

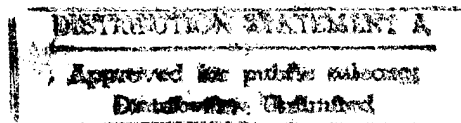
On the Jacobi Polynomials $P_n^{(\alpha, -k)}$

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It is well known that Jacobi polynomials $P_n^{(\alpha,\beta)}$ form complete orthogonal systems in the weighted space $L_{\alpha,\beta}^2[-1,1]$ whenever $\alpha, \beta > -1$. On the other hand, in certain physical applications, (for example, in the angular momentum calculations in quantum mechanics), there naturally occur polynomials $P_n^{(\alpha,-k)}$ with integer α and k . We show that for any $\alpha > -1$ and integer $k \geq 1$ the Jacobi polynomials $P_n^{(\alpha,-k)}$ ($n = k, k+1, \dots$) form complete orthogonal systems in $L_{\alpha,-k}^2[-1,1]$ with the weight $w_{\alpha,-k}(x) = (1-x)^\alpha \cdot (1+x)^{-k}$. In addition, for $\alpha \geq 0$ and $n \geq 1$ we obtain an upper bound on $[-1,1]$ for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha,-1)}(x)$, which is similar to the well known bound for the function $((1-x)/2)^{\alpha/2+1/4} \cdot P_n^{(\alpha,0)}(x)$.

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1 Introduction

Jacobi polynomials $P_n^{(\alpha,\beta)}$ play an important role in pure and applied mathematics, numerical analysis, physics, and engineering. It is well known that all other classical polynomials orthogonal on $[-1, 1]$ with appropriately chosen weights (Legendre, Chebyshev, Gegenbauer), as well as certain combinations of elementary functions, are particular cases of Jacobi polynomials (see, for example, Askey [2]).

Normally one assumes that $\alpha, \beta > -1$, in part due to the well known facts (see, for example Chapt. 3 and 4 of Szegő [6], Lecture 2 of Askey [2]) summarized in Theorems 1.1 and 1.2 below. In Theorem 1.2 and elsewhere in the paper $L_{\alpha,\beta}^2[-1, 1]$ denotes the weighted space with the weight function $w_{\alpha,\beta} : [-1, 1] \rightarrow \mathbb{R}$ defined by the formula

$$w_{\alpha,\beta}(x) \stackrel{\text{def}}{=} (1-x)^\alpha(1+x)^\beta. \quad (1)$$

Theorem 1.1. *For any integer $n, m \geq 0$ and arbitrary real $\alpha, \beta \geq -1$,*

$$\int_{-1}^1 w_{\alpha,\beta}(x) \cdot P_n^{(\alpha,\beta)}(x) \cdot P_m^{(\alpha,\beta)}(x) dx = \delta_{nm} h_{\alpha,\beta}^n, \quad (2)$$

where δ_{nm} is Kronecker's delta, and

$$h_{\alpha,\beta}^n \stackrel{\text{def}}{=} \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}. \bullet \quad (3)$$

Theorem 1.2. *Let $\alpha, \beta > -1$ and suppose, that for an arbitrary function $f \in L_{\alpha,\beta}^2[-1, 1]$ the coefficients f_n are defined by the formula*

$$f_n \stackrel{\text{def}}{=} \int_{-1}^1 w_{\alpha,\beta}(x) P_n^{(\alpha,\beta)}(x) f(x) dx. \quad (4)$$

Then in $L_{\alpha,\beta}^2[-1, 1]$,

$$f = \sum_{n=0}^{\infty} \frac{f_n}{h_{\alpha,\beta}^n} P_n^{(\alpha,\beta)}. \bullet \quad (5)$$

On the other hand, polynomials $P_n^{(\alpha,-k)}$ (for integer α and k) occur naturally in certain physical applications (see, for example, Chapt. 3 and 5 of Biedenharn and Louck [3] and references herein), which stimulates interest in this type of Jacobi polynomials.

In this paper we generalize Theorems 1.1 and 1.2 for the case of negative integer β . Namely we show that for all $k \geq 1$ and $\alpha > -1$ any sequence $\{P_n^{(\alpha, -k)}\}$ ($n = k, k+1, \dots$) forms an orthogonal complete system in $L^2_{\alpha, -k}[-1, 1]$ (see Theorems 1.4 and 1.5 below). This property is a consequence of the observation that the polynomials $\{P_n^{(\alpha, -k)}\}$ for all $n \geq k$ have a zero of exactly k -th order at $x = -1$ (see Theorem 1.3 and Corollary 1.1 below), so that the relevant inner products involving $w_{\alpha, -k}$ and $P_n^{(\alpha, -k)}$ (i. e. analogues of integrals (2) and (4)) exist.

In addition, we obtain a uniform upper bound for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha, -1)}(x)$ on $[-1, 1]$ (see Theorem 1.6 below). This bound is similar to the well known Szegő's bound for the function $((1-x)/2)^{\alpha/2+1/4} \cdot P_n^{(\alpha, 0)}(x)$ (see formula (23) below). Upper bounds for Jacobi polynomials are of significant interest in certain applications (see, for example, Chap. 7 of Szegő [6], Nevai, Erdélyi, and Magnus [5], Elbert and Laforgia [4], and references herein). Note that most upper bounds for the polynomials $P_n^{(\alpha, \beta)}$ have been derived for the case $\alpha, \beta \geq -1/2$.

The plan and main results of the paper are as follows.

Section 2 contains relevant mathematical facts to be used in the remainder of the paper.

In Section 3 we establish a formula connecting the polynomials $P_n^{(\alpha, -k)}$ and $P_{n-k}^{(\alpha, k)}$, and prove the completeness of the system $\{P_n^{(\alpha, -k)}\}$. The main results of this section are Theorems 1.3, 1.4 and 1.5, and Corollary 1.1 below.

Theorem 1.3. *For any integer n and k such that $n \geq k$, and arbitrary real $\alpha > -1$,*

$$P_n^{(\alpha, -k)}(x) = \frac{\Gamma(n + \alpha + 1)\Gamma(n - k + 1)}{\Gamma(n - k + \alpha + 1)\Gamma(n + 1)} \cdot \left(\frac{1+x}{2}\right)^k \cdot P_{n-k}^{(\alpha, k)}(x). \bullet \quad (6)$$

Corollary 1.1. *For all $\alpha > -1$ and $k \leq n$, $P_n^{(\alpha, -k)}$ has the zero of k -th order at $x = -1$. The remaining $n - k$ zeroes of $P_n^{(\alpha, -k)}$ are located in the interior of the interval $[-1, 1]$. •*

Remark 1.1. The formula (6) for integer α is known and widely used in the rotation group computations in quantum mechanics (see, for example, Chap. 3 of Biedenharn and Louck [3]) •.

Remark 1.2. Combining (6) with the formula (21) below we can rewrite the relation

(6) in the form

$$P_n^{(\alpha, -k)}(x) = \left(\frac{1+x}{2}\right)^k \cdot \frac{P_n^{(\alpha, k)}(1)}{P_{n-k}^{(\alpha, k)}(1)} \cdot P_{n-k}^{(\alpha, k)}(x). \bullet \quad (7)$$

Theorem 1.4. For any integer k, n , and m such that $n, m \geq k$, and arbitrary $\alpha > -1$,

$$\int_{-1}^1 w_{\alpha, -k}(x) \cdot P_n^{(\alpha, -k)}(x) \cdot P_m^{(\alpha, -k)}(x) = \delta_{nm} h_{\alpha, -k}^n, \quad (8)$$

where the function $w_{\alpha, \beta}$ is defined in (1), and

$$h_{\alpha, -k}^n = h_{\alpha, \beta}^n \Big|_{\beta=-k} = \frac{2^{\alpha-k+1}}{2n + \alpha - k + 1} \cdot \frac{\Gamma(n + \alpha + 1)(n - k)!}{\Gamma(n - k + \alpha + 1)n!}. \bullet \quad (9)$$

Theorem 1.5. Let n and k be integers such that $n \geq k$, and suppose that $\alpha > -1$. Suppose further, that for an arbitrary function $f \in L_{\alpha, -k}^2[-1, 1]$ the coefficients f_n are defined by the formula

$$f_n \stackrel{\text{def}}{=} \int_{-1}^1 w_{\alpha, -k}(x) P_n^{(\alpha, -k)}(x) f(x) dx. \quad (10)$$

Then in $L_{\alpha, -k}^2[-1, 1]$,

$$f = \sum_{n=k}^{\infty} \frac{f_n}{h_{\alpha, -k}^n} P_n^{(\alpha, -k)}. \bullet \quad (11)$$

In Section 4 we obtain an upper bound for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha, -1)}(x)$ on $[-1, 1]$. The main result of this section is Theorem 1.6 below.

Theorem 1.6. For all $\alpha \geq 0$ and integer $n \geq 1$,

$$\max_{x \in [-1, 1]} \left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha, -1)}(x)| \leq 8 \left(\frac{e}{3\pi}\right)^{1/2}. \bullet \quad (12)$$

Remark 1.3. Substituting $x = \cos \theta$ we can rewrite (12) in the form

$$\max_{\theta \in [0, \pi]} \left(\sin \frac{\theta}{2}\right)^{\alpha} |P_n^{(\alpha, -1)}(\cos \theta)| \leq 8 \left(\frac{e}{3\pi}\right)^{1/2}. \bullet \quad (13)$$

2 Relevant Mathematical Facts

All formulae of this section that are given without a reference can be found in Chaps. 6 and 22 of Abramowitz and Stegun [1].

2.1 The Gamma Function

The gamma function $\Gamma: \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function for all arguments $x \in \mathbb{C}$, save for the points $x = 0, -1, -2, \dots$, where it has simple poles. The function $1/\Gamma: \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function for all arguments $x \in \mathbb{C}$, and

$$1/\Gamma(-n) = 0 \quad \text{for all } n = 0, 1, \dots \quad (14)$$

For all $x > 0$ the function Γ can be written in the form

$$\Gamma(1+x) = (2\pi)^{1/2} \cdot x^{x+1/2} \exp(-x + \vartheta/x), \quad (15)$$

where $0 < \vartheta < 1/12$.

On $[1, 2]$ the function Γ has the unique minimum

$$\gamma_0 \stackrel{\text{def}}{=} \min_{x \in [1, 2]} \Gamma(x) = 0.8856031 \dots \quad (16)$$

2.2 Jacobi Polynomials

Jacobi polynomials $P_n^{(\alpha, \beta)}$ can be defined by the Rodrigues formula

$$P_n^{(\alpha, \beta)}(x) \stackrel{\text{def}}{=} \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left((1-x)^{\alpha+n} (1+x)^{\beta+n} \right), \quad (17)$$

and for $\alpha, \beta > -1$ they have the following explicit form:

$$P_n^{(\alpha, \beta)}(x) = \left(\frac{x-1}{2} \right)^n \times \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha-m+1)\Gamma(m+\beta+1)m!(n-m)!} \left(\frac{x+1}{x-1} \right)^m. \quad (18)$$

Below we give certain relevant well known equalities for Jacobi polynomials.

$$(2n+\alpha+\beta)P_n^{(\alpha, \beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha, \beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha, \beta)}(x), \quad (19)$$

$$\frac{d}{dx} \left((1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) \right) = -2(n+1)(1-x)^{\alpha-1} (1+x)^{\beta-1} P_{n+1}^{(\alpha-1, \beta-1)}(x), \quad (20)$$

$$P_n^{(\alpha,\beta)}(1) = (-1)^n P_n^{(\beta,\alpha)}(-1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)n!}. \quad (21)$$

Finally we cite three inequalities for the polynomials $P_n^{(\alpha,\beta)}$; they are valid for all integer $n \geq 0$, $\alpha, \beta \geq -1/2$, and $-1 \leq x \leq 1$.

$$|P_n^{(\alpha,\beta)}(x)| \leq \max \left[|P_n^{(\alpha,\beta)}(1)|, |P_n^{(\alpha,\beta)}(-1)| \right], \quad (22)$$

$$\left(\frac{1-x}{2} \right)^{\alpha/2+1/4} |P_n^{(\alpha,0)}(x)| \leq 1, \quad (23)$$

$$(1-x)^{\alpha/2+1/4} (1+x)^{\beta/2+1/4} |P_n^{(\alpha,\beta)}(x)| \leq \left(\frac{2e}{\pi} \right)^{1/2} \left(2 + (\alpha^2 + \beta^2)^{1/2} \right)^{1/2} (h_{\alpha,\beta}^n)^{1/2}, \quad (24)$$

where $h_{\alpha,\beta}^n$ is defined in (3). The inequality (22) is standard (see, for example, Chap. 22 of Abramowitz and Stegun [1]), the inequality (23) can be found in Chap. 7 of Szegő [6], and the inequality (24) was recently obtained by Nevai, Erdélyi, and Magnus [5].

3 Orthogonality and Completeness of Systems of Polynomials $\{P_n^{(\alpha,-k)}\}$

In this section we prove Theorems 1.3, 1.4, and 1.5. Note that Corollary 1.1 immediately follows from (6) and Theorem 3.3.1 of Szegő [6].

Proof of Theorem 1.3. We begin with the observation that for any integer k and m such that $k \leq n$ and $m < k$,

$$\frac{\Gamma(n-k+1)}{\Gamma(m-k+1)(n-m)!} = 0, \quad (25)$$

which is an immediate consequence of (14).

Next, an inspection of the formula (18) shows that for any fixed $x \in \mathbb{C}$ and $\alpha > -1$, Jacobi polynomials are analytic functions of $\beta \in \mathbb{C}$. Therefore by the principle of

analytic continuation we can substitute $\beta = -k$ into (18) which in combination with (25) yields

$$P_n^{(\alpha, -k)}(x) = \left(\frac{x-1}{2}\right)^n \times \sum_{m=k}^n \frac{\Gamma(n+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha-m+1)\Gamma(m-k+1)m!(n-m)!} \left(\frac{x+1}{x-1}\right)^m. \quad (26)$$

Substituting $m = k + l$ into (26) we have

$$P_n^{(\alpha, -k)}(x) = \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k \times \sum_{l=0}^{n-k} \frac{\Gamma(n+\alpha+1) \cdot \Gamma(n-k+1)}{\Gamma(n-k+\alpha-l+1) \cdot \Gamma(l+k+1) \cdot l! \cdot (n-k-l)!} \left(\frac{x+1}{x-1}\right)^l. \quad (27)$$

Now (6) is an immediate consequence of (18) and (27). •

Lemma 3.1 below can be easily proven by combining (3), (9), and (21); this result will be used in the proofs of Theorems 1.4 and 1.5.

Lemma 3.1 For any $k \leq n$ and $\alpha > -1$,

$$h_{\alpha, -k}^{n+k} = \left(\frac{P_{n+k}^{(\alpha, k)}(1)}{2^k P_n^{(\alpha, k)}(1)}\right)^2 h_{\alpha, k}^n. \quad (28)$$

Proof of Theorem 1.4. Substituting (7) into the left hand side of (8) and using (2) we have

$$\begin{aligned} & \int_{-1}^1 w_{\alpha, -k}(x) \cdot P_n^{(\alpha, -k)}(x) \cdot P_m^{(\alpha, -k)}(x) dx = \\ & \frac{1}{2^{2k}} \frac{P_n^{(\alpha, k)}(1)}{P_{n-k}^{(\alpha, k)}(1)} \cdot \frac{P_m^{(\alpha, k)}(1)}{P_{m-k}^{(\alpha, k)}(1)} \int_{-1}^1 w_{\alpha, k}(x) \cdot P_{n-k}^{(\alpha, k)}(x) \cdot P_{m-k}^{(\alpha, k)}(x) dx = \\ & \left(\frac{P_n^{(\alpha, k)}(1)}{2^k P_{n-k}^{(\alpha, k)}(1)}\right)^2 h_{\alpha, k}^{n-k} \delta_{nm}. \end{aligned} \quad (29)$$

Combining (28) and (29) we immediately obtain (8). •

Proof of Theorem 1.5. We begin with an observation that if $f \in L^2_{\alpha,-k}[-1, 1]$ then $\tilde{f} \in L^2_{\alpha,k}[-1, 1]$, where the function \tilde{f} is defined by the formula

$$\tilde{f}(x) \stackrel{\text{def}}{=} \frac{f(x)}{(1+x)^k}. \quad (30)$$

By Theorem 1.2 we have in $L^2_{\alpha,k}[-1, 1]$

$$\tilde{f} = \sum_{n=0}^{\infty} \frac{\tilde{f}_n}{h_{\alpha,k}^n} P_n^{(\alpha,k)}, \quad (31)$$

where

$$\tilde{f}_n \stackrel{\text{def}}{=} \int_{-1}^1 w_{\alpha,k}(x) P_n^{(\alpha,k)}(x) \tilde{f}(x) dx. \quad (32)$$

Combining (32) with (7) and (30) we obtain

$$\tilde{f}_n = 2^k \frac{P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} \int_{-1}^1 w_{\alpha,-k}(x) P_{n+k}^{(\alpha,-k)}(x) f(x) dx = 2^k \frac{P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} f_{n+k}. \quad (33)$$

Next, the substitution of (33) into (31) in combination with (7) and (30) yields

$$f = \sum_{n=0}^{\infty} \left(\frac{2^k P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} \right)^2 \frac{a_{n+k}}{h_{\alpha,k}^n} P_{n+k}^{(\alpha,-k)}, \quad (34)$$

and now (11) immediately follows from (28) and (34). •

4 An Upper Bound for Functions

$$((1-x)/2)^{\alpha/2} P_n^{(\alpha,-1)}(x)$$

In this section we prove Theorem 1.6. It is carried out by means of considering five regions of parameters n , x , and α , obtaining an upper bound for each region separately, and finally choosing the largest such bound as a uniform upper bound for the function $((1-x)/2)^{\alpha/2} |P_n^{(\alpha,-1)}(x)|$ ($\alpha \geq 0$, $n \geq 1$) on $[-1, 1]$. Throughout the proof of the theorem we will use the notation

$$x_0 \stackrel{\text{def}}{=} 1 - \left(\frac{2 + \alpha}{2n + \alpha - 1} \right)^2. \quad (35)$$

We begin with two preliminary results summarized in Lemmas 4.1 and 4.2. below. Their proofs are immediate consequences of (19) and (24), respectively.

Lemma 4.1. For any $x \in \mathbb{C}$, $n \geq 1$ and $\alpha > -1$,

$$|P_n^{(\alpha, -1)}(x)| \leq \frac{n + \alpha}{2n + \alpha} \left(|P_n^{(\alpha, 0)}(x)| + |P_{n-1}^{(\alpha, 0)}(x)| \right). \quad (36)$$

Lemma 4.2. For any $0 \leq x \leq 1$ and $\alpha \geq -1/2$,

$$\left(\frac{1-x}{2} \right)^{\alpha/2} |P_n^{(\alpha, 0)}(x)| \leq 2 \left(\frac{e}{\pi} \right)^{1/2} \left(\frac{2 + \alpha}{2n + \alpha + 1} \right)^{1/2} \frac{1}{(1-x)^{1/4}} \quad (37)$$

Proof of Theorem 1.6.

Region 1. We define this region by the inequalities

$$\alpha \geq 0, \quad (38)$$

$$n \geq 1, \quad (39)$$

$$-1 \leq x \leq 0. \quad (40)$$

From (23) and (40) we have

$$\left(\frac{1-x}{2} \right)^{\alpha/2} |P_n^{(\alpha, 0)}(x)| \leq \left(\frac{1-x}{2} \right)^{-1/4} \leq 2^{1/4}, \quad (41)$$

which in combination with (36) yields

$$\left(\frac{1-x}{2} \right)^{\alpha/2} |P_n^{(\alpha, -1)}(x)| \leq 2^{5/4} = 2.3784 \dots \quad (42)$$

