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The Response of a Beaded Spring

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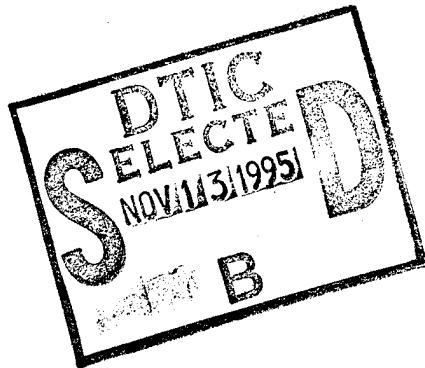19951109 081

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Abstract

Two distinct approaches are employed to derive the coplanar response of a beaded string. In the first approach the beads are modeled by the localized impedances that they present to the string. In the second approach the portion of the string between adjacent beads is assigned to be a one-dimensional dynamic system. Adjacent dynamic systems are coupled at the location of the bead they share. The two approaches derive expressions for the response that do not reconcile term by term. However, the expressions are shown to yield identical results by performing a number of challenging computer experiments using the two models and then comparing corresponding data.

I Introduction

A string is employed as a generic master structure to which beads are attached at localized positions as generic appendages [1, 2]. A bead is defined in terms of the impedance that the string perceives in the bead at the position of attachment. It is assumed a priori that the motion of the beaded string and the external drive that generates the motion are coplanar so that this motion can be described in terms of a spatially one-dimensional equation of motion. This equation of motion may be stated in the impedance operator form

$$z(x, \omega)\, v(x, \omega) = p_e(x, \omega) \quad , \tag{1}$$

where x is the spatial variable, ω is the frequency variable, $z(x, \omega)$ is the impedance operator for the beaded string, and $v(x, \omega)$ and $p_e(x, \omega)$ are the response and the external drive on the beaded string, respectively. The impedance operator of the unbeaded string is designated $z_0(x, \omega)$ so that

$$z(x, \omega) = z_0(x, \omega) + z_s(x, \omega) \quad , \tag{2a}$$

$$z_s(x, \omega) = \sum_j z_{sj}(\omega)\, \delta(x - x_j) \quad , \tag{2b}$$

where $z_{sj}(\omega)$ is the impedance of the (j)th bead as perceived by string. The string is considered infinite and uniform so that

$$z_0(x, \omega) = i\omega m\, [1 + (k_p)^{-2}\, \partial^2/\partial x^2] \quad , \tag{3}$$

where m is the mass per unit length and k_p is the free wavenumber

$$k_p = k_{p0}(1 - i\eta_p) \quad ; \qquad k_{p0} = (\omega/c_p) \quad , \tag{4}$$

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with c_p the speed of propagation and η_p the loss factor in the string. The impedances $z_{sj}(\omega)$ of the beads may be simple; e.g., composed of a mass-spring system in parallel, complex; e.g., composed of several mass-spring systems in various combinations, or even compounded; e.g., composed of wave-bearing components. The bead must, however, present itself to a string coplanarly and locally as stated in Eq. (2). The definition and description of the beaded string just proposed can be modeled in two distinct ways. The first closely resembles the manner of the introduction just presented and is sketched in Fig. 1a. In the second, the string between adjacent beads is defined to be a dynamic system. Then the beaded string is modeled in terms of a cascade of coupled one-dimensional dynamic systems as is sketched in Fig. 1b. The authors, over a period of several years, used one or the other model to estimate the response of complex structural systems; e.g., ribbed panels and networks composed of coupled one-dimensional dynamic systems [2-7]. Notably similar developments were initiated and pursued by Hodges and Woodhouse. Many of their efforts in this area are summarized and referenced in Reference 8. In the present paper a reasonable structural system is doubly modelized to accommodate the two approaches so they can be examined under the same mantle and then compared. The formalism pertaining to each approach will be only briefly presented in the paper. For details the reader may consult References 2-8. It is found that each approach yields a distinct expression for the impulse response function of the beaded string; however, these expressions are a response solution to the same structure and, therefore, they must possess commonality. A particular effort in this paper is finding the commonality that the two formalisms, developed in the two approaches, may possess with respect to this simple generic appended structure. Commonality of this kind can often be used to decipher the advantages and disadvantages that one approach may hold over the other.

There exist two analytical ways of showing that a commonality is present between two distinct expressions. In the first, various terms in the expressions are manipulated until the terms in one expression reconcile, term by term, with those in the other. In the second, a master expression is established which can, by an appropriate operational choice, be converted into one or

the other expression.[1] In a one-and a two-beaded string one may, by manipulating the various terms, readily show the actual identity of the two derived expressions. However, the task of this procedure becomes increasingly more complex as the number of beads increases. Straightforward attempts by the authors have not been successful in systematically reconciling the expressions term by term; it is thus concluded that the expressions are characteristically different. Analogeously the authors have not been successful in finding a master expression that would reduce under certain choices to one or the other of the derived expressions. Although a term by term reconcilation and/or a master expression are helpful, their immediate absence is not a strange occurrence; many different approaches to deriving a solution in the field of physics yield identical results but the expressions for the solution may fail either to reconcile term by term or to multiply emanate from a master expression [11]. With the advent of large computers, in such circumstances the commonality of the two expressions may be shown by conducting computer experiments. In the present paper this procedure is employed; the experiments are selected to be computationally simple but rich in diversity of data, thereby rendering the comparisons challenging.

[1] The Sommerfeld-Watson transformation is a prime example in which a common solution relates two different expressions that constitute an identical solution to the response of a model of a physical system [9]. One expression is modal in character and the other wavy. The master expression in the Sommerfeld-Watson transformation can be converted into the one or the other expression by choosing one or the other path of contour integration to derive the desired expression [10]. It transpires that the two expressions that are derived in these conversions cannot be reconciled term by term; indeed, the two expressions are not even close in satisfying such a reconciliation, notwithstanding that the very essence of the transformation is the derivation of two different expressions to the same solution.

II First approach: Response of a beaded string

Starting with Eqs. (1) and (2) the response behavior of an unbeaded string is tackled first. In terms of the impedance operator $z_0(x, \omega)$ of the unbeaded string the response $v_0(x, \omega)$ generated by the external drive $p_e(x, \omega)$ is expressed

$$z_0(x, \omega) \, v_0(x, \omega) = p_e(x, \omega) \quad . \tag{5}$$

Equation (5) may be inverted to read

$$v_0(x, \omega) = \int g_0(x \mid x', \omega) \, dx' \, p_e(x', \omega) \quad , \tag{6}$$

where $g_0(x \mid x', \omega)$ is the impulse response function of the unbeaded string. From Eq. (3) this impulse response function is derived in the form

$$g_0(x \mid x', \omega) = (2\pi)^{-1/2} \int dk \, Z_0(k, \omega) \, (2\pi)^{-1/2} \exp\left[-ik(x - x')\right] \quad , \tag{7}$$

where $Z_0(k, \omega)$ is the eigenvalue of $z_0(x, \omega)$ with respect to the Fourier eigenfunction $(2\pi)^{-1/2} \exp(-ikx)$; namely

$$\left[z_0(x, \omega) = Z_0(k, \omega)\right] (2\pi)^{-1/2} \exp(-ikx) \quad . \tag{8}$$

It is assumed that the impulse response function is proper and known; the unbeaded string plays the role of a generic master structure in this paper [1, 2]. Indeed, for an infinite uniform string

$$g_0(x \mid x', \omega) = (2mc_p)^{-1} \exp\left[-ik_p|x - x'|\right] \quad , \tag{9}$$

$$\bar{g}_0(x \mid x', \omega) = g_0(x \mid x', \omega) \left[g_0(x' \mid x', \omega)\right]^{-1} = \exp\left[-ik_p|x - x'|\right] \quad , \tag{10}$$

where the free wavenumber k_p, the speed c_p, and m are defined in Eq. (4). Inverting the impedance operator stated in Eq. (2), Eq. (1) may be cast in a corresponding impulse response function form

$$v(x, \omega) = \int g(x \mid x', \omega) \, dx' \, p_e(x', \omega) \quad , \tag{11a}$$

where the impulse response function $g(x \mid x', \omega)$ may be derived in the proper form

$$g(x \mid x', \omega) = g_0(x \mid x', \omega) - g_s(x \mid x', \omega) \quad ;$$

$$g_s(x \mid x', \omega) = \sum_n \sum_m \overline{g}_0(x \mid x_n, \omega) \, \overline{z}_n(\omega) \, c_{nm}(\omega) \, g_0(x_m \mid x', \omega) \quad , \tag{12a}$$

$$\overline{z}_n(\omega) = \overline{z}_{sn}(\omega) \, [1 + \overline{z}_{sn}(\omega)]^{-1} \quad ; \quad \left. \begin{array}{c} \overline{z}_{sn}(\omega) \\ \\ \overline{z}_n(\omega) \end{array} \right\} = g_0(x_n \mid x_n, \omega) \left\{ \begin{array}{c} z_{sn}(\omega) \\ \\ z_n(\omega) \end{array} \right. \quad , \tag{13}$$

$$\underset{\sim}{c}(\omega) = \left(c_{nm}(\omega) \right) = \left(\delta_{ji} + \overline{g}_0(x_j \mid x_i, \omega) \, \overline{z}_i(\omega) \, (1 - \delta_{ji}) \right)^{-1} \quad ;$$

$$\overline{g}_0(x \mid x', \omega) = g_0(x \mid x', \omega) \, [g_0(x' \mid x', \omega)]^{-1} \quad , \tag{14a}$$

and the summations are over the beads [2-4]. Clearly, the impulse response function $g(x \mid x', \omega)$ is proper in the sense that it is a functional of quantities and parameters that describe the properties of the string and beads only; it is independent of the response $v(x, \omega)$ and the external drive $p_e(x, \omega)$ to which the string is subjected [2-7]. In particular, in the absence of the beads $g_s(x \mid x', \omega) \equiv 0$; the impulse response function $g_s(x \mid x', \omega)$ describes the modification of the impulse response function from $g_0(x \mid x', \omega)$ to $g(x \mid x', \omega)$ caused by the introduction of beads. The quantity $\overline{z}_n(\omega)$ is the coupling coefficient of the (n)th bead to the uniform string with Le Chatelier's principle in place [2, 12]. This coupling coefficient is to be distinguished from the coupling coefficient

$c_{nm}(\omega)$, notwithstanding that $c_{nm}(\omega)$ is a functional of the coupling coefficients $\bar{z}_i(\omega)$. The coupling coefficient $c_{nm}(\omega)$ describes the interaction of the (m)th bead, via the string, with the (n)th bead in the insitu presence of all the beads.

In previous works and subsequent discussions it emerges that the coupling coefficient $\bar{z}_n(\omega)$, that describes the coupling of the (n)th bead to the uniform string, is equivalently the reflection coefficient R_n for the "pressure" response at that bead. However, the reflection coefficient R_n^v for the "velocity" response at that bead is equal to $(-R_n)$ and, therefore, $R_n^v = -\bar{z}_n(\omega)$. In terms of the reflection coefficient R_n^v, Eq. (12a) and the first of Eq. (14a) may be written in the forms

$$g(x \mid x', \omega) = g_0(x \mid x', \omega) + g_s^v(x \mid x', \omega) \quad ;$$

$$g_s^v(x \mid x', \omega) = \sum_n \sum_m \bar{g}_0(x \mid x_n, \omega) \, R_n^v \, c_{nm}(\omega) \, g_0(x_m \mid x', \omega) \quad , \tag{12b}$$

$$\underset{\sim}{c}(\omega) = \left(c_{nm}(\omega) \right) = \left(\delta_{ji} - \bar{g}_0(x_j \mid x_i, \omega) \, R_i^v \, (1 - \delta_{ji}) \right)^{-1} \quad , \tag{14b}$$

respectively. Equation (11a) is repeated in the form

$$v_1(x, \omega) = \int g_1(x \mid x', \omega) \, dx' \, p_e(x', \omega) \quad , \tag{11b}$$

where the unit subscript in $g_1(x \mid x', \omega)$ indicates that in Eq. (11b), Eqs. (12b) and (14b) are to be used. The unit subscript in $v_1(x, \omega)$, of Eq. (11b), emphasizes that the response is derived on the basis of the first approach.

III Second approach: Response of coupled dynamic systems

The modeling of the beaded string in terms of a cascade of coupled dynamic systems is briefly depicted in Fig. 1b. However, it may be conducive to formulate here, once again, the response of a general ensemble of coupled one-dimensional dynamic systems [6, 7, 13]. Then the manner of reducing the general formalism to the specific model of a beaded string can be examined as part of the effort in this paper; this reduction is by no means a trivial task. To define a dynamic system; the (j)th, one needs to specify the two terminal positions $x_{\alpha j}$, the two terminal reflection coefficients $\Lambda_{\alpha j j}$ and the two propagation functions $t_j^\alpha(x_j \mid x_j', \omega)$ describing transfer of response toward termination (junction) α from the position x_j' to x_j, where $\alpha = r$ or q, r defining one selected junction and q the other. In a string the propagation function in one direction is the same as for the other, however for the sake of generality, in the definition of the propagation function the distinction of propagation toward one or the other junction is retained. Explicitly then

$$t_j^\alpha(x_j \mid x_j', \omega) = t_{0j}^\alpha(x_j \mid x_j', \omega)\, U\left[(x_j - x_j')\, S(x_{\alpha j} - x_{\beta j})\right] \quad , \tag{15}$$

where

$$U(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \quad ; \quad S(x) = \text{Sign}(x) \quad , \tag{16}$$

and $t_{0j}^\alpha(x_j \mid x_j')$ is the propagation function were the (j)th dynamic system to be extrapolated, toward junction α, uniformly to infinity. A vectorial propagation function $\hat{t}_j(x_j \mid x_j')$ can then be defined in terms of two directional unit vectors $\hat{\alpha}$ in the form

$$\hat{t}_j(x_j \mid x_j') = \sum_{r,q} t_j^\alpha(x_j \mid x_j')\, \hat{\alpha} \quad . \tag{17}$$

The external drive needs to be compatibly defined. The external drive is described in terms of two functions; one pertains to initiating response that propagates toward junction r and the other toward

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junction q. Thus compatibly with Eq. (17), the vectorial form of the external drive is expressed in the form

$$\widehat{v}_{ej}(x_j, \omega) = \sum_{r,q} \widehat{\alpha} \, v_{e\alpha j}(x_j, \omega) \quad , \tag{18}$$

where the component of the external drive $v_{e\alpha j}(x_j, \omega)$ initiates a "velocity" response that propagates toward junction α. A notational convention emerges: Quantities that describe propagation toward junction α are designated by the <u>superscript</u> α and those associated with the external drive that initiates response toward junction α are designated by the <u>subscript</u> α; e.g., the quantity A^α propagates toward junction α and the quantity A^α_β propagates toward junction α and is initiated by the external drive that is directed toward junction β. In addition, the carat over a quantity designates it to be a two-vector; each component is associated with one or the other of the two directional vectors; e.g., $\widehat{B} = \sum_{r,q} B^\alpha \widehat{\alpha}$ or $\widehat{C} = \sum_{r,q} \widehat{\alpha} \, C_\alpha$. The directional vectors are orthogonal in the sense that

$$[\widehat{\alpha} \, \widehat{\beta}] = \delta_{\alpha\beta} \quad . \tag{19}$$

The direct response $v_{dj}(x_j, \omega)$, in the (j)th dynamic system is given by

$$v_{dj}(x_j, \omega) = \int [\widehat{t}_j(x_j \mid x'_j, \omega) \, dx'_j \, \widehat{v}_{ej}(x'_j, \omega)] \quad , \tag{20}$$

The direct response is free, by definition, of any response components that result from interaction with the junctions. From Eqs. (17) - (19) it is derived that

$$v_{dj}(x_j, \omega) = \sum_{r,q} v^\alpha_{dj}(x_j, \omega) \quad ; \quad v^\alpha_{dj}(x_j, \omega) = \int t^\alpha_j(x_j \mid x'_j, \omega) \, dx'_j \, v_{e\alpha j}(x'_j, \omega) \quad . \tag{21}$$

To obtain the response beyond the direct components, it is necessary to account also for components resulting from the interactions with the junctions. In preparation for this accounting a matrix representation of the coupled dynamic systems is constructed as depicted in Fig. 2. This figure plays an essential role in the development of the formalism. The two-vector representation in each dynamic system is supplemented by the vector and matrix representation with respect to the multiplicity of dynamic systems. Thus, the terminal position $x_{\alpha j}$ and the directional unit vector $\hat{\alpha}$ are constructed into a vector and a diagonal matrix quantities of the forms

$$\underline{x}_\alpha = \{x_{\alpha j}\} \quad ; \quad \underset{\approx}{\alpha} = \left(\hat{\alpha}\,\delta_{ji}\right) \quad ; \quad [\underset{\approx}{\alpha}\,\underset{\approx}{\beta}] = \left(\delta_{\alpha\beta}\delta_{ji}\right) \quad . \tag{22}$$

The reflection coefficients at the junctions are now replaced by junction matrices of the forms

$$\underset{\approx}{\Lambda}_\alpha = \left(\Lambda_{\alpha j i}\right) \quad , \tag{23}$$

where $\Lambda_{\alpha j i}$ is the transmission coefficient at junction α from the (i)th to the (j)th dynamic system and the self transmission coefficient $\Lambda_{\alpha j j}$ is the reflection coefficient at junction α in the (j)th dynamic system. From Eq. (15) a propagation matrix is constructed in the form

$$\underset{\approx}{t}^\alpha (\underline{x} \mid \underline{x}', \omega) = \left(t_j^\alpha(x_j \mid x_j', \omega)\,\delta_{ji}\right) \quad . \tag{24}$$

Using Fig. 2 one finds the relationship

$$\underset{\approx}{\hat{g}}^\alpha (\underline{x} \mid \underline{x}', \omega) = \underset{\approx}{t}^\alpha(\underline{x} \mid \underline{x}', \omega)\,\underset{\approx}{\alpha} + \underset{\approx}{\hat{g}}^{\beta-} (\underline{x} \mid \underline{x}', \omega) \quad . \tag{25}$$

The first term on the right of Eq. (25) is recognized as the direct term that is directed toward junction α. The factor $\underset{\approx}{\alpha}$ in this term ensures that it is generated by the term in the external drive

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that initiates a response that is similarly directed. The second term is also necessarily directed toward junction $\alpha\,(\beta - \equiv \alpha)$, but it results from a superposition of terms that experience at least one interaction with either of the junctions. This term can be split further into two terms, one initiated by the term in the external drive that is directed toward junction α and the other by the term in the external drive that is directed toward junction β; namely

$$\hat{\underset{\approx}{g}}^{\beta-}(\underline{x}\mid\underline{x}',\omega) = \overline{\underset{\approx}{g}}_{\alpha}^{\alpha}(\underline{x}\mid\underline{x}',\omega)\,\underset{\approx}{\alpha} + \overline{\underset{\approx}{g}}_{\beta}^{\alpha}(\underline{x}\mid\underline{x}',\omega)\,\underset{\approx}{\beta} \quad . \tag{26}$$

From Fig. 2 one also finds that

$$\hat{\underset{\approx}{g}}^{\beta-}(\underline{x}\mid\underline{x}',\omega) = \underset{\approx}{t}^{\alpha}(\underline{x}\mid\underline{x}_{\beta},\omega)\,\underset{\approx}{A}_{\beta}\,\hat{\underset{\approx}{g}}^{\beta}(\underline{x}_{\beta}\mid\underline{x}',\omega) \quad . \tag{27}$$

Utilizing Eqs. (25) - (27), and after straightforward algebraic manipulations, one obtains

$$\overline{\underset{\approx}{g}}_{\alpha}^{\alpha}(\underline{x}\mid\underline{x}',\omega) = \underset{\approx}{t}^{\alpha}(\underline{x}\mid\underline{x}_{\beta},\omega)\,\underset{\approx}{A}_{\beta}\,\underset{\approx}{D}_{\beta}\,\overline{\underset{\approx}{A}}_{\alpha}\,\underset{\approx}{t}^{\alpha}(\underline{x}_{\alpha}\mid\underline{x}',\omega) \quad , \tag{28a}$$

$$\overline{\underset{\approx}{g}}_{\beta}^{\alpha}(\underline{x}\mid\underline{x}',\omega) = \underset{\approx}{t}^{\alpha}(\underline{x}\mid\underline{x}_{\beta},\omega)\,\underset{\approx}{A}_{\beta}\,\underset{\approx}{D}_{\beta}\,\underset{\approx}{t}^{\beta}(\underline{x}_{\beta}\mid\underline{x}',\omega) \quad , \tag{28b}$$

where

$$\overline{\underset{\approx}{A}}_{\alpha} = \underset{\approx}{t}^{\beta}(\underline{x}_{\beta}\mid\underline{x}_{\alpha})\,\underset{\approx}{A}_{\alpha} \quad ; \quad \underset{\approx}{D}_{\beta} = [\,\underset{\approx}{1} - \overline{\underset{\approx}{A}}_{\alpha}\,\overline{\underset{\approx}{A}}_{\beta}\,]^{-1} \quad , \tag{29}$$

and $\underset{\approx}{1}$ is the unit matrix [6,7]. From Eqs. (25) and (26) the impulse response matrix for an ensemble of coupled dynamics systems is derived in the form

$$\widehat{\underset{\approx}{g}} (\underline{x} \mid \underline{x}', \omega) = \sum_{r,q} \widehat{\underset{\approx}{g}}^{\alpha}(\underline{x} \mid \underline{x}', \omega) \quad ;$$

$$\widehat{\underset{\approx}{g}}^{\alpha} (\underline{x} \mid \underline{x}', \omega) = \underset{\sim}{f}^{\alpha}(\underline{x} \mid \underline{x}, \omega)\, \underset{\approx}{\alpha} + \underset{\approx}{g}^{\alpha}_{\alpha} (\underline{x} \mid \underline{x}', \omega)\, \underset{\approx}{\alpha} + \underset{\approx}{g}^{\alpha}_{\beta} (\underline{x} \mid \underline{x}', \omega)\, \underset{\approx}{\beta} \quad , \qquad (30)$$

and from Eq. (28) it is clear that $\widehat{\underset{\approx}{g}}^{\alpha}(\underline{x} \mid \underline{x}', \omega)$ and, therefore, also $\widehat{\underset{\approx}{g}} (\underline{x} \mid \underline{x}', \omega)$, is proper

depending only on the properties of the dynamic systems and the couplings at the two junctions.

The impulse response matrix consists of six terms, a set of three for each of the two propagation

directionalities. The first of the three terms in the set is associated with the direct component and

the other two with the non-direct (reverberant) components in the response that is directed toward

one of the two junctions, one generated by an external drive that initiates propagation toward the

same junction and the other toward the opposite junction. In this connection the external drive that

is compatible with Eq. (28) is derived from Eq. (18) in the form

$$\widehat{\underset{\sim}{v}}_e (\underline{x}, \omega) = \sum_{r,q} \underset{\approx}{\alpha}\, \underset{\sim}{v}_{e\alpha}(\underline{x}, \omega) \quad ; \quad \underset{\sim}{v}_{e\alpha}(\underline{x}, \omega) = \left\{ v_{e\alpha j}(x_j, \omega) \right\} \quad . \qquad (31)$$

[cf. Eqs. (17) and (24).] A superposition of the two propagation directionalities yield an impulse

response matrix as is stated in Eq. (30). On the other hand, an anti-superposition of the two

propagation directionalities yield an impulse response matrix of the form

$$\widehat{\underset{\approx}{g}}^{-} (\underline{x} \mid \underline{x}', \omega) = \sum_{r,q} \underset{\approx}{S}(\underline{x}_\alpha - \underline{x}_\beta)\, \widehat{\underset{\approx}{g}}^{\alpha}(\underline{x} \mid \underline{x}', \omega) \quad ; \quad \underset{\approx}{S}(\underline{x}) = \left(S(x_j)\, \delta_{ji} \right) \quad . \qquad (32)$$

[cf. Eqs. (15) and (16).] There are measurements that utilize the form of the impulse response

matrix as stated in Eq. (32). However, in this paper the impulse response matrix as stated in Eq.

(30) is the only one that is subsequently employed; Eq. (32) in that sense is mentioned only in

passing. From Eqs. (30) and (31) the "velocity" response vector $\underset{\sim}{v}(\underline{x}, \omega)$ is expressed in the form

$$\underset{\sim}{v} (\underline{x}, \omega) = \int \left[\widehat{\underset{\approx}{g}} (\underline{x} \mid \underline{x}', \omega)\, \underset{\approx}{dx}'\, \widehat{\underset{\sim}{v}}_e (\underline{x}', \omega) \right] \quad ; \quad \underset{\approx}{dx}' = \left(dx_j\, \delta_{ji} \right) \quad . \qquad (33)$$

From Eqs. (22), (30), (31), and (33) one obtains

$$\underline{v}\,(\underline{x},\,\omega) = \sum_{r,q}\; \left\{ \underline{v}_d^\alpha(\underline{x},\,\omega) + \underline{v}_\alpha^\alpha(\underline{x},\,\omega) + \underline{v}_\beta^\alpha(\underline{x},\,\omega) \right\} \qquad , \qquad (34)$$

where

$$\underline{v}_d^\alpha(\underline{x},\,\omega) \;=\; \int\; \left[\underline{\underline{t}}^\alpha(\underline{x}\mid \underline{x}',\,\omega)\; d\underline{\underline{x}}'\; \underline{v}_{e\alpha}(\underline{x}',\,\omega) \right] \quad ;$$

$$\underline{v}_d^\alpha(\underline{x},\,\omega) = \left\{ v_{\alpha j}^\alpha\,(x_j,\,\omega) \right\} \qquad , \qquad (35a)$$

$$\underline{v}_\xi^\gamma(\underline{x},\,\omega) \;=\; \int\; \left[\underline{\underline{g}}_\xi^\gamma(\underline{x}\mid \underline{x}',\,\omega)\; d\underline{\underline{x}}'\; \underline{v}_{e\xi}(\underline{x}',\,\omega) \right] \quad ;$$

$$\underline{v}_\xi^\gamma(\underline{x},\,\omega) = \left\{ v_{\xi j}^\gamma\,(x_j,\,\omega) \right\} \qquad . \qquad (35b)$$

[cf. Eqs. (20) and (21) and a conventional note post Eq. (18).]

IV Commonality in the two approaches

The two approaches, if they address the same model of the beaded string, must yield the same response if subjected to an identical external drive. The second approach is derived, however, by reducing a more versatile and compounded formalism to fit a model of a beaded string in which the coupled dynamic systems are cascaded and possess equal propagation functions. A question arises whether the reduction results in an expression that can be analytically reconciled term by term with the expression for the response derived on the basis of the first approach. To answer this question the reduction needs to be executed so that the expressions derived in the two approaches are cast in the same terminology. With this in mind, the external drive in the second approach, designated by $\widehat{v}_e(\underline{x}, \omega)$ and stated in Eq. (31), is identified with the external drive $p_e(x, \omega)$ in the first approach by setting

$$\underline{v}_{e\alpha}(\underline{x}, \omega) = \underline{v}_{e\beta}(\underline{x}, \omega) \quad ; \quad \underline{v}_{e\alpha}(\underline{x}, \omega) = \underline{g}_0(\underline{x} \mid \underline{x}, \omega)\, \underline{p}_e(\underline{x}, \omega) \quad ;$$

$$\underline{p}_e(\underline{x}, \omega) = \{p_{ei}(x_i, \omega)\} \quad , \tag{36}$$

where

$$\underline{g}_0(\underline{x} \mid \underline{x}', \omega) = \left(\left(g_0(x_j \mid x_j', \omega)\, \delta_{ji}\right)\right) \quad , \tag{37a}$$

and $g_0(x_j \mid x_j', \omega)$ is defined in Eqs. (6) and (7). From Eqs. (15) and (24a) the propagation matrix may be cast in the convenient matrix product

$$\underline{t}^{\alpha}(\underline{x} \mid \underline{x}', \omega) = \underline{t}_0^{\alpha}(\underline{x} \mid \underline{x}', \omega)\, \underline{U}^{\alpha}(\underline{x} - \underline{x}') \quad , \tag{38}$$

where

$$\underset{\approx}{t}_0^\alpha (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \left(t_{0j}^\alpha (x_j \mid x_j', \omega) \, \delta_{ji} \right) \quad , \tag{39a}$$

$$\underset{\approx}{U}^\alpha (\underset{\sim}{x} - \underset{\sim}{x}') = \left(U (x_j - x_j') \, S (x_{\alpha j} - x_{\beta j}) \, \delta_{ji} \right) \quad . \tag{39b}$$

To render Eqs. (38) and (39) compatible with a beaded string one then recognizes that

$$\underset{\approx}{t}_0^\alpha (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \underset{\approx}{t}_0^\beta (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \quad ; \quad \underset{\approx}{t}_0^\alpha (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \underset{\approx}{\bar{g}}_0 (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \quad , \tag{40}$$

where

$$\underset{\approx}{\bar{g}}_0 (\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \left(\bar{g}_0 (x_j \mid x_j', \omega) \, \delta_{ji} \right) \quad , \tag{37b}$$

and $\bar{g}_0 (x_j \mid x_j', \omega)$ is defined in Eq. (14a). The junction matrices $\underset{\approx}{\Lambda}_\alpha$ are reduced next. The matrices are constructed for the beaded string with the assistance of Fig. 1b. For a cascade of coupled dynamic systems these matrices are typically

$$\underset{\approx}{\Lambda}_r = \begin{pmatrix} & \Lambda_{rj-1j-1} & \Lambda_{rj-1j} & \\ & \Lambda_{rjj-1} & \Lambda_{rjj} & O \\ & & O & \Lambda_{rj+1j+1} \end{pmatrix} , \tag{41a}$$

$$\underset{\approx}{\Lambda}_q = \begin{pmatrix} & \Lambda_{qj-1j-1} & O & \\ & O & \Lambda_{qjj} & \Lambda_{qjj+1} \\ & & \Lambda_{qj+1j} & \Lambda_{qj+1j+1} \end{pmatrix} , \tag{41b}$$

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where, for example, Λ_{rjj} is the reflection coefficient at junction r with respect to the (j)th dynamic system and Λ_{rj-1j} is the transmission coefficient at junction r from the (j)th to the (j-1)th dynamic system. If the (n)th bead is in junction r then

$$\Lambda_{r\,n\,n-1} = \Lambda_{r\,n-1\,n} = [1 + \bar{z}_{sn}(\omega)]^{-1} = T_n \quad , \tag{42}$$

where $\bar{z}_{sn}(\omega)$ is defined in Eq. (13) and T_n designates the transmission coefficient across the (n)th bead [6, 7]. Since the reflection coefficients are with respect to the "velocity" response it follows that

$$\Lambda_{r\,n\,n} = (\Lambda_{r\,n\,n-1} - 1) = - \bar{z}_n(\omega) = R_n^v \quad ; \qquad R_n^v = (T_n - 1) \quad , \tag{43}$$

where $\bar{z}_n(\omega)$ is also defined in Eq. (13). In particular, if $|\bar{z}_{sn}(\omega)| \to \infty$ then $R_n^v \to -1$ which is commensurate with a zero "velocity" response at the junction at $x_n = x_{r\,n}$. This is an appropriate boundary condition at this position in the junction. A "pressure" response, on the other hand, is characterized by a reflection coefficient R_n [$=(1 - T_n)$] that tends toward unity when $|\bar{z}_{sn}(\omega)| \to \infty$, which is commensurate with the well known "pressure doubling" at the junction at $x_n = x_{r\,n}$ [3, 14]. Returning to Eqs. (25) - (29) one obtains

$$\underset{\approx}{t}^{\alpha}(\underline{x} \mid \underline{x}', \omega) = \bar{g}_0(\underline{x} \mid \underline{x}', \omega) \, \underset{\approx}{U}^{\alpha}(\underline{x} - \underline{x}') \quad , \tag{44a}$$

$$\underset{\approx}{\bar{g}}^{\alpha}_{\alpha}(\underline{x} \mid \underline{x}', \omega) = \bar{g}_0(\underline{x} \mid \underline{x}_{\beta}, \omega) \, \underset{\approx}{\Delta}_{\beta} \, \underset{\approx}{D}_{\beta} \, \underset{\approx}{\bar{\Delta}}_{\alpha} \, \bar{g}_0(\underline{x}_{\alpha} \mid \underline{x}', \omega) \quad , \tag{44b}$$

$$\underset{\approx}{\bar{g}}^{\alpha}_{\beta}(\underline{x} \mid \underline{x}', \omega) = \bar{g}_0(\underline{x} \mid \underline{x}_{\beta}, \omega) \, \underset{\approx}{\Delta}_{\beta} \, \underset{\approx}{D}_{\beta} \, \bar{g}_0(\underline{x}_{\beta} \mid \underline{x}', \omega) \quad , \tag{44c}$$

$$\underset{\approx}{\bar{\Delta}}_{\alpha} = \bar{g}_0(\underline{x}_{\beta} \mid \underline{x}_{\alpha}, \omega) \, \underset{\approx}{\Delta}_{\alpha} \quad ; \qquad \underset{\approx}{D}_{\beta} = \left[\underset{\approx}{1} - \bar{g}_0(\underline{x}_{\beta} \mid \underline{x}_{\alpha}, \omega) \, \underset{\approx}{\Delta}_{\alpha} \, \bar{g}_0(\underline{x}_{\alpha} \mid \underline{x}_{\beta}, \omega) \, \underset{\approx}{\Delta}_{\beta}\right]^{-1} \quad , \tag{45}$$

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and it is noted that

$$\underset{\sim}{U}^{\alpha}(\underset{\sim}{x} - \underset{\sim}{x}_{\beta}) = \underset{\sim}{U}^{\alpha}(\underset{\sim}{x}_{\alpha} - \underset{\sim}{x}) = \underset{\sim}{1} \quad , \tag{46}$$

In addition, from Eq. (14a), (31), (36), and (37) one obtains

$$\overline{\underset{\sim}{g}}_0(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \underset{\sim}{g}_0(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \left[\underset{\sim}{g}_0(\underset{\sim}{x}' \mid \underset{\sim}{x}', \omega)\right]^{-1} \quad , \tag{47}$$

$$\widehat{\underset{\sim}{v}}_e(\underset{\sim}{x}, \omega) = \underset{\sim}{g}_0(\underset{\sim}{x} \mid \underset{\sim}{x}, \omega) \sum_{r,q} \underset{\sim}{\alpha} \, \underset{\sim}{p}_e(\underset{\sim}{x}, \omega) \quad . \tag{48}$$

From Eqs. (30), (31), (33) and (44) - (48), and after some cumbersome algebraic manipulations, one may cast the vectorial response of a beaded string in the form

$$\underset{\sim}{v}(\underset{\sim}{x}, \omega) = \int \underset{\sim}{g}(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \, d\underset{\sim}{x}' \, \underset{\sim}{p}_e(\underset{\sim}{x}', \omega) \quad ; \quad \underset{\sim}{v}(\underset{\sim}{x}, \omega) = \left\{ v_j(x_j, \omega) \right\} \quad , \tag{49}$$

where $v_j(x_j, \omega)$ is the response of the (j)th dynamic system,

$$\underset{\sim}{g}(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \underset{\sim}{g}_d(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) + \underset{\sim}{g}_s^v(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \quad ;$$

$$\underset{\sim}{g}_d(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \left(g_{dj}(x_j \mid x_j', \omega) \, \delta_{ji} \right) \quad ;$$

$$\underset{\sim}{g}_s^v(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \left(g_{sji}^v(x_j \mid x_i', \omega) \right) \quad , \tag{50}$$

$$\underset{\sim}{g}_d(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \sum_{r,q} \underset{\sim}{g}_0(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) \, \underset{\sim}{U}^{\alpha}(\underset{\sim}{x} - \underset{\sim}{x}') \quad , \tag{51}$$

$$\underset{\sim}{g}_s^v(\underset{\sim}{x} \mid \underset{\sim}{x}', \omega) = \sum_{r,q} \overline{\underset{\sim}{g}}_0(\underset{\sim}{x} \mid \underset{\sim}{x}_{\beta}, \omega) \, \underset{\sim}{\Lambda}_{\beta} \, \underset{\sim}{D}_{\beta} \left\{ \overline{\underset{\sim}{g}}_0(\underset{\sim}{x}_{\beta} \mid \underset{\sim}{x}_{\alpha}, \omega) \, \underset{\sim}{\Lambda}_{\alpha} \, \underset{\sim}{g}_0(\underset{\sim}{x}_{\alpha} \mid \underset{\sim}{x}', \omega) + \underset{\sim}{g}_0(\underset{\sim}{x}_{\beta} \mid \underset{\sim}{x}', \omega) \right\} . \tag{52}$$

The resemblance between Eq. (52) and (12b) is beginning to emerge even though the former formalism is expressed in matrix form. This matrix form, however, can be commuted to a scalar form so that comparison between the two approaches can be further facilitated. For this purpose the following constructs are formed from Eq. (50):

$$g_d(x \mid x', \omega) = \sum_j h_j(x_j) \, g_{dj}(x_j \mid x_j', \omega) \, h_j(x_j') \quad , \tag{53a}$$

$$g_s^v(x \mid x', \omega) = \sum_j \sum_i h_j(x_j) \, g_{sji}^v(x_j \mid x_i', \omega) \, h_i(x_i') \quad , \tag{53b}$$

where the span function $h_j(x_j)$ is defined

$$h_j(x_j) \, h_i(x_i') = h_j(x_j) \, h_j(x_j') \, \delta_{ji} + h_j(x_j) \, h_i(x_i') \, (1 - \delta_{ji}) \quad ;$$

$$h_j(x_j) = \begin{cases} 1 \ , & x_{\beta j} < x_j < x_{\alpha j} \\ \\ 0 \ , & x_j < x_{\beta j} \text{ and } x_j > x_{\alpha j} \end{cases} \quad . \tag{54}$$

Similarly, from Eq. (36):

$$p_e(x, \omega) = \sum_k h_k(x_k) \, p_{ek}(x_k, \omega) \quad . \tag{55}$$

The summations in Eqs. (53) and (55) are over the dynamic systems. [cf. Eq. (12) where the summations are over the beads.] With these constructs Eq. (49) may be written in the scalar form

$$v_2(x, \omega) = \int g_2(x \mid x', \omega) \, dx' \, p_e(x, \omega) \quad , \tag{56}$$

with the imposition that the external drive is as stated in Eq. (55) and the impulse response function is as derived from Eq. (53); namely

$$g_2(x \mid x', \omega) = g_d(x \mid x', \omega) + g_s^v(x \mid x', \omega) \qquad . \qquad (57)$$

The subscript 2 in $v_2(x, \omega)$ and $g_2(x \mid x', \omega)$ of Eqs. (56) and (57) is meant to emphasize that these quantities are derived for the beaded string on the basis of the second approach.

The two approaches that are formulated in this paper model the beaded string differently. However, the basic parameters that describe this dynamic system are made common to both the formalisms. It follows, therefore, that the expressions derived for the normalized impulse response functions, in Eqs. (11b) and (12b) and in Eqs. (56) and (57) need to be universally identical; namely

$$\bar{g}_1(x \mid x', \omega) \equiv \bar{g}_2(x \mid x', \omega) \qquad , \qquad (58)$$

and if the external drives are identical, it follows that

$$v_1(x, \omega) \equiv v_2(x, \omega) \qquad . \qquad (59)$$

Nonetheless, the authors have found no direct relationship between $\bar{g}_1(x \mid x', \omega)$ and $\bar{g}_2(x \mid x', \omega)$; the expressions for these two quantities cannot be reconciled term by term. Clearly $g_d(x \mid x', \omega)$ is not identical with $g_0(x \mid x', \omega)$ and, therefore to start with the leading terms in Eqs. (12b) and (57), respectively, do not strictly reconcile. With identical external drives it follows that neither is $v_d(x, \omega)$ reconciled with $v_0(x, \omega)$. One may be tempted to redefine the direct response so that some specific interactions with the junctions be counted into the camp of the direct response, thereby achieving reconciliation at least in the leading terms in the respective expressions, notwithstanding that in a one- and two-beaded string one may, by manipulating the compositions of various terms, readily show the actual identity of the two derived expressions. However, the task of this procedure proves increasingly more complex as the number of beads increases. Straightforward

attempts by the authors have not been successful either in systematically reconciling the expressions term by term or in finding a master expression that would reduce, under certain choices, to one and to the other of the derived expressions. These failures render the use of computations, to show the validity of the identity stated in Eq. (58), a reasonable proposition.

V Computer experiments

The comparisons between the two approaches are conducted in terms of Eq. (58). The normalized impulse response function $\bar{g}(x \mid x', \omega)$ for each of the two approaches are assigned the same quantities and parameters that define the common beaded string structure. In the displays in this paper the number N of beads is 14 and the number (N+1) of coupled dynamic systems is 15, with the two end dynamic systems being spatially semi-infinite, as shown in Fig. 3. The computations of the absolute values of the normalized impulse response functions are displayed on the $\{(x/b), (\omega/\omega_c)\}$-plane in a waterfall format, where b is a spatial linear scale and ω_c is a frequency scale. In these forms a normalized impulse response function is displayed as a function of (x/b) at equal increments of (ω/ω_c) which are then regularly displaced along the (ω/ω_c) axis to prevent curves from confusingly overlapping [3-5]. Standard displays are shown first in Fig. 4; the displays are standard in the sense that standard conditions and values are chosen for the quantities and parameters that define the beaded string. These standard conditions and values are:

$$b_j = \mid x_{rj} - x_{qj} \mid \;\; ; \;\; b = (N-1)^{-1} \sum_j^N b_j \;\; ; \;\; b_j = b \;\; ; \;\; (x'/b) = 0.2 \quad , \tag{60a}$$

$$(b\omega_c/c_p) = (bk_c) = 16 \;\; ; \;\; c_p \neq c_p(\omega) \quad , \tag{60b}$$

$$z_{sn}(\omega) = (i\omega M_n) \left[1 - (\omega/\omega_n)^2 (1 - i\eta_n)\right]^{-1} \;\; ; \;\; \omega_n^2 = [K_n(1 + \eta_n^2)/M_n] \;\; ;$$

$$z_{sn}(\omega) = z_s(\omega) \;\; ; \;\; M_n = M \;\; ; \;\; (\omega_n/\omega_c)^2 >> 1 \quad , \tag{60c}$$

$$(M/mb) = 0.3 \;\; ; \;\; \eta_p = 5 \times 10^{-3} \;\; ; \;\; \eta_c = 10^{-i} \quad , \tag{60d}$$

and the range for the displays is $\{0, 0\} \leq \{(x/b), (\omega/\omega_c)\} \leq \{13, 0.8\}$. In Eq. (60c) it is assumed that the beads are simply composed of mass-spring system in parallel; M_n is the mass, K_n is the stiffness, and η_n is the loss factor associated with the (n)th bead. Figure 4 is computed on

the basis that the beaded string is defined in terms of the standard conditions and values stated in Eq. (60); in Fig. 4a, $|\bar{g}_1(x \,|\, x', \omega)|$ is depicted, and in Fig. 4b, $|\bar{g}_2(x \,|\, x', \omega)|$ is depicted in waterfall forms. The data displayed in Fig. 4 is rich, the phenomena of "aliasings" and of "pass and stop bands" are clearly visible in the patterns of the figures. Such patterns call for phase and amplitude interferences of considerable exactness [5]. Within the accuracy of the computations, Figs. 4a and 4b are deemed identical. Figure 4 is repeated in Fig. 5, except that now the loss factor η_p in the string is changed by an order of magnitude, from its standard value of 5×10^{-3} to 5×10^{-2}. [cf. Eq. (60d).] The weakening of the phenomenon of pass and stop bands and the increase in attenuation away from the position of initiation at (x'/b) is clearly visible in comparing Fig. 5 with Fig. 4. Evidently the weakening of the phenomenon of pass and stop bands and the increase in attenuation are tracked in Figs. 5a and 5b. The patterns in these figures are deemed identical within the accuracy of the computations. Again Fig. 4 is repeated in Fig. 6, except that in the latter figure the identity of the separations between adjacent beads are disturbed by some $\pm 10\%$. This disturbance tends to destroy the phenomena of pass and stop bands and of aliasings, especially at the higher frequency range where $(bk_{p0})^2 > 1$ [5]. The degree of this destruction becomes evident in comparing Fig. 6 with Fig. 4. The disturbance of the identity of the separations between adjacent beads, however, is as effective in changing the patterns in Fig. 6a as it is in Fig. 6b. Indeed, Figs. 6a and 6b match within the accuracy of the computations; within that accuracy, these figures are deemed identical. Figure 4 is repeated, once again, in Fig. 7, except that in the latter figure the standard value of $(\omega_n/\omega_c)^2$ is changed by four orders of magnitude, from 10^3 to 10^{-1}, which brings (ω_n/ω_c) centrally into the range of frequency depicted in the figures. In this case, the impedance $z_s(\omega)$ of a bead is "mass controlled" in the range of frequency $(\omega/\omega_c) \lesssim 3 \times 10^{-1}$, is "resistance controlled" in the range $(\omega/\omega_c) \simeq 3 \times 10^{-1}$, and is "stiffness controlled" in the range $(\omega/\omega_c) \gtrsim 3 \times 10^{-1}$ [15]. These changes in the impedance characteristics of the beads are clearly visible in the patterns of Fig. 7 when compared with those in Fig. 4. The patterns in the two figures are substantially the same for the range $(\omega/\omega_c) \lesssim 3 \times 10^{-1}$, the patterns are weak in the range $(\omega/\omega_c) \simeq 3\text{-}4 \times 10^{-1}$, indicating diminishing response by damping that is

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amply provided by the beads, the weakness continues and becomes severe by a continuous decrease in the impedance $z_s(\omega)$ with increase of frequency in the frequency range $(\omega/\omega_c) \gtrsim (5\text{-}6)\text{x}10^{-1}$. The drastic changes in the patterns in Figs. 7a and 7b, from that of Figs. 4a and 4b, respectively, are, however, the same so that Figs. 7a and 7b are substantially identical, at least to within the accuracy of the computations. Finally, Fig. 4 is repeated in Fig. 8, except that in the latter figure the two end-beads are assigned impedances that render the string well nigh finite, provided the external drive is confined to within the beaded region of the string. The finiteness of the string is apparent in Fig. 8; the impulse response function on either side, beyond the end-beads, is negligible. This feature is present in both Figs. 8a and 8b; it is not present in Figs. 4a and 4b though.

The range and richness of the data displayed in Figs. 4a-8a and in Figs. 4b-8b, demonstrate that the impulse response functions $\overline{g}_1(x \mid x', \omega)$ and $\overline{g}_2(x \mid x', \omega)$ are identical. This identity is established computationally in this paper. The results of computations, employing each approach, are compared side by side. The comparisons show that within the computational accuracy, the patterns displayed in the two sets of figures match well enough to deem them identical. The purpose for the exercise pursued in this paper is thus accomplished.

Appendix A

For the sake of simplicity in the text the master structure is defined in terms of a string that is infinite and uniform as defined in Eqs. (3) and (7). This results in an impulse response function for the master structure that is as stated in Eqs. (9) and (10). Of course, the master structure may be defined, for example, in terms of an impulse response function which includes some, but not all, of the beads. In this case Eqs. (11)-(14) need to be modified. These modifications are infused as follows: The master structure <u>incorporates</u> a few of the beads and therefore its impulse response function $g_\infty(x \mid x', \omega)$ is of the form

$$g_\infty(x \mid x', \omega) = g_0(x \mid x', \omega) - g_{s_0}(x \mid x', \omega) \quad ;$$

$$g_{s_0}(x \mid x', \omega) = \sum_n^{inco} \sum_m \overline{g}_0(x \mid x_n, \omega) \, \overline{z}_n(\omega) \, c_{nm}(\omega) \, g_0(x_m \mid x', \omega) \quad . \qquad (A1)$$

The remaining beads are now appendaged to this new master structure [2]. From Eqs. (12a) and (A1) one derives for the impulse response function of the beaded spring

$$g_1(x \mid x', \omega) = g_\infty(x \mid x', \omega) - g_{s_\infty}(x \mid x', \omega) \quad ;$$

$$g_{s_\infty}(x \mid x', \omega) = \sum_n^{appe} \sum_m g_\infty(x \mid x', \omega) \, \overline{z}_n(\omega) \, c_{nm}(\omega) \, g_\infty(x_m \mid x', \omega) \quad , \qquad (A2)$$

where the parameters and the summations in Eq. (A1) are in reference and over beads that are incorporated into the original master structure. On the other hand, the parameters and the summations in Eq. (A2) are in reference and over beads that are not so incorporated; they are merely "appendaged" to the new master structure [2]. The expression derived on the basis of the second approach remains unaltered. Comparing $g_1(x \mid x', \omega)$ as expressed in Eq. (A2) with $g_2(x \mid x', \omega)$ as expressed in Eqs. (53) and (57) indicates even greater disparity in the leading terms than previously existed in the two respective expressions. Clearly $g_\infty(x \mid x', \omega)$ in Eq. (A1) contains terms arising from interactions with beads (junctions) that $g_0(x \mid x', \omega)$ does not include

and that are totally omitted in $g_d(x \mid x', \omega)$. Reconciling the leading terms becomes a task even more complex than appears in the text. In particular, were the two end beads the only ones incorporated in the master structure, and these beads were to be made equivalently rigid, and the external drive were confined to within these end beads, the Sommerfeld-Watson approximation may be used to cast $g_\infty(x \mid x', \omega)$ in the form [9,10]

$$g_\infty(x \mid x', \omega) = \sum_h \left\{ Z_h(\omega) \right\}^{-1} \psi_h(x) \, \psi_h^*(x') \quad ,$$

(A3)

where $\psi_h(x)$ are the eigenfunctions of the finite string between the two end beads and

$$\int \psi_h(x) \, \psi_h^*(x) \, dx = \delta_{hk} \quad ; \quad \sum_h \psi_h(x) \, \psi_h^*(x') = \delta(x - x') \quad ,$$

(A4)

$$[z_\infty(x, \omega) = Z_h(\omega)] \, \psi_h(x) \quad .$$

(A5)

The insertion of $g_\infty(x \mid x', \omega)$, as defined in Eq. (A3), in Eq. (A2) renders the reconciling of Eq. (A2) and Eqs. (53) and (57) term by term well nigh impossible. Yet, there is no dispute that $g_1(x \mid x', \omega)$ as expressed in Eq. (A2) is identical to $g_2(x \mid x', \omega)$ as expressed in Eqs. (53) and (57), as long as they address an identical beaded string structure. The computational identity of $g_1(x \mid x', \omega)$ and $g_2(x \mid x', \omega)$ in this extreme case is depicted in Fig. 8. At the other extreme is a situation in which most of a multitude of beads are incorporated and the remaining few are appendaged. Starting with this kind of a new master structure the influence of the few attached appendages may be considered as perturbations. The two extreme cases, as derived by inserting Eq. (A1) in Eq. (A2), yield expressions for $g_1(x \mid x', \omega)$ that cannot be readily reconciled term by term. Indeed, if there is a multiplicity of beads so that they can be separated into a group of "inco" and a group of "appe," the subsequent terms in the resulting expression are unique to the specific grouping. Equation (12) is then the expression when all the beads are in the "appe" group. Thus, even within the first approach there are many variations on the theme that originally yielded

Eq. (12). For certain purposes one or the other expression for the response of a multiply beaded string may prove the more suitable. Variations on the theme can be similarly devised for the second approach that originally yield Eqs. (53) and (57). One may then be led to remark: the selection of a successful expression for the impulse response function of a complex structural system is not dictated a priori, but rather, is often merely a choice in the hands of the skilled manipulator [9-11]. The more the variations on a theme, the better are his choices and "zeh meshubach."

References

1. C. Soize, "Probabilistic structural modeling in linear dynamic analysis of complex mechanical systems: I. Theoretical elements," La Recherche Aerospatiale, 5, 23-48 (1986).

2. G. Maidanik and J. Dickey, "An impulse response function for a fuzzy structure, Part I: Accepted for publication in JASA. To appear in March 1995 issue.

3. G. Maidanik and J. Dickey, "Velocity distributions on unloaded finitely and regularly ribbed membranes," J. Sound Vib. 149, 43-70 (1991).

4. G. Maidanik and J. Dickey, "Response of regularly ribbed fluid loaded panels," J. Sound Vib. 155, 481-495 (1992).

5. G. Maidanik and J. Dickey, "Localization and delocalization in periodic one-dimensional dynamic systems," Acustica 73, 119-128 (1991).

6. G. Maidanik and L. J. Maga, "Response of coupled basic one-dimensional dynamic systems," J. Sound Vib. 111, 361-375 (1987).

7. G. Maidanik and J. Dickey, "Response of coupled one-dimensional dynamic systems," J. Sound Vib. 121, 187-194 (1988).

8. C. H. Hodges and J. Woodhouse, "Theories of noise and bivration transmission in complex structures," Rep. Prog. Phys. 49, 107-170 (1986).

9. Arnold Sommerfeld, Partial Differential Equations in Physics, Volume VI, (Academic Press, New York, 1967), translated by Ernst G. Straus.

10. Albert J. Tucker and G. Maidanik, "Dual representation of the green function of a vibrating finite string," J. Acous. Soc. Am. 49, 927(L) (1971).

11. L. B. Felsen, J. M. Ho, and I. T. Lu, "Three-dimensional Green's function for fluid-loaded thin elastic cylindrical shell: alternative representations and ray acoustic forms," J. Acoust. Soc. Am. 87, 554-569 (1990); (E) 89, 1463-1464 (1991).

12. D. N. Lapedes (Ed.) <u>Dictionary of Scientific and Technical Terms,</u> (McGraw-Hill New York 1978) 2nd ed.; P. B. Gove (Ed.), <u>Webster's Third Revision International Dictionary Unabridged</u> (Merriam, Springfield, 1967); J. B. Brackenridge and R. M. Rosenberg, <u>The Principle of Physics and Chemistry,</u> (McGraw-Hill, New York,1970).

13. J. Dickey and G. Maidanik, "Temporal response of coupled one-dimensional dynamic systems," J. Acous. Soc. Am. <u>89</u>, 1-9 (1991).

14. A. P. Dowling and J. E. Ffowcs Williams, <u>Sound and Sources of Sound,</u> (John Wiley and Sons, New York, 1983).

15. P. M. Morse, <u>Vibration and Sound,</u> (McGraw-Hill, New York, 1948), 2nd ed.
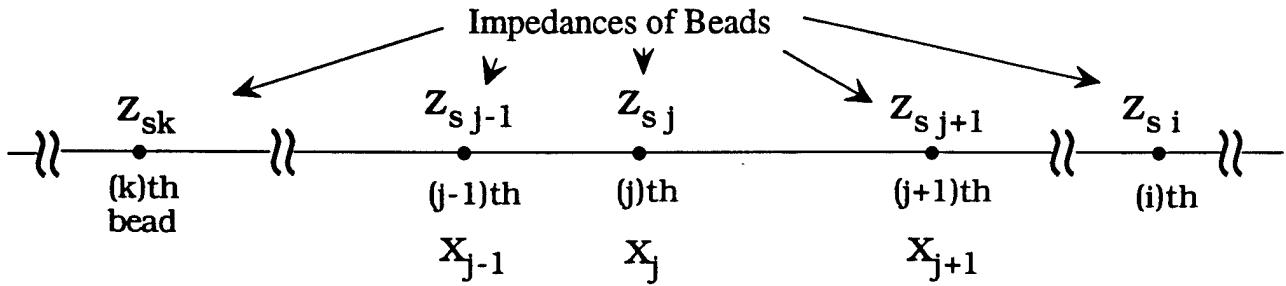
Fig. 1a. Model of a beaded string for the first approach: $z_{sj}(\omega)$ and x_j are the impedance of the (j)th bead as perceived by the stirng of the (j)th bead as perceived by the string and the position of the (j)th bead on the string, respectively.
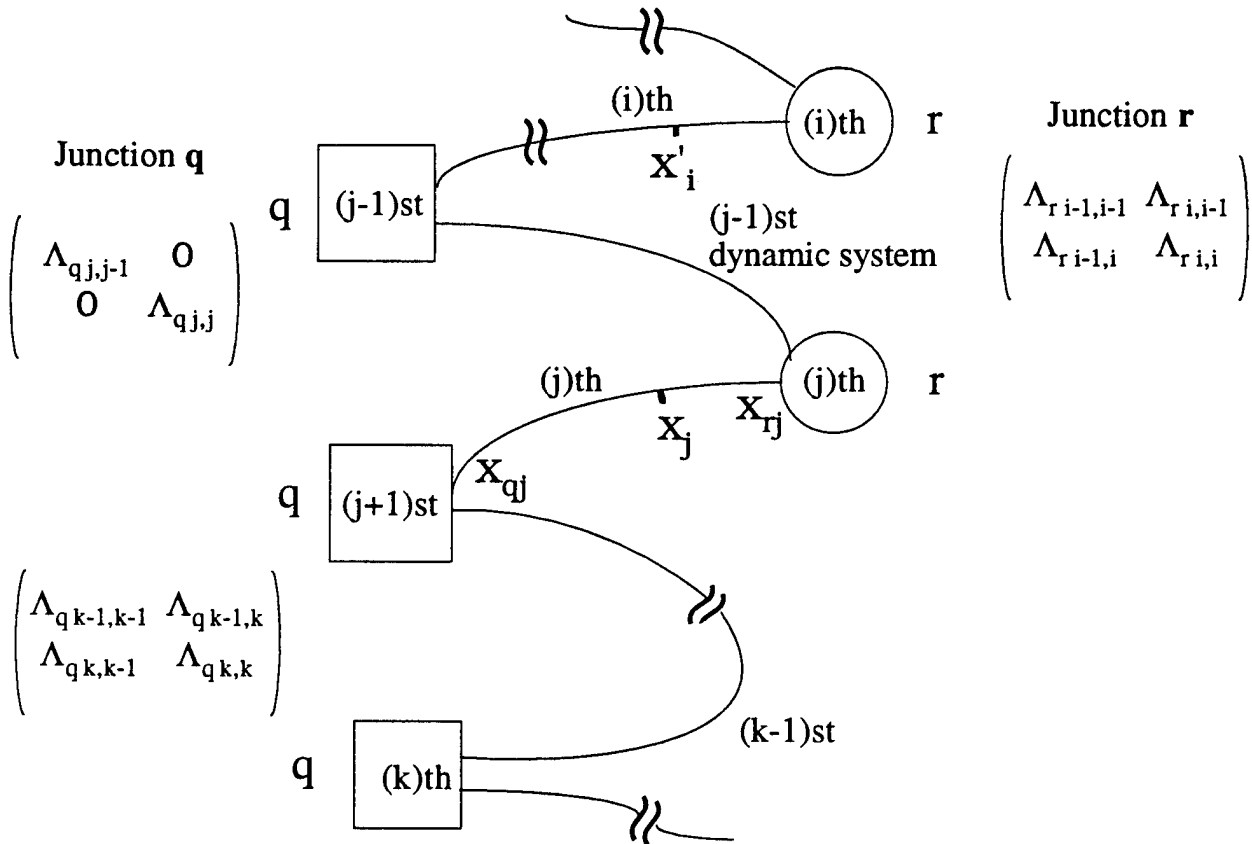
Fig. 1b. Model of a beaded string for the second approach: Adjacent beads define a one-dimensional dynamic system and the coupling between two adjacent dynamic systems is determined by the bead they share. [cf. Fig. 1a.] The beads are alternately placed in junction r and junction q and the junctions are defined in terms of junction matrices $\underset{\sim}{\Lambda}_r$ and $\underset{\sim}{\Lambda}_q$. These matrices are diagonally composed of 2 x 2 matrix elements.

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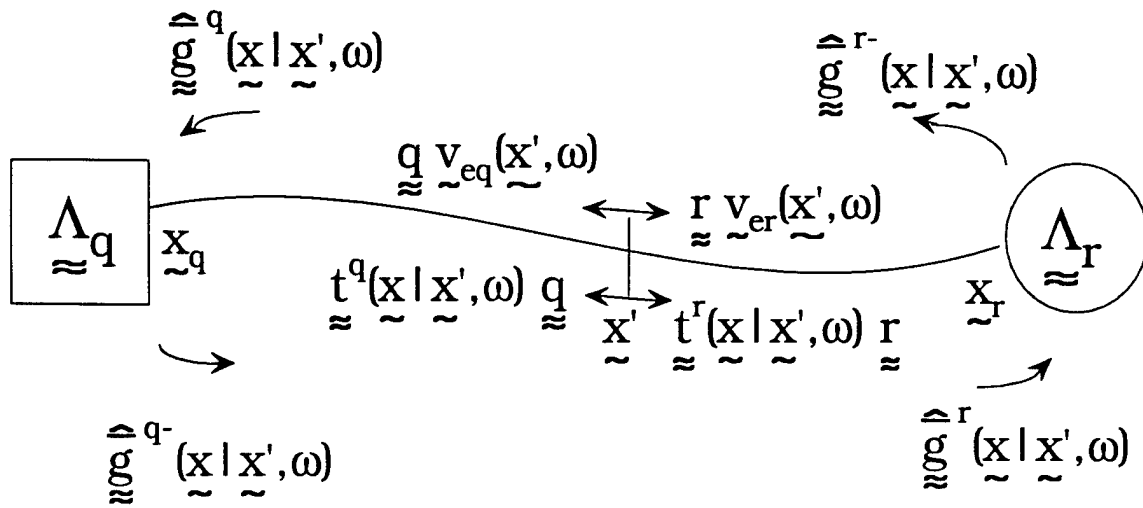
Fig. 2. A general matrix representation of an ensemble of coupled one-dimensional dynamic systems. The external "velocity" drive $\hat{\underline{v}}_e(\underline{x}, \omega)$ is composed of two-vector with each term (component) being a multi-component vector of a rank equal to the number of dynamic systems. The dynamic systems are defined in terms of two terminal vectors $\underline{x}_\alpha$ and two propagation matrices $\underline{t}^\alpha(\underline{x} \mid \underline{x}, \omega)$ which are assigned directionality with respect to propagation toward junction α; $\alpha = r$ or q. The two junctions are defined in terms of the junction matrices $\underline{\underline{\Lambda}}_\alpha$. The transfer matrix $\hat{\underline{\underline{g}}}^\alpha(\underline{x} \mid \underline{x}', \omega)$ relates to the response that is directed toward junction α.
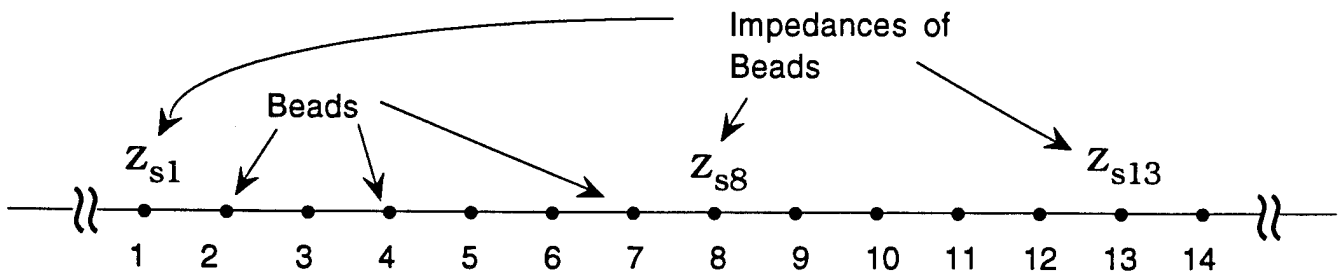
a) Fig. 3a. A model of a - fourteen beaded string that befits the first approach. [cf. Fig. 1a.]
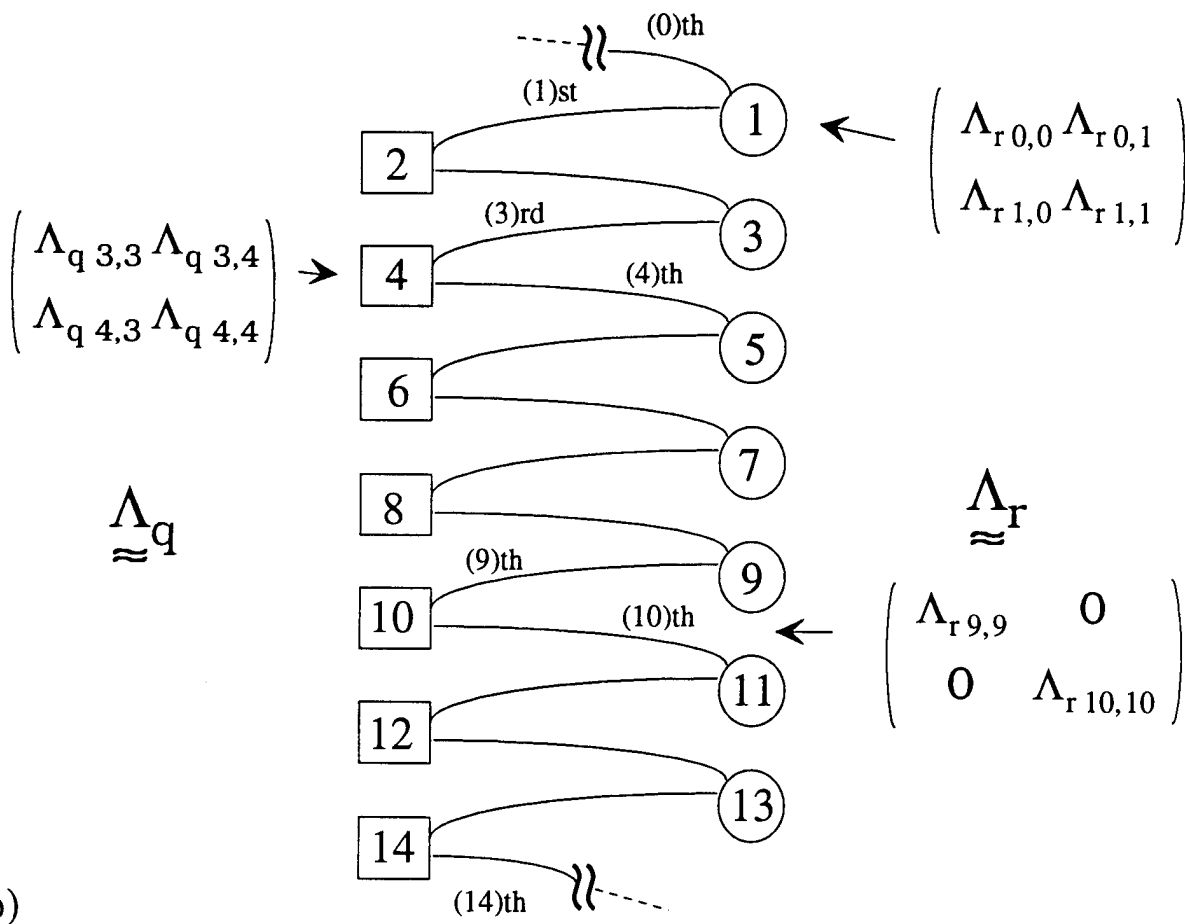
b)

Fig. 3b. A model of a - fourteen beaded string that befits the second approach. [cf. Fig. 1b.]
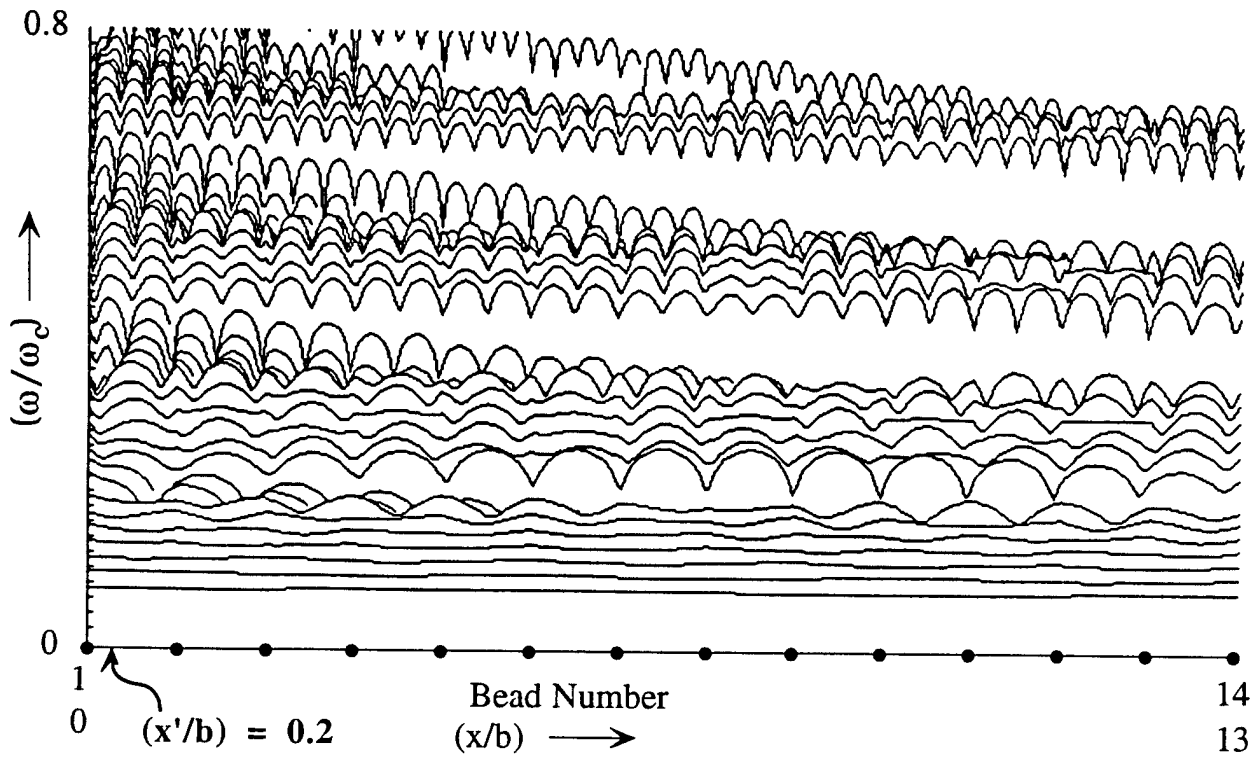
Fig. 4a. A waterfall display of $|\, g_1(\underline{x}\,|\,\underline{x}',\,\omega)\,|$, for the model defined in Fig. 3a, under standard conditions and values. [cf. Eq. (60).]
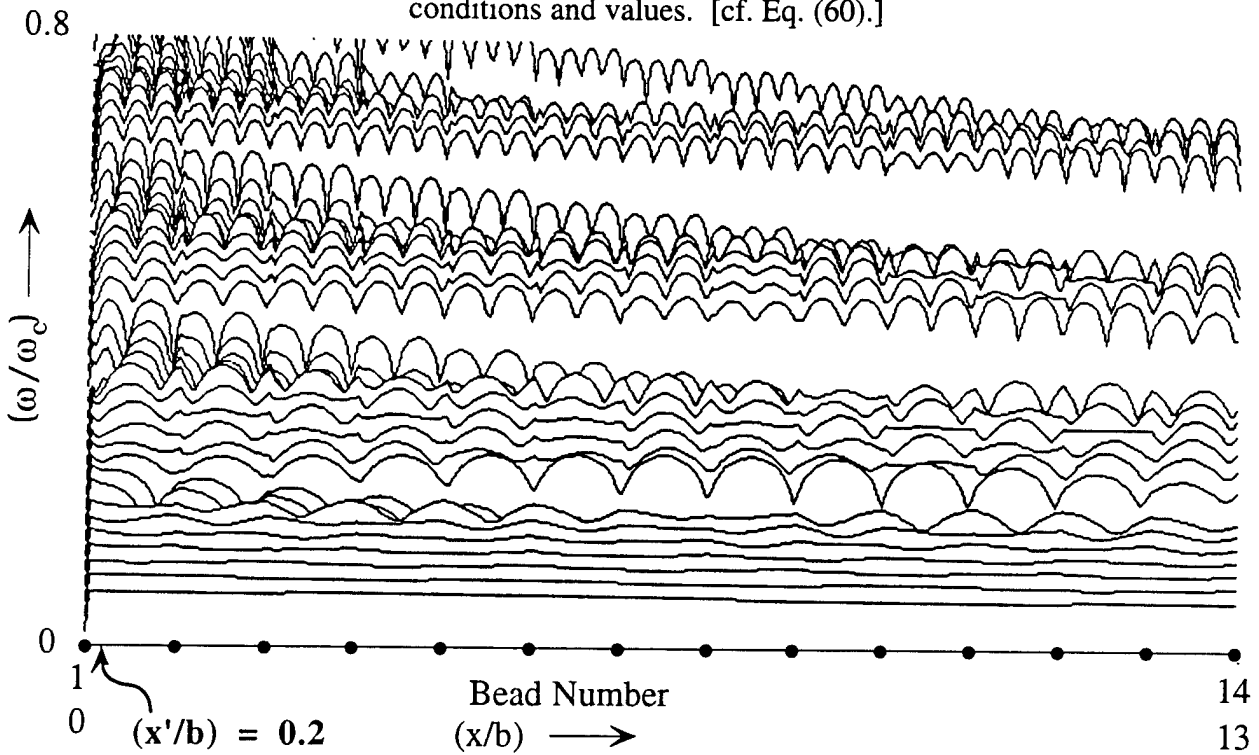
Fig. 4b. A waterfall display of $|\, g_2(\underline{x}\,|\,\underline{x}',\,\omega)\,|$, for the model defined in Fig. 3b, under standard conditions and values. [cf. Eq. (60).]

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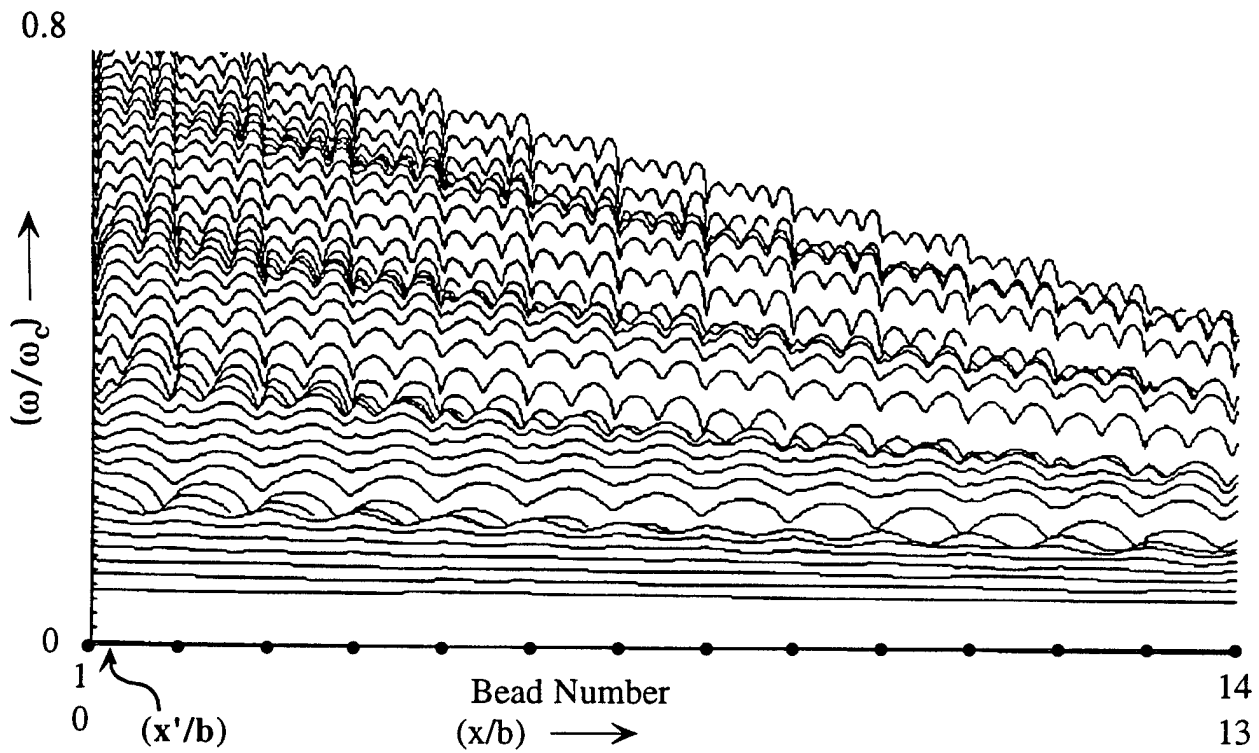
a)

Fig. 5a. A repeat of Fig. 4a except that $\eta_p = 5 \times 10^{-2}$.
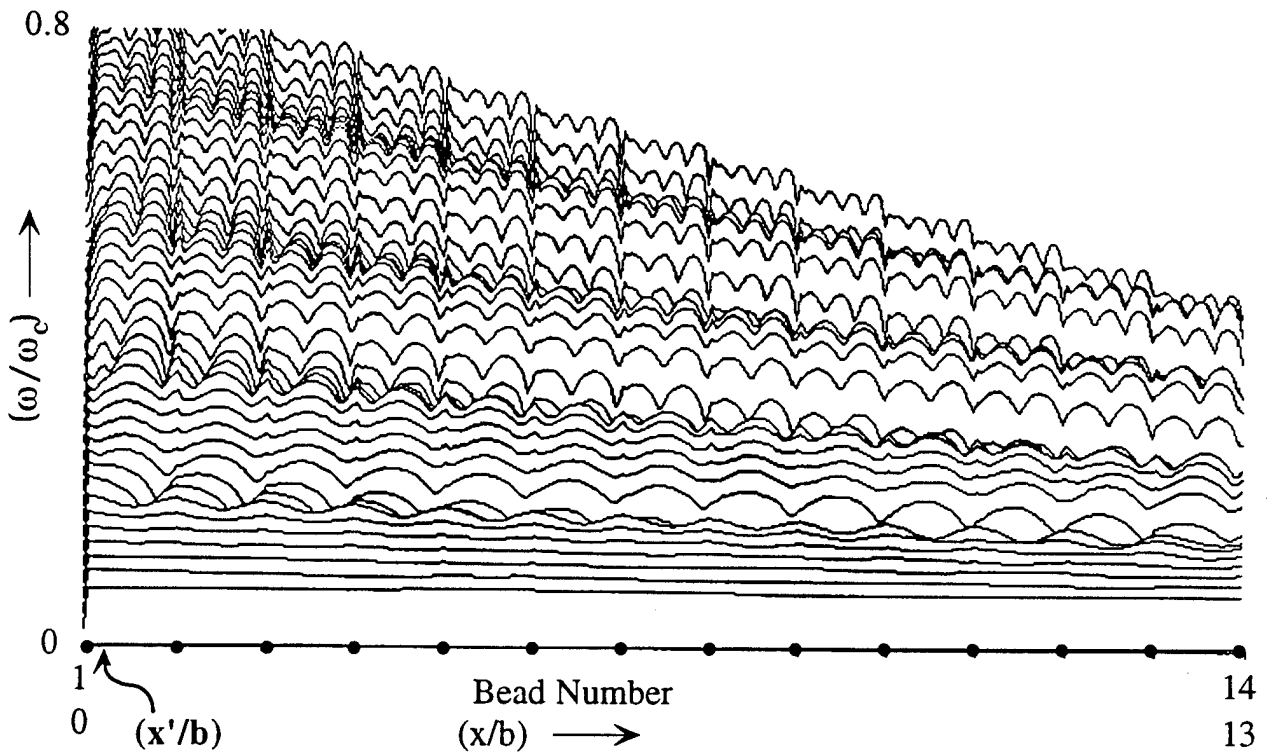
b)

Fig. 5b. A repeat of Fig. 4b except that $\eta_p = 5 \times 10^{-2}$.

Figure 5
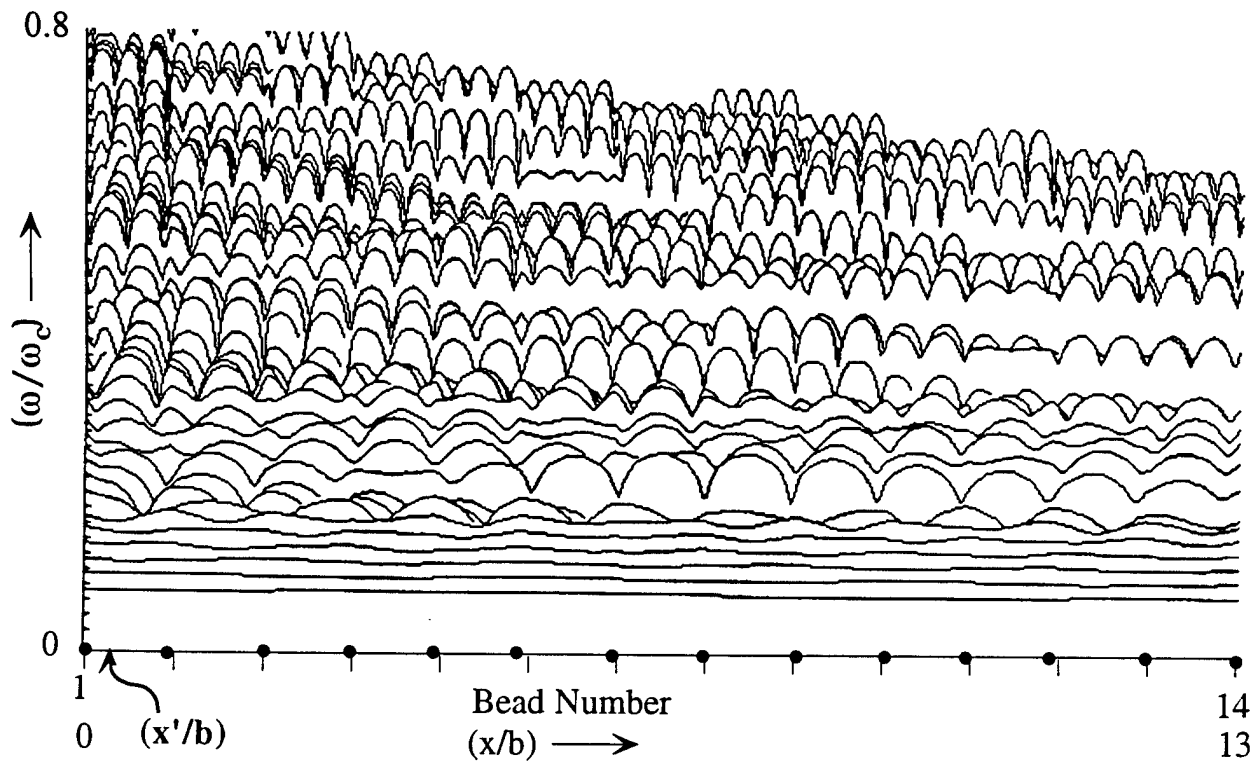
Fig. 6a. A repeat of Fig. 4a except that $[-\, b \times 10^{-1} \lesssim \Delta b_j \lesssim b \times 10^{-1}]$.

a)
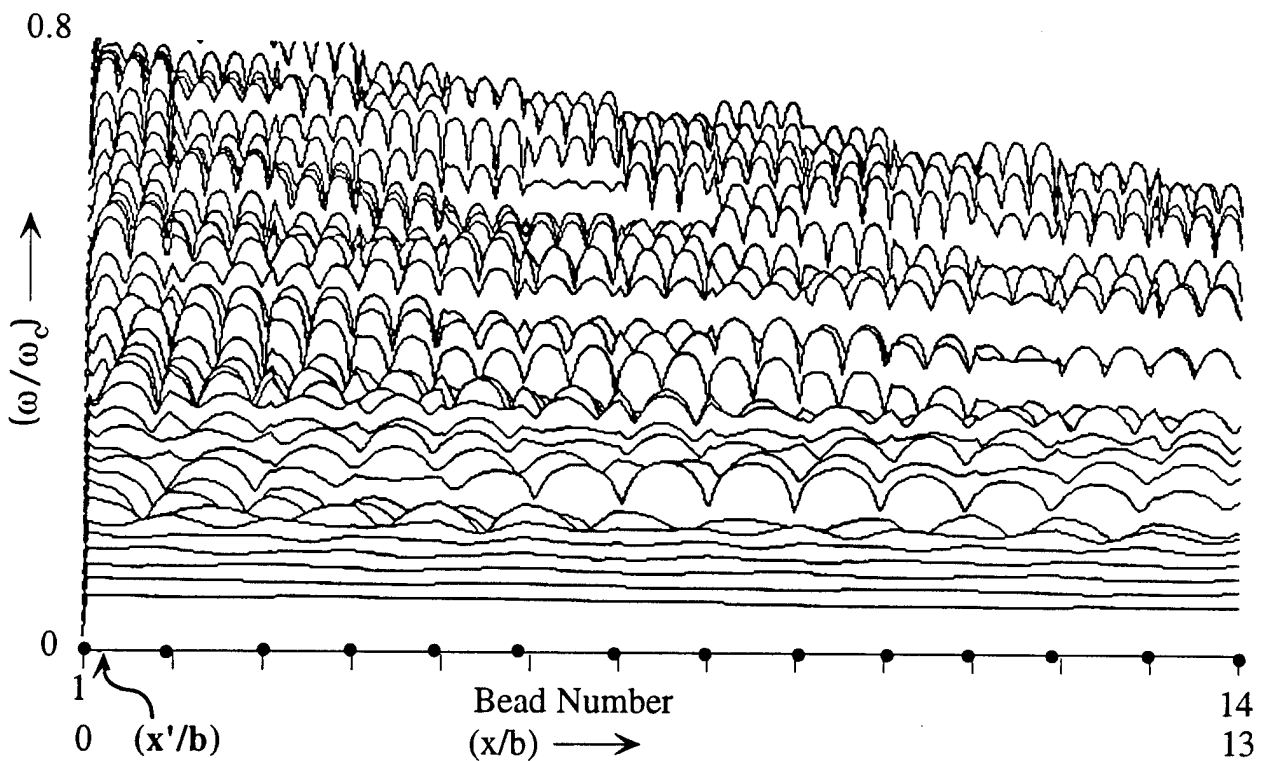
b)

Fig. 6b. A repeat of Fig. 4b except that $[-\, b \times 10^{-1} \lesssim \Delta b_j \lesssim b \times 10^{-1}]$.
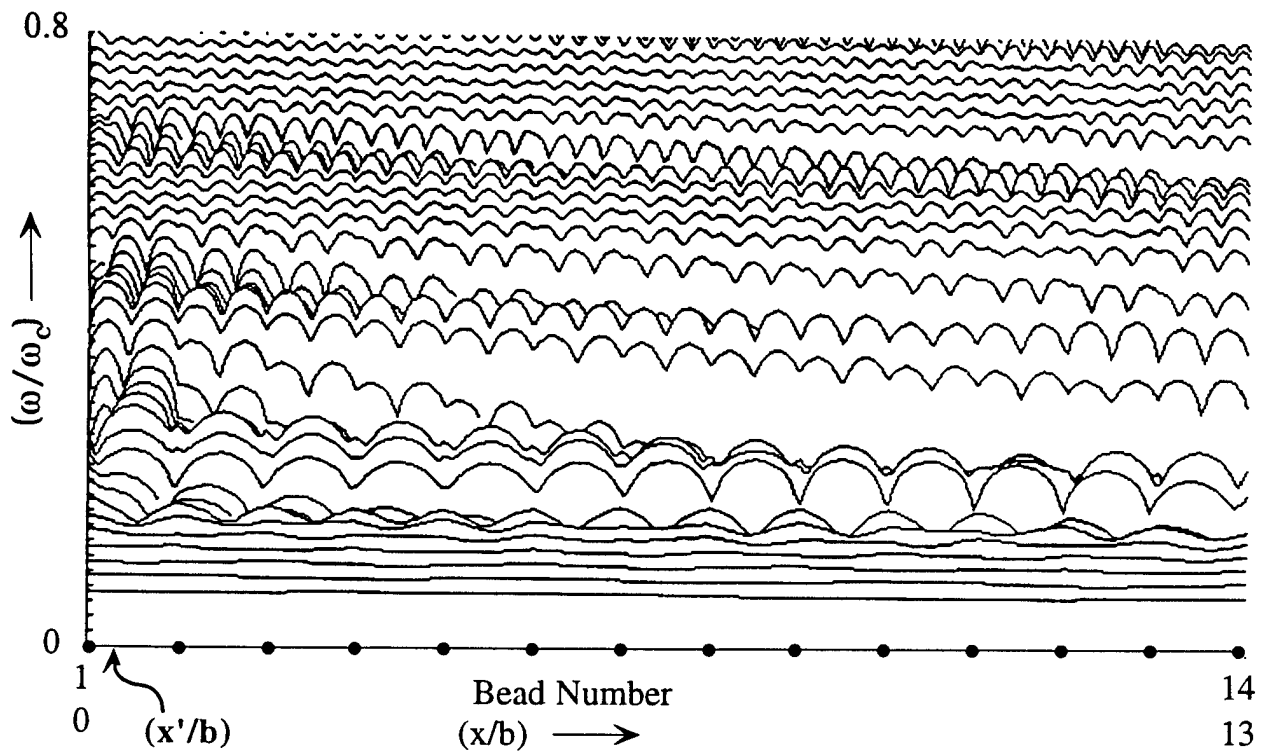
Fig. 7a. A repeat of Fig. 4a except that $(\omega_n/\omega_c)^2 = 10^{-1}$.
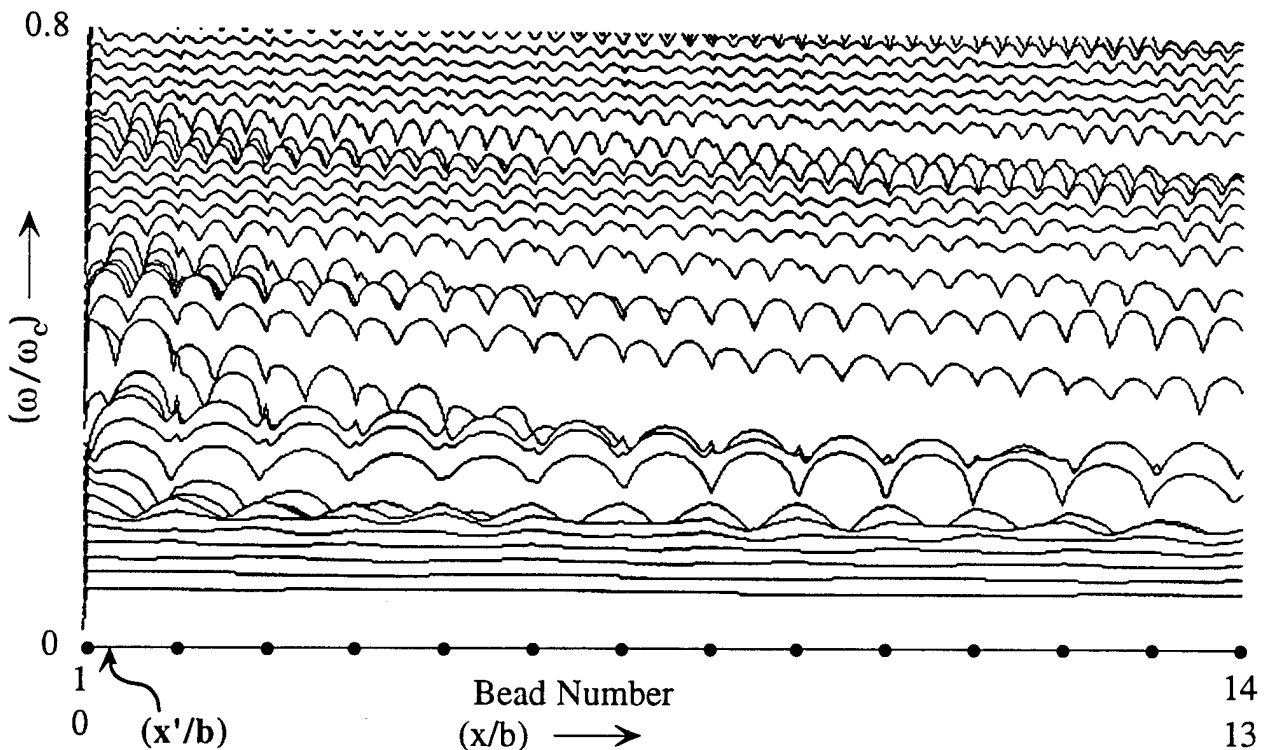
b)

Fig. 7b. A repeat of Fig. 4b except that $(\omega_n/\omega_c)^2 = 10^{-1}$.
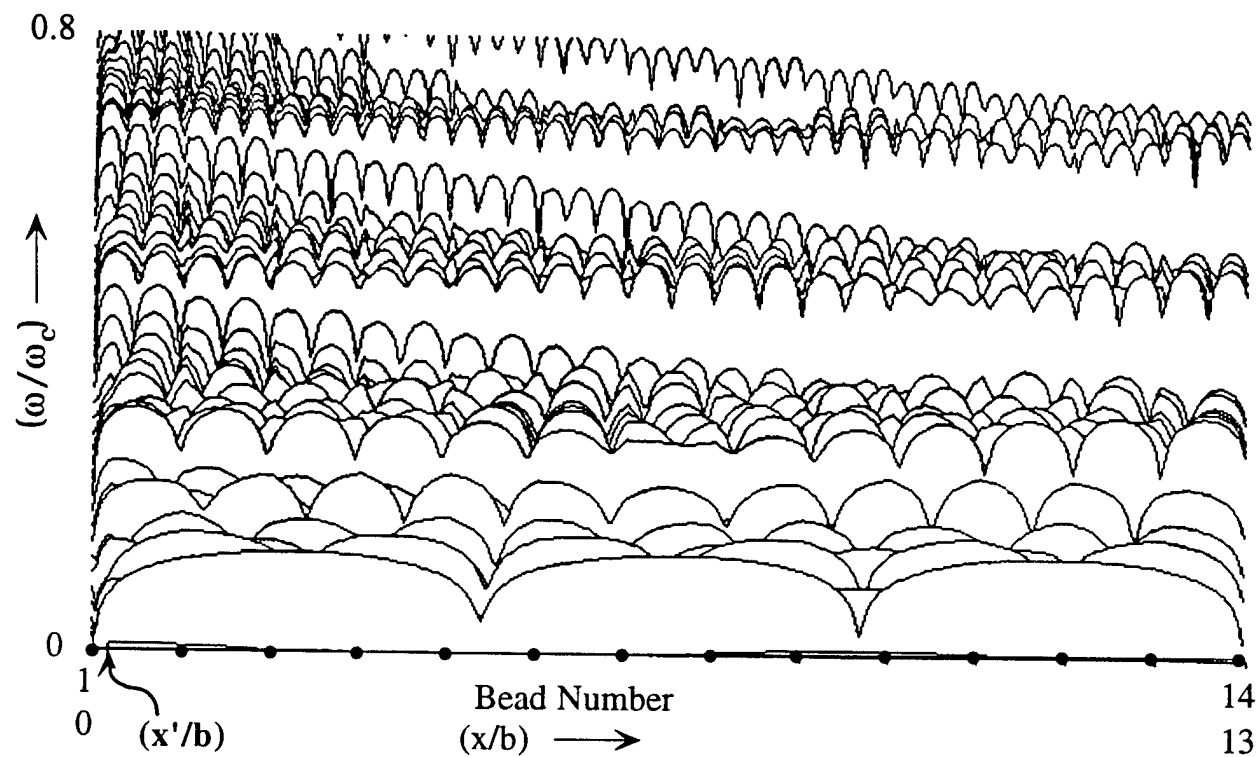
Fig. 8a. A repeat of Fig. 4a except that $(M_1/mb) = (M_{14}/mb) = 10^2$.
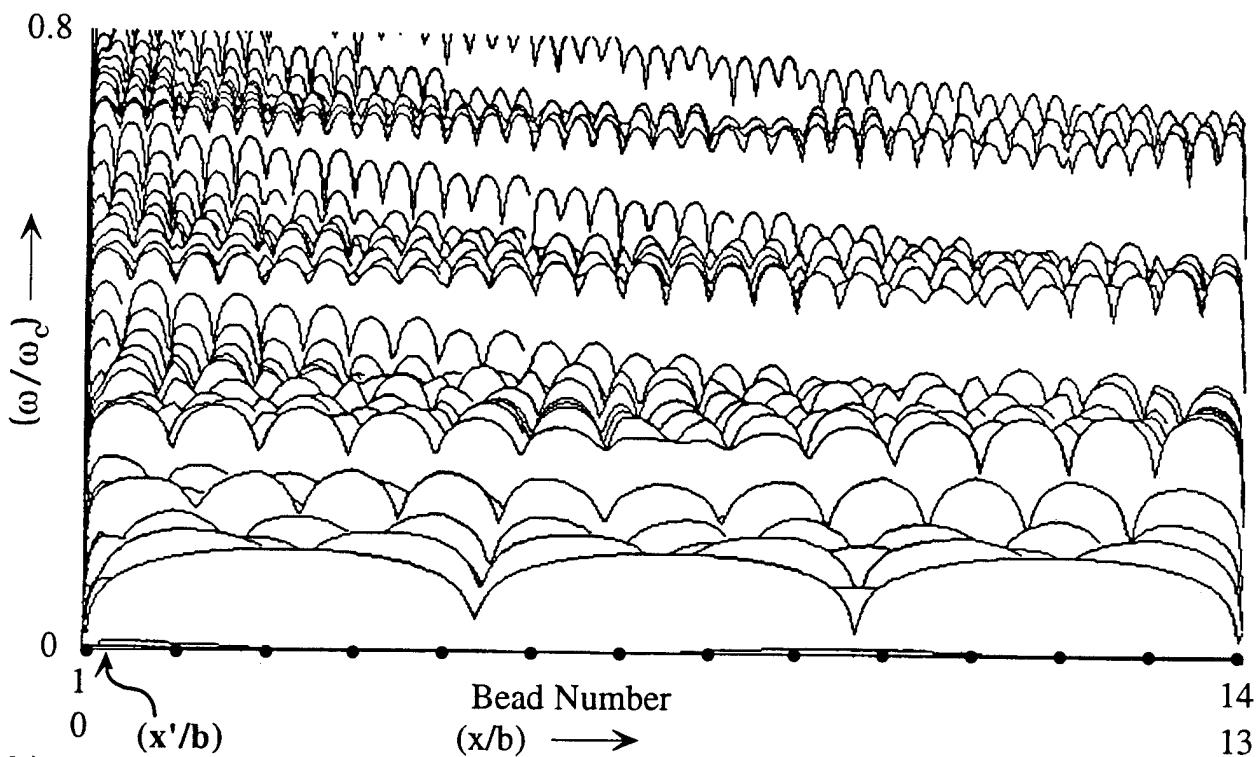
Fig. 8b. A repeat of Fig. 4b except that $(M_1/mb) = (M_{14}/mb) = 10^2$.

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13. ABSTRACT (Maximum 200 words)
Two distinct approaches are employed to derive the coplanar response of a beaded string. In the first approach the beads are modeled by the localized impedances that they present to the string. In the second approach the portion of the string between adjacent heads is assigned to be a one–dimensional dynamic system. Adjacent dynamic systems are coupled at the location of the bead they share. The two approaches derive expressions for the response that do not reconcile term by term. However, the expressions are shown to yield identical results by performing a number of challenging computer experiments using the two models and then comparing corresponding data.

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