weighted continuous function spaces, and the relations between the weighted Sobolev spaces and countably weighted Sobolev spaces in Cartesian coordinates and in the spherical and cylindrical coordinates.

These well-defined spaces are the foundations for comprehensive study of the regularity theory of elliptic problem with piecewise analytic date in $\mathbb{R}^3$, which are essential for the design of effective computation and the analysis of the $h-p$ version of the finite element method for solving elliptic problems in three-dimensional nonsmooth domain arising from mechanics and engineering.

Keywords: Piecewise analytic date, polyhedral domain, neighbourhoods of edges and vertices, weighted Sobolev space, weighted continuous function space, countably normed spaces.

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1. INTRODUCTION

In engineering applications many problems in $\mathbb{R}^3$ are characterized by partial differential equations with piecewise analytic data such as nonsmooth domains, abruptly changes of types of boundary conditions, piecewise analytic coefficients and boudary conditions, etc., for instance, the physical domains of structral mechanical problems often have edges and vertices, interfaces between different materials and material cracks. The solutions of these problems have strong singularities at the edges and vertices and around the cracks, which make the conventional numerical approximation extremely difficult and inefficient. Hence comprehensive study on the regularity of the solutions of elliptic problems in $\mathbb{R}^3$ with piecewise analytic data is of great significance not only for theoretical reasons but also for the desigh of effective computations and the optimal convergence of numerical method for these problems.

The regularities of the solutions on nonsmooth domains are typically described in terms of usual Sobolev spaces and the asymptotic expansions where the solutions are decomposited into regular and singular parts (see [13,14,15,17,18,19,20,32,33,34,36,37,38,41]). Recently the classical weighted Sobolev spaces $\mathbf{W}_\beta^k$ and $\mathbf{V}_\beta^k$ of finite order with Kondrat'ev-type and Maz'ya-type weight, respectively, were used to investigate the regularities of high-order derivatives of the solutions (see [35,40,42]). These regularity results are important and useful for the regularity theory for elliptic problems on nonsmooth domains and for solving these problems by conventional numerical approaches. But these results do not characterize sufficiently the class of solutions of the problems in applications. The solutions $u(x)$ of many practical problems on polygonal and polyhedral domains may be analytic except at the vertices and edges, and their derivative of order $k \geq 1$ may grow rapidly as x tends to the vertices or edges and as k increases. The regularity described by usual Sobolev spaces and the classical weighted Sobolev spaces $\mathbf{W}_\beta^k$ and $\mathbf{V}_\beta^k$ is unable to reflect these natures of singularity, and the quantitative features of the growth of the derivatives of high order are totally neglected. These features are extremely important for numerical analysis and effective computations. Hence we need a new regularity theory for elliptic problems with piecewise analytic data, which allows us to construct a maximally effective numerical method and to achieve the optimal rate of convergence. It has been proved by the approximation theory of the $h-p$ version of the finite element methods and confirmed by computational practices that the optimal rate is the exponetial rate with respect to the number of degree of freedom.

We have found that the most proper regularity theory which best serve the goal of numerical analysis is the one described in the frame of countably normed spaces which provide us with qualitative as well as quantitative analysis of the solutions and their derivatives of any order. Based upon this regularity theory it has been shown theoretically and computationally that the exponential convergence of the $h-p$ version of the finite/boundary element method can be achieved. The regulairty theory of this type for two-dimensional problems on nonsmooth domains have been well established in 1980's. Here we refer to [4,5] and [27] for the boundary value problems of scalar equation and elasticity equations, to [10] for the eigen value problems and to [25] for the interface problems. These regularity results have successfully led to the proof of the exponential convergence of the $h-p$ version for problems on polygonal domains. For the $h-p$ version of the finite element and boundary element method we refer to [6,11,22,23,24,29,30], and refer to [2,8,9,21] for the $h-p$ version in three dimensions, which has been addressed recently in 1990's. Since the regularity results in two-dimensions can not be directly and easily generalized to the three dimensional case, we have to establish a regularity theory for three-dimensional problems on polyhedral domains, which is much more complicated. The complexity of the singularity for the three dimensional problems is caused not only by the higher dimension, but also by the totally different characters of singularities, namely, edge singularity, vertex singularity and vertex-edge singularity. Hence we have to find proper weight functions and proper countably normed spaces in neighborhoods of edges and vertices separately so that these spaces can characterize precisely and sufficiently the singular feature in different neighborhoods of the domain. It is worth indicating that the structure of the dynamical weights used in this series is different from ones of Kondrat'ev-type and Maz'ya-type. The power of the weight for the m-th derivaties, $0 \leq m \leq k$, is fixed or decrease as m increase for functions belonging to the classical spaces $\mathbf{V}_\beta^k$ and $\mathbf{W}_\beta^k$, respectively. Consequently the spaces $\mathbf{V}_\beta^k$ and $\mathbf{W}_\beta^k$ must be of finite order. On the contrary the power of the dynamical weight associated with the spaces $\mathbf{H}_\beta^{k,l}$ increases as m increases, which allow us to introduce the countably normed spaces $\mathbf{B}_\beta^l$ and $\mathbf{C}_\beta^l$ to precisely reflects the nature of singularities on finite polygonal and polyhedral domains. Hence the regularity theory given in our series has obvious advantages in engineering applications.

This series consisting of three papers is devoted to the analysis of regularity of the solutions of elliptic problems on nonsmooth domains in $\mathbb{R}^3$ in the frame of countably normed spaces. The first paper concentrates on establishing the theory of the countably normed spaces $\mathbf{B}_\beta^l$ and $\mathbf{C}_\beta^l$ and the weighted Sobolev spaces furnished with the dynamical weights over polyhedral domains. The second one deals with the existence and uniqueness of the weak solution for elliptic problem with data given in the weighted Sobolev spaces $\mathbf{H}_\beta^{k,l}$, and analyzes the regularity of the solution in neighborhoods of edges of polyhedral domains. The regularities in neighborhoods of vertex-edges and inner-neighborhood of vertices are addressed in the third paper.

In present paper (Part I) we introduce countably normed spaces with weighted Sobolev norms and weighted $\mathbf{C}^k$ - norms. As a framework for comprehensive study of regularity theory we shall explore the structures of these spaces qualitatively as well as quantitatively as to be sure that these spaces meet our theoretical and numerical purposes. In Section 2 we define various neighborhoods of edges, vertex-edges and inner-neighborhoods of vertice, and the weighted Sobolev spaces $\mathbf{H}_\beta^{k,l}$, and the countably normed spaces $\mathbf{B}_\beta^l$ in Cartesian coordinates on these neighborhoods and the whole polyhedral domain. Section 3 addresses the imbeddings of the weighted Sobolev spaces $\mathbf{H}^{k,l}_{\beta}$ into usual Sobolev spaces with fraction order and continuous function spaces. The weighted Sobolev spaces $\mathcal{H}^{k,l}_{\beta}$ and the countably normed spaces $\mathcal{B}^{l}_{\beta}$ in cylindrical coordinates on neighborhoods of edges and in spherical coordinates on neighborhoods of vertices are given in Section 4, and the relation between these spaces and those in Cartesian coordinates are established there. We introduce in Section 5 the countably normed spaces $\mathbf{C}^{l}_{\beta}$ in weighted continuous function norm, and their relation with those with weighted Sobolev norm are fully addressed in this section.

2. PRELIMINARY

2.1. The Neighbourhoods Of Edges, Vertices And Vertex-edges.

Let Ω be a polyhedral domain in $\mathbb{R}^3$, and let Γ_i, $i \in \mathcal{I} = \{1, 2, 3 \ldots, I\}$ be the faces (open), Λ_{ij} be the edges, which are the intersections of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$, and A_m, $m \in \mathcal{M} = \{1, 2, \ldots, M\}$ be the vertices of Ω. By $\mathcal{I}_m$ we denote a subset $\{j \in \mathcal{I} \mid A_m \in \bar{\Gamma}_j\}$ of $\mathcal{I}$ for $m \in \mathcal{M}$. Let $\mathcal{L} = \{ij \mid i, j \in \mathcal{I}, \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$, and let $\mathcal{L}_m$ denote a subset of $\mathcal{L}$ such that $\mathcal{L}_m = \{ij \in \mathcal{L} \mid A_m \in \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$. We denote by ω_{ij} the interior angle between Γ_i and Γ_j for $ij \in \mathcal{L}$. Let $\Gamma^{\circ} = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^1 = \bigcup_{i \in \mathcal{N}} \Gamma_i$ where $\mathcal{D}$ is a subset of $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \setminus \mathcal{D}$. For $m \in \mathcal{M}$, $\mathcal{D}_m = \mathcal{D} \cap \mathcal{I}_m$ and $\mathcal{N}_m = \mathcal{N} \cap \mathcal{I}_m$.

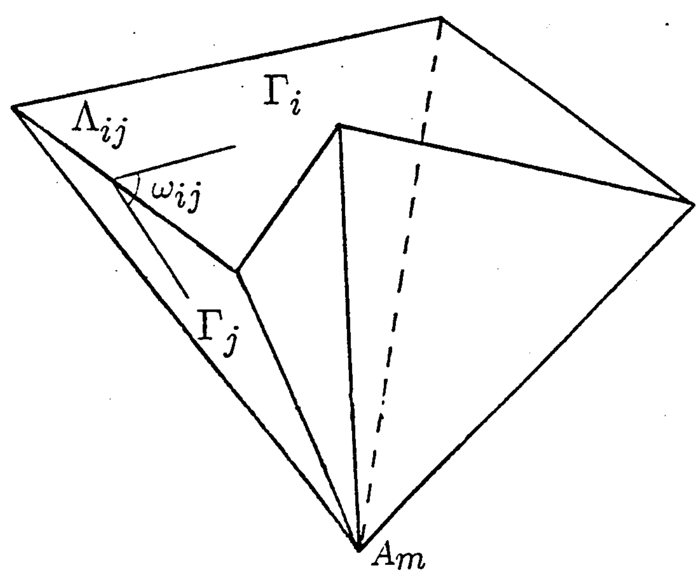

Figure 2.1 Polyhedral Domain Ω

For effectively studying the regularity of the solution of elliptic problems on polyhedral domain we shall decompose the domain into various neighbourhoods of low-dimensional manifolds.

We define a neighbourhood $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$ of the edge Λ_{ij}, shown in Figure 2.2 and assume that $\Lambda_{ij} = \{x = (x_1, x_2, x_3) \mid x_1 = x_2 = 0, a \le x \le b\}$, as follows:

$$\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) = \{x \in \Omega \mid 0 < r = \operatorname{dist}(x, \Lambda_{ij}) < \varepsilon_{ij}, a + \delta_{ij} < x_3 < b - \delta_{ij}\}.$$

It can be written as $Q_{\varepsilon_{ij}} \times I_{\delta_{ij}}$ with $Q_{\varepsilon_{ij}} = \{(r,\theta) \mid 0 < r < \varepsilon, 0 < \theta < \omega_{ij}\}$ and $I_{\delta_{ij}} = (a+\delta_{ij}, b-\delta_{ij})$, where (r,θ,x_3) are cylindrical coordinate with respect to the edge Λ_{ij}, ε_{ij} and δ_{ij} are selected such that $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \cap \bar{\Gamma}_\ell = \emptyset$ for $\ell \in \mathcal{I}$, $\ell \neq i, j$.

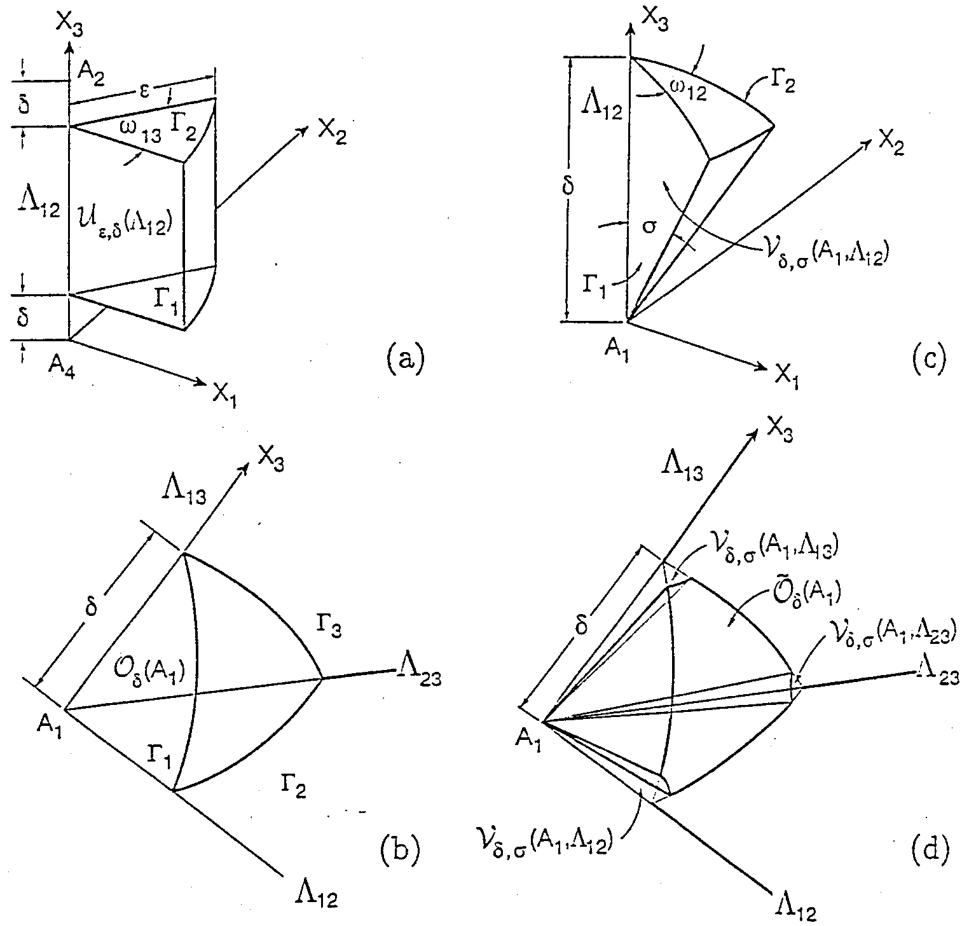

Figure 2.2 Neighbourhoods of Edges and Vertices

(a) the neighborhood $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$; (b) the neighborhood $\mathcal{O}_{\delta_m}(A_m)$;

(c) the neighborhood $\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij})$; (d) the inner neighborhood $\tilde{\mathcal{O}}_{\delta_m(A_m)}$.

By $\mathcal{O}_{\delta_m}(A_m)$ we denote a neighbourhood of the vertex A_m, shown in Figure 2.2,

$$\mathcal{O}_{\delta_m}(A_m) = \{x \in \Omega \mid 0 < \rho = \operatorname{dist}(x, A_m) < \delta_m\}.$$

Here we assume that A_m is in the origin and $0 < \delta_m < 1$ such that $\mathcal{O}_{\delta_m}(A_m) \cap \bar{\Gamma}_\ell = \emptyset$ for any $\ell \in (\mathcal{I} \setminus \mathcal{I}_m)$. We need further to decompose $\mathcal{O}_{\delta_m}(A_m)$ into several neighborhoods of vertex-edge and an inner-neighbourhood of vertex.

We introduce a neighbourhood $\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{iz})$, shown in Figure 2.2, by

$$\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) = \{x \in \mathcal{O}_{\delta_m}(A_m) \mid 0 < \phi < \sigma_{ij}\}$$

where ϕ is the angle between the edge Λ_{ij}, $ij \in \mathcal{L}_m$ and the radial from A_m to the point x. We always assume that the vertex A_m is at the origin and the edge Λ_{ij} lies along the positive x_3-axis. Let (ϕ, θ, ρ) be the spherical coordinates with respect to the vertex A_m and the edge Λ_{ij}, then $\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) = S_{\sigma_{ij}} \times I_{\delta_m}$ with $I_{\delta_m} = (0, \delta_m)$ and $S_{\sigma_{ij}} = \{(\phi, \theta) \mid 0 < \phi < \sigma_{ij}, 0 < \theta < \omega_{ij}\}$ $\sigma_{ij} \in (0, \frac{\pi}{6})$ is selected such that

$$\bar{\mathcal{V}}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) \cap \bar{\mathcal{V}}_{\delta_m,\sigma_{k,\ell}}(A_m, \Lambda_{k,\ell}) = A_m \quad \text{for all } k\ell \in \mathcal{L}_m,\ k\ell \neq ij.$$

Next we define an inner-neighbourhood $\tilde{\mathcal{O}}_{\delta_m}(A_m)$ of the vertex A_m, by

$$\tilde{\mathcal{O}}_{\delta_m}(A_m) = \mathcal{O}_{\delta_M}(A_m) \setminus \bigcup_{ij \in \mathcal{L}_m} \bar{\mathcal{V}}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij})$$

which is shown in Figure 2.2.

Let $\delta_{ij} < \frac{1}{2}\delta_m \cos\sigma_{ij}$, and $\varepsilon_{ij} > \frac{1}{2}\delta_m \sin\sigma_{ij}$ for $ij \in \mathcal{L}_m$, $m \in \mathcal{M}$. Then $\Omega_0 = \Omega \setminus \{\bigcup_{m \in \mathcal{M}}\{\mathcal{O}_{\delta_m/2}(A_m) \bigcup_{ij \in \mathcal{L}_m} \mathcal{U}_{\varepsilon_{ij}/2, \delta_{ij}/2}(\Lambda_{ij})\}$ contain no vertices and edges of the polyhedral domain Ω, which is called the regular region of Ω, and $\Omega_0 \cap \mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \neq \emptyset$, $\Omega_0 \cap \tilde{\mathcal{O}}_{\delta_m}(A_m) = \emptyset$, $\Omega_0 \cap \mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) \neq \emptyset$ for any $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$. Meanwhile we note that $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \cap \mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) \neq \emptyset$ for $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$.

For the sake of simplicity we shall write $\mathcal{U}_{ij}$ or $\mathcal{U}(\Lambda_{ij})$ $\mathcal{V}_{m,ij}$ or $\mathcal{V}(A_m, \Lambda_{ij})$, $\tilde{\mathcal{O}}_m$ or $\tilde{\mathcal{O}}(A_m)$ instead of $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$, $\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij})$ and $\tilde{\mathcal{O}}_{\delta_m}(A_m)$.

2.2. The Weighted Sobolev Spaces $\mathbf{H}^{k,\ell}_\beta(\Omega)$ And Countably Normed Spaces $\mathbf{B}^\ell_\beta(\Omega)$.

By $\mathbf{H}^k(\Omega)$, $k \geq 0$ integer, we denote the usual Sobolev space on Ω with the norm

$$\|u\|^2_{\mathbf{H}^k(\Omega)} = \sum_{\sigma \leq |\alpha| \leq k} \|D^\alpha u\|^2_{\mathbf{L}^2(\Omega)}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha u = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} D^{\alpha_3}_{x_3} u = u_{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}$ is the weak (or distributional) partial derivative. As usual, $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$, $\mathbf{H}^1_0(\Omega) = \{u \in \mathbf{H}^2(\Omega) \mid u = 0 \text{ on } \Gamma^0\}$,

$$|u|^2_{\mathbf{H}^k(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha u\|^2_{\mathbf{L}^2(\Omega)}$$

and

$$|D^k u|^2 = \sum_{|\alpha| = k} |D^\alpha u|^2.$$

It is well known that the solutions of elliptic problems in polyhedral domains may be very singular. The usual Sobolev spaces are not sufficient to describe the natures of regularities of high-order derivatives of the solutions. Hence we shall introduce the weighted normed spaces and countably normed spaces which are defined in individual neighbourhoods of edges, vertex-edges and inner-neighbourhoods of vertices.

Let $r = r(x) = \mathrm{dist}(x, \Lambda_{ij})$ for $x \in \mathcal{U}_{ij}$ and $\beta_{ij} \in (0,1)$. The weight function is defined by

$$\Phi^{\alpha,\ell}_{\beta_{ij}}(x) = \begin{cases} r(x)^{\beta_{ij}+|\alpha'|-\ell} & \text{for } |\alpha'| = \alpha_1+\alpha_2 \geq \ell \\ 1 & \text{for } |\alpha'| < \ell. \end{cases} \tag{2.1}$$

The weighted Sobolev space $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ and the countably normed space $\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ with integer k and ℓ, $k \geq \ell \geq 0$ are defined as

$$\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \Big\{u \mid \|u\|_{\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} = \sum_{0\leq|\alpha|\leq k} \|\Phi^{\alpha,\ell}_{\beta_{ij}}(x)D^\alpha u\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} < \infty\Big\}$$

and

$$\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \{u \in \mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \text{ for all } k \geq \ell, \text{ and } \|\Phi^{\alpha,\ell}_{\beta_{ij}}(x)D^\alpha u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq Cd^\alpha \alpha!\}.$$

Hereafter $d^\alpha = d_1^{\alpha_1}d_2^{\alpha_2}d_3^{\alpha_3}$ and $\alpha! = \alpha_1!\alpha_2!\alpha_3!$, the constants $C \geq 1$ and $d_i \geq 1$ are independent of α.

Next, let $\rho = \rho(x)$ and $\phi = \phi(x)$ be as before for $x \in \mathcal{V}_{m,ij}$. We define a weight function for integer $\ell \geq 0$ and a pair of real numbers $\beta_{m,ij} = (\beta_m, \beta_{ij})$ with $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$

$$\Phi^{\alpha,\ell}_{\beta_{m,ij}}(x) = \begin{cases} \rho^{\beta_m+|\alpha|-\ell}(\sin\phi)^{\beta_{ij}+|\alpha'|-\ell}, & \text{for } |\alpha'| = \alpha_1+\alpha_2 \geq \ell \\ \rho^{\beta_m+|\alpha|-\ell}, & \text{for } |\alpha'| < \ell \leq |\alpha| \\ 1, & \text{for } |\alpha| < \ell, \end{cases}$$

and we have the weighted Sobolev space with integer $k \geq \ell$ over the neighbourhood $\mathcal{V}_{m,ij}$

$$\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) = \Big\{u \mid \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} = \sum_{|\alpha|\leq k} \|\Phi^{\alpha,\ell}_{\beta_{m,ij}}(x)D^\alpha u\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})} < \infty\Big\}$$

and the countably normed space

$$\mathbf{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) = \Big\{u \in \mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) \text{ for all } k \geq \ell, \text{ and } \|\Phi^{\alpha,\ell}_{\beta_{ij}}(x)D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \leq Cd^\alpha\alpha!\Big\}.$$

We now introduce a weight function in inner-neighbourhood $\tilde{\mathcal{O}}_m$ of the vertex A_m with an integer $\ell \geq 0$ and a real number $\beta_m \in (0, \frac{1}{2})$

$$\Phi^{\alpha,\ell}_{\beta_m}(x) = \begin{cases} \rho^{\rho_m+|\alpha|-\ell}, & \text{for } |\alpha| \geq \ell \\ 1, & \text{for } |\alpha| < \ell \end{cases}$$

with $\rho = \rho(x)$ being defined as before and define the weighted Sobolev space on $\tilde{\mathcal{O}}_m$ with integer $k \geq \ell$

$$\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m) = \Big\{u \mid \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)} = \sum_{|\alpha|\leq k} \|\Phi^{\alpha,\ell}_{\beta_m}(x)D^\alpha u\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} < \infty\Big\}$$

and the Countably normed space

$$\mathbf{B}^{\ell}_{\beta_m}(\tilde{\mathcal{O}}_m) = \Big\{ u \in \mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m) \text{ for all } k \geq \ell, \text{ and } \|\Phi^{\alpha,\ell}_{\beta_m}(x) D^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C d^\alpha \alpha! \Big\}$$

By β we denote a multi-index $(\beta_m, \beta_{ij}, m \in \mathcal{M}, ij \in \mathcal{L})$ with $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$. Then the weighted Sobolev space $\mathbf{H}^{k,\ell}_{\beta}(\Omega)$ over Ω is a set of functions such that their restrictions belong to $\mathbf{H}^k(\Omega_0)$, $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, $\mathbf{H}^{k,\ell}_{\beta_m}(\mathcal{O}_m)$ and $\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ for all $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$, furnished with the norm

$$\begin{aligned} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta'}(\Omega)} = \|u\|^2_{\mathbf{H}^k(\Omega_0)} + \sum_{ij \in \mathcal{L}} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} + \sum_{m \in \mathcal{M}} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)} \\ + \sum_{m \in \mathcal{M}} \sum_{ij \in \mathcal{L}_m} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}. \end{aligned}$$

The countably normed space $\mathbf{B}^{\ell}_{\beta}(\Omega)$ consists of all functions such that their restrictions belong to $\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, $\mathbf{B}^{\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$ and $\mathbf{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ for any $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$, and their restrictions on $\bar{\Omega}_0$ are analytic.

Although β is a multi-index, it has local interpretations in the individual neighbourhoods, namely, $\beta = \beta_m$ in the inner-neighbourhood $\tilde{\mathcal{O}}_m$, $\beta = \beta_{ij}$ in the neighbourhood $\mathcal{U}_{ij}$ and $\beta = \beta_{m,ij} = (\beta_m, \beta_{ij})$ in the neighbourhood $\mathcal{V}_{m,ij}$. Consequently, we shall write $\mathbf{H}^{k,\ell}_{\beta}(\tilde{\mathcal{O}}_m) = \mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$, $\mathbf{H}^{k,\ell}_{\beta}(\mathcal{V}_{m,ij}) = \mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, $\mathbf{H}^{k,\ell}_{\beta}(\mathcal{U}_{ij}) = \mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, $\mathbf{B}^{\ell}_{\beta}(\tilde{\mathcal{O}}_m) = \mathbf{B}^{\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$, $\mathbf{B}^{\ell}_{\beta}(\mathcal{V}_{m,ij}) = \mathbf{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, $\mathbf{B}^{\ell}_{\beta}(\mathcal{U}_{ij}) = \mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, *etc.*

3. IMBEDDINGS OF $\mathbf{H}^{k,\ell}_{\beta}(\Omega)$

In this section we shall prove imbeddings of $\mathbf{H}^{k,\ell}_{\beta}(\Omega)$ into spaces of continuous function and fractional-order Sobolev spaces. These imbedding theorems are of great importance not only for the regularity of the solutions for elliptic problems on polyhedral domains, but also for the numerical approximation for these problems (see [8,9,21]).

3.1. Imbedding Of $\mathbf{H}^{k,\ell}_{\beta}(\Omega)$ Into Fractional Order Sobolev Spaces.

For non-integer $s \geq 0$, the space $\mathbf{H}^s(\Omega)$ is defined as a fractional order Sobolev space (see [1]).

Lemma 3.1. *Let $u \in \mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})$ with $\beta_{ij} \in (0,1)$, and $u(x) = 0$ for $x = (x_1, x_2, x_3) \in \mathcal{U}_{ij}$ with $r = (x_1^2 + x_2^2)^{\frac{1}{2}} \geq \frac{1}{2}\varepsilon_{ij}$. Then $u \in \mathbf{H}^{1+\theta}(\mathcal{U}_{ij})$ for $\theta = 1 - \beta_{ij} - \varepsilon$, $\varepsilon > 0$ arbitrary, and*

$$\|u\|_{\mathbf{H}^{1+\theta}(\mathcal{U}_{ij})} \leq C \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}. \tag{3.1}$$

Proof. Let $\mathcal{U}_{ij} = \mathcal{U}_{\varepsilon_{ij},\delta_{ij}} = Q_{\varepsilon_{ij}} \times I_{\delta_{ij}}$ with $I_{\delta_{ij}} = (a + \delta_{ij}, b - \delta_{ij})$ and $Q_{\varepsilon_{ij}} = \{\bar{x} = (x_1, x_2) \mid 0 < r = (x_1^2 + x_2^2)^{\frac{1}{2}} < \varepsilon_{ij}, 0 < \theta < \omega_{ij}\}$. For $\bar{x}_0 = (x_{0,1}, x_{0,2}) = \big(r_0 \cos(\omega_{ij}/2), r_0 \sin(\omega_{ij}/2)\big)$ with

$r_0 \in (0, \varepsilon_{ij}/2)$ we let $\mathcal{U}_{ij}^{r_0} = \{x = (x_1 + x_{0,1}, x_2 + x_{0,2}, x_3) \in \mathcal{U}_{ij} \mid (x_1^2 + x_2^2)^{\frac{1}{2}} < \varepsilon_{ij}/2\}$. Then obviously $\mathcal{U}_{ij}^{r_0} \subset \mathcal{U}_{ij}$ for any $r_0 \in (0, \varepsilon_{ij}/2)$.

Define now $v_0(x) = u(x_1 + x_{0,1}, x_2 + x_{0,2}, x_3)$ for $x \in Q_{\varepsilon_{ij}/2} \times I_{\delta_{ij}}$ and $v_0(x) = 0$ for $x \in \mathcal{U}_{ij} \setminus (S_{\varepsilon_{ij}/2} \times I_{\delta_{ij}})$, and $w_0(x) = u(x) - v_0(x)$. Then $v_0 \in \mathbf{H}^2(\mathcal{U}_{ij})$ and $w_0 \in \mathbf{H}^1(\mathcal{U}_{ij})$. Further we have

$$\|v_0(x)\|_{\mathbf{H}^1(\mathcal{U}_{ij})} \leq C\|u\|_{\mathbf{H}^1(\mathcal{U}_{ij})} \tag{3.2}$$

and

$$\begin{aligned} \|D^2 v_0\|^2_{L^2(\mathcal{U}_{ij})} &= \sum_{|\alpha|=2} \int_{\mathcal{U}_{ij}} \sum |D^\alpha v_0|^2 \, dX \\ &\leq \int_{\mathcal{U}_{ij}^{r_0}} |D^2 u|^2 \, dx \\ &\leq r_0^{-2\beta_{ij}} \int_{\mathcal{U}_{ij}^{r_0}} |D^2 u|^2 r^{2\beta_{ij}} \, dx. \end{aligned}$$

which together with (3.2) implies

$$\|v_0\|_{\mathbf{H}^2(\mathcal{U}_{ij})} \leq C r_0^{-\beta_{ij}} \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}. \tag{3.3}$$

Let now $x \in \mathcal{U}_{ij}$, $x_\tau = (x_1 + \tau x_{0,1}, x_2 + \tau x_{0,2}, x_3)$, $0 \leq \tau \leq 1$. Then

$$-w_0(x) = \int_0^1 \frac{\partial}{\partial \tau} u(x_\tau) \, d\tau = \int_0^2 \bar{x}_0 \cdot (\nabla_{\bar{x}} \mathcal{U})(x_\tau) \, d\tau$$

where $\nabla_{\bar{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. Hence we have by Schwartz's inequality, for $0 \leq s < \frac{1}{2}$

$$\begin{aligned} |w_0(x)| &\leq r_0 \int_0^1 |(\nabla_{\bar{x}} u)(x_\tau)| \, dt \\ &\leq C r_0 \Big(\int_0^1 |(\nabla_{\bar{x}} u)(\bar{x}_\tau)|^2 \tau^{2s} \, d\tau \Big)^{\frac{1}{2}}, \end{aligned}$$

therefore

$$\begin{aligned} \|w_0\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} &\leq C r_0^2 \int_{\mathcal{U}_{ij}} dx \int_0^1 |(D^1 u)(x_\tau)|^2 \tau^{2s} \, d\tau \\ &= C r_0^2 \int_0^1 \tau^{2s} \, d\tau \int_{\mathcal{U}_{ij}^{\tau r_0}} |D^1 u|^2 \, dx. \end{aligned}$$

Selecting $s = 0$ we get

$$\|w_0\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq C r_0 \|u\|_{\mathbf{H}^1(\mathcal{U}_{ij})} \leq C r_0 \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \tag{3.4}$$

with C independent of r_0.

Note that $r = (x_1^2 + x_2^2)^{\frac{1}{2}} \geq \tau r_0$ for $x \in \mathcal{U}_{ij}^{\tau r_0}$. Then we have

$$\|D^1 w_0\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq C r_0^2 \int_0^1 \tau^{2s}\, d\tau \int_{\mathcal{U}_{ij}^{\tau r_0}} |D^2 u|^2\, dx$$
$$\leq C r_0^2 \int_0^1 \tau^{2s} (\tau r_0)^{-2\beta_{ij}}\, d\tau \int_{\mathcal{U}_{ij}^{\tau r_0}} |D^2 u|^2 r^{2\beta_{ij}}\, dx$$

Due to the assumption that $\beta_{ij} < 1$, we can select $s = \frac{\beta_{ij}}{2} < \frac{1}{2}$, which leads to

$$\|D^1 w_0\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq C r_0^{(1-\beta_{ij})} \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \tag{3.5}$$

with c depending on β_{ij}, but not on r_0. Combining (3.4) and (3.5) we have

$$\|w_0\|_{\mathbf{H}^1(\mathcal{U}_{ij})} \leq C r_0^{(1-\beta_{ij})} \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}. \tag{3.6}$$

Using the K-method for interpolation, (see [12]) we define

$$K(u,t) = \inf_{\substack{\psi \in \mathbf{H}^1(\mathcal{U}_{ij}) \\ \phi \in \mathbf{H}^2(\mathcal{U}_{ij}) \\ \varphi + \psi = u}} (\|\psi\|_{\mathbf{H}^1(\mathcal{U}_{ij})} + t\|\varphi\|_{\mathbf{H}^2(\mathcal{U}_{ij})})$$

Obviously by selecting $\psi = u$ and $\varphi = 0$ we get

$$K(u,t) \leq \|u\|_{\mathbf{H}^1(\mathcal{U}_{ij})} \leq \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(U_{ij})}. \tag{3.7}$$

Selecting $\varphi = 0$ and $\psi = w_0$ gives us

$$K(u,t) \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} (r_0^{(1-\beta_{ij})} + t r_0^{-\beta_{ij}}). \tag{3.8}$$

The norm of the space $\mathbf{H}^{1+\theta}(\mathcal{U}_{ij})$ is defined (see [12])

$$\|u\|^2_{\mathbf{H}^{1+\theta}(\mathcal{U}_{ij})} = \int_0^\infty |t^{-\theta} K(u,t)|^2 \frac{dt}{t}.$$

By (3.7) and (3.8) we have

$$\int_1^\infty |t^{-\theta} K(u,t)|^2 \frac{dt}{t} \leq \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \int_1^\infty t^{-1-2\theta}\, dt$$
$$\leq \frac{1}{2\theta} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}$$

and

$$\int_0^1 |t^{-\theta} K(u,t)|^2 \frac{dt}{t} \leq C\|u\|^2_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \int_0^1 (t^{-2\theta-1} r_0^{2(1-\beta_{ij})} + t^{-2\theta+1} r_0^{-2\beta_{ij}})\, dt.$$

By selecting $r_0 = t$ and $\theta = 1 - \beta_{ij} - \varepsilon$, $\varepsilon > 0$ arbitrary, we obtain (3.1). ∎

The lemma now enables us to prove the imbedding theorem.

Theorem 3.1. $\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})$ **is imbedded into** $\mathbf{H}^{1+\theta}(\mathcal{U}_{ij})$ **for** $\beta_{ij} \in (0,1)$ **and** $\theta = 1-\beta_{ij}-\varepsilon$, $\varepsilon > 0$, **arbitrary.**

Proof. Let $\varphi(\xi) \in C^\infty(\mathbb{R}^+_1)$ such that $\varphi(\xi) = 1$ for $0 < \xi < \frac{1}{2}$ and $\varphi(\xi) = 0$ for $\xi > 1$. For $u \in \mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})$, define $v(x) = u(x)\varphi(\frac{2r}{\varepsilon_{ij}})$ and $w(x) = u(x) - v(x)$. Then $v(x) = 0$ for $x \in \mathcal{U}_{ij}$ with $r > \frac{1}{2}\varepsilon_{ij}$, and

$$\|v\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \leq C\|u\|_{\mathbf{H}^{2,2}(\mathcal{U}_{ij})}.$$

Then applying (3.1) to v implies that $v \in \mathbf{H}^{1+\theta}(\mathcal{U}_{ij})$ for $\theta = 1-\beta_{ij}-\varepsilon$, $\varepsilon > 0$ arbitrary, and

$$\|v\|_{\mathbf{H}^{1+\theta}(\mathcal{U}_{ij})} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}. \tag{3.9}$$

Note that Supp.$w \subset \mathcal{U}_{\varepsilon_{ij}\delta_{ij}} \setminus \mathcal{U}_{\varepsilon_{ij}/2,\delta_{ij}}$, and

$$\|w\|_{\mathbf{H}^2(\mathcal{U}_{ij})} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}.$$

Hence $u \in \mathbf{H}^{1+\theta}(\mathcal{U}_{ij})$ with $\theta = 1-\beta_{ij}-\varepsilon$, $\varepsilon > 0$, arbitrary, and the proof is completed. ∎

The arguments can be carried out for the space $\mathbf{H}^{\ell,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ with any integer $\ell \geq 1$, and we have the Corollary 3.1.

Corollary 3.1. *$\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ is imbedded into $\mathbf{H}^{\ell-1+\theta}(\mathcal{U}_{ij})$ for $k \geq \ell \geq 1$, $\beta_{ij} \in (0,1)$ and $\theta = 1-\beta_{ij}-\varepsilon$, $\varepsilon > 0$, arbitrary.*

Next, let us consider the imbeddings of $\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ into a fractional order Sobolev space.

Lemma 3.2. *Let $u \in \mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ with $\beta_m \in (0,\frac{1}{2})$ and $u(x) = 0$ for $x = (x_1,x_2,x_3) \in \tilde{\mathcal{O}}_m$ with $\rho = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}} > \frac{1}{2}\delta_m$. Then $u \in \mathbf{H}^{1+\theta}(\tilde{\mathcal{O}}_m)$ with $\theta = 1-\beta_m-\varepsilon$, $\varepsilon > 0$ arbitrary, and*

$$\|u\|_{\mathbf{H}^{1+\theta}(\tilde{\mathcal{O}}_m)} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)} \tag{3.10}$$

Proof. $\tilde{\mathcal{O}}_m$ is a star-shape domain with the center at the vertex A_m (the origin). Let $x_0 = (x_{0,1},x_{0,2},x_{0,3}) \in \tilde{\mathcal{O}}_m$ with $\rho_0 = (x_{0,1}^2+x_{0,2}^2+x_{0,3}^2)^{\frac{1}{2}} \in (0,\frac{1}{2}\delta_m)$. Then analogously as in Lemma 3.1 we define

$$\tilde{\mathcal{O}}^{\rho_0}_m = \Big\{x = \bar{x}+x_0 \mid \bar{x} \in \tilde{\mathcal{O}}_m, |\bar{x}| \leq \frac{1}{2}\delta_m\Big\}.$$

Then $\tilde{\mathcal{O}}^{\rho_0}_m \subset \tilde{\mathcal{O}}_m$ for any $\rho_0 < \frac{1}{2}\delta_m$. Now we will proceed very similarly as in Lemma 3.1. Define $v_0(x) = u(x+x_0)$ and $w_0(x) = u(x)-v_0(x)$. Then $v_0 \in \mathbf{H}^2(\tilde{\mathcal{O}}_m)$ and $w_0(x) = \mathbf{H}^1(\tilde{\mathcal{O}}_m)$. It can be proved in the same way

$$\|v_0\|_{\mathbf{H}^2(\tilde{\mathcal{O}}_m)} \leq C\rho_0^{-\beta_m}\|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)} \tag{3.11}$$

and for $l = 0,1$ and $s \in [0,\frac{1}{2})$

$$\|D^l w_0\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\rho_0^2\int_0^1 \tau^{2s}\,d\tau \int_{\tilde{\mathcal{O}}^{\tau\rho_0}_m} |D^{l+1}u|^2\,dx.$$

Noting that $\rho = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \geq \tau\rho_0$ for $x \in \tilde{\mathcal{O}}_m^{\tau\rho_0}$ and $\beta_m < \frac{1}{2}$ we have for $s = 0$

$$\|w_0\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\rho_0^2 \|u\|^2_{\mathbf{H}^1(\tilde{\mathcal{O}}_m)} \leq C\rho_0^2 \|u\|^2_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)} \tag{3.12}$$

and

$$\begin{aligned} \|D^1 w_0\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} &\leq C\rho_0^{2(1-\beta_m)} \int_0^1 \tau^{-2\beta_m}\, d\tau \int_{\tilde{\mathcal{O}}_m^{\tau\rho_0}} |D^2 u|^2 \rho^{2\beta_m}\, dx \\ &\leq C\rho_0^{2(1-\beta_m)} \cdot \|u\|^2_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \end{aligned} \tag{3.13}$$

(3.12) and (3.13) yield

$$\|w_0\|_{\mathbf{H}^1(\tilde{\mathcal{O}}_m)} \leq C\rho_0^{1-\beta_m} \|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \tag{3.14}$$

Defining the fractional order space $\mathbf{H}^{1+\theta}(\tilde{\mathcal{O}}_m)$ by K-method and arguing as in Lemma 3.1 we get $u \in \mathbf{H}^{1+\theta}(\tilde{\mathcal{O}}_m)$ for $\theta = 1 - \beta_m - \varepsilon$, $\varepsilon > 0$ arbitrary, and (3.10) holds. ∎

Analogously, Lemma 3.2 leads to Theorem 3.2 and Corollary 3.2.

Theorem 3.2. *$\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ is imbedded into $\mathbf{H}^{1+\theta}(\tilde{\mathcal{O}}_m)$ for $\beta_m \in (0, \frac{1}{2})$ and $\theta = 1 - \beta_m - \varepsilon$, $\varepsilon > 0$ arbitrary.* ∎

Corollary 3.2. *$\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$ is imbedded into $\mathbf{H}^{\ell-1+\theta}(\tilde{\mathcal{O}}_m)$ for $\beta_m(0, \frac{1}{2})$, $k \geq \ell \geq 1$ and $\theta = 1 - \beta_m - \varepsilon$, $\varepsilon > 0$ arbitrary.* ∎

We now address imbeddings of $\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ into $\mathbf{H}^{1+\theta}(\mathcal{V}_{m,ij})$ with $0 < \theta < 1$.

Lemma 3.3. *Let $u \in \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ with $\beta_{m,ij} = (\beta_m, \beta_{ij})$, $\beta_m \in (0, \frac{1}{2})$, $\beta_{ij} \in (0,1)$ and $u(x) = 0$ for $x = (x_1, x_2, x_3) \in \mathcal{V}_{m,ij})$ with $\rho = (x_1^2, +x_2^2 + x_3^2)^{\frac{1}{2}} > \frac{1}{2}\delta_m$. Then $u \in \mathbf{H}^{1+\theta}(\mathcal{V}_{m,ij})$ with $\theta = 1 - \max(\beta_{ij}, \beta_m) - \varepsilon$, $\varepsilon > 0$ arbitrary, and*

$$\|u\|_{\mathbf{H}^{1+\theta}(\mathcal{V}_{m,ij})} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}. \tag{3.15}$$

Proof. The domain $\mathcal{V}_{m,ij}$ is a star-shape domain with the center at the vertex A_m (the origin). Let $x_0 = (x_{0,1}, x_{0,2}, x_{0,3}) \in \mathcal{V}_{m,ij}$ with $\rho_0 = (x_{0,1}^2 + x_{0,2}^2 + x_{0,3}^2)^{\frac{1}{2}} \in (0, \frac{1}{2}\delta_m)$. Then analogously as in Lemma 3.2 we define

$$\mathcal{V}^{\rho_0}_{m,ij} = \Big\{ x = \bar{x} + x_0 \mid \bar{x} \in \mathcal{V}_{m,ij}, \rho(x) < \frac{1}{2}\delta_m \Big\}.$$

The $\mathcal{V}^{\rho_0}_{m,ij} \subset \mathcal{V}_{m,ij}$ for any $\rho_0 < \frac{1}{2}\delta_m$. Let us define $v_0 = u(x + x_0)$ and $w_0 = u(x) - v_0$. Then we have

$$\|v_0\|_{\mathbf{H}^1(\mathcal{V}_{m,ij})} \leq \|u\|_{\mathbf{H}^2(\mathcal{V}_{m,ij})} \tag{3.16}$$

and

$$\|D^2 v_0\|^2_{L^2(\mathcal{V}_{m,ij})} = \int_{\mathcal{V}_{m,ij}} \sum_{|\alpha|=2} |D^\alpha v_0|^2\, dx \le \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx. \tag{3.17}$$

Let us note that $\rho = \rho(x) \ge \rho_0$ for any $x \in \mathcal{V}^{\rho_0}_{m,ij}$ and $r = r(x) = (x_1^2 + x_2^2)^{\frac{1}{2}} = \rho(x)\sin\phi(x) \ge r_0 = r(x_0) = \rho_0 \sin\phi_0$, where $\phi = \phi(x)$ and ϕ_0 are the angles between the edge Λ_{ij} and the radial from the origin to x and x_0, respectively. If $\beta_m \ge \beta_{ij}$ we have

$$\begin{aligned}
\int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2 (\sin\phi)^{2\beta_{ij}} \rho^{2\beta_m}\, dx &= \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2 \left(\frac{r}{\rho}\right)^{2\beta_{ij}} \rho^{2\beta_m}\, dx \\
&\ge r_0^{2\beta_{ij}} \int_{\mathcal{V}^{\rho_0}_{m,ij}} \rho^{2(\beta_m - \beta_{ij})} |D^2 u|^2\, dx \\
&\ge r_0^{2\beta_{ij}} \rho_0^{2(\beta_m - \beta_{ij})} \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx \\
&= \rho_0^{2\beta_m} (\sin\phi_0)^{2\beta_{ij}} \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx.
\end{aligned} \tag{3.18}$$

Analogously we get for $\beta_m < \beta_{ij}$

$$\begin{aligned}
\int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2 (\sin\phi)^{2\beta_{ij}} \rho^{2\beta_m}\, dx &= \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2 r^{2\beta_{ij}} \rho^{2(\beta_m - \beta_{ij})}\, dx \\
&\ge r_0^{2\beta_{ij}} \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx \\
&\ge \rho_0^{2\beta_{ij}} (\sin\phi_0)^{2\beta_{ij}} \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx.
\end{aligned} \tag{3.19}$$

In either cases we always have that for $\gamma = \max(\beta_{ij}, \beta_m)$

$$\begin{aligned}
\|v_0\|^2_{\mathbf{H}^2(\mathcal{V}^{\rho_0}_{m,ij})} &\le \int_{\mathcal{V}^{\rho_0}_{m,ij}} |D^2 u|^2\, dx \\
&\le C\rho_0^{-2\gamma} (\sin\phi_0)^{-2\beta_{ij}} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}
\end{aligned} \tag{3.20}$$

Analogously as in the proof of Lemma 3.2 we get

$$\|D^l w_0\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\rho_0^2 \int_0^1 \tau^{2s}\, d\tau \int_{\mathcal{V}^{\tau\rho_0}_{m,ij}} |D^{l+1} u|^2\, dx$$

with $s \in [0, \frac{1}{2})$ and $l = 0, 1$. Therefore, for $l = 0$ and $s = 0$ we have

$$\|w\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\rho_0^2 \|u\|^2_{\mathbf{H}^2(\mathcal{V}_{m,ij})},$$

and for $l = 1$ and $s = \beta_{ij}/2 < \frac{1}{2}$ we have by (3.20)

$$\begin{aligned}
\|D^1 w_0\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le C\rho_0^2 \int_0^1 \tau^{2s} (\tau\rho_0)^{-2\gamma} (\sin\phi_0)^{-2\beta_{ij}} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}\, d\tau \\
&\le C\rho_0^{2(1-\gamma)} (\sin\phi_0)^{-2\beta_{ij}} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.
\end{aligned}$$

Theorem 3.3. $\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ *is imbedded into* $\mathbf{H}^{1+\theta}(\mathcal{V}_{m,ij})$ *for* $\beta_{m,ij} = (\beta_m, \beta_{ij})$ $\beta_m \in (0, \frac{1}{2})$, $\beta_{ij} \in (0,1)$ *and* $\theta = 1 - \max(\beta_m, \beta_{ij}) - \varepsilon$, $\varepsilon > 0$ *arbitrary.*

Proof. For $u \in \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, defined $v(x) = \varphi(\frac{2\rho}{\delta_m})u(x)$ and $w(x) = u - v(x)$ when $\varphi(\xi)$ is a C^∞-function as before. Then $v(x) \subset \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and vanishes for $x \in \mathcal{V}_{m,ij}$ with $\rho(x) \geq \frac{1}{2}\delta_m$. By Lemma 3.3 $v(x) \in \mathbf{H}^{1+\theta}(\mathcal{V}_{m,ij})$ with $\theta = 1 - \max(\beta_m, \beta_{ij}) - \varepsilon$, $\varepsilon > 0$ arbitrary, and (3.15) holds for $v(x)$. Further note that Supp.$w(x) \subset \mathcal{V}_{\frac{1}{2}} = \mathcal{V}_{\sigma_{ij},\delta_m} \setminus \mathcal{V}_{\sigma_{ij},\delta_m/2}$. Then $w(x) \in \mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{V}_{\frac{1}{2}})$ and

$$\|w\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{V}_{\frac{1}{2}})} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.$$

Due to Theorem 3.1 $w(x) \in \mathbf{H}^{1+\theta'}(\mathcal{V}_{\frac{1}{2}})$ with $\theta' = 1 - \beta_{ij} - \varepsilon$, $\varepsilon > 0$ arbitrary, and

$$\|w\|_{\mathbf{H}^{1+\theta'}(\mathcal{V}_{\frac{1}{2}})} \leq C\|w\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{V}_{\frac{1}{2}})} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.$$

This establishes the desired imbeddings. ■

Corollary 3.3. $\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ *is imbedded into* $\mathbf{H}^{\ell-1+\theta}(\mathcal{V}_{m,ij})$ *for* $k \geq \ell \geq 1$ *and* $\theta = 1 - \max(\beta_{ij}, \beta_m) - \varepsilon$, $\varepsilon > 0$ *arbitrary.* ■

Let us note that $\Omega = \Omega_0 \bigcup (\bigcup_{ij\in\mathcal{L}} \mathcal{U}_{ij}) \cup (\bigcup_{m\in\mathcal{M}} \tilde{\mathcal{O}}_m) \cup (\bigcup_{\substack{ij\in\mathcal{L}_m \\ m\in\mathcal{M}}} \mathcal{V}_{m,ij})$ and that $\Omega_0 \cap \mathcal{U}_{ij} \neq \emptyset$, $\Omega_0 \cap \tilde{\mathcal{O}}_m \neq \emptyset$ and $\Omega_0 \cap \mathcal{V}_{m,ij} \neq \emptyset$ for all $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$. Due to the definition of the space $\mathbf{H}^{2,2}_\beta(\Omega)$, $u \in \mathbf{H}^2(\Omega_0) \subset \mathbf{H}^{1+\theta}(\Omega_0)$ for any $\theta \in (0,1)$. Combining the theorems above together we have

Theorem 3.4. $\mathbf{H}^{2,2}_\beta(\Omega)$ *is imbedded into* $\mathbf{H}^{1+\theta}(\Omega)$ *with* $\theta = \min_{ij,m}(1 - \beta_m, 1 - \beta_{ij}) - \varepsilon$, $\varepsilon > 0$ *arbitrary.*

Corollary 3.4. $\mathbf{H}^{k,\ell}_\beta(\Omega)$ *is imbedded into* $\mathbf{H}^{\ell-1+\theta}(\Omega)$ *with* $k \geq \ell \geq 1$ *and* $\theta = \min_{ij,m}(1 - \beta_m, 1 - \beta_{ij}) - \varepsilon$, $\varepsilon > 0$ *arbitrary.* ■

3.2. Imbedding Of $\mathbf{H}^{k,\ell}_\beta(\Omega)$ Into The Space Of Continuous Functions.

The continuity is a very important property of the solution, and it is essential for approximability of the solution by numerical method. The regularity of the solution in terms of Sobolev space $\mathbf{H}^s(\Omega)$ implies the continuity in the three dimension if $s > \frac{3}{2}$. The solutions of elliptic problems on polyhedral domains belong to $\mathbf{H}^{k,2}_\beta(\Omega)$ but not necessarily to $\mathbf{H}^s(\Omega)$ with $s > \frac{3}{2}$. Therefore the imbedding results of $\mathbf{H}^{k,\ell}_\beta(\Omega)$, $k \geq \ell \geq 2$ into spaces of continuous function are of great importance to the regularity theory of the solution of elliptic problems in polyhedral domains and the approximation theory of numerical methods.

