

TOLERANCE THEORY OF PERIODIC SURFACES  
DISSERTATION

James A. Godsey, Major, USAF

AFIT/DSG/ENG/95S-02

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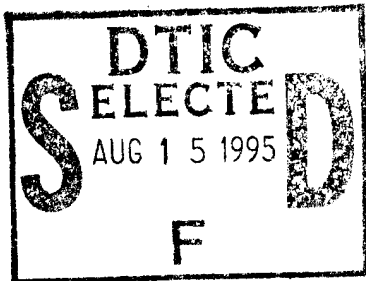
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TOLERANCE THEORY OF PERIODIC SURFACES

DISSERTATION

Presented to the Faculty of the Graduate School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

James A. Godsey, B.S., M.S.

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July 1995

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TOLERANCE THEORY OF PERIODIC SURFACES

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James A. Godsey

*Dedicated to my sons, A.R. and Alex, in the hopes that their thirst for knowledge and truth will never be quenched.*

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## Abstract

Periodic surfaces are the microwave analogs of optical diffraction gratings. While diffraction gratings serve as spectrum analyzers, periodic surfaces have found many practical applications as frequency selective surfaces which transmit or reflect a preselected band of frequencies. In examining the electromagnetic scattering from a periodic surface, one generally assumes the surface is planar and is truly periodic which implies the surface is infinite in extent. Practical considerations make deviations from the ideal unavoidable. For example, infinite surfaces are impossible and true periodicity may not be achieved due to manufacturing tolerances and imperfections. The effects of such deviations on the performance of a given periodic surface thus are matters of practical as well as theoretical interest.

The purpose of this work is to characterize the degradation due to deviations from strict periodicity, on electromagnetic scattering for plane wave incidence upon a periodic surface. To render the problem tractable and still yield useful results, the two-dimensional case of a strip grating consisting of infinitely thin conducting metallic strips with parallel edges, separated by gaps or slits, was chosen to initiate investigations into the effects of the variations. The variations are assumed to be small random perturbations in the widths of the strips and slits as compared to an ideal periodic strip grating.

Prior works in random linear antenna arrays treat the element location as a random point with distributions such as Poisson or uniform. This treatment is not suitable for use as a basis for this study because a distribution specifying the distance between successive points is more appropriate in specifying the random errors in the width of each strip and slit. Using this specification, the average power factor of a linear array of point sources is calculated and is equal to the power spectral density of the random process specifying the inter element spacing. The variance and standard deviation of the average power factor

are also calculated and as the number of points approaches infinity, the standard deviation approaches the mean.

Random variations in geometry preclude exact analytical solutions using techniques such as the Wiener-Hopf method. Hence, formulations for the far-field statistical average power pattern of a strip grating with errors in the widths of the strips and the slits are presented using two approximate methods. The first formulation utilizes the Born approximation in which the unknown aperture fields are replaced by the incident fields. The second formulation utilizes an approximation for the unknown fields which satisfies the edge condition, i.e. has singularities at the edges of the slits/ strips. Approximations for the scattered fields are first derived using perfect electrical conducting (PEC) surface equivalence for a TE<sub>z</sub> polarized plane wave incident upon a strip grating consisting of an infinite PEC screen cut by a number of infinitely long slits (infinite in the z-direction). Babinet's principle is then used to obtain approximations for the fields obtained when a TM<sub>z</sub> polarized plane wave is incident upon the complementary grating formed by interchanging the slits and strips of the original grating, i.e. a number of infinitely long strips in free space. Expressions for the average power pattern are developed in terms of the following variables: number of slits or strips, desired width- and spacing-to-wavelength ratios, and the characteristic functions of the probability density functions of the width and spacing errors. Examples are presented showing the effects of the above variables for both uniform distributed errors and errors with a density function based upon a cosine function. The variance and standard deviation for both approximations are examined, and, like the power factor of the array of point sources, as the number of points approaches infinity, the standard deviation approaches the mean. The results are compared to and agree with the average of a number of patterns computed for realizations of gratings containing actual randomly generated errors in the width of each strip and slit.

# **TOLERANCE THEORY OF PERIODIC SURFACES**

## **I. Introduction**

### **Purpose**

The purpose of this research effort was to develop analytical methods to characterize the degradation in electromagnetic scattering when a plane wave is incident upon a periodic surface having small but random variations in geometry. The characteristics of interest were the reflection and transmission properties as functions of angle of incidence, frequency, and polarization. To render the problem tractable and yet yield useful results, the analysis considered a two-dimensional periodic surface composed of perfect electrical conducting (PEC) strips separated by gaps or slits, i.e. a strip grating. In the ideal case, all strips have equal widths and all slits have equal widths. This research investigated the effects of small changes in the widths of the strips and slits from the ideal case.

### **Background**

Periodic surfaces are the microwave analogs of optical diffraction gratings. While diffraction gratings serve as spectrum analyzers, periodic surfaces have found many practical applications as frequency selective surfaces which transmit or reflect a preselected band of frequencies. In formulating the problem of reflection and transmission of a periodic surface within the framework of Maxwell's equations, one generally assumes the surface is planar and is truly periodic which implies the surface is infinite in extent. Practical considerations make deviations from the ideal unavoidable. For example, infinite surfaces are impossible and true periodicity may not be achieved due to manufacturing

tolerances and imperfections. The effects of such deviations on the performance of a given periodic surface thus become matters of practical as well as theoretical interest.

Similar problems arise in antenna array theory and have been investigated by several authors, notably Lo (14), Ruze (23), Skolnik (25), and Steinberg (26). In antenna array analysis, the location of each element is treated as random and the problem can be solved using well established time series formulations if one replaces distance by time.

A number of authors have investigated different mode matching, variational, and Wiener-Hopf solutions to the problem of the exact periodic strip grating. Several of these include Agronovich, Marchenko, and Shestopalov (1), Baldwin and Heins (5), Chen (8), Daniele, Gilli and Viterbo (9), Ishimaru (11:189-194), Kieburz and Ishimaru (13), Luk'yanov (16), Luneberg and Westpfahl (17), Miles (18), Vanblaricum and Mittra (28), Weinstein (29:267-281), and Wu (30). No work has been done, however, on gratings with random variations in geometry.

## **Overview**

The purpose of this work was to conduct a thorough and detailed investigation of the effects of deviations from strict periodicity on the scattering from periodic surfaces. To make the problem tractable and at the same time reflect prevailing manufacturing standards, the deviations were assumed to be small random perturbations in an otherwise periodic system. The two-dimensional case of a strip grating consisting of an infinitely thin, planar PEC screen cut by an infinite number of periodically spaced, infinitely long slits with parallel edges formed the basis for this research. The effects were characterized in terms of the expected value of the far-field power pattern of the scattered fields.

Before starting on the strip grating problem, two chapters are included to provide important background information and theory. First, the average power pattern of a linear

array of point sources with random spacing between points is determined in Chapter II. Associated with this chapter is an appendix which presents the calculation of the variance of an array with an infinite number of point sources. The methods developed in this chapter and the associated appendix for dealing with random spacings in the array serve as the basis for analyzing the random errors introduced into the strip grating in later chapters. Second, basic electromagnetics theory used in the formulation of the scattered fields is presented in Chapter III. Here, PEC surface equivalence is used to derive a general expression, in terms of the unknown equivalent magnetic surface current, for the fields produced when a TE<sub>z</sub> polarized field is incident upon a strip grating with the slits and strips infinite in the z-direction. Babinet's principle is then used to derive an expression for the fields, in terms of the unknown equivalent electric surface current, produced with TM<sub>z</sub> incidence upon the complementary grating. The complementary grating is obtained by interchanging the slits and strips in the original grating. As a result of Babinet's principle, the equivalent magnetic surface current for TE<sub>z</sub> incidence upon a strip grating is the same as the equivalent electric surface current for TM<sub>z</sub> incidence upon the complementary grating. Consequently, the same integral equation is used to represent the scattered fields for both cases.

The next two chapters utilize two different approximations for the unknown equivalent surface currents. Chapter IV presents the results of utilizing the Born approximation in which the unknown fields are replaced by the incident fields while Chapter V presents the results of utilizing an approximation which satisfies the edge condition. For both methods, expressions for the far-field average power pattern are derived for both gratings with no errors and gratings with independent, identically distributed errors in width (i.e. the length of each slit for the TE<sub>z</sub> mode or the length of each strip for the TM<sub>z</sub> mode) and spacing (i.e. the length of each strip for the TE<sub>z</sub> mode

or the length of each slit for the TMz mode). The probability density functions (PDF) of the errors in width may or may not be equal to the PDF of the errors in spacing.

The far-field average power pattern is computed and presented in graph format using uniformly distributed PDFs with zero mean for the following conditions:

- 1) Width errors alone, spacing errors alone, and both width and spacing errors
- 2) Width equal to spacing, width less than spacing, width greater than spacing
- 3) Maximum error values (expressed as a percentage of desired width or spacing)  
of: 0 (i.e. no error), 5, 10, 15 and 20%

Graphs are presented for gratings with a finite number of slits/strips to allow comparison between the patterns of gratings with errors to gratings with no errors. In addition, graphs are presented for gratings with an infinite number of slits/ strips. Finally, to provide a more realistic picture of the average power pattern, patterns are provided for PDFs based on a cosine function which distributes most of the errors around the mean value of zero. Associated with these two chapters are appendices which present the calculation of the variance of a strip grating with an infinite number of slits/ strips for both the Born approximation and the edge condition approximation.

To validate the expressions for the average power pattern derived in Chapters IV and V, the results of a number of trial realizations are presented in Chapter VI. Here, errors in a number of gratings are randomly generated using the uniform or cosine density functions. The average power pattern is computed for each grating and the results averaged together. This average of the realizations is then compared to the statistical average computed using the expressions derived in Chapters IV and V. Plots showing the standard deviation of the realizations are also provided.

Finally, the results of this research are summarized in Chapter VII which outlines the major accomplishments and provides observations and conclusions arising as a result of this effort

## II. Linear Array of Point Sources

### Overview

Before looking at the problem of the strip grating, it is insightful to calculate the average power pattern of a linear aperiodic array of point sources. This is equivalent to the average power pattern of a linear aperiodic antenna array. As stated previously, several authors have investigated the problem of random errors in antenna arrays. In the works referenced in Chapter I, the authors calculate the average power pattern of a linear aperiodic array where the location of each element is treated as random. The distribution of the element locations is typically specified by normal or Poisson PDFs. These calculations, however, are not useful in this investigation because they consider the location of each element as the random quantity, not the spacing between elements.

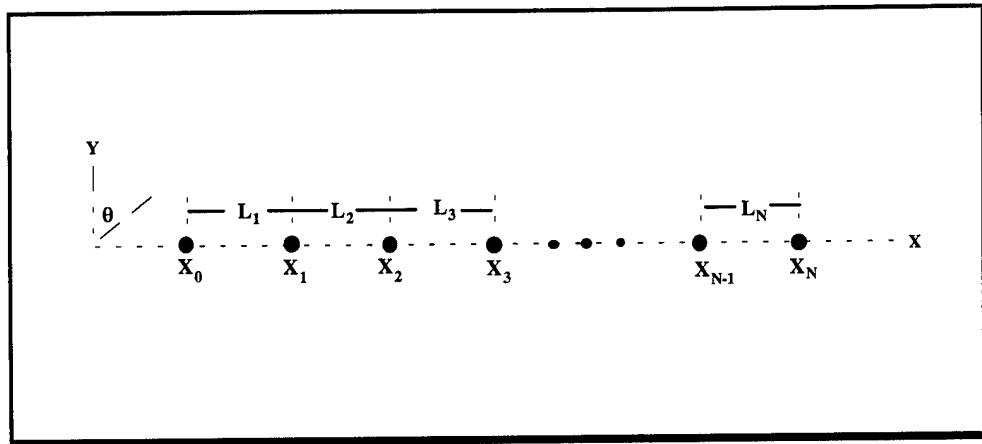
### Average Power Pattern of a Linear Array of Point Sources

Let  $N+1$  point sources be arranged on a line with the location of the  $n^{\text{th}}$  source denoted by  $X_n$  as shown in Figure 2.1. Define the distance between the point sources to be given by:

$$L_n = X_n - X_{n-1}, n = 1, 2, \dots, N \quad (2-1)$$

The array factor is given by (15:11-9):

$$f(u) = \sum_{n=0}^N e^{jkuX_n} \quad (2-2)$$



**Figure 2.1 Linear Array of Point Sources**

where

$$u = \sin \theta$$

$$\theta = \text{angle measured with respect to the normal to the array} \quad (2-3)$$

$$k = 2\pi / \lambda, \lambda = \text{wavelength}$$

The average power factor will be defined as:

$$\overline{|f(u)|^2} = \frac{1}{N+1} \left| \sum_{n=0}^N e^{jkuX_n} \right|^2 \quad (2-4)$$

For the case where the spacing is periodic with period T, then:

$$L_n = T \quad \text{for all } n \quad (2-5)$$

and the average power factor becomes:

$$\begin{aligned}
\overline{|f_{\text{exact}}(u)|^2} &= \frac{1}{N+1} \left| \sum_{n=0}^N e^{jkunT} \right|^2 \\
&= \frac{1}{N+1} \left\{ \frac{\sin \left[ (N+1) \frac{kTu}{2} \right]}{\sin \left( \frac{kTu}{2} \right)} \right\}^2
\end{aligned} \tag{2-6}$$

In the limit as the number of point sources approaches infinity, the average power factor becomes:

$$\lim_{N \rightarrow \infty} \overline{|f_{\text{exact}}(u)|^2} = \frac{\lambda}{T} \sum_{n=-\infty}^{\infty} \delta \left( u - \frac{n\lambda}{T} \right) \tag{2-7}$$

Now let the array have random errors in spacing such that the  $L_n$  are independent, identically distributed random variables with a PDF denoted by  $p(L)$ . The average power factor is now given by:

$$\overline{|f(u)|^2} = \frac{1}{N+1} E \left\{ |f(u)|^2 \right\} \tag{2-8}$$

where  $E \left\{ |f(u)|^2 \right\}$  denotes the expected value of  $|f(u)|^2$  given by (21:138):

$$E \left\{ |f(u)|^2 \right\} = \int_{-\infty}^{+\infty} |f(u)|^2 p(L) dL = \int_0^{+\infty} |f(u)|^2 p(L) dL \tag{2-9}$$

The last equality arises from the fact that  $p(L) = 0$  for  $L < 0$ , i.e. the distance between sources is always a positive number. Thus

$$\begin{aligned}
\overline{|f(u)|^2} &= E \left\{ \left| \frac{1}{N+1} \sum_{n=0}^N e^{jkuX_n} \right|^2 \right\} \\
&= \frac{1}{N+1} E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jkuX_n} e^{-jkuX_m} \right\} \\
&= \frac{1}{N+1} E \left\{ (N+1) + 2 \operatorname{Re} \sum_{n=1}^N \sum_{m=0}^{N-n} e^{jku(X_{n+m} - X_m)} \right\}
\end{aligned} \tag{2-10}$$

Where "Re" denotes the real part. But,

$$X_{n+m} - X_m = L_{m+1} + L_{m+2} + \dots + L_{m+n} \quad (n \text{ terms}) \tag{2-11}$$

and

$$\begin{aligned}
\overline{|f(u)|^2} &= \frac{1}{N+1} E \left\{ (N+1) + 2 \operatorname{Re} \sum_{n=1}^N \sum_{m=0}^{N-n} e^{jku(L_{m+1} + L_{m+2} + \dots + L_{m+n})} \right\} \\
&= 1 + E \left\{ \frac{2}{N+1} \operatorname{Re} \sum_{n=1}^N \sum_{m=0}^{N-n} e^{jku(L_{m+1} + L_{m+2} + \dots + L_{m+n})} \right\} \\
&= 1 + \frac{2}{N+1} \operatorname{Re} \sum_{n=1}^N \sum_{m=0}^{N-n} E \left\{ e^{jku(L_{m+1} + L_{m+2} + \dots + L_{m+n})} \right\}
\end{aligned} \tag{2-12}$$

Where the last equality arises from the fact that the expected value of a sum of independent random variables is the sum of the expected value of each separate random variable. Since all the  $L_n$  are independent but identically distributed,

$$\begin{aligned}
E \left\{ e^{jku(L_{m+1} + L_{m+2} + \dots + L_{m+n})} \right\} &= E \left\{ e^{jkuL_{m+1}} \right\} E \left\{ e^{jkuL_{m+2}} \right\} \dots E \left\{ e^{jkuL_{m+n}} \right\} \\
&= E^n \left\{ e^{jkuL} \right\}
\end{aligned} \tag{2-13}$$

and

$$\begin{aligned}
|\overline{f(u)}|^2 &= 1 + \frac{2}{N+1} \operatorname{Re} \sum_{n=1}^N \sum_{m=0}^{N-n} E^n \{ e^{jkuL} \} \\
&= 1 + \frac{2}{N+1} \operatorname{Re} \sum_{n=1}^N (N+1-n) E^n \{ e^{jkuL} \}
\end{aligned} \tag{2-14}$$

Now define the characteristic function  $\Phi(u)$  as:

$$\Phi(u) = \int_{-\infty}^{+\infty} p(L) e^{jkuL} dL = \int_0^{+\infty} p(L) e^{jkuL} dL \tag{2-15}$$

The characteristic function has the following general properties (21:153-154):

1.  $\Phi(0) = \int_{-\infty}^{+\infty} p(L) dL = 1$
2.  $|\Phi(u)| = \left| \int_{-\infty}^{+\infty} e^{jkuL} p(L) dL \right| \leq \left| \int_{-\infty}^{+\infty} p(L) dL \right| = 1 \Rightarrow |\Phi(u)| \leq 1$
3.  $|\Phi(u)| < 1$  for  $u \neq 0$  unless  $L$  takes on values forming an arithmetic progression (i.e. periodic)

Substituting for the characteristic function, the average power factor becomes:

$$|\overline{f(u)}|^2 = 1 + 2 \operatorname{Re} \sum_{n=1}^N \Phi^n(u) - \frac{2}{N+1} \operatorname{Re} \sum_{n=1}^N n \Phi^n(u) \tag{2-17}$$

Now use the relations:

$$\sum_{n=1}^N \Phi^n = \frac{\Phi - \Phi^{N+1}}{1 - \Phi} \tag{2-18}$$

and

$$\sum_{n=1}^N n \Phi^n = \frac{\Phi + N\Phi^{N+2} - (N+1)\Phi^{N+1}}{(1 - \Phi)^2} \tag{2-19}$$

to obtain:

$$\overline{|f(u)|}^2 = 1 + 2\operatorname{Re}\left\{\frac{\Phi(u) - \Phi(u)^{N+1}}{1 - \Phi(u)}\right\} - \frac{2}{N+1}\operatorname{Re}\left\{\frac{\Phi(u) + N\Phi(u)^{N+2} - (N+1)\Phi(u)^{N+1}}{[1 - \Phi(u)]^2}\right\} \quad (2-20)$$

Application of L'Hopital's rule yields:

$$\overline{|f(0)|}^2 = N+1 \quad (2-21)$$

For the case of an infinite number of point sources, i.e.  $N \rightarrow \infty$ , use the following relations for  $|\Phi| < 1$ :

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \Phi^n = \frac{\Phi}{1 - \Phi} \quad (2-22)$$

and

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N n\Phi^n = \frac{\Phi}{(1 - \Phi)^2} \quad (2-23)$$

to obtain:

$$\begin{aligned} \overline{|f(u)|}^2 &= \lim_{N \rightarrow \infty} \left\{ 1 + 2\operatorname{Re}\left\{\frac{\Phi(u)}{1 - \Phi(u)}\right\} - \frac{2}{N+1}\operatorname{Re}\left\{\frac{\Phi(u)}{[1 - \Phi(u)]^2}\right\} \right\} \\ &= 1 + 2\operatorname{Re}\left\{\frac{\Phi(u)}{1 - \Phi(u)}\right\} \\ &= \frac{1 - |\Phi(u)|^2}{|1 - \Phi(u)|^2} \end{aligned} \quad (2-24)$$

For the case where  $u = 0$ , application of a Taylor series expansion around  $u = 0$  can be used as follows:

Let

$$u \rightarrow \mu + j\omega \quad (2-25)$$

then retaining only the first order terms of a Taylor series expansion for small  $\mu$  and  $\omega$ ,

$$e^{jkuL} \rightarrow 1 + jk(\mu + j\omega)L = 1 - \omega kL + jk\mu L \quad (2-26)$$

and

$$\begin{aligned} \Phi(\mu + j\omega) &\rightarrow \int_0^\infty p(L)[1 - \omega kL + jk\mu L]dL \\ &= 1 - \omega k\bar{L} + jk\mu\bar{L} \end{aligned} \quad (2-27)$$

where

$$\bar{L} = E\{L\} = \int_0^\infty p(L)L dL \quad (2-28)$$

Substituting Equation (2-27) into Equation (2-24) yields:

$$\begin{aligned} \overline{|f(\mu + j\omega)|^2} &= \frac{1 - |1 - \omega k\bar{L} + jk\mu\bar{L}|^2}{|1 - (1 - \omega k\bar{L} + jk\mu\bar{L})|^2} = \frac{1 - [(1 - \omega k\bar{L})^2 + (k\mu\bar{L})^2]}{[(\omega k\bar{L})^2 + (k\mu\bar{L})^2]} \\ &= \frac{2\omega - k\bar{L}(\omega^2 + \mu^2)}{k\bar{L}(\omega^2 + \mu^2)} \end{aligned} \quad (2-29)$$

or retaining only the first order terms:

$$\overline{|f(\mu + j\omega)|^2} = \frac{2\omega}{k\bar{L}(\omega^2 + \mu^2)} \quad (2-30)$$

Using the following relation (27:166):

$$\frac{2\omega}{\omega^2 + \mu^2} \rightarrow 2\pi\delta(\mu) \quad \text{as } \omega \rightarrow 0 \quad (2-31)$$

Equation (2-24) becomes:

$$\overline{|f(\mu)|^2} = \frac{2\pi\delta(\mu)}{k\bar{L}} = \frac{\lambda\delta(\mu)}{\bar{L}} \quad \text{for } \mu = 0 \quad (2-32)$$

which is the same as for an array with exact periodic spacing.

The variance and standard deviation ( $\sigma$ ) of the average power pattern of an array with an infinite number of point sources is calculated in Appendix A with the following results:

$$\sigma = \begin{cases} \overline{|f(u)|^2} & \text{for } u \neq 0 \\ 0 & \text{for } u = 0 \end{cases} \quad (2-33)$$

Comparing Equation (2-6) to Equation (2-20), we see that the average power pattern for the array with random spacing errors is related to the average power pattern with no spacing errors via terms involving the characteristic function of the spacing errors. Similar expressions will be developed for the strip grating.

### III. Electromagnetic Theory

#### Overview

This chapter presents basic electromagnetics theory required to develop the expressions for the fields scattered by a strip grating. Expressions for the scattered fields are derived in terms of integral equations involving unknown equivalent magnetic or electric scattering currents.

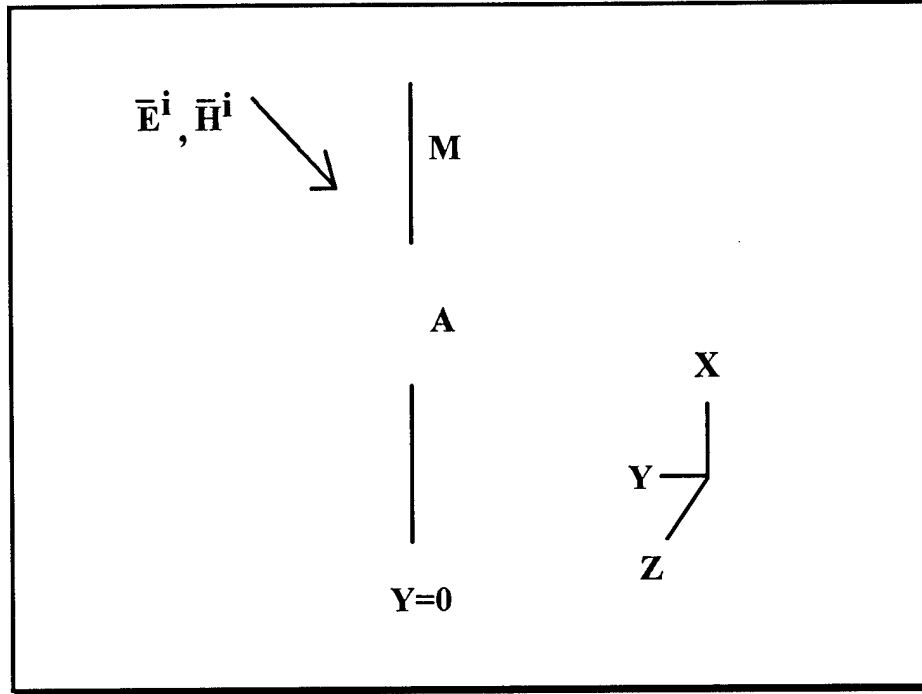
#### Integral Equations for an Aperture in a Conducting Screen

The following forms of the free space Maxwell's equations expressed in Gaussian units with  $e^{j\omega t}$  time convention will be used in all derivations:

$$\begin{aligned}\nabla \times \bar{H} &= jk\bar{E} \\ \nabla \times \bar{E} &= -jk\bar{H}\end{aligned}\tag{3-1}$$

Consider a plane wave incident upon a planar PEC screen containing one or more apertures as shown in Figure 3.1. Let A denote the aperture(s) and M denote the PEC surface. An equivalent problem can be obtained by wrapping a Huygen surface around the screen and using PEC surface equivalence with the appropriate equivalent electric and magnetic surface currents (4:329-334). The equivalent surface currents are given by (4:330):

$$\begin{aligned}\bar{J}_{\text{Seq}} &= \hat{n} \times \bar{H} \\ \bar{M}_{\text{Seq}} &= \bar{E} \times \hat{n}\end{aligned}\tag{3-2}$$



**Figure 3.1 Plane Wave Incident upon PEC Screen with Aperture**

where  $\hat{n}$  is the outward normal from the Huygen surface and  $\bar{E}$  and  $\bar{H}$  are the total fields on the surface of the screen. Since the screen has been replaced with a PEC plane, the  $\bar{J}_{seq}$  electric currents do not radiate. Also,  $\bar{M}_{seq} = 0$  on the original PEC surface, leaving a non-zero  $\bar{M}_{seq}$  magnetic current only in the location of the former aperture as shown in Figure 3.2.  $\bar{M}_{seq1}$  and  $\bar{M}_{seq2}$  are given by (4:330):

$$\begin{aligned}\bar{M}_{seq1} &= -\hat{n}_1 \times \bar{E}_A = -\hat{y} \times \bar{E}_A \\ \bar{M}_{seq2} &= -\hat{n}_2 \times \bar{E}_A = \hat{y} \times \bar{E}_A = -\bar{M}_{seq1}\end{aligned}\tag{3-3}$$

where

$$\bar{E}_A = \text{total } \bar{E} \text{ in the aperture}$$

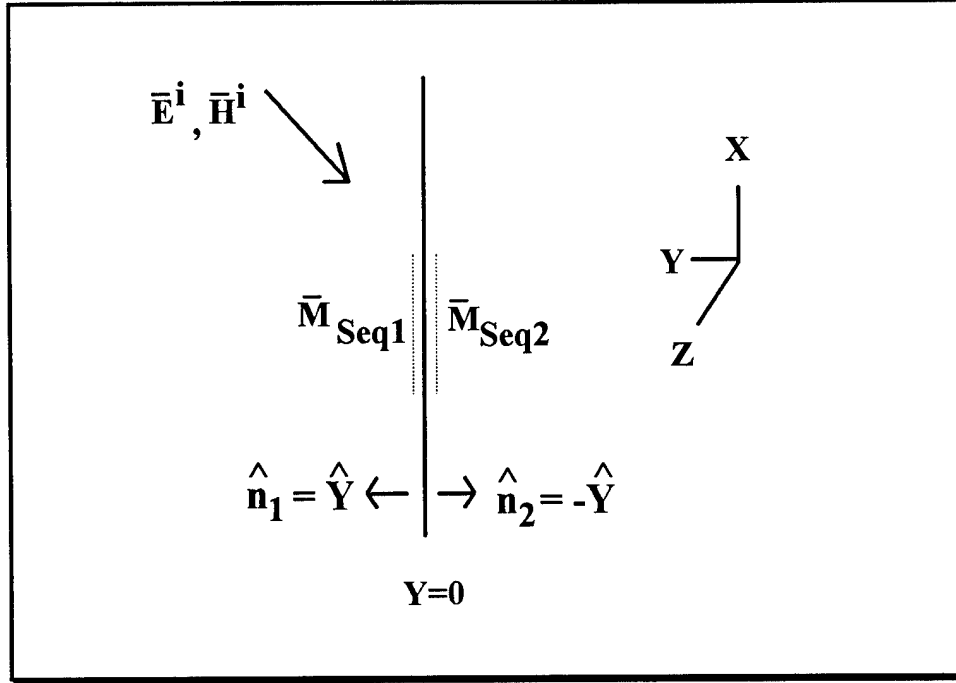


Figure 3.2 Equivalent Problem

The vector potential  $\bar{F}$  is given by (4:279):

$$\bar{F}(\bar{R}) = \int_A \bar{M}_s(\bar{R}') G_o(\bar{R}, \bar{R}') da' \quad (3-4)$$

where

$$G_o(\bar{R}, \bar{R}') = \frac{e^{-jk|\bar{R}-\bar{R}'|}}{4\pi|\bar{R}-\bar{R}'|} \quad (3-5)$$

is the free space Green's Function and

$$|\bar{R} - \bar{R}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \quad (3-6)$$

The scattered fields due to the vector potential are given by (4:259):

$$\begin{aligned}\bar{E}^s &= -\nabla \times \bar{F} \\ \bar{H}^s &= -\frac{1}{jk}(\nabla \times \bar{E}_s) = \frac{j}{k}\nabla \times (-\nabla \times \bar{F}) = -jk\bar{F} - \frac{j}{k}\nabla(\nabla \cdot \bar{F})\end{aligned}\quad (3-7)$$

The total fields are given by, using image theory (7:97):

$$\bar{E} = \begin{cases} \bar{E}^i + \bar{E}^r + \bar{E}^s(2\bar{M}_{Seq1}) & , y > 0 \\ \bar{E}^s(2\bar{M}_{Seq2}) & , y < 0 \end{cases} \quad (3-8)$$

$$\bar{H} = \begin{cases} \bar{H}^i + \bar{H}^r + \bar{H}^s(2\bar{M}_{Seq1}) & , y > 0 \\ \bar{H}^s(2\bar{M}_{Seq2}) & , y < 0 \end{cases} \quad (3-9)$$

where  $\bar{E}^s(2\bar{M}_{Seq1})$  and  $\bar{H}^s(2\bar{M}_{Seq1})$  are the scattered fields due to  $2\bar{M}_{Seq1}$  and  $\bar{E}^s(2\bar{M}_{Seq2})$  and  $\bar{H}^s(2\bar{M}_{Seq2})$  are the scattered fields due to  $2\bar{M}_{Seq2}$ . The factor of two is due to the image of the magnetic current in the presence of the PEC plane.  $\bar{E}^i$  and  $\bar{H}^i$  are the incident fields while  $\bar{E}^r$  and  $\bar{H}^r$  are the fields reflected from the PEC plane. From image theory, the components of the reflected fields are given by (3:158):

$$\begin{aligned}E_x^r(x, y, z) &= -E_x^i(x, -y, z) & H_x^r(x, y, z) &= H_x^i(x, -y, z) \\ E_y^r(x, y, z) &= E_y^i(x, -y, z) & H_y^r(x, y, z) &= -H_y^i(x, -y, z) \\ E_z^r(x, y, z) &= -E_z^i(x, -y, z) & H_z^r(x, y, z) &= H_z^i(x, -y, z)\end{aligned} \quad (3-10)$$

### Boundary Conditions for an Aperture in a Conducting Screen

Due to the symmetry of the problem and the vector potential, the following relations apply to the scattered fields (12:436), (7:97), (6:559):

$$\begin{aligned}
E_x^s(x, y, z) &= E_x^s(x, -y, z) & H_x^s(x, y, z) &= -H_x^s(x, -y, z) \\
E_y^s(x, y, z) &= -E_y^s(x, -y, z) & H_y^s(x, y, z) &= H_y^s(x, -y, z) \\
E_z^s(x, y, z) &= E_z^s(x, -y, z) & H_z^s(x, y, z) &= -H_z^s(x, -y, z)
\end{aligned} \tag{3-11}$$

On the PEC surface, the tangential E-field components must be zero. Thus, on the PEC surface M,

$$E_x = E_z = 0 \tag{3-12}$$

In the aperture A, continuity of the fields requires that

$$\begin{aligned}
E_{x2}^s &= E_x^i + E_x^r + E_{x1}^s & H_{x2}^s &= H_x^i + H_x^r + H_{x1}^s \\
E_{y2}^s &= E_y^i + E_y^r + E_{y1}^s & H_{y2}^s &= H_y^i + H_y^r + H_{y1}^s \\
E_{z2}^s &= E_z^i + E_z^r + E_{z1}^s & H_{z2}^s &= H_z^i + H_z^r + H_{z1}^s
\end{aligned} \tag{3-13}$$

Using Equation (3-10) and the fact that  $x=0$  in the aperture, Equation (3-13) becomes

$$\begin{aligned}
E_{x2}^s &= E_{x1}^s & H_{x2}^s &= 2H_x^i + H_{x1}^s \\
E_{y2}^s &= 2E_y^i + E_{y1}^s & H_{y2}^s &= H_{y1}^s \\
E_{z2}^s &= E_{z1}^s & H_{z2}^s &= 2H_z^i + H_{z1}^s
\end{aligned} \tag{3-14}$$

Using Equation (3-11) at  $x=0$ , we see that in the aperture,

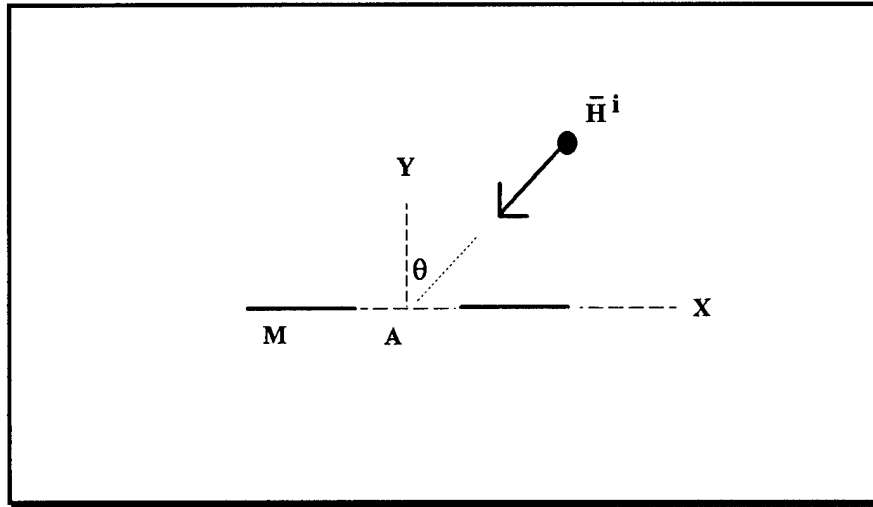
$$\begin{aligned}
H_{x2}^s &= H_x^i \\
H_{z2}^s &= H_z^i \\
E_{y2}^s &= E_y^i
\end{aligned} \tag{3-15}$$

### Equations for TEz Mode Incident on Screen with Infinite Slits

Now consider a TEz polarized wave incident upon a conducting screen in which the aperture or apertures are infinite in the z-direction. For the geometry shown in Figure 3.3, the TEz incident wave is given by:

$$\vec{H}^i = H_z^i \hat{z} = e^{jk(x\sin\theta + y\cos\theta)} \hat{z} \quad (3-16)$$

Since the incident magnetic field has only a z-directed component which is independent of z and the screen has no variations in z, the scattered and total magnetic fields also have only a z-directed component which is independent of z.



**Figure 3.3 Plane Wave Incident upon PEC Screen with Slit, TEz Mode**

The scattered magnetic fields can be found using Equations (3-4) through (3-7) assuming  $\vec{M}_s$  has only a z-directed component which is independent of z (4:698-699). Thus

$$\vec{F}(x, y) = \hat{z} \int_A M_{sz}(x') \frac{e^{-jk\sqrt{(x-x')^2 + y^2 + (z-z')^2}}}{4\pi\sqrt{(x-x')^2 + y^2 + (z-z')^2}} dx' dz' \quad (3-17)$$

Use the following integral:

$$\int_{-\infty}^{+\infty} \frac{e^{-jk\sqrt{a^2+z^2}}}{\sqrt{a^2+z^2}} dz = -j\pi H_0^{(2)}(ka) \quad (3-18)$$

to obtain

$$\bar{F}(x, y) = -\hat{z} \frac{j}{4} \int_{L1} M_{Sz}(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (3-19)$$

where  $L1$  is the set of lines on the  $x$ -axis where the apertures cut the  $z = 0$  plane.

Substituting Equation (3-19) into Equations (3-7) and (3-9) yields

$$\bar{H} = H_z \hat{z} \quad (3-20)$$

where

$$H_z = \begin{cases} H_z^i + H_z^r - \frac{k}{2} \int_{L1} M_S(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' & , y > 0 \\ \frac{k}{2} \int_{L1} M_S(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' & , y < 0 \end{cases} \quad (3-21)$$

and

$$\begin{aligned} M_S &= M_{Seq1z} = -M_{Seq2z} = E_{Ax} \\ &= \text{the } x \text{ component of the } E \text{ field in the aperture} \end{aligned} \quad (3-22)$$

The  $E$ -field components are found by substituting Equation (3-20) into Equation (3-1) to obtain:

$$\begin{aligned}
E_x &= \frac{1}{jk} \frac{\partial H_z}{\partial y} \\
E_y &= -\frac{1}{jk} \frac{\partial H_z}{\partial x} \\
E_z &= 0
\end{aligned} \tag{3-23}$$

The unknown quantity  $M_s$  can be found by solving the following integral equation obtained by substituting Equation (3-21) into the boundary condition for the tangential component of the H-field in the aperture (Equation (3-15)) :

$$\int_{L1} M_s(x') H_0^{(2)}(k|x-x'|) dx' = \frac{2}{k} H_z^i(x, 0) \tag{3-24}$$

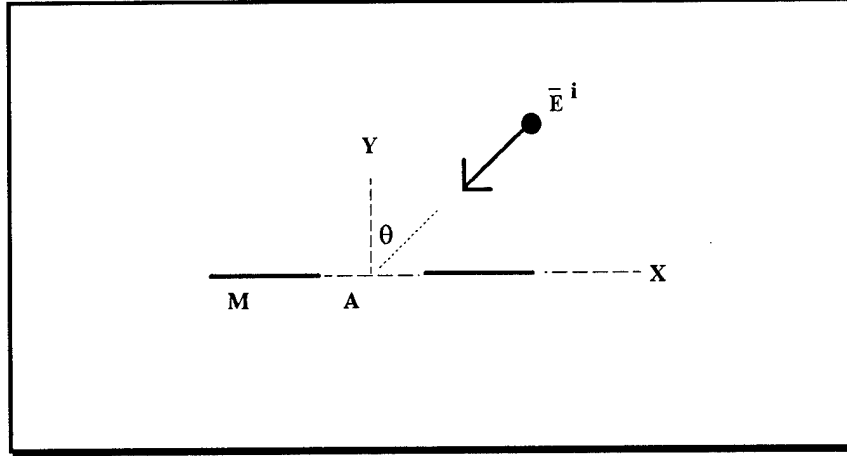
which is valid for points in the slit regions.

### Equations for TMz Mode Incident on Screen with Infinite Slits

Now consider a TMz polarized wave incident upon a conducting screen in which the aperture or apertures are infinite in the z-direction. For the geometry shown in Figure 3.4, the TMz incident wave is given by:

$$\vec{E}^i = E_z^i \hat{z} = e^{jk(x \sin \theta + y \cos \theta)} \hat{z} \tag{3-25}$$

Since the incident electric field has only a z-directed component which is independent of z and the screen has no variations in z, the scattered and total electric fields also have only a z-directed component which is independent of z. The scattered electric fields can be found using Equations (3-4) through (3-7) assuming  $\vec{M}_s$  has only a x-directed component which is independent of z. This leads to a solution for the scattered electric field in the form of:



**Figure 3.4 Plane Wave Incident upon PEC Screen with Slit, TMz Mode**

$$\vec{E}_z = \hat{z} \frac{\partial F_x}{\partial y} \quad (3-26)$$

An alternative solution can be found directly from the TEz mode solution using Babinet's principle. Babinet's principle is as follows (3:166) and (6:560):

- 1) Let  $\vec{E}_1^i = \vec{F}$  and  $\vec{H}_1^i = \vec{G}$  be incident on the screen from the +y side and  $\vec{E}_1$  and  $\vec{H}_1$  be the resulting fields on the -y side.
- 2) Let  $\vec{E}_2^i = -\vec{G}$  and  $\vec{H}_2^i = \vec{F}$  be incident on the complementary screen from the +y side and  $\vec{E}_2$  and  $\vec{H}_2$  be the resulting fields on the -y side. The complementary screen is formed by interchanging the PEC surfaces and the apertures.

Then:

$$\begin{aligned} \vec{E}_1 + \vec{H}_2 &= \vec{F} \\ \vec{H}_1 - \vec{E}_2 &= \vec{G} \end{aligned} \quad (3-27)$$

Babinet's principle can be used as follows:

Let  $\bar{E}_1, \bar{E}_1^i, \bar{H}_1, \bar{H}_1^i$  be the TMz fields and  $\bar{E}_2, \bar{E}_2^i, \bar{H}_2, \bar{H}_2^i$  be the TEz fields. Then,

$$\begin{aligned}\bar{E}_1 &= E_{z1} \hat{z} \\ \bar{H}_2 &= H_{z2} \hat{z}\end{aligned}\tag{3-28}$$

and

$$\bar{E}_1^i = \bar{H}_2^i = F_z \hat{z}\tag{3-29}$$

From Equation (3-27),

$$E_{z1} = E_{z1}^i - \frac{k}{2} \int_{L2} M_S(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \tag{3-30}$$

where  $L2$  is the set of lines on the  $x$ -axis where the PEC surfaces cut the  $z=0$  plane. Although  $M_S$  is numerically equal to the magnetic scattering current found in the TEz case, it now can be considered to represent the electric current density induced on the surface of the strips. This leads to introducing an electric scattering current  $J_S$  by the following substitution:

$$M_S \rightarrow J_S \tag{3-31}$$

With this notation, Equation (3-30) becomes

$$E_{z1} = E_{z1}^i - \frac{k}{2} \int_{L2} J_S(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \tag{3-32}$$

Thus, for the TMz mode,

$$\bar{E} = E_z \hat{z} \quad (3-33)$$

where

$$\begin{aligned} E_z &= E_z^i - \frac{k}{2} \int_{L2} J_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad \text{for } y < 0 \\ &= E_z^i + E_z^{s,tot} \end{aligned} \quad (3-34)$$

The  $E_z^{s,tot}$  notation is used to differentiate the field scattered by the complementary grating for the TMz case from the field scattered by the original grating for the TEz case as given in Equation (3-8). To solve for the electric field for  $y > 0$ , superimpose the image problem of  $-E_z^i(x, -y, z)$  incident on the negative side of the screen which yields  $-E_z(x, -y, z)$  (3:163-164). Now

$$\begin{aligned} E_z(x, y, z) - E_z(x, -y, z) &= E_z^i(x, y, z) + E_z^{s,tot}(x, y, z) - E_z^i(x, -y, z) - E_z^{s,tot}(x, -y, z) \\ &= E_z^i(x, y, z) - E_z^i(x, -y, z) \end{aligned} \quad (3-35)$$

since  $E_z^{s,tot}$  is an even function in  $y$ . Thus for  $y > 0$ ,

$$\begin{aligned} E_z(x, y, z) &= E_z(x, -y, z) + E_z^i(x, y, z) - E_z^i(x, -y, z) \\ &= E_z^i(x, -y, z) - \frac{k}{2} \int_{L2} J_s H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' + E_z^i(x, y, z) - E_z^i(x, -y, z) \\ &= E_z^i(x, y, z) - \frac{k}{2} \int_{L2} J_s H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx', \quad y > 0 \end{aligned} \quad (3-36)$$

Equations (3-34) and (3-36) have the same form, thus

$$\begin{aligned} E_z &= E_z^i - \frac{k}{2} \int_{L2} J_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad \text{for all } y \\ &= E_z^i + E_z^{s,tot} \end{aligned} \quad (3-37)$$

Note the similarity in the form of the integrals for the TMz case, Equation (3-37) and the TEz case Equation (3-21). The H-field components are found by substituting Equation (3-33) into Equation (3-1) to obtain:

$$\begin{aligned} H_x &= -\frac{1}{jk} \frac{\partial E_z}{\partial y} \\ H_y &= \frac{1}{jk} \frac{\partial E_z}{\partial x} \\ H_z &= 0 \end{aligned} \quad (3-38)$$

The unknown quantity  $J_s$  can be found by solving the following integral equation obtained by substituting Equation (3-37) into the boundary condition for the tangential component of the E-field on the (PEC) surface (Equation (3-12)) :

$$\int_{L_2} J_s(x') H_0^{(2)}(k|x-x'|) dx' = \frac{2}{k} E_z^i(x, 0) \quad (3-39)$$

which is valid for points on the original PEC surfaces. Note that Equation (3-39) has the same form as Equation (3-24) obtained for the TEz case. In fact, since

$$\begin{aligned} E_z^i \text{ for the original grating} &= H_z^i \text{ for complementary grating} \\ L_2 \text{ for the original grating} &= L_1 \text{ for the complementary grating} \end{aligned} \quad (3-40)$$

then as expected from Babinet's principle,

$$J_s \text{ for the original grating} = M_s \text{ for complementary grating} \quad (3-41)$$

and in the region behind the grating, i.e.  $y < 0$ ,

$$-E_z^{s, \text{tot}} \text{ for the original grating} = H_z^s \text{ for complementary grating} \quad (3-42)$$

### Equations for TEz and TMz Modes Incident on Strip Grating

The integrals for the TEz mode in Equation (3-21) and the TMz mode in Equation (3-37) both are of the form:

$$S = \frac{k}{2} \int_L I_s H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (3-43)$$

where  $S$  is the scattered field,  $I_s = M_s$  or  $J_s$ , and  $L = L_1$  or  $L_2$  as appropriate. Thus both can be evaluated in the same manner. Now consider the screen to contain a number of strips and slits infinite in the  $z$ -direction as shown in Figure 3.5.

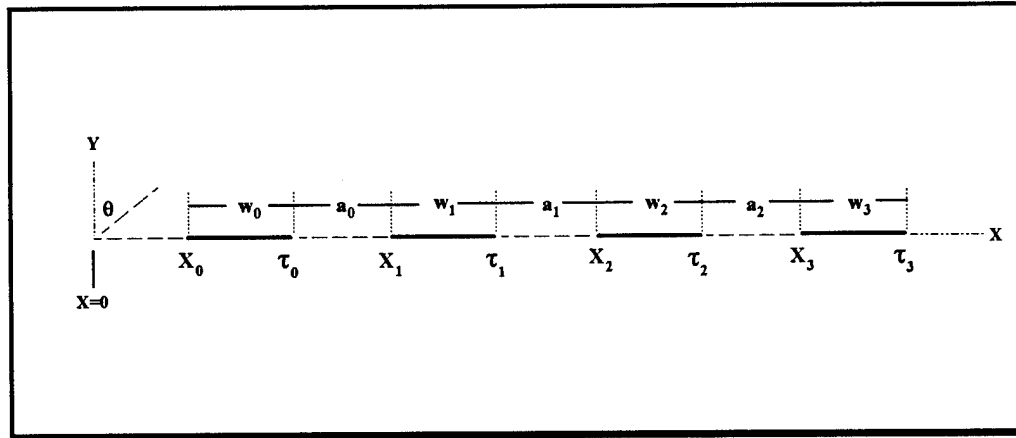


Figure 3.5 Infinite Strip Grating

For the TMz case, the  $w_n$  represent the width of the strips and the  $a_n$  represent the width of the slits. For the TEz case, the  $a_n$  represent the width of the strips and the  $w_n$  represent the width of the slits. For future discussion, the term width will refer to the  $w_n$ .

while the term spacing will refer to the  $a_n$ . Using these specifications for the grating, Equation (3-43) becomes:

$$S = \sum_n \frac{k}{2} \int_{x_n}^{\tau_n} I_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (3-44)$$

where  $n$  ranges from 0 to  $+\infty$ . The integral equations incorporating the boundary conditions, Equation (3-24) and Equation (3-39), become:

$$\sum_n \frac{k}{2} \int_{x_n}^{\tau_n} I_s(x') H_0^{(2)}(k|x-x'|) dx' = e^{jkx \sin \theta} \quad (3-45)$$

Equation (3-45) is valid for  $X_n < x < \tau_n$ .

Referring to Figure 3.5, the following relations hold for a grating with no errors:

$$\begin{aligned} a_n &= A \\ w_n &= W \\ A + W &= T \end{aligned} \quad (3-46)$$

where  $T$  is the period of the grating. Setting the origin at  $X_0$ , i.e.  $X_0 = 0$ , Equation (3-44) becomes:

$$S = \sum_n \frac{k}{2} \int_{nT}^{nT+W} I_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (3-47)$$

Using the change of variables

$$x'' = x' - nT \quad (3-48)$$

Equation (3-47) becomes:

$$S = \sum_n \frac{k}{2} \int_0^w I_s(x'' + nT) H_0^{(2)}(k\sqrt{(x - x'' - nT)^2 + y^2}) dx'' \quad (3-49)$$

Since the grating is periodic, the current in the slits or strips is related according to Floquet's theorem by (24:14):

$$I_s(x'' + nT) = I_s(x'') e^{jknT \sin \theta} \quad (3-50)$$

This leads to

$$S = \sum_n \frac{k}{2} \int_0^w I_s(x'') e^{jknT \sin \theta} H_0^{(2)}(k\sqrt{(x - x'' - nT)^2 + y^2}) dx'' \quad (3-51)$$

The equation for the boundary conditions, Equation (3-45) becomes:

$$\sum_n \frac{k}{2} \int_0^w I_s(x'') e^{jknT \sin \theta} H_0^{(2)}(k|x - x'' - nT|) dx'' = e^{jkx \sin \theta} \quad (3-52)$$

For the purposes of this analysis, errors in geometry of an ideal grating will be introduced as perturbations from the desired values of each strip and slit width. This can be expressed as:

$$\begin{aligned} a_n &= A + \epsilon_n \\ w_n &= W + \xi_n \end{aligned} \quad (3-53)$$

subject to the conditions:

$$E\{\epsilon\} = E\{\xi\} = 0 \quad (3-54)$$

where the  $\epsilon_n$  are independent, identically distributed random variables with a PDF denoted by  $p_\epsilon(\epsilon)$ . Similarly, the  $\xi_n$  are independent, identically distributed random variables with a PDF denoted by  $p_\xi(\xi)$ . The PDFs  $p_\epsilon(\epsilon)$  and  $p_\xi(\xi)$  may or may not be equal to each other.

## IV. Born Approach

### Overview

This chapter presents results obtained when the unknown fields in the integral field equations are replaced by the incident fields. This is an application of the Born approximation (20:1073) and is equivalent to a physical optics approximation. It is valid in the high frequency region where the wavelength of the incident wave is much smaller than the widths of all the slits and strips. Consequently, it is especially useful in problems of optical diffraction.

### Far Field Pattern for Grating with No Errors

For a grating with no errors, the scattered fields are given by (Equation (3-47)):

$$S = \sum_n \frac{k}{2} \int_{nT}^{nT+W} I_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (4-1)$$

Using the relation

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-j(x-\pi/4)} \text{ as } x \rightarrow \infty \quad (4-2)$$

the scattered fields in the far zone become:

$$S \approx \sum_n \frac{k}{2} \int_{nT}^{nT+W} I_s(x') \sqrt{\frac{2}{\pi k\sqrt{(x-x')^2 + y^2}}} e^{-j(k\sqrt{(x-x')^2 + y^2} - \pi/4)} dx' \quad (4-3)$$

Next, use the Fraunhofer approximation:

$$\sqrt{(x-x')^2 + y^2} \approx \begin{cases} R & \text{for amplitude purposes} \\ R + x' \sin \theta & \text{for phase purposes} \end{cases} \quad (4-4)$$

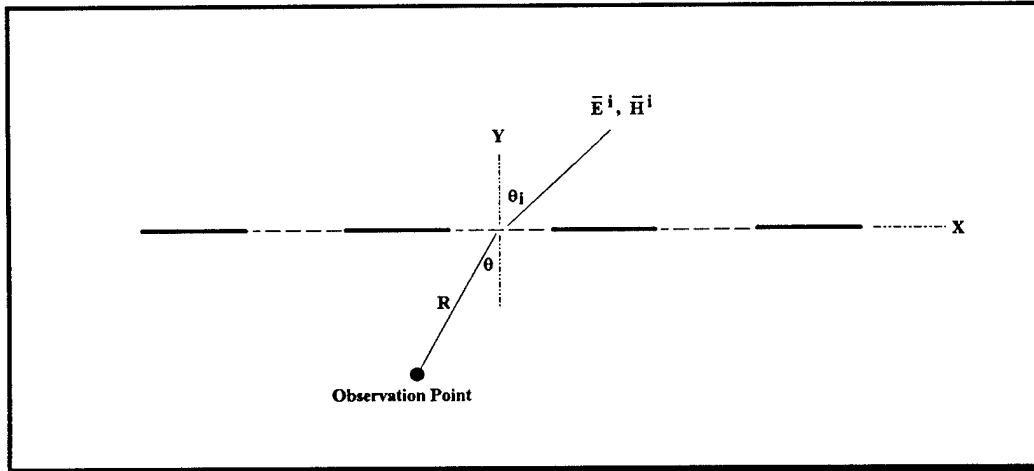
where

$$\sqrt{x^2 + y^2} = R \quad (4-5)$$

and  $\theta$  is the observation angle as defined in Figure 4.1 to obtain:

$$S \approx S^f = \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{nT}^{nT+W} I_s(x') e^{-jkx' \sin \theta} dx' \quad (4-6)$$

where the superscript "f" refers to the far-field approximation for the scattered fields.



**Figure 4.1 Incidence and Observation Angles**

Consider TE<sub>z</sub> incidence with the incident electric field defined by:

$$H_z^i = e^{jk(x \sin \theta_i + y \cos \theta_i)} \quad (4-7)$$

then,

$$\begin{aligned}
\bar{I}_s(x) &= \bar{M}_s = -\hat{y} \times \bar{E} \Big|_{y=0} \\
&\approx -\hat{y} \times \bar{E}^i \Big|_{y=0} = -\hat{y} \times \frac{1}{jk} \left( \frac{\partial H_z}{\partial y} \hat{x} - \frac{\partial H_z}{\partial x} \hat{y} \right) \Big|_{y=0} \\
&= \frac{1}{jk} \frac{\partial H_z}{\partial y} \Big|_{y=0} \hat{z} = \cos \theta_i e^{jkx \sin \theta_i} \hat{z}
\end{aligned} \tag{4-8}$$

and

$$I_s(x') = \cos \theta_i e^{jkx' \sin \theta_i} \tag{4-9}$$

for both the TMz and TEz cases. Inserting this into Equation (4-6) yields:

$$\begin{aligned}
S^f &= \cos \theta_i \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{nT}^{nT+W} e^{jkx' \sin \theta_i} e^{-jkx' \sin \theta} dx' \\
&= \cos \theta_i \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{nT}^{nT+W} e^{jkx'(\sin \theta_i - \sin \theta)} dx' \\
&= \cos \theta_i \sqrt{\frac{2}{\pi k}} e^{j\pi/4} e^{jkWu/2} \frac{e^{-jkR}}{\sqrt{R}} \frac{\sin\left(\frac{kWu}{2}\right)}{u} \sum_n e^{jknTu}
\end{aligned} \tag{4-10}$$

where

$$u = \sin \theta_i - \sin \theta \tag{4-11}$$

Let  $n$  range from 0 to  $N$ , then Equation (4-10) becomes:

$$\begin{aligned}
S^f &= \cos \theta_i \sqrt{\frac{2}{\pi k}} e^{j\pi/4} e^{jkWu/2} \frac{e^{-jkR}}{\sqrt{R}} \frac{\sin\left(\frac{kWu}{2}\right)}{u} \frac{\sin\left[\frac{(N+1)kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)} \\
&= W \cos \theta_i \sqrt{\frac{k}{2\pi}} e^{j\pi/4} e^{jkWu/2} \frac{e^{-jkR}}{\sqrt{R}} \frac{\sin\left(\frac{kWu}{2}\right)}{\left(\frac{kWu}{2}\right)} \frac{\sin\left[\frac{(N+1)kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)}
\end{aligned} \tag{4-12}$$

The average power pattern for the grating with no errors is given by:

$$\begin{aligned}
 \overline{|S_{\text{exact}}^f(u)|^2} &= \frac{1}{N+1} |S^f(u)|^2 \\
 &= \frac{1}{N+1} \frac{k(W \cos \theta_i)^2}{2\pi R} \left[ \frac{\sin\left(\frac{kWu}{2}\right)}{\left(\frac{kWu}{2}\right)} \right]^2 \left\{ \frac{\sin\left[(N+1)\frac{kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)} \right\}^2 \\
 &= \frac{1}{N+1} \frac{k(W \cos \theta_i)^2}{2\pi R} \left[ \frac{\sin\left(\frac{kWu}{2}\right)}{\left(\frac{kWu}{2}\right)} \right]^2 \left\{ \frac{\sin\left[(N+1)\frac{kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)} \right\}^2
 \end{aligned} \tag{4-13}$$

where

$$I_0 = \frac{k(W \cos \theta_i)^2}{2\pi R} \tag{4-14}$$

is the power intensity for a single slit or strip with  $u=0$ . With the exception of the  $1/(N+1)$  factor, Equation (4-13) is the familiar Fraunhofer diffraction pattern of an array of  $N+1$  slits found in optics textbooks (10:410).

To find the average power pattern for the infinite strip grating, use Equation (2-7):

$$\begin{aligned}
 \overline{|S_{\text{exact}}^f(u)|^2} &= I_0 \left[ \frac{\sin\left(\frac{kWu}{2}\right)}{\left(\frac{kWu}{2}\right)} \right]^2 \frac{\lambda}{T} \sum_{n=-\infty}^{\infty} \delta\left(u - \frac{n\lambda}{T}\right) \\
 &= \frac{I_0 \lambda}{T} \sum_{n=-\infty}^{\infty} \left[ \frac{\sin\left(\frac{n\pi W}{T}\right)}{\left(\frac{n\pi W}{T}\right)} \right]^2 \delta\left(u - \frac{n\lambda}{T}\right)
 \end{aligned} \tag{4-15}$$

## Grating with Errors

Now let the width of the strips and slits be defined as in Equation (3-53). The far zone scattered fields in Equation (4-10) become:

$$\begin{aligned}
S^f &= \cos\theta_i \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{x_n}^{\tau_n} e^{jkux'} dx' \\
&= -\frac{j \cos\theta_i}{u\sqrt{2\pi k}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n (e^{jku\tau_n} - e^{jkux_n})
\end{aligned} \tag{4-16}$$

Again let  $n$  range from 0 to  $N$ . The average power pattern is given by:

$$\overline{|S^f(u)|^2} = \frac{1}{N+1} \frac{\cos^2 \theta_i}{2\pi k u^2 R} E \left\{ \sum_{n=0}^N \sum_{m=0}^N (e^{jku\tau_n} - e^{jkux_n}) (e^{-jku\tau_m} - e^{-jkux_m}) \right\} \tag{4-17}$$

Let

$$\begin{aligned}
D &= \sum_{n=0}^N \sum_{m=0}^N (e^{jku\tau_n} - e^{jkux_n}) (e^{-jku\tau_m} - e^{-jkux_m}) \\
&= \sum_{n=0}^N \sum_{m=0}^N [e^{jku(\tau_n - \tau_m)} + e^{jku(x_n - x_m)} - e^{jku(\tau_n - x_m)} - e^{jku(x_n - \tau_m)}]
\end{aligned} \tag{4-18}$$

Separating the terms for which  $n=m$ , this becomes:

$$\begin{aligned}
D &= (N+1) + (N+1) - \sum_{n=0}^N e^{jku(\tau_n - x_n)} - \sum_{n=0}^N e^{-jku(\tau_n - x_n)} \\
&\quad + 2 \operatorname{Re} \left\{ \sum_{n=1}^N \sum_{m=0}^{N-n} [e^{jku(\tau_{n+m} - \tau_m)} + e^{jku(x_{n+m} - x_m)} - e^{jku(\tau_{n+m} - x_m)} - e^{jku(x_{n+m} - \tau_m)}] \right\}
\end{aligned} \tag{4-19}$$

or

$$D = 2(N+1) - 2 \operatorname{Re} \left( \sum_{n=0}^N e^{jku(\tau_n - x_n)} \right) + 2 \operatorname{Re} \left\{ \sum_{n=1}^N \sum_{m=0}^{N-n} \left[ e^{jku(\tau_{n+m} - \tau_m)} + e^{jku(x_{n+m} - x_m)} - e^{jku(\tau_{n+m} - x_m)} - e^{jku(x_{n+m} - \tau_m)} \right] \right\} \quad (4-20)$$

Now the following relations are true:

$$\tau_n - x_n = w_n \quad (4-21)$$

$$\begin{aligned} \tau_{n+m} - \tau_m &= a_m + w_{m+1} + a_{m+1} + w_{m+2} + \cdots + a_{n+m-1} + w_{n+m} \\ &= \sum_{k=m}^{n+m-1} (a_k + w_{k+1}) \end{aligned} \quad (4-22)$$

$$\begin{aligned} x_{n+m} - x_m &= w_m + a_m + w_{m+1} + a_{m+1} + \cdots + w_{n+m-1} + a_{n+m-1} \\ &= \sum_{k=m}^{n+m-1} (a_k + w_k) \end{aligned} \quad (4-23)$$

$$\begin{aligned} \tau_{n+m} - x_m &= w_m + a_m + \cdots + w_{n+m-1} + a_{n+m-1} + w_{n+m} \\ &= w_{n+m} + \sum_{k=m}^{n+m-1} (a_k + w_k) \end{aligned} \quad (4-24)$$

$$\begin{aligned} x_{n+m} - \tau_m &= a_m + w_{m+1} + a_{m+1} + \cdots + w_{n+m-1} + a_{n+m-1} \\ &= a_m + \sum_{k=m+1}^{n+m-1} (a_k + w_k) \end{aligned} \quad (4-25)$$

Also define the following characteristic functions:

$$\begin{aligned} \Theta(u) &= E \{ e^{jkuw} \} \\ \Phi(u) &= E \{ e^{jkua} \} \end{aligned} \quad (4-26)$$

For the types of errors given by Equation (3-53), these become:

$$\begin{aligned}\Theta(u) &= E\{e^{jkuw}\} = E\{e^{jku(W+\xi)}\} = e^{jkuW} E\{e^{jku\xi}\} \\ \Phi(u) &= E\{e^{jkua}\} = E\{e^{jku(A+\epsilon)}\} = e^{jkuA} E\{e^{jku\epsilon}\}\end{aligned}\quad (4-27)$$

Then

$$\begin{aligned}E\{D\} &= 2(N+1) - 2\operatorname{Re}\left\{\sum_{n=0}^N \Theta\right\} + 2\operatorname{Re}\left\{\sum_{n=1}^N \sum_{m=0}^{N-n} [\Phi^n \Theta^n + \Phi^n \Theta^n - \Phi^n \Theta^{n+1} - \Phi^n \Theta^{n-1}]\right\} \\ &= 2(N+1) - 2(N+1)\operatorname{Re}\{\Theta\} \\ &\quad + 2\operatorname{Re}\left\{\sum_{n=1}^N (N+1-n)[\Phi^n \Theta^n + \Phi^n \Theta^n - \Phi^n \Theta^{n+1} - \Phi^n \Theta^{n-1}]\right\}\end{aligned}\quad (4-28)$$

or

$$\begin{aligned}E\{D\} &= 2(N+1)\left[1 - \operatorname{Re}\{\Theta\} + \operatorname{Re}\left\{\sum_{n=1}^N (\Phi\Theta)^n \left[2 - \Theta - \frac{1}{\Theta}\right]\right\}\right] \\ &\quad - 2\operatorname{Re}\left\{\sum_{n=1}^N n(\Phi\Theta)^n \left[2 - \Theta - \frac{1}{\Theta}\right]\right\}\end{aligned}\quad (4-29)$$

Use Equations (2-18) and (2-19) to obtain:

$$\begin{aligned}E\{D\} &= 2(N+1)\left[1 - \operatorname{Re}\{\Theta\} + \operatorname{Re}\left\{\left[\frac{(\Phi\Theta) - (\Phi\Theta)^{N+1}}{1 - (\Phi\Theta)}\right] \left[2 - \Theta - \frac{1}{\Theta}\right]\right\}\right] \\ &\quad - 2\operatorname{Re}\left\{\left[\frac{(\Phi\Theta) + N(\Phi\Theta)^{N+2} - (N+1)(\Phi\Theta)^{N+1}}{[1 - (\Phi\Theta)]^2}\right] \left[2 - \Theta - \frac{1}{\Theta}\right]\right\}\end{aligned}\quad (4-30)$$

Substituting Equation (4-30) into Equation (4-17), the average power pattern becomes:

$$\begin{aligned} \overline{|S^f(u)|^2} = \frac{\cos^2 \theta_i}{\pi k u^2 R} & \left[ 1 - \text{Re}\{\Theta\} + \text{Re} \left\{ \left[ \frac{(\Phi\Theta) - (\Phi\Theta)^{N+1}}{1 - (\Phi\Theta)} \right] \left[ 2 - \Theta - \frac{1}{\Theta} \right] \right\} \right. \\ & \left. - \frac{1}{N+1} \text{Re} \left\{ \left[ \frac{(\Phi\Theta) + N(\Phi\Theta)^{N+2} - (N+1)(\Phi\Theta)^{N+1}}{[1 - (\Phi\Theta)]^2} \right] \left[ 2 - \Theta - \frac{1}{\Theta} \right] \right\} \right] \end{aligned} \quad (4-31)$$

or using Equation (4-14),

$$\begin{aligned} \overline{|S^f(u)|^2} = I_0 \frac{2}{(kuW)^2} & \left[ 1 - \text{Re}\{\Theta\} + \text{Re} \left\{ \left[ \frac{(\Phi\Theta) - (\Phi\Theta)^{N+1}}{1 - (\Phi\Theta)} \right] \left[ 2 - \Theta - \frac{1}{\Theta} \right] \right\} \right. \\ & \left. - \frac{1}{N+1} \text{Re} \left\{ \left[ \frac{(\Phi\Theta) + N(\Phi\Theta)^{N+2} - (N+1)(\Phi\Theta)^{N+1}}{[1 - (\Phi\Theta)]^2} \right] \left[ 2 - \Theta - \frac{1}{\Theta} \right] \right\} \right] \end{aligned} \quad (4-32)$$

Note the similarity of Equation (4-32) to Equation (2-20). Substituting  $u=0$  into the first equation in Equation (4-16) yields:

$$\begin{aligned} \overline{|S^f(0)|^2} &= \frac{1}{N+1} \frac{k \cos^2 \theta_i}{2\pi R} E \left\{ \sum_{n=0}^N \sum_{m=0}^N w_n w_m^* \right\} \\ &= \frac{1}{N+1} \frac{I_0}{W^2} \left[ (N+1) E\{w^2\} + (N^2 + N) E^2\{w\} \right] \\ &= \frac{I_0}{W^2} \left[ E\{w^2\} + N E^2\{w\} \right] \end{aligned} \quad (4-33)$$

or

$$\overline{|S^f(0)|^2} = \frac{I_0}{W^2} \left[ (N+1) W^2 + E\{\xi^2\} \right] \quad (4-34)$$

For the case of an infinite number of slits or strips, i.e.  $N \rightarrow \infty$ , Equations (2-22) and (2-23) can be utilized to obtain:

$$\overline{|S^f(u)|^2} = I_0 \frac{2}{(kuW)^2} \left[ 1 - \text{Re}\{\Theta\} + \text{Re}\left\{ \left[ \frac{(\Phi\Theta)}{1-(\Phi\Theta)} \right] \left[ 2 - \Theta - \frac{1}{\Theta} \right] \right\} \right] \quad (4-35)$$

To examine the behavior at  $u = 0$ , again let  $u \rightarrow \mu + j\omega$  and use the first order terms of the following Taylor series expansions:

$$\begin{aligned} \Phi(\mu + j\omega) &\approx \int_0^\infty p_a(a) [1 - \omega ka + jk\mu a] da \\ &= 1 - \omega k\bar{a} + jk\mu\bar{a} \\ \Theta(\mu + j\omega) &\approx \int_0^\infty p_w(w) [1 - \omega kw + jk\mu w] dw \\ &= 1 - \omega k\bar{w} + jk\mu\bar{w} \end{aligned} \quad (4-36)$$

With the types of errors given by Equations (3-53) and (3-54),

$$\begin{aligned} \bar{a} &= A + E\{\varepsilon\} = A \\ \bar{w} &= W + E\{\xi\} = W \end{aligned} \quad (4-37)$$

and Equation (4-36) becomes:

$$\begin{aligned} \Phi(\mu + j\omega) &\approx 1 - \omega kA + jk\mu A \approx e^{jk(\mu + j\omega)A} \\ \Theta(\mu + j\omega) &\approx 1 - \omega kW + jk\mu W \approx e^{jk(\mu + j\omega)W} \end{aligned} \quad (4-38)$$

for small  $\mu$  and  $\omega$ . Substitution into Equation (4-35) yields:

$$\begin{aligned} \overline{|S^f(\mu + j\omega)|}^2 = I_0 \frac{2}{[k(\mu + j\omega)^2 W]^2} & \left[ 1 - \operatorname{Re}\{e^{jk(\mu + j\omega)W}\} \right. \\ & \left. + \operatorname{Re}\left\{ \frac{\left( e^{jk(\mu + j\omega)A} e^{jk(\mu + j\omega)W} \right)}{1 - \left( e^{jk(\mu + j\omega)A} e^{jk(\mu + j\omega)W} \right)} \right\} \left[ 2 - e^{jk(\mu + j\omega)W} - e^{-jk(\mu + j\omega)W} \right] \right] \end{aligned} \quad (4-39)$$

or

$$\begin{aligned} \overline{|S^f(\mu + j\omega)|}^2 = I_0 \frac{2}{[k(\mu + j\omega)^2 W]^2} & \left[ 1 - \cos[k(\mu + j\omega)W] \right. \\ & \left. + \operatorname{Re}\left\{ \frac{\left( e^{jk(\mu + j\omega)(A+W)} \right)}{1 - e^{jk(\mu + j\omega)(A+W)}} \right\} \left[ 2 - 2 \cos[k(\mu + j\omega)W] \right] \right] \end{aligned} \quad (4-40)$$

which becomes:

$$\begin{aligned} \overline{|S^f(\mu + j\omega)|}^2 = I_0 \frac{2}{[k(\mu + j\omega)^2 W]^2} & 2 \left\{ \sin\left[ \frac{k(\mu + j\omega)W}{2} \right] \right\}^2 \\ & \times \left[ 1 + \operatorname{Re}\left\{ \frac{\left( e^{jk(\mu + j\omega)(A+W)} \right)}{1 - e^{jk(\mu + j\omega)(A+W)}} \right\} \right] \end{aligned} \quad (4-41)$$

or

$$\overline{|S^f(\mu + j\omega)|}^2 = I_0 \left\{ \frac{\sin\left[ \frac{k(\mu + j\omega)W}{2} \right]}{\frac{k(\mu + j\omega)W}{2}} \right\}^2 \frac{1 - |e^{jk(\mu + j\omega)T}|^2}{|1 - e^{jk(\mu + j\omega)T}|^2} \quad (4-42)$$

Using Equation (2-31), this becomes as  $\omega \rightarrow 0$ :

$$\left| \overline{S^f(\mu)} \right|^2 = \frac{I_0 \lambda}{T} \delta(\mu) \quad \text{for } \mu = 0 \quad (4-43)$$

which is the same as for the case of the grating with no errors.

### Grating with Uniformly Distributed Errors

Suppose  $\epsilon$  is uniformly distributed over the range:  $-cA \leq \epsilon \leq cA$  where  $c$  is the maximum spacing error expressed as a percentage of the desired spacing  $A$ . The PDF of  $\epsilon$  is given by:

$$P_\epsilon(\epsilon) = \begin{cases} \frac{1}{2cA} & -cA \leq \epsilon \leq cA \\ 0 & \text{otherwise} \end{cases} \quad (4-44)$$

The characteristic function  $\Phi$  is given by:

$$\begin{aligned} \Phi(u) &= e^{jkuA} E\{e^{jku\epsilon}\} \\ &= \frac{e^{jkuA}}{2cA} \int_{-cA}^{cA} e^{jku\epsilon} d\epsilon \\ &= e^{jkuA} \frac{\sin(kucA)}{kucA} \end{aligned} \quad (4-45)$$

Now let  $\xi$  be uniformly distributed over:  $-bW \leq \xi \leq bW$  where  $b$  is the maximum width error expressed as a percentage of the desired width  $W$ . The PDF of  $\xi$  is given by:

$$P_\xi(\xi) = \begin{cases} \frac{1}{2bW} & -bW \leq \xi \leq bW \\ 0 & \text{otherwise} \end{cases} \quad (4-46)$$

The characteristic function  $\Theta$  is given by:

$$\begin{aligned}
\Theta(u) &= e^{jkuW} E\{e^{jku\xi}\} \\
&= \frac{e^{jkuW}}{2bW} \int_{-bW}^{bW} e^{jku\xi} d\xi \\
&= e^{jkuW} \frac{\sin(kubW)}{kubW}
\end{aligned} \tag{4-47}$$

Graphs comparing the average power patterns, normalized by dividing by  $I_0$  (denoted by  $I/I_0$ ), for exact gratings and gratings with errors are shown below. To allow a clearer picture of the effects of the random errors, a finite number of slits/ strips is used for most of the graphs. Plots for an infinite number of slits/ strips are included to show the degradation to the impulse functions of the ideal grating. The following notation is used in the graphs to indicate the values of the appropriate variables:

$u$ :  $\sin \theta_i - \sin \theta$

$N+1$ : Number of Slits or Strips.

$W/\lambda$ : Desired Width (Ratio of desired slit (TEz) or strip (TMz) width ( $W$ ) to Wavelength)

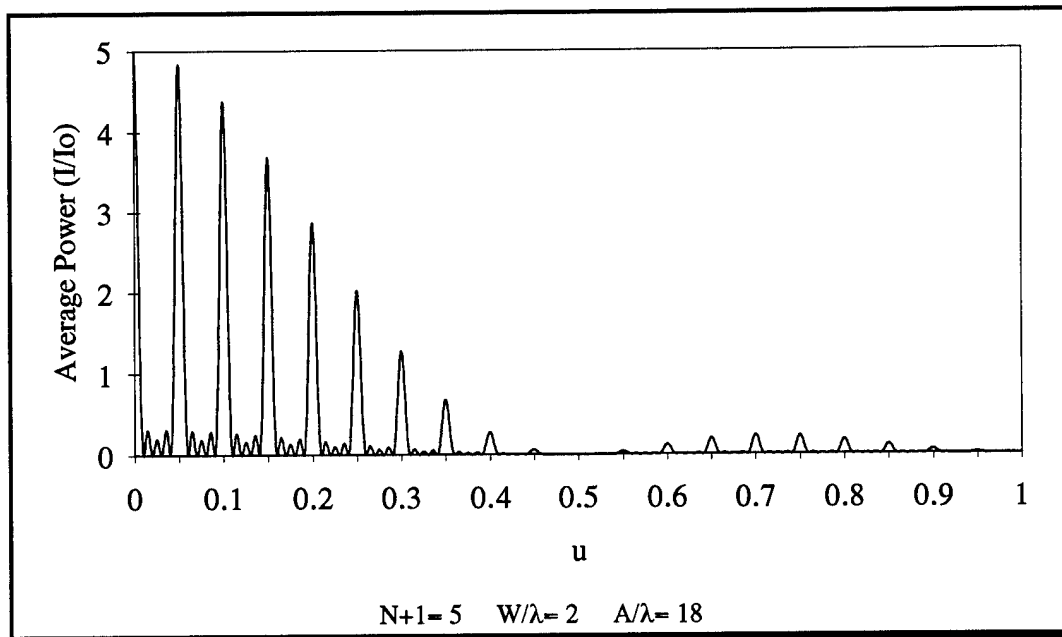
$A/\lambda$ : Desired Spacing (Ratio of desired strip (TEz) or slit (TMz) width ( $A$ ) to Wavelength)

Uniform Dist: Values for grating with errors computed using uniform distributions for the errors

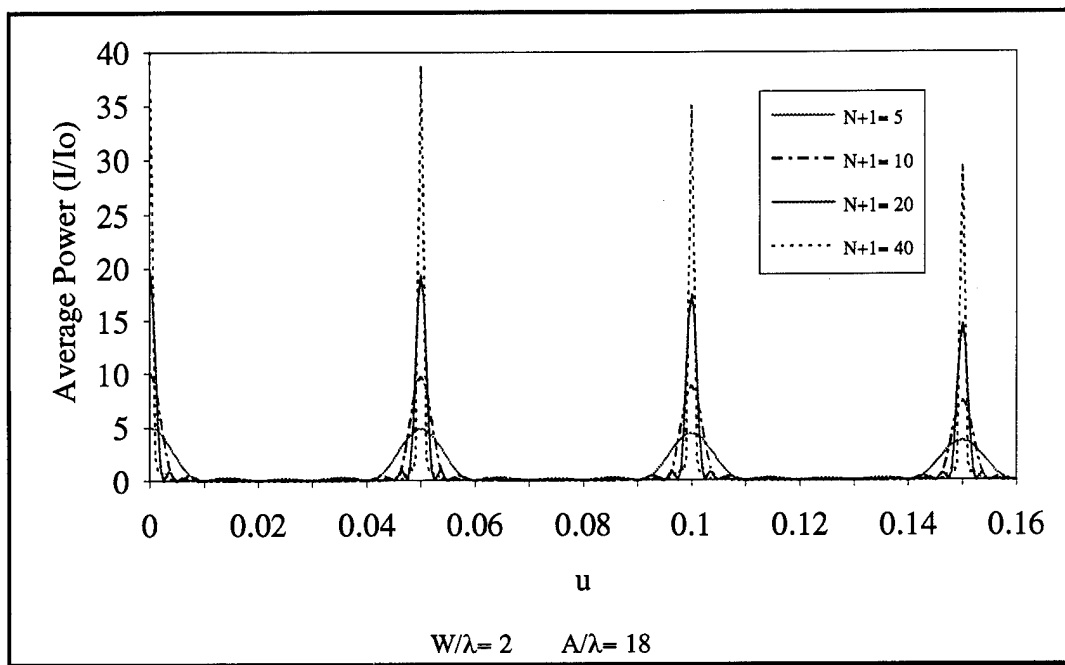
%W Tol: The maximum value allowed of the error in width ( $\xi$ ) expressed as a percentage of width  $W$ . This corresponds to the variable " $b$ " in Equation (4-46)

%A Tol: The maximum value allowed of the error in width ( $\epsilon$ ) expressed as a percentage of width  $A$ . This corresponds to the variable " $c$ " in Equation (4-44)

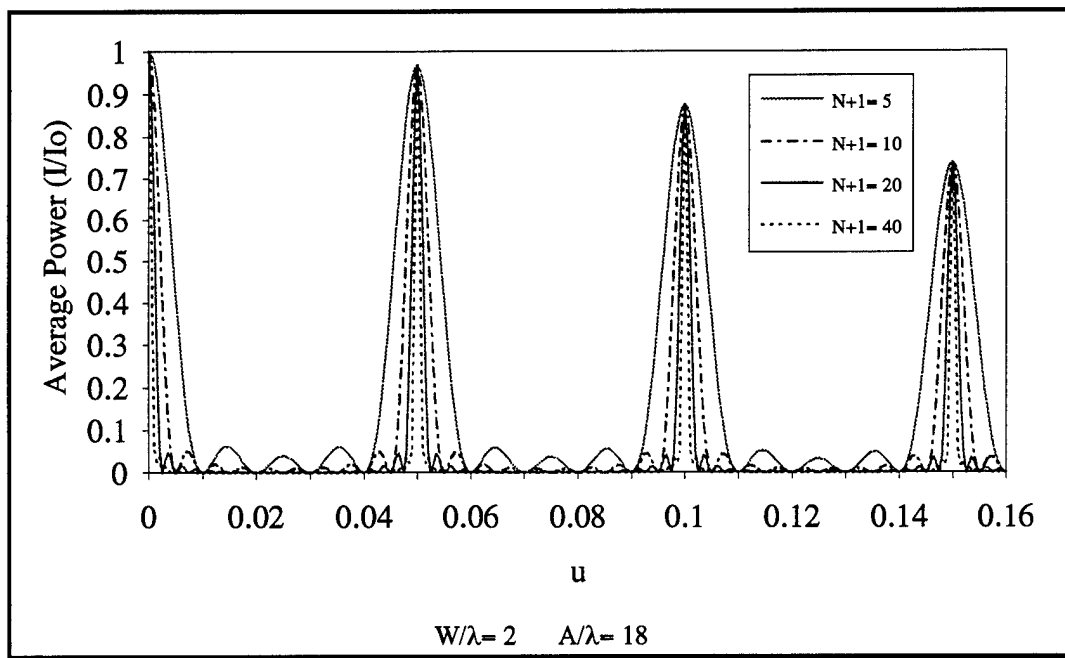
Figure 4.2 shows the average power pattern for an exact grating (i.e. a grating with no errors) with a finite number of slits or strips. The plots consist of a series of large spikes located  $\arcsin(\lambda/T)$  apart in angle. The plot clearly shows the sinc envelope of the average power pattern which is governed by the  $w/T$  ratio. On an expanded scale, Figure 4.3 shows the effects of increasing the total number of slits or strips. As the number of slits/ strips increases, the amplitudes of the spikes increase and while the widths of the spikes decrease and the pattern begins to resemble a series of impulse functions as given in Equation (4-15). Figure 4.4 presents the same pattern with the values normalized so that the maximum value (at  $u = 0$ ) equals one, again showing the sinc envelope to the ratios of the main spikes.



**Figure 4.2 Far Field Born Approximation - Grating with no Errors**



**Figure 4.3 Far Field Born Approximation - Effects of Total Number of Slits/ Strips on Grating with no Errors**

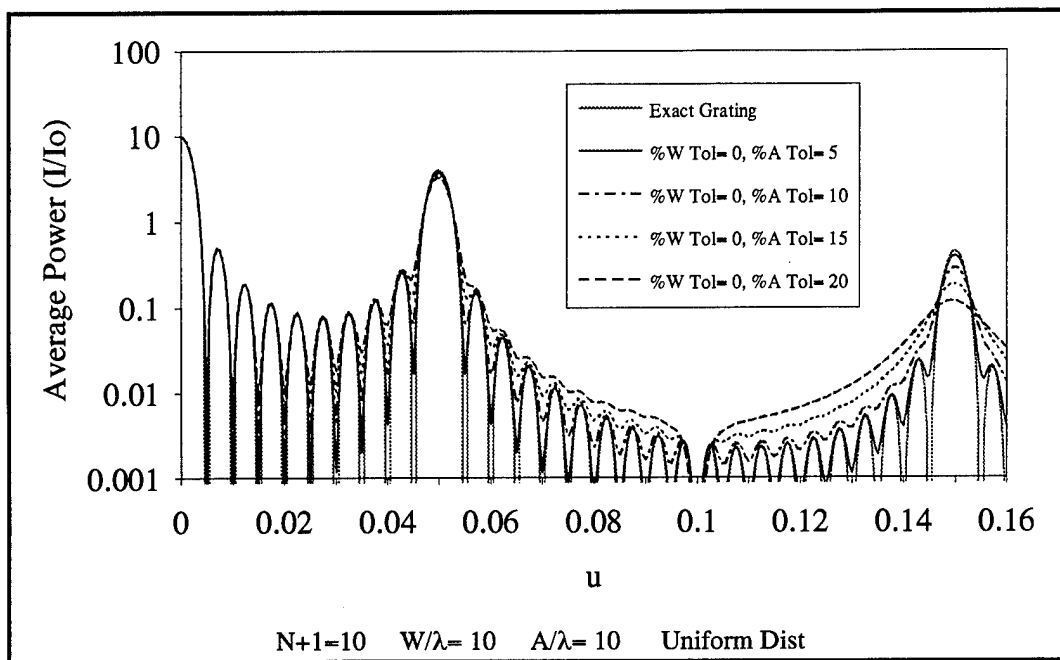


**Figure 4.4 Normalized Far Field Born Approximation - Effects of Total Number of Slits/ Strips on Grating with no Errors**

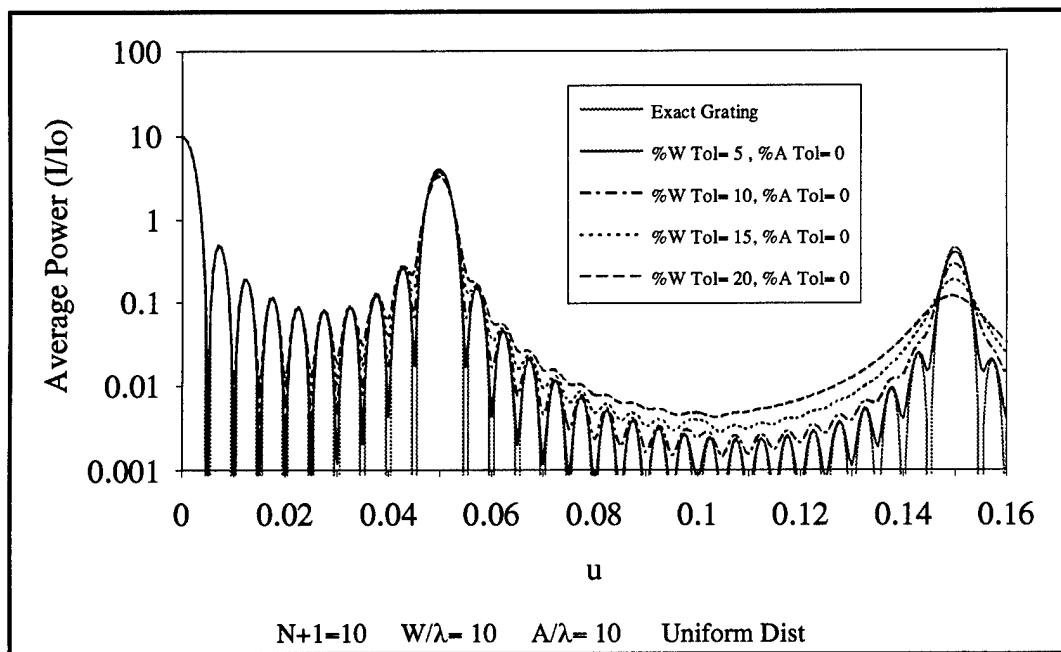
Figures 4.5 through 4.8 show the effects of errors in spacing alone, width alone, and both spacing and width respectively for a finite grating with desired width equal to desired spacing, i.e.  $W = A$ . The plots show the overall effect is to lower the amplitude of the main peaks and broaden the width of the spikes. The values between the main spikes also increase slightly. The effects become more pronounced as the parameter  $u$  is increased. A comparison of Figure 4.5 to Figure 4.6 shows that the effects of error in width alone are on the order of the errors in spacing alone for a given tolerance level. Figures 4.7 and 4.8 show the combined effects of errors in both spacing and width.

Figures 4.9 through 4.12 show the effects of errors in spacing alone, width alone, and both spacing and width respectively for a finite grating with desired width less than the desired spacing, i.e.  $W < A$ . The plots again show the overall effect is to lower the amplitude of the main peaks and broaden the width of the spikes. A comparison of Figure 4.9 to Figure 4.10 shows that, for this case, the effects of error in spacing alone are greater than the effects due to errors in width alone for a given percent tolerance level. In terms of absolute tolerance in wavelengths, however, the errors are similar, i.e. the %A Tol = 5 errors in Figure 4.9 are similar to the %W Tol = 15 errors in Figure 4.10. Figures 4.11 and 4.12 show the combined effects of errors in both spacing and width.

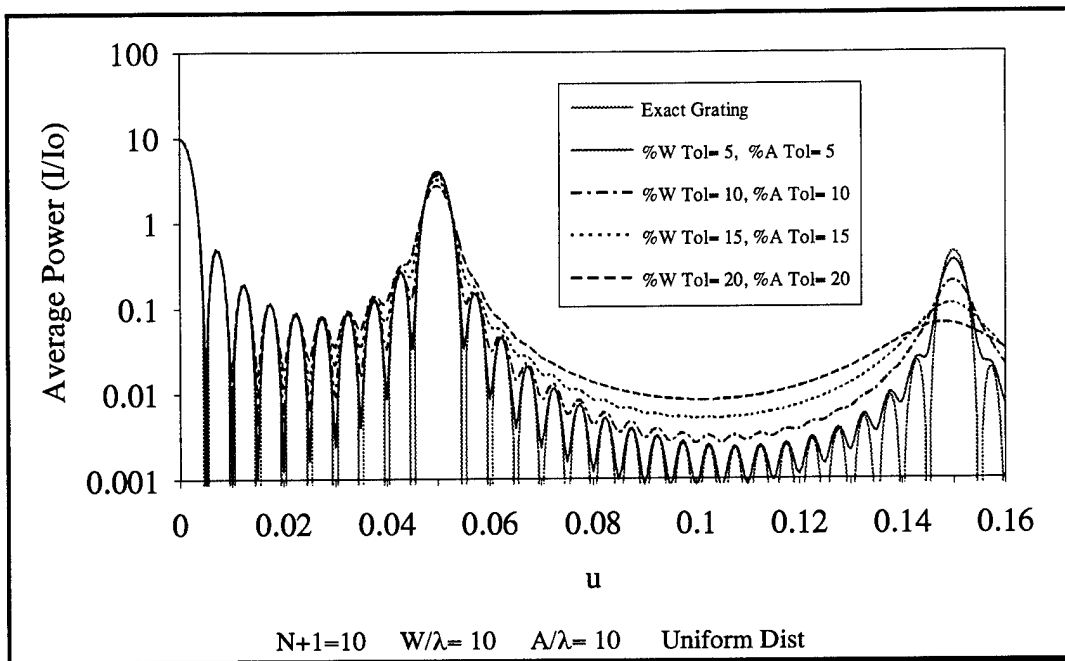
Figures 4.13 through Figure 4.16 show the effects of errors in spacing alone, width alone, and both spacing and width respectively for a finite grating with desired width greater than the desired spacing, i.e.  $W > A$ . A comparison of Figure 4.13 to Figure 4.14 shows that, opposite to the previous case, the effects of error in width alone are greater than the effects due to errors in spacing alone for a given percent tolerance level. In terms of absolute tolerance in wavelengths, however, the errors are similar, i.e. the %A Tol = 15 errors in Figure 4.13 are similar to the %W Tol = 5 errors in Figure 4.14. Figures 4.15 and 4.16 show the combined effects of errors in both spacing and width.



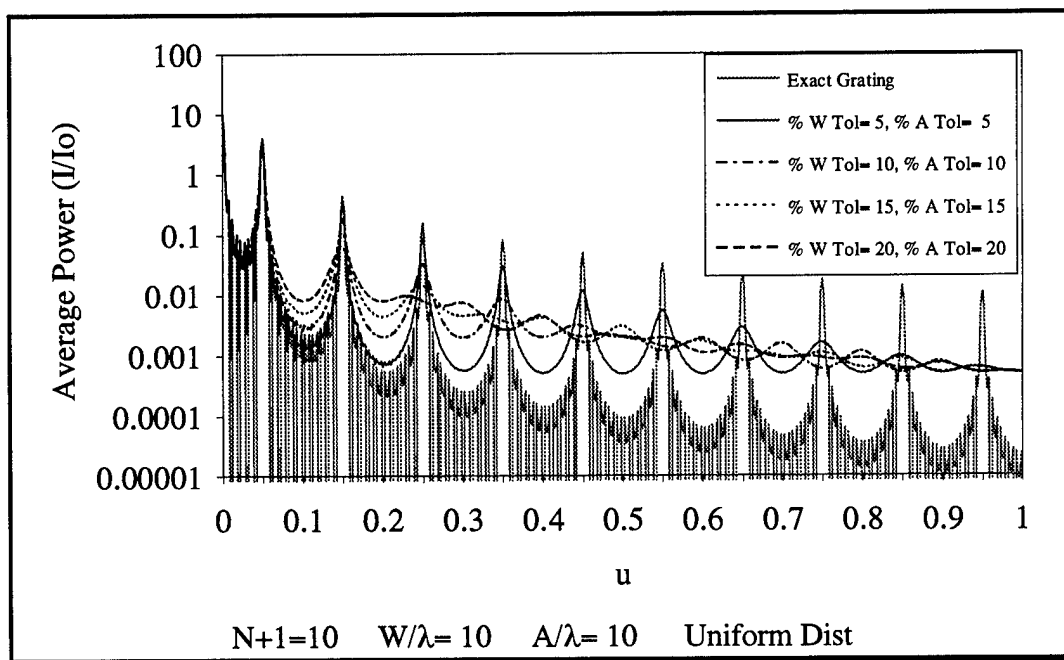
**Figure 4.5 Far Field Born Approximation - Effects of Spacing Errors on Grating with Desired Width Equal to Desired Spacing**



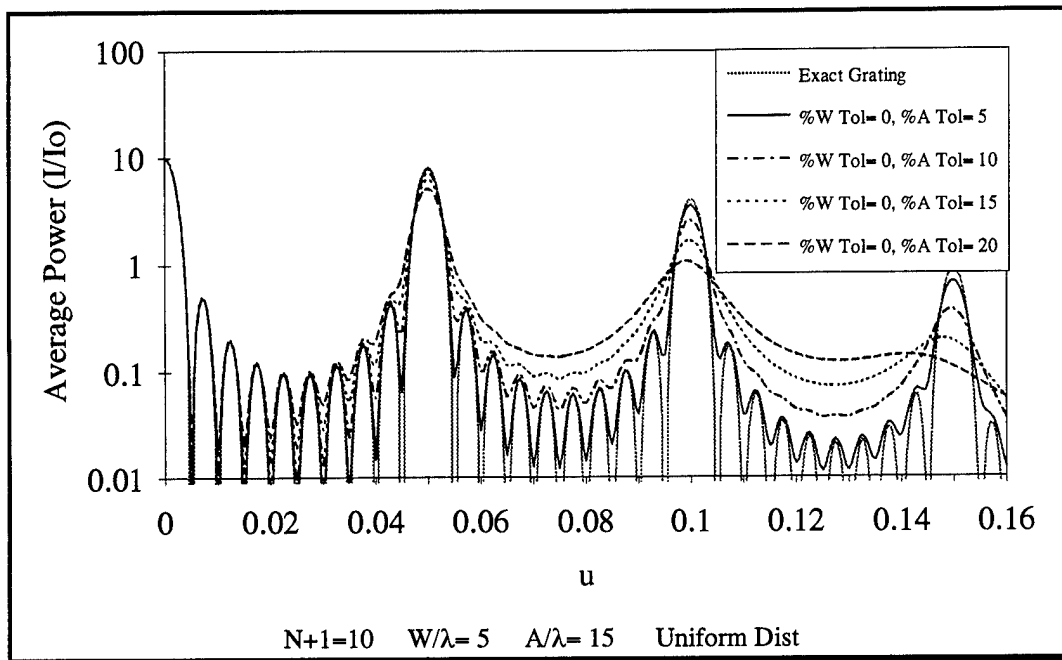
**Figure 4.6 Far Field Born Approximation - Effects of Width Errors on Grating with Desired Width Equal to Desired Spacing**



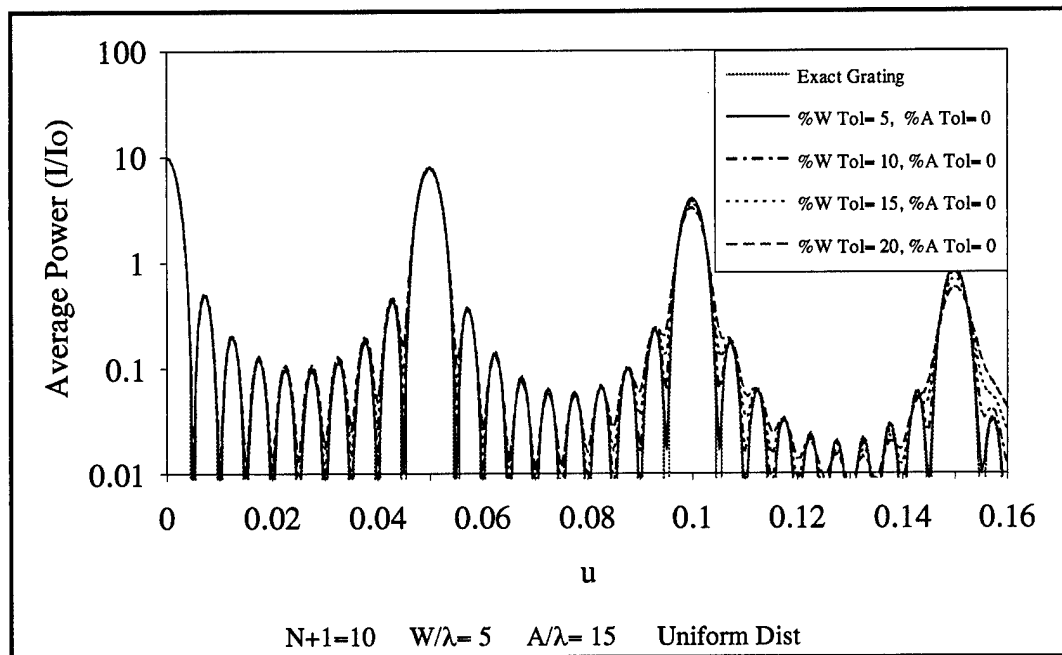
**Figure 4.7 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Equal to Desired Spacing**



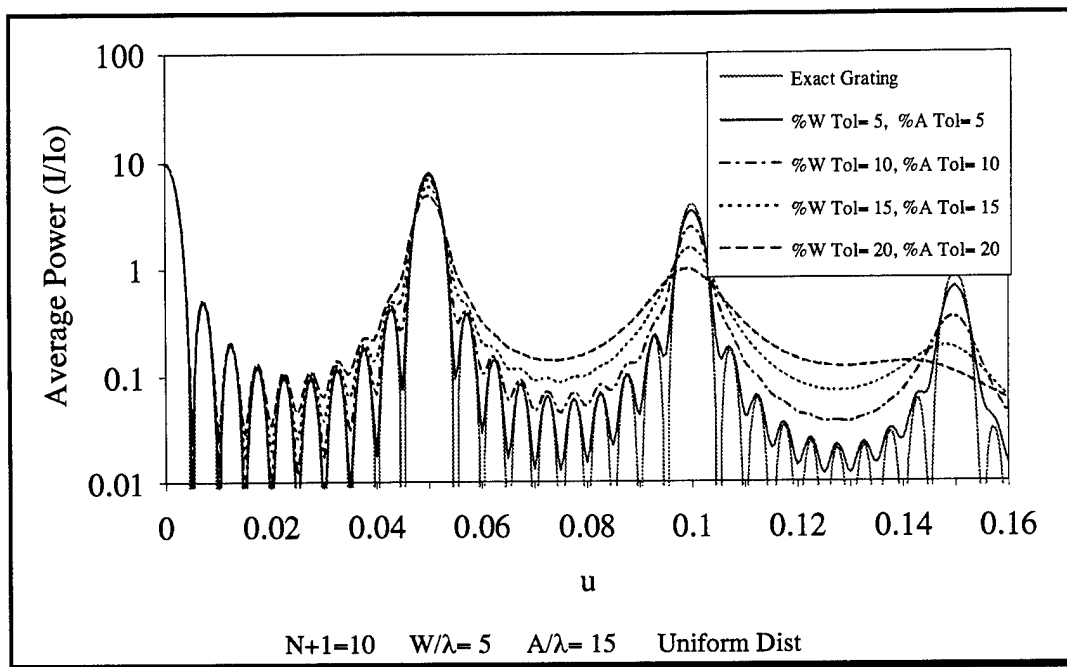
**Figure 4.8 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Equal to Desired Spacing**



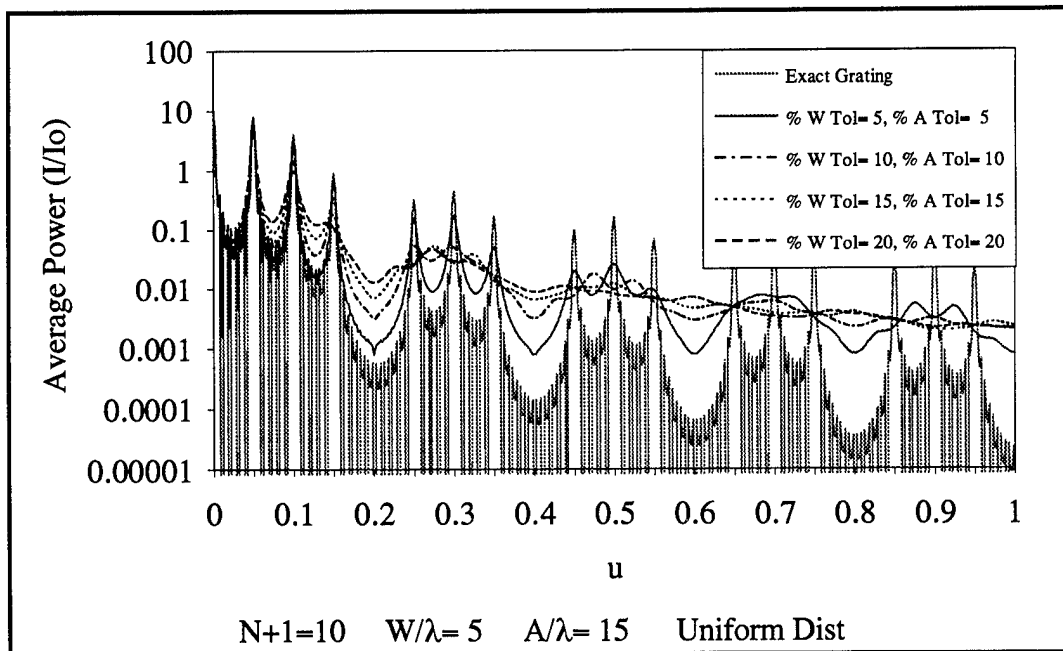
**Figure 4.9 Far Field Born Approximation - Effects of Spacing Errors on Grating with Desired Width Less Than Desired Spacing**



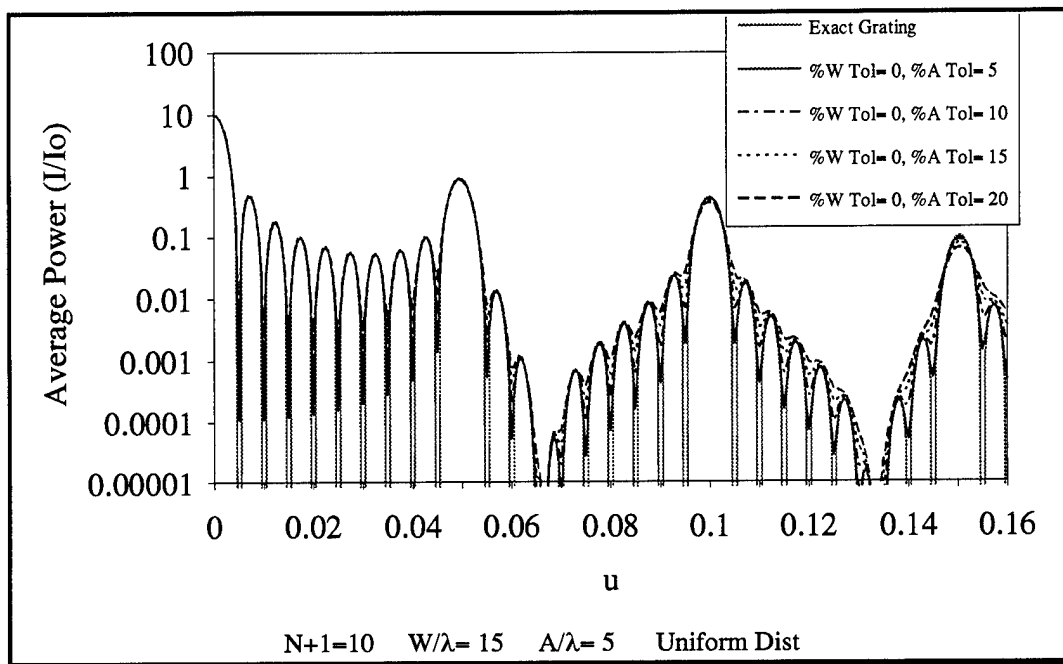
**Figure 4.10 Far Field Born Approximation - Effects of Width Errors on Grating with Desired Width Less Than Desired Spacing**



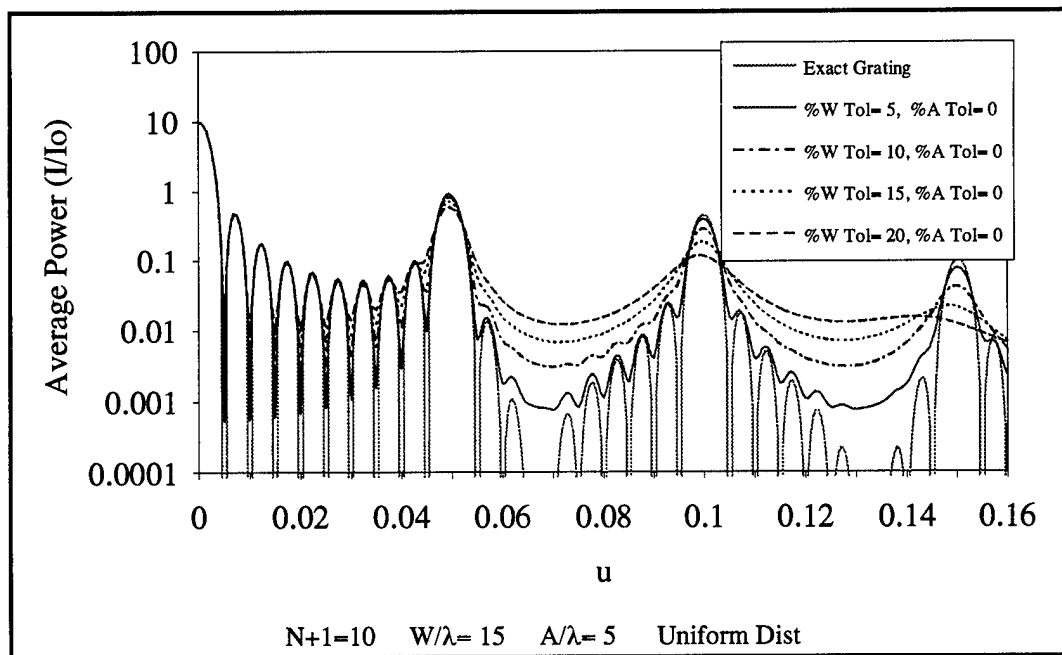
**Figure 4.11 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Less Than Desired Spacing**



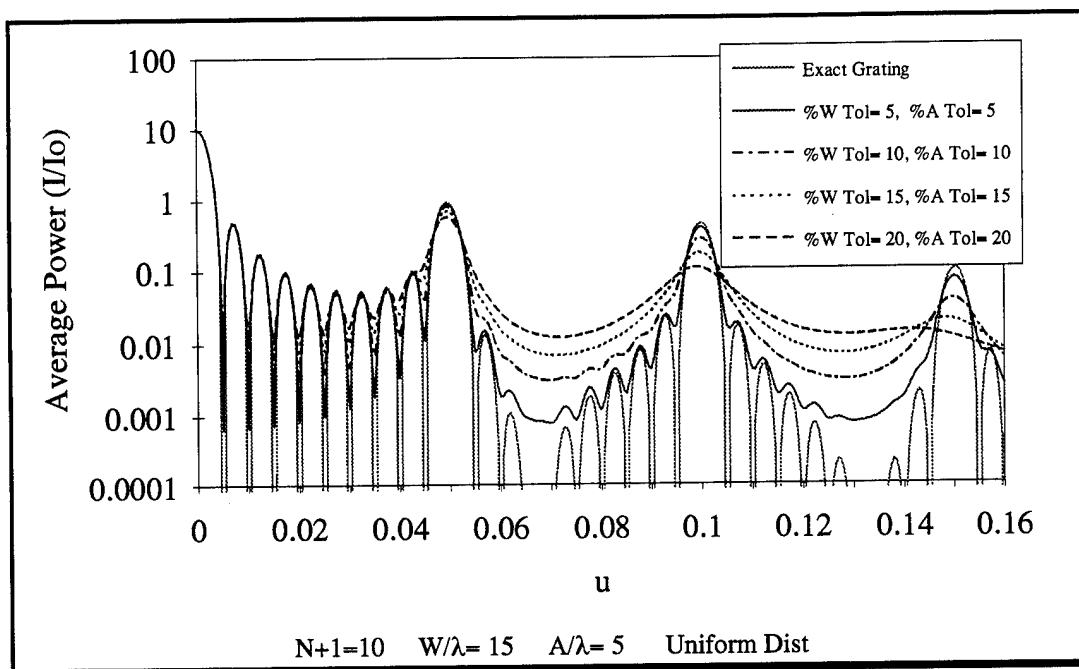
**Figure 4.12 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Less Than Desired Spacing**



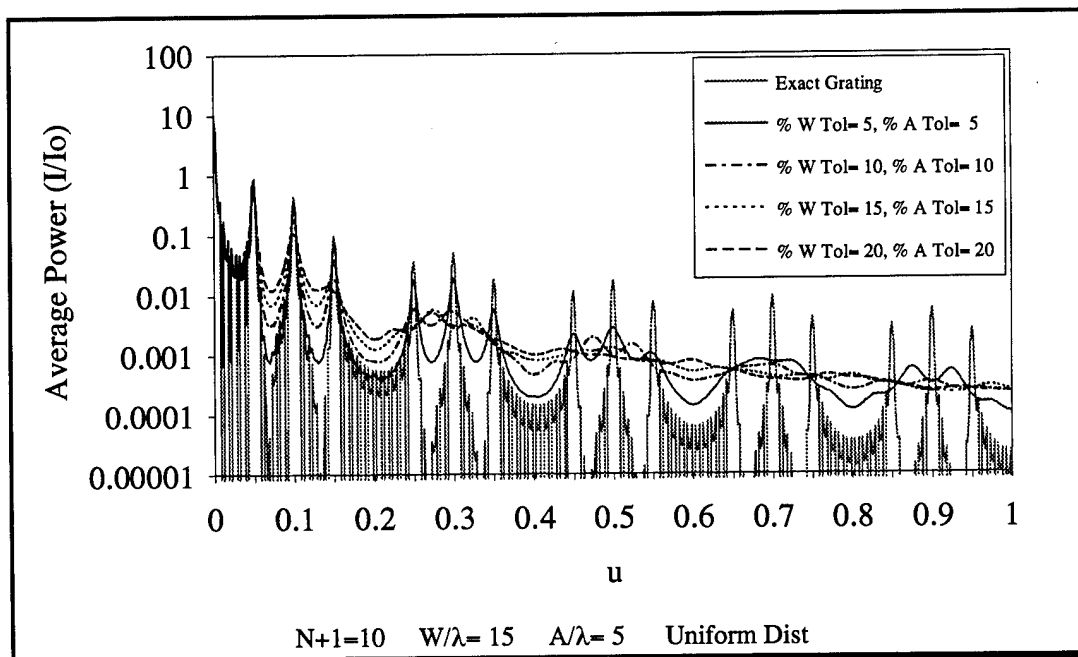
**Figure 4.13 Far Field Born Approximation - Effects of Spacing Errors on Grating with Desired Width Greater Than Desired Spacing**



**Figure 4.14 Far Field Born Approximation - Effects of Width Errors on Grating with Desired Width Greater Than Desired Spacing**



**Figure 4.15 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Greater Than Desired Spacing**

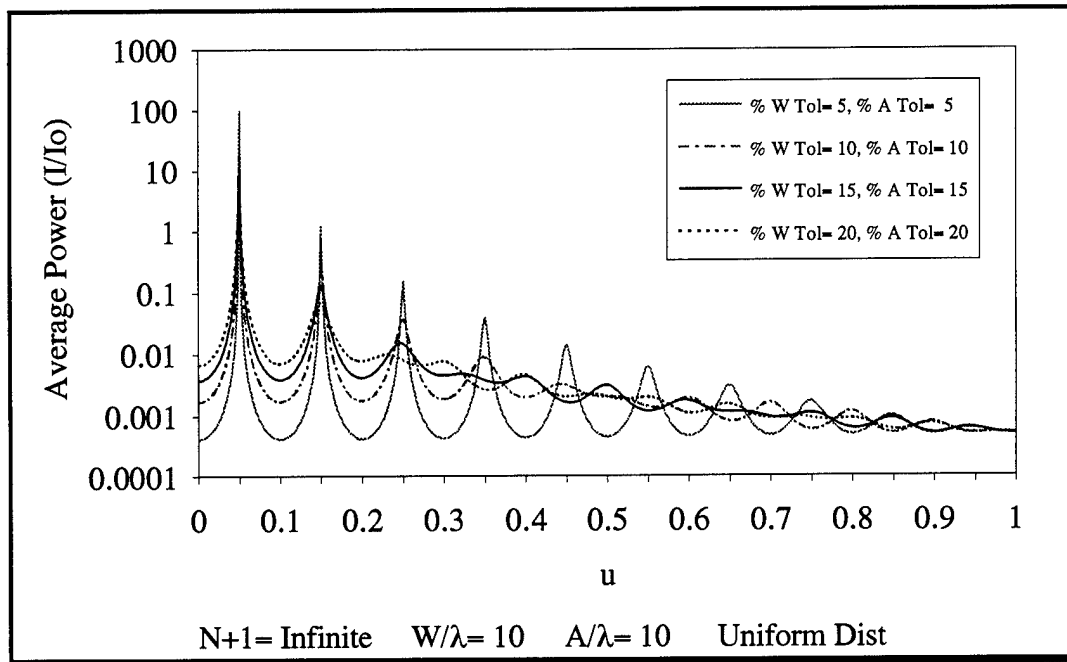


**Figure 4.16 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Greater Than Desired Spacing**

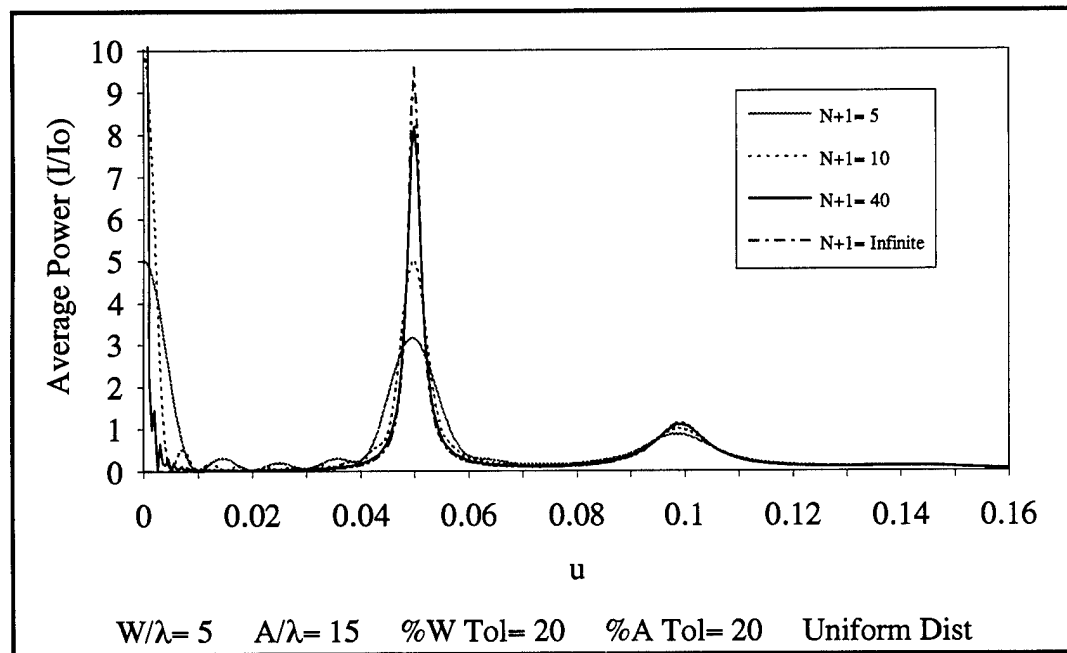
Figure 4.17 shows the average power pattern for a grating with an infinite number of slits/ strips for varying amounts of equal width and spacing tolerances. The effects due to increasing the amount of errors follow the same trends as for the finite cases. Although not shown on the graph, the patterns still contain an impulse at the origin ( $u = 0$ ) which is not plotted due to its infinite amplitude. Using a ray analysis approach, the presence of the impulse can be regarded as being due to the fact that all rays travel equal optical path lengths for  $u = 0$ . The graph also shows significant reduction in amplitude and an increase in beam widths as compared to a grating with no errors. The figure also shows non-zero values at locations between the main spikes. These effects increase as the parameter  $u$  is increased, essentially washing out the grating lobe pattern of the grating with no errors.

Figure 4.18 shows the progression of increasing the number of slits or strips from a small finite value up to an infinite number. Here the build up to the impulse at the origin is seen as well as the to the finite values at the other main spike locations.

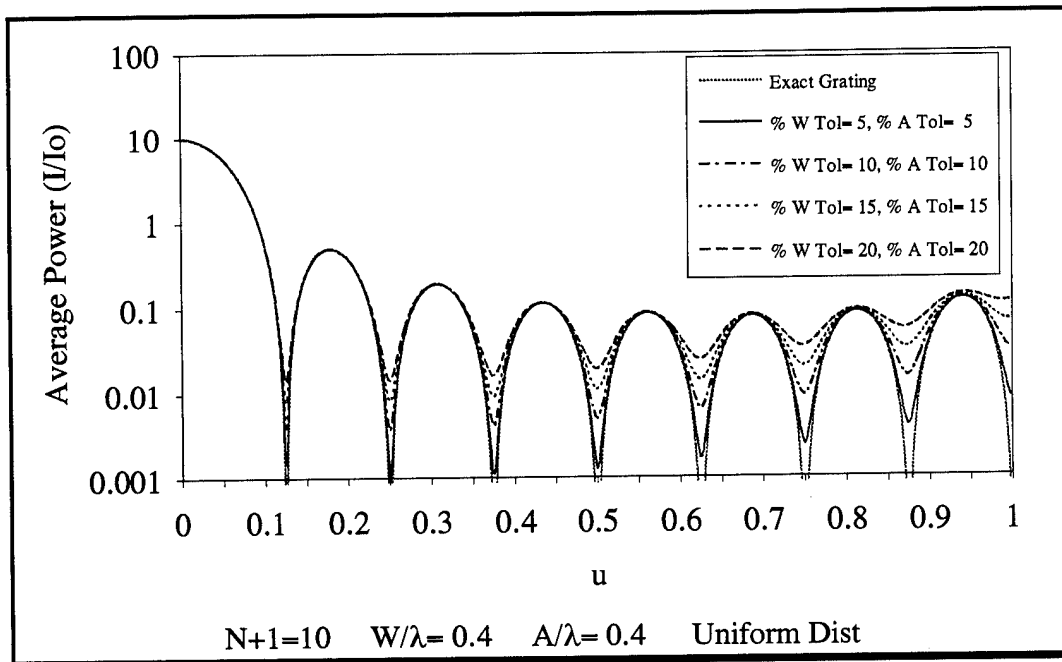
For completeness and later reference, Figures 4.19 and 4.20 show the average power pattern for low frequency scattering, i.e.  $T/\lambda < 1$ . Figure 4.19 presents the results for a grating with a finite number of slits or strips. The main effect of the errors is to raise the value of the nulls in the floor of the pattern while the main lobe is not significantly affected. Figure 4.20 presents the results for a grating with an infinite number of slits or strips.



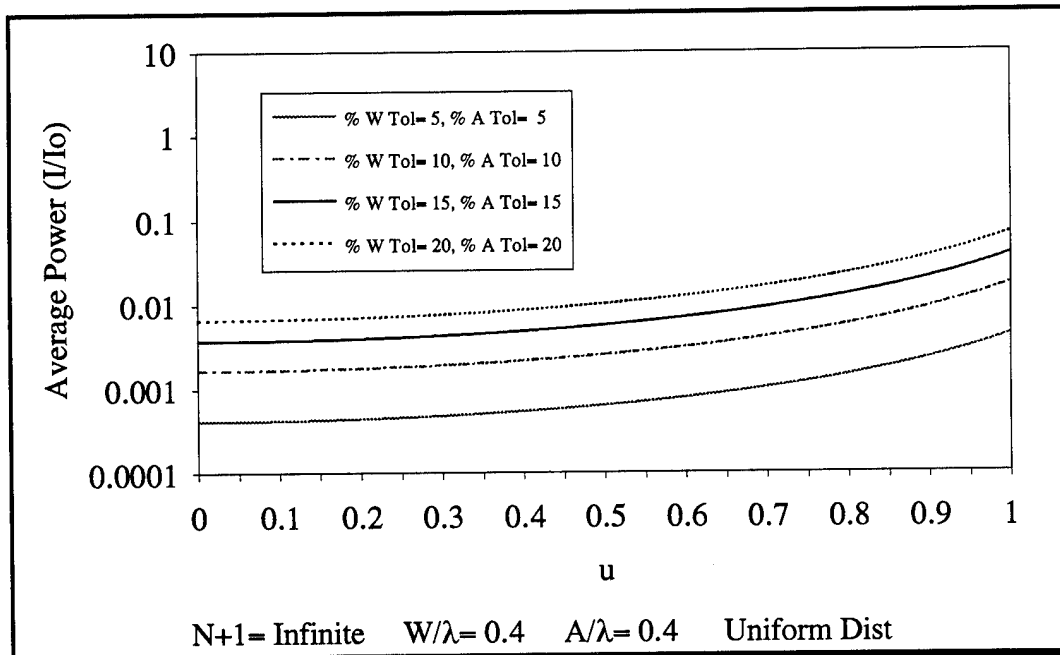
**Figure 4.17 Far Field Born Approximation - Effects of Width and Spacing Errors on Grating with Infinite Number of Slits or Strips**



**Figure 4.18 Far Field Born Approximation - Effects of Total Number of Slits/ Strips on Grating with Errors**



**Figure 4.19 Far Field Born Approximation - Effects of Errors on Grating with Finite Number of Slits or Strips, Low Frequency Case**



**Figure 4.20 Far Field Born Approximation - Effects of Errors on Grating with Infinite Number of Slits or Strips, Low Frequency Case**

## Grating with Cosine Function Distributed Errors

The uniformly distributed errors can be regarded as representing a worst case distribution of the errors and may not be representative of typical manufacturing errors. A PDF based upon a cosine function is more representative of actual errors. Let the PDF of  $\epsilon$  be given by:

$$P_{\epsilon}(\epsilon) = \begin{cases} \frac{1}{2cA} \left[ 1 + \cos\left(\frac{\pi\epsilon}{cA}\right) \right] & -cA \leq \epsilon \leq cA \\ 0 & \text{otherwise} \end{cases} \quad (4-48)$$

This PDF distributes most of the errors around zero while the density tapers off to zero at  $\epsilon = \pm cA$ . The characteristic function  $\Phi$  is given by:

$$\begin{aligned} \Phi(u) &= e^{jkuA} E\{e^{jku\epsilon}\} \\ &= \frac{e^{jkuA}}{2cA} \int_{-cA}^{cA} e^{jku\epsilon} \left[ 1 + \cos\left(\frac{\pi\epsilon}{cA}\right) \right] d\epsilon \end{aligned} \quad (4-49)$$

Using the integral :

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} \quad (4-50)$$

the characteristic function becomes:

$$\Phi(u) = \frac{e^{jkuA}}{1 - \left(\frac{kucA}{\pi}\right)^2} \frac{\sin(kucA)}{kucA} \quad (4-51)$$

Use of L'Hopital's rule yields:

$$\Phi(u) = \frac{1}{2} e^{j\frac{\pi}{c}u} \quad \text{for } u = \frac{\pi}{kcA} \quad (4-52)$$

Similarly, let the PDF of  $\xi$  be given by:

$$P_{\xi}(\xi) = \begin{cases} \frac{1}{2bW} \left[ 1 + \cos\left(\frac{\pi\xi}{cA}\right) \right] & -bW \leq \xi \leq bW \\ 0 & \text{otherwise} \end{cases} \quad (4-53)$$

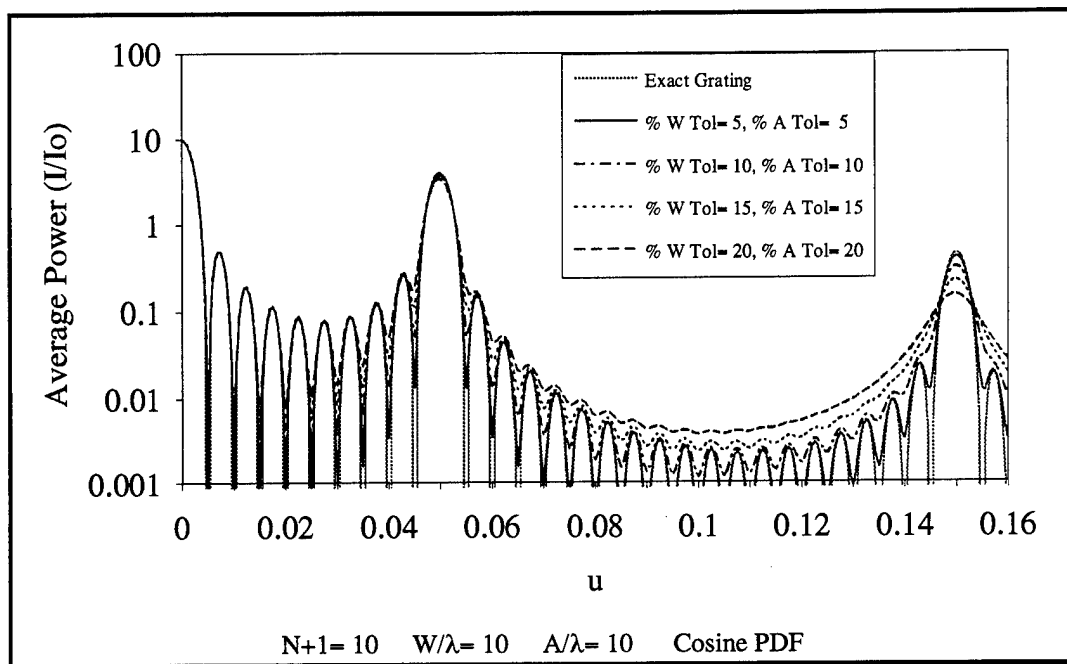
The characteristic function  $\Theta$  is given by:

$$\begin{aligned} \Theta(u) &= e^{jkuW} E\{e^{jku\xi}\} \\ &= \frac{e^{jkuW}}{1 - \left(\frac{kubW}{\pi}\right)^2} \frac{\sin(kubW)}{kubW} \end{aligned} \quad (4-54)$$

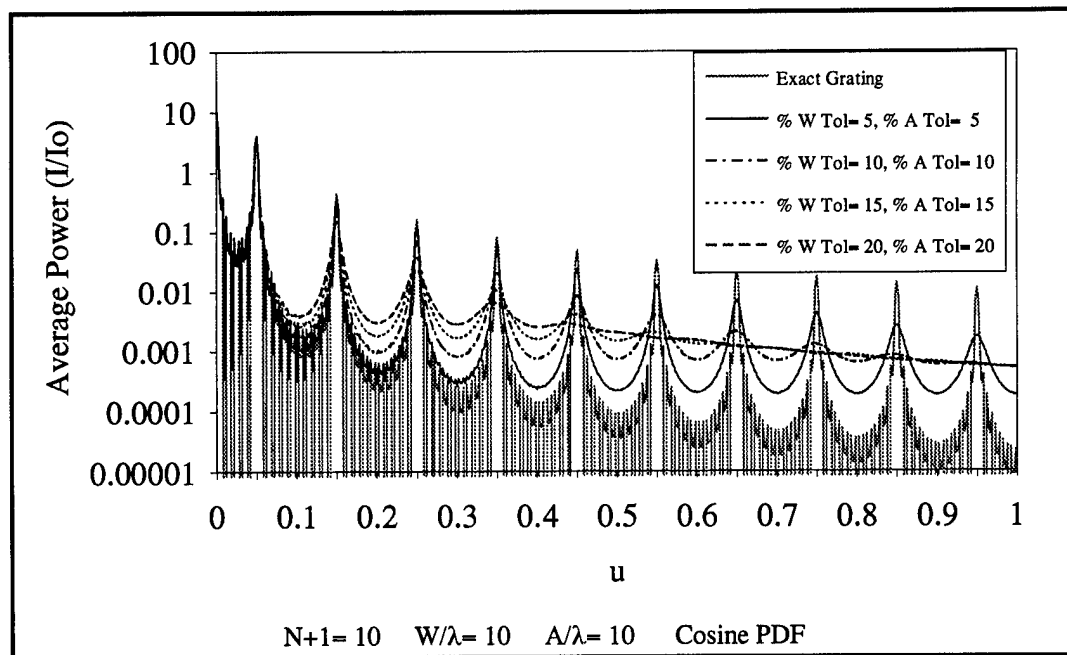
and

$$\Theta(u) = \frac{1}{2} e^{j\frac{\pi}{b}u} \quad \text{for } u = \frac{\pi}{kbW} \quad (4-55)$$

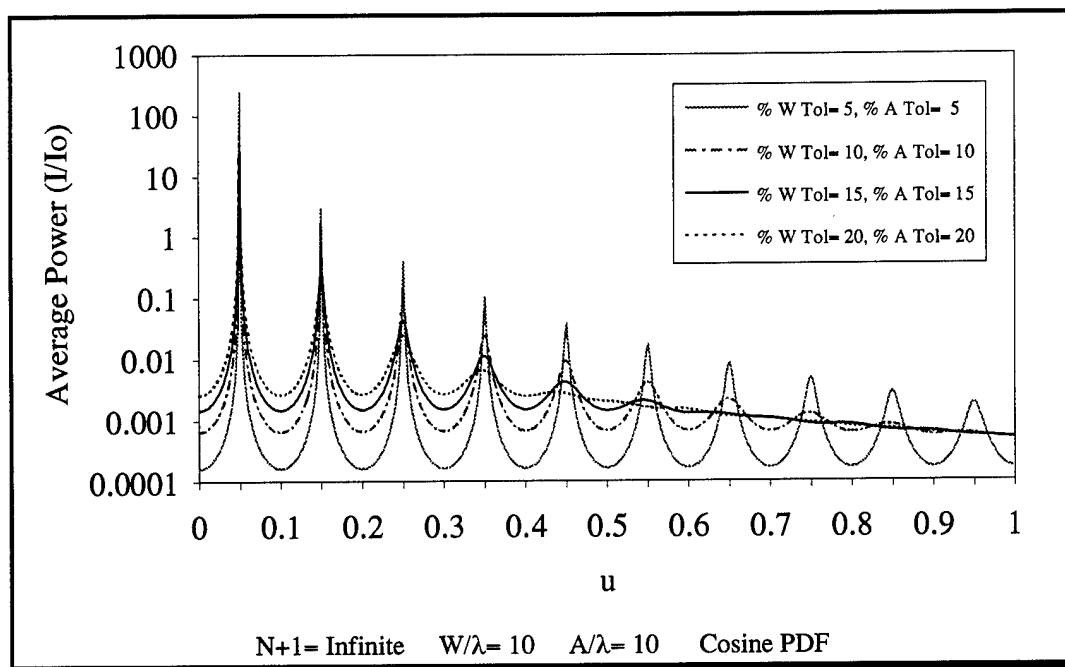
Graphs showing the average power pattern of gratings with errors using the cosine PDF are shown below. Figures 4.21 and 4.22 show the pattern for a grating with a finite number of slits/ strips while Figure 4.23 shows the pattern for a grating with an infinite number of slits/ strips. The overall effects are similar to the gratings with the uniformly distributed PDFs. As expected, for a given maximum error (i.e. % Tol), the errors in spacing and width have less of an effect upon the average power pattern of a grating with cosine distributed errors than to a grating with uniformly distributed errors.



**Figure 4.21 Far Field Born Approximation - Cosine PDF, Finite Number of Slits/Strips**



**Figure 4.22 Far Field Born Approximation - Cosine PDF, Finite Number of Slits/Strips**



**Figure 4.23 Far Field Born Approximation - Cosine PDF, Infinite Number of Slits/Strips**

## V. Edge Condition Approach

### Overview

This chapter presents results obtained when the unknown current in the integral field equation is replaced by an approximation which satisfies the edge condition, i.e. the current behaves as  $\rho^{-0.5}$  where  $\rho$  is the distance from the edge (11:577-579). The approximation is first obtained for the case of a grating with no errors and an infinite number of slits/strips and then modified for the case of a grating with width and spacing errors. This approach is especially valid in the low frequency region where the wavelength is much larger than the widths of all the slits and strips.

### Far Field Pattern for Grating with No Errors

Setting the origin at the mid point of the first slit or strip, i.e.  $1/2 (X_0 + \tau_0) = 0$ , the scattered fields for a grating with no errors are given by (Equation (3-44)):

$$S = \sum_n \frac{k}{2} \int_{nT-W/2}^{nT+W/2} I_s(x') H_0^{(2)}(k\sqrt{(x-x')^2 + y^2}) dx' \quad (5-1)$$

which becomes in the far field:

$$S^f = \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{nT-W/2}^{nT+W/2} I_s(x') e^{-jkx'\sin\theta} dx' \quad (5-2)$$

The unknown current  $I_s$  can be approximated by the following expression (11:192):

$$I_{sn}(x') = \frac{C e^{jkx'\sin\theta_i}}{\left[ \left( \frac{w}{2} \right)^2 - (x' - nT)^2 \right]^{1/2}} \quad (5-3)$$

where the subscript "n" refers to the current in the n<sup>th</sup> slit or strip and "C" is a constant. The value for the constant "C" can be obtained using a modal expansion for the scattered field as provided in Appendix B. This results in:

$$C = \left\{ k\pi \sum_{m=0}^{\infty} \frac{1}{\rho_m T} \left[ J_0 \left( \frac{n\pi W}{T} \right) \right]^2 \right\}^{-1} \quad (5-4)$$

where

$$\rho_m = \left[ k^2 - \left( k \sin \theta_i + \frac{2m\pi}{T} \right)^2 \right]^{1/2} \quad (5-5)$$

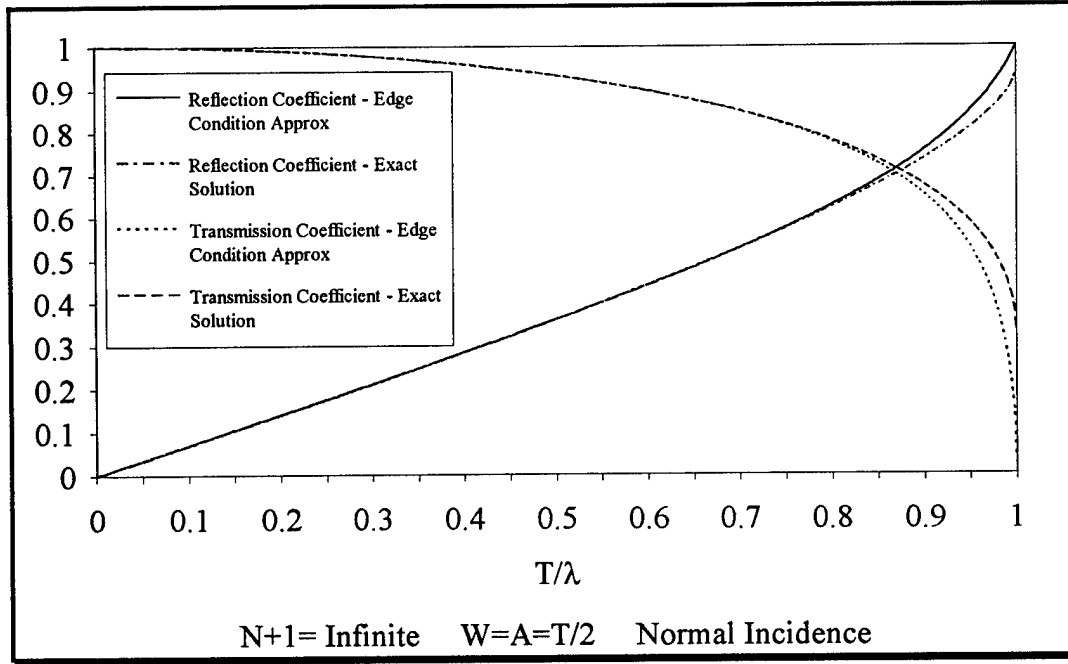
Figure 5.1 compares the reflection and transmission coefficients as a function of the  $T/\lambda$  ratio computed using this approximation to the coefficients obtained from an exact solution using the Wiener-Hopf method (29:267-281). The coefficients are calculated for a grating with no errors, an infinite number of slits/strips, normal incidence TEz mode, and the width equal to the spacing, i.e.  $W = A = T/2$ . For the range of  $T/\lambda$  ratios shown, corresponding to only one propagating scattered mode, the approximation agrees very well with the exact solution.

Substituting Equation (5-3) into Equation (5-2) results in:

$$S^f = \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{nT-W/2}^{nT+W/2} \frac{C e^{jkx' \sin \theta_i}}{\left[ \left( \frac{W}{2} \right)^2 - (x' - nT)^2 \right]^{1/2}} e^{-jkx' \sin \theta} dx' \quad (5-6)$$

Using the change of variables:

$$x'' = x' - nT \quad (5-7)$$



**Figure 5.1 Comparison of Approximation Using the Edge Condition to Exact Solution Obtained via the Wiener-Hopf Method**

Equation (5-6) becomes:

$$\begin{aligned}
 S^f &= \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{-W/2}^{W/2} \frac{C e^{jk(\sin\theta_i - \sin\theta)(x'' + nT)}}{[(W/2)^2 - (x'')^2]^{1/2}} dx'' \\
 &= C \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n e^{jkunT} \int_{-W/2}^{W/2} \frac{e^{jkux''}}{[(W/2)^2 - (x'')^2]^{1/2}} dx''
 \end{aligned} \tag{5-8}$$

The integral is simplified in Appendix C with the following results:

$$\int_{-W/2}^{W/2} \frac{e^{jkux''}}{[(W/2)^2 - (x'')^2]^{1/2}} dx'' = \pi J_0\left(\frac{kuW}{2}\right) \tag{5-9}$$

Which leads to:

$$S^f = C \sqrt{\frac{k\pi}{2}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} J_0\left(\frac{kuW}{2}\right) \sum_n e^{jkunT} \quad (5-10)$$

The average power pattern for the grating with no errors is given by:

$$\begin{aligned} \overline{|S_{\text{exact}}^f(u)|^2} &= \frac{1}{N+1} \frac{C^2 k\pi}{2R} [J_0\left(\frac{kuW}{2}\right)]^2 \left\{ \frac{\sin\left[(N+1)\frac{kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)} \right\}^2 \\ &= \frac{I_0}{N+1} [J_0\left(\frac{kuW}{2}\right)]^2 \left\{ \frac{\sin\left[(N+1)\frac{kTu}{2}\right]}{\sin\left(\frac{kTu}{2}\right)} \right\}^2 \end{aligned} \quad (5-11)$$

where

$$I_0 = \frac{C^2 k\pi}{2R} \quad (5-12)$$

To find the average power pattern for the infinite strip grating, use Equation (2-7):

$$\overline{|S_{\text{exact}}^f(u)|^2} = \frac{I_0 \lambda}{T} \sum_{n=-\infty}^{\infty} [J_0\left(\frac{n\pi W}{T}\right)]^2 \delta\left(u - \frac{n\lambda}{T}\right) \quad (5-13)$$

### Grating with Errors

Now let the width of the strips and slits be defined as in Equation (3-53). Using

$$\begin{aligned} \tau_n &= \frac{x_n + \tau_n + W_n}{2} \\ x_n &= \frac{x_n + \tau_n - W_n}{2} \end{aligned} \quad (5-14)$$

The far zone scattered field becomes:

$$S^f = \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{\frac{x_n+\tau_n-w_n}{2}}^{\frac{x_n+\tau_n+w_n}{2}} I_s(x') e^{-jkx' \sin \theta} dx' \quad (5-15)$$

The approximation for the unknown scattering current given in Equation (5-3) is modified to retain the appropriate edge condition behavior as follows:

$$I_{sn}(x') = \frac{C e^{jkx' \sin \theta_i}}{\left[ \left( w_n/2 \right)^2 - \left( x' - \frac{(x_n+\tau_n)}{2} \right)^2 \right]^{1/2}} \quad (5-16)$$

Substituting this into Equation (5-15), the scattered field becomes:

$$S^f = \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{\frac{x_n+\tau_n-w_n}{2}}^{\frac{x_n+\tau_n+w_n}{2}} C e^{jkux'_i} \left[ \left( w_n/2 \right)^2 - \left( x' - \frac{(x_n+\tau_n)}{2} \right)^2 \right]^{-1/2} dx' \quad (5-17)$$

Using the change of variables:

$$x'' = x' - \frac{x_n + \tau_n}{2} \quad (5-18)$$

The scattered field becomes:

$$\begin{aligned} S^f &= \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n \int_{-\frac{w_n}{2}}^{\frac{w_n}{2}} C e^{jku(x'' + \frac{x_n+\tau_n}{2})} \left[ \left( w_n/2 \right)^2 - (x'')^2 \right]^{-1/2} dx'' \\ &= C \sqrt{\frac{k}{2\pi}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n e^{jku(x_n+\tau_n)/2} \int_{-\frac{w_n}{2}}^{\frac{w_n}{2}} e^{jkux''} \left[ \left( w_n/2 \right)^2 - (x'')^2 \right]^{-1/2} dx'' \end{aligned} \quad (5-19)$$

Using Equation (5-9), this becomes:

$$S^f = C \sqrt{\frac{\pi k}{2}} e^{j\pi/4} \frac{e^{-jkR}}{\sqrt{R}} \sum_n e^{j\frac{ku}{2}(x_n + \tau_n)} J_0\left(\frac{kuw_n}{2}\right) \quad (5-20)$$

Letting  $n$  range from 0 to  $N$ , the average power pattern is given by:

$$\overline{|S^f(u)|^2} = \frac{C^2 \pi k}{(N+1)2R} E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{j\frac{ku}{2}(x_n + \tau_n)} J_0\left(\frac{kuw_n}{2}\right) e^{-j\frac{ku}{2}(x_m + \tau_m)} J_0\left(\frac{kuw_m}{2}\right) \right\} \quad (5-21)$$

Separating terms for which  $n=m$ ,

$$\begin{aligned} \overline{|S^f(u)|^2} = & \frac{C^2 \pi k}{(N+1)2R} E \left\{ \left[ \sum_{n=0}^N J_0\left(\frac{kuw_n}{2}\right) \right]^2 \right\} \\ & + 2 \operatorname{Re} \left\{ \sum_{n=1}^N \sum_{m=0}^{N-n} e^{j\frac{ku}{2}(x_{n+m} + \tau_{n+m} - x_m - \tau_m)} J_0\left(\frac{kuw_{n+m}}{2}\right) J_0\left(\frac{kuw_m}{2}\right) \right\} \end{aligned} \quad (5-22)$$

or

$$\begin{aligned} \overline{|S^f(u)|^2} = & \frac{C^2 \pi k}{2R} \left[ E \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} \right. \\ & \left. + \frac{2}{N+1} \operatorname{Re} \left\{ E \left\{ \sum_{n=1}^N \sum_{m=0}^{N-n} e^{j\frac{ku}{2}(x_{n+m} + \tau_{n+m} - x_m - \tau_m)} J_0\left(\frac{kuw_{n+m}}{2}\right) J_0\left(\frac{kuw_m}{2}\right) \right\} \right\} \right] \end{aligned} \quad (5-23)$$

Let

$$B = \sum_{n=1}^N \sum_{m=0}^{N-n} e^{j\frac{ku}{2}(x_{n+m} + \tau_{n+m} - x_m - \tau_m)} J_0\left(\frac{kuw_{n+m}}{2}\right) J_0\left(\frac{kuw_m}{2}\right) \quad (5-24)$$

From Equations (4-22) and (4-23), the following is true:

$$\begin{aligned}
 x_{n+m} - x_m + \tau_{n+m} - \tau_m &= \sum_{k=m}^{n+m-1} (a_k + w_k) + \sum_{k=m}^{n+m-1} (a_k + w_{k+1}) \\
 &= 2a_m + w_m + w_{n+m} + 2 \sum_{k=m+1}^{n+m-1} (a_k + w_k)
 \end{aligned} \tag{5-25}$$

Using the characteristic functions:

$$\begin{aligned}
 \Theta(u) &= E\{e^{jkuw}\} = E\{e^{jku(W+\xi)}\} = e^{jkuW} E\{e^{jku\xi}\} \\
 \Phi(u) &= E\{e^{jku a}\} = E\{e^{jku(A+\epsilon)}\} = e^{jkuA} E\{e^{jku\epsilon}\}
 \end{aligned} \tag{5-26}$$

Then

$$\begin{aligned}
 E\{B\} &= \sum_{n=1}^N \sum_{m=0}^{N-n} (\Theta\Phi)^{(n-1)} \Phi E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \\
 &= E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \sum_{n=1}^N (N+1-n) (\Theta\Phi)^n \Theta^{-1} \\
 &= E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \Theta^{-1} \left[ (N+1) \sum_{n=1}^N (\Theta\Phi)^n - \sum_{n=1}^N n (\Theta\Phi)^n \right]
 \end{aligned} \tag{5-27}$$

Use Equations (2-18) and (2-19) to obtain:

$$\begin{aligned}
 E\{B\} &= E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \Theta^{-1} \left[ (N+1) \frac{\Theta\Phi - (\Theta\Phi)^{(N+1)}}{1 - \Theta\Phi} \right. \\
 &\quad \left. - \frac{+N(\Theta\Phi)^{(N+2)} - (N+1)(\Theta\Phi)^{(N+1)}}{[1 - \Theta\Phi]^2} \right]
 \end{aligned} \tag{5-28}$$

Inserting Equation (5-28) into Equation (5-23), the average power pattern becomes:

$$\begin{aligned} \overline{|S^f(u)|^2} = \frac{C^2 \pi k}{2R} & \left[ E \left\{ \left[ J_0 \left( \frac{kuw}{2} \right) \right]^2 \right\} + 2 \operatorname{Re} \left[ E^2 \left\{ e^{j \frac{kuw}{2}} J_0 \left( \frac{kuw}{2} \right) \right\} \Theta^{-1} \left\{ \frac{\Theta \Phi - (\Theta \Phi)^{(N+1)}}{1 - \Theta \Phi} \right\} \right] \right. \\ & \left. - 2 \operatorname{Re} \left[ E^2 \left\{ e^{j \frac{kuw}{2}} J_0 \left( \frac{kuw}{2} \right) \right\} \Theta^{-1} \left\{ \frac{+N(\Theta \Phi)^{(N+2)} - (N+1)(\Theta \Phi)^{(N+1)}}{(N+1)[1 - \Theta \Phi]^2} \right\} \right] \right] \end{aligned} \quad (5-29)$$

Substituting  $u=0$  into Equation (5-21) yields:

$$\overline{|S^f(0)|^2} = \frac{C^2 \pi k}{2R} (N+1) = I_0 (N+1) \quad (5-30)$$

For the case of an infinite number of slits or strips, i.e.  $N \rightarrow \infty$ , Equations (2-22) and (2-23) can be utilized to obtain:

$$\overline{|S^f(u)|^2} = \frac{C^2 \pi k}{2R} \left\{ \left[ E \left\{ \left[ J_0 \left( \frac{kuw}{2} \right) \right]^2 \right\} + 2 \operatorname{Re} \left[ E^2 \left\{ e^{j \frac{kuw}{2}} J_0 \left( \frac{kuw}{2} \right) \right\} \left\{ \frac{\Phi}{1 - \Theta \Phi} \right\} \right] \right] \right\} \quad (5-31)$$

To examine the behavior at  $u = 0$ , again let  $u \rightarrow \mu + j\omega$  and use the first order terms of the following Taylor series expansions for  $u + j\omega \approx 0$ :

$$J_0 \left[ \frac{k(\mu + j\omega)w}{2} \right] \approx 1 \quad (5-32)$$

$$e^{j \frac{k(\mu + j\omega)w}{2}} \approx 1 - \frac{\omega kw}{2} + j \frac{k\mu w}{2} \quad (5-33)$$

which leads to:

$$e^{j\frac{k(\mu+j\omega)w}{2}} J_0\left[\frac{k(\mu+j\omega)w}{2}\right] \approx 1 - \frac{\omega kw}{2} + j\frac{k\mu w}{2} \quad (5-34)$$

and

$$\begin{aligned} E\left\{e^{j\frac{k(\mu+j\omega)w}{2}} J_0\left[\frac{k(\mu+j\omega)w}{2}\right]\right\} &\approx \int_0^\infty p_w(w) \left[1 - \frac{\omega kw}{2} + j\frac{k\mu w}{2}\right] dw \\ &\approx 1 - \frac{\omega k\bar{w}}{2} + j\frac{k\mu\bar{w}}{2} \end{aligned} \quad (5-35)$$

$$E\left\{\left[J_0\left[\frac{k(\mu+j\omega)w}{2}\right]\right]^2\right\} \approx 1 \quad (5-36)$$

With the types of errors given by Equations (3-53) and (3-54),

$$\begin{aligned} \bar{a} &= A + E\{\varepsilon\} = A \\ \bar{w} &= W + E\{\xi\} = W \end{aligned} \quad (5-37)$$

Equation (5-35) becomes:

$$E\left\{e^{j\frac{k(\mu+j\omega)w}{2}} J_0\left[\frac{k(\mu+j\omega)w}{2}\right]\right\} \approx 1 - \frac{\omega kW}{2} + j\frac{k\mu W}{2} \approx e^{j\frac{k(\mu+j\omega)W}{2}} \quad (5-38)$$

for small  $\mu$  and  $\omega$ . Using Equation (4-38),

$$\begin{aligned} \Theta(\mu + j\omega)\phi(\mu + j\omega) &\approx e^{jk(\mu+j\omega)W} e^{jk(\mu+j\omega)A} \\ &\approx e^{jk(\mu+j\omega)T} \end{aligned} \quad (5-39)$$

Substitution into Equation (5-31) yields:

$$\overline{|S^f(\mu + j\omega)|^2} = \frac{C^2 \pi k}{2R} \left[ 1 + 2 \operatorname{Re} \left\{ \left( e^{j \frac{k(\mu + j\omega)W}{2}} \right)^2 \left( \frac{e^{jk(\mu + j\omega)A}}{1 - e^{jk(\mu + j\omega)T}} \right) \right\} \right] \quad (5-40)$$

or

$$\begin{aligned} \overline{|S^f(\mu + j\omega)|^2} &= \frac{C^2 \pi k}{2R} \left[ 1 + 2 \operatorname{Re} \left\{ \frac{e^{jk(\mu + j\omega)T}}{1 - e^{jk(\mu + j\omega)T}} \right\} \right] \\ &= \frac{C^2 \pi k}{2R} \frac{1 - |e^{jk(\mu + j\omega)T}|^2}{|1 - e^{jk(\mu + j\omega)T}|^2} \end{aligned} \quad (5-41)$$

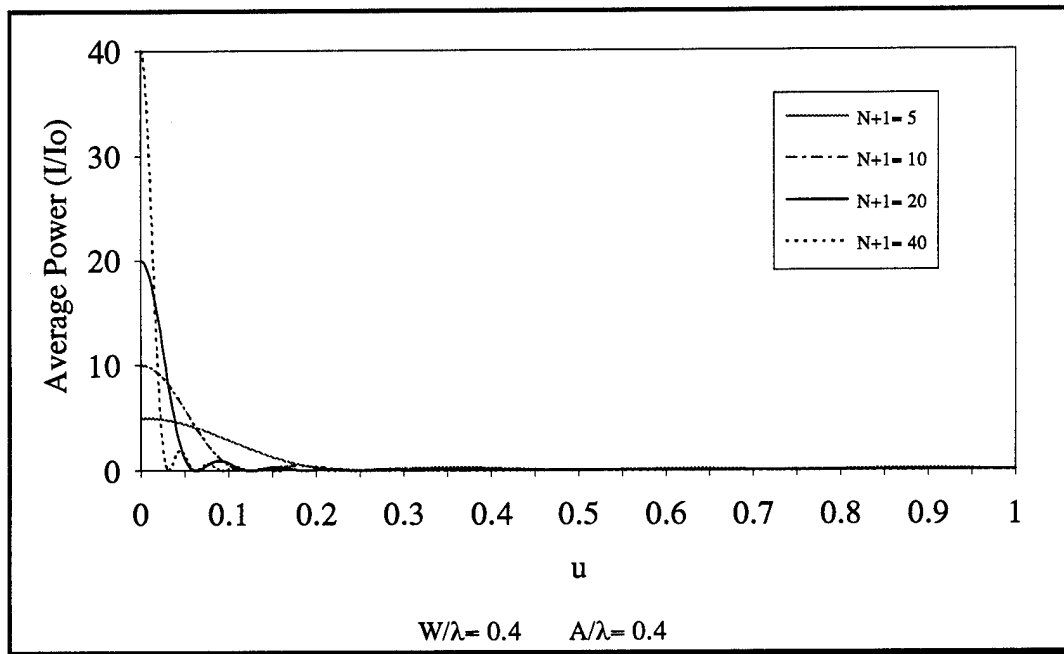
Using Equation (2-31), as  $\omega \rightarrow 0$  this becomes:

$$\overline{|S^f(\mu + j\omega)|^2} = \frac{C^2 \pi k}{2R} \frac{\lambda}{T} \delta(\mu) = \frac{I_0 \lambda}{T} \delta(\mu) \quad \text{for } \mu = 0 \quad (5-42)$$

### Grating with Uniformly Distributed Errors

Let  $\varepsilon$  and  $\xi$  have uniform PDFs as given in Equation (4-44) and Equation (4-46). The corresponding characteristic functions  $\Phi$  and  $\Theta$  are given by Equation (4-45) and Equation (4-47). Graphs comparing the average power patterns (normalized by dividing by  $I_0$ ) for exact gratings and gratings with errors are shown below. The majority of the graphs present results for low frequency scattering i.e.  $\lambda > T$  since the edge condition approach is more accurate in this regime. The expected values for terms involving the Bessel function ( $J_0$ ) were computed numerically. To allow comparison of gratings with no errors to gratings with errors, a number of graphs were computed using a finite number of slits or strips. This permits the graphs to show the progressive changes in the average power pattern of a grating with errors to a grating with no errors.

Figure 5.2 shows the effects of increasing the total number of slits or strips for a grating with no errors and  $T = 0.8\lambda$ . As expected, as the number of slits or strips increases, the pattern begins to resemble an impulse function at the origin. This is consistent with only one propagating mode being scattered from a grating with no errors and an infinite number of slits or strips when  $\lambda > T$ .



**Figure 5.2 Far Field Edge Condition Approximation - Effects of Total Number of Slits/ Strips on Grating with no Errors**

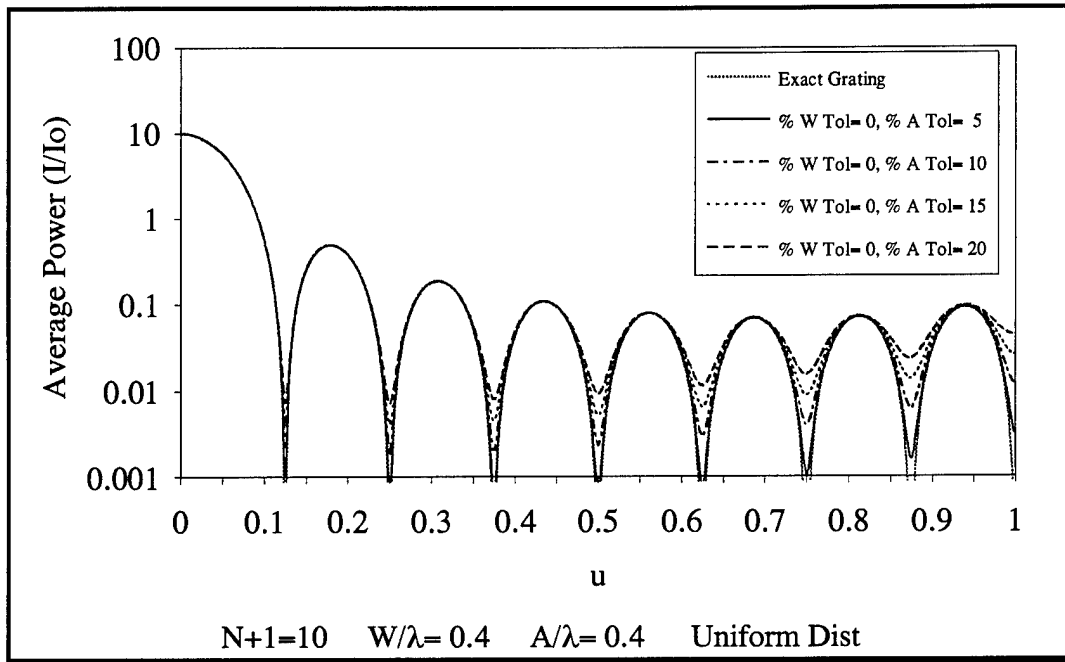
Figures 5.3 to 5.11 show the effects of errors in spacing alone, width alone, and both spacing and width for finite gratings with various ratios of desired width to desired spacing. The desired period for the graphs is  $T = 0.8\lambda$ . The plots show the overall

effects are similar to the effects seen using the zero order approximation. Most significantly, the main lobe is not significantly degraded and the nulls in the floor are raised.

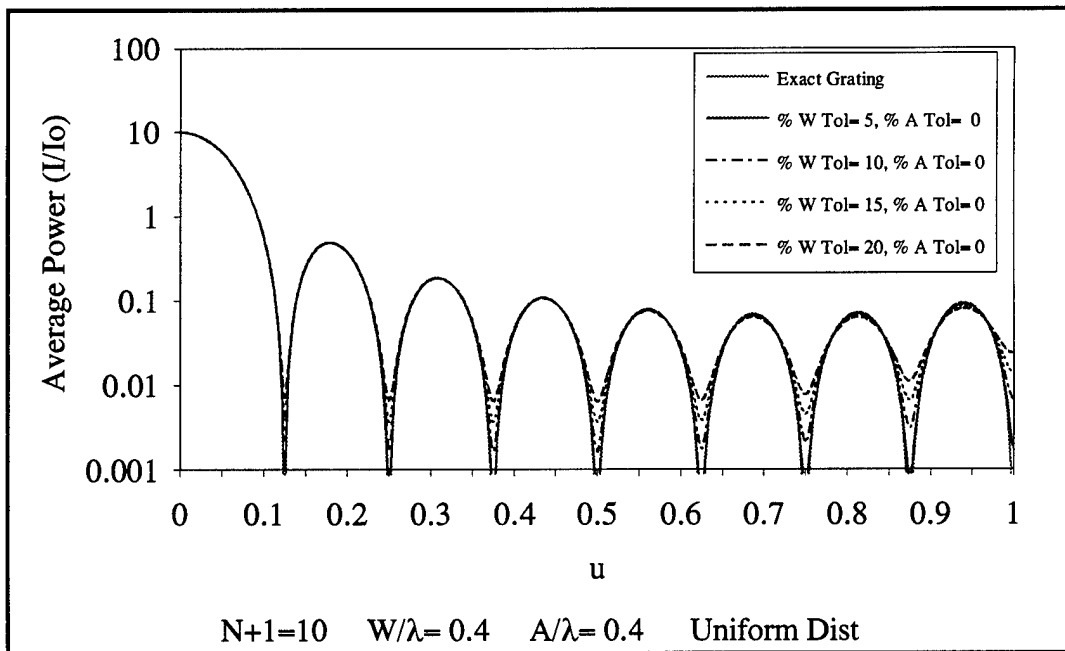
Figures 5.3 to 5.5 show the effects of errors in spacing alone, width alone, and both spacing and width for a finite grating with desired width equal to desired spacing, i.e.  $W = A$ . A comparison of Figure 5.3 to Figure 5.4 shows that, as in the results obtained using the zero order approximation, the effects of error in width alone are on the order of the errors in spacing alone for a given tolerance level. Figure 5.5 shows the combined effects of errors in both spacing and width. Note the similarity of this graph to Figure 4.19, obtained using the zero order approach.

Figures 5.6 through 5.8 show the effects of errors in spacing alone, width alone, and both spacing and width respectively for a finite grating with desired width less than the desired spacing, i.e.  $W < A$ . A comparison of Figure 5.6 to Figure 5.7 shows that, for this case, the effects of error in spacing alone are again greater than the effects due to errors in width alone for a given percent tolerance level. Figure 5.8 shows the combined effects of errors in both spacing and width.

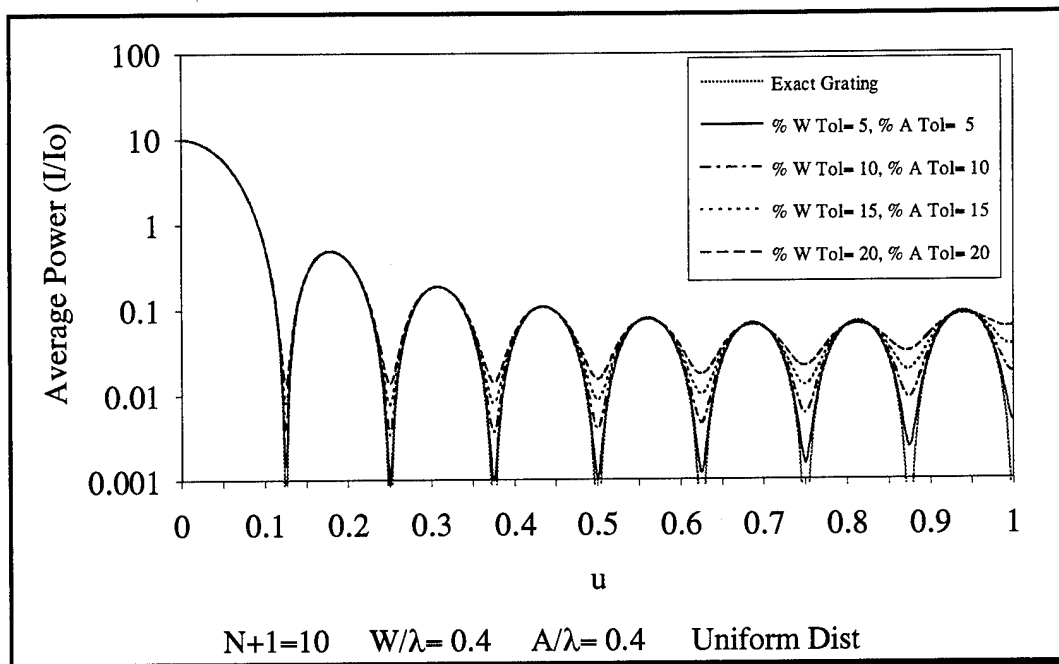
Figures 5.9 through 5.11 show the effects of errors in spacing alone, width alone, and both spacing and width respectively for a finite grating with desired width greater than the desired spacing, i.e.  $W > A$ . A comparison of Figure 5.9 to Figure 5.10 shows that, opposite to the previous case, the effects of error in width alone are again greater than the effects due to errors in spacing alone for a given percent tolerance level. Figure 5.11 shows the combined effects of errors in both spacing and width.



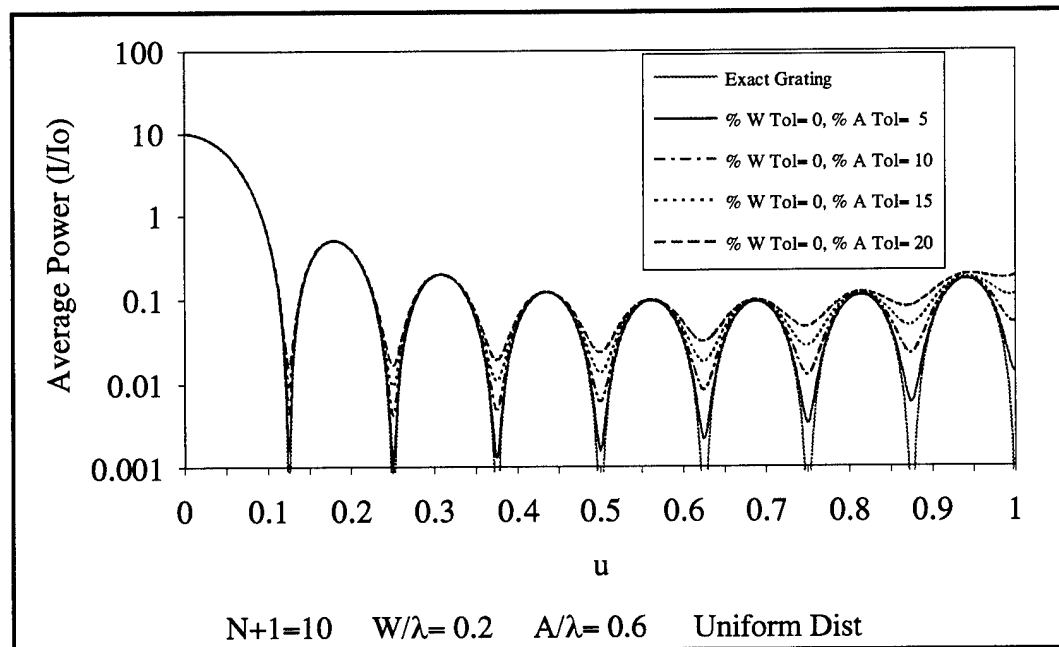
**Figure 5.3 Far Field Edge Condition Approximation - Effects of Spacing Errors on Grating with Desired Width Equal to Desired Spacing**



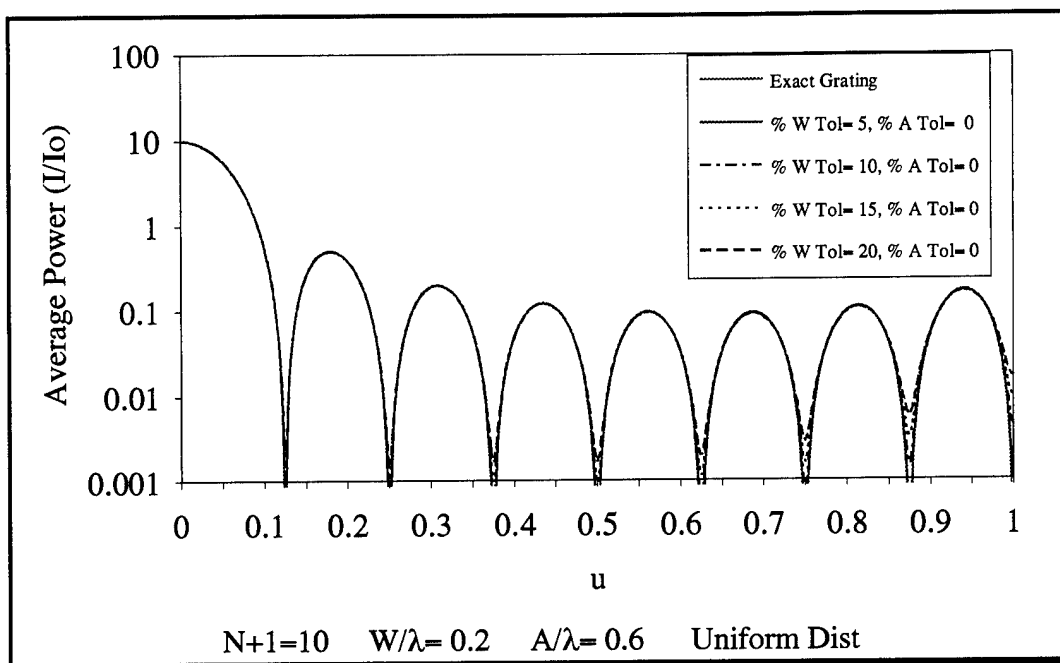
**Figure 5.4 Far Field Edge Condition Approximation - Effects of Width Errors on Grating with Desired Width Equal to Desired Spacing**



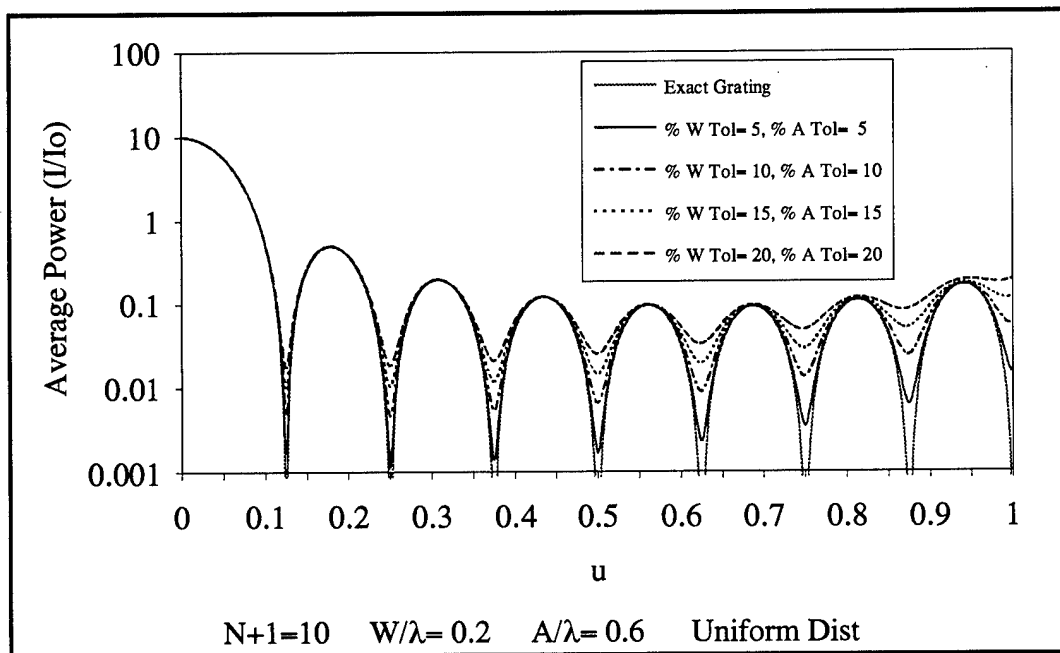
**Figure 5.5 Far Field Edge Condition Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Equal to Desired Spacing**



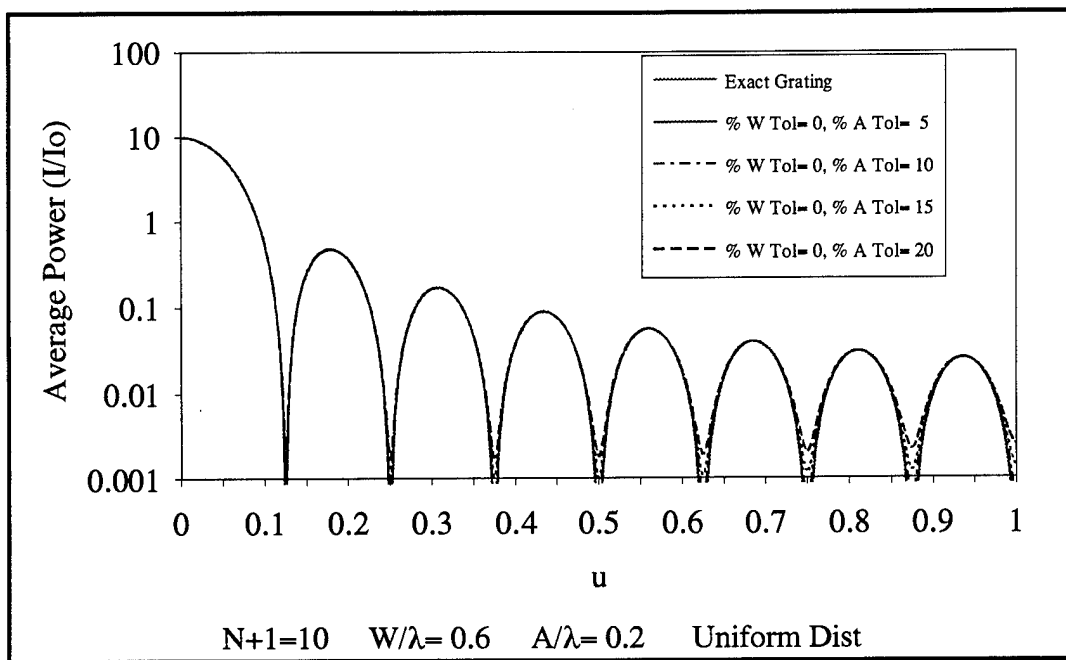
**Figure 5.6 Far Field Edge Condition Approximation - Effects of Spacing Errors on Grating with Desired Width Less Than Desired Spacing**



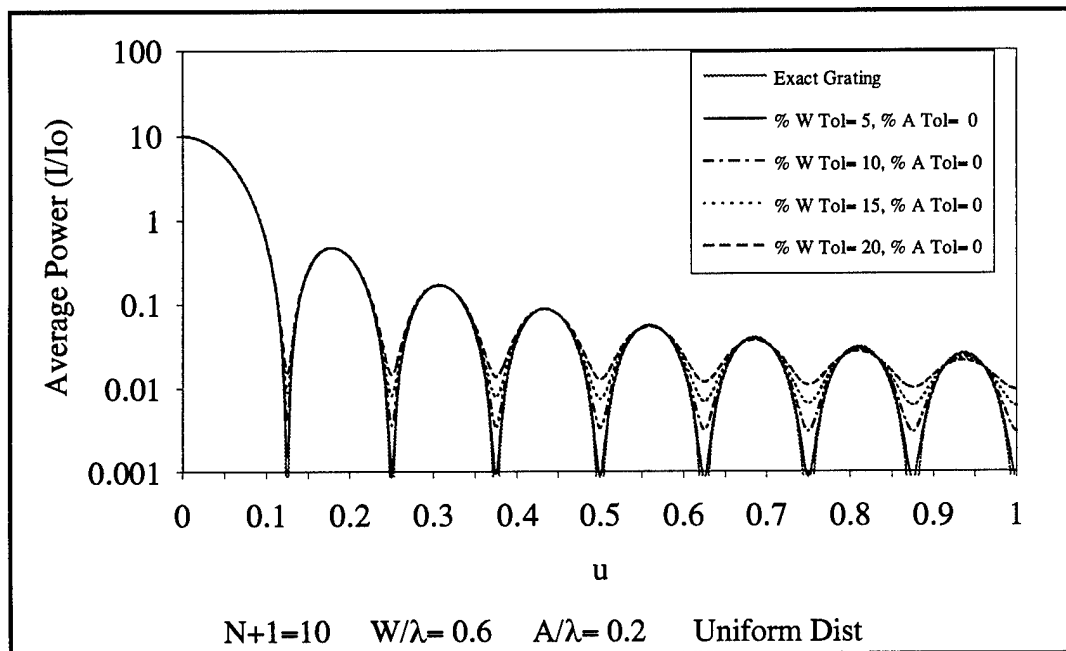
**Figure 5.7 Far Field Edge Condition Approximation - Effects of Width Errors on Grating with Desired Width Less Than Desired Spacing**



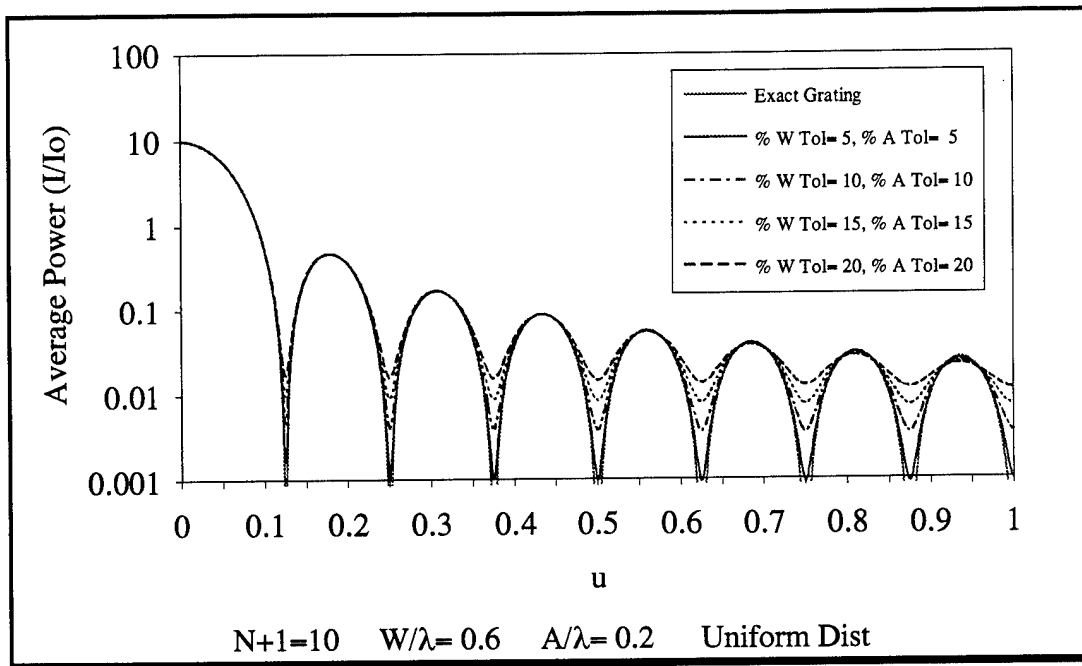
**Figure 5.8 Far Field Edge Condition Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Less Than Desired Spacing**



**Figure 5.9 Far Field Edge Condition Approximation - Effects of Spacing Errors on Grating with Desired Width Greater Than Desired Spacing**



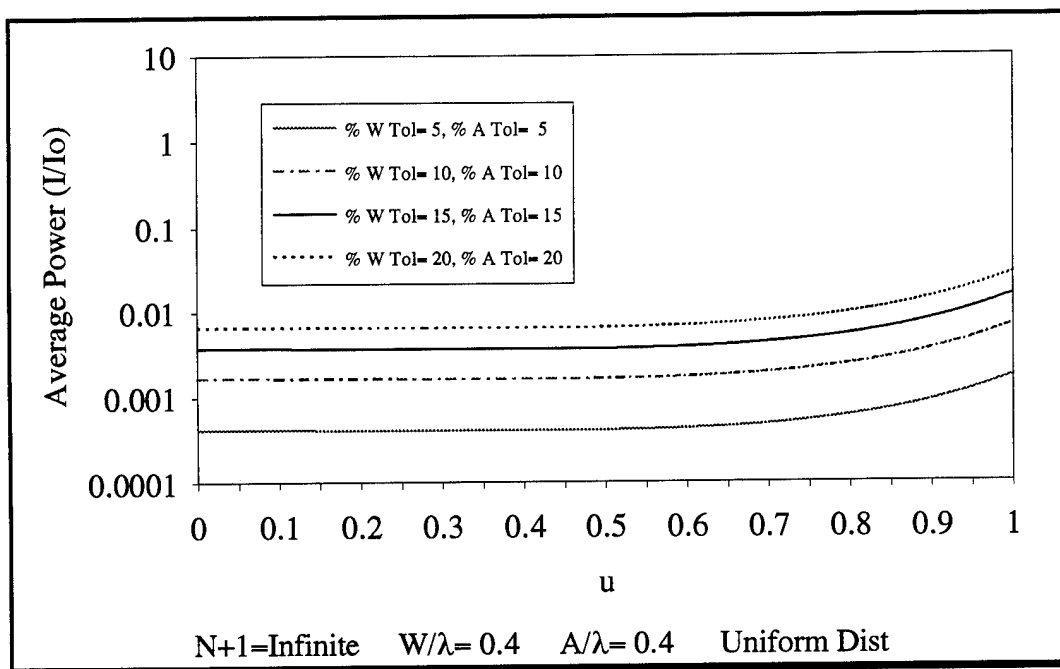
**Figure 5.10 Far Field Edge Condition Approximation - Effects of Width Errors on Grating with Desired Width Greater Than Desired Spacing**



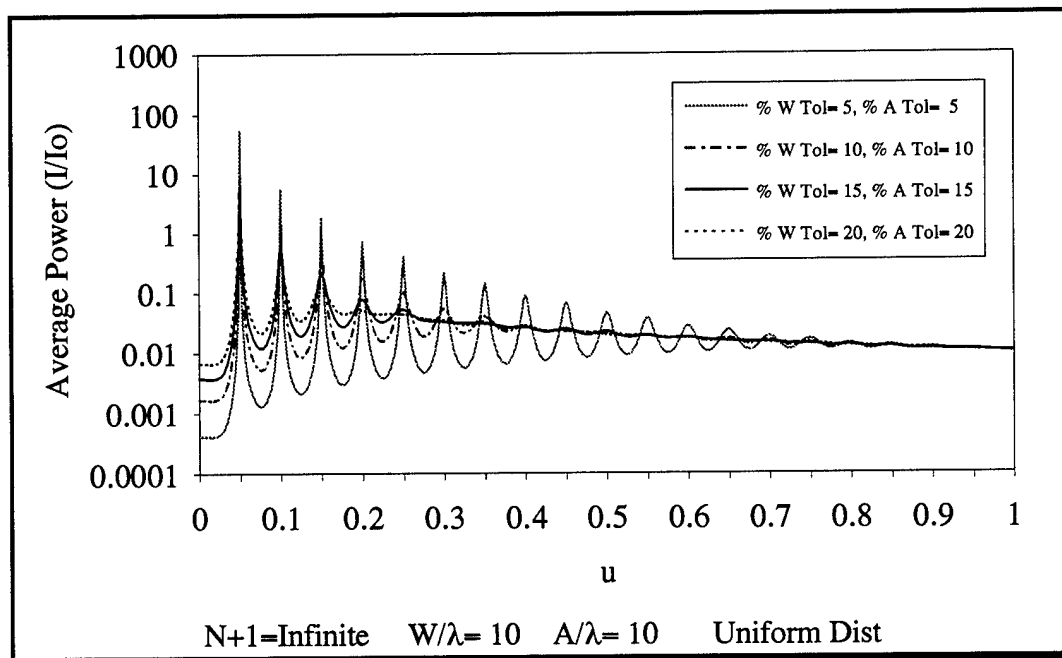
**Figure 5.11 Far Field Edge Condition Approximation - Effects of Width and Spacing Errors on Grating with Desired Width Greater Than Desired Spacing**

Figure 5.12 shows the average power pattern for a grating with an infinite number of slits/ strips for varying amounts of equal width and spacing tolerances and a desired period  $T = 0.8\lambda$ . Although not shown on the graph, the patterns still contain an impulse at the origin ( $u = 0$ ) which is not plotted due to its infinite amplitude. Whereas a grating with no errors would have a power pattern equal to zero for  $u \neq 0$ , the graph shows small but non-zero values for  $u \neq 0$ . Note the similarity of this graph to Figure 4.20.

Figure 5.13 shows the average power pattern for a grating with an infinite number of slits/ strips for varying amounts of equal width and spacing tolerances and a desired period  $T = 20\lambda$ . Although not shown on the graph, the patterns still contain an impulse at the origin ( $u = 0$ ) which is not plotted due to its infinite amplitude. The graph is very similar to Figure 4.17 which has the same width and spacing values.



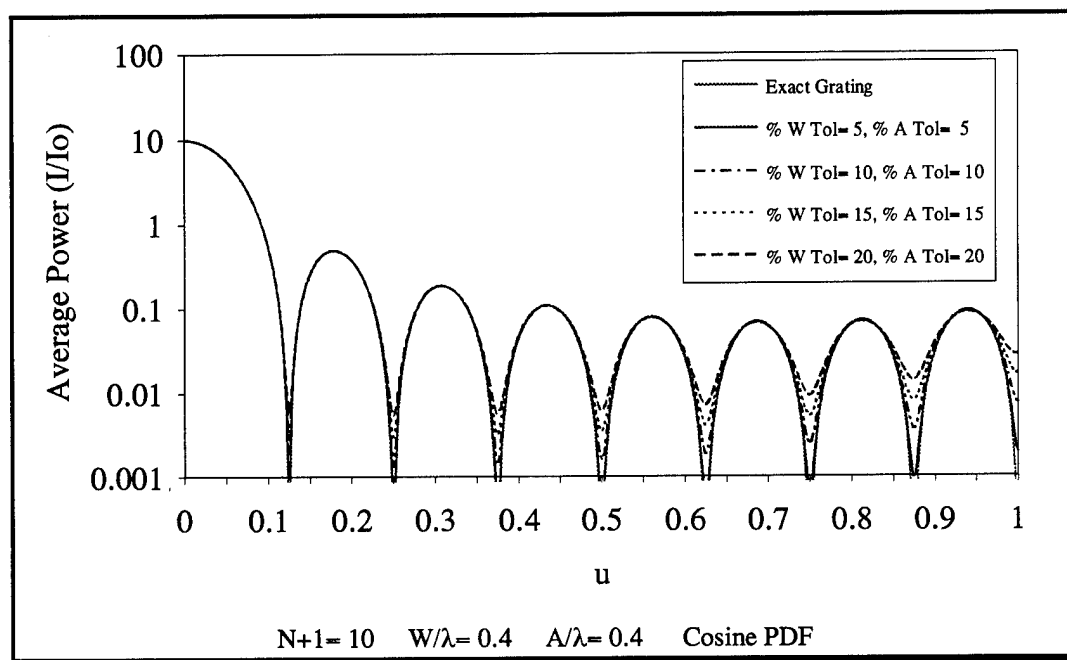
**Figure 5.12 Far Field Edge Condition Approximation - Effects of Width and Spacing Errors on Grating with Infinite Number of Slits or Strips**



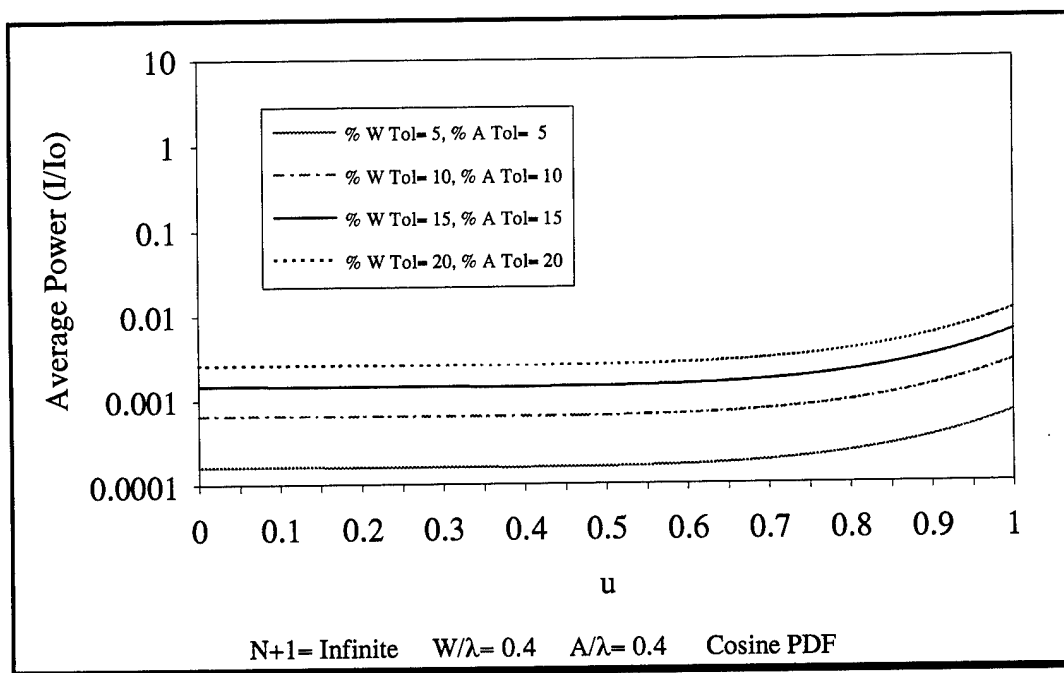
**Figure 5.13 Far Field Edge Condition Approximation - Effects of Width and Spacing Errors on Grating with Infinite Number of Slits or Strips**

## Grating with Cosine Function Distributed Errors

Let  $\epsilon$  and  $\xi$  have cosine function PDFs as given in Equation (4-48) and Equation (4-53). The corresponding characteristic functions  $\Phi$  and  $\Theta$  are given in Equation (4-51) and Equation (4-54). Graphs showing the average power pattern of gratings with errors using the cosine PDF are shown below. Figure 5.14 shows the pattern for a grating with a finite number of slits/ strips while Figure 5.15 shows the pattern for a grating with an infinite number of slits/ strips. The overall effects are similar to the gratings with the uniformly distributed PDFs. As expected, for a given maximum error (i.e. % Tol), the errors in spacing and width have less of an effect upon the average power pattern of a grating with cosine distributed errors than to a grating with uniformly distributed errors.



**Figure 5.14 Far Field Edge Condition Approximation - Cosine PDF, Finite Number of Slits/ Strips**



**Figure 5.15 Far Field Edge Condition Approximation - Cosine PDF, Infinite Number of Slits/ Strips**

## **VI. Realizations and Validation**

### **Overview**

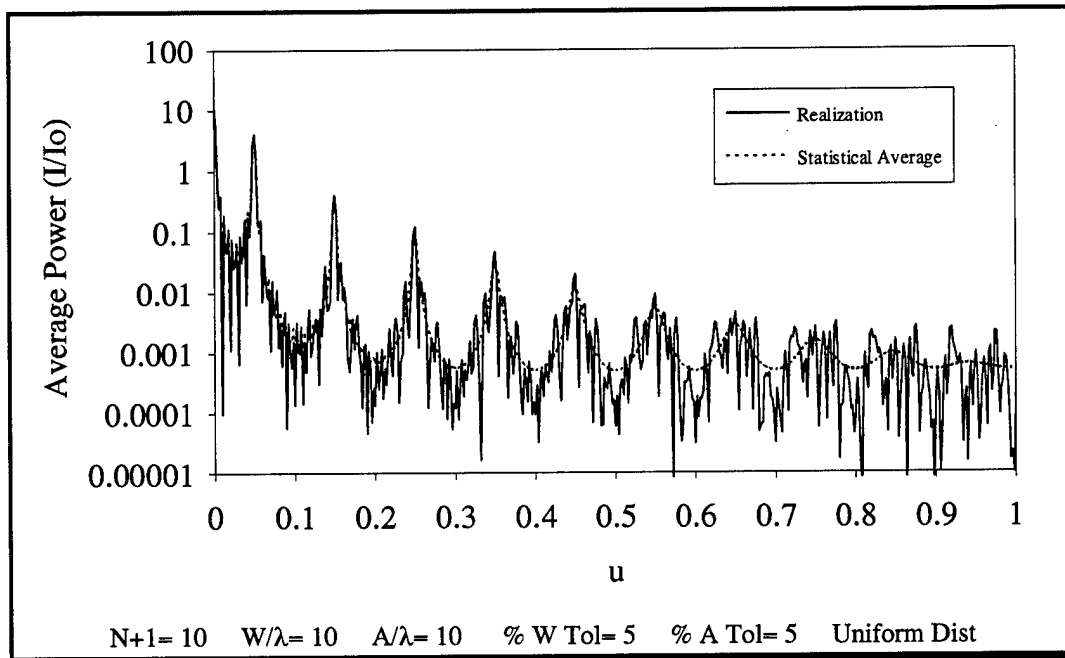
This chapter compares the statistical average power pattern computed using the expressions developed in the previous chapters to the average of a number of trial realizations. To compute the average power pattern for a realization, the errors in a grating are randomly generated using the uniform or cosine function PDFs. For all cases, a total of twenty realizations are computed for each grating. The average power pattern is then computed for each grating and the results averaged together. Plots showing the standard deviation of the realizations are also provided.

### **Realizations Using the Born Approximation**

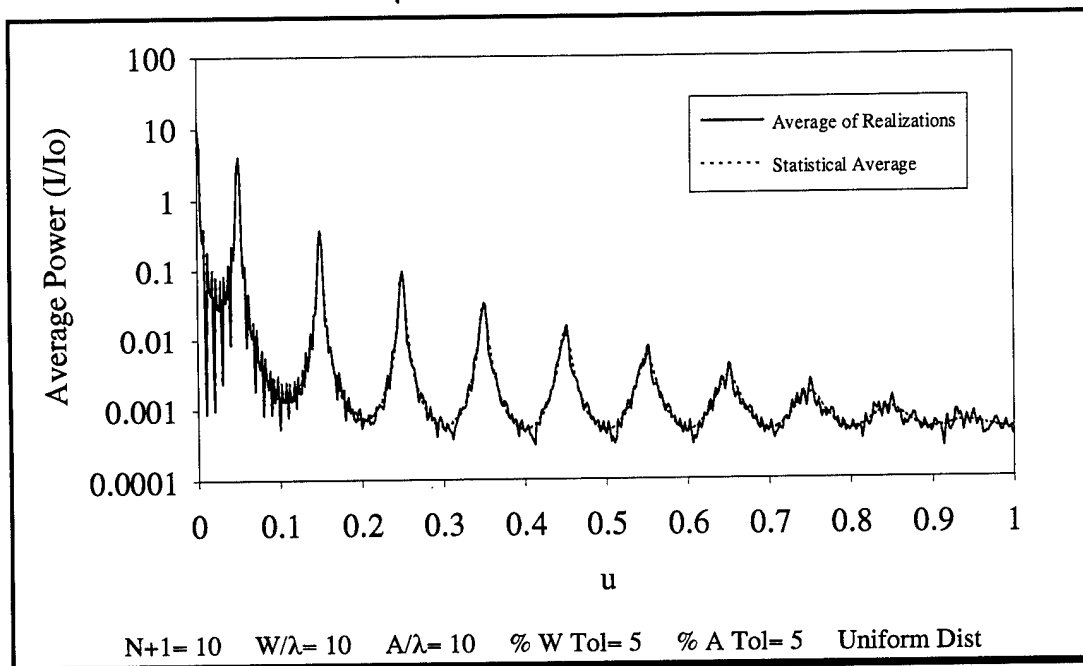
Figures 6.1 through 6.12 present the results of trial realizations using the Born approximation. Figure 6.1 shows a typical realization of the average power pattern for a grating with  $N+1=10$  slits/ strips and uniformly distributed errors with maximum width and spacing errors equal to five percent. The graph also shows the statistical average power pattern computed using the formulas developed in Chapter IV. As shown, the realization tends to follow the statistical average. Figure 6.2 shows the results of averaging the power pattern for twenty trial realizations. Even for only twenty realizations, the statistical average and the average of the realizations show excellent agreement. The standard deviation and the standard deviation/ mean ratio of the twenty realizations are shown in Figure 6.3. The standard deviation at the origin ( $u = 0$ ) is small ( $\sim 0.14$ ). For values of  $u$  greater than zero, the standard deviation resembles the average power pattern. In fact, the ratio of the standard deviation to the mean shows this as the curve starts out at a small value at  $u = 0$  and then oscillates about a ratio equal to one.

Figures 6.4 through 6.6 present results for the same gratings with maximum width and spacing errors equal to ten percent. The same trends are apparent and the statistical average power pattern again agrees with the average of the realizations. The number of slits/ strips is increased to  $N+1= 100$  for Figures 6.7 through 6.9. Again the statistical average power pattern agrees with the average of the realizations and the ratio of the standard deviation to the mean starts out at a small value at the origin and then oscillates about a ratio equal to one. This behavior of the standard deviation is consistent with the expression developed in Appendix D for the standard deviation of a grating with an infinite number of slits/ strips.

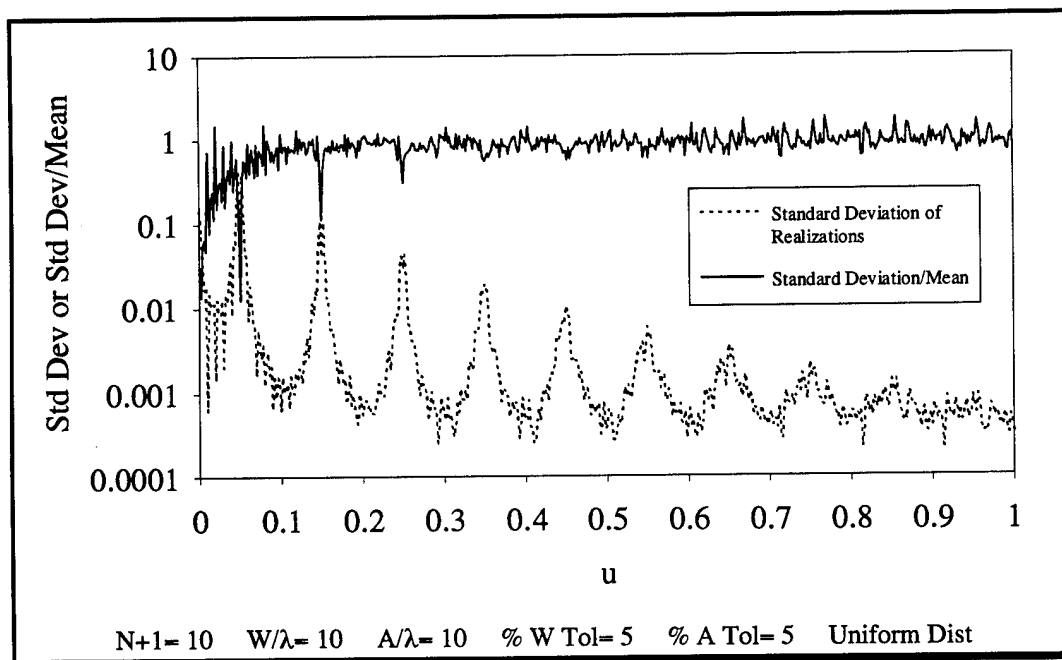
Finally, Figures 6.10 through 6.12 present the results for a grating with  $N+1= 10$  slits/ strips with errors generated using the cosine function PDF. Again, the same trends appear as for the gratings with the uniformly distributed errors.



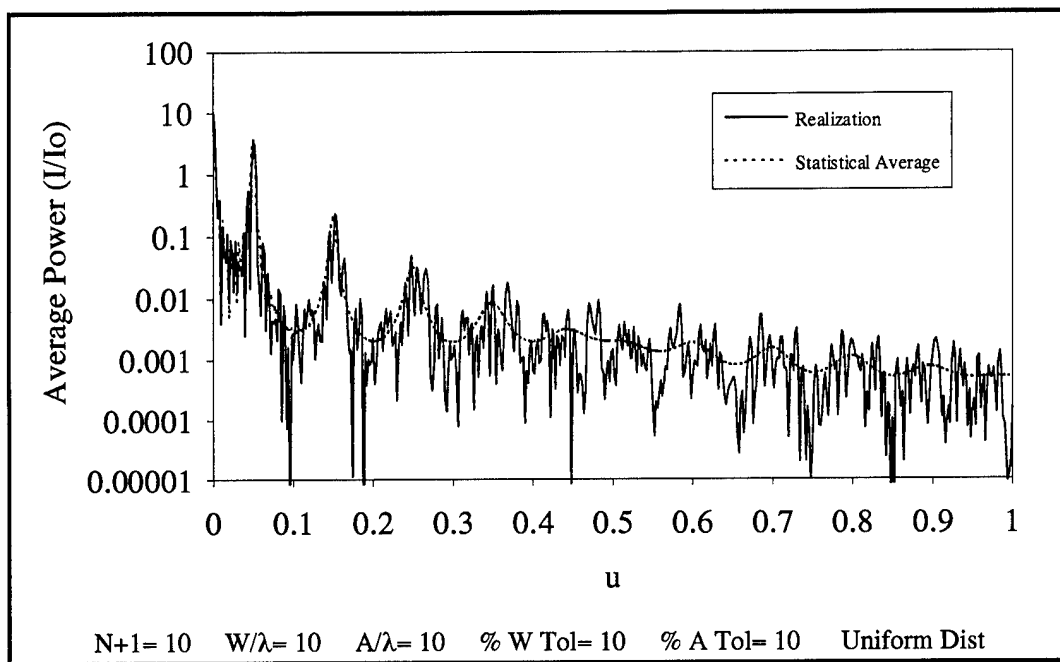
**Figure 6.1 Realization of Far Field Average Power Pattern - Born Approximation with Uniformly Distributed Errors**



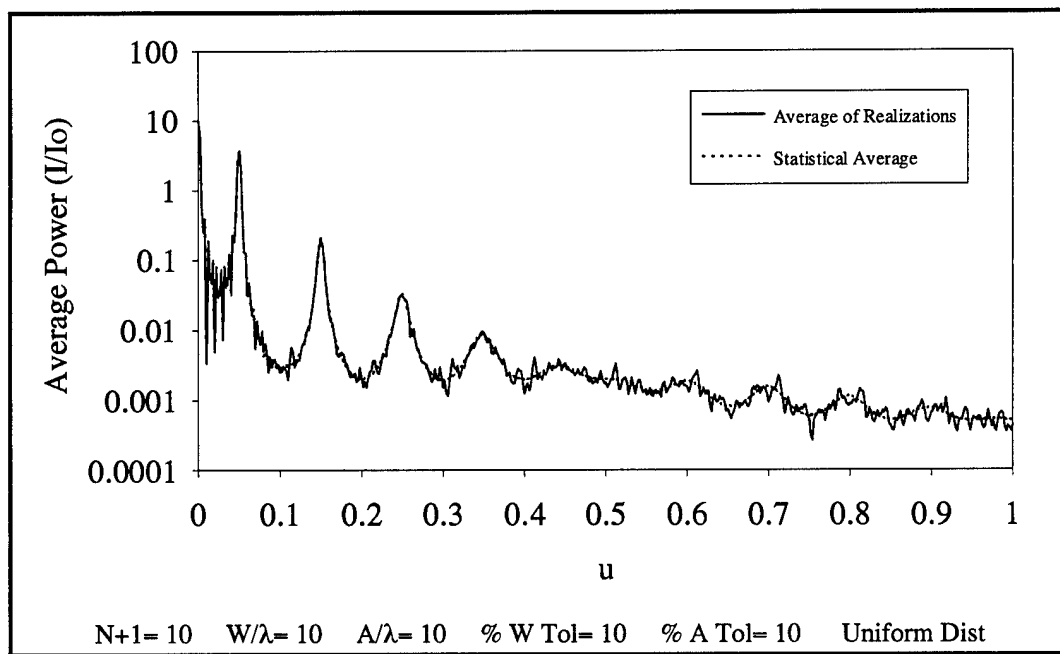
**Figure 6.2 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Born Approximation with Uniformly Distributed Errors**



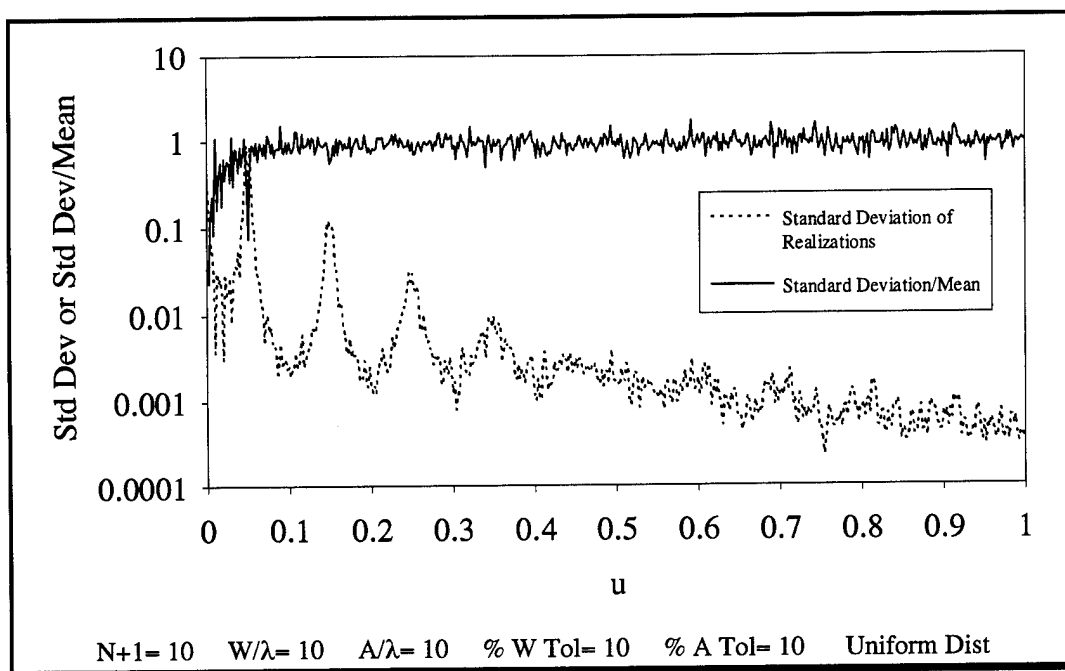
**Figure 6.3 Far Field Average Power Pattern - Standard Deviation of Realizations - Born Approximation with Uniformly Distributed Errors**



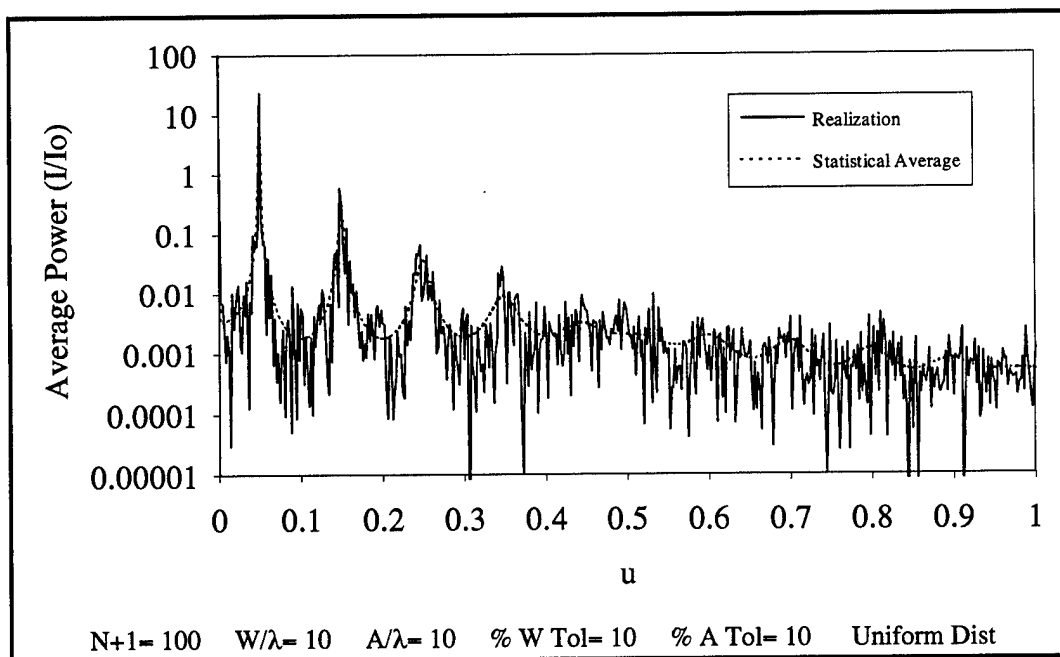
**Figure 6.4 Realization of Far Field Average Power Pattern - Born Approximation with Uniformly Distributed Errors**



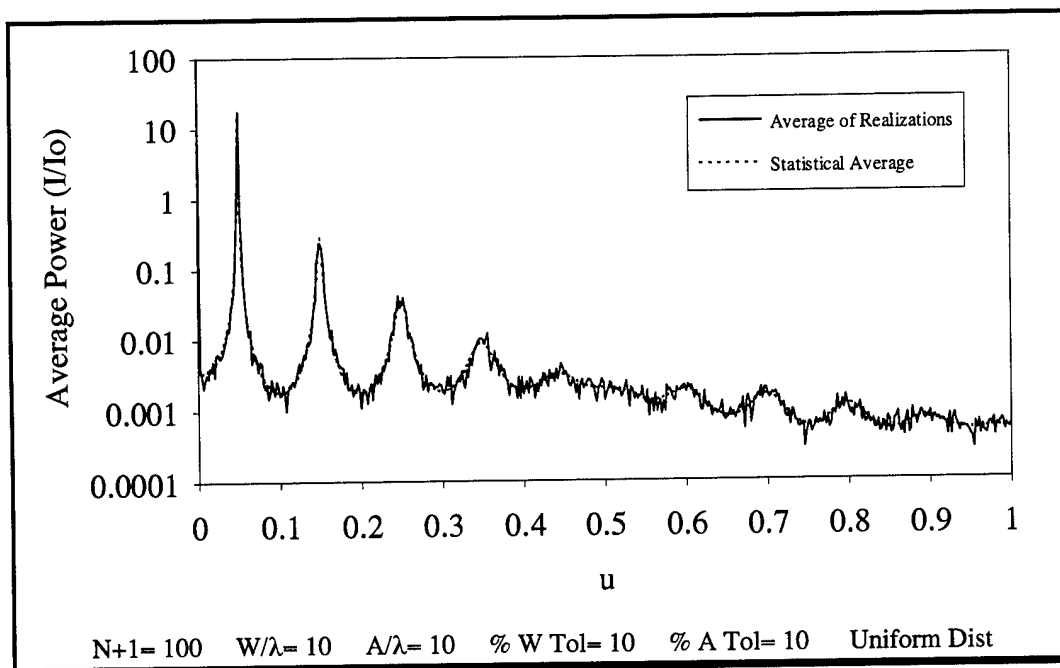
**Figure 6.5 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Born Approximation with Uniformly Distributed Errors**



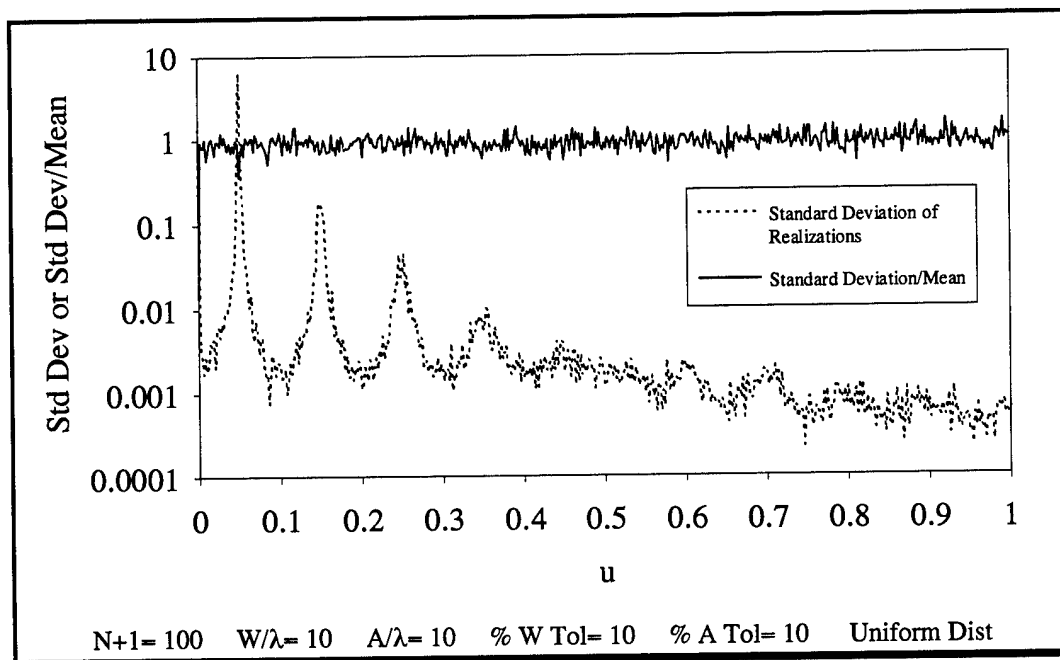
**Figure 6.6 Far Field Average Power Pattern - Standard Deviation of Realizations - Born Approximation with Uniformly Distributed Errors**



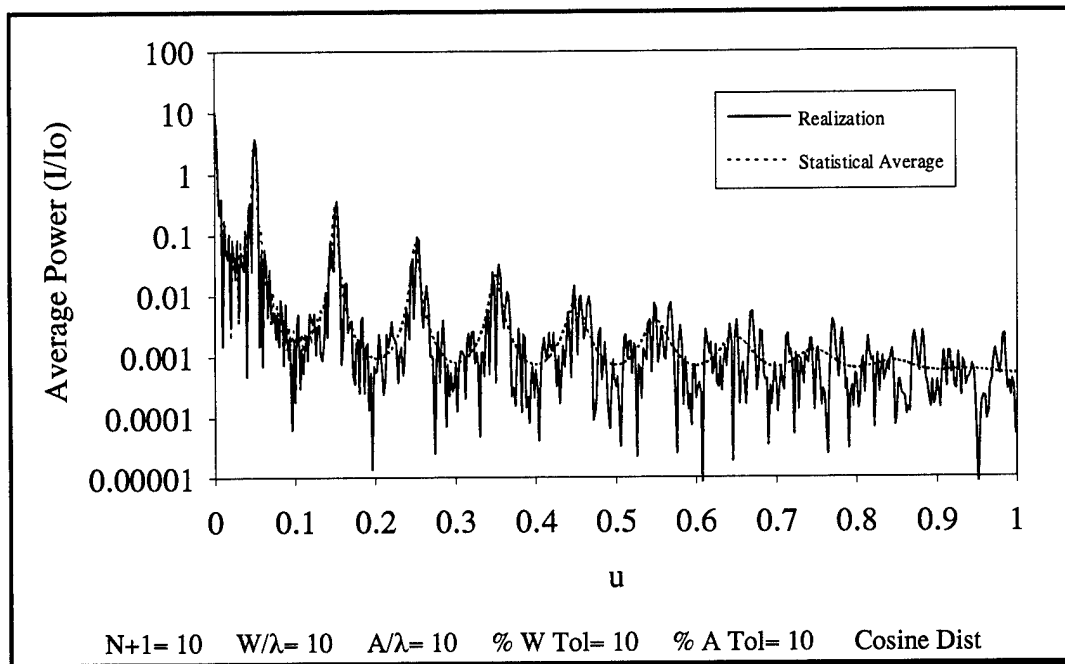
**Figure 6.7 Realization of Far Field Average Power Pattern - Born Approximation with Uniformly Distributed Errors**



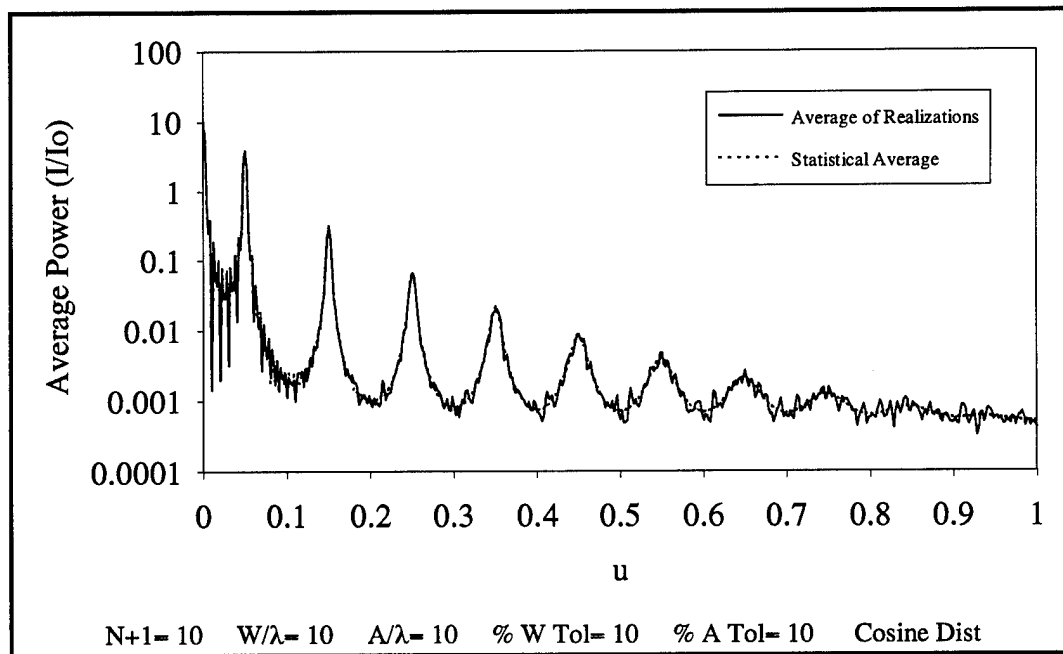
**Figure 6.8 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Born Approximation with Uniformly Distributed Errors**



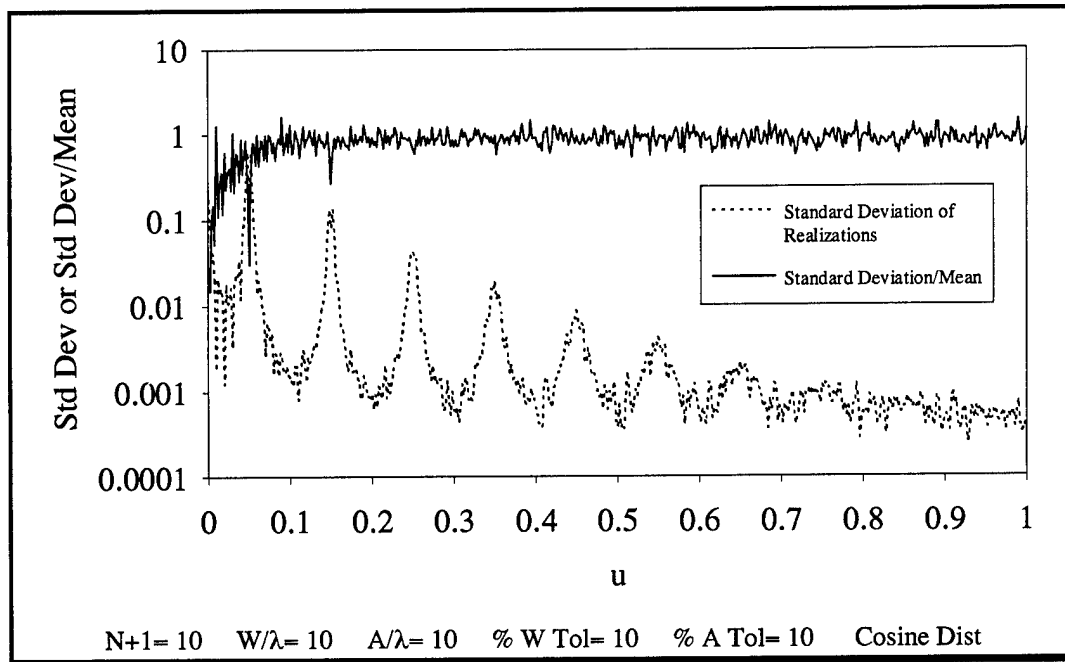
**Figure 6.9 Far Field Average Power Pattern - Standard Deviation of Realizations - Born Approximation with Uniformly Distributed Errors**



**Figure 6.10 Realization of Far Field Average Power Pattern - Born Approximation with Cosine Distributed Errors**



**Figure 6.11 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Born Approximation with Cosine Distributed Errors**

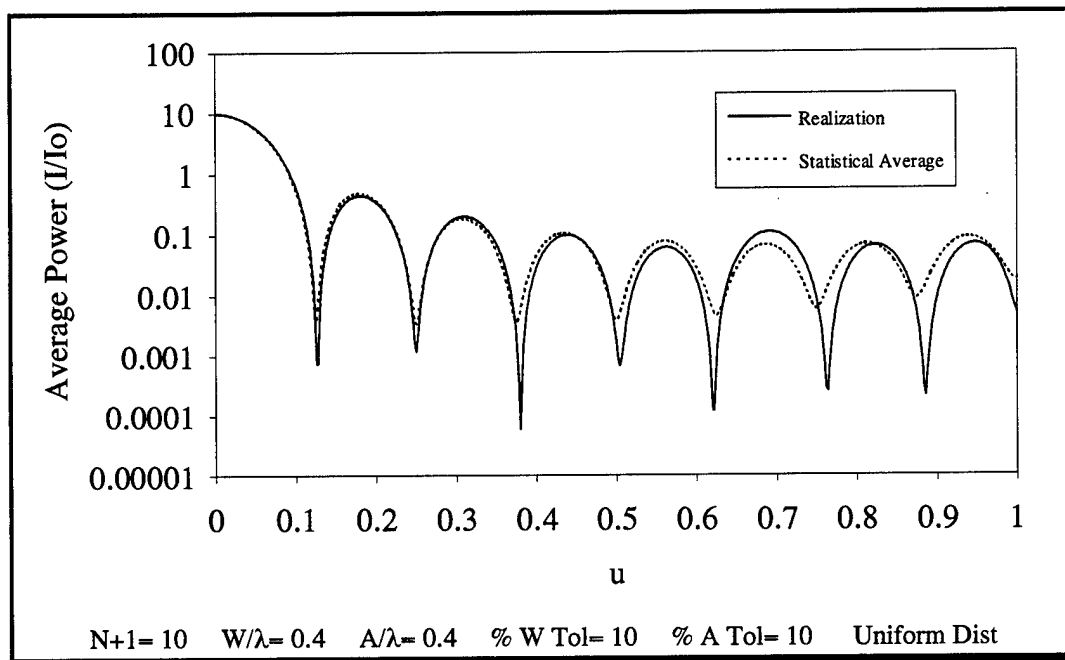


**Figure 6.12 Far Field Average Power Pattern - Standard Deviation of Realizations - Born Approximation with Cosine Distributed Errors**

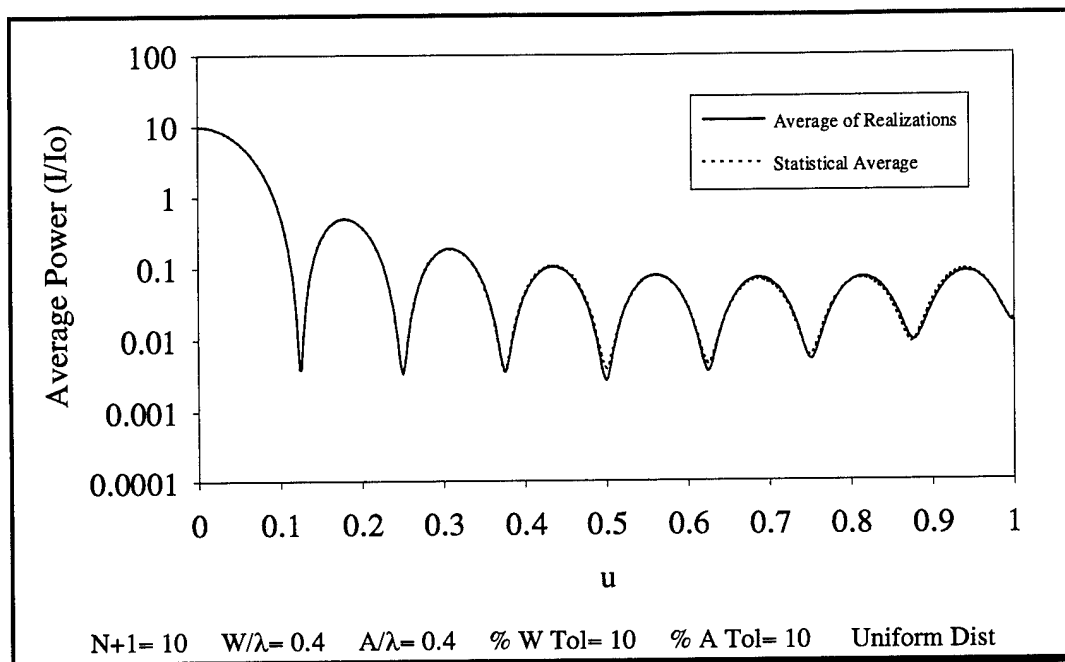
### Realizations Using the Edge Condition Approximation

Figures 6.13 through 6.15 present the results of trial realizations using the Edge Condition approximation. Figure 6.13 shows a typical realization of the average power pattern for a grating with  $N+1= 10$  slits/ strips and uniformly distributed errors with maximum width and spacing errors equal to ten percent. The graph also shows the statistical average power pattern computed using the formulas developed in Chapter V. As shown, the realization tends to follow the statistical average. Figure 6.14 shows the results of averaging the power pattern for twenty trial realizations. The statistical average and the average of the realizations show excellent agreement. The standard deviation and the standard deviation/ mean ratio are shown in Figure 6.15. As a consequence of using the edge condition approximation, the standard deviation at the origin is zero. For values of  $u$  greater than zero, the standard deviation resembles the average power pattern. The

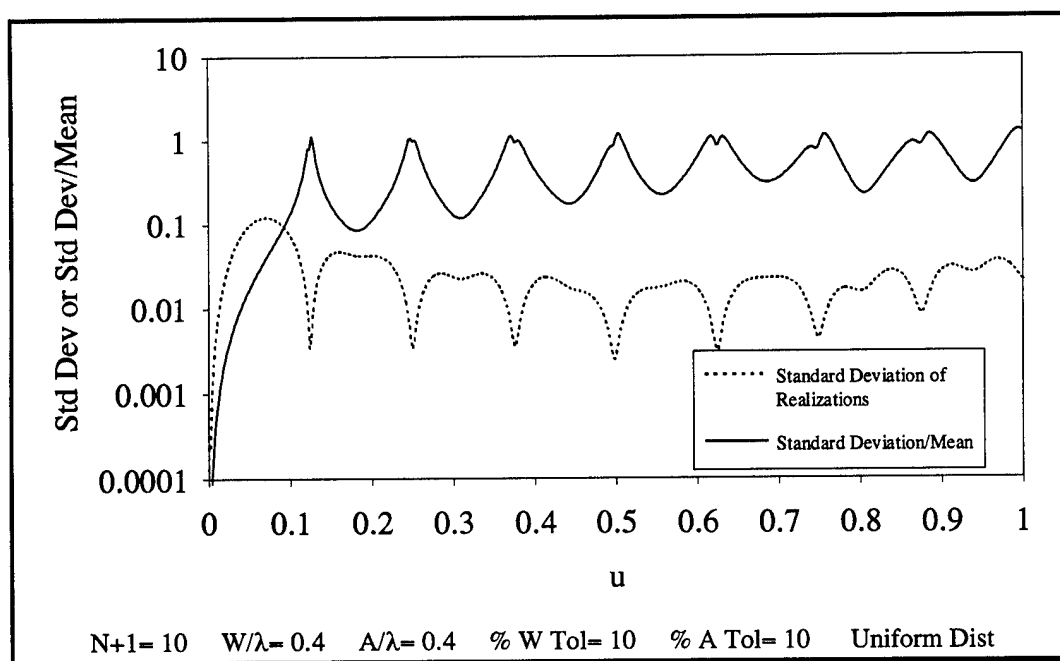
ratio of the standard deviation to the mean follows the same trend has for the previous cases although the ratio does not quite reach a value of one about which to oscillate. The number of slits/ strips is increased to  $N+1= 100$  for Figures 6.16 through 6.18. Again the statistical average power pattern agrees with the average of the realizations. As shown in Figure 6.18, the ratio of the standard deviation to the mean does reach a value of one about which to oscillate. Again, the behavior of the standard deviation is consistent with the expression developed in Appendix E for the standard deviation of a grating with an infinite number of slits/ strips.



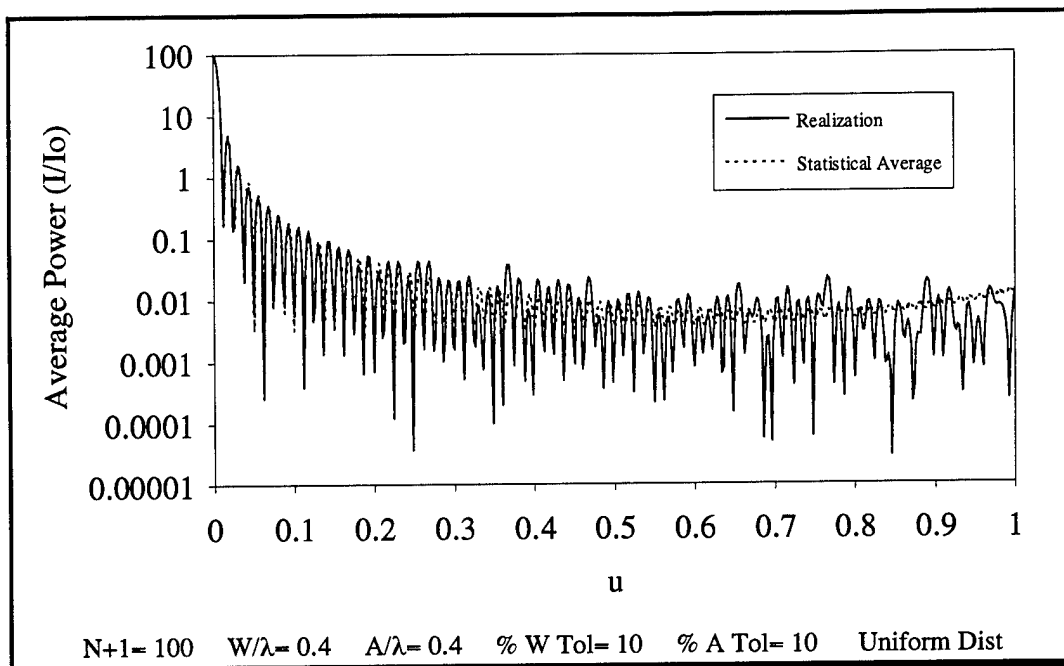
**Figure 6.13 Realization of Far Field Power Pattern - Edge Condition Approximation with Uniformly Distributed Errors**



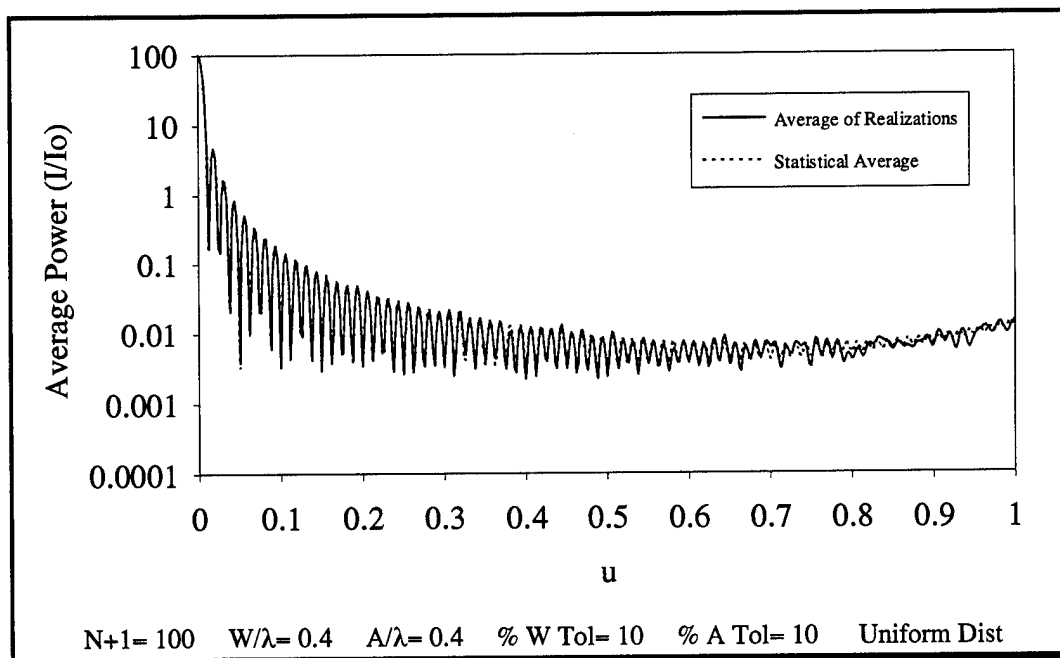
**Figure 6.14 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Edge Condition Approximation with Uniformly Distributed Errors**



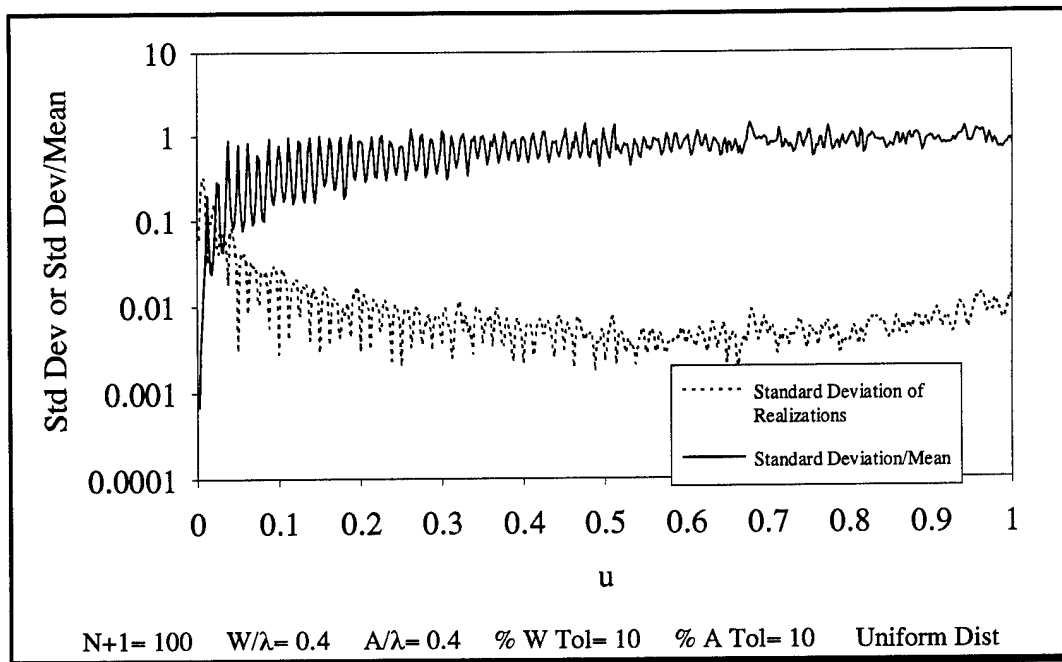
**Figure 6.15 Far Field Average Power Pattern - Standard Deviation of Realizations - Edge Condition Approximation with Uniformly Distributed Errors**



**Figure 6.16 Realization of Far Field Power Pattern - Edge Condition Approximation with Uniformly Distributed Errors**



**Figure 6.17 Far Field Average Power Pattern - Comparison of Average of Realizations to Statistical Average - Edge Condition Approximation with Uniformly Distributed Errors**



**Figure 6.18 Far Field Average Power Pattern - Standard Deviation of Realizations - Edge Condition Approximation with Uniformly Distributed Errors**

## **VII. Conclusions**

### **Overview**

This chapter summarizes the accomplishments of this research effort, provides observations and conclusions regarding the results, and suggests several areas for possible follow on investigation.

### **Accomplishments**

The first accomplishment of this effort is the formulation for the statistical average power pattern of a linear array of point sources with errors in spacing. This is equivalent to the average power pattern of a linear antenna array. Other authors have investigated the problem of the random array by treating the location of each element as random. The formulation developed in this work treats the spacing between the elements as random or as a desired spacing plus or minus a random error. This approach is more practical from a manufacturing standpoint than the random location approach. Although the development of the average power pattern of a linear array of point sources is presented to provide a basis for the formulation of the grating problem, its applicability to antenna array analysis makes it a noteworthy accomplishment in its own right. This is especially true when one considers the ease with which random amplitude and phase errors could be added to the formulation.

The major accomplishment of this effort is the development of two formulations for the far-field statistical average power pattern of a strip grating with errors in the widths of the strips and the slits as compared to an ideal, exact periodic grating. The first formulation utilizes the Born approximation in which the unknown aperture fields are replaced by the incident fields. The second formulation utilizes an approximation for the unknown fields which satisfies the edge condition. Approximations for the scattered fields

are first derived using PEC surface equivalence for a TE<sub>z</sub> polarized plane wave incident upon a strip grating consisting of an infinite PEC screen cut by a number of infinitely long slits (infinite in the z-direction). Babinet's principle is then used to obtain approximations for the fields obtained when a TM<sub>z</sub> polarized plane wave is incident upon the complementary grating formed by interchanging the slits and strips of the original grating, i.e. a number of infinitely long strips in free space. Expressions for the average power pattern are developed in terms of the following variables: number of slits or strips, desired width- and spacing-to-wavelength ratios, and the characteristic functions of the PDFs of the width and spacing errors.

The formulation of the far-field statistical average power pattern is unique in that it is based upon independent but identically distributed errors in the widths of each slit/ strip and independent but identically distributed errors in the spacings between them. The use of the independent errors leads to expressions incorporating geometric series involving the characteristic functions of the width and spacing PDFs which can be evaluated in closed form even for an infinite number of terms. This allows the average power pattern of a grating with errors to be calculated for gratings with an infinite as well as a finite number of strips and slits.

## **Observations and Conclusions**

A number of observations and conclusions can be drawn from the graphs in Chapters IV and V which show the far-field average power patterns of gratings with different geometries. Three observations are immediately apparent when looking at the patterns of gratings with a finite number of slits/ strips. First and possibly the most significant, the main lobe of a grating with errors is not significantly different from that of a grating with no errors. Second, the grating lobes are reduced in magnitude and broadened slightly. Third, the level of the sidelobes between the grating lobes and the

nulls between the sidelobes are raised. Furthermore, the effects increase as the observation angle moves away from the main lobe.

The graphs also show the relative effects of the errors are consistent with the ratio of the desired width to desired spacing. For the same amount of maximum percent error, the effects of spacing errors are more pronounced on a grating with desired spacing greater than desired width than the effects of width errors. Similarly, for the same amount of maximum percent error, the effects of width errors are more pronounced on a grating with desired width greater than desired spacing than the effects of spacing errors. For a grating with desired width equal to desired spacing, the effects of errors in width are on the order of the effects due to errors in spacing.

The same observations can be made regarding gratings with an infinite number of slits/ strips. For a periodic grating with no errors and an infinite number of slits and strips, the power pattern consists of a number of impulse functions located  $\arcsin(\lambda/T)$  apart in angle. These impulse functions correspond to plane waves. As shown in Chapters IV and V, the impulse function at the origin (i.e. the main lobe) for an infinite strip grating with errors is unchanged from that of a grating with no errors. The remaining grating lobes are significantly reduced in amplitude and broadened in width as evidenced by the fact they are no longer impulse functions. This means that, except for the main lobe, the scattered waves are no longer discrete plane waves. The power patterns of the infinite strip gratings also show non-zero values between the grating lobes.

It is interesting to note that the average power pattern for a grating computed using the Born approximation is similar to the average power pattern for a grating computed using the edge condition approximation. This leads to the conclusion that the effects would be similar if an exact expression for the unknown fields could be obtained and utilized.

The power patterns of a number of strip gratings containing random errors in spacing and width were computed and the results averaged together. As shown in Chapter VI, the average of these realizations agrees with the computed statistical average for both the Born approximation and the edge condition approximation. This result validates the formulations developed in Chapters IV and V for the statistical average power pattern. The standard deviation of the realizations was also computed and the results agree with the formulations developed in Appendices A, D, and E for the case of a grating with an infinite number of slits/ strips.

The behavior of the standard deviation approaching the average power pattern as the number of slits/ strips goes to infinity is quite interesting. As a possible explanation, consider the case of the array of point sources. By invoking the central limit theorem for probability, the PDF of the array factor approaches gaussian. As a result, the PDF of the square of the magnitude of the array factor, i.e. the power factor, approaches a negative exponential. One property of a process with a negative exponential PDF is that the standard deviation is equal to the mean.

It is obvious that the formulations developed in this work can be used to determine the average power pattern for a strip grating for a given distribution of errors in width and spacing. Conversely, they can be used to determine allowed tolerance limits to meet specific design criteria. Additionally, the formulation developed for the two-dimensional case of a strip grating can be expanded to the three-dimensional case of square or rectangular apertures in a PEC plane or its complement.

## **Recommendations**

Several areas exist where follow on work relating to this research effort can be investigated. The first area involves an extension of the analysis of the linear array of point sources. The applicability of this analysis to antenna array theory makes the addition

of random phase and amplitude errors a worthy effort. The additional effort in this area could be strengthened by extending the analysis to two-dimensional arrays.

Second, as mentioned above, the formulations presented for the average power pattern of a strip grating using the Born and edge condition approximations can be expanded to include square or rectangular apertures on a PEC plane (or its complement) as opposed to infinitely long slits and strips. The examinations of the quadruple summations found in the Appendices A, D, and E will be helpful in these areas.

A third area of possible additional effort involves the investigation into a purely analytical formulation. Diffraction from a strip grating with no errors has been investigated by many authors. Most of the analytical solutions have involved use of the Wiener-Hopf method. As outlined in Appendix F, several problems exist which make the use of the Wiener-Hopf method inapplicable to the problem of a grating with errors in slit and strip widths. Nevertheless, a concentrated effort to develop an analytical formulation, whether based upon the Wiener-Hopf method or some other method would be worthwhile.

## Appendix A. Variance of Linear Array of Point Sources

This appendix derives an analytical expression for the variance of the average power pattern of a linear array of point sources. To keep the number of terms at a manageable level and simplify the expressions, the variance is calculated for an array with an infinite number of point sources.

As given in Equation (2-2), the array factor for the array of point sources is given by:

$$f(u) = \sum_{n=0}^N e^{jkuX_n} \quad (A-1)$$

The variance of the average power pattern is given by:

$$\begin{aligned} \sigma^2 &= E \left\{ \left( \frac{1}{(N+1)} |f(u)|^2 \right)^2 \right\} - E^2 \left\{ \left( \frac{1}{(N+1)} |f(u)|^2 \right) \right\} \\ &= \frac{1}{(N+1)^2} E \left\{ |f(u)|^4 \right\} - \left[ \overline{|f(u)|^2} \right]^2 \end{aligned} \quad (A-2)$$

The second term is the square of the average power pattern, which for an infinite number of slits/ strips, is given in Equation (2-24):

$$\begin{aligned} \overline{|f(u)|^2}^2 &= 1 + 2 \operatorname{Re} \left\{ \frac{\Phi(u)}{1 - \Phi(u)} \right\} \\ &= \frac{1 - |\Phi(u)|^2}{|1 - \Phi(u)|^2} \end{aligned} \quad (A-3)$$

where  $\Phi$  is the characteristic function and

$$\left(\overline{[f(u)]^2}\right)^2 = 1 + 4\operatorname{Re}\left\{\frac{\Phi(u)}{1-\Phi(u)}\right\} + 4\left[\operatorname{Re}\left\{\frac{\Phi(u)}{1-\Phi(u)}\right\}\right]^2 \quad (\text{A-4})$$

The first term in Equation (A-2) is given by:

$$\begin{aligned} \frac{1}{(N+1)^2} E\left\{|f(u)|^4\right\} &= \frac{1}{(N+1)^2} E\left\{\sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N \sum_{n=0}^N e^{jkuX_p} e^{-jkuX_l} e^{jkuX_m} e^{-jkuX_n}\right\} \\ &= \frac{1}{(N+1)^2} E\left\{\sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N \sum_{n=0}^N e^{jku(X_p - X_l + X_m - X_n)}\right\} \end{aligned} \quad (\text{A-5})$$

The quadruple summation can be expressed as:

$$E\left\{\sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N \sum_{n=0}^N e^{jku(X_p - X_l + X_m - X_n)}\right\} = S_1 + S_2 + \cdots + S_{15} \quad (\text{A-6})$$

where

$$S_1 = N + 1 \quad [\text{for indices } p = l = m = n] \quad (\text{A-7})$$

$$S_2 = N(N+1) \quad [\text{for indices } (p = l) \neq (m = n)] \quad (\text{A-8})$$

The notation  $(p = l) \neq (m = n)$  refers to indices such that  $p = l$  and  $m = n$  but  $p \neq m$  or  $n$  and  $l \neq m$  or  $n$ . Continuing with the summations:

$$S_3 = N(N+1) \quad [\text{for indices } (p = n) \neq (l = m)] \quad (\text{A-9})$$

$$S_4 = E \left\{ \sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N \sum_{n=0}^N e^{jku(X_p - X_l + X_m - X_n)} \right\}, p \neq l \neq m \neq n \quad [\text{for indices } p \neq l \neq m \neq n] \quad (\text{A-10})$$

$$S_5 = (N-1)E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jku(X_m - X_n)} \right\}, n \neq m \quad [\text{for indices } (p=1) \neq m \neq n] \quad (\text{A-11})$$

The notation  $(p=1) \neq m \neq n$  refers to indices such that  $p=1$ ,  $m \neq n$ , and  $p \neq m$  or  $n$ .

$$S_6 = (N-1)E \left\{ \sum_{m=0}^N \sum_{l=0}^N e^{jku(X_m - X_l)} \right\}, m \neq l \quad [\text{for indices } (p=n) \neq m \neq l] \quad (\text{A-12})$$

$$S_7 = (N-1)E \left\{ \sum_{p=0}^N \sum_{l=0}^N e^{jku(X_m - X_l)} \right\}, p \neq l \quad [\text{for indices } (m=n) \neq p \neq l] \quad (\text{A-13})$$

$$S_8 = (N-1)E \left\{ \sum_{p=0}^N \sum_{n=0}^N e^{jku(X_p - X_n)} \right\}, p \neq n \quad [\text{for indices } (l=m) \neq p \neq n] \quad (\text{A-14})$$

$$S_9 = E \left\{ \sum_{m=0}^N \sum_{n=0}^N e^{jku2(X_m - X_n)} \right\}, m \neq n \quad [\text{for indices } (p=m) \neq (l=n)] \quad (\text{A-15})$$

$$S_{10} = E \left\{ \sum_{p=0}^N \sum_{l=0}^N \sum_{n=0}^N e^{jku(2X_p - X_l - X_n)} \right\}, p \neq l \neq n \quad [\text{for indices } (p=m) \neq l \neq n] \quad (\text{A-16})$$

$$S_{11} = E \left\{ \sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N e^{jku(X_p + X_m - 2X_l)} \right\}, p \neq l \neq m \quad [\text{for indices } (l=n) \neq p \neq m] \quad (\text{A-17})$$

$$S_{12} = E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jku(X_m - X_n)} \right\}, n \neq m \quad [\text{for indices } (p=1=m) \neq n] \quad (\text{A-18})$$

$$S_{13} = E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jku(X_m - X_n)} \right\}, n \neq m \quad [\text{for indices } (p = l = n) \neq m] \quad (\text{A-19})$$

$$S_{14} = E \left\{ \sum_{p=0}^N \sum_{l=0}^N e^{jku(X_p - X_l)} \right\}, p \neq l \quad [\text{for indices } (p = m = n) \neq l] \quad (\text{A-20})$$

$$S_{15} = E \left\{ \sum_{p=0}^N \sum_{l=0}^N e^{jku(X_p - X_l)} \right\}, p \neq l \quad [\text{for indices } (l = m = n) \neq p] \quad (\text{A-21})$$

Examining the above equations, the following are true:

$$S_5 = S_6 = S_7 = S_8 \quad (\text{A-22})$$

$$S_{11} = S_{10}^* \quad (\text{A-23})$$

and

$$S_{12} = S_{13} = S_{14} = S_{15} \quad (\text{A-24})$$

The double summations can be expressed as:

$$\begin{aligned} S_{12} &= E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jku(X_m - X_n)} \right\}, n \neq m = \left[ E \left\{ \sum_{n=0}^N \sum_{m=0}^N e^{jku(X_m - X_n)} \right\} \right] - (N+1) \\ &= 2(N+1) \text{Re} \left\{ \frac{\Phi(u) - \Phi(u)^{N+1}}{1 - \Phi(u)} \right\} - 2 \text{Re} \left\{ \frac{\Phi(u) + N\Phi(u)^{N+2} - (N+1)\Phi(u)^{N+1}}{[1 - \Phi(u)]^2} \right\} \end{aligned} \quad (\text{A-25})$$

For the case of  $N \rightarrow \infty$ , the following is true:

$$\lim_{N \rightarrow \infty} \frac{S_{12}}{(N+1)^2} = 0 \quad (\text{A-26})$$

and thus there are no contributions from  $S_{12}$ ,  $S_{13}$ ,  $S_{14}$ , and  $S_{15}$ . A similar argument shows that:

$$\lim_{N \rightarrow \infty} \frac{S_9}{(N+1)^2} = 0 \quad (\text{A-27})$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{S_5}{(N+1)^2} &= \lim_{N \rightarrow \infty} \frac{(N-1)S_{12}}{(N+1)^2} \\ &= \lim_{N \rightarrow \infty} \frac{2(N-1)(N+1)}{(N+1)^2} \operatorname{Re} \left\{ \frac{\Phi(u)}{1-\Phi(u)} \right\} \\ &= 2 \operatorname{Re} \left\{ \frac{\Phi(u)}{1-\Phi(u)} \right\} \end{aligned} \quad (\text{A-28})$$

Thus,

$$\lim_{N \rightarrow \infty} \frac{S_5 + S_6 + S_7 + S_8}{(N+1)^2} = 8 \operatorname{Re} \left\{ \frac{\Phi(u)}{1-\Phi(u)} \right\} \quad (\text{A-29})$$

Also for the infinite case:

$$\lim_{N \rightarrow \infty} \frac{S_1}{(N+1)^2} = 0 \quad (\text{A-30})$$

and

$$\lim_{N \rightarrow \infty} \frac{S_2}{(N+1)^2} = \frac{S_3}{(N+1)^2} = 1 \quad (\text{A-31})$$

Now consider  $S_4$ . It can be broken into 24 cases as follows:

Case 1:  $p > l > m > n$

Case 13:  $l > p > n > m$

Case 2:  $p > l > n > m$

Case 3:  $p > m > l > n$

Case 4:  $p > m > n > l$

Case 5:  $p > n > l > m$

Case 6:  $p > n > m > l$

Case 7:  $m > l > p > n$

Case 8:  $m > l > n > p$

Case 9:  $m > p > l > n$

Case 10:  $m > p > n > l$

Case 11:  $m > n > l > p$

Case 12:  $m > n > p > l$

Case 14:  $l > p > m > n$

Case 15:  $l > n > p > m$

Case 16:  $l > n > m > p$

Case 17:  $l > m > p > n$

Case 18:  $l > m > n > p$

Case 19:  $n > p > l > m$

Case 20:  $n > p > m > l$

Case 21:  $n > l > p > m$

Case 22:  $n > l > m > p$

Case 23:  $n > m > p > l$

Case 24:  $n > m > l > p$

First, consider case 1,  $p > l > m > n$ . The following relations are true:

$$X_n = X_0 + L_1 + L_2 + \cdots + L_n \quad (A-32)$$

$$X_m = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_m \quad (A-33)$$

$$X_l = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_m + L_{m+1} + \cdots + L_l \quad (A-34)$$

$$X_p = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_m + L_{m+1} + \cdots + L_l + L_{l+1} + \cdots + L_p \quad (A-35)$$

and

$$X_p - X_l + X_m - X_n = \underbrace{L_{l+1} + \cdots + L_p}_{p-l \text{ terms}} + \underbrace{L_{n+1} + \cdots + L_m}_{m-n \text{ terms}} \quad (A-36)$$

The summation becomes:

$$S_{4,1} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-1)+(m-n)} \quad (A-37)$$

The notation  $S_{4,1}$  signifies case 1 of summation 4. Next consider case 2,  $p > l > n > m$ .

For this case, the following are true:

$$X_m = X_0 + L_1 + L_2 + \cdots + L_m \quad (A-38)$$

$$X_n = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_n \quad (A-39)$$

$$X_l = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_n + L_{n+1} + \cdots + L_l \quad (A-40)$$

$$X_p = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_n + L_{n+1} + \cdots + L_l + L_{l+1} + \cdots + L_p \quad (A-41)$$

and

$$X_p - X_l + X_m - X_n = \underbrace{L_{l+1} + \cdots + L_p}_{p-l \text{ terms}} - \underbrace{(L_{m+1} + \cdots + L_n)}_{n-m \text{ terms}} \quad (A-42)$$

The summation becomes:

$$S_{4,2} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{n=1}^{l-1} \sum_{m=0}^{n-1} \Phi(u)^{(p-1)} [\Phi^*(u)]^{(n-m)} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-1)} [\Phi^*(u)]^{(m-n)} \quad (A-43)$$

Now consider case 3,  $p > m > l > n$ . For this case, the following are true:

$$X_n = X_0 + L_1 + L_2 + \cdots + L_n \quad (\text{A-44})$$

$$X_l = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_l \quad (\text{A-45})$$

$$X_m = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_l + L_{l+1} + \cdots + L_m \quad (\text{A-46})$$

$$X_p = X_0 + L_1 + L_2 + \cdots + L_n + L_{n+1} + \cdots + L_l + L_{l+1} + \cdots + L_m + L_{m+1} + \cdots + L_p \quad (\text{A-47})$$

and

$$X_p - X_l + X_m - X_n = \underbrace{2(L_{l+1} + \cdots + L_m)}_{m-l \text{ terms}} + \underbrace{(L_{n+1} + \cdots + L_l)}_{l-n \text{ terms}} + \underbrace{(L_{m+1} + \cdots + L_p)}_{p-m \text{ terms}} \quad (\text{A-48})$$

The summation becomes:

$$\begin{aligned} S_{4,3} &= \sum_{p=3}^N \sum_{m=2}^{p-1} \sum_{l=1}^{m-1} \sum_{n=0}^{l-1} \Phi(2u)^{(m-1)} \Phi(u)^{(p-m)} \Phi(u)^{(l-n)} \\ &= \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(2u)^{(l-m)} \Phi(u)^{(p-l+m-n)} \end{aligned} \quad (\text{A-49})$$

For case 4,  $p > m > n > l$ , the following are true:

$$X_l = X_0 + L_1 + L_2 + \cdots + L_l \quad (\text{A-50})$$

$$X_n = X_0 + L_1 + L_2 + \cdots + L_l + L_{l+1} + \cdots + L_n \quad (\text{A-51})$$

$$X_m = X_0 + L_1 + L_2 + \cdots + L_l + L_{l+1} + \cdots + L_1 + L_{l+1} + \cdots + L_m \quad (A-52)$$

$$X_p = X_0 + L_1 + L_2 + \cdots + L_l + L_{l+1} + \cdots + L_n + L_{n+1} + \cdots + L_m + L_{m+1} + \cdots + L_p \quad (A-53)$$

$$X_p - X_l + X_m - X_n = \underbrace{2(L_{n+1} + \cdots + L_m)}_{m-n \text{ terms}} + \underbrace{(L_{l+1} + \cdots + L_n)}_{n-l \text{ terms}} + \underbrace{(L_{m+1} + \cdots + L_p)}_{p-m \text{ terms}} \quad (A-54)$$

The summation becomes:

$$\begin{aligned} S_{4,4} &= \sum_{p=3}^N \sum_{m=2}^{p-1} \sum_{n=1}^{m-1} \sum_{l=0}^{n-1} \Phi(2u)^{(m-n)} \Phi(u)^{(n-l)} \Phi(u)^{(p-m)} \\ &= \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(2u)^{(l-m)} \Phi(u)^{(p-l+m-n)} \end{aligned} \quad (A-55)$$

which is the same as case 3, i.e.

$$S_{4,4} = S_{4,3} \quad (A-56)$$

For case 5,  $p > n > l > m$ , the following relations are true:

$$X_m = X_0 + L_1 + L_2 + \cdots + L_m \quad (A-57)$$

$$X_l = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_l \quad (A-58)$$

$$X_n = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_l + L_{l+1} + \cdots + L_n \quad (A-59)$$

$$X_p = X_0 + L_1 + L_2 + \cdots + L_m + L_{m+1} + \cdots + L_l + L_{l+1} + \cdots + L_n + L_{n+1} + \cdots + L_p \quad (A-60)$$

$$X_p - X_l + X_m - X_n = \underbrace{(L_{n+1} + \dots + L_k)}_{k-n \text{ terms}} - \underbrace{(L_{m+1} + \dots + L_1)}_{n-1 \text{ terms}} \quad (\text{A-61})$$

The summation becomes:

$$S_{4,5} = \sum_{p=3}^N \sum_{n=2}^{p-1} \sum_{l=1}^{n-1} \sum_{m=0}^{l-1} \Phi(u)^{(p-n)} [\Phi^*(u)]^{(l-m)} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-1)} [\Phi^*(u)]^{(m-n)} \quad (\text{A-62})$$

which is the same as case 2, i.e.

$$S_{4,5} = S_{4,2} \quad (\text{A-63})$$

For case 6,  $p > n > m > 1$ . The following relations are true:

$$X_l = X_0 + L_1 + L_2 + \dots + L_l \quad (\text{A-64})$$

$$X_m = X_0 + L_1 + L_2 + \dots + L_l + L_{l+1} + \dots + L_m \quad (\text{A-65})$$

$$X_n = X_0 + L_1 + L_2 + \dots + L_l + L_{l+1} + \dots + L_m + L_{m+1} + \dots + L_n \quad (\text{A-66})$$

$$X_p = X_0 + L_1 + L_2 + \dots + L_l + L_{l+1} + \dots + L_m + L_{m+1} + \dots + L_n + L_{n+1} + \dots + L_p \quad (\text{A-67})$$

and

$$X_p - X_l + X_m - X_n = \underbrace{L_{n+1} + \dots + L_p}_{p-n \text{ terms}} + \underbrace{L_{l+1} + \dots + L_m}_{m-l \text{ terms}} \quad (\text{A-68})$$

The summation becomes:

$$S_{4,6} = \sum_{p=3}^N \sum_{n=2}^{p-1} \sum_{m=1}^{n-1} \sum_{l=0}^{m-1} \Phi(u)^{(p-n)+(m-l)} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-l)+(m-n)} \quad (A-69)$$

which is the same as case 1, i.e.

$$S_{4,6} = S_{4,1} \quad (A-70)$$

By symmetry, the cases when "m" is the highest index, i.e. cases 7 - 12, give identical results. Similarly, the cases when "l" and "n" are the highest indices, i.e. cases 13 - 24, result in the complex conjugates of the above cases. The final result is:

$$S_4 = 8 \operatorname{Re} \left\{ \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-l)+(m-n)} + \Phi(u)^{(p-l)} [\Phi^*(u)]^{(m-n)} + \Phi(2u)^{(l-m)} \Phi(u)^{(p-l+m-n)} \right\} \quad (A-71)$$

Let

$$S_4 = 8 \operatorname{Re} \{ S_{4a} + S_{4b} + S_{4c} \} \quad (A-72)$$

where

$$S_{4a} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{p-l+m-n} \quad (A-73)$$

$$S_{4b} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(u)^{(p-l)} [\Phi^*(u)]^{(m-n)} \quad (A-74)$$

and

$$S_{4c} = \sum_{p=3}^N \sum_{l=2}^{p-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} \Phi(2u)^{(l-m)} \Phi(u)^{(p-l+m-n)} \quad (A-75)$$

The following relations will be utilized in examining these summations:

$$\sum_{m=0}^{n-1} t^{-m} = \sum_{m=0}^{n-1} \frac{1}{t^m} = \frac{1 - \frac{1}{t^n}}{1 - \frac{1}{t}} = \frac{t}{1-t} (t^{-n} - 1) \quad (A-76)$$

$$\sum_{m=1}^N t^m = t \sum_{m=0}^{N-1} t^m = \frac{t(1-t^N)}{1-t} \quad (A-77)$$

$$\sum_{m=2}^N t^m = \frac{t(1-t^N)}{1-t} - t = \frac{t^2}{1-t} (1-t^{N-1}) \quad (A-78)$$

Let

$$\begin{aligned} \Phi(u) &= A \\ \Phi(2u) &= B \end{aligned} \quad (A-79)$$

Returning to  $S_4$ ,  $S_{4a}$  can be written as:

$$\begin{aligned} S_{4a} &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} \sum_{m=1}^{l-1} A^m \sum_{n=0}^{m-1} A^{-n} = \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} \sum_{m=1}^{l-1} A^m s1 \\ &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} s2 = \sum_{p=3}^N A^p s3 \end{aligned} \quad (A-80)$$

where

$$s1 = \sum_{n=0}^{m-1} A^{-n} = \frac{A}{1-A} (A^{-m} - 1) \quad (A-81)$$

$$\begin{aligned}
s_2 &= \sum_{m=1}^{l-1} \frac{A}{1-A} (1-A^m) = \frac{A}{1-A} \left[ \sum_{m=0}^{l-1} 1 - \sum_{m=0}^{l-1} A^m \right] \\
&= \frac{Al}{1-A} - \frac{A}{(1-A)^2} + \frac{A^{l+1}}{(1-A)^2}
\end{aligned} \tag{A-82}$$

$$\begin{aligned}
s_3 &= \frac{A}{1-A} \sum_{l=2}^{p-1} lA^{-l} - \frac{A}{(1-A)^2} \sum_{l=2}^{p-1} A^{-l} + \frac{A}{(1-A)^2} \sum_{l=2}^{p-1} 1 \\
&= \frac{A}{1-A} \left\{ \sum_{l=1}^{p-1} lA^{-l} - \frac{1}{A} \right\} - \frac{A}{(1-A)^2} \left\{ \sum_{l=0}^{p-1} A^{-l} - 1 - \frac{1}{A} \right\} + \frac{A(p-2)}{(1-A)^2} \\
&= \frac{A^2 p}{(1-A)^2} \left( \frac{1}{A} \right)^p + \frac{Ap}{(1-A)^2} + \frac{2A^2}{(1-A)^3} - \frac{2}{(1-A)^3} \left( \frac{1}{A} \right)^{p-2}
\end{aligned} \tag{A-83}$$

and

$$\begin{aligned}
S_{4a} &= \frac{A^2}{(1-A)^2} \sum_{p=3}^N p + \frac{A}{(1-A)^2} \sum_{p=3}^N pA^p + \frac{2A^2}{(1-A)^3} \sum_{p=3}^N A^p - \frac{2A^2}{(1-A)^3} \sum_{p=3}^N 1 \\
&= \frac{A^2}{(1-A)^2} \left\{ \sum_{p=1}^N p - 1 - 2 \right\} + \frac{A}{(1-A)^2} \left\{ \sum_{p=1}^N pA^p - A - 2A^2 \right\} \\
&\quad + \frac{2A^2}{(1-A)^3} \left\{ \sum_{p=1}^N A^p - 1 - A^2 \right\} - \frac{2A^2(N-3)}{(1-A)^3}
\end{aligned} \tag{A-84}$$

or

$$\begin{aligned}
S_{4a} &= \frac{A^2}{(1-A)^2} \left[ \frac{N(N+1)}{2} - 3 \right] + \frac{A}{(1-A)^2} \left[ \frac{A + NA^{(N+2)} - (N+1)A^{(N+1)}}{(1-A)^2} - A - 2A^2 \right] \\
&\quad + \frac{2A^2}{(1-A)^3} \left[ \frac{A - A^{(N+1)}}{1-A} - 1 - A^2 \right] - \frac{2A^2(N-3)}{(1-A)^3}
\end{aligned} \tag{A-85}$$

For the infinite case,

$$\lim_{N \rightarrow \infty} \frac{S_{4a}}{(N+1)^2} = \frac{A^2}{2(1-A)^2} \quad (A-86)$$

Next,  $S_{4b}$  can be expressed as:

$$\begin{aligned} S_{4b} &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} \sum_{m=1}^{l-1} A^{*m} \sum_{n=0}^{m-1} A^{*-n} = \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} \sum_{m=1}^{l-1} A^{*m} s_1 \\ &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} A^{-l} s_2 = \sum_{p=3}^N A^p s_3 \end{aligned} \quad (A-87)$$

where

$$s_1 = \sum_{n=0}^{m-1} A^{*-n} = \frac{A^*}{1-A^*} (A^{*-m} - 1) \quad (A-88)$$

$$\begin{aligned} s_2 &= \sum_{m=1}^{l-1} \frac{A^*}{1-A^*} (1 - A^{*m}) = \frac{A^*}{1-A^*} \left[ \sum_{m=0}^{l-1} 1 - \sum_{m=0}^{l-1} A^{*m} \right] \\ &= \frac{A^* l}{1-A^*} - \frac{A^*}{(1-A^*)^2} + \frac{A^{*(l+1)}}{(1-A^*)^2} \end{aligned} \quad (A-89)$$

$$\begin{aligned} s_3 &= \frac{A^*}{1-A^*} \sum_{l=2}^{p-1} l A^{-l} - \frac{A^*}{(1-A^*)^2} \sum_{l=2}^{p-1} A^{-l} + \frac{A^*}{(1-A^*)^2} \sum_{l=2}^{p-1} 1 \\ &= \frac{A^*}{1-A^*} \left\{ \sum_{l=1}^{p-1} l A^{-l} - \frac{1}{A} \right\} - \frac{A^*}{(1-A^*)^2} \left\{ \sum_{l=0}^{p-1} A^{-l} - 1 - \frac{1}{A} \right\} + \frac{A^*(p-2)}{(1-A^*)^2} \\ &= \frac{AA^*p}{|1-A|^2} \left( \frac{1}{A} \right)^p + \frac{A^*p}{|1-A|^2} + \frac{2AA^*}{|1-A|^2(1-A)} - \frac{AA^*(2-A-A^*)}{|1-A|^4} \left( \frac{1}{A} \right)^p + \frac{A^*A^{-1}(A^*-A)}{|1-A|^4} \end{aligned} \quad (A-90)$$

and

$$S_{4b} = \frac{AA^*}{|1-A|^2} \sum_{p=3}^N p + \frac{A^*}{|1-A|^2} \sum_{p=3}^N pA^p + \left[ \frac{2AA^*}{|1-A|^2(1-A)} + \frac{A^*A^{-1}(A^*-A)}{|1-A|^4} \right] \sum_{p=3}^N A^p - \frac{AA^*(2-A-A^*)}{|1-A|^4} \sum_{p=3}^N 1 \quad (A91)$$

which becomes:

$$S_{4b} = \frac{AA^*}{|1-A|^2} \left\{ \sum_{p=1}^N p - 1 - 2 \right\} + \frac{A^*}{|1-A|^2} \left\{ \sum_{p=1}^N pA^p - A - 2A^2 \right\} + \left[ \frac{2AA^*}{|1-A|^2(1-A)} + \frac{A^*A^{-1}(A^*-A)}{|1-A|^4} \right] \left\{ \sum_{p=1}^N A^p - 1 - A^2 \right\} - \frac{AA^*(2-A-A^*)(N-3)}{|1-A|^4} \quad (A-92)$$

or

$$S_{4b} = \frac{AA^*}{|1-A|^2} \left[ \frac{N(N+1)}{2} - 3 \right] + \frac{A^*}{|1-A|^2} \left[ \frac{A + NA^{(N+2)} - (N+1)A^{(N+1)}}{(1-A)^2} - A - 2A^2 \right] + \left[ \frac{2AA^*}{|1-A|^2(1-A)} + \frac{A^*A^{-1}(A^*-A)}{|1-A|^4} \right] \left[ \frac{A - A^{(N+1)}}{1-A} - 1 - A^2 \right] - \frac{AA^*(2-A-A^*)(N-3)}{|1-A|^4} \quad (A-93)$$

For the infinite case,

$$\lim_{N \rightarrow \infty} \frac{S_{4b}}{(N+1)^2} = \frac{AA^*}{2|1-A|^2} \quad (A-94)$$

Referring to Equations (A-84) and (A-91), the only contribution as  $N \rightarrow \infty$  is from terms involving  $\sum_p$ . Next,  $S_{4c}$  can be expressed as:

$$\begin{aligned} S_{4c} &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} \left(\frac{B}{A}\right)^l \sum_{m=1}^{l-1} \left(\frac{B}{A}\right)^{-m} \sum_{n=0}^{m-1} A^{-n} = \sum_{p=3}^N A^p \sum_{l=2}^{p-1} \left(\frac{B}{A}\right)^l \sum_{m=1}^{l-1} \left(\frac{B}{A}\right)^{-m} s1 \\ &= \sum_{p=3}^N A^p \sum_{l=2}^{p-1} \left(\frac{B}{A}\right)^l s2 = \sum_{p=3}^N A^p s3 \end{aligned} \quad (A-95)$$

where

$$s1 = \sum_{n=0}^{m-1} A^{-n} = \frac{A}{1-A} (A^{-m} - 1) \quad (A-96)$$

$$\begin{aligned} s2 &= \sum_{m=1}^{l-1} \left(\frac{B}{A}\right)^{-m} \frac{A}{1-A} (A^{-m} - 1) = \frac{A}{1-A} \left[ \sum_{m=0}^{l-1} B^{-m} - \sum_{m=0}^{l-1} \left(\frac{B}{A}\right)^{-m} \right] \\ &= \frac{A}{1-A} \left\{ \frac{B}{1-B} (B^{-1} - 1) - \frac{B/A}{1-B/A} \left[ \left(\frac{B}{A}\right)^{-1} - 1 \right] \right\} \end{aligned} \quad (A-97)$$

$$\begin{aligned} s3 &= \frac{A}{1-A} \sum_{l=2}^{p-1} \left(\frac{B}{A}\right)^l \left\{ \frac{B}{1-B} (B^{-1} - 1) - \frac{B/A}{1-B/A} \left[ \left(\frac{B}{A}\right)^{-1} - 1 \right] \right\} \\ &= \frac{A}{1-A} \sum_{l=2}^{p-1} \left\{ \frac{B}{1-B} \left[ \left(\frac{1}{A}\right)^l - \left(\frac{B}{A}\right)^l \right] - \frac{B/A}{1-B/A} \left[ 1 - \left(\frac{B}{A}\right)^l \right] \right\} \end{aligned} \quad (A-98)$$

or

$$\begin{aligned} s3 &= \frac{A}{1-A} \frac{B}{1-B} \left\{ \frac{\left(\frac{1}{A}\right)^2}{1-\frac{1}{A}} \left[ 1 - \left(\frac{1}{A}\right)^{p-2} \right] - \frac{\left(\frac{B}{A}\right)^2}{1-\frac{B}{A}} \left[ 1 - \left(\frac{B}{A}\right)^{p-2} \right] \right\} \\ &\quad - \frac{A}{1-A} \frac{B/A}{1-B/A} \left\{ p-2 - \frac{\left(\frac{B}{A}\right)^2}{1-\frac{B}{A}} \left[ 1 - \left(\frac{B}{A}\right)^{p-2} \right] \right\} \end{aligned} \quad (A-99)$$

which becomes:

$$s_3 = \frac{A \left[ \left( \frac{B}{A} \right)^2 - \left( \frac{B}{A} \right)^p \right]}{(1-A)(1-B/A)} \left[ \frac{B/A}{1-B/A} - \frac{B}{1-B} \right] + \frac{AB}{(1-A)(1-B)(1-1/A)} \left[ \left( \frac{1}{A} \right)^2 - \left( \frac{1}{A} \right)^p \right] - \frac{B(p-2)}{(1-A)(1-B/A)} \quad (A-100)$$

and

$$s_4 = \frac{A}{(1-A)(1-B/A)} \left[ \frac{B/A}{1-B/A} - \frac{B}{1-B} \right] \left\{ \left( \frac{B}{A} \right)^2 \sum_{p=3}^N A^p - \sum_{p=3}^N B^p \right\} + \frac{AB}{(1-A)(1-B)(1-1/A)} \left\{ \left( \frac{1}{A} \right)^2 \sum_{p=3}^N A^p - \sum_{p=3}^N 1 \right\} - \frac{B}{(1-A)(1-B/A)} \left\{ \sum_{p=3}^N p A^p - 2 \sum_{p=3}^N A^p \right\} \quad (A-101)$$

While these summations can be calculated, the above expression does not contain a term of the form  $\sum_p p$  and:

$$\lim_{N \rightarrow \infty} \frac{S_{4c}}{(N+1)^2} = 0 \quad (A-102)$$

Thus,

$$\lim_{N \rightarrow \infty} \frac{S_4}{(N+1)^2} = \lim_{N \rightarrow \infty} 8 \operatorname{Re} \frac{S_{4a} + S_{4b} + S_{4c}}{(N+1)^2} = \frac{4A^2}{(1-A)^2} + \frac{4AA^*}{4|1-A|^2} \quad (A-103)$$

Now consider  $S_{10}$ . It can be broken into six cases as follows:

Case 1:  $p > l > m$

Case 4:  $p > m > l$

Case 2:  $l > p > m$

Case 5:  $m > p > l$

Case 3:  $l > m > p$

Case 6:  $m > l > p$

First, consider case 1,  $p > l > m$ . The following relations are true:

$$2X_p = 2(X_0 + L_1 + \dots + L_m) + 2(L_{m+1} + \dots + L_l) + 2(L_{l+1} + \dots + L_p) \quad (\text{A-104})$$

$$X_l = X_0 + L_1 + \dots + L_m + L_{m+1} + \dots + L_l \quad (\text{A-105})$$

$$X_m = X_0 + L_1 + L_2 + \dots + L_m \quad (\text{A-106})$$

and

$$2X_p - X_l - X_m = \underbrace{2(L_{l+1} + \dots + L_p)}_{p-l \text{ terms}} + \underbrace{(L_{m+1} + \dots + L_l)}_{l-m \text{ terms}} \quad (\text{A-107})$$

The summation becomes:

$$\begin{aligned} S_{10,1} &= \sum_{p=2}^N \sum_{l=1}^{p-1} \sum_{m=0}^{l-1} A^{l-m} B^{p-1} = \sum_{p=2}^N B^p \sum_{l=1}^{p-1} \left(\frac{A}{B}\right)^l \sum_{m=0}^{l-1} A^{-m} \\ &= \sum_{p=2}^N B^p \sum_{l=1}^{p-1} \left(\frac{A}{B}\right) s_1 = \sum_{p=2}^N B^p s_2 \end{aligned} \quad (\text{A-108})$$

where

$$s_1 = \sum_{m=0}^{l-1} A^{-m} = \frac{A}{1-A} (A^{-1} - 1) \quad (\text{A-109})$$

$$\begin{aligned} s_2 &= \sum_{l=1}^{p-1} \left(\frac{B}{A}\right)^{-1} \frac{A}{1-A} (A^{-1} - 1) = \frac{A}{1-A} \left[ \sum_{l=1}^{p-1} B^{-1} - \sum_{l=1}^{p-1} \left(\frac{B}{A}\right)^{-1} \right] \\ &= \frac{A}{1-A} \left\{ \frac{B}{1-B} (B^{-p} - 1) - \frac{B/A}{1-B/A} \left[ (B/A)^{-p} - 1 \right] \right\} \end{aligned} \quad (\text{A-110})$$

and

$$\begin{aligned}
S_{10,1} &= \frac{A}{1-A} \sum_{p=2}^N B^p \left\{ \frac{B}{1-B} (B^{-p} - 1) - \frac{B/A}{1-B/A} \left[ \left( \frac{B}{A} \right)^{-p} - 1 \right] \right\} \\
&= \frac{A}{1-A} \sum_{p=2}^N \left\{ \frac{B}{1-B} [1 - B^p] - \frac{B/A}{1-B/A} [A^p - B^p] \right\}
\end{aligned} \tag{A-111}$$

or

$$\begin{aligned}
S_{10,1} &= \frac{A}{1-A} \frac{B}{1-B} \left[ (N-1) - \frac{B^2}{1-B} (1 - B^{N-1}) \right] \\
&\quad - \frac{A}{1-A} \frac{B/A}{1-B/A} \left[ \frac{A^2}{1-A} (1 - A^{N-1}) - \frac{B^2}{1-B} (1 - B^{N-1}) \right]
\end{aligned} \tag{A-112}$$

which becomes

$$\begin{aligned}
S_{10,1} &= \frac{AB(N-1)}{(1-A)(1-B)} + \frac{AB^2(1-B^{N-1})}{(1-A)(1-B)} \left( \frac{B}{A-B} - \frac{A}{1-A} \right) \\
&\quad - \frac{A^3B(1-A^{N-1})}{(1-A)^2(A-B)}
\end{aligned} \tag{A-113}$$

and

$$\lim_{N \rightarrow \infty} \frac{S_{10,1}}{(N+1)^2} = 0 \tag{A-114}$$

Now, consider case 2,  $1 > p > m$ . The following relations are true:

$$2X_p = 2(X_0 + L_1 + \dots + L_m) + 2(L_{m+1} + \dots + L_p) \tag{A-115}$$

$$X_1 = X_0 + L_1 + \dots + L_m + L_{m+1} + \dots + L_p + L_{p+1} + \dots + L_1 \tag{A-116}$$

$$X_m = X_0 + L_1 + L_2 + \dots + L_m \tag{A-117}$$

and

$$2X_p - X_l - X_m = \underbrace{L_{m+1} + \dots + L_p}_{p-m \text{ terms}} - \underbrace{(L_{p+1} + \dots + L_l)}_{l-p \text{ terms}} \quad (\text{A-118})$$

The summation becomes:

$$\begin{aligned} S_{10,2} &= \sum_{l=2}^N \sum_{p=1}^{l-1} \sum_{m=0}^{p-1} A^{p-m} A^{*(l-p)} = \sum_{l=2}^N A^{*l} \sum_{p=1}^{l-1} \left( \frac{A}{A^*} \right)^p \sum_{m=0}^{p-1} A^{-m} \\ &= \sum_{l=2}^N A^{*l} \sum_{p=1}^{l-1} \left( \frac{A}{A^*} \right)^p s1 = \sum_{l=2}^N A^{*l} s2 \end{aligned} \quad (\text{A-119})$$

where

$$s1 = \sum_{m=0}^{p-1} A^{-m} = \frac{A}{1-A} (A^{-p} - 1) \quad (\text{A-120})$$

$$\begin{aligned} s2 &= \sum_{p=1}^{l-1} \left( \frac{A}{A^*} \right)^p \frac{A}{1-A} (A^{-p} - 1) = \frac{A}{1-A} \left[ \sum_{p=1}^{l-1} \left( \frac{1}{A^*} \right)^p - \sum_{p=1}^{l-1} \left( \frac{A}{A^*} \right)^p \right] \\ &= \frac{A}{1-A} \left\{ \frac{A^*}{1-A^*} (A^{*-1} - 1) - \frac{A^*/A}{1-A^*/A} \left[ \left( A^*/A \right)^{-1} - 1 \right] \right\} \end{aligned} \quad (\text{A-121})$$

and

$$\begin{aligned} S_{10,2} &= \frac{A}{1-A} \sum_{l=2}^N A^{*l} \left\{ \frac{A^*}{1-A^*} (A^{*-1} - 1) - \frac{A^*/A}{1-A^*/A} \left[ \left( A^*/A \right)^{-1} - 1 \right] \right\} \\ &= \frac{A}{1-A} \sum_{l=2}^N \left\{ \frac{A^*}{1-A^*} [1 - A^{*l}] - \frac{A^*/A}{1-A^*/A} [A^l - A^{*l}] \right\} \end{aligned} \quad (\text{A-122})$$

or

$$S_{10,2} = \frac{A}{1-A} \frac{A^*}{1-A^*} \left[ (N-1) - \frac{A^{*2}}{1-A^*} (1-A^{*N-1}) \right] - \frac{A}{1-A} \frac{A^*/A}{1-A^*/A} \left[ \frac{A^2}{1-A} (1-A^{N-1}) - \frac{A^{*2}}{1-A^*} (1-A^{*N-1}) \right] \quad (A-123)$$

which becomes

$$S_{10,2} = \frac{AA^*(N-1)}{(1-A)(1-A^*)} + \frac{AA^{*3}(1-A^{*N-1})}{(1-A)(1-A^*)} \left( \frac{1}{A-A^*} - \frac{1}{1-A^*} \right) - \frac{A^3A^*(1-A^{N-1})}{(1-A)^2(A-A^*)} \quad (A-124)$$

and

$$\lim_{N \rightarrow \infty} \frac{S_{10,2}}{(N+1)^2} = 0 \quad (A-125)$$

Next consider case 3,  $l > m > k$ . The following relations are true:

$$2X_p = 2(X_0 + L_1 + \dots + L_p) \quad (A-126)$$

$$X_l = X_0 + L_1 + \dots + L_p + L_{p+1} + \dots + L_m + L_{m+1} + \dots + L_l \quad (A-127)$$

$$X_m = X_0 + L_1 + L_2 + \dots + L_p + L_{p+1} + \dots + L_m \quad (A-128)$$

and

$$2X_p - X_l - X_m = - \underbrace{2(L_{p+1} + \dots + L_m)}_{m-p \text{ terms}} - \underbrace{(L_{m+1} + \dots + L_l)}_{l-m \text{ terms}} \quad (A-129)$$

The summation becomes:

$$\begin{aligned}
 S_{10,3} &= \sum_{l=2}^N \sum_{m=1}^{l-1} \sum_{p=0}^{m-1} A^{*(l-m)} B^{*(m-p)} = \sum_{l=2}^N A^{*l} \sum_{m=1}^{l-1} \left( \frac{B^*}{A^*} \right)^m \sum_{p=0}^{m-1} B^{*-p} \\
 &= \sum_{l=2}^N A^{*l} \sum_{m=1}^{l-1} \left( \frac{B^*}{A^*} \right)^m s1 = \sum_{l=2}^N A^{*l} s2
 \end{aligned} \tag{A-130}$$

where

$$s1 = \sum_{p=0}^{m-1} B^{*-p} = \frac{B^*}{1-B^*} (B^{*-m} - 1) \tag{A-131}$$

$$\begin{aligned}
 s2 &= \sum_{m=1}^{l-1} \left( \frac{B^*}{A^*} \right)^m \frac{B^*}{1-B^*} (B^{*-m} - 1) = \frac{B^*}{1-B^*} \left[ \sum_{m=1}^{l-1} \left( \frac{1}{A^*} \right)^m - \sum_{m=1}^{l-1} \left( \frac{B^*}{A^*} \right)^m \right] \\
 &= \frac{B^*}{1-B^*} \left\{ \frac{A^*}{1-A^*} (A^{*-1} - 1) - \frac{A^*/B^*}{1-A^*/B^*} \left[ \left( A^*/B^* \right)^{-1} - 1 \right] \right\}
 \end{aligned} \tag{A-132}$$

and

$$\begin{aligned}
 S_{10,3} &= \frac{B^*}{1-B^*} \sum_{l=2}^N A^{*l} \left\{ \frac{A^*}{1-A^*} (A^{*-1} - 1) - \frac{A^*/B^*}{1-A^*/B^*} \left[ \left( A^*/B^* \right)^{-1} - 1 \right] \right\} \\
 &= \frac{B^*}{1-B^*} \sum_{l=2}^N \left\{ \frac{A^*}{1-A^*} [1 - A^{*1}] - \frac{A^*/B^*}{1-A^*/B^*} [B^{*1} - A^{*1}] \right\}
 \end{aligned} \tag{A-133}$$

or

$$\begin{aligned}
 S_{10,3} &= \frac{B^*}{1-B^*} \frac{A^*}{1-A^*} \left[ (N-1) - \frac{A^{*2}}{1-A^*} (1 - A^{*N-1}) \right] \\
 &\quad - \frac{B^*}{1-B^*} \frac{A^*/B^*}{1-A^*/B^*} \left[ \frac{B^{*2}}{1-B^*} (1 - A^{*N-1}) - \frac{A^{*2}}{1-A^*} (1 - A^{*N-1}) \right]
 \end{aligned} \tag{A-134}$$

which becomes

$$S_{10,3} = \frac{A^* B^* (N-1)}{(1-A^*)(1-B^*)} + \frac{B^* A^{*2} (1-A^{*N-1})}{(1-A^*)(1-B^*)} \left( \frac{A^*}{B^* - A^*} - \frac{A^*}{1-A^*} \right) - \frac{B^{*3} A^* (1-A^{N-1})}{(1-B^*)^2 (B^* - A^*)} \quad (A-135)$$

and

$$\lim_{N \rightarrow \infty} \frac{S_{10,3}}{(N+1)^2} = 0 \quad (A-136)$$

Thus

$$\lim_{N \rightarrow \infty} \frac{S_{10}}{(N+1)^2} = 0 \quad (A-137)$$

and there are no contributions from  $S_{10}$ . Since  $S_{11}$  is the complex conjugate of  $S_{10}$ ,

$$\lim_{N \rightarrow \infty} \frac{S_{11}}{(N+1)^2} = 0 \quad (A-138)$$

Collecting the non-zero terms,

$$\frac{1}{(N+1)^2} E\left\{|f(u)|^4\right\} = 2 + 4 \operatorname{Re} \frac{\Phi^2}{(1-\Phi)^2} + 4 \operatorname{Re} \frac{\Phi \Phi}{(1-\Phi)(1-\Phi^*)} + 8 \operatorname{Re} \frac{\Phi}{1-\Phi} \quad (A-139)$$

The variance is given by:

$$\begin{aligned}
\sigma^2 &= \frac{1}{(N+1)^2} E\left\{|f(u)|^4\right\} - \left[\overline{|f(u)|^2}\right]^2 \\
&= 2 + 4\operatorname{Re} \frac{\Phi^2}{(1-\Phi)^2} + 4\operatorname{Re} \frac{\Phi\Phi^*}{(1-\Phi)(1-\Phi^*)} + 8\operatorname{Re} \frac{\Phi}{1-\Phi} \\
&\quad - \left\{1 + 4\operatorname{Re}\left\{\frac{\Phi}{1-\Phi}\right\} + 4\left[\operatorname{Re}\left\{\frac{\Phi}{1-\Phi}\right\}\right]^2\right\}
\end{aligned} \tag{A-140}$$

or

$$\sigma^2 = 1 + 4\operatorname{Re} \frac{\Phi^2}{(1-\Phi)^2} + 4\operatorname{Re} \frac{\Phi\Phi^*}{(1-\Phi)(1-\Phi^*)} + 4\operatorname{Re} \frac{\Phi}{1-\Phi} - 4\left[\operatorname{Re}\left\{\frac{\Phi}{1-\Phi}\right\}\right]^2 \tag{A-141}$$

Simplifying,

$$\begin{aligned}
\sigma^2 &= 1 + 4\operatorname{Re} \frac{\Phi^2 - \Phi^2\Phi^* + \Phi\Phi^* - \Phi^2\Phi^* + \Phi - \Phi^2 - \Phi\Phi^* + \Phi^2\Phi^*}{(1-\Phi)^2(1-\Phi^*)} \\
&\quad - 4\left[\operatorname{Re}\left\{\frac{\Phi}{1-\Phi}\right\}\right]^2 \\
&= 1 + 4\operatorname{Re} \frac{\Phi(1-\Phi\Phi^*)}{(1-\Phi)^2(1-\Phi^*)} - 4\left[\operatorname{Re}\left\{\frac{\Phi}{1-\Phi}\right\}\right]^2
\end{aligned} \tag{A-142}$$

Now,

$$\begin{aligned}
4\operatorname{Re} \frac{\Phi(1-\Phi\Phi^*)}{(1-\Phi)^2(1-\Phi^*)} &= 4\operatorname{Re} \frac{\Phi(1-\Phi\Phi^*)(1-\Phi^*)}{(1-\Phi)^2(1-\Phi^*)^2} = 4\operatorname{Re} \frac{\Phi(1-\Phi\Phi^*) - \Phi\Phi^* + \Phi\Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} \\
&= 4\operatorname{Re} \frac{\frac{1}{2}(\Phi + \Phi^*)(1-\Phi\Phi^*) - \Phi\Phi^* + \Phi\Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} \\
&= \frac{2\Phi + 2\Phi^* - 2\Phi^2\Phi^* - 2\Phi\Phi^{*2} - 4\Phi\Phi^* + 4\Phi^2\Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2}
\end{aligned} \tag{A-143}$$

and

$$\operatorname{Re} \frac{\Phi}{1-\Phi} = \operatorname{Re} \frac{\Phi(1-\Phi^*)}{(1-\Phi)(1-\Phi^*)} = \frac{\frac{1}{2}(\Phi + \Phi^*) - \Phi\Phi^*}{(1-\Phi)(1-\Phi^*)} \quad (\text{A-144})$$

which leads to:

$$\begin{aligned} 4 \left[ \operatorname{Re} \frac{\Phi}{1-\Phi} \right]^2 &= 4 \left[ \frac{\frac{1}{2}(\Phi + \Phi^*) - \Phi\Phi^*}{(1-\Phi)(1-\Phi^*)} \right]^2 \\ &= \frac{\Phi^2 + \Phi^{*2} + 2\Phi\Phi^* - 4\Phi^2\Phi^* - 4\Phi\Phi^{*2} + 4\Phi^2\Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} \end{aligned} \quad (\text{A-145})$$

Substituting Equations (A-143) and (A-145) into Equation (A-142) yields:

$$\begin{aligned} \sigma^2 &= 1 + \frac{2\Phi + 2\Phi^* + 2\Phi^2\Phi^* + 2\Phi\Phi^{*2} - 6\Phi\Phi^* - \Phi^2 - \Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} \\ &= \frac{(1-\Phi)^2(1-\Phi^*)^2 + 2\Phi + 2\Phi^* + 2\Phi^2\Phi^* + 2\Phi\Phi^{*2} - 6\Phi\Phi^* - \Phi^2 - \Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} \end{aligned} \quad (\text{A-146})$$

or

$$\begin{aligned} \sigma^2 &= \frac{1 - 2\Phi\Phi^* + \Phi^2\Phi^{*2}}{(1-\Phi)^2(1-\Phi^*)^2} = \frac{(1-\Phi\Phi^*)^2}{[(1-\Phi)(1-\Phi^*)]^2} \\ &= \left[ \frac{1 - |\Phi|^2}{1 - \Phi|^2} \right]^2 = \left[ \overline{|f(u)|^2} \right]^2 \end{aligned} \quad (\text{A-147})$$

Thus the variance is equal to the square of the average power pattern and the standard deviation is equal to the average power pattern, i.e.

$$\sigma = \overline{|f(u)|}^2 \quad (\text{A-148})$$

Equations (A-147) and (A-148) are valid for  $u \neq 0$ . For the case of  $u = 0$ ,

$$\begin{aligned} \frac{1}{(N+1)^2} E\{|f(0)|^4\} &= \frac{1}{(N+1)^2} E\left\{\sum_{p=0}^N \sum_{l=0}^N \sum_{m=0}^N \sum_{n=0}^N 1\right\} \\ &= \frac{1}{(N+1)^2} E\{(N+1)^4\} = (N+1)^2 \end{aligned} \quad (\text{A-149})$$

$$\overline{|f(0)|}^2 = \frac{1}{N+1} E\left\{\sum_{n=0}^N \sum_{m=0}^N 1\right\} = \frac{1}{N+1} (N+1)^2 = N+1 \quad (\text{A-150})$$

and

$$\begin{aligned} \sigma^2 &= \frac{1}{(N+1)^2} E\{|f(0)|^4\} - \left[\overline{|f(0)|^2}\right]^2 \\ &= (N+1)^2 - (N+1)^2 \\ &= 0, \text{ for } u = 0 \end{aligned} \quad (\text{A-151})$$

## Appendix B. Modal Formulation for Infinite Grating with No Errors

### Conversion of Integral Equation to Modal Equation

Setting the origin at the mid point of the first slit or strip, i.e.  $1/2 (X_0 + \tau_0) = 0$ , the scattered fields for a grating with no errors are given by (Equation (3-51)):

$$\begin{aligned} S &= \frac{k}{2} \sum_{n=0}^{\infty} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') e^{jnkT \sin \theta_i} H_0^{(2)}(k\sqrt{(x-x''-nT)^2 + y^2}) dx'' \\ &= \frac{k}{2} \sum_{n=0}^{\infty} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \left[ \frac{1}{2\pi} \int e^{j\omega x} \mathcal{F} \left\{ e^{jnkT \sin \theta_i} H_0^{(2)}(k\sqrt{(x-x''-nT)^2 + y^2}) \right\} d\omega \right] dx'' \end{aligned} \quad (B-1)$$

where

$$\mathcal{F}\{f(x)\} = \int f(x) e^{-j\omega x} dx \quad (B-2)$$

and

$$f(x) = \frac{1}{2\pi} \int \mathcal{F}\{f(x)\} e^{j\omega x} d\omega \quad (B-3)$$

are the Fourier and inverse Fourier transform relationships. Using the Fourier Transform (2:58):

$$\mathcal{F}\left\{H_0^{(2)}(k\sqrt{x^2 + y^2})\right\} = \frac{2e^{-j|y|\sqrt{k^2 - \omega^2}}}{\sqrt{k^2 - \omega^2}} \quad (B-4)$$

the scattered fields become:

$$\begin{aligned}
S &= \frac{k}{2} \sum_{n=0}^{\infty} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \left\{ \frac{1}{2\pi} \int e^{j\omega x} \left[ e^{jnkT \sin \theta_i} \frac{2e^{-j|y|\sqrt{k^2 - \omega^2}} e^{-j\omega(x'' + nT)}}{\sqrt{k^2 - \omega^2}} \right] d\omega dx'' \right\} \\
&= \frac{k}{2\pi} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \int \frac{e^{j\omega(x-x'')} e^{-j|y|\sqrt{k^2 - \omega^2}}}{\sqrt{k^2 - \omega^2}} \sum_{n=0}^{\infty} e^{jnkT \sin \theta_i} e^{-j\omega nT} d\omega dx''
\end{aligned} \tag{B-5}$$

or

$$\begin{aligned}
S &= \frac{k}{2\pi} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \int \frac{e^{j\omega(x-x'')} e^{-j|y|\sqrt{k^2 - \omega^2}}}{\sqrt{k^2 - \omega^2}} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \left(k \sin \theta_i + \frac{2\pi n}{T}\right)\right) d\omega dx'' \\
&= \frac{k}{T} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \int \frac{e^{j\omega(x-x'')} e^{-j|y|\sqrt{k^2 - \omega^2}}}{\sqrt{k^2 - \omega^2}} \sum_{n=-\infty}^{\infty} \delta(\omega - \beta_n) d\omega dx''
\end{aligned} \tag{B-6}$$

where

$$\beta_n = k \sin \theta_i + \frac{2\pi n}{T} \tag{B-7}$$

Using the sifting property of the delta function, the scattered fields become:

$$\begin{aligned}
S &= \frac{k}{T} \sum_{n=-\infty}^{\infty} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \frac{e^{j\beta_n(x-x'')} e^{-j|y|\sqrt{k^2 - \beta_n^2}}}{\sqrt{k^2 - \beta_n^2}} dx'' \\
&= \frac{k}{T} \sum_{n=-\infty}^{\infty} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') \frac{e^{j\beta_n(x-x'')} e^{-j|y|\rho_n}}{\rho_n} dx''
\end{aligned} \tag{B-8}$$

where

$$\rho_n = \sqrt{k^2 - \beta_n^2} \tag{B-9}$$

Equation (B-8) can be rewritten as:

$$S = \begin{cases} \sum_n B_n e^{j(\beta_n x - \rho_n y)} & \text{for } y > 0 \\ \sum_n B_n e^{j(\beta_n x + \rho_n y)} & \text{for } y < 0 \end{cases} \quad (\text{B-10})$$

where

$$B_n = \frac{k}{\rho_n T} \int_{-\frac{w}{2}}^{\frac{w}{2}} I_s(x'') e^{-j\beta_n x''} dx'' \quad (\text{B-11})$$

Equations (B-10) and (B-11) are equivalent to modal formulations found in many references ((11:189-194) for example). For the TE<sub>z</sub> mode, coefficients of reflection (R) and transmission (Γ) are given by:

$$\begin{aligned} R &= 1 - B_0 \\ \Gamma &= B_0 \end{aligned} \quad (\text{B-12})$$

and for the TM<sub>z</sub> mode:

$$\begin{aligned} R &= -B_0 \\ \Gamma &= 1 - B_0 \end{aligned} \quad (\text{B-13})$$

### Born Approximation

Using the Born approximation as given in Equation (4-9):

$$I_s(x'') = \cos \theta_i e^{jkx'' \sin \theta_i} \quad (\text{B-14})$$

then,

$$\begin{aligned}
B_n &= \frac{k \cos \theta_i}{\rho_n T} \int_{-\frac{w}{2}}^{\frac{w}{2}} e^{j(k \sin \theta_i - \beta_n)x''} dx'' \\
&= \frac{k \cos \theta_i}{\rho_n T} \sum_n \int_{-\frac{w}{2}}^{\frac{w}{2}} e^{-j\frac{2\pi n}{T}x''} dx'' \\
&= \frac{wk \cos \theta_i}{\rho_n T} \frac{\sin\left(\frac{n\pi w}{T}\right)}{\frac{n\pi w}{T}}
\end{aligned} \tag{B-15}$$

with

$$B_0 = \frac{w}{T} \tag{B-16}$$

### Edge Condition Approximation

Using the edge condition approximation as given in Equation (5-3):

$$I_{sn}(x'') = \frac{C e^{jkx' \sin \theta_i}}{\left[ \left( \frac{w}{2} \right)^2 - (x'' - nT)^2 \right]^{1/2}} \tag{B-17}$$

then

$$\begin{aligned}
B_n &= \frac{k}{\rho_n T} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{C e^{jkx'' \sin \theta_i}}{\left[ \left( \frac{w}{2} \right)^2 - (x'')^2 \right]^{1/2}} e^{-j\beta_n x''} dx'' \\
&= \frac{Ck}{\rho_n T} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{e^{j\frac{2\pi n}{T}x''}}{\left[ \left( \frac{w}{2} \right)^2 - (x'')^2 \right]^{1/2}} dx'' \\
&= \frac{Ck}{\rho_n T} \pi J_0\left(\frac{n\pi w}{T}\right)
\end{aligned} \tag{B-18}$$

To determine the value for the constant "C", use the integral equation relating the boundary conditions:

$$\frac{2}{k} e^{jkx \sin \theta_i} = S|_{y=0} = \sum_n B_n e^{-j\beta_n x} \quad (B-19)$$

Multiply both sides by  $I_s^*$  and integrate over  $-w/2$  to  $w/2$  to obtain:

$$\begin{aligned} C \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx}{\left[ \left( \frac{w}{2} \right)^2 - (x)^2 \right]^{1/2}} &= \frac{Ck\pi}{T} \sum_n \frac{J_0\left(\frac{n\pi w}{T}\right)}{\rho_n} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{C e^{jkx'' \sin \theta_i} e^{-j\beta_n x''}}{\left[ \left( \frac{w}{2} \right)^2 - (x'')^2 \right]^{1/2}} dx'' \\ &= \frac{Ck\pi}{T} \sum_n \frac{J_0\left(\frac{n\pi w}{T}\right)}{\rho_n} C\pi J_0\left(\frac{n\pi w}{T}\right) \end{aligned} \quad (B-20)$$

or

$$\begin{aligned} C\pi &= \frac{C^2 k \pi^2}{T} \sum_n \frac{1}{\rho_n} \left[ J_0\left(\frac{n\pi w}{T}\right) \right]^2 \\ C &= \left\{ \frac{k\pi}{T} \sum_n \frac{1}{\rho_n} \left[ J_0\left(\frac{n\pi w}{T}\right) \right]^2 \right\}^{-1} \end{aligned} \quad (B-21)$$

## Appendix C. Simplification of Integral Used in Edge Condition Approximation

This appendix simplifies the following integral used in the edge condition approximation:

$$I = \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{e^{j\Delta x} dx}{\left[\left(\frac{w}{2}\right)^2 - x^2\right]^{1/2}} \quad (C-1)$$

Let

$$u = \frac{2x}{w} \Rightarrow \begin{cases} x = \frac{wu}{2} \\ dx = \frac{w}{2} du \end{cases} \quad (C-2)$$

Then

$$I = \int_{-1}^1 \frac{e^{j\frac{\Delta w u}{2}} \frac{w}{2} du}{\left[\left(\frac{w}{2}\right)^2 - \left(\frac{wu}{2}\right)^2\right]^{1/2}} = \int_{-1}^1 \frac{e^{j\frac{\Delta w u}{2}} \frac{w}{2} du}{(1 - u^2)^{1/2}} \quad (C-3)$$

Now, let

$$v = \cos^{-1} u \Rightarrow \begin{cases} u = \cos v \\ du = -\sin v dv \\ (1 - u^2)^{1/2} = \sin v \end{cases} \quad (C-4)$$

then

$$\begin{aligned}
 I &= \int_0^{\pi} \frac{e^{j\frac{\Delta w \cos v}{2}} \sin v \, dv}{\sin v} \\
 &= \int_0^{\pi} e^{j\frac{\Delta w \cos v}{2}} \, dv \\
 &= \pi J_0\left(\frac{\Delta w}{2}\right)
 \end{aligned}
 \tag{C-5}$$

## Appendix D. Variance of the Average Power Pattern of a Strip Grating - Born Approximation

This appendix derives an analytical expression for the variance of the average power pattern of a strip grating computed using the Born approximation. As in the case of the array of point sources, the variance is calculated for a grating with an infinite number of slits/ strips to keep the number of terms at a manageable level and simplify the expressions.

The variance of the average power pattern is given by:

$$\begin{aligned}\sigma^2 &= E\left\{\left(\frac{1}{(N+1)}|S(u)|^2\right)^2\right\} - E^2\left\{\left(\frac{1}{(N+1)}|S(u)|^2\right)\right\} \\ &= \frac{1}{(N+1)^2} E\{|S(u)|^4\} - \left[\overline{|S(u)|^2}\right]^2\end{aligned}\quad (D-1)$$

The second term is the square of the average power pattern, which for an infinite number of slits/ strips and using the Born approximation , is given in Equation (4-35):

$$\begin{aligned}\overline{|S(u)|^2} &= I_0 \frac{2}{(kuW)^2} \left[ 1 - \text{Re}\{\Theta\} + \text{Re}\left\{\left[\frac{(\Phi\Theta)}{1-(\Phi\Theta)}\right] \left[2 - \Theta - \frac{1}{\Theta}\right]\right\} \right] \\ &= I_0 \frac{4}{(kuW)^2} \left[ E\left\{\sin^2\left(\frac{kuw}{2}\right)\right\} + 2\text{Re}E^2\left\{e^{\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi}{1-(\Phi\Theta)} \right]\end{aligned}\quad (D-2)$$

The first term in Equation (D-1) is given by:

$$\begin{aligned}
\frac{1}{(N+1)^2} E\{|S(u)|^4\} &= \frac{1}{(N+1)^2} E\left\{\left|\frac{\sqrt{I_0}}{kuW} \sum_{n=0}^N (e^{jku\tau_n} - e^{jkuX_n})\right|^4\right\} \\
&= \frac{1}{(N+1)^2} \frac{16I_0^2}{(kuW)^4} E\left\{\left|\sum_{n=0}^N e^{jkuX_n} e^{j\frac{kuw_n}{2}} \sin\left(\frac{kuw_n}{2}\right)\right|^4\right\}
\end{aligned} \tag{D-3}$$

Following the same procedure used in Appendix A for the array of point sources, this becomes for the case of  $N \rightarrow \infty$ :

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{1}{(N+1)^2} E\{|S(u)|^4\} &= \frac{16I_0^2}{(kuW)^4} \left[ 2E^2\left\{\sin^2\left(\frac{kuw}{2}\right)\right\} \right. \\
&\quad + 4\operatorname{Re} E^4\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi^2}{(1-\Theta\Phi)^2} \\
&\quad + 4\operatorname{Re} E^2\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} E^2\left\{e^{-j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi\Phi^*}{|1-\Theta\Phi|^2} \\
&\quad \left. + 8\operatorname{Re} E^2\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} E\left\{\sin^2\left(\frac{kuw}{2}\right)\right\} \frac{\Phi}{(1-\Theta\Phi)} \right]
\end{aligned} \tag{D-4}$$

Substituting Equations (D-2) and (D-4) into Equation (D-1), the variance becomes:

$$\begin{aligned}
\sigma^2 &= \frac{16I_0^2}{(kuW)^4} \left[ E^2\left\{\sin^2\left(\frac{kuw}{2}\right)\right\} + 4\operatorname{Re} E^4\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi^2}{(1-\Theta\Phi)^2} \right. \\
&\quad + 4\operatorname{Re} E^2\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} E^2\left\{e^{-j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi\Phi^*}{|1-\Theta\Phi|^2} \\
&\quad + 4\operatorname{Re} E^2\left\{e^{j\frac{kuw_n}{2}} \sin\left(\frac{kuw_n}{2}\right)\right\} E\left\{\sin^2\left(\frac{kuw_n}{2}\right)\right\} \frac{\Phi}{(1-\Theta\Phi)} \\
&\quad \left. - 4\left[\operatorname{Re} E^2\left\{e^{j\frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right)\right\} \frac{\Phi}{1-(\Phi\Theta)}\right]^2 \right]
\end{aligned} \tag{D-5}$$

To simplify, let:

$$\alpha = E^2 \left\{ e^{j \frac{kuw}{2}} \sin\left(\frac{kuw}{2}\right) \right\} \quad (D-6)$$

The second and third terms of Equation (D-5) become:

$$\begin{aligned} 4 \operatorname{Re} \left[ \frac{\alpha^2 \Phi^2}{(1 - \Theta \Phi)^2} + \frac{\alpha \alpha^* \Phi \Phi^*}{|1 - \Theta \Phi|^2} \right] &= 4 \operatorname{Re} \frac{\alpha^2 \Phi^2 (1 - \Theta^* \Phi^*) + \alpha \alpha^* \Phi \Phi^* (1 - \Theta \Phi) (1 - \Theta^* \Phi^*)}{(1 - \Theta \Phi)^2 (1 - \Theta^* \Phi^*)} \\ &= 4 \frac{\alpha \Phi (1 - \Theta^* \Phi^*) + \alpha^* \Phi^* (1 - \Theta \Phi)}{|1 - \Theta \Phi|^4} \operatorname{Re} (\alpha \Phi - \alpha \Theta^* \Phi \Phi^*) \\ &= 2 \frac{\alpha \Phi (1 - \Theta^* \Phi^*) + \alpha^* \Phi^* (1 - \Theta \Phi)}{|1 - \Theta \Phi|^4} \left[ (\alpha \Phi + \alpha^* \Phi^*) - (\alpha \Theta^* + \alpha^* \Theta) \Phi \Phi^* \right] \\ &= 2 \frac{[\alpha \Phi (1 - \Theta^* \Phi^*) + \alpha^* \Phi^* (1 - \Theta \Phi)]^2}{|1 - \Theta \Phi|^4} \end{aligned} \quad (D-7)$$

The fifth term of Equation (D-5) can be expressed as:

$$\begin{aligned} 4 \left[ \operatorname{Re} \frac{\alpha \Phi}{1 - \Phi \Theta} \frac{1 - \Phi^* \Theta^*}{1 - \Phi^* \Theta^*} \right]^2 &= 4 \left[ \frac{(\alpha \Phi + \alpha^* \Phi^*) - (\alpha \Theta^* + \alpha^* \Theta) \Phi \Phi^*}{2 |1 - \Phi \Theta|^2} \right]^2 \\ &= \frac{[\alpha \Phi (1 - \Theta^* \Phi^*) + \alpha^* \Phi^* (1 - \Theta \Phi)]^2}{|1 - \Theta \Phi|^4} \end{aligned} \quad (D-8)$$

Substituting Equations (D-7) and (D-8) into Equation (D-5), the variance becomes:

$$\sigma^2 = \frac{16I_0^2}{(kuW)^4} \left[ E^2 \left\{ \sin^2 \left( \frac{kuw}{2} \right) \right\} + 4 \operatorname{Re} E^2 \left\{ e^{j \frac{kuw_n}{2}} \sin \left( \frac{kuw_n}{2} \right) \right\} E \left\{ \sin^2 \left( \frac{kuw_n}{2} \right) \right\} \frac{\Phi}{(1 - \Theta \Phi)} \right. \\ \left. + \frac{[\alpha \Phi (1 - \Theta^* \Phi^*) + \alpha^* \Phi^* (1 - \Theta \Phi)]^2}{|1 - \Theta \Phi|^4} \right] \quad (D-9)$$

or after substituting Equation (D-8) for the last term:

$$\sigma^2 = \frac{16I_0^2}{(kuW)^4} \left[ E^2 \left\{ \sin^2 \left( \frac{kuw}{2} \right) \right\} + 4 \operatorname{Re} E^2 \left\{ e^{j \frac{kuw_n}{2}} \sin \left( \frac{kuw_n}{2} \right) \right\} E \left\{ \sin^2 \left( \frac{kuw_n}{2} \right) \right\} \frac{\Phi}{(1 - \Theta \Phi)} \right. \\ \left. + 4 \left[ \operatorname{Re} E^2 \left\{ e^{j \frac{kuw}{2}} \sin \left( \frac{kuw}{2} \right) \right\} \frac{\Phi}{1 - (\Phi \Theta)} \right]^2 \right] = [S(u)]^2 \quad (D-10)$$

and the standard deviation is equal to the mean, i.e.

$$\sigma = \overline{[S(u)]^2} \quad (D-11)$$

The above equations are valid for  $u \neq 0$ . Substituting  $u = 0$  into Equation (D-3) yields:

$$\frac{1}{(N+1)^2} E \{ |S(0)|^4 \} = \frac{I_0^2}{W^4 (N+1)^2} \left[ (N+1) E \{ w^4 \} + 4N(N+1) E \{ w^3 \} E \{ w \} \right. \\ \left. + 6N(N+1)(N-1) E^2 \{ w \} E \{ w^2 \} + 3N(N+1) E^2 \{ w^2 \} \right. \\ \left. + N(N+1)(N-1)(N-2) E^4 \{ w \} \right] \quad (D-12)$$

and from Equation (4-33):

$$\overline{[S(0)]^2} = \frac{I_0}{W^2} \left[ E \{ w^2 \} + N E^2 \{ w \} \right] \quad (D-13)$$

Sustitution of the last two equations into Equation (D-1) yields:

$$\begin{aligned} \sigma(0)^2 = \frac{I_0^2}{W^4} & \left[ \frac{2N-4N^2}{N+1} E^4\{w\} + \frac{2N-1}{N+1} E^2\{w^2\} + \frac{1}{N+1} E\{w^4\} \right. \\ & \left. + \frac{4N^2-8N}{N+1} E^2\{w\}E\{w^2\} + \frac{4N}{N+1} E\{w^3\}E\{w\} \right] \end{aligned} \quad (D-14)$$

which goes to infinity as  $N \rightarrow \infty$ . On the other hand:

$$\lim_{N \rightarrow \infty} \frac{\sigma(0)^2}{\left[ \overline{|S(0)|^2} \right]^2} = 0 \Rightarrow \lim_{N \rightarrow \infty} \frac{\sigma(0)}{\overline{|S(0)|}} = 0 \quad (D-15)$$

## Appendix E. Variance of the Average Power Pattern of a Strip Grating - Edge Condition Approximation

This appendix derives an analytical expression for the variance of the average power pattern of a strip grating computed using the Edge condition approximation. As in the cases of the array of point sources and the grating using the Born approximation, the variance is calculated for a grating with an infinite number of slits/ strips to keep the number of terms at a manageable level and simplify the expressions.

As given in Equation (D-1) The variance of the average power pattern is given by:

$$\begin{aligned}\sigma^2 &= E\left\{\left(\frac{1}{(N+1)}|S(u)|^2\right)^2\right\} - E^2\left\{\left(\frac{1}{(N+1)}|S(u)|^2\right)\right\} \\ &= \frac{1}{(N+1)^2} E\{|S(u)|^4\} - \left[\overline{|S(u)|^2}\right]^2\end{aligned}\quad (E-1)$$

The second term is the square of the average power pattern, which for an infinite number of slits/ strips and using the edge condition approximation, is given in Equation (5-32):

$$\overline{|S(u)|^2} = I_0 \left[ E\left\{\left[J_0\left(\frac{kuw}{2}\right)\right]^2\right\} + 2\text{Re}E^2\left\{e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right)\right\} \frac{\Phi}{1-(\Phi\Theta)} \right] \quad (E-2)$$

The first term in Equation (E-1) is given by:

$$\begin{aligned}\frac{1}{(N+1)^2} E\{|S(u)|^4\} &= \frac{1}{(N+1)^2} E\left\{\left|\sqrt{I_0} \sum_{n=0}^N e^{j\frac{ku}{2}(X_n+\tau_n)} J_0\left(\frac{kuw_n}{2}\right)\right|^4\right\} \\ &= \frac{I_0^2}{(N+1)^2} E\left\{\left|\sum_{n=0}^N e^{jkuX_n} e^{j\frac{kuw_n}{2}} J_0\left(\frac{kuw_n}{2}\right)\right|^4\right\}\end{aligned}\quad (E-3)$$

Following the same procedures used in Appendix A for the array of point sources and Appendix D for the grating using the Born approximation, this becomes for the case of  $N \rightarrow \infty$ :

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{1}{(N+1)^2} E\{|S(u)|^4\} = I_0^2 & \left[ 2E^2 \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} \right. \\
& + 4 \operatorname{Re} E^4 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi^2}{(1-\Theta\Phi)^2} \\
& + 4 \operatorname{Re} E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} E^2 \left\{ e^{-j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi\Phi^*}{|1-\Theta\Phi|^2} \\
& \left. + 8 \operatorname{Re} E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} E \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} \frac{\Phi}{(1-\Theta\Phi)} \right] \quad (E-4)
\end{aligned}$$

Substituting Equations (E-2) and (E-4) into Equation (E-1), the variance becomes:

$$\begin{aligned}
\sigma^2 = I_0^2 & \left[ E^2 \left\{ J_0^2\left(\frac{kuw}{2}\right) \right\} + 4 \operatorname{Re} E^4 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi^2}{(1-\Theta\Phi)^2} \right. \\
& + 4 \operatorname{Re} E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} E^2 \left\{ e^{-j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi\Phi^*}{|1-\Theta\Phi|^2} \\
& + 4 \operatorname{Re} E^2 \left\{ e^{j\frac{kuw_n}{2}} J_0\left(\frac{kuw_n}{2}\right) \right\} E \left\{ \left[ J_0\left(\frac{kuw_n}{2}\right) \right]^2 \right\} \frac{\Phi}{(1-\Theta\Phi)} \\
& \left. - 4 \left[ \operatorname{Re} E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi}{1-(\Phi\Theta)} \right]^2 \right] \quad (E-5)
\end{aligned}$$

To simplify, let:

$$\alpha = E^2 \left\{ e^{j\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \quad (E-6)$$

The second and third terms of Equation (E-5) become, using Equation (D-7):

$$4\text{Re}\left[\frac{\alpha^2\Phi^2}{(1-\Theta\Phi)^2} + \frac{\alpha\alpha^*\Phi\Phi^*}{|1-\Theta\Phi|^2}\right] = 2\frac{[\alpha\Phi(1-\Theta^*\Phi^*) + \alpha^*\Phi^*(1-\Theta\Phi)]^2}{|1-\Theta\Phi|^4} \quad (\text{E-7})$$

The fifth term of Equation (E-5) can be expressed as, using Equation (D-8):

$$4\left[\text{Re}\frac{\alpha\Phi}{1-\Phi\Theta}\frac{1-\Phi^*\Theta^*}{1-\Phi^*\Theta^*}\right]^2 = \frac{[\alpha\Phi(1-\Theta^*\Phi^*) + \alpha^*\Phi^*(1-\Theta\Phi)]^2}{|1-\Theta\Phi|^4} \quad (\text{E-8})$$

Substituting Equations (E-7) and (E-8) into Equation (E-5), the variance becomes:

$$\begin{aligned} \sigma^2 = I_0^2 & \left[ E^2 \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} + 4\text{Re} E^2 \left\{ e^{j\frac{kuw_n}{2}} J_0\left(\frac{kuw_n}{2}\right) \right\} E \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} \frac{\Phi}{(1-\Theta\Phi)} \right. \\ & \left. + \frac{[\alpha\Phi(1-\Theta^*\Phi^*) + \alpha^*\Phi^*(1-\Theta\Phi)]^2}{|1-\Theta\Phi|^4} \right] \quad (\text{E-9}) \end{aligned}$$

or after substituting Equation (E-8) for the last term:

$$\begin{aligned} \sigma^2 = I_0^2 & \left[ E^2 \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} + 4\text{Re} E^2 \left\{ e^{j\frac{kuw_n}{2}} J_0\left(\frac{kuw_n}{2}\right) \right\} E \left\{ \left[ J_0\left(\frac{kuw}{2}\right) \right]^2 \right\} \frac{\Phi}{(1-\Theta\Phi)} \right. \\ & \left. + 4 \left[ \text{Re} E^2 \left\{ e^{\frac{kuw}{2}} J_0\left(\frac{kuw}{2}\right) \right\} \frac{\Phi}{1-(\Phi\Theta)} \right]^2 \right] = [S(u)]^2 \quad (\text{E-10}) \end{aligned}$$

and the standard deviation is equal to the mean, i.e.

$$\sigma = \overline{|S(u)|}^2 \quad (\text{E-11})$$

The above equations are valid for  $u \neq 0$ . Substituting  $u = 0$  into Equation (E-3) yields:

$$\frac{1}{(N+1)^2} E\{|S(0)|^4\} = I_0^2 (N+1)^2 \quad (\text{E-12})$$

and from Equation (5-30):

$$\overline{|S(0)|}^2 = I_0 (N+1) \quad (\text{E-13})$$

Substituting the last two equations into Equation (E-1) yields:

$$\sigma(0)^2 = 0 \quad (\text{E-14})$$

## **Appendix F. Problems with Finding an Analytical Solution**

As stated in Chapter I, the problem of diffraction by a strip grating has been investigated by many authors. The pioneering works in solving this problem analytically was accomplished by Baldwin and Heins (5) and Weinstein (29) using the Wiener-Hopf method. Several problems exist which make use of the Wiener-Hopf approach inapplicable to the problem of a strip grating with random errors in strip width and spacing. First, the mathematical formulation of the problem of diffraction by a grating containing an infinite number of strips and slits results in an infinite set of integral equations. The periodic property of the solution allows the infinite set to be replaced by a single integral equation valid over one period. Once the solution over one period is obtained, the solution to other periods is obtained through phase adjustment (Floquet's theorem). Second, an analytical solution using the Wiener-Hopf method is possible only when the width of the strips is equal to the width of the apertures. As a result, use of the Wiener-Hopf approach is inapplicable to the problem of finding an analytical solution to the problem of diffraction from a grating with random strip and slit widths.

## Bibliography

1. Z.S. Agronovich *et al.* "The Diffraction of Electromagnetic Waves from Plane Metallic Lattices," *Soviet Physics - Technical Physics*, 7: 277-286 (1962)
2. Harry Bateman. *Tables of Integrals and Transforms* (vol. I). New York: McGraw-Hill, 1954.
3. Bevan Baker and E. T. Copson. *The Mathematical Theory of Huygen's Principle* (Second Edition). London: Clarendon Press, 1950.
4. Constantine A. Balanis. *Advanced Engineering Electromagnetics*. New York: John Wiley and Sons, 1989.
5. George L. Baldwin and Albert E. Heins. "On the Diffraction of a Plane Wave by an Infinite Plane Grating," *Math Scand*, 2: 103-118 (1954).
6. Max Born and Emil Wolf. *Principles of Optics* (Fourth Edition). New York: Pergamon Press, 1970.
7. Laurent Cairo and Theo Kahan. *Variational Techniques in Electromagnetism*. New York: Gordon and Breach, 1965.
8. Chao-Chun Chen. "Transmission Through a Conducting Screen Perforated Periodically with Apertures" *IEEE Transactions on Microwave Theory and Techniques*, MTT-18: 627-632 (Sep 1970).
9. Vito Daniele *et al.* "Diffraction of a Plane Wave by a Strip Grating," *Electromagnetics*, 10: 245-269 (1990).
10. Eugene Hecht. *Optics*. Reading, Mass: Addison-Wesley, 1990.
11. Akira Ishimaru. *Electromagnetic Wave Propagation, Radiation, and Scattering*. New Jersey: Prentice Hall, 1991.
12. John D. Jackson. *Classical Electrodynamics* (Second Edition). New York: John Wiley and Sons, 1975.

13. R. B. Kieburtz and A. Ishimaru. "Scattering by a Periodically Apertured Conducting Screen" *IRE Transactions on Antennas and Propagation*, AP-9: 506-514 (Nov 1961).
14. Yuen T. Lo. "Aperiodic Arrays," *Antenna Handbook*, edited by Yuen T. Lo and Shung-Wu Lee. New York: Van Nostrand Reinhold Company, 1988.
15. Yuen T. Lo. "Array Theory," *Antenna Handbook*, edited by Yuen T. Lo and Shung-Wu Lee. New York: Van Nostrand Reinhold Company, 1988.
16. V. D. Luk'yanov. "Exact Solution of the Problem of Diffraction of an Obliquely Incident Plane Wave at a Grating," *Soviet Physics Doklady*, 25: 905-906 (Nov 1980).
17. E. Luneberg and K. Westpfahl. "Diffraction of Plane Waves by an Infinite Strip Grating," *Annalen der Physik*, 7. Folge, Band 27, Heft 3: 257-288 (1971).
18. John W. Miles. "The Diffraction of a Plane Wave Through a Grating," *Quarterly of Applied Mathematics*, VII:45-64 (Jan 1949).
19. R. Mittra and S. W. Lee. *Analytical Techniques in the Theory of Guided Waves*. New York: Collier-Macmillan Limited, 1971.
20. Philip M. Morse and Herman Feshbach. *Methods of Theoretical Physics (part II)*. New York: McGraw-Hill Book Company, 1953.
21. Athanasios Papoulis. *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill Book Company, 1965.
22. Vittal P. Pyati. Class Handout Distributed in EENG 726, Advanced Electromagnetics II. School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH, Aug 1993.
23. John Ruze. "Antenna Tolerance Theory - A Review" *Proceedings of the IEEE*, 54: 633-640 (April 1966).

24. J. Paul Skinner. Class Handout Distributed in EENG 899, Special Study on Frequency Selective Surfaces. School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH, Jan 1994.
25. Merrill I. Skolnik. "Nonuniform Arrays," *Antenna Theory (part 1)*, edited by Robert E. Collin and Francis J. Zucker. New York: McGraw-Hill, 1969.
26. Bernard D. Steinberg. *Principles of Aperture and Array System Design*. New York: John Wiley and Sons, 1976.
27. R. L. Stratonovich. *Topics in the Theory of Random Noise (volume 1)*, New York: Gordon and Breach, 1963,
28. Glenn F. Vanblaricum, Jr. and Raj Mittra. "A Modified Residue-Calculus Technique for Solving a Class of Boundary Value Problems - Parts I and II," *IEEE Transactions on Microwave Theory and Techniques*, MTT-17: 302-319 (Jun 1969).
29. Lev Albertovich Weinstein. *The Theory of Diffraction and the Factorization Method*. Boulder: Golem Press, 1969.
30. C. P. Wu. "Variational and Iterative Methods for Waveguides and Arrays," *Computer Techniques for Electromagnetics*, edited by R. Mittra. New York: Springer-Verlag, 1987.

## **Vita**

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