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The research accomplishments in this project include: parallel implementation of the *p*-version, development of multi-*p* methods and multi-*p* conditioning, study of the conditioning of global stiffness matrices, investigation of the problem of model selection, numerical analysis of material interface singularities in two dimensions, and extraction of generalized stress intensity factors using the principle of complementary energy.

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Title:
Research and Development of the
p-version of the Finite Element Method

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Contents

1 Introduction

1.1 Impact and Influence of the p-version

Research and development of the p-version of the finite element method at Washington University have been supported by AFOSR since 1977.

The p-version of the finite element method, together with the h–p version, have now been as widely accepted as the classical h-version, as efficient computational approaches to the solution of a wide variety of engineering and scientific problems. One current computer implementation of the p-version is the code MSC/PROBE, which is marketed by the MacNeal-Schwendler Corporation. p-version capabilities have been incorporated into MSC/NASTRAN and are now available to the general user. Thus, the p-version which until recently has only been licensed for use by a large number of aerospace companies and government laboratories on an individual basis, is now accessible to all users of MSC/NASTRAN.

PROBE contains many unique features including fracture mechanics extraction procedures which were developed specifically for the p-version of the finite element method. These procedures, along with explicit quality control features, provide an excellent tool for performing fracture mechanics analyses and verifying the accuracy of the results [1].

Several other computer codes which implement the p-version are now available for engineering use. These codes include:

- ANSYS, marketed by Swanson Analysis, Houston, PA.
- IDEAS, marketed by Structural Dynamics Research Corp., Cincinnati, OH.
- COSMOS/M, marketed by the Structural Research and Analysis Corporation, Santa Monica, CA.
- Mechanica, marketed by Rasna Corporation, San Jose, CA.
- NISA/P-Adapt, marketed by Engineering Mechanics Research Corporation, Troy, MI.

(for some details on these codes, see Machine Design, July 25, 1991, pp. 73–77, and Mechanical Engineering, September, 1991, pp. 83–84).

Most important, perhaps, is a new software structure, PEGASYS, marketed by ESRD (Engineering Software Research and Development).

PEGASYS: A Software Infrastructure for R & D

PEGASYS is an advanced software infrastructure, designed with two objectives in mind: To support research and to provide a framework for technology deployment in numerical simulation.

Essentially, PEGASYS is an open system of software modules with well-documented interface specifications. PEGASYS currently consists of approximately 250,000 lines of code, written in the FORTRAN and C languages. PEGASYS is maintained in the popular workstations, such as Hewlett Packard, Silicon Graphics, SPARCstation, IBM RS6000 and VAXstation. Using the software modules of PEGASYS, it is possible for researchers and developers to assemble simulation software products designed to meet specific engineering and scientific objectives.

A number of finite element procedures concerned with element formulation, mapping, assembly, extraction procedures, error estimation and adaptivity have been completed. The currently active R & D projects utilizing PEGASYS include the following:

- Development of hierarchic models for structural plates and shells;
- Development of hierarchic models for laminated composites;
- Superconvergent methods for the computation of stress intensity factors, including generalized stress intensity factors for multi-material interface problems;
- Investigation of crack propagation phenomena;
- Mechanics of filament-wound composites;
- Advanced solution methods for elastic-plastic problems;
- Development of models for structural connections;
- Error estimation and quality control procedures;
- Implementation of advanced finite element software on parallel computers;
- Design sensitