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EFFECT OF THE TEMPERATURE OF THE MODERATOR ON THE VELOCITY DISTRIBUTION OF NEUTRONS WITH NUMERICAL CALCULATIONS FOR H AS MODERATOR

by

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EFFECT OF THE TEMPERATURE OF THE MODERATOR ON THE VELOCITY DISTRIBUTION OF NEUTRONS WITH NUMERICAL CALCULATIONS FOR H AS MODERATOR

By E. P. Wigner and J. E. Wilkins, Jr.

ABSTRACT

In this paper we set up an integral equation governing the energy distribution of neutrons that are being slowed down uniformly throughout the entire space by a uniformly distributed moderator whose atoms are in motion with a Maxwellian distribution of velocities. The effects of chemical binding and crystal reflection are ignored. When the moderator is hydrogen, the integral equation is reduced to a differential equation and solved by numerical methods. In this manner we obtain a refinement of the dv/v^2 law.

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1. There are two particularly simple problems in connection with the energy distribution of neutrons that are present in a medium of finite temperature. In the first problem the slowing down is uniform throughout the entire space that is itself uniformly filled with the slowing down material. In this case the neutron distribution is evidently the same over all space. In the second problem the neutrons enter a half space from one side with uniform intensity and diffuse into it. The question in this case is the density distribution of neutrons at large distances from the boundary plane of the half space and the exponential relaxation length of the neutron density. We shall be interested only in the first problem.

There are three phenomena which make the calculation of the energy distribution of the neutrons somewhat complicated. These are the finite velocity of the particles with which they collide, the effect of the chemical binding on the scattering cross section, and finally the effects of crystal reflection. We shall disregard the last two effects, and the calculation to be given will, therefore, be valid only in a monoatomic gas. Seitz and Goldberger are engaged in the study of the crystal effect, and Teller has made considerations on the effect of the chemical binding.

To derive an equation satisfied by $N(v)$, where $N(v)dv$ is the number of neutrons per cm^3 whose velocity lies between v and $v + dv$, we proceed as follows. Let us denote the probability that a neutron with velocity v_1 will acquire in unit length of time a velocity between v and $v + dv$ by collision with an atom, by $N\sigma_s P(v,v_1)\, dv$. Here N is the number of atoms per cm^3 and σ_s is the scattering cross section of the moderator. Hence the number of neutrons acquiring a velocity between v and $v + dv$ in unit length of time is

$$N dv \int_0^\infty \sigma_s P(v, v_1)\, N(v_1)\, dv_1 .$$

Let $N\sigma_s V(v)$ be the probability that a neutron with velocity v will be scattered in unit time, and let $N\sigma_s\gamma$ be the probability that a neutron with velocity v will be absorbed in unit time. If we adopt the $1/v$ law, this second probability will be independent of the velocity of the neutron. We further assume

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that the scattering cross section is independent of the velocity of the neutron. This is a valid assumption for a moderator gas, since our results will be applied only in a reasonably close neighborhood of thermal energies. It is now clear that N(v) satisfies the following integral equation:

$$\int_0^\infty \underline{P}(v,v_1)\, N(v_1)\, dv_1 = [V(v) + \gamma]\, N(v). \tag{1}$$

Before calculating $\underline{P}$ and V we shall derive a property of the above equation that simplifies calculations greatly and also can serve as a check. Evidently equation (1) is satisfied by the Maxwell distribution if $\gamma = 0$. This together with the principle of detailed balance permits us to give equation (1) a symmetric form. If we denote the Maxwell distribution of velocity by M(v)

$$M(v)\, dv = \frac{4\beta^3}{\sqrt{\pi}}\, v^2\, e^{-\beta^2 v^2}\, dv, \quad \beta^2 = \frac{1}{2kT} \tag{2}$$

the principle of detailed balance reads

$$P(v,v_1)\, M(v_1) = \underline{P}(v_1,v)\, M(v). \tag{3}$$

Herein the mass of the neutron is used as the unit of mass. It follows that

$$\frac{1}{\sqrt{M(v)}}\, \underline{P}(v,v_1)\, \sqrt{M(v_1)} = \frac{1}{\sqrt{M(v_1)}}\, \underline{P}(v_1,v)\, \sqrt{M(v)} = S(v,v_1) = S(v_1,v) \tag{3a}$$

is a symmetric kernel. Thus if we introduce the new quantity $\nu(v) = N(v)/\sqrt{M(v)}$ into the equation (1), we get the equation

$$\int_0^\infty S(v,v_1)\, \nu(v_1)\, dv_1 = [V(v) + \gamma]\, \nu(v) \tag{4}$$

which has a symmetric kernel.

2. The next task is to calculate V and the kernel $\underline{P}$. Both quantities depend of course on the temperature of the gas and also on the mass of its atoms. The calculation, although quite laborious, is entirely straightforward and gives for $v_1 < v$

$$\underline{P}(v,v_1) = \frac{2\theta^2 v}{\sqrt{\pi}\, v_1}\, e^{\beta^2(v_1^2 - v^2)}\, [I(\beta\theta v_1 - \beta\zeta v) + I(\beta\theta v_1 + \beta\zeta v)] + \frac{2\theta^2 v}{\sqrt{\pi}\, v_1}\, [I(\beta\theta v - \beta\zeta v_1) - I(\beta\zeta v_1 + \beta\theta v)]\ . \tag{5}$$

For $v_1 > v$ it gives

$$\underline{P}(v,v_1) = \frac{2\theta^2 v}{\sqrt{\pi}\, v_1}\, [I(\beta\theta v - \beta\zeta v_1) + I(\beta\theta v + \beta\zeta v_1)] + \frac{2\theta^2 v}{\sqrt{\pi}\, v_1}\, e^{\beta^2(v_1^2 - v^2)}\, [I(\beta\theta v_1 - \beta\zeta v) - I(\beta\zeta v + \beta\theta v_1)]\ . \tag{5a}$$

Herein

$$\theta = \frac{m+1}{2\sqrt{m}}, \quad \zeta = \frac{m-1}{2\sqrt{m}}, \quad \beta^2 = \frac{1}{2kT} \tag{5b}$$

where m is the mass of the atoms of the moderator in units of the neutron mass. I is the odd function

$$I(x) = \int_0^x e^{-x^2} dx. \tag{5c}$$

For V we obtain*

$$V(v) = \frac{1}{\sqrt{\pi}} (2v + 1/\beta^2 mv)\, I(\beta \sqrt{m} v) + e^{-\beta^2 mv^2} / \beta \sqrt{m\pi} \tag{6}$$

Equations (5) and (6) determine the quantities which occur in the integral equation (1). In the derivation it was assumed that the scattering cross section σ_s is independent of the velocity of the neutrons. This is a valid assumption for a moderator gas since (1) will be applied only in a reasonably close neighborhood of thermal energies. It was further assumed that the scattering is spherically symmetric in the center of mass coordinate system. The physical significance of γ is $\gamma = \sigma_a v/\sigma_s$ where σ_a is the absorption cross section for the relative velocity v between neutron and atom — γ is independent of the latter.

The equations (5) and (6) can be simplified considerably by measuring the velocity in terms of the thermal velocity. Let us denote therefore

$$\beta v = x, \quad \beta v_1 = x_1 \tag{7}$$

and

$$\Gamma = \beta\gamma = \frac{\sigma_a v}{\sigma_s \sqrt{2kT}}. \tag{7a}$$

Then Γ is the probability of absorption of the neutron while it goes one mean free path with the velocity corresponding to the energy kT.

We then have

$$\underline{P}(x,x_1) = \frac{2\theta^2}{\sqrt{\pi}} \frac{x}{x_1} e^{(x_1^2 - x^2)} [I(\theta x_1 - \zeta x) + I(\theta x_1 + \zeta x)]$$
$$+ \frac{2\theta^2}{\sqrt{\pi}} \frac{x}{x_1} [I(\theta x - \zeta x_1) - I(\zeta x_1 + \theta x)] \text{ for } x_1 < x \tag{8}$$

and

$$\underline{P}(x,x_1) = \frac{2\theta^2}{\sqrt{\pi}} \frac{x}{x_1} [I(\theta x - \zeta x_1) + I(\theta x + \zeta x_1)]$$
$$+ \frac{2\theta^2}{\sqrt{\pi}} \frac{x}{x_1} e^{(x_1^2 - x^2)} [I(\theta x_1 - \zeta x) - I(\zeta x + \theta x_1)] \tag{8a}$$

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* V is also given by G. Jager in Winkelmann's Handbuch der Physik (Barth, 1906), vol. 3, p. 698.

for $x_1 > x$. We further have

$$V(x) = \frac{1}{\sqrt{\pi}} (2x + 1/mx) I(m^{\frac{1}{2}}x) + e^{-mx^2}/\sqrt{m\pi} \tag{8b}$$

and the integral equation becomes

$$\int_0^\infty S(x,x_1)\, \nu(x_1)\, dx_1 = [\, V(x) + \Gamma\,]\, \nu(x). \tag{9}$$

Although the derivation of the expressions for $\underline{P}$ and V is quite straightforward, it may be worthwhile to give a few of the intermediate steps to facilitate checking the equations.

V(v) is the probability of a collision of the neutron with an atom of unit cross section and the velocity distribution

$$M_m(v_2)\, dv_2 = \frac{4m^{3/2}\beta^3}{\sqrt{\pi}} v_2^2\, e^{-\beta^2 m v_2^2}\, dv_2. \tag{10}$$

The probability for a collision with an atom of velocity v_2 is, if the neutron velocity is v, and the cosine of the angle between the directions of motion is μ, simply

$$v_r = \sqrt{v^2 + v_2^2 - 2vv_2\mu} \tag{10a}$$

which is the relative velocity of the two particles. The number of atoms within unit velocity range at v_2 is given by (10), the probability that μ lies between μ and $\mu + d\mu$ is $\frac{1}{2}d\mu$ so that

$$V(v) = \int_{-1}^{1} \tfrac{1}{2}d\mu \int_0^\infty dv_2\, M_m(v_2) \sqrt{v^2 + v_2^2 - 2vv_2\mu}\ . \tag{10b}$$

The integration over μ is elementary and gives

$$\tfrac{1}{2} \int_0^\infty \frac{M(v_2)}{3vv_2} \left[(v + v_2)^3 - |v - v_2|^3 \right] dv_2. \tag{10c}$$

When integrating this expression one must proceed with the two cases $v_2 < v$ and $v_2 > v$ separately. This kind of disjunction is characteristic also for the calculation of $\underline{P}$. With this proviso, the integration of (10c) can be carried out easily, and it gives (6).

The calculation of $\underline{P}$ is more cumbersome. Collision of a neutron with the velocity v_1 with an atom of velocity v_2 gives a velocity between v and v + dv for the neutron with the probability

$$\begin{aligned} \underline{P}(v;\, v_1,\, v_2,\, \mu) &= 0 && \text{if } v < v_{min} \\ &= \frac{2v}{v_{max}^2 - v_{min}^2} && \text{if } v_{min} < v < v_{max} \\ &= 0 && \text{if } v > v_{max} \end{aligned} \tag{11}$$

These equations express the familiar fact that, after the collision, the probability is uniform in energy scale between the minimum and maximum energies. Herein, μ is again the cosine between the directions of motion of neutron and atom before the collision,

$$v_{min} = v_c - mv_r/(m+1),$$
$$v_{max} = v_c + mv_r/(m+1), \qquad (11a)$$

and

$$v_c = \frac{1}{m+1}\sqrt{v_1^2 + m^2v_2^2 + 2mv_1v_2\mu} \qquad (11b)$$

is the velocity of the center of mass of neutron plus atom, $mv_r/(m+1)$ is the velocity of the neutron in the center of mass coordinate system. One derives (11) most easily by a geometrical argument. Now

$$\underline{P}(v,v_1) = \int_{-1}^{1} \tfrac{1}{2}d\mu \int_0^\infty dv_2\, M_m(v_2)v_r\, \underline{P}(v;\, v_1,\, v_2,\, \mu). \qquad (11c)$$

$$= \frac{(1+m)^2 v}{4m} \iint d\mu\, dv_2\, M_m(v_2)\, (v_1^2 + m^2v_2^2 + 2mv_1v_2\mu)^{-\frac{1}{2}}$$

where the integration is to be extended over the region limited by $-1 < \mu < 1$ and that $v_{min} < v < v_{max}$. The integration itself is quite elementary. However, the domain of integration is quite complicated and is bounded in general by two straight lines and two parts of a hyperbola, if one introduces the variable $2\, mv_1v_2\mu$ for μ. A number of cases has to be distinguished and the integration carried out separately over different parts of the domain of integration before one obtains (5) and (5a).

3. The kernel S is greatly simplified in the case m = 1, i.e. when the moderator is hydrogen gas. We then have

$$\sqrt{\pi}\, S(x,x_1) = 4I(x)e^{\frac{1}{2}(x^2 - x_1^2)} \qquad (x < x_1), \qquad (12a)$$

$$\sqrt{\pi}\, S(x,x_1) = 4I(x_1)e^{\frac{1}{2}(x_1^2 - x^2)} \qquad (x > x_1), \qquad (12b)$$

$$\sqrt{\pi}\, V(x) = (2x + 1/x)\, I(x) + e^{-x^2} \qquad (13)$$

and the integral equation becomes

$$4e^{-\frac{1}{2}x^2}\int_0^x dx_1\, I(x_1)e^{\frac{1}{2}x_1^2}\, \nu(x_1) + 4I(x)\, e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}x_1^2}\, \nu(x_1)\, dx_1$$
$$= \left[(2x + \frac{1}{x})\, I(x) + e^{-x^2} + \sqrt{\pi}\quad \Gamma\right] \nu(x). \qquad (14)$$

The $S(x,x_1)$ of (12) has the form of a Green's function of an ordinary second order differential equation. The integral operator of (14) is therefore the reciprocal of a second order differential operator. If one applies this operator to equation (14), it will itself be transformed into a second order differential equation.

Without using the theory of the Green's function, one can proceed as follows. Let us find a second order differential operator

$$L_1 = \frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x) \qquad (15)$$

which gives zero if applied either to $I(x)\, e^{\frac{1}{2}x^2}$ or to $e^{-\frac{1}{2}x^2}$. This condition gives two linear equations for the two unknowns a and b. One obtains

$$a = \frac{-2I(x)}{e^{-x^2} + 2xI(x)}, \qquad b = \frac{e^{-x^2}}{e^{-x^2} + 2xI(x)} - x^2 \tag{15a}$$

In order to bring L_1 into a self-adjoint form, one may multiply it with P^{-1} where

$$P = e^{-x^2} + 2xI(x). \tag{16}$$

This gives

$$L = \frac{1}{P}\frac{d^2}{dx^2} - \frac{2I}{P^2}\frac{d}{dx} + \frac{e^{-x^2}}{P^2} - \frac{x^2}{P} = \frac{d}{dx}\frac{1}{P}\frac{d}{dx} + \frac{e^{-x^2}}{P^2} - \frac{x^2}{P} \tag{15b}$$

If one applies either L_1 or L to (14), the integral will give rise to two types of terms. Differentiation of the function before the integral sign gives zero. The other terms will contain no integral. Hence the integral equation reduces to a second order differential equation, the second derivative coming from applying L_1 or L to the right side of (14). By virtue of these remarks, one sees that any solution of the integral equation (14) also satisfies the differential equation

$$-\frac{d}{dx}\frac{1}{P}\frac{d}{dx}(V + \Gamma)\,\nu(x) + \left[W(V + \Gamma) - 4/\pi^{\frac{1}{2}}\right]\nu(x) = 0 \tag{17}$$

where

$$W(x) = x^2/P - e^{-x^2}/P^2. \tag{18}$$

Since $S(0,x_1) = 0$, we also have $\nu(0) = 0$. Conversely, any solution $\nu(x)$ of the differential equation (17) such that $\nu(0) = 0$ is a solution of the integral equation (14). The other solutions of (17) are, however, not solutions of (14). The two solutions of (17) behave, at x = 0, like a constant and x itself. The former solution would give an N(v) that goes to zero as v, the latter—the one which we have to use—gives an N(v) that goes at v = 0 to zero as v^2, which is the same way as the Maxwell distribution M(v) goes to zero.

4. It would be possible to discuss (17) in a general way and to use it directly to obtain numerical results. However, we found it more expedient to proceed as follows. We place

$$\mu(x) = \left[V(x) + \Gamma\right]\nu(x), \tag{19}$$

so that we have

$$-(\mu'/P)' + \left\{W(x) - 4\pi^{-\frac{1}{2}}\left[V(x) + \Gamma\right]^{-1}\right\}\mu = 0,\ \mu(0) = 0, \tag{19a}$$

observing that for $x \geqq 0$, $V(x) > 0$, so that $V(x) + \Gamma \neq 0$. We reduce the second order linear differential equation (19a) to a Riccati equation by making the substitution

$$y = \mu'/\mu P, \tag{20}$$

which leads to

$$y = W(x) - 4\pi^{-\frac{1}{2}} \left[V + \Gamma \right]^{-1} - Py^2. \tag{20a}$$

The boundary condition $\mu(0) = 0$, together with $\mu(x) \not\equiv 0$, implies that we have

$$\lim_{x=0} \left\{ y(x) - x^{-1} \right\} = 0. \tag{20b}$$

Conversely we obtain a solution $\mu(x)$ from a solution $y(x)$ of (20a) by setting

$$\mu(x) = \mu'(0)x \exp \int_0^x \left[P(t)y(t) - t^{-1} \right] dt.$$

The solution $y(x)$ of (20a) and (20b) may be written in the form

$$y(x) = \frac{1}{x} - \frac{4(1+\delta)x}{3(1+2\delta)} + \frac{(103 + 380\delta + 364\delta^2)x^3}{90(1+2\delta)^2} \tag{21}$$

$$\frac{-(1163 + 6666\delta + 13200\delta^2 + 8864\delta^3)x^5}{945(1+2\delta)^3} + \dots ,$$

where $\delta = \frac{1}{4}\sqrt{\pi}\,\Gamma$. We use this series to compute the values of $y(x)$ for $\delta = 0.1$, 0.2 and 1 and for x= 0.1, 0.2, 0.3, 0.4, and 0.5. With these values of y we compute successively μ, ν and N, normalizing the functions so that $N''(0) = 2$.

With the value of $y(\frac{1}{2})$ thus obtained we continue the function y(x) by numerical integration of (20a). In order to decide how far to carry the numerical integration, we now investigate the behavior of the solutions y of (20a) as x approaches ∞ We use the following expansion of I(x)

$$I(x) = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2} + \dots\right).$$

Neglecting terms involving e^{-x^2} we see that

$$P(x) \approx \sqrt{\pi}x, \qquad V(x) \approx x + \frac{1}{2x}, \qquad W(x) \approx \frac{x}{\sqrt{\pi}} \tag{22}$$

Hence (20a) becomes

$$\sqrt{\pi}y' = x - \frac{4x}{x^2 + \frac{1}{2} + \Gamma x} - \pi x y^2 . \tag{22a}$$

This has two solutions which are finite at ∞ and which can be expanded in terms of inverse powers of x:

$$\sqrt{\pi}y_1 \approx 1 - \frac{2}{x^2} + \frac{2\Gamma}{x^3} - \frac{(2\Gamma^2+3)}{x^4} + \frac{(2\Gamma^3+5\Gamma)}{x^5} - \ldots , \tag{23a}$$

$$\sqrt{\pi}y_2 \approx -1 + \frac{2}{x^2} - \frac{2\Gamma}{x^3} + \frac{(2\Gamma^2-1)}{x^4} - \frac{(2\Gamma^3-\Gamma)}{x^5} + \ldots . \tag{23b}$$

These two series give the asymptotic expansions of two linearly independent solutions of (19a):

$$\mu_1 = \exp\int y_1 \, P dx \approx \exp\left(\tfrac{1}{2}x^2 - 2\ln x - 2\Gamma/x + \ldots\right) , \tag{24a}$$

$$\mu_2 = \exp\int y_2 \, P dx \approx \exp\left(-\tfrac{1}{2}x^2 + 2\ln x + 2\Gamma/x - \ldots\right) , \tag{24b}$$

For the asymptotic behavior of the two linearly independent solutions of (17), this gives because of (19)

$$\nu_1 \approx \frac{1}{x^3} \frac{e^{(\frac{1}{2}x^2 - 2\Gamma/x)}}{1 + \Gamma x^{-1} + \frac{1}{2}x^{-2}} \tag{25a}$$

$$\nu_2 \approx x \frac{e^{(-\frac{1}{2}x^2 + 2\Gamma/x)}}{1 + \Gamma x^{-1} + \frac{1}{2}x^{-2}} \tag{25b}$$

and for N

$$N_1 \approx \frac{1}{x^2} \frac{e^{-2\Gamma/x}}{1 + \Gamma/x \ldots} \tag{26a}$$

$$N_2 \approx x^2 \frac{e^{-x^2} + 2\Gamma/x}{1 + \Gamma/x \ldots} \tag{26b}$$

It is evident from physical considerations that if $\gamma > 0$ the slowing down at high energies, Nv^2, cannot be zero. Hence in the asymptotic expansion of N at $x = \infty$ the coefficient of N_1 cannot be zero. Hence the coefficient of μ_1 in the asymptotic expansion of μ also is not ·ero. As a result, μ_1 gives the asymptotic behavior of μ because for large x we have $\mu_2 \ll \mu_1$. It follows that it is necessary to carry out our numerical integration only to a point x_0 at which the series (23a) gives the same value as the numerical integration. For $x \geq x_0$ the series (23a) may be used to obtain values of y. In practice it turns out that $x_0 = 5$ is a suitable choice.

Having computed y in this manner for $x \geq 0.5$ we can compute $\mu(x)$ by the formula

$$\mu(x) = \mu(\tfrac{1}{2}) \exp\int_{\frac{1}{2}}^{x} P(t)\, y(t)\, dt$$

and then we can compute $\nu(x)$ and N(x).

The asymptotic form of N at large x or large v is given by (26a). Replacing $e^{-2\Gamma/x}$ in this by $(1 + \Gamma/x)^{-2}$ —which is within the accuracy of the expression in the denominator—we obtain, reintroducing ϕ_s and the life time τ of the neutrons,

$$N(v)\,dv \approx \frac{Cv\,dv}{(N\sigma_s\,\tau\,v + 1)^3} \tag{27}$$

where C is a constant. This shows that the deviation from the dv/v^2 law arises at high energies not from the positive temperature of the atoms but from the absorption. It causes N(v) to become lower than the dv/v^2 law would indicate. At lower energies the Figures show that N(v) becomes higher than $1/v^2$. This is the result of the finite temperature of the atoms. For T = 0, (27) is correct for all v. The constant C occurring in (27) is of some importance and may be evaluated directly from the values of N(x) previously calculated. If this be done it is possible to obtain a check on the computations used to find N(x) by calculating C in another manner. The number of neutrons which are slowed down to velocities below v_0 is given, when v_0 is very large, by the formula

$$N\,\sigma_s \int_{v_0}^{\infty} \frac{vN(v)v_0^2}{v^2}\,dv \sim \frac{N\sigma_s}{2} \lim_{v=\infty} v^2 N(v) = \frac{C}{2N^2\,\sigma_s^{\,2}\,\tau^3} \tag{28}$$

Since v_0 is large, this is approximately equal to the number of neutrons that are absorbed at velocities below v_0 and in the limit as v_0, approaches infinity, we have equality. Thus

$$N\sigma_s\gamma \int_0^{\infty} N(v)dv = \frac{C}{2N^2\sigma_s^{\,2}\,\tau^3} \tag{28a}$$

Hence the constant C of the asymptotic expression for N(v) can also be obtained by integrating N(v). The integral was evaluated as follows. Replace v by $x = \beta v$, so that

$$\Gamma \int_0^{\infty} N(x)\,dx = \frac{C\,\beta^2}{2N^3\sigma_s^{\,3}\,\tau^3}$$

The integral $\int_0^6 N(x)\,dx = A$ may be computed by numerical integration of the values of N(x) previously computed. To compute $B = \int_6^{\infty} N(x)\,dx$, we replace N(x) by an asymptotic expansion of the form

$$N(x) = \frac{C\,\beta^2}{2N^3\,\sigma_s^{\,3}\,\tau^3}\left[\frac{1}{x^2} + \ldots\ldots\right] \tag{29}$$

(which may be obtained by making more precise the reasoning which leads to 26a) and integrating directly. In this manner we get a linear equation

$$A + B = \frac{C\,\beta^2}{2N^3\sigma_s^{\,3}\,\tau^3}\,, \tag{30}$$

which may be solved for C. The constant C was computed in this manner for the three values of γ previously mentioned, and it was found that they agreed satisfactorily (about 1%) with the values of C computed directly.

Our numerical results are summarized in the attached Figure which shows the graphs of $M(x) = x^2 e^{-x^2}$ ($\gamma = 0$) and N(x) for $\delta = \frac{1}{4}\sqrt{\pi}\,\beta\,\gamma = 0.1$, 0.2 and 1 respectively. This corresponds to σ_a/σ_s values of 0.226, 0.452, 2.257 at kT energy. In the three latter cases, the straight line graph of $\frac{1}{x^2} \lim_{x=\infty} x^2 N(x)$ is also included, so that the deviation from the dv/v^2 law may be visualized.

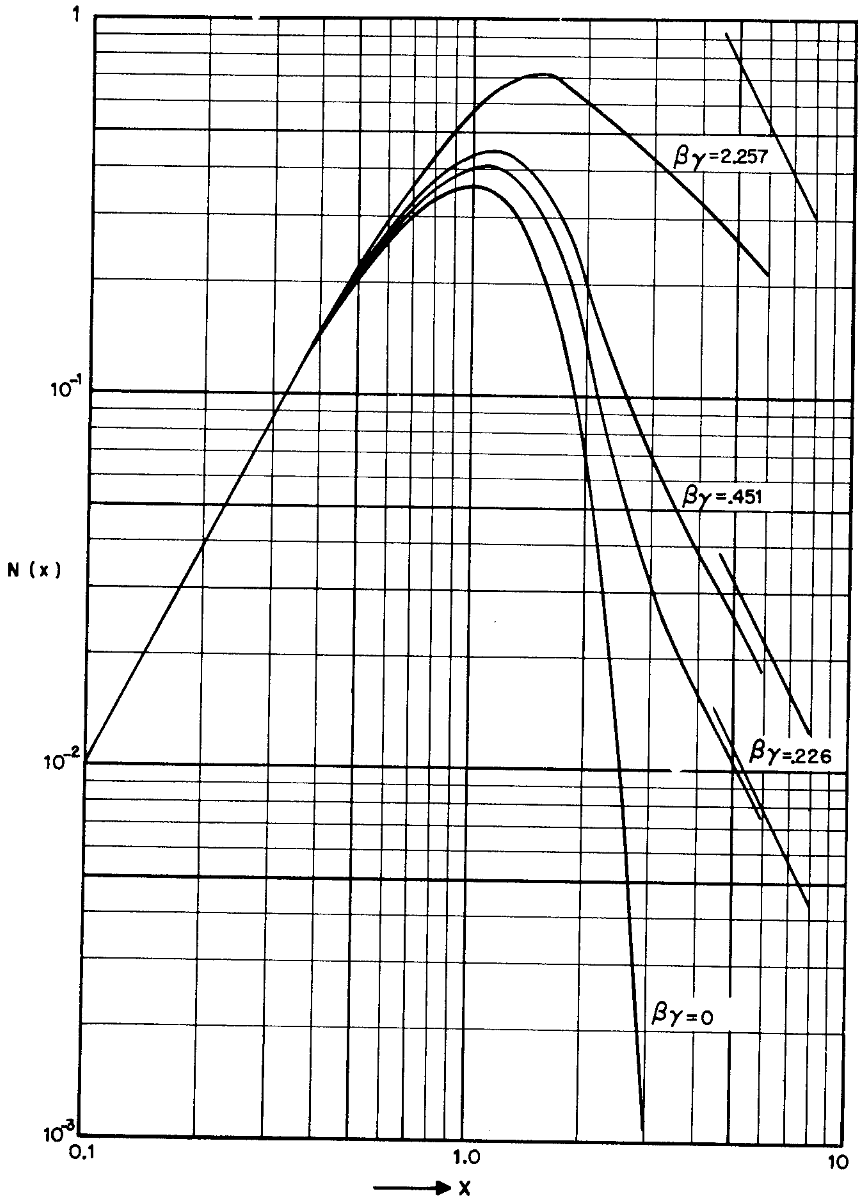

Figure 1. Graphs showing N(x) for $\beta\gamma = 0$, 0.226, 0.451, and 2.257, and its asymptotic behavior for large X for $\beta\gamma = 0.226$, 0.451, and 2.257. x is proportional to the velocity, X = 1 corresponding to kT.

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