Region 2. We define this region by the inequalities

$$\alpha \geq 0, \quad (43)$$

$$n = 1, \quad (44)$$

$$0 < x \leq 1. \quad (45)$$

The formulae (6) and (18) yield

$$P_1^{(\alpha,-1)}(x) = (1+\alpha) \left(\frac{1+x}{2}\right) P_0^{(\alpha,1)}(x) = (1+\alpha) \left(\frac{1+x}{2}\right), \quad (46)$$

and now combining (46) with (43) and (45) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_1^{(\alpha,-1)}(x)| = (1+\alpha) \left(\frac{1+x}{2}\right) \left(\frac{1-x}{2}\right)^{\alpha/2} \leq \max_{\alpha \geq 0} (1+\alpha) 2^{-\alpha/2} = 1.5011 \dots \quad (47)$$

Region 3. We define this region by the inequalities

$$\alpha \geq 0, \quad (48)$$

$$n \geq 2, \quad (49)$$

$$0 < x \leq x_0, \quad (50)$$

where x_0 is defined in (35).

Combining (37) and (50) we can write

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,0)}(x)| \leq 2 \left(\frac{e}{\pi}\right)^{1/2} \left(\frac{2+\alpha}{2n+\alpha+1}\right)^{1/2} \frac{1}{(1-x_0)^{1/4}}, \quad (51)$$

and substituting (35) into (51) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,0)}(x)| \leq 2 \left(\frac{e}{\pi}\right)^{1/2}. \quad (52)$$

Similarly,

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_{n-1}^{(\alpha,0)}(x)| \leq 2 \left(\frac{e}{\pi}\right)^{1/2}. \quad (53)$$

Now substituting (52) and (53) into (36) we obtain

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,-1)}(x)| \leq 4 \left(\frac{e}{\pi}\right)^{1/2} = 3.7207 \dots \quad (54)$$

Region 4. We define this region by the inequalities

$$0 \leq \alpha \leq 1, \quad (55)$$

$$n \geq 2, \quad (56)$$

$$x_0 < x \leq 1. \quad (57)$$

Combining (21), (22), and (57) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,0)}(x)| \leq \left(\frac{1-x_0}{2}\right)^{\alpha/2} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}. \quad (58)$$

Substituting (15) and (35) into (58) and using (16) we have

$$\begin{aligned} \left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,0)}(x)| &\leq \frac{1}{\gamma_0} \left(\frac{2+\alpha}{2^{1/2}}\right)^\alpha \left(\frac{\alpha+n}{2n+\alpha-1}\right)^\alpha \left(1+\frac{\alpha}{n}\right)^{n+1/2} \times \\ &\exp(-\alpha) \exp(\vartheta_1/(\alpha+n) - \vartheta_2/n), \end{aligned} \quad (59)$$

where $0 < \vartheta_1, \vartheta_2 < 1/12$.

Next, one can easily verify that for all $n \geq 2$ and $0 \leq \alpha \leq 1$,

$$\left(\frac{2+\alpha}{2^{1/2}}\right)^\alpha \leq \frac{3}{2^{1/2}}, \quad (60)$$

$$\left(\frac{\alpha+n}{2n+\alpha-1}\right)^\alpha \leq 1, \quad (61)$$

$$\left(1+\frac{\alpha}{n}\right)^n \leq \exp(\alpha), \quad (62)$$

$$\left(1+\frac{\alpha}{n}\right)^{1/2} \leq \left(\frac{3}{2}\right)^{1/2}, \quad (63)$$

$$\exp(\vartheta_1/(\alpha+n) - \vartheta_2/n) \leq \exp(1/24), \quad (64)$$

and

$$\frac{n+\alpha}{2n+\alpha} \leq \frac{3}{5}. \quad (65)$$

Now combining (36) with (59-65) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} \left|P_n^{(\alpha,-1)}(x)\right| \leq \frac{3^{5/2}}{10\gamma_0} \exp(1/24) = 1.8350 \dots \quad (66)$$