As in previous sections we address the imbeddings in each neighbourhoods of edges and vertex-edges and inner neighbourhoods of vertices, and we start with the imbedding of $\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})$ into $\mathbf{C}^0(\bar{\mathcal{U}}_{ij})$.

By $\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)$, $0 < \theta < 1$ we denote the completion of C^∞-functions with the norm

$$\|v\|^2_{\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)} = \int_{-\infty}^{\infty} \|v(\cdot, x_3)\|^2_{\mathbf{H}^{1+\theta}(\mathbb{R}^2)}\, dx_3 + \sum_{|\alpha|=1} \|D^\alpha v_{x_3}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} + \|v\|^2_{\mathbf{H}^1(\mathbb{R}^3)}.$$

Then we have the following lemma

Lemma 3.5. $\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)$ *is imbedded in* $\mathbf{C}^0(\mathbb{R}^3)$.

Proof. Since $\mathbf{C}_0^\infty(\mathbb{R}^3)$ is dense in $\mathbf{W}_{1+\theta,z}(\mathbb{R}^3)$ it is sufficient to show that for $v(x) \in \mathbf{C}_0^\infty(\mathbb{R}^3)$

$$\|v\|_{\mathbf{C}^0(\mathbb{R}^3)} \leq C\|v\|_{\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)}. \tag{3.23}$$

Let $V(\xi)$ denote the Fourier transform of $v(x)$, *i.e.*

$$V(\xi) = F(v) = \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} v(x) e^{-i\xi\cdot x}\, dx$$

and let $\rho^2 = \sum_{i=1}^3 \xi_i^2$, $r^2 = \sum_{i=1}^2 \xi_i^2$, and $\psi(\xi) = 1 + \rho^2 + r^{2(1+\theta)} + r^2\xi_3^2 + \xi_3^4$. Then

$$v(x) = \frac{1}{(\sqrt{2\pi})^3} \int_{\mathbb{R}^3} V(\xi) e^{-ix\cdot\xi}\, d\xi$$

and

$$\|v\|^2_{\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)} = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} |V(\xi)|^2 |\psi(\xi)|^2\, d\xi.$$

By Schwartz's inequality we have

$$|v(x)|^2 \leq C\|v\|^2_{\mathbf{W}_{1+\theta,2}(\mathbb{R}^3)} \int_{\mathbb{R}^3} \frac{1}{|\psi(\xi)|^2}\, d\xi.$$

Let B_1 be a ball centered at the origin with radius $= 1$ and $Q_1 = \{\xi \mid |\xi| \geq 1, |\xi_3| \geq ar\}$ and $S_1 = \{\xi \mid |\xi| \geq 1, |\xi_3| < ar\}$ with some $a > 1$. Then we have

$$\int_{B_1} \frac{1}{\psi(\xi)^2}\, dx \leq C$$

and

$$\int_{Q_1} \frac{1}{\psi^2}\, d\xi \leq C \int_1^\infty \frac{1}{\rho^2}\, d\rho \leq C.$$

For $\xi \in S_1$, $\psi^2 \geq \xi_3^4 + r^2\xi_3^2 + r^{2(1+\theta)} \geq \frac{1}{2a^2}(\xi_3^4 + r^2\xi_3^2 + r^{2\theta}\xi_3^2 + r^{2(1+\theta)})$, then

$$\begin{aligned}\int_{S_1} \frac{1}{\psi^2}\, d\xi &\leq C \int_{\frac{1}{\sqrt{1+a^2}}}^\infty \left(\int_0^a \frac{d\xi_3}{(\xi_3^2 + r^2)(\xi_3^2 + r^{2\theta})}\right) r\, dr \\ &\leq C \int_{\frac{1}{\sqrt{1+a^2}}}^\infty r^{-1-\theta}\, dr \leq C.\end{aligned}$$

This completes the proof. ∎

Theorem 3.5. $\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})$ *is imbedded in* $\mathbf{C}^0(\bar{\mathcal{U}}_{ij})$.

Proof. Let $\mathcal{U}_{ij} = Q_{\varepsilon_{ij}} \times I_{\delta_{ij}}$, and assume without losing generality that $I_{\delta_{ij}} = I = (0,1)$. By Fubini's Theorem $u \in \mathbf{H}^{2,2}_{\beta_{ij}}(Q_{\varepsilon_{ij}})$ for almost every $x_3 \in I$. According to the extension theorem of [3] we can extend u to a function in $\mathbf{H}^{2,2}_{\beta_{ij}}(\mathbb{R}^2)$ with compact support $Q \subset \mathbb{R}^2$, and

$$\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathbb{R}^2)} = \|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(Q)} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(Q_{\varepsilon_{ij}})} \tag{3.22}$$

which implies the extended function denoted by u again belongs to $\mathbf{H}^{2,2}_{\beta_{ij}}(\mathbb{R}^2 \times I)$ with preserving the norm. Using the technique of "finite-order reflection" (see *e.g.* [16,36]) we can extend the function u to $\mathbb{R}^2 \times (-1,1)$, then extend it to $\mathbb{R}^2 \times I^*$, $I^* = (-1,2)$ in the same way with preserving the norm. Set $v(x) = \varphi(x_3)u(x)$ with $\varphi(x_3)$ being a C^∞-function such that $\varphi(x_3) \equiv 1$ for $x_3 \in I$ and $\varphi(x_3) \equiv 0$ for $x_3 \notin I^*$. Then $v(x) \in \mathbf{H}^{2,2}_{\beta_{ij}}(\mathbb{R}^3)$ with support contained in $Q \times I^*$, and by (3.22)

$$\|v(x)\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathbb{R}^3)} = \|v(x)\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(Q\times I^*)} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}.$$

By Theorem 3.1 $v \in \mathbf{H}^{1+\theta}(Q \times I^*)$ with $\theta = 1 - \beta_{ij} - \varepsilon$, $\varepsilon > 0$ arbitrary, which implies $v \in W_{1+\theta,2}(\mathbb{R}^3)$, and

$$\|v\|_{W_{1+\theta,2}(\mathbb{R}^3)} \leq C\|u\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}.$$

By Lemma 3.5 $v \in \mathbf{C}^0(\mathbb{R}^3)$, and (3.23) holds. Note that $v(x) = u(x)$ for $x \in \mathcal{U}_{ij}$. Then the imbedding follows at once. ■

We next consider the imbedding of $\mathbf{H}^{2,2}_{\beta}(\mathcal{O}_m)$ into $\mathbf{C}^0(\bar{\mathcal{O}}_m)$. To this end we shall define an extension operator which continuously maps $\mathbf{H}^{2,2}_{\beta}(\mathcal{O}_m)$ into $\mathbf{H}^{2,2}_{\beta}(\mathbb{R}^3)$. The Stein extension may serve this purpose, hence we follow closely the notation and arguments of [43, Chapter 6, Section 3] in three-dimensional setting.

Let $\varphi(x_1,x_2)\colon \mathbb{R}^2 \to \mathbb{R}^1$ be function which satisfies the Lipschitz condition:

$$|\varphi(\tilde{x}) - \varphi(\tilde{x}^*)| < M|\tilde{x} - \tilde{x}^*|, \quad \text{for } \tilde{x} = (x_1,x_2) \in \mathbb{R}^2 \text{ and } \tilde{x}^* = (x_1^*,x_2^*) \in \mathbb{R}^2,$$

and let $D = \{x = (\tilde{x},x_3) \in \mathbb{R}^3, x_3 > \varphi(\tilde{x})\}$ be an open set in $\mathbb{R}^3$ which is called a *special Lipschitz domain.* We assume that $\varphi(0) = 0$. For $x \notin \bar{D}$, we let $\delta(x)$ denote the distance from x to D, and let $\Delta(x)$ be the regularized distance as constructed in [43, p. 171] such that

$$C_1\delta(x) \leq \Delta(x) \leq C_2\delta(x), \quad \text{for } x \in {}^c\bar{D} = \mathbb{R}^2|\bar{D} \tag{3.24}$$

where the positive constants C_1 and C_2 depend only on M, and $\Delta(x) \in C^\infty({}^c\bar{D})$ satisfying

$$|D^\alpha\Delta(x)| \leq B_\alpha|\delta(x)|^{1-|\alpha|} \tag{3.25}$$

with B_α independent of x.

For any $x_0 \in {}^c\bar{D}$ and a real number $K > m$, $\Gamma_{K,x_0} = \{x = (\tilde{x}, x_3) : x_3 < \varphi(\tilde{x}_0), |x_3 - \varphi(\tilde{x}_0)| \geq K|\tilde{x} - \tilde{x}_0|\}$ denote a lower cone with the vertex at $(\tilde{x}_0, \varphi(\tilde{x}_0))$. Then $\Gamma_{K,x_0} \cap \bar{D} = \{(\tilde{x}_0, \varphi(\tilde{x}_0)\}$, and for $x \in {}^c\bar{D}$

$$\delta(x) \geq (1 + K^{-2})^{-1/2}\big(\varphi(\tilde{x}) - x_3\big). \tag{3.26}$$

Hence due to (3.24)

$$\varphi(\tilde{x}) - x_3 \leq (1 + K^{-2})^{\frac{1}{2}}\delta(x) \leq \frac{1}{C_1}(1 + K^{-2})^{\frac{1}{2}}\Delta(x) = C_3\Delta(x).$$

According to [42] $C_1 = \frac{1}{4}$ and $C_3 = 5(1 + K^{-2})^{\frac{1}{2}}$. Let $\delta^*(x) = 2C_3\Delta(x)$. Then

$$\delta^*(\tilde{x}, x_3) \geq 2\big(\varphi(\tilde{x}) - x_3\big)$$

and

$$\delta^*(\tilde{x}, x_3) \geq 2|x_3|, \quad \text{if } \varphi(\tilde{x}) = 0. \tag{3.27}$$

Let ψ be a smooth function defined on $[1, \infty)$ which satisfies

$$\int_1^\infty \psi(x)\, d\lambda = 1, \int_1^\infty \psi(\lambda)\lambda^k\, d\lambda = 0, \quad k = 1, 2, \ldots. \tag{3.28}$$

Let $f(x)$ be defined on $\bar{D}$. We then define the extension Ef by $Ef(x) = f(x)$, $x \in \bar{D}$ and

$$Ef(x) = \int_1^\infty f\big(\tilde{x}, x_3 + \lambda\delta^*(x)\big)\psi(\lambda)\, d\lambda, \quad x \notin \bar{D}. \tag{3.29}$$

Let $\tilde{x}^0 \in \mathbb{R}^2$, and suppose that $\varphi(\tilde{x}^0) = 0$. Then for $x = (\tilde{x}^0, x_3) \in {}^c\bar{D}$, $x_3 < 0$ and due to (3.24) we have for $\lambda \geq 1$

$$x_3 + \lambda\delta^*(x) \geq x_3 + \delta^*(x) \geq x_3 + 2|x_3| = |x_3|, \tag{3.30}$$

and $\delta(x) = \operatorname{dist}(x, \bar{D}) \leq \operatorname{dist}\Big(x, \big(\tilde{x}^0, \varphi(\tilde{x}^0)\big)\Big) = \varphi(\tilde{x}^0) - x_3$. Then we have

$$\delta^*(x) = 2C_3\Delta(x) \leq 2C_3C_2\delta(x) \leq 2C_3C_2\big(\varphi(\tilde{x}^0) - x_3\big) \leq 2C_3C_2|x_3| = a|x_3| \tag{3.31}$$

with $a = 2C_2C_3$. Here we used the assumption that $\varphi(\tilde{x}^0) = 0$. Letting $s = y + \lambda\delta^*(x)$ and using the fact that $|\psi(\lambda)| \leq \frac{A}{\lambda^2}$ (see [43, p. 187]) we have for $x = (\tilde{x}^0, x_3) \in {}^c\bar{D}$ with $\varphi(\tilde{x}^0) = 0$

$$|Ef(\tilde{x}^0, x_3)| \leq A\int_{x_3+\lambda\delta^*(x)}^\infty |f(\tilde{x}^0, s)|\frac{\delta^*(x)}{(s - x_3)^2}\, ds \tag{3.32}$$

by (3.30) and (3.31)

$$\leq A^*|x_3| \int_{|x_3|}^{\infty} |f(\tilde{x}^0, s)| \frac{1}{s^2}\, ds$$

where $A^* = aA$. This estimate is an important property of the extended function $Ef(x)$, which will be used in the proof of Lemma 3.6 and 3.7.

Let $\mathbf{H}^{\ell,\ell}_{\beta_1}(D)$, $\beta_1 \in (0, \frac{1}{2})$, $0 \leq \ell \leq 2$ be the space of functions with the norm

$$\|u\|^2_{\mathbf{H}^{\ell,\ell}_{\beta_1}(D)} = \sum_{0 \leq |\alpha| \leq \ell} \|\Phi^{\alpha,\ell}_{\beta_1} D^\alpha u\|^2_{L^2(D)}$$

when the weight function is given as before

$$\Phi^{\alpha,\ell}_{\beta_1}(x) = \begin{cases} \rho(x)^{\beta_1}, & \text{for } |\alpha| = \ell, \\ 1 & \text{for } |\alpha| < \ell, \end{cases}$$

with $\rho(x) = (\sum_{1=i}^3 |x_i|^2)^{\frac{1}{2}}$.

__Lemma 3.6__. *E is bounded map: $\mathbf{H}^{\ell,\ell}_{\beta_1}(D) \to \mathbf{H}^{\ell,\ell}_{\beta_1}(\mathrm{I\!R}^3)$ for $0 \leq \ell \leq 2$.*

Proof. Let us fix $\tilde{x}^0 \in \mathrm{I\!R}^2$ and assume that $\varphi(\tilde{x}_0) = 0$. For $x = (\tilde{x}^0, x_3) \in {}^c\bar{D}$

$$\rho(\tilde{x}^0, x_3) = |\tilde{x}^0|^2 + |x_3|^2 \leq |\tilde{x}^0|^2 + s^2 = \rho(\tilde{x}^0, s) \quad \text{for } s > |x_3|, \text{ and}$$
$$|\rho^{\beta_1}(\tilde{x}^0, x_3) Ef(\tilde{x}^0, x_3)| \leq A^*|x_3| \int_{|x_3|}^{\infty} \rho^{\beta_1}(\tilde{x}^0, s)|f(\tilde{x}^0, s)| \frac{ds}{s^2}.$$

Therefore

$$\begin{aligned} \int_{-\infty}^{0} \rho^{2\beta_1}(\tilde{x}^0, x_3)|Ef(\tilde{x}^0, x_3)|^2\, dx_3 &\leq A^{*2} \int_{-\infty}^{0} \Big(\int_{|x_3|}^{\infty} \rho^{2\beta_1}(\tilde{x}_0, s) f(\tilde{x}^0, s) \frac{ds}{s^2} \Big)^2 |x_3|^2\, dx_3. \\ &= A^{*2} \int_{0}^{\infty} \Big(\int_{|x_3|}^{\infty} \rho^{2\beta_1}(\tilde{x}_0, s) f(\tilde{x}_0, s) \frac{ds}{s^2} \Big) |x_3|^2\, dx_3. \end{aligned} \tag{3.33}$$

Now we quote the Hardy's inequality from [43, p. 272]

$$\Big(\int_0^\infty \Big(\int_t^\infty F(s)\, ds \Big)^p t^{\gamma-1}\, dt \Big)^{\frac{1}{p}} \leq \frac{p}{\gamma} \Big(\int_0^\infty |sF(s)|^p s^{\gamma-1}\, ds \Big)^{\frac{1}{p}}. \tag{3.34}$$

Letting $F(s) = \rho^{\beta_1}(\tilde{x}^0, s) f(\tilde{x}^0, s)/s^2$, $p = 2$, $\gamma = 3$, and applying (3.34) to the righthand side of (3.33) we get

$$\begin{aligned} \int_{-\infty}^{0} \rho^{2\beta_1}(\tilde{x}^0, x_3)|Ef(\tilde{x}^0, x_3|\, dx_3 &\leq A^{*2} \int_0^\infty \Big(\frac{s\rho^{\beta_1}(\tilde{x}^0, s) f(\tilde{x}^0, s)}{s^2} \Big)^2 s^2\, ds \\ &\leq A^{*2} \int_0^\infty \rho^{2\beta_1}(\tilde{x}^0, s)|f(\tilde{x}^0, s)|^2\, ds. \end{aligned} \tag{3.35}$$

If $\varphi(\tilde{x}^0) \neq 0$ by a simple translation in x_3 we get

$$\int_{-\infty}^{\varphi(\tilde{x}^0)} \rho^{2\beta_1}(\tilde{x}^0, x_3)|Ef(\tilde{x}_0, x_3)|^2\, dx_3 \leq A^{*2} \int_{\varphi(\tilde{x}^0)}^{\infty} \rho^{2\beta_1}(\tilde{x}_0, s)|f(\tilde{x}_0, s)|^2\, ds \tag{3.36}$$

which implies

$$(3.37) \qquad \int_{-\infty}^{\infty} \rho^{2\beta_1}(\tilde{x}^0, x_3)|Ef(\tilde{x}_0, x_3)|^2\, dx_3 \leq A^{*2} \int_{\varphi(\tilde{x}^0)}^{\infty} \rho^{2\beta_1}(\tilde{x}_0, s)|f(\tilde{x}_0, s)^2\, ds.$$

Integrating both sides over $\tilde{x}^0 \in \mathbb{R}^2$ gives

$$\|\rho^{\beta_1} Ef(x)\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C\|\rho^{\beta_1} f(x)\|_{\mathbf{L}^2(D)}$$

which is the desired result for $\ell = 0$.

Now we turn to the case $\ell = 2$. It has been shown in [43] that E is a bounded map: $\mathbf{H}^1(D) \to \mathbf{H}^1(\mathbb{R}^3)$. It remains to estimate the second derivative of $Ef(x)$. Differentiating $Ef(x)$ for $x \in {}^c\bar{D}$ gives us

$$(3.38) \qquad \begin{aligned} \frac{\partial^2 Ef(x)}{\partial x_1^2} = &\int_1^{\infty} f_{x_1^2}(\tilde{x}, x_3 + \lambda\delta^*(x))\psi(x)\, d\lambda + \delta^*_{x_1}(x)\int_1^{\infty} f_{x_1 x_3}(\tilde{x}, x_3 + \lambda\delta^*(x))\lambda\psi(\lambda)\, d\lambda \\ &+ \delta^*_{x_1}(x)^2 \int_1^{\infty} f_{x_3^2}(\tilde{x}, x_3 + \lambda\delta^*(x))\lambda^2\psi(\lambda)\, d\lambda \\ &+ \delta^*_{x_1^2}(x)\int_1^{\infty} f_{x_3}(\tilde{x}, x_3 + \lambda\delta^*(x))\lambda\psi(\lambda)\, d\lambda. \end{aligned}$$

Let $\tilde{x}^0 \in \mathbb{R}^2$ be fixed and $\varphi(\tilde{x}^0) = 0$. Using the facts that $|\psi(\lambda)| \leq \frac{1}{\lambda^2}, \frac{A}{\lambda^3}, \frac{A}{\lambda^4}$ for $\lambda \geq 1$, and handling the first three terms of (3.38) in the same way for $Ef(x)$ before we have for $\alpha_1 + \alpha_3 = 2$

$$\left|\int_1^{\infty} f_{x_1^{\alpha_1} x_3^{\alpha_3}}(\tilde{x}^0, x_3 + \lambda\delta^*(x))\lambda^{\alpha_3}\psi(\lambda)\, d\lambda\right| \leq A|x_3| \int_{|x_3|}^{\infty} |f_{x_1^{\alpha_1} x_3^{\alpha_3}}(\tilde{x}^0, s)|\frac{ds}{s^2}.$$

For the fourth term of (3.38), using the orthogonality given in (3.28) and the techniques of [43] we have the estimate

$$\left|\int_1^{\infty} f_{x_3}(\tilde{x}^0, x_3 + x\delta(x))\lambda\psi(\lambda)\, d\lambda\right| \leq A|x_3|^2 \int_{|x_3|}^{\infty} |f_{x_3^2}(\tilde{x}^0, s)\frac{ds}{s^2}.$$

Due to (3.25) and (3.26) we have $|\delta^*_{x_1^2}(x)| \leq C|x_3|^{-1}$, which implies that

$$\left|\frac{\partial^2 Ef(x)}{\partial x_1^2}\right| \leq C|x_3| \sum_{\alpha_1+\alpha_3=2} \int_{|x_3|}^{\infty} |f_{x_1^{\alpha_1} x_3^{\alpha_3}}(\tilde{x}^0, s)|\frac{ds}{s^2}.$$

Then arguing as for (3.35)–(3.37) we obtain

$$\left\|\rho^{\beta_1}\frac{\partial^2 Ef(x)}{\partial x_1^2}\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C\|\rho^{\beta_1} D^2 f\|_{\mathbf{L}^2(D)}.$$

Carrying out the arguments above for general terms $D^{\alpha}Ef(x)$ with $|\alpha| = 2$ completes the proof for $\ell = 2$. The theorem for $\ell = 1$ can be proved in similar way. Thus the desired extension is established. ∎

Now we consider another weighted Sobolev space $\mathbf{H}^{\ell,\ell}_{\beta_1,\beta_2}(D)$ over D, $0 \le \ell \le 2$, $\beta_1 \in (0, \frac{1}{2})$ and $\beta_2 \in (0,1)$, with the norm

$$\|u\|_{\mathbf{H}^{\ell,\ell}_{\beta_1,\beta_2}}(D) = \Big(\sum_{0\le|\alpha|\le\ell} \|\Phi^{\alpha,\ell}_{\beta_1,\beta_2} D^\alpha u\|^2_{\mathbf{L}^2(D)} \Big)^{\frac{1}{2}}$$

where

$$\Phi^{\alpha,\ell}_{\beta_1,\beta_2}(x) = \begin{cases} \rho^{\beta_1}(\sin\phi)^{\beta_2}, & \text{for } \alpha_1+\alpha_2 = \ell \\ \rho^{\beta_1} & \text{for } \alpha_1+\alpha_2 < \ell \le |\alpha| \\ 1 & \text{for } |\alpha| < \ell \end{cases}$$

with $\rho = \rho(x) = (\sum_{i=1}^3 x_i^2)^{\frac{1}{2}}$ and $\sin\phi = \sin\phi(x) = (x_1^2 + x_2^2)^{\frac{1}{2}}/\rho$.