Region 5. We define this region by the inequalities

$$\alpha > 1, \quad (67)$$

$$n \geq 2, \quad (68)$$

$$x_0 < x \leq 1. \quad (69)$$

The formula (20) yields

$$(1-x)^\alpha(1+x)P_n^{(\alpha,1)}(x) = 2(n+1) \int_x^1 (1-t)^{\alpha-1} P_{n+1}^{(\alpha-1,0)}(t) dt, \quad (70)$$

while from (6) we have

$$P_n^{(\alpha,1)}(x) = \frac{2(n+1)}{(1+x)(n+\alpha+1)} P_{n+1}^{(\alpha,-1)}(x). \quad (71)$$

Combining (70) and (71) we obtain

$$(1-x)^\alpha P_n^{(\alpha,-1)}(x) = (n+\alpha) \int_x^1 (1-t)^{\alpha-1} P_n^{(\alpha-1,0)}(t) dt. \quad (72)$$

Next, combining (37) and (72) we have

$$\begin{aligned} & \left(\frac{1-x}{2}\right)^{\alpha/2} \left|P_n^{(\alpha,-1)}(x)\right| \leq \\ & 2^{-1/2}(1-x)^{-\alpha/2}(n+\alpha) \int_x^1 \left[\left(\frac{1-t}{2}\right)^{\alpha/2-1/2} \left|P_n^{(\alpha-1,0)}(t)\right| \right] (1-t)^{\alpha/2-1/2} dt \leq \\ & 2^{1/2}(1-x)^{-\alpha/2}(n+\alpha) \left(\frac{e}{\pi}\right)^{1/2} \left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2} \int_x^1 (1-t)^{\alpha/2-3/4} dt. \end{aligned} \quad (73)$$

Integration in (73) in combination with (69) produces

$$\begin{aligned} \left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,-1)}(x)| &\leq 2^{1/2} \left(\frac{e}{\pi}\right)^{1/2} \frac{n+\alpha}{\alpha/2+1/4} \left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2} (1-x)^{1/4} \leq \\ &2^{1/2} \left(\frac{e}{\pi}\right)^{1/2} \frac{n+\alpha}{\alpha/2+1/4} \left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2} (1-x_0)^{1/4}, \end{aligned} \quad (74)$$

which after the substitution of x_0 from (35) becomes

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,-1)}(x)| \leq 2^{1/2} \left(\frac{e}{\pi}\right)^{1/2} \frac{n+\alpha}{\alpha/2+1/4} \left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2} \left(\frac{2+\alpha}{2n+\alpha-1}\right)^{1/2}. \quad (75)$$

For all $n \geq 2$ and $\alpha > 1$ we have

$$\frac{(1+\alpha)^{1/2}(2+\alpha)^{1/2}}{\alpha/2+1/4} \leq \frac{4}{3}6^{1/2}, \quad (76)$$

and

$$\frac{n+\alpha}{(2n+\alpha)^{1/2}(2n+\alpha-1)^{1/2}} \leq 1. \quad (77)$$

Finally, substituting (76) and (77) into (75) we obtain

$$\left(\frac{1-x}{2}\right)^{\alpha/2} |P_n^{(\alpha,-1)}(x)| \leq 8 \left(\frac{e}{3\pi}\right)^{1/2} = 4.2963\dots \quad (78)$$

Now the conclusion of the theorem is a consequence of (42), (47), (54), (66), and (78). •

5 Conclusions and Generalizations

We have proven the orthogonality and completeness of the system $\{P_n^{(\alpha,-k)}\}$ and obtained an upper bound for the function $((1-x)/2)^{\alpha/2} P_n^{(\alpha,-1)}(x)$. These results can be extended in the following two directions.

First, it appears that there exist analogues of Theorems 1.3, 1.4, and 1.5 for Laguerre polynomials L_n^α with $\alpha = -1, -2, \dots$.

Second, one can try to obtain an inequality sharper than (12). Our numerical experiments indicate that the constant $8(e/3\pi)^{1/2} = 4.29\dots$ in (12) is not optimal.

This work is currently in progress and its results will be reported at a later date.

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