Lemma 3.7. *Assume that $\phi(x) \le \sigma_0 < \frac{\pi}{2}$. Then $E: \mathbf{H}^{\ell,\ell}_{\beta_1,\beta_2}(D) \to \mathbf{H}^{\ell,\ell}_{\beta_1,\beta_2}(\mathbb{R}^3)$ is a bounded map for $0 \le \ell \le 2$.*

Proof. Let $\tilde{x}^0 \in \mathbb{R}^2$ be fixed and $\varphi(\tilde{x}^0) = 0$. If $\beta_i \ge \beta_2$, then for $s > |x_3|$ we have for $x = (\tilde{x}^0, x_3) \in {}^c\bar{D}$

$$\rho^{\beta_1}(x)\big(\sin\phi(x)\big)^{\beta_2} = \rho^{\beta_1-\beta_2}(x)|\tilde{x}^0|^{\beta_2} \le \rho^{\beta_1-\beta_2}(\tilde{x}^0,s)|\tilde{x}^0|^{\beta_2}. \tag{3.39}$$

Therefore we have by (3.32)

$$\big|\rho^{\beta_1}(x)\big(\sin\phi(x)\big)^{\beta_2} Ef(x)\big| \le A^*|x_3||\tilde{x}^0|^{\beta_2} \int_{|x_3|}^\infty \rho^{\beta_1-\beta_2}(\tilde{x}^0,s)|f(\tilde{x}^0,s)|\frac{ds}{s^2}$$

Applying the Hardy's inequality (3.34) and arguing as before we get

$$\begin{aligned}\int_{-\infty}^0 \rho^{2\beta_1}(\sin\phi)^{2\beta_2}|Ef(\tilde{x}^0,x_3)|^2\,dx_3 &\le C\int_0^\infty \rho^{2(\beta_1-\beta_2)}(\tilde{x}^0,s)|\tilde{x}_0|^{2\beta_2}|f(\tilde{x}^0,s)|^2\,ds \\ &= C\int_0^\infty \rho^{2\beta_1}(\tilde{x}^0,s)\big(\sin\phi(\tilde{x}^0,s)\big)^{2\beta_2}|f(\tilde{x}^0,s)|^2\,ds.\end{aligned}$$

If $\beta_1 < \beta_2$, then for $x = (\tilde{x}^0, x_3) \in {}^c\bar{D}$ we get

$$\begin{aligned}\rho^{\beta_1}(x)\big(\sin\phi(x)\big)^{\beta_2}|Ef(x)| = \rho^{\beta_1-\beta_2}(x)|\tilde{x}^0|^{\beta_2}|Ef(x)| &\le |x_3|^{\beta_1-\beta_2}|\tilde{x}^0|^{\beta_2}|Ef(x)| \\ &\le C|x_3|^{\beta_1-\beta_2+1}|\tilde{x}^0|^{\beta_2}\int_{|x_3|}^\infty |f(\tilde{x}_0,s)|\frac{ds}{s^2}.\end{aligned}$$

Applying the Hardy's inequality (3.38) and arguing as before we have

$$\begin{aligned}\int_{-\infty}^0 \rho^{2\beta_1}(x)\big(\sin\phi(x)\big)^{2\beta_2}|Ef(x)|^2\,dx_3 &\le C|\tilde{x}^0|^{2\beta_2}\int_0^\infty \Big(\frac{sf(\tilde{x}^0,s)}{s^2}\Big)^2 s^{2(1+\beta_1-\beta_2)}\,ds \\ &= C|\tilde{x}^0|^{2\beta_2}\int_0^\infty s^{2(\beta_1-\beta_2)}|f(\tilde{x}^0,s)|^2\,ds \\ &\le C\int_0^\infty \rho^{2\beta_1}(\tilde{x}^0,s)\Big(\frac{|\tilde{x}^0|}{\rho(\tilde{x}_0,s)}\Big)^{2\beta_2}\Big(\frac{\rho(\tilde{x}_0,s)}{s}\Big)^{2\beta_2}|f(\tilde{x}^0,s)|^2\,ds \\ &\le C\int_0^\infty \rho^{2\beta_1}(\tilde{x}^0,s)\big(\sin\phi(\tilde{x}_0,s)\big)^{2\beta_2}|f(\tilde{x}^0,s)|^2\,ds.\end{aligned}$$

Here we used the assumption that $\phi(x) \le \sigma_0 < \frac{\pi}{2}$ for $x \in \bar{D}$. Arguing as for (3.37) we get for $\varphi(\tilde{x}^0) \ne 0$

$$\int_{-\infty}^{\infty} \rho^{2\beta_1}(\sin\phi)^{2\beta_2}|Ef(\tilde{x}^0, x_3)|^2\, dx_3 \le C \int_{\varphi(\tilde{x}^0)}^{\infty} \rho^{2\beta_1}(\sin\phi)^{2\beta_2}|f(\tilde{x}^0, s)|\, ds.$$

Then integrating both sides above over $\tilde{x}^0 \in \mathbb{R}^2$ gives the theorem for $\ell = 0$.

The argument used above for the second derivatives of $Ef(x)$ in the proof of Lemma 3.6 can be carried out here for $\ell = 1, 2$. Hence the lemma is completed. ∎

Let us remark the assumption that $\phi(x) \le \sigma_0 < \frac{\pi}{2}$. First of all, the assumption holds in most practical application. Secondly we can drop this assumption by modifying the weight function

$$\tilde{\Phi}^{\alpha,\ell}_{\beta_1,\beta} = \begin{cases} \Phi^{\alpha,\ell}_{\beta_1,\beta_2}(x) & \text{for } x \in D \text{ with } \phi(x) \le \sigma_0 < \frac{\pi}{2} \\ 1 & \text{for } \phi(x) > \sigma_0. \end{cases}$$

Then all arguments in the proof of Lemma 3.7 can be carried out, and it will be sufficient for the imbedding of $\mathbf{H}^{2,2}_{\beta}(\mathcal{O}_m)$ into $\mathbf{C}^0(\bar{\mathcal{O}}_m)$.

It is worth indicating that the extension operator E defined in (3.29) is a bounded map from $\mathbf{H}^{k,\ell}_{\beta_1}(D)$ to $\mathbf{H}^{k,\ell}_{\beta_1}(\mathbb{R}^3)$, and from $\mathbf{H}^{k,\ell}_{\beta_1,\beta_2}(D)$ to $\mathbf{H}^{k,\ell}_{\beta_1,\beta_2}(\mathbb{R}^3)$ for any $k \ge \ell \ge 0$. But we will not elaborate it further here because it does not serve our goal of establishing desired imbeddings of $\mathbf{H}^{k,\ell}_{\beta}(\mathcal{O}_m)$ into $\mathbf{C}^{\ell-2}(\bar{\mathcal{O}}_m)$, $\ell \ge 2$.

We now consider an infinite polyhedron $\mathcal{O}$ which coincides with polyhedral domain Ω in a neighbourhood $\mathcal{O}_m$ of the vertex A_m. We assume that A_m is the origin and one edge Λ_{ij} of $\mathcal{O}_m$ is on the positive x_3-axis. Let $\mathcal{V}_{\sigma_{ij},\infty} = S_{\sigma_{ij}} \times \mathbb{R}^1$ and $\tilde{\mathcal{O}}_\infty = \mathcal{O} \setminus \bigcup_{ij \in \mathcal{L}_m} \bar{\mathcal{V}}_{\sigma_{ij},\infty}$. Both $\mathcal{V}_{\sigma_{ij},\infty}$ and $\tilde{\mathcal{O}}_\infty$ are the special Lipschitz domains.

<u>**Lemma 3.8**</u>. *There is a bounded map* $E: \mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m) \to \mathbf{H}^{2,2}_{\beta_m}(\mathbb{R}^3)$.

Proof. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be an open covering of $\tilde{\mathcal{O}}_m$, and let $\{\phi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to this covering. Each function $u_i = \phi_i u \in \mathbf{H}^{2,2}_{\beta_m}(\mathcal{O}_\infty)$ with compact support contained in $\mathcal{O}_m$. Applying Lemma 3.6 to u_i we get an extension $Eu_i \in \mathbf{H}^{2,2}_{\beta_m}(\mathbb{R}^3)$ with preserving the norm. Set $Eu = \sum_{i=1}^{\infty} Eu_i$. Then $Eu \in \mathbf{H}^{2,2}_{\beta_m}(\mathbb{R}^3)$, and

$$\|Eu\|_{\mathbf{H}^{2,2}_{\beta_m}(\mathbb{R}^3)} \le C\|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}$$

Since $Eu(x) = u(x)$ for $x \in \tilde{\mathcal{O}}_m$, we complete the proof. ∎

<u>**Lemma 3.9**</u>. *There is a bounded map* $E: \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) \to \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathbb{R}^3)$.

Proof. The proof is the same as that of the previous lemma, except that Lemma 3.7 is used instead of Lemma 3.6.

We now are ready to establish the imbedding results of $\mathbf{H}^{2,2}_{\beta}(\mathcal{O}_m)$ into $\mathbf{C}^0(\bar{\mathcal{O}}_m)$.

Theorem 3.6. *For $\beta_m \in (0, \frac{1}{2})$, $\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ is imbedded into $\mathbf{C}^0(\overline{\tilde{\mathcal{O}}}_m)$.*

Proof. By Lemma 3.6 we extend u to $\mathbb{R}^3$ preserving the weighted Sobolev norm. Let $\varphi(\rho)$ be a C^∞-function such that $\varphi \equiv 1$ for $\rho < \rho_0/2$ and $\varphi \equiv 0$ for $\rho > \rho_0$ with $\rho_0 > 2\delta_m$. Set $v = \varphi(\rho)u$ and $f = \Delta v$. Then v and f have a compact support $B_{\rho_0} = \{x \mid |x| < \rho_0\}$. Define

$$V(x) = \frac{1}{2\pi}\int_{B_{\rho_0}} f(y)\frac{1}{|x-y|}\,dy.$$

Using the fact that $\beta_m < \frac{1}{2}$ we get by Schwartz's inequality

$$|V(x)| \le C\Big(\int_{B_{\rho_0}} |f(y)|^2\rho^{2\beta_m}\Big)^{\frac{1}{2}}\Big(\int_{B_{\rho_0}} \rho^{-2\beta_m}\frac{1}{|x-y|^2}\,dy\Big)^{\frac{1}{2}}$$
$$\le C\|v\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)} \le C\|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}.$$

We obviously have $\Delta V = f = \Delta v$, and it is easily seen that $V(x) \to 0$ as $|x| \to \infty$. Because $v(x)$ has compact support, the standard uniqueness argument gives us that $V = v$. Therefore

$$\|v\|_{\mathbf{C}^0(\bar{\mathcal{O}}_m)} \le C\|u\|_{\mathbf{H}^{2,2}_{\beta_m}(\bar{\mathcal{O}}_m)}.$$

Since $v(x) = u(x)$ for $x \in \tilde{\mathcal{O}}_m$, the theorem is proved. ■

Corollary 3.6. *$\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$ is imbedded into $\mathbf{C}^{\ell-2}(\overline{\tilde{\mathcal{O}}}_m)$ for $k \ge \ell \ge 2$.*

Theorem 3.7. *$\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ with $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$ is imbedded into $\mathbf{C}^0(\bar{\mathcal{V}}_{m,ij})$.*

Proof. The proof is the same as that for the previous theorem except that the estimate

$$|V(x)| \le \Big(\int_{B_{\rho_0}} |f(y)|^2|\rho^{2\beta_m}(\sin\phi)^{2\beta_{ij}}\Big)^{\frac{1}{2}}\Big(\int_{B_{\rho_0}} \rho^{-2\beta_m}(\sin\phi)^{-2\beta_{ij}}\frac{1}{|x-y|^2}\,dy\Big)^{\frac{1}{2}}$$
$$\le C\|v\|_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}$$

is used. Here we used the fact that $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$. ■

Corollary 3.7. *$\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ is imbedded into $\mathbf{C}^{\ell-2}(\bar{\mathcal{V}}_{m,ij})$ for $k \ge \ell \ge 2$.*

Combining Theorem 3.5–3.7 and Corollary 3.5–3.7 we have

Theorem 3.8. *Let $\beta_m \in (0, \frac{1}{2})$ for $m \in \mathcal{M}$ and $\beta_{ij} \in (0,1)$ for $ij \in \mathcal{L}$. Then $\mathbf{H}^{2,2}_{\beta}(\Omega)$ is imbedded into $\mathbf{C}^0(\bar{\Omega})$.* ■

Corollary 3.8. *$\mathbf{H}^{k,\ell}_{\beta}(\Omega)$ is imbedded into $\mathbf{C}^{\ell-2}(\bar{\Omega})$ for $k \ge \ell \ge 2$, $\beta_m \in (0, \frac{1}{2})$, $m \in \mathcal{M}$ and $\beta_{ij} \in (0,1)$, $ij \in \mathcal{L}$.*

4. WEIGHTED SOBOLEV SPACES AND COUNTABLE NORMED SPACES IN CYLINDRICAL AND SPHERICAL COORDINATES

The weighted Sobolev spaces and countably normed spaces in Cartesian coordinates are defined in Section 2 which are of great significance to the theory of regularities of solutions of elliptic problem on polyhedral domain and to the applications of numerical analysis (see [8,9,21]). Due to the natures of singularity in various neighbourhoods of edges and vertices it would be much easier for investigation if we use these spaces in cylindrical coordinates on neighbourhoods of edges and in spherical coordinates on neighbourhoods of vertices. Hence the relations between these spaces in Cartesian coordinates and in cylindrical and spherical coordinates are extremely important.

The weighted Sobolev spaces and countably normed spaces in cylindrical and spherical coordinates will be defined in this section and the relations between these spaces and those in Cartesian coordinates will be the focal points.

4.1. Weighted Sobolev Spaces And Countably Normed Spaces Over Neighbourhoods Of Edges In Cylindrical Coordinates

Let $\mathcal{U}_{ij}$ denote a neighbourhood of the edge Λ_{ij} as in previous sections, namely, $\Lambda_{ij} = \{x = (0,0,x_3) \mid a+\delta_{ij} < x_3 < b-\delta_{ij}\}$ lies on the x_3-axis. $x = (x_1,x_2,x_3)$ and $x = (r,\theta,x_3)$ are the Cartesian and cylindrical coordinates for $x \in \mathcal{U}_{ij}$ with respect to the edge Λ_{ij}. We write $\mathcal{D}^\alpha u = u_{r^{\alpha_1}\theta^{\alpha_2}x_3^{\alpha_3}}$ and

$$|\mathcal{D}^k u|^2 = \sum_{|\alpha|=k} |r^{-\alpha_2}\mathcal{D}^\alpha u|^2$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha' = (\alpha_1,\alpha_2)$ and $|\alpha'| = \alpha_1+\alpha_2$, $|\alpha| = |\alpha'|+\alpha_3$.

By $\Phi^{\alpha,\ell}_{\beta_{ij}}(x)$ with $\beta_{ij} \in (0,1)$ and integer $\ell \geq 0$ we denote the same weight function defined in Section 2. Then the weighted Sobolev space $\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ with integers $k \geq \ell$ is defined as

$$\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \left\{u \mid \|u\|^2_{\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} = \sum_{|\alpha|\leq k} \|\Phi^{\alpha,\ell}_{\beta_{ij}} r^{-\alpha_2}\mathcal{D}^\alpha u\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} < \infty\right\}$$

and the countably normed space $\mathcal{B}^\ell_{\beta_{ij}}(\mathcal{U}_{ij})$, $\ell \geq 0$ is introduced as

$$\mathcal{B}^\ell_{\beta_{ij}}(\mathcal{U}_{ij}) = \{u \mid u \in \mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \text{ and } \|\Phi^{\alpha,\ell}_{\beta_{ij}} r^{-\alpha_2}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq Cd^\alpha\alpha! \text{ for any } k \geq \ell,\}.$$

<u>**Theorem 4.1**</u>. *For $0 \leq \ell \leq 2$ and $k \geq \ell$, $\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ is equivalent to $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$. Moreover, if for $\tilde{\alpha}$ with $|\tilde{\alpha}| \leq k$*

$$\|\Phi^{\tilde{\alpha},\ell}_{\beta_{ij}} r^{-\alpha_2}\mathcal{D}^{\tilde{\alpha}} u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq \tilde{C}\tilde{d}^{\tilde{\alpha}}\tilde{\alpha}!, \tag{4.1}$$

then for $|\alpha| = k$

$$\|\Phi^{\alpha,\ell}_{\beta_{ij}}, D^\alpha u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq Cd^\alpha\alpha!, \tag{4.2}$$

vice versa, (4.1) stands for $|\tilde{\alpha}| = k$ if (4.2) holds for $|\alpha| \leq k$.

Proof. For the proof is analogous to those in two dimensions, we refer the reader to [4]. ■

As a consequence of (4.1) and (4.2) we have

Corollary 4.1. *For $0 \le \ell \le 2$, $\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ are $\mathcal{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ are equivalent.* ∎

Remark 4.1. For any $k \ge \ell \ge 0$, it is always true that $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \subset \mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, and $\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \subset \mathcal{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$. ∎

4.2. Weighted Sobolev Spaces And Countably Normed Spaces Over Neighbourhoods Of Vertex-edges In Spherical Coordinates

Let $\mathcal{V}_{m,ij}$ be a neighbourhood of the vertex A_m and edge Λ_{ij}, and we assume as usual that A_m is at the origin and Λ_{ij} lies on the positive x_3-axis. $x = (x_1, x_2, x_3)$ and $x = (\phi, \theta, \rho)$ denote the Cartesian and spherical coordinate of $x \in \mathcal{V}_{m,ij}$ with respect to A_m and Λ_{ij}. We write $\mathcal{D}^{\alpha}u = u_{\phi^{\alpha_1}\theta^{\alpha_2}\rho^{\alpha_3}}$ and

$$|\mathcal{D}^k u|^2 = \sum_{|\alpha|=k} |\rho^{-|\alpha'|}(\sin\phi)^{-\alpha_2}\mathcal{D}^{\alpha}u|^2$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha' = (\alpha_1, \alpha_2)$ are the same as before.

The weight function $\Phi^{\alpha,\ell}_{\beta_{mij}}(x)$ with $\beta_{m,ij} = (\beta_m, \beta_{ij})$, $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$, is defined by (2.2). Then the weighted Sobolev space $\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ with integer $k \ge \ell$ and the countably normed space $\mathcal{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ with $\ell \ge 0$ are introduced as

$$\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) = \Big\{u \mid \|u\|^2_{\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} = \sum_{|\alpha|\le k} \|\Phi^{\alpha,\ell}_{\beta_{m,ij}}(x)\rho^{-|\alpha'|}(\sin\phi)^{-\alpha_2}\mathcal{D}^{\alpha}u\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})} < \infty\Big\}$$

and

$$\begin{aligned}&\mathcal{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})\\ &\quad = \{u \mid u \in \mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) \text{ for any } k \ge \ell, \text{ and } \|\Phi^{\alpha,\ell}_{\beta_{m,ij}}\rho^{-|\alpha'|}(\sin\phi)^{-\alpha_2}\mathcal{D}^{\alpha}u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le Cd^{\alpha}d!\}.\end{aligned}$$

The following lemmas are essential to establishing desired imbedding theorems.

Lemma 4.1. *Let $\mathcal{V}_{\sigma,\delta} = S_{\sigma} \times I_{\delta}$ with $S_{\sigma} = \{(\phi,\theta) \mid \sigma_0 < \phi < \sigma, 0 < \theta < \omega\}$ and $I_{\delta} = (0,\delta)$. Then for $\sigma_0 \ge 0$ and $\beta_m \in (0, \frac{1}{2})$*

$$\|\rho^{\beta_m - 1}u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} \le C\Big\{\sum_{|\alpha|=1} \|\rho^{\beta_m+\alpha_3-1}\mathcal{D}^{\alpha}u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} + \|u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})}\Big\} \tag{4.3}$$

and for $\ell = 0, 1$

$$\|\rho^{\beta_m-1}D^{\ell}u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} \le C\Big\{\|\rho^{\beta_m}D^{\ell+1}u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} + \|D^{\ell}u\|^2_{\mathbf{H}^1(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})}\Big\} \tag{4.4}$$

provided the right hand sides are finite.

Proof. Let

$$\bar{u}(\rho) = \frac{1}{|S_{\sigma}|}\int_{S_{\sigma}} u(\phi,\theta,\rho)\,ds.$$

It is easily seen that

$$\frac{d\bar{u}}{d\rho} = \frac{1}{|S_\sigma|}\int_{S_0} u_\rho(\phi,\theta,\rho)\,ds$$

and

$$\int_0^\delta \rho^{2(\beta_m+1)}\Big|\frac{d\bar{u}}{d\rho}\Big|^2\,d\rho \le C\int_{\mathcal{V}_{\sigma,\delta}} \rho^{2\beta_1}|u_\rho|^2\,dx = \|\rho^{\beta_m}u_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}.$$

Noting that $0<\beta_m<\frac{1}{2}$, we have by Lemma A.1 of [4]

$$\begin{aligned}\int_0^\delta \rho^{2\beta_m}|\bar{u}(\rho)-a|^2\,d\rho &\le C\int_0^\delta \rho^{2\beta_m+2}\Big|\frac{d\bar{u}}{d\rho}\Big|^2\,d\rho\\ &\le C\|\rho^{\beta_m}u_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}.\end{aligned}$$

where $a=\bar{u}(1)$, and by the imbedding theorem of the Sobolev space(see[1])

$$|a|^2 \le C\int_{\delta/2}^\delta \Big(\Big|\frac{d\bar{u}}{d\rho}\Big|^2+|\bar{u}|^2\Big)\,d\rho \le c(\|u_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})}+\|u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})})$$

which leads to

$$\int_0^\delta \rho^{2\beta_m}|\bar{u}|^2\,d\rho \le C(\|\rho^{\beta_m}u_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}+\|u\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})})$$

and

$$\|\rho^{\beta_m-1}\bar{u}\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} \le C((\|\rho^{\beta_m}u_\rho\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}+\|u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,,\delta}\setminus\mathcal{V}_{\sigma,\delta/2})}). \tag{4.5}$$

Further for almost every ρ we get

$$\begin{aligned}u(\phi,\theta,\rho)-\bar{u}(\rho) &= \frac{1}{|S_\sigma|}\int_{S_\sigma}\big(u(\phi,\theta,\rho)-u(\tilde{\phi},\tilde{\theta},\rho)\big)\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}\\ &= \frac{1}{|S_\sigma|}\int_{s_\sigma}\Big\{\int_{\tilde{\phi}}^{\phi}\frac{\partial u}{\partial\phi}(\hat{\phi},\theta,\rho)\,d\hat{\phi}+\int_{\tilde{\theta}}^{\theta}\frac{\partial u}{\partial\theta}(\tilde{\phi},\hat{\theta},\rho)\,d\hat{\theta}\Big\}\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}.\end{aligned}$$

Further we have

$$\begin{aligned}\int_{S_\sigma}\Big|\int_{\tilde{\theta}}^{\theta}\frac{\partial u}{\partial\theta}(\tilde{\phi},\hat{\theta},\rho)\,d\hat{\theta}\Big|\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta} &\le C\int_{S_\sigma}\Big(\int_0^\omega\Big|\frac{\partial u}{\partial\theta}(\tilde{\phi},\hat{\theta},\rho)\Big|\,d\hat{\theta}\Big)\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}\\ &\le C\Big(\int_{S_\sigma}\Big|\frac{\partial u}{\partial\theta}\Big|^2\,dS\Big)^{\frac{1}{2}},\end{aligned} \tag{4.6}$$

and by Schwartz's inequality

$$\begin{aligned}&\int_{S_\sigma}\Big|\int_{\tilde{\phi}}^{\phi}\frac{\partial u}{\partial\phi}(\hat{\phi},\theta,\rho)\,d\hat{\phi}\Big|\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}\\ &\le \int_{S_\sigma}\Big\{\Big(\int_{\tilde{\phi}}^{\phi}\Big|\frac{\partial u}{\partial\phi}\phi(\hat{\phi},\theta,\rho)\Big|^2\sin\hat{\phi}\,d\hat{\phi}\Big)^{\frac{1}{2}}\Big|\int_{\tilde{\phi}}^{\phi}\frac{1}{\sin\hat{\phi}}\,d\hat{\phi}\Big|^{\frac{1}{2}}\Big\}\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}\\ &\le \Big(\int_{\sigma_0}^{\sigma}\Big|\frac{\partial u}{\partial\phi}(\hat{\phi},\theta,\rho)\Big|^2\sin\hat{\phi}\,d\hat{\phi}\Big)^{\frac{1}{2}}\int_{S_\sigma}\Big|\int_{\tilde{\phi}}^{\phi}\frac{1}{\sin\hat{\phi}}\,d\hat{\phi}\Big|^{\frac{1}{2}}\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta}.\end{aligned} \tag{4.7}$$

Note that

$$\int_{S_\sigma}\Big|\int_{\tilde\phi}^{\phi}\frac{1}{\sin\hat\phi}\,d\hat\phi\Big|^{\frac12}\sin\tilde\phi\,d\tilde\phi\,d\tilde\theta \le C_1\int_{\sigma_0}^{\sigma}|\ln\phi-\ln\tilde\phi|^{\frac12}\sin\tilde\phi\,d\tilde\phi \tag{4.8}$$
$$\le C_1(|\ln\phi|+C_2)^{\frac12}.$$

Substituting (4.8) into (4.7) we obtain

$$\int_{S_\sigma}\Big|\int_{\tilde\phi}^{\phi}\frac{\partial u}{\partial\phi}(\hat\phi,\theta,\rho)\,d\hat\phi\Big|\sin\tilde\phi\,d\tilde\phi\,d\tilde\theta \le C_1(|\ln\phi|+C_2)^{\frac12}\Big(\int_{\sigma_0}^{\sigma}\Big|\frac{\partial u}{\partial\phi}(\hat\phi,\theta,\rho)\Big|^2\sin\hat\phi\,d\hat\phi\Big)^{\frac12}. \tag{4.9}$$

The combination of (4.6) and (4.9) leads to

$$\begin{aligned}\int_{S_\sigma}|u(\phi,\theta,\rho)-\bar u(\rho)|^2\,dS &\le C\Big\{\int_{S_\sigma}\Big|\frac{\partial u}{\partial\theta}\Big|^2 dS+\int_{S_\sigma}C_1(|\ln\phi|+C_2)\Big(\int_{\sigma_0}^{\sigma}\Big|\frac{\partial u}{\partial\phi}(\hat\phi,\theta,\rho)\Big|^2\sin\hat\phi\,d\hat\theta\Big)\,dS\Big\}\\ &\le C\Big\{\int_{S_\sigma}\Big|\frac{\partial u}{\partial\theta}\Big|^2 dS+\int_0^{\omega_{ij}}\Big[\Big(\int_{\sigma_0}^{\sigma}(C_1|\ln\phi|+C_2)\sin\phi\,d\phi\Big)\\ &\qquad\cdot\Big(\int_{\sigma_0}^{\sigma}\Big|\frac{\partial u}{\partial\phi}(\hat\phi,\theta,\rho)\Big|^2\sin\hat\phi\,d\hat\phi\Big)\Big]\,d\theta\Big\}\\ &\le C\int_{S_\sigma}\Big(\Big|\frac{\partial u}{\partial\theta}\Big|^2+\Big|\frac{\partial u}{\partial\phi}\Big|^2\Big)\,dS.\end{aligned} \tag{4.10}$$

Therefore

$$\|\rho^{\beta_m-1}(u-\bar u)\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2\beta_m}\Big(\frac{1}{\rho^2}|u_\phi|^2+\frac{1}{\rho^2}|u_\theta|^2\Big)\,dx. \tag{4.11}$$

Combining (4.5) and (4.11) we obtain (4.3).

Since $|\mathcal{D}^1u|^2=|D^1u|^2$ we get (4.4) for $\ell=0$ from (4.3). Setting $v=D^\alpha u$ with $|\alpha|=1$, and applying (4.3) to v we obtain (4.4) for $\ell=1$. ∎

Lemma 4.2. *Let $\mathcal{V}_{\sigma,\delta}=S_\sigma\times I_\delta$ be the same as that in Lemma 4.2 with $\sigma_0=0$, and let $\beta_{\bar m,ij}=(\beta_m,\beta_{ij})$ with $\beta_m\in(0,\frac12)$ and $\beta_{ij}\in(0,1)$.*

(i) If $u\in\mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})$, then

$$\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1}u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\|u\|_{\mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})}=C\|u\|_{\mathcal{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})}. \tag{4.12}$$

(ii) If $u\in\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})$, then

$$\big\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-2}\big(u-u(0,0,x_3)\big)\big\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\|u\|_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})}. \tag{4.13}$$

(iii) If $u\in\mathbf{H}^{\ell,\ell}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})$ with $\ell\ge2$, and if $D^\alpha u$ vanishes along the edge Λ_{ij} for all α with $0\le|\alpha|\le\ell-2$, then

$$\|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}-\ell}u\|_{\mathbf{L}^2(\mathcal{V}_\sigma)}\le C\|u\|_{\mathbf{H}^{\ell,\ell}_{\beta_{m,ij}}(\mathcal{V}_{\sigma,\delta})}. \tag{4.14}$$

Proof. By Lemma A.2 of [4] we have for almost every $\rho \in I_\delta$

$$(4.15)\quad \int_{S_\sigma} (\sin\phi)^{2(\beta_{ij}-1)}|u|^2\,d\xi \le C\Big\{\int_{S_\sigma} (\sin\phi)^{2\beta_{ij}}\Big(|u_\phi|^2+\frac{1}{\sin^2\phi}|u_\theta|^2\Big)\,dS+\int_{S_\sigma\setminus S_{\sigma/2}}|u|^2\,dS\Big\}$$

where C is a constant independent of ρ. Multiplying (4.15) with $\rho^{2\beta_m}$ and integrating over ρ, we get

$$(4.16)\quad \begin{aligned}&\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1}u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\\ &\qquad\le C\Big\{\sum_{|\alpha\|=1}\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}+\alpha_1-1}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}+\|\rho^{\beta_m-1}\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma/2,\delta})}\Big\}.\end{aligned}$$

Applying Lemma 4.1 with $\sigma_0=\sigma/2$ we get

$$\begin{aligned}\|\rho^{\beta_m-1}u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma/2,\delta})}&\le C\Big\{\sum_{|\alpha|=1}\|\rho^{\beta_m+\alpha_3-1}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma/2,\delta})}+\|u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta}\setminus\mathcal{V}_{\sigma/2,\delta}\setminus\mathcal{V}_{\sigma/2,\delta/2})}\Big\}\\ &\le C\|u\|_{\mathcal{H}^{1,1}_{\beta_m,ij}(\mathcal{V}_{\sigma,\delta})}\end{aligned}$$

which together with (4.16) leads to (4.12)

Furthermore, by the arguments of Lemma A.1 and A.2 of [4] it can be shown that for almost every $\rho \in I_\delta$

$$\int_{s_\sigma}(\sin\phi)^{2(\beta_{ij}-2)}|u-a|^2\,ds \le C\int_{s_\sigma}(\sin\phi)^{2(\beta_{ij}-1)}\Big(|u_\phi|^2+\frac{1}{\sin^2\phi}|u_\theta|^2\Big)^2\,ds$$

where $a=u(0,\theta,\rho)=u(0,0,x_3)$. This implies that

$$\begin{aligned}\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2(\beta_m-2)}(\sin\phi)^{2(\beta_{ij}-2)}|u-a|^2\,dx &\le C\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2(\beta_m-2)}(\sin\phi)^{2(\beta_{ij}-1)}\Big(|u_\phi|^2+\frac{1}{\sin^2\phi}|u_\theta|^2\Big)\,dx\\ &\le C\int_{\mathcal{V}_{\sigma,\sigma}}\rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^1u|^2\,dx.\end{aligned}$$

Applying (4.12) to $D^\alpha u$ with $|\alpha|=1$ we obtain (4.13) immediately.

Now let $u\in\mathbf{H}^{k,\ell}_{\beta_m,ij}(\mathcal{V}_{m,ij})$ with $\ell\ge 2$ such that $D^\alpha u(0,0,x_3)=0$ for $x_3\in I_\delta$ and $|\alpha|\le l-2$. Then the arguments above can be carried out for $\ell\ge 2$, namely

$$\int_{s_\sigma}(\sin\phi)^{2(\beta_{ij}-\ell)}|u|^2\,ds\le C\int_{s_\sigma}(\sin\phi)^{2(\beta_{ij}+1-\ell)}\Big(|u_\phi|^2+\frac{1}{\sin^2\phi}|u_\theta|^2\Big)\,ds$$

and

$$\begin{aligned}\|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}-\ell}u\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})}&\le C\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2(\beta_m+1-\ell)}(\sin\phi)^{2(\beta_{ij}+1-\ell)}|D^1u|^2\,dx\\ &\le C\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^{\ell-1}u|^2\,dx\\ &\le C\big(\|\rho^{\beta_m}(\sin\phi)^{\beta_{ij}}D^\ell u\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})}+\|D^{\ell-1}u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\big)\end{aligned}$$

which yields the desired result (4.14). ∎

Lemma 4.3. *Let $V_{\sigma,\delta} = S_\sigma \times I_\delta$ as before with $\sigma_0 \geq 0$. Then for $\beta_m \in (0, \frac{1}{2})$*

(4.16)
$$\|\rho^{\beta_m-2}(u_\phi - a)\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} \leq C\Big\{\sum_{|\alpha'|=1} \|\rho^{\beta_m+\alpha_3-2}(\sin\phi)^{\beta_{ij}+\alpha_1-1}\mathcal{D}^{\alpha'}u\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} + \|\rho^{\beta_m-1}u_{\phi\rho}\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\Big\}$$

with

$$a = \lim_{\rho\to 0} \frac{1}{|S_\sigma|}\int_{S_\sigma} u_\phi(\phi,\theta,\rho)\,dS \tag{4.17}$$

provided the right hand side of (4.16) is finite.

Proof. Let $v = u_\phi$ and

$$\bar{v}(\rho) = \frac{1}{|S_\sigma|}\int_{S_\sigma} v(\phi,\theta,\rho)\,dS.$$

Then

$$\frac{d\bar{v}}{d\rho} = \frac{1}{|S_\sigma|}\int_{S_\sigma} v_\rho(\phi,\theta,\rho)\,dS$$

and

$$\int_\sigma^\delta |\bar{v}|^2\,d\rho \leq C\int_{\mathcal{V}_{m,ij}} \left|\frac{1}{\rho}u_\phi\right|^2 dx,$$
$$\int_\sigma^\delta \rho^{2\beta_m}\left|\frac{d\bar{v}}{d\rho}\right|^2 d\rho \leq \int_{\mathcal{V}_{m,ij}} \rho^{2\beta_m}\left|\frac{1}{\rho}u_{\phi\rho}\right|^2 dx.$$

Therefore $\bar{v} \in \mathbf{H}^{1,1}_{\beta_m}(I_\delta) \subset \mathbf{C}^0(\bar{I}_\delta)$ due to Lemma 4.1 of [5]. Letting $a = \bar{v}(0) = \lim_{\rho\to 0}\frac{1}{|S_\sigma|}\int_{S_\sigma} u_\phi(\phi,\theta,\rho)\,dS$, we have by Lemma A.1 of [4]

$$\int_0^\delta \rho^{2\beta_m-2}|\bar{v}-a|^2\,d\rho \leq C\int_0^\delta \rho^{2\beta_m}\left|\frac{d\bar{v}}{d\rho}\right|^2 d\rho.$$

Here we used the fact that $\beta_m \in (0,\frac{1}{2})$. This implies

$$\begin{aligned}\|\rho^{\beta_m-2}(\bar{v}-a)\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} &\leq C\|\rho^{\beta_m-1}v_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\\ &\leq C\|\rho^{\beta_m-1}u_{\rho\phi}\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}.\end{aligned} \tag{4.18}$$

Arguing as in the proof of Lemma 4.1 we have

$$\begin{aligned}\|\rho^{\beta_m-2}(v-\bar{v})\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})} &\leq C\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-1+\alpha_1}\mathcal{D}^{\alpha'}v\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\\ &\leq C\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-1+\alpha_1}\mathcal{D}^{\alpha'}u_\phi\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}.\end{aligned} \tag{4.19}$$

The proof of (4.19) is the same as that given for (4.11) of Lemma 4.1 except that the inequalities

$$\int_{S_\sigma}\left|\int_{\tilde{\theta}}^{\theta}\frac{\partial v}{\partial\theta}(\tilde{\phi},\hat{\theta},\rho)\,d\hat{\theta}\right|\sin\tilde{\phi}\,d\tilde{\phi}\,d\tilde{\theta} \leq C\Big(\int_{S_\sigma}(\sin\tilde{\phi})^{2(\beta_{ij}-1)}\left|\frac{\partial v}{\partial\theta}(\tilde{\phi},\tilde{\theta},\rho)\right|^2\sin\tilde{\phi}\,d\tilde{\phi}\,d\theta\Big)^{\frac{1}{2}},$$

$$\int_{S_\sigma}\Big|\int_{\tilde\phi}^{\phi}\frac{\partial v}{\partial\hat\theta}(\hat\phi,\theta,\rho)\,d\hat\phi\Big|\sin\tilde\phi\,d\tilde\phi\,d\tilde\theta$$

$$\le C\int_{S_\sigma}\Big(\int_{\tilde\phi}^{\phi}(\sin\hat\phi)^{2\beta_{ij}}\Big|\frac{\partial v}{\partial\hat\phi}(\hat\phi,\theta,\rho)\Big|^2\sin\hat\phi\,d\hat\phi\Big)^{\frac12}\Big(\int_{\tilde\phi}^{\phi}(\sin\hat\phi)^{-(1+2\beta_{ij})}\,d\hat\phi\Big)^{\frac12}\sin\tilde\phi\,d\tilde\phi\,d\theta$$

$$\le C\big((\sin\phi)^{-2\beta_{ij}}+c_1\big)^{\frac12}\Big(\int_{\sigma_0}^{\sigma}(\sin\hat\phi)^{2\beta_{ij}}\Big|\frac{\partial v}{\partial\hat\phi}(\hat\phi,\theta,\rho)\Big|^2\sin\hat\phi\,d\theta\Big)^{\frac12}$$

and

$$\int_{S_\sigma}|v-\bar v|^2\,ds\le C\Big\{\int_{S_\sigma}(\sin\phi)^{2(\beta_{ij}-1)}\Big|\frac{\partial v}{\partial\theta}(\phi,\theta,\rho)\Big|^2\,dS$$
$$+\int_{\sigma_0}^{\sigma}\big((\sin\phi)^{-2\beta_{ij}}+c_1\big)\sin\phi\,d\phi\int_{S_\sigma}(\sin\phi)^{2\beta_{ij}}\Big|\frac{\partial v}{\partial\phi}(\phi,\theta,\rho)\Big|^2\,dS\Big\}$$
$$\le C\sum_{|\alpha'|=1}\int_{S_\sigma}(\sin\phi)^{2(\beta_{ij}+\alpha_1-1)}\Big|\mathcal{D}^{\alpha'}\frac{\partial v}{\partial\phi}\Big|^2\,dS$$

are used instead of (4.6), (4.7) and (4.10) respectively. The combination of (4.18) and (4.19) yields (4.16). ■

Lemma 4.4. *Let $V_{\sigma,\delta}=S_\sigma\times I_\delta$ as before with $\sigma_0=0$. Then for $\beta_m\in(0,\frac12)$ and $\beta_{ij}\in(0,1)$*

$$\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-1}(u_\phi-a)\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}$$
$$\le C\Big\{\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}+\alpha_1-1}\mathcal{D}^{\alpha'}u_\phi\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}+\|\rho^{\beta_m-1}u_\phi\rho\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\Big\}\tag{4.20}$$

where a is given by (4.17), and

$$\|\rho^{\beta_m-2}(\sin\phi)^{\beta_m-2}u_\theta\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}+\alpha_1-2}\mathcal{D}^{\alpha'}u_\theta\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}.\tag{4.21}$$

Proof. By Lemma A.2 of [4] we have

(4.22)
$$\int_{S_\sigma}(\sin\phi)^{2\beta_{ij}-2}|u_\phi-a|^2\,dS\le C\int_{S_\sigma}(\sin\phi)^{2\beta_{ij}}\Big(|u_{\phi\phi}|+\frac{1}{\sin^2\phi}|u_{\theta\theta}|^2\Big)\,dS+\int_{S_\sigma\setminus S_{\sigma/2}}|u_\phi-a|^2\,dS$$

with a indicated above. Multiplying (4.22) with $\rho^{2\beta_m-2}$ and integrating over ρ we get

$$\|\rho^{\beta_m-2}(\sin\phi)^{\beta_j-1}(u_\phi-a)\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\Big\{\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}+\alpha_1-1}\mathcal{D}^{\alpha'}u_\phi\|^2_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}$$
$$+\int_{\mathcal{V}_{\sigma,\delta}}\rho^{2\beta_m-4}|u_\phi-a|^2\,dx\Big\}.\tag{4.23}$$

By Lemma 4.3

$$\|\rho^{\beta_m-2}(u_\phi-a)\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\le C\{\|\rho^{\beta_m-1}u_\phi\rho\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}$$
$$+\sum_{|\alpha'|=1}\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}+\alpha_1-1}\mathcal{D}^{\alpha'}u_\phi\|_{\mathbf{L}^2(\mathcal{V}_{\sigma,\delta})}\}.\tag{4.24}$$

Combining (4.23) and (4.24) we get (4.20).

Analogously, by the arguments of Lemma A.2 of [4] we have

$$\int_{S_\sigma} (\sin\phi)^{2(\beta_{ij}-2)}|u_\theta|^2\,dS \le C\int_{S_\sigma}(\sin\phi)^{2(\beta_{ij}-1)}\Big(|u_{\phi\theta}|^2+\frac{1}{\sin^2\phi}|u_{\theta\theta}|^2\Big)\,dS$$

which completes the proof of (4.21), and hence the lemma. ∎

Theorem 4.2. *If $u \in \mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, $0 \le \ell \le 2$, $k \ge \ell$, $\beta = (\beta_m, \beta_{ij})$ with $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$, then $u \in \mathcal{H}^{k,\ell}_{\beta_m}(\mathcal{V}_{m,ij})$, and for any $\tilde{\alpha}$ with $|\tilde{\alpha}| \le k$*

$$\|\Phi^{\tilde{\alpha},\ell}_{\beta_{m,ij}}\rho^{-|\tilde{\alpha}'|}(\sin\phi)^{-\tilde{\alpha}_2}\mathcal{D}^{\tilde{\alpha}}u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\|u\|_{\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} \tag{4.25}$$

moreover, if for α with $|\alpha| \le k$

$$\|\Phi^{\alpha,\ell}_{\beta_{m,ij}}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le Cd^\alpha \alpha!, \tag{4.26}$$

then for $|\tilde{\alpha}| = k$

$$\|\Phi^{\tilde{\alpha},\ell}_{\beta_{m,ij}}\rho^{-|\tilde{\alpha}'|}(\sin\phi)^{-\tilde{\alpha}_2}\mathcal{D}^{\tilde{\alpha}}u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\tilde{d}^{\tilde{\alpha}}\tilde{\alpha}!. \tag{4.27}$$

Proof. Let us note that

$$\begin{cases} u_\rho = u_{x_1}\sin\phi\cos\theta + u_{x_2}\sin\phi\sin\theta + u_{x_3}\cos\phi \\ \frac{1}{\rho}u_\phi = u_{x_1}\cos\phi\cos\theta + u_{x_2}\cos\phi\sin\theta - u_{x_3}\sin\phi. \\ \frac{1}{\rho\sin\phi}u_\theta = -u_{x_1}\sin\theta + u_{x_2}\cos\theta \end{cases} \tag{4.28}$$

which implies that $|D^1u|^2 = |\mathcal{D}^1u|^2$,

$$\|\mathcal{D}^1u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} = \|D^1u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \tag{4.29}$$

and

$$\|u\|_{\mathcal{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} = \|u\|_{\mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}. \tag{4.20}$$

For higher-order derivatives we will prove (4.25) and (4.27) for $\tilde{\alpha} = (k,0,0)$, $\tilde{\alpha} = (0,k,0)$ and $\tilde{\alpha} = (0,0,k)$ with $k \ge \ell$. The general term $\mathcal{D}^\alpha u$ can be treated in a similar way.

Note that

$$u_{\rho^k} = \sum_{|\alpha|=k}\frac{(|\alpha|)!}{\alpha_1!\,\alpha_2!\,\alpha_3!}D^\alpha u(\sin\phi\cos\theta)^{\alpha_1}(\sin\phi\sin\theta)^{\alpha_2}(\cos\theta)^{\alpha_3} \tag{4.31}$$

which implies that for $k \ge \ell = 0$

$$\begin{aligned} \|\rho^{k+\beta_m}(\sin\phi)^{\beta_{ij}}u_{\rho^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le \sum_{|\alpha|=k}\frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\|\rho^{|\alpha|+\beta_m}(\sin\phi)^{|\alpha'|+\beta_{ij}}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &< C(k)\sum_{|\alpha|=k}\|\Phi^{\alpha,\ell}_{\beta_{m,ij}}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}. \end{aligned} \tag{4.32}$$

Then (4.25) for $\tilde{\alpha} = (0,0,k)$ stands. If (4.26) holds for $\ell = 0$, then

$$\begin{aligned}\|\rho^{k+\beta_m}(\sin\phi)^{\beta_{ij}} u_\rho\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le C \sum_{|\alpha|=k} \frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!} d^\alpha \alpha! \\ &= C\,\tilde{d}_3^k k!\end{aligned}$$

with $\tilde{d}_3 = 3\max(d_1, d_2, d_3)$, which is (4.27) for $\ell = 0$ and $\tilde{\alpha} = (0,0,k)$.

For $k \ge \ell \ge 1$ and $\beta_{ij} \in (0,1)$, we get from (4.31)

$$\begin{aligned}\|\rho^{k-\ell+\beta_m} u_\rho k\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le \sum_{|\alpha|=k} \frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!} \|\rho^{|\alpha|-\ell+\beta_m}(\sin\phi)^{|\alpha'|-\ell+\beta_{ij}} D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &\le C(k) \sum_{|\alpha|=k} \|\Phi^{\alpha,\ell}_{\beta_{m,ij}} D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\end{aligned}$$

which together with (4.32) implies (4.25) for $1 \le \ell \le 2$ and $\alpha = (0,0,k)$. Similarly (4.27) can be proven for $\alpha = (0,0,k)$ with $k \ge \ell$ if (4.25) holds for $\ell = 1, 2$.

Next we consider u_{θ^k}, $k \ge \ell$. Arguing as in [4] we have

$$u_{\theta^k} = \sum_{n=1}^{k} (\rho\sin\phi)^n \sum_{j=0}^{n} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\ge 0}} a^{(k)}_{n,j,n_1,n_2} (\sin\theta)^{n_1}(\cos\theta)^{n_2} u_{x_1^{n-j} x_2^j} \tag{4.33}$$

and

$$A^{(k)}_n = \sum_{j=0}^{n} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\ge 0}} |a^{(k)}_{n,j,n_1,n_2}| \le 4^k \frac{k!}{n!}. \tag{4.34}$$

Then it follows from (4.33) that

$$\begin{aligned}&\|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}-\ell} u_{\theta^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &\le \sum_{n=1}^{k}\sum_{j=0}^{n} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\ge 0}} |a^{(k)}_{n,j,n_1,n_2}|\, \|\rho^{n+\beta_m-\ell}(\sin\phi)^{\beta_{ij}+n-\ell} u_{x_1^{n-j} x_2^j}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &\le C(k) \sum_{\substack{\ell\le|\alpha|\le k \\ \alpha_3=0}} \|\Phi^{\alpha,\ell}_{\beta_{m,ij}} D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}.\end{aligned} \tag{4.35}$$

Here we applied (4.12) of Lemma 4.2 to the first term of the summation if $\ell = 2$, namely,

$$\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1} u_{x_1^{n-j} x_2^j}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C \|\rho^{\beta_m}(\sin\phi)^{\beta_{ij}} D^1(u_{x_1^{n-j} x_2^j})\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}.$$

Hence (4.25) stands for $\tilde{\alpha} = (0,k,0)$ and $\ell \ge 0$. Furthermore (4.26) and (4.33)–(4.34) lead to

$$\begin{aligned}\|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}-\ell} u_{\theta^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le \sum_{n=1}^{k}\sum_{j=0}^{n} \sum_{\substack{n_1,n_2\ge 0 \\ n_1+n_2=n}} |a^{(k)}_{n,j,n_1,n_2}|\, d_1^{n-j} d_2^j (n-j)!\, j! \\ &\le \sum_{n=1}^{k} \left(\frac{1}{4}\tilde{d}_2\right)^n n!\, A^{(k)}_n \\ &\le C\tilde{d}_2^k k!\end{aligned}$$

when $\tilde{d}_2 = 4\max(d_1, d_2, d_3)$. This is (4.27) for $\tilde{\alpha} = (0, k, 0)$ with $k \geq \ell$, $0 \leq \ell \leq 2$.

Now we consider u_{ϕ^k}, $k \geq \ell$. To this end we have to derive by the induction:

$$(4.36) \quad u_{\phi^k} = \sum_{n=1}^{k} \rho^n \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2=n \\ n_3+n_4=|\alpha'| \\ n_i \geq 0}} a^{(k)}_{\alpha,n_1,n_2,n_3,n_4} (\sin\phi)^{n_1} (\cos_2\phi)^{n_2} (\sin\theta)^{n_3} (\cos\theta)^{n_4} D^\alpha u,$$

$$(4.37) \qquad A^{(k)}_n = \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2=n \\ n_3+n_4=|\alpha'| \\ n_i \geq 0}} |a^{(k)}_{\alpha,n_1,n_2,n_3,n_4}| \leq 5^k \frac{k!}{n!}, \quad \text{for } 1 \leq n \leq k,$$

and

$$(4.38) \qquad n_i \geq 0 \quad \text{for } 1 \leq i \leq 4, \; n_1 + n_2 = |\alpha| = n, \text{ and } n_3 + n_4 = |\alpha'|.$$

Obviously, (4.36)–(4.38) holds for $k = 1$ due to (4.28). Suppose that they are true up to $(k-1)$, then

$$\begin{aligned} u_{\phi^k} = \sum_{n=1}^{k-1} \rho^n \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2=n \\ n_3+n_4=|\alpha'| \\ n_i \geq 0}} & a^{(k-1)}_{\alpha,n_1,n_2,n_3,n_4} \{(\sin\phi)^{n_1} (\cos\phi)^{n_2} (\sin\theta)^{n_3} (\cos\theta)^{n_4} \\ & \cdot \rho[\cos\phi\cos\theta D^\alpha u_{x_1} + \cos\phi\sin\theta D^\alpha u_{x_2} - \sin\phi D^\alpha u_{x_3}] \\ & + (\sin\theta)^{n_3} (\cos\theta)^{n_4} D^\alpha u [n_1 (\sin\phi)^{n_1-1} (\cos\phi)^{n_2+1} - n_2 (\sin\phi)^{n_1+1} (\cos\phi)^{n_2-1}]\}. \end{aligned}$$

from which we get for $|\alpha| = n = k$

$$(4.39) \qquad \begin{aligned} a^{(k)}_{\alpha,n_1,n_2,n_3,n_4} &= a^{(k-1)}_{\alpha-e_1,n_1,n_2-1,n_3,n_4-1} + a^{(k-1)}_{\alpha-e_2,n,n_2-1,n_3-1,n_4} \\ &\quad - a^{(k-1)}_{\alpha-e_3,n_1-1,n_2,n_3,n_4} \end{aligned}$$

and for $|\alpha| = n < k$

$$(4.40) \qquad \begin{aligned} a^{(k)}_{\alpha,n_1,n_2,n_3,n_4} &= a^{(k-1)}_{\alpha-e_1,n_1,n_2-1,n_3,n_4-1} + a^{(k-1)}_{\alpha-e_2,n_1,n_2-1,n_3-1,n_4} \\ &\quad - a^{(k-1)}_{\alpha-\ell_3,n_1-1,n_2,n_3,n_4} + (n_1+1) a^{(k-1)}_{\alpha,n_1+1,n_2-1,n_3,n_4} \\ &\quad - (n_2+1) a^{(k-1)}_{\alpha,n_1-1,n_2+1,n_3,n_4} \end{aligned}$$

where $\alpha - e_1 = (\alpha_1 - 1, \alpha_2, \alpha_3)$, $\alpha - e_2 = (\alpha_1, \alpha_2 - 1, \alpha_3)$ and $\alpha - e_3 = (\alpha_1, \alpha_2, \alpha_3 - 1)$. The terms on the right hand sides of (4.39) and (4.40) are absent if any of their sub-index is negative. The assumption of induction up to $(k-1)$ and (4.39)–(4.40) imply (4.38) for $n = k$, and

$$A^{(k)}_k = 3A^{(k-1)}_{k-1} \leq 3 \cdot 5^{k-1} \leq 5^k$$

and for $n < k$,

$$\begin{aligned} A_n^{(k)} &\le 3A_{n-1}^{(k-1)} + 2(n+1)A_n^{(k-1)} \\ &\le 3 \cdot 5^{k-1}\frac{(k-1)!}{(n-1)!} + 2(n+1)5^{k-1}\frac{(k-1)!}{n!} \\ &\le 5^k \frac{k!}{n!}. \end{aligned}$$

This completes the induction and establises (4.36)–(4.38).

Now, it follows directly from (4.36) that for $k \ge \ell$ and $0 \le \ell \le 1$.

$$\begin{aligned} &\|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}+k-\ell}u_{\phi k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &\le \sum_{h=1}^{k}\sum_{|\alpha|=n}\sum_{\substack{n_1+n_2=n\\ n_3+n_4=|\alpha'|\\ n_i\ge 0}} |a^{(k)}_{\alpha,n_1,n_2,n_3,n_4}|\,\|\rho^{\beta_m+n-\ell}(\sin\phi)^{\beta_{ij}+k-\ell}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ &\le \sum_{n=1}^{p}\sum_{|\alpha|=n}\sum_{\substack{n_1+n_2=n\\ n_3+n_4=|\alpha'|\\ n_i\ge 0}} |a^{(k)}_{\alpha,n_1,n_2,n_3,n_4}|\,\|\Phi^{\alpha,\ell}_{\beta_{m,ij}}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}. \end{aligned} \tag{4.41}$$

If $\ell = 2$, we have for those α in (4.41) with $|\alpha| = n = 1$

$$\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le \|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \tag{4.42}$$

by (4.12) of Lemma 4.2

$$\le C\|\rho^{\beta_m}(\sin\phi)^{\beta_{ij}}D^1(D^\alpha u)\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}.$$

Hence we obtain (4.41) for $k \ge \ell = 2$, which leads to (4.25) for $0 \le \ell \le 2$ and $\tilde\alpha = (k,0,0)$ with $k \ge \ell$.

If (4.26) holds we have by (4.37) and (4.41)

$$\begin{aligned} \|\rho^{\beta_m-\ell}(\sin\phi)^{\beta_{ij}+k-\ell}u_{\phi^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\le \sum_{n=1}^{k}\sum_{|\alpha|=n}\sum_{\substack{n_1+n_2=n\\ n_3+n_4=|\alpha'|\\ n_i\ge 0}} |a^{k}_{\alpha,n_1,n_2,n_3,n_4}|\,d^\alpha\alpha! \\ &\le C\sum_{n=1}^{k}\Big(\frac{1}{5}\tilde d_2\Big)^n n!\,A_n^{(k)} \le C\tilde d_2^k k! \end{aligned}$$

where $\tilde d_2 = 5\max(d_1, d_2, d_3)$. Thus (4.27) holds for $\tilde\alpha = (k,0,0)$ with $k \ge \ell$, $0 \le \ell \le 2$.

Since the arguments for u_{ρ^k}, u_{ϕ^k} and u_{θ^k} with $k \ge \ell$ can be carried out for general terms $\mathcal{D}^{\tilde\alpha}u$ with $|\tilde\alpha| \le k$, we obtained the desired results. ∎

From (4.26) and (4.27) of Theorem 4.2 we have immediately

Corollary 4.2. $\mathbf{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) \subset \mathcal{B}^{\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$. ■

Unlike the relation between $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ and $\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ with $0 \le \ell \le 2$ the converse of Theorem 4.2 does not hold. There is a concrete example which will give us a hint about the difference between $\mathcal{H}^{k,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and $\mathbf{H}^{k,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$. Consider $w = \phi$. Obviously $w \in \mathcal{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ but $w_{x_1^2} = -\rho^{-2}(2\sin\phi\cos\phi\cos^2\theta + \cot\phi\sin^2\theta) \notin \mathbf{L}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ for any $\beta_m \in (0,\frac{1}{2})$. The next theorem deals with the difference.

Theorem 4.3. *If $u \in \mathcal{H}^{k,\ell}_{\beta_{m,ij}}$ with $k \ge \ell \ge 0$ and $0 \le \ell \le 2$, $\beta_m \in (0,\frac{1}{2})$ and $\beta_{ij} \in (0,1)$ then $(u - \chi(\phi)) \in \mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ where $\chi(\phi) = 0$ for $0 \le \ell \le 1$ and $\chi(\phi) = a\phi$ for $\ell = 2$ with a given by*

$$a = \lim_{\rho\to 0} \frac{1}{|S_{\sigma_{ij}}|}\int_{S_{\sigma_{ij}}} u_\phi(\phi,\theta,\rho)\,dS, \tag{4.43}$$

and

$$\|u-\chi\|_{\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} \le C\|u\|_{\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}. \tag{4.44}$$

Moreover, if for any $|\alpha| \le k$

$$\|\Phi^{\alpha,\ell}_{\beta_{m,ij}}\rho^{|\alpha'|}(\sin\phi)^{-\alpha_2}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le Cd^\alpha\alpha!, \tag{4.45}$$

then for $|\tilde\alpha| = k$

$$\|\Phi^{\tilde\alpha,\ell}_{\beta_{m,ij}} D^{\tilde\alpha}(u-\chi)\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le \tilde C\tilde d^{\tilde\alpha}\tilde\alpha!. \tag{4.46}$$

Proof. We shall first prove by the induction the following:

$$u_{x_1^k} = \sum_{n=1}^{k}\sum_{|\alpha|=n}\rho^{-(k-\alpha_3)}(\sin\phi)^{-(k-\alpha_1-\alpha_3)}\sum_{\substack{n_1+n_2\le n\\ n_3+n_4=k\\ n_i\ge 0}} b^{(k)}_{\alpha,n_1,n_2,n_3n_4} \tag{4.47}$$

$$(\sin\phi)^{n_1}(\cos\phi)^{n_2}(\sin\theta)^{n_3}(\cos\theta)^{n_4}\mathcal{D}^\alpha u$$

$$B^{(k)}_n = \sum_{|\alpha|=n}\sum_{\substack{n_1+n_2=n\\ n_3+n_4=k\\ n_i\ge 0}} |b^{(k)}_{b_\alpha,n_1,n_2,n_3,n_4}| \le 7^k\frac{k!}{n!} \tag{4.48}$$

$$0 \le n_i \le \quad \text{for } 1 \le i \le 4,\ n_1+n_2 \le k,\ n_3+n_4 = k. \tag{4.49}$$

It is trivial that

$$\begin{cases} u_{x_1} = u_\rho\sin\phi\cos\theta + \frac{1}{\rho}u_\phi\cos\phi\cos\theta - \frac{1}{\rho\sin\phi}u_\theta\sin\theta\\ u_{x_2} = u_\rho\sin\phi\sin\theta + \frac{1}{\rho}u_\phi\cos\phi\sin\theta + \frac{1}{\rho\sin\phi}u_\theta\cos\theta\\ u_{x_3} = u_\rho\cos\phi - \frac{1}{\rho}u_\phi\sin\phi \end{cases}$$

Then (4.47)–(4.49) holds for $k = 1$. Suppose that (4.47)–(4.49) holds up to $(k-1)$. Differentiating (4.47) we get by straightful calculation for $|\alpha| = k$

$$b^{(k)}_{\alpha,n_1,n_2,n_3,n_4} = b^{(k-1)}_{\alpha-e_1,n_1,n_2-1,n_3,n_4-1} - b^{(k-1)}_{\alpha-e_2,n_1,n_2,n_3-1,n_4} + b^{(k-1)}_{\alpha-e_3,n_1-1,n_2,n_3,n_4-1}$$

and for $|\alpha| < k$

$$\begin{aligned} b^{(k)}_{\alpha,n_1,n_2,n_3,n_4} = {} & b^{(k-1)}_{\alpha-e_1,n_1,n_2-1,n_3,n_4-1} - b^{(k-1)}_{\alpha-e_2,n_1,n_2,n_3-1,n_4} + b^{(k-1)}_{\alpha-e_3,n_1-1,n_2,n_3,n_4-1} \\ & - (k-1-\alpha_3) b^{(k-1)}_{\alpha,n_1-2,n_2,n_3,n_4-1}, -(k-1-\alpha_1-\alpha_3) b^{(k-1)}_{\alpha,n_1,n_2-2,n_3,n_4-1} \\ & + n_1 b^{(k-1)}_{\alpha,n_1,n_2-2,n_3,n_4-1} - n_2 b^{(k-1)}_{\alpha,n_1-2,n_2,n_3,n_4-1} + n_3 b^{(k-1)}_{\alpha,n_1,n_2,n_3,n_4-1} \\ & - n_4 b^{(k-1)}_{\alpha,n_1,n_2,n_3-2,n_4+1}. \end{aligned}$$

where $\alpha - e_i$, $i = 1,2,3$ are the same with those in the proof of Theorem 4.2. Obviously (4.47) and (4.49) hold due to the hypothesis of induction and

$$\begin{aligned} B^{(k)}_n & \le 3B^{(k-1)}_{n-1} + [(k-i-\alpha_3) + (k-1-\alpha_3-\alpha_3) + (n_1+n_2+n_3+n_4)] B^{(k-1)}_n \\ & \le 3B^{(k-1)}_{n-1} + 4kB^{(k-1)}_n \\ & \le 3 \cdot 7^{k-1} \frac{(k-1)!}{(n-1)!} + 4 \cdot 7^{k-1} \frac{(k-1)!k}{n!} \le 7^k \frac{k!}{n!}. \end{aligned}$$

Hence (4.48) holds for k. Thus (4.47)–(4.49) are proved by the induction.

It follows from (4.47) that

$$\begin{aligned} & \|\rho^{k-\ell+\beta_m} (\sin\phi)^{k-\ell+\beta_{ij}} (u-\chi)_{x_1^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \\ & \le C \sum_{n=1}^{k} \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2\le k \\ n_3+n_4=k \\ n_i \ge 0}} b^{(k)}_{\alpha,n_1,n_2,n_3,n_4} \|\rho^{\alpha_3-\ell+\beta_m} (\sin\phi)^{\alpha_3+\alpha_1-\ell+\beta_{ij}} \mathcal{D}^\alpha (u-\chi)\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}. \end{aligned} \tag{4.50}$$

For $\ell = 0,1$ (4.44) follows from (4.50) immediately. For $\ell = 2$ we have by (4.12) of Lemma 4.2

$$\|\rho^{\beta_m-1}(\sin\phi)^{\beta_{ij}-1} u_\rho\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\|u\|_{\mathcal{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})},$$

and by (4.19) of Lemma 4.4

$$\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-1}(u_\phi - a)\|_{\mathbf{L}^2)(\mathcal{V}_{mij})} \le C\|u\|_{\mathcal{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}$$

and by (4.20) of Lemma 4.4

$$\|\rho^{\beta_m-2}(\sin\phi)^{\beta_{ij}-2} u_\theta\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \le C\|u\|_{\mathcal{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{mij})}.$$

Substituting those above into (4.50) we obtain

$$\begin{aligned}
&\|\rho^{k-2+\beta_m}(\sin\phi)^{k-2+\beta_{ij}}(u-\chi)_{x_1^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\\
&\qquad\le C\Big\{\|u\|_{\mathcal{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}+\sum_{n=2}^{k}\sum_{|\alpha|=n}\|\Phi^{\alpha,2}_{\beta_{m,ij}}\mathcal{D}^\alpha(u-\chi)\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\Big\}\\
&\qquad\le C\|u\|_{\mathcal{H}^{k,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.
\end{aligned}$$

The arguments above can be carried out for general term $D^\alpha(u-\chi)$ for α with $|\alpha|\le k$. Hence we get the desired results (4.44).

Furthermore, if (4.45) holds for $|\alpha|\le k$, we get for $k\ge\ell$ from (4.50)

$$\begin{aligned}
\|\rho^{k-\ell+\beta_m}(\sin\phi)^{k-\ell+\beta_{ij}}(u-\chi)_{x_1^k}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}&\le C\sum_{n=1}^{k}\sum_{|\alpha|=n}\sum_{\substack{n_1+n_2\le k\\ n_3+n_4=k\\ n_i\ge 0}}|b^{(k)}_{\alpha,n_1,n_2,n_3,n_4}|d^\alpha\alpha!\\
&\le C\sum_{n=1}^{k}B^{(k)}_n\Big(\frac{1}{7}\tilde{d}_1\Big)^n n!
\end{aligned}$$

by (4.48)

$$\begin{aligned}
&\le Ck!\,7^k\sum_{n=1}^{k}\Big(\frac{1}{7}\tilde{d}_1\Big)^n\\
&\le cd_1^k k!
\end{aligned}$$

where $\tilde{d}_1=7\max\{d_1,d_2,d_3\}$. Thus (4.46) holds for $\tilde{\alpha}=(k,0,0)$ with $k\ge\ell$ and $0\le\ell\le 2$. For general term $D^{\tilde{\alpha}}u$ with $|\tilde{\alpha}|=k\ge\ell$ (4.46) can be treated in the same way. The proof of the theorem is completed. ∎

Due to (4.45) and (4.46) we immediately obtain the following corollary.

Corollary 4.3. *If $u\in\mathcal{B}^\ell_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ for $0\le\ell\le 2$, then $(u-\chi)\in\mathbf{B}^\ell_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$.*

Summarizing Theorem 4.2–4.3 and Corollary 4.2–4.3 we have

Theorem 4.4. *(i) For $\ell=0,1$ $\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and $\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ are equivalent and the same are $\mathbf{B}^\ell_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and $\mathcal{B}^\ell_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$;*

(ii) $\mathbf{H}^{k,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ (resp. $\mathbf{B}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$) is equivalent to the quotient space $\mathcal{H}^{k,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})\setminus\mathcal{P}$ (resp. $\mathcal{B}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})\setminus\mathcal{P}$), where $\mathcal{P}=\{a\phi, a\in\mathbb{R}^1\}$.

4.3. Weighted Sobolev Space And Countably Normed Spaces Over Inner-neighbourhood Of Vertices In Spherical Coordinates.

Let $\tilde{\mathcal{O}}_m$ be an inner neighbourhood of the vertex A_m, and we assume that A_m is at the origin and one of the edge Λ_{ij} connecting A_m is on the positive x_3-axis. $x=(x_1,x_2,x_3)$ and $x=(\phi,\theta,\rho)$

denote the extension and spherical coordinates of $x \in \tilde{\mathcal{O}}_m$. There is a $\sigma_0 \geq 0$ such that $\phi(x) \geq \sigma_0$ for $x \in \tilde{\mathcal{O}}_m$. Let $\mathcal{D}^\alpha u = \mathcal{D}^{\alpha'} u_{\rho^{\alpha_3}} = u_{\phi^{\alpha_1}\theta^{\alpha_2}\rho^{\alpha_3}}$ with α and α' being the same as before.

Using the weight function $\Phi^{\alpha,\ell}_{\beta_m}(x)$ given by (2.3) we define for $\beta_m \in (0, \frac{1}{2})$ and integers k and ℓ, $k \geq \ell \geq 0$

$$\mathcal{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m) = \left\{ u \mid \|u\|^2_{\mathcal{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)} = \sum_{|\alpha| \leq k} \|\Phi^{\alpha,\ell}_{\beta_m} \rho^{-|\alpha'|} \mathcal{D}^\alpha u\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} < \infty \right\}$$

and

$$\mathcal{B}^\ell_{\beta_m}(\tilde{\mathcal{O}}_m) = \{ u \mid u \in \mathcal{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m) \text{ for all } k \geq \ell, \|\Phi^{\alpha,\ell}_{\beta_m} \rho^{-|\alpha'|} \mathcal{D}^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\, d^\alpha \alpha! \}$$

<u>**Theorem 4.5**</u>. *If $u \in \mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$, $0 \leq \ell \leq 2$, $k \geq \ell$, $\beta_m \in (0, \frac{1}{2})$, then $u \in \mathcal{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$, and for α with $|\alpha| \leq k$*

$$\|\Phi^{\alpha,\ell}_{\beta_m} \rho^{-|\alpha'|} \mathcal{D}^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C \|u\|_{\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \tag{4.51}$$

Moreover, for α with $|\alpha| = k$

$$\|\Phi^{\tilde{\alpha},\ell}_{\beta_m} \rho^{-|\tilde{\alpha}'|} \mathcal{D}^{\tilde{\alpha}} u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C \tilde{d}^{\tilde{\alpha}} \tilde{\alpha}! \tag{4.52}$$

if for α with $|\alpha| \leq k$

$$\|\Phi^{\alpha,\ell}_{\beta_m} D^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C d^\alpha \alpha!. \tag{4.53}$$

Proof. Using (4.31), (4.33)–(4.34) and (4.36)–(4.38) and noting that $\sin\phi \geq \sin\sigma_0 > 0$, we can prove (4.51) for $\tilde{\alpha} = (0,0,k)$, $\tilde{\alpha} = (k,0,0)$ and $\tilde{\alpha} = (0,k,0)$ in the same way except applying (4.3) of Lemma 4.1 for the cases $\alpha = (k,0,0)$ and $\alpha = (0,k,0)$, instead of (4.12) of Lemma 4.2.

Analogously (4.51) can be argued for general term $\mathcal{D}^\alpha u$ for α with $|\alpha| \leq k$. (4.52) can be proved in a similar way as (4.26) if (4.53) holds. ∎

Like the relation between $\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and $\mathcal{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ the converse of Theorem 4.4 is not true. There are two concrete examples: $u_1 = \phi$ and $u_2 = \theta$. Both are not in $\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ for any $\beta_m \in (0, \frac{1}{2})$, meanwhile $u_i \in \mathcal{H}^{k,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ for any $k \geq 2$ and any $\beta_m \in (0, \frac{1}{2})$, $i = 1, 2$. It is worth indicating $|\mathcal{D}^1 u_2| \in \mathbf{L}^2(\tilde{\mathcal{O}}_m)$, but $|\mathcal{D}^1 u_2| \notin \mathbf{L}^2(\mathcal{V}_{m,ij})$.

Let S be the intersection of the unit sphere and the infinite polyhedron which coincides with Ω at the neighbourhood $\mathcal{O}_m$ of A_m, and let $\tilde{S}_\sigma = S \setminus \bigcup_{ij \in \mathcal{L}_m} S_{\sigma_{ij}}$ with $\sigma_{ij} = \sigma$, $ij \in \mathcal{L}_m$.

<u>**Lemma 4.5**</u>. *For $\tilde{\mathcal{O}}_m = \tilde{S}_\sigma \times I_\delta$ we have*

$$\|\rho^{\beta_m - 2}(u_\phi - a)\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C \sum_{|\alpha|=1} \|\rho^{\beta_m + \alpha_3 - 2} \mathcal{D}^\alpha u_\phi\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \tag{4.54}$$

and

$$\|\rho^{\beta_m - 2}(u_\theta - b)\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C \sum_{|\alpha|=1} \|\rho^{\beta_m + \alpha_3 - 2} \mathcal{D}^\alpha u_\theta\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \tag{4.55}$$

when

$$a = \lim_{\rho \to 0} \frac{1}{|\tilde{S}_\sigma|} \int_{\tilde{S}_\sigma} u_\phi(\phi, \theta, \rho)\, dS, \tag{4.56}$$

and

$$b = \lim_{\rho \to 0} \frac{1}{|\tilde{S}_\sigma|} \int_{\tilde{S}_\sigma} u_\theta(\phi, \theta, \rho)\, dS. \tag{4.57}$$

Proof. Let $v = u_\phi$ and

$$\bar{v}(\rho) = \frac{1}{|\tilde{S}_\sigma|} \int_{\tilde{S}_\sigma} u_\phi(\phi, \theta, \rho)\, dS.$$

Then arguing in the same way as that for Lemma 4.3 we have

$$\begin{aligned} \|\rho^{\beta_m - 2}(\bar{v} - a)^2\|_{\mathbf{L}^2(\tilde{O}_m)} &\le C\|\rho^{\beta_m - 1} v_\rho\|_{\mathbf{L}^2(\tilde{O}_m)} \\ &\le C\|\rho^{\beta_m - 1} u_{\phi\rho}\|_{\mathbf{L}^2(\tilde{O}_m)} \end{aligned} \tag{4.58}$$

and arguing analogously as for (4.11) of Lemma 4.1 we have

$$\begin{aligned} \|\rho^{\beta_m - 2}(v - \bar{v})\|_{\mathbf{L}^2(\tilde{O}_m)} &\le C \sum_{|\alpha'|=1} \|\rho^{\beta_m - 2}\mathcal{D}^{\alpha'} v\| \\ &= C \sum_{|\alpha'|=1} \|\rho^{\beta_m - 2}\mathcal{D}^{\alpha'} u_\phi\|. \end{aligned} \tag{4.59}$$

Then (4.54) follows from (4.58) and (4.59).

Next let $w = u_\theta$, and

$$\bar{w}(\rho) = \frac{1}{|\tilde{S}_\sigma|} \int_{\tilde{S}_\sigma} u_\theta(\phi, \theta, \rho)\, ds.$$

Analogously we have

$$\|\rho^{\beta_m - 2}(\bar{w} - b)\|_{\mathbf{L}^2(\tilde{O}_m)} \le C\|\rho^{\beta_m - 1} u_{\theta\rho}\|_{\mathbf{L}^2(\tilde{O}_m)}$$

and

$$\|\rho^{\beta_m - 2}(w - \bar{w})\|_{\mathbf{L}^2(\tilde{O}_m)} \le C \sum_{|\alpha'|=1} \|\rho^{\beta_m - 2}\mathcal{D}^{\alpha'} u_\theta\|_{\mathbf{L}^2(\tilde{O}_m)}$$

which yields (4.55). ∎

Theorem 4.6. *If $u \in \mathcal{H}^{k,\ell}_{\beta_m}(\tilde{O}_m)$, $0 \le \ell \le 2$ and $\beta_m \in (0, \frac{1}{2})$, then $\big(u - \chi(\phi,\theta)\big) \in \mathbf{H}^{k,\ell}_{\beta_m}(\tilde{O}_m)$ where $\chi = 0$ for $\ell = 0, 1$, and $\chi(\phi,\theta) = a\phi + b\theta$ for $\ell = 2$, a and b are given by (4.56) and (4.57) respectively, and for α with $|\alpha| \le k$*

$$\|\Phi^{\alpha,\ell}_{\beta_m} D^\alpha (u - \chi)\|_{\mathbf{L}^2(\tilde{O}_m)} \le C\|u\|_{\mathcal{H}^{k,\ell}_{\beta_m}(\tilde{O}_m)}. \tag{4.60}$$

Moreover, if for α with $|\alpha| \le k$

$$\|\Phi^{\alpha,\ell}_{\beta_m} \rho^{-|\alpha|} \mathcal{D}^\alpha u\|_{\mathbf{L}^2(\tilde{O}_m)} \le C d^\alpha \alpha! \tag{4.61}$$

then for $|\tilde{\alpha}| = k$

$$\|\Phi^{\tilde{\alpha},\ell}_{\beta_m} D^{\tilde{\alpha}}(u-\chi)\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq \tilde{C} d^{\tilde{\alpha}} \tilde{\alpha}!. \tag{4.62}$$

Proof. We will prove (4.60) and (4.61) for $\alpha = (k, 0, 0)$. The proof for general α with $|\alpha| = k$ is the same. Due to (4.47) we have

(4.63)

$$\|\rho^{k-\ell+\beta_m}(u-\chi)_{x_1^k}\|_{\mathbf{L}^2(\mathcal{O}_m)} \leq \sum_{n=1}^{k} \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2\leq k\\ n_3+n_4=k\\ n_i\geq 0}} (\sin\sigma_0)^{-k+n} |b^{(k)}_{\alpha,n_1,n_1,n_3,n_4}| \|\rho^{\alpha_3-\ell+\beta_m}\mathcal{D}^{\alpha}(u-\chi)\|_{\mathbf{L}^2(\mathcal{O}_m)}.$$

For $\ell = 0, 1$ (4.60) for $\tilde{\alpha} = (k, 0, 0)$ with $k \geq \ell$ follows immediately. For $\ell = 2$ and $|\alpha| = 1$, we have by (4.3) of Lemma 4.1

$$\|\rho^{\beta_m-1}u_\rho\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\|u\|_{\mathcal{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}$$

and by (4.54) and (4.55) of Lemma 4.5 we have

$$\|\rho^{\beta_m-2}(u_\phi - a)\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\|u\|_{\mathcal{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}$$

and

$$\|\rho^{\beta_m-2}(u_\theta - b)\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C\|u\|_{\mathcal{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}.$$

Then (4.60) stands for $\tilde{\alpha} = (k, 0, 0)$ with $k \geq \ell \geq 2$.

If (4.61) holds for α with $|\alpha| \leq k$ then it follows from (4.47)-(4.48) and (4.61) that

$$\begin{aligned}
\|\rho^{\beta_m+k-\ell}(u-\chi)_{x_1^k}\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} &\leq \sum_{n=1}^{k} \sum_{|\alpha|=n} \sum_{\substack{n_1+n_2\leq k\\ n_3+n_4=k\\ n_i\geq 0}} (\sin\sigma_0)^{-k+n} |b^{(k)}_{\alpha,n_1,n_2,n_3,n_4}| \alpha!\, d^{\alpha} \\
&\leq \sum_{n=1}^{k} (\sin\sigma_0)^{-k+n} B^{(k)}_n n! d^n \\
&\leq C(\sin\sigma_0)^{-k} 7^k k! \sum_{n=1}^{k} \Big(\frac{d_1^* \sin\sigma_0}{7}\Big)^n \\
&\leq C\tilde{d}_1^k k!
\end{aligned}$$

with $d_1^* = \max(d_1, d_2, d_3)$ and $\tilde{d}_1 = \max(d_1^*, 7/\sin\sigma_0)$. Thus we complete the proof. ∎

(4.61) and (4.62) in Theorem 4.5 give the following corollary.

Corollary 4.6. *If* $u \in \mathcal{B}^{\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$, $0 \leq \ell \leq 2$, *then* $(u - \chi) \in \mathbf{B}^{\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)$. ∎

Summarizing Theorem 4.5–4.6 and Corollary 4.5–4.6 we have

Theorem 4.7. *(i) For $\ell = 0, 1$ $\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{O}_m)$ and $\mathbf{B}^{\ell}_{\beta_m}(\tilde{O}_m)$ are equivalent to $\mathcal{H}^{k,\ell}_{\beta_m}(\tilde{O}_m)$ and $\mathcal{B}^{\ell}_{\beta_m}(\tilde{O}_m)$, respectively;*

(ii) $\mathbf{H}^{k,2}_{\beta_m}(\tilde{O}_m)$ and $\mathbf{B}^{2}_{\beta_m}(\tilde{O}_m)$ are equivalent to the quotient space $\mathcal{H}^{k,2}_{\beta_m}(\tilde{O}_m)\backslash\mathcal{P}$ and $\mathcal{B}^{2}_{\beta_m}(\tilde{O}_m)\backslash\mathcal{P}$, with $\mathcal{P} = \{a\phi + b\theta, a, b \in \mathbb{R}^1\}$. ∎

5. COUNTABLY WEIGHTED CONTINUOUS FUNCTION SPACE $\mathbf{C}^2_\beta(\Omega)$

The countably normed spaces $\mathbf{B}^2_\beta(\Omega)$ defined in Section 2 give the description of quantity of the derivatives of functions of any order in weighted Sobolev norm, which will be used in Part II and Part III of this series of our papers to describe the regularities of the solutions of elliptic problems in nonsmooth domains. In many applications, for instance, the error estimates of the p and $h-p$ versions of the finite element solutions, we prefer to use the pointwise estimates of high-order derivatives of solutions (see [8,9,21]). The imbedding of $\mathbf{H}^{k,\ell}_\beta(\Omega)$ into $\mathbf{C}^{\ell-2}(\bar{\Omega})$, $\ell \geq 2$ tell us only the continuities of the derivatives, but it gives no quantitative information of the high-order derivatives of the solutions. We shall introduce a countably weighted space $\mathbf{C}^2_\beta(\Omega)$ with weighted $\mathbf{C}^k-$ norm in this section and establish the relation between the space $\mathbf{B}^2_\beta(\Omega)$ and $\mathbf{C}^2_\beta(\Omega)$. Then combining the regularity theorems in the frame of the space $\mathbf{B}^2_\beta(\Omega)$, which will be given in Part II and Part III, we shall have pointwise estimates of the high-order derivatives of the solutions of elliptic problems in nonsmooth domains in $\mathbb{R}^3$.

5.1. Countably Normed Space $\mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$.

Let $\mathcal{U}_{ij} = Q_{\varepsilon_{ij}} \times I_{\delta_{ij}}$ be the neighbourhood of the edge Λ_{ij} which lies on x_3-axis, and let $r(x) = r = \mathrm{dist}(x, \Lambda_{ij})$ for $x \in \mathcal{U}_{ij}$. We write $Q = Q_{\varepsilon_{ij}}$ and $I = I_{\delta_{ij}}$, and assume that $I = (0,1)$.

By $\mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$, $0 < \beta_{ij} < 1$, we denote a set of functions $u \in \mathbf{C}^0(\bar{\mathcal{U}}_{ij})$ such that for $|\alpha| \geq 0$

$$(5.1) \qquad \|r_{ij}^{\beta_{ij}+\alpha_1+\alpha_2-1} D^\alpha\big(u(x) - u(0,0,x_3)\big)\|_{\mathbf{C}^0(\bar{\mathcal{U}}_{ij})} \leq C d^\alpha \alpha!$$

and for $k \geq 0$

$$(5.2) \qquad \Big\| \frac{d^k}{dx_3^k} u(0,0,x_3) \Big\|_{\mathbf{C}^0(\bar{\mathbf{I}}_{\delta_{ij}})} \leq C d_3^k k!.$$

It follows from (5.1) that for $x \in \mathcal{U}_{ij}$ and any α

$$(5.1') \qquad \big|D^\alpha\big(u(x) - u(0,0,x_3)\big)\big| \leq C d^\alpha \alpha!\, r^{-(\beta_{ij}+\alpha_1+\alpha_2-1)}(x),$$

and (5.2) implies that $v(x_3) = u(0,0,x_3)$ is an analytic function of x_3 on I. Furthermore, the definitions of the space $\mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$ and $\mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij})$ imply

Theorem 5.1. $\mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij}) \subset \mathbf{B}^2_{\beta_{ij}+\varepsilon}(\mathcal{U}_{ij})$ *for $\varepsilon > 0$, arbitrary.*

For the converse of Theorem 5.1 we need a lemma which follows directly from Lemma A.2 and A.3 of [4].

Lemma 5.1. *Let $\mathcal{U}_\varepsilon = \{x = (x_1, x_2, x_3) \mid (x_1, x_2) \in Q_\varepsilon, x_3 \in I\}$ with $Q_\varepsilon = \{(r,\theta) \mid 0 < r < \varepsilon, 0 < \theta < \omega\}$ and $I = (0,1)$. (x_1, x_2, x_3) and (r, θ, x_3) are the Cartesian and cylindrical coordinates, respectively. Then for $\beta \in (0,1)$ we have*

$$
\begin{aligned}
\|r^{\beta-1}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} &\le C\Big\{\sum_{|\alpha'|=1} \|r^{\alpha_1-1+\beta}\mathcal{D}^{\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} + \|u\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_{\frac{1}{2}})}\Big\} \\
&\le C\Big\{\sum_{|\alpha'|=1} \|r^{\beta}D^{\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} + \|u\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_{\frac{1}{2}})}\Big\},
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\|r^{\beta-2}(u-u(0,0,x_3))\|^2_{\mathbf{L}^2(\mathcal{U}_1)} &\le C\Big(\sum_{|\tilde\alpha'|=2} \|r^{\alpha_1-2+\beta}\mathcal{D}^{\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} + \|u_r\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_2)}\Big\} \\
&\le C\Big\{\sum_{|\tilde\alpha'|=2} \|r^{\beta}D^{\alpha}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} + \sum_{|\tilde\alpha'|=1} \|D^{\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_{\frac{1}{2}})}\Big\}
\end{aligned}
\tag{5.4}
$$

and for α with $|\alpha'| = 1$

$$
\|r^{\beta+\alpha_1-2}\mathcal{D}^{\alpha}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} \le C\Big\{\sum_{|\tilde\alpha'|=1} \|r^{\tilde\alpha_1-2+\beta}\mathcal{D}^{\tilde\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} + \sum_{|\tilde\alpha'|\le 1} \|D^{\tilde\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_{\frac{1}{2}})}\Big\}
\tag{5.5}
$$

and

$$
\|r^{\beta-1}D^{\alpha}u\|^2_{\mathbf{L}^2(\mathcal{U}_1)} \le C\Big\{\sum_{|\tilde\alpha'|=2} \|r^{\beta}D^{\tilde\alpha'}u\|_{\mathbf{L}^2(\mathcal{U}_1)} + \sum_{|\tilde\alpha'|\le 1} \|D^{\tilde\alpha'}u\|^2_{\mathbf{L}^2(\mathcal{U}_1\setminus\mathcal{U}_{\frac{1}{2}})}\Big\}
\tag{5.6}
$$

provided the right sides of (5.3)–(5.6) are finite. ∎

Theorem 5.2. $\mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij}) \subset \mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$.

Proof. Let $u \in \mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij})$. By Theorem 3.5, $u \in \mathbf{C}^0(\bar{\mathcal{U}}_{ij})$, and by the definition of $\mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij})$

$$
\|r^{\beta_{ij}+\alpha_1+\alpha_2-2}D^{\alpha}u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \le C_0\, d^{\alpha}\alpha!, \quad \text{for } \alpha \text{ with } \alpha_1+\alpha_2 \ge 2,
\tag{5.7}
$$

and

$$
\|D^{\alpha}u\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \le C_0\, d^{\alpha}\alpha!, \quad \text{for } \alpha \text{ with } \alpha_1+\alpha_2 < 2.
\tag{5.8}
$$

For arbitrary $x^0 \in \bar{\mathcal{U}}_{ij}$ there is a cylinder $\tilde{D}(x^0) = \{x \mid \sum_{i=1,2}|x_i - x_i^0|^2 < R(x_0), x_3 \in I\}$ with radius $R(x^0) = \frac{1}{2}\operatorname{dist}(x^0, \Lambda_{ij}) = \frac{1}{2}r(x^0)$.

Let $D(x^0) = \tilde{D}(x^0) \cap \mathcal{U}_{ij}$. Then for any $x \in D(x^0)$

$$\frac{1}{2}r(x^0) \le r(x^0) - R(x^0) \le r(x) \le r(x^0) + R(x^0) = \frac{3}{2}r(x^0). \tag{5.9}$$

Let M be a linear mapping of the cylinder $\tilde{D}_0 = \{\xi = (\xi_1, \xi_2, \xi_3) \mid 0 < \tilde{R} = (\xi_1^2 + \xi_3^2)^{\frac{1}{2}} < 1, \xi_3 \in I\}$ onto $\tilde{D}(x^0)$, and it maps $D_0 = \{\xi \in D_0 \mid 0 < \text{Ⓗ} < \omega_{ij}\}$ onto $D(x^0)$, where $(\tilde{R}, \text{Ⓗ}, \xi_3)$ are the cylindrical coordinates of the point $\xi \in \tilde{D}_0$. Set $v(x) = D^\alpha u(x)$ for α with $|\alpha| \ge \alpha_1 + \alpha_2 \ge 1$, and $V(\xi) = v\big(M(\xi)\big)$. Then

$$\begin{aligned} \|V\|^2_{\mathbf{H}^2(D_0)} &\le C \sum_{|\gamma| \le 2} R^{2(\gamma_1+\gamma_2-1)}(x^0) \int_{D(x^0)} |D^{\alpha+\gamma} u|^2 \, dx \\ &\le C \sum_{|\gamma|\le 2} r^{2(\gamma_1+\gamma_2-1)}(x^0) \int_{D(x^0)} |D^{\alpha+\gamma}u|^2 \, dx \end{aligned} \tag{5.10}$$

where $\gamma = (\gamma', \gamma_3) = (\gamma_1, \gamma_2, \gamma_3)$ and $|\gamma| = |\gamma'| + \gamma_3 = \gamma_1 + \gamma_2 + \gamma_3$.

If $|\alpha'| + |\gamma'| = \sum_{s=1}^2 (\alpha_s + \gamma_s) \ge 2$ we have by (5.9)

$$\begin{aligned} \int_{D(x^0)} |D^{\alpha+\gamma}u|^2 \, dx &\le C\left(\frac{r(x^0)}{2}\right)^{-2(\beta_{ij}+|\gamma'|+|\alpha'|-2)} \int_{D(x^0)} r^{2(\beta_{ij}+|\alpha'|+|\gamma'|-2)}(x)|D^{\alpha+\gamma}u|^2 \, dx \\ &\le CC_0^2 r^{-2(\beta_{ij}+|\alpha'|+|\gamma'|-2)}(x^0)\big((2d)^{\alpha+\gamma}(\alpha+\gamma)!\big)^2. \end{aligned} \tag{5.11}$$

If $|\alpha'| + |\gamma'| = 1$, we have by (5.9) and Lemma A.1 of [4]

$$\begin{aligned} \int_{D(x^0)} |D^{\alpha+\gamma}u|^2 \, dx &\le Cr^{-2(\beta_{ij}-1)}(x^0) \int_{D(x^0)} r^{2(\beta_{ij}-1)}(x)|D^{\alpha+\gamma}u|^2 \, dx \\ &\le Cr^{-2(\beta_{ij}-1)}(x^0) \int_{\mathcal{U}_{ij}} r^{2\beta_{ij}}(x)\big(|D^1(D^{\alpha+\gamma}u)|^2 + |D^{\alpha+\gamma}u|^2\big) \, dx \\ &\le CC_0^2 r^{-2(\beta_{ij}-1)}(x^0)(d^\alpha \alpha!)^2. \end{aligned} \tag{5.12}$$

Combining (5.10)–(5.12) we get for $\alpha_1 + \alpha_2 \ge 1$

$$\|V\|_{\mathbf{H}^2(D_0)} \le \tilde{C} r^{-(\beta_{ij}+\alpha_1+\alpha_2-1)}(x^0)\tilde{d}^\alpha \alpha!$$

with $\tilde{C} = CC_0 > C_0$ and $\tilde{d} > d$. The Sobolev imbedding theorem implies that for $|\alpha| \ge \alpha_1 + \alpha_2 \ge 1$

$$\begin{aligned} \|D^\alpha u\|_{\mathbf{C}^0(\bar{D}(x^0))} &= \|V\|_{\mathbf{C}^0(\bar{D}_0)} \\ &\le \tilde{C} r^{-(\beta_{ij}+|\alpha'|-1)}(x^0)\tilde{d}^\alpha \alpha!. \end{aligned} \tag{5.13}$$

Now consider the case for α with $|\alpha'| = \alpha_1 + \alpha_2 = 0$. Let $v = D^\alpha\big(u(x) - u(0,0,x_3)\big)$ and $V(\xi) = v\big(M(\xi)\big)$. Analogously we have for γ with $|\gamma'| = \gamma_1 + \gamma_2 = 2$

$$\begin{aligned} \|D^\gamma V\|_{\mathbf{L}^2(\tilde{D}_0)} &= R^{2(|\gamma'|-1)}(x^0) \int_{D(x^0)} |D^{\alpha+\gamma}u|^2 \, dx \\ &\le Cr^{2(1-\beta_{ij})}(x^0) \int_{D(x^0)} r(x)^{2(\beta_{ij}+|\gamma'|-2)}|D^{\alpha+\gamma}u|^2 \, dx \\ &\le CC_0 r^{2(1-\beta_{ij})}(x^0) d^{\alpha+\gamma}(\alpha+\gamma)! \end{aligned} \tag{5.14}$$

and for γ with $|\gamma| \le 2$ and $|\gamma'| = \gamma_1 + \gamma_2 = 1$ we have by (5.9)

$$\|D^\gamma V\|_{\mathbf{L}^2(D_0)} = \int_{D(x^0)} |D^{\alpha+\gamma} u|^2\, dx \tag{5.15}$$
$$\le C r^{-2(\beta_{ij}-1)}(x^0) \int_{D(x^0)} r^{2(\beta_{ij}-1)} |D^{\alpha+\gamma} u|^2\, dx$$

by (5.3) of Lemma 5.1

$$\le C r^{-2(\beta_{ij}-1)}(x^0) \int_{D(x^0)} r^{2\beta_{ij}} |D^1(D^{\alpha+\gamma} u)|^2\, dx$$
$$\le C C_0 r^{-2(\beta_{ij}-1)}(x^0) d^{\alpha+\gamma+1}(\alpha+\gamma+1)!$$

and for γ with $|\gamma| \le 2$ and $\gamma_1 + \gamma_2 = 0$ we have by (5.9)

$$\|D^\gamma V\|^2_{\mathbf{L}^2(D_0)} = R(x^0)^{-2} \int_{D(x^0)} \left|D^{\alpha+\gamma}\big(u - u(0,0,x_3)\big)\right|^2 dx \tag{5.16}$$
$$\le C r^{-2(\beta_{ij}-1)}(x^0) \int_{D(x^0)} r^{2(\beta_{ij}-2)} \left|D^{\alpha+\gamma}\big(u - u(0,0,x_3)\right|^2 dx$$

by (5.6) of Lemma 5.1

$$\le C r^{-2(\beta_{ij}-1)}(x^0) \|D^{\alpha+\gamma} u\|^2_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})}$$
$$\le C C_0 r^{-2(\beta_{ij}-1)}(x^0) d^{\alpha+\gamma+2}(\alpha+\gamma+2)!$$

Combining (5.14)–(5.16) we obtain for some $\tilde{C} = CC_0 \ge C_0$ and $\tilde{d} > d$

$$\|V\|_{\mathbf{H}^2(D_0)} \le \tilde{C} r^{-2(\beta_{ij})}(x^0) \tilde{d}^\alpha \alpha!.$$

The Sobolev imbedding theorem (see [1]) further leads for α with $\alpha_1 + \alpha_2 = 0$ to

$$\left\|D^\alpha\big(u - u(0,0,x_3)\big)\right\|_{\mathbf{C}^0(\bar{D}(x^0))} = \|V\|_{\mathbf{C}^0(\bar{D}_0)} \tag{5.17}$$
$$\le \tilde{C} r^{-(\beta_{ij}-1)}(x^0) \tilde{d}^\alpha \alpha!.$$

Note that the constants C and $\tilde{d}$ above are independent of x^0, and x^0 is an arbitrary point in $\mathcal{U}_{ij}$. Hence (5.1) follows from (5.13) and (5.17) at once.

To prove (5.2) we let $w(x) = u_{x_3^k}(x)$. By Theorem 3.5 $w(x) \in \mathbf{C}^0(\bar{\mathcal{U}}_{ij})$, and for some $\tilde{d} > d$

$$\|w(x)\|_{\mathbf{C}^0(\bar{\mathcal{U}}_{ij})} \le C\|w(x)\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} = C\|u_{x_3^k}\|_{\mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}_{ij})} \tag{5.18}$$
$$\le C C_0 d_3^k (k+2)! (\max\{d_i\})^2 \le \tilde{C} \tilde{d}^\alpha \alpha!$$

where $\tilde{C} > CC_0 \ge C_0$ and $\tilde{d} \ge d$. This leads (5.2) and completes the theorem. ■

5.2. Countably Normed Space $\mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$.

Let $\tilde{\mathcal{O}}_m$ be the inner-neighbourhood of A_m as before. It is assumed that A_m is located in the origin, and one of the edge Λ_{ij}, $ij \in \mathcal{L}_m$ lies on the positive x_3-axis. Let $\rho(x) = \rho = \operatorname{dist}(x, A_m)$ for $x \in \tilde{\mathcal{O}}_m$.

By $\mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$, $\beta_m \in (0, \frac{1}{2})$ we denote a set of functions $u(x) \in \mathbf{C}^0(\tilde{\mathcal{O}}_m)$ such that for $|\alpha| \geq 0$

$$\left\|\rho^{\beta_m+|\alpha|-\frac{1}{2}} D^\alpha\big(u(x) - u(A_m)\big)\right\|_{\mathbf{C}^0(\overline{\tilde{\mathcal{O}}}_m)} \leq C d^\alpha \alpha! \tag{5.19}$$

which is equivalent to

$$\left|D^\alpha\big(u(x) - u(A_m)\big)\right| \leq C d^\alpha \alpha! \rho^{-(\beta_m+|\alpha|-\frac{1}{2})}(x). \tag{5.19'}$$

This shows how the derivatives grow as α increases and x tends to the vertex A_m.

Due to the definition $\mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$ and $\mathbf{B}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$ we immediately have the following theorem.

Theorem 5.3. *$\mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}}_m) \subset \mathbf{B}^2_{\beta_m+\varepsilon}(\tilde{\mathcal{O}}_m)$ with $\varepsilon > 0$, arbitrary.*

For the converse we introduce a lemma.

Lemma 5.2. *Let $u \in \mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$. Then*

$$\int_{\tilde{\mathcal{O}}_m} \rho^{2(\beta_m-2)} |u - u(A_m)|^2 \, dx \leq C \|u\|^2_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \tag{5.20}$$

Proof. Due to the imbedding of $\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)$ (Theorem 3.6) $u \in \mathbf{C}^0(\overline{\tilde{\mathcal{O}}}_m)$. We can prove

$$\int_{\tilde{\mathcal{O}}_m} \rho^{2(\beta_m-2)} |u - u(A_m)|^2 \, dx \leq C \|\rho^{\beta_m-1} |D^1 u|\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)}$$

in the same way as that for (4.3) of Lemma 4.1 except that

$$\int_0^\delta \rho^{2\beta_m-2} |\bar{u}(\rho) - a|^2 \, d\rho \leq C \|\rho^{\beta_m-1} u_\rho\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)}$$

is used with $a = \bar{u}(0) = u(A_m)$, instead of $\bar{u}(1)$. Then applying (4.4) of Lemma 4.1 we obtain (5.20). ∎

Theorem 5.4. $\mathbf{B}^2_{\beta_m}(\tilde{\mathcal{O}}_m) \subset \mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}})$.

Proof. Let $u \in \mathbf{B}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$. Then $u \in \mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m) \subset \mathbf{C}^0(\overline{\tilde{\mathcal{O}}}_m)$ by Theorem 3.6, and for α with $|\alpha| \geq 2$

$$\|\rho_m^{|\alpha|-2+\beta_m} D^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C d^\alpha \alpha!$$

and for α with $|\alpha| \leq 1$

$$\|D^\alpha u\|_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} \leq C.$$

Fix an arbitrary point $x_0 \in \overline{\tilde{\mathcal{O}}}_m$ but $x_0 \neq A_m$. There exists a ball $\tilde{D}(x^0)$ centered at x^0 with radius $R(x^0) = \frac{1}{2}\operatorname{dist}(x^0, A_m) = \frac{1}{2}\rho(x^0)$. By $D(x^0)$ we denote $D(x^0) \cap \tilde{\mathcal{O}}_m^0$. For any $x \in D(x^0)$ we have

$$\frac{1}{2}\rho(x^0) \leq \rho(x^0) - R(x^0) = \rho(x) \leq \rho(x^0) + R(x^0) \leq \frac{3}{2}\rho(x^0). \tag{5.21}$$

Let M be a linear mapping of the unit ball $\tilde{D}_0 = \{\xi = (\xi_1, \xi_2, \xi_3) \mid (\sum_{i=1}^{3} \xi_i^2)^{\frac{1}{2}} < 1\}$ onto $\tilde{D}(x^0)$, which also maps D_0 onto $D(x^0)$. Set $v(x) = D^\alpha\big(u - u(A_m)\big)$, $|\alpha| \geq 0$ and $V(\xi) = v\big(M(\xi)\big)$. Then by (5.21)

$$\begin{aligned} \|V\|_{\mathbf{H}^2(D_0)} &\leq C \sum_{|\gamma| \leq 2} R^{2(|\gamma| - \frac{3}{2})}(x^0) \int_{D(x^0)} |\, D^\gamma V|^2 \, dx \\ &\leq C \sum_{|\gamma| \leq 2} \rho^{2(|\gamma| - \frac{3}{2})}(x^0) \int_{D(x^0)} |D^{\alpha+\gamma} u|^2 \, dx \end{aligned} \tag{5.22}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $\gamma' = (\gamma_1, \gamma_2)$. If $|\alpha + \gamma| \geq 2$ we have

$$\begin{aligned} \int_{D(x^0)} |D^{\alpha+\gamma} u|^2 \, dx &\leq C \left(\frac{\rho(x^0)}{2}\right)^{-2(|\alpha+\gamma| - 2 + \beta_m)} \int_{D(x^0)} \rho^{2(|\alpha+\gamma| - 2 + \beta_m)} |D^{\alpha+\gamma} u|^2 \, dx \\ &\leq C \rho^{-2(|\alpha+\gamma| - 2 + \beta_m)}(x^0) \big((\alpha d)^{\alpha+\gamma} (\alpha+\gamma)!\big)^2. \end{aligned} \tag{5.23}$$

If $|\alpha + \gamma| = 1$, then by (4.4) of Lemma 4.1

$$\begin{aligned} \int_{D(x^0)} |D^{\alpha+\gamma} u| \, dx^2 &\leq C \rho^{2(1-\beta_m)}(x^0) \int_{D(x^0)} \rho^{2(\beta_m - 1)} |D^{\alpha+\gamma} u|^2 \, dx \\ &\leq C \rho^{2(1-\beta_m)}(x^0) \|u\|^2_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \end{aligned} \tag{5.24}$$

If $|\alpha + \gamma| = 0$, then by (5.20) of Lemma 5.2 we have

$$\begin{aligned} \int_{D(x^0)} |u - u(A_m)|^2 \, dx &\leq C \rho^{2(2-\beta_m)}(x^0) \int_{D(x^0)} \rho^{2(\beta_m - 2)} |u - u(A_m)|^2 \, dx \\ &\leq C \rho^{2(2-\beta_m)}(x^0) \|u\|^2_{\mathbf{H}^{2,2}_{\beta_m}(\tilde{\mathcal{O}}_m)}. \end{aligned} \tag{5.25}$$

Combining (5.22)–(5.25) we obtain for some $\tilde{d} > 2d$

$$\|V\|_{\mathbf{H}^2(D_0)} \leq C \rho^{-(\beta_m + |\alpha| - \frac{1}{2})}(x^0) \tilde{d}^\alpha \alpha!.$$

By Sobolev imbedding theorem

$$\|V\|_{\mathbf{C}^0(\bar{D}_0)} \leq C \|V\|_{\mathbf{H}^2(D_0)} \leq C \rho(x^0)^{-(\beta_m + |\alpha| - \frac{1}{2})} \tilde{d}^\alpha \alpha!.$$

Note that x_0 is arbitrary and that C and $\tilde{d}$ are independent of x_0 and α. Hence (5.19) follows immediately. ∎

5.3. Countably Normed Space $\mathbf{C}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$.

Let $\mathcal{V}_{m,ij} = S_{\sigma_{ij}} \times I_{\delta_m}$ be the neighbourhood of A_m and Λ_{ij} as before, and we assume that A_m is at the origin and Λ_{ij} lies on the positive x_3-axis. Let $\rho(x) = \rho = \text{dist}(x, A_m)$, $r(x) = r = \text{dist}(x, \Lambda_{ij})$, and $\sin\phi = \sin\phi(x) = \frac{r(x)}{\rho(x)}$. $\phi = \phi(x)$ is the angle between the radial $A_m x$ and the edge Λ_{ij}.

By $\mathbf{C}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, $\beta_m \in (0, \frac{1}{2})$ and $\beta_{ij} \in (0,1)$ we denote a set of functions $u(x) \in \mathbf{C}^0(\bar{\mathcal{V}}_{m,ij})$ such that for α with

$$\left\|\rho^{\beta_m+|\alpha|-\frac{1}{2}}(\sin\phi)^{\beta_{ij}+\alpha_1+\alpha_2-1} D^\alpha\big(u(x) - u(0,0,x_3)\big)\right\|_{\mathbf{C}^0(\bar{\mathcal{V}}_{m,ij})} \le C d^\alpha \alpha! \tag{5.26}$$

and

$$\left\| |x_3|^{\beta_m+|\alpha|-\frac{1}{2}} \frac{d^k}{dx_3^k}\big(u(0,0,x_3) - u(A_m)\big)\right\|_{\mathbf{C}^0(\bar{\mathbf{I}}_{\delta_m})} \le C d_3^k k!, \tag{5.27}$$

(5.26) is equivalent to the estimates

$$\left|D^\alpha\big(u(x) - u(0,0,x_3)\big)\right| \le C d^\alpha \alpha! \rho^{-(\beta_m+|\alpha|+\frac{1}{2})}(x)(\sin\phi(x))^{-(\beta_{ij}+\alpha_1+\alpha_2-1)} \tag{5.26$'$}$$

which indicates the growth of the derivatives with respect to $\rho(x)$, $\phi(x)$ and α, and (5.27) is equivalent

$$\left|\frac{d^k}{dx_3^k}\big(u(0,0,x_3) - u(A_m)\big)\right| \le C d_3^k k! |x_3|^{-(\beta_m+k-\frac{1}{2})}. \tag{5.27$'$}$$

which tells that the trace of $u(x)$ on the edge Λ_{ij} belongs to the countably normed space $\mathbf{C}^2_{\beta_m}(I_{\delta_{ij}})$ with respect to the vertex A_m (see [5]).

Then by the definition of the space $\mathbf{C}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and the space $\mathbf{B}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ we immediately conclude

Theorem 5.5. *$\mathbf{C}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) \subset \mathbf{B}^2_{\beta_{m,ij}+\varepsilon}(\mathcal{V}_{m,ij})$ with $\beta_{m,ij} + \varepsilon = (\beta_m + \varepsilon, \beta_{ij} + \varepsilon)$, $\varepsilon > 0$ arbitrary.* ■

For the converse theorem we need a lemma.

Lemma 5.3 Let $u \in \mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, *then*

$$\int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-2)}|u - u(A_m)|^2 \, dx \le C\|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}. \tag{5.28}$$

Proof. Due to the imbedding of $\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ into $C^0(\mathcal{V}_{m,ij})$ (Theorem 3.7), $u \in C^0(\mathcal{V}_{m,ij})$. It can be proved that

$$\int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-2)}|u - u(A_m)|^2 \, dx \le C\|\rho^{\beta_m-1}D^1 u\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})}$$

in the same way as that for (4.3) of Lemma 4.1 except that

$$\int_0^\delta \rho^{2(\beta_m-2)}|\bar{u}(\rho)-a|^2 \, d\rho \le C\|\rho^{\beta_m-1}u_\rho\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})}$$

is used with $a=\bar{u}(0)=u(A_m)$, instead of $\bar{u}(1)$. Furhter

$$\int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-1)}|D^1u|^2 \, dx \le C\int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^1u|^2 \, dx$$

by (4.12) of Lemma 4.2

$$\le C\|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}$$

which leads to (5.28).

<u>Theorem 5.6</u>. $\mathbf{B}^2_{\beta_{m,ij}}(\bar{\mathcal{V}}_{m,ij})\subset \mathbf{C}^2_{\beta_{m,ij}}(\bar{\mathcal{V}}_{m,ij})$.

Proof. We assume that $\sigma_{ij}=\sigma<\frac{\pi}{4}$. For a fixed point $x^0\in\bar{\mathcal{V}}_{m,ij}$ but $x^0\notin\bar{\Lambda}_{ij}$, there exists an ellipsoid $\tilde{D}(x^0)=\{x=(x_1,x_2,x_3)\mid \sum_{i=1,2}\frac{(x_i-x_i^0)^2}{a^2}+\frac{(x_3-x_3^0)^2}{b^2}=1$ with $a=\frac{1}{2}\operatorname{dist}(x_0,\Lambda_{ij})=\frac{1}{2}r(x^0)$ and $b=\frac{1}{2}\operatorname{dist}(x_0,A_m)=\frac{1}{2}\rho(x^0)\}$. Let $D(x^0)=\tilde{D}(x^0)\cap\mathcal{V}_{m,ij}$. Then obviously we have

$$\frac{1}{2}r(x^0)\le r(x^0)-a\le r(x)\le r(x^0)+a\le\frac{3}{2}r(x^0) \tag{5.29}$$

and

$$\frac{1}{2}\rho(x^0)\le \rho(x^0)-b\le \rho(x)\le \rho(x^0)+b\le\frac{3}{2}\rho(x^0). \tag{5.30}$$

Let $u(x)\in\mathbf{B}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$. Then $u(x)\in\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})\subset\mathbf{C}^0(\bar{\mathcal{V}}_{m,ij})$ by the imbedding theorem (Theorem 3.7), and for $|\alpha|\ge\alpha_1+\alpha_2\ge2$

$$\|\rho^{\beta_m+|\alpha|-2}(\sin\phi)^{\beta_{ij}+\alpha_1+\alpha_2-2}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\le Cd^\alpha\alpha!$$

and for $|\alpha|\ge2$ and $\alpha_1+\alpha_2\le2$

$$\|\rho^{\beta_m+|\alpha|-2}D^\alpha u\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}\le Cd^\alpha\alpha!.$$

Now let M be a linear mapping of the unit ball $\tilde{D}_0=\{\xi=(\xi_1,\xi_2,\xi_3)\mid\sum_{i=1}^3\xi_i^2\le1\}$ onto the ellipsoid $\tilde{D}(x^0)$, then M maps D_0 onto $D(x^0)$. For $|\alpha|\ge\alpha_1+\alpha_2\ge1$ set $v=D^\alpha u$ and $v(\xi)=V\big(M(\xi)\big)$. Then

$$\|V\|^2_{\mathbf{H}^2(D_0)}=\sum_{|\gamma|\le2}a^{2(\gamma_1+\gamma_2-1)}b^{2(\gamma_3-\frac{1}{2})}\int_{D(x^0)}|D^{\alpha+\gamma}u|^2\,dx. \tag{5.31}$$

If $|\alpha+\gamma| \geq \sum_{s=1}^{2}(\alpha_s+\gamma_s) \geq 2$ we have

(5.32)
$$\begin{aligned}\int_{D(x^0)} |D^{\alpha+\gamma}u|^2\,dx \leq C\Big(\frac{\rho(x^0)}{2}\Big)^{-2(\beta_m+|\alpha+\gamma|-2)}&\Big(\frac{r(x_0)}{3\rho(x^0)}\Big)^{-2(\beta_{ij}+|\alpha'|+|\gamma'|-2)}\\ &\cdot\int_{D(x^0)} \rho^{2(\beta_m+|\alpha+\gamma|-2)}(\sin\phi)^{2(\beta_{ij}+|\alpha'|+|\gamma'|-2)}|D^{\alpha+\gamma}u|^2\,dx\\ &\leq C\rho(x^0)^{-2(\beta_m+|\alpha+\gamma|-2)}\big(\sin\phi(x^0)\big)^{-2(\beta_{ij}+|\alpha'|+|\gamma'|-2)}\big((6d)^{\alpha+\gamma}(\alpha+\gamma)!\big)^2.\end{aligned}$$

If $|\alpha+\gamma| \geq 2$ and $\alpha_1+\alpha_2=1$, $\gamma_1=\gamma_2=0$, we have

$$\begin{aligned}\int_{D(x^0)} |D^{\alpha+\gamma}u|^2\,dx \leq C\Big(\frac{3}{2}\rho(x^0)\Big)^{2(1-\beta_m)}&\Big(\frac{3r(x^0)}{\rho(x^0)}\Big)^{2(1-\beta_{ij})}\\ &\cdot\int_{\frac{1}{2}\rho(x^0)}^{\delta}\int_{S_{\sigma_{ij}}} \rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^{\alpha+\gamma}u|^2\rho^2\,dSd\rho\end{aligned}\tag{5.33}$$

By the arguments similar to those for (4.15) and (4.16) we have

$$\begin{aligned}&\int_{\frac{1}{2}\rho(x^0)}^{\delta}\int_{S_{\sigma_{ij}}} \rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^{\alpha+\gamma}u|^2\rho^2\,dSd\rho\\ \leq& C\int_{\frac{1}{2}\rho(x^0)}^{\delta}\int_{S_{\sigma_{ij}}}\Big(\rho^{2\beta_m}(\sin\phi)^{2\beta_{ij}}\sum_{i=1}^{3}|D^{\alpha+\gamma}u_{x_i}|^2+|D^{\alpha+\gamma}u|^2\Big)\rho^2\,dSd\rho\\ \leq& C(\frac{\rho(x^0)}{2})^{-2(|\alpha+\gamma|-1)}\cdot\Big(\int_{\mathcal{V}_{m,ij}}\rho^{2(\beta_m+|\alpha+\gamma|-1)}(\sin\phi)^{2\beta_{ij}}\sum_{i=1}^{3}|D^{\alpha+\gamma}u_{x_i}|^2\,dx\\ &+\int_{\mathcal{V}_{m,ij}}\rho^{2(\beta_m+|\alpha+\gamma|-1)}|D^{\alpha+\gamma}u|^2\,dx\Big).\end{aligned}\tag{5.34}$$

Combination of (5.33) and (5.34) gives for $|\alpha+\gamma| \geq 2, \alpha_1+\alpha_2=1$ and $\gamma_1=\gamma_2=0$

$$\begin{aligned}\int_{D(x^0)} |D^{\alpha+\gamma}u^2\,dx \leq& C\rho^{-2(\beta_m+|\alpha+\gamma|-2)}(\sin\phi(x^0))^{-2(\beta_{ij}-1)}d^{\alpha+\gamma}(\alpha+\gamma)!\\ &\cdot\Big(\sum_{i=1}^{3}(\alpha_i+1)d_i+1\Big).\end{aligned}\tag{5.35}$$

If $|\alpha|=\alpha_1+\alpha_2=1$, $|\gamma|=0$, we have

$$\begin{aligned}\int_{D(x^0)} |D^{\alpha}u|^2\,dx \leq C\Big(\frac{3}{2}\rho(x^0)\Big)^{2(1-\beta_m)}&\Big(\frac{3r(x^0)}{\rho(x^0)}\Big)^{2(1-\beta_{ij})}\\ &\cdot\int_{D(x^0)}\rho^{2(\beta_m-1)}(\sin\phi)^{2(\beta_{ij}-1)}|D^{\alpha}u|^2\,dx\end{aligned}\tag{5.36}$$

by (4.12) of Lemma 4.2

$$\leq C\rho^{2(1-\beta_m)}(x^0)\Big(\sin\phi(x^0)\Big)^{2(1-\beta_{ij})}\|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.$$

Combining (5.31) and (5.35))-(5.36) we get for α with $\alpha_1+\alpha_2\geq 1$ and $\tilde{d}\geq 6d$

$$\|V\|_{\mathbf{H}^2(D_0)}\leq C\rho^{-(\beta_m+|\alpha|-\frac{1}{2})}(x^0)\big(\sin\phi(x^0)\big)^{-(\beta_{ij}+\alpha_1+\alpha_2-1)}\tilde{d}^\alpha\alpha!. \tag{5.37}$$

We next consider the case that $|\alpha'|=\alpha_1+\alpha_2=0$. Set $v=u-u(0,0,x_3)$ and $V=v\big(M(\xi)\big)$. Then analogously we have

$$\|V\|_{\mathbf{H}^2(D_0)}\leq C\rho(x^0)^{-(\beta_m+|\alpha|-\frac{1}{2})}\big(\sin\phi(x^0)\big)^{-(\beta_{ij}-1)}\tilde{d}^\alpha\alpha!. \tag{5.38}$$

By Sobolev imbedding theorem

$$\|V\|_{\mathbf{C}^0(\bar{D}_0)}\leq C\|V\|_{\mathbf{H}^2(D_0)}$$

which together with (5.37) and (5.38) implies (5.26).

In order to prove (5.27) we let $x^0=(0,0,x_3^0)$ with $x_3^0\in I_{\delta_m}$ and $\mathcal{U}_{x^0}=\{x\in\mathcal{V}_{m,ij}\mid 0<r<\frac{1}{2}x_3^0\tan\sigma,\frac{1}{2}x_3^0<x_3<x_3^0\}$. There is a mapping M:

$$\begin{cases} x_1=\frac{1}{2}x_3^0\tan\sigma\,\xi_1\\ x_2=\frac{1}{2}x_3^0\tan\sigma\,\xi_2\\ x_3=x_3^0\xi_3\end{cases}$$

which maps $\mathcal{U}^0=\{\xi=(\xi_1,\xi_2,x_3)\mid 0<\sqrt{\xi_1^2+\xi_2^2}<1,\frac{1}{2}<\xi_3<1\}$ onto $\mathcal{U}_{x^0}$. Let $U(\xi)=u\big(M(\xi)\big)-u(0,0,0)$. Then for α with $\alpha_1+\alpha_2\geq 2$

$$\int_{\mathcal{U}^0}|D^\alpha U|^2(\xi_1^2+\xi_2^2)^{\beta_{ij}+\alpha_1+\alpha_2-2}\,d\xi\leq C{x_3^0}^{2(\alpha_3+\frac{1}{2}-\beta_{ij})}\int_{\mathcal{U}_{x^0}}|D^\alpha u|^2r^{2(\beta_{ij}+\alpha_1+\alpha_2-2)}\,dx \tag{5.39}$$

and

$$\int_{\mathcal{U}_{x^0}}|D^\alpha u|^2r^{2(\beta_{ij}+\alpha_1+\alpha_2-2)}\,dx\leq CA(x^0)\int_{\mathcal{U}_{x^0}}|D^\alpha u|^2\rho^{2(\beta_m+|\alpha|-2)}(\sin\phi)^{2(\beta_{ij}+\alpha_1+\alpha_2-2)}\,dx \tag{5.40}$$

where

$$A(x^0)=\Big(\frac{x_3^0}{2}\Big)^{-2(\beta_m-\beta_{ij}+\alpha_3)}\quad\text{if }\beta_m-\beta_{ij}+\alpha_3\geq 0$$

and

$$A(x^0)=\Big(\frac{x_3^0}{\cos\sigma}\Big)^{-2(\beta_m-\beta_{ij}+\alpha_3)}\quad\text{if }\beta_m-\beta_{ij}+\alpha_3<0$$

(5.39) and (5.40) yield for $\alpha_1+\alpha_2\geq 2$

$$\begin{aligned}\int_{\mathcal{U}^0}|D^\alpha U|^2(\xi_1^2+\xi_2^2)^{\beta_{ij}+\alpha_1+\alpha_2-2}\,d\xi&\leq C2^{\alpha_3}|x_3^0|^{-2(\beta_m-\frac{1}{2})}d^\alpha\alpha!\\&\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})}(2d)^\alpha\alpha!.\end{aligned} \tag{5.41}$$

Here we used the assumption that $\sigma < \frac{\pi}{4}$.

For α with $\alpha_1 + \alpha_2 < 2$ and $|\alpha| \geq 2$ we have

$$
\begin{aligned}
\int_{\mathcal{U}^0} |D^\alpha U|^2 \, d\xi &\leq C|x_3^0|^{2(|\alpha|-\frac{3}{2})} \int_{\mathcal{V}_{m,ij}} |D^\alpha u|^2 \, dx \\
&\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} 2^{|\alpha|} \int_{\mathcal{U}_{x^0}} |D^\alpha u|^2 \rho^{2(|\alpha|-2+\beta_m)} \, dx \\
&\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} (2d)^\alpha \alpha!.
\end{aligned} \tag{5.42}
$$

For α with $|\alpha| = 1$

$$
\begin{aligned}
\int_{\mathcal{U}^0} |D^\alpha U|^2 \, d\xi &\leq C|x_3^0|^{-1} \int_{\mathcal{U}_{x^0}} |D^\alpha u|^2 \, dx \\
&\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} \int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-1)} |D^\alpha u|^2 \, dx
\end{aligned}
$$

by (4.3) of Lemma 4.1

$$
\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})},
$$

and

$$
\begin{aligned}
\int_{\mathcal{U}^0} |U(\xi)|^2 \, d\xi &\leq C|x_3^0|^{-3} \int_{\mathcal{U}_{x^0}} |u(x) - u(0,0,0)|^2 \, dx \\
&\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} \int_{\mathcal{V}_{m,ij}} \rho^{2(\beta_m-2)} |u(x) - u(0,0,0)|^2 \, dx
\end{aligned} \tag{5.43}
$$

by Lemma 5.3

$$
\leq C|x_3^0|^{-2(\beta_m-\frac{1}{2})} \|u\|^2_{\mathbf{H}^{2,2}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.
$$

Therefore $\mathcal{U}(\xi) \in \mathbf{H}^{2,2}_{\beta_{ij}}(\mathcal{U}^0)$ due to (5.41)-(5.43). Then applying (5.18) of Theorem 5.2 we have that for $\xi_3 \in [\frac{1}{2}, 1]$ and $\tilde{d} = Kd$ with some $K > 1$

$$
\left| \frac{d^k}{d\xi_3^k} \big(U(0,0,\xi_3) - U(0,0,0) \big) \right| \leq C|x_3^0|^{-(\beta_m-\frac{1}{2})} \tilde{d}^\alpha \alpha!
$$

which implies

$$
\left| \frac{d^k}{dx_3^k} \big(u(0,0,x_3^0) - u(0,0,0) \big) \right| \leq C|x_3^0|^{-(\beta_m+k-\frac{1}{2})} \tilde{d}^\alpha \alpha!.
$$

Since x_3^0 is arbitrary, this completes the proof of (5.26), and hence the theorem.

5.4. Countably Normed Space $\mathbf{C}^2_\beta(\Omega)$.

By $\mathbf{C}^2_\beta(\Omega)$ we denote the countably weighted continuous function space, namely for $u \in \mathbf{C}^2_\beta(\Omega)$, there hold:

(i) $u \in \mathbf{C}^0(\bar{\Omega})$;

(ii) $u|_{\mathcal{U}_{ij}} \in \mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$ for any $ij \in \mathcal{L}$;

(iii) $u|_{\tilde{\mathcal{O}}_m} \in \mathbf{C}^2_{\beta_m}(\tilde{\mathcal{O}}_m)$ for any $m \in \mathcal{M}$;

(iv) $u|_{\mathcal{V}_{m,ij}} \in \mathbf{C}^2_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ for any $ij \in \mathcal{L}_m$ and $m \in \mathcal{M}$;

(v) $u|_{\bar{\Omega}_0} \in \mathbf{C}^\infty(\bar{\Omega}_0)$, and for $x \in \bar{\Omega}_0$

$$|D^\alpha u(x)| \le C d^\alpha \alpha!.$$

Due to the definition of $\mathbf{B}^2_\beta(\Omega)$ and $\mathbf{C}^2_\beta(\Omega)$ and Theorem 5.1–5.6 we have the following conclusion.

Theorem 5.7. $\mathbf{B}^2_\beta(\Omega) \subset \mathbf{C}^2_\beta(\Omega) \subset \mathbf{B}^2_{\beta+\varepsilon}(\Omega)$ *with* $\varepsilon > 0$, *arbitrary.* ■

Remark 5.1. *Analogously the spaces* $\mathbf{C}^\ell_\beta(\Omega)$ *for* $\ell > 2$ *and* $0 \le \ell < 2$ *can be defined, then similar results for these spaces will be valid, namely,* $\mathbf{B}^\ell_\beta(\Omega) \subset \mathbf{C}^\ell_\beta(\Omega) \subset \mathbf{B}^\ell_{\beta+\varepsilon}(\Omega)$ *with* $\varepsilon > 0$, *arbitrary.* ■

We now have established the theory of the countably normed spaces and the dynamical weighted Sobolev spaces in $\mathbb{R}^3$,which will be the foundation to study the regularity of solutions for elliptic problems on polyhedral domains in the forth coming papers[26,27]. The theorems on imbedding and the equavilence of spaces in different coordinates and in different weighted norms will precisely characterize the behaviours of the solutions in various neighborhoods of polyhedral domains. The theory can be generalized further for the problems on nonsmooth domains in $\mathbb{R}^3$ with surface boundaries. Nevertheless we will not elaborate it here although this case is important in engineering applications.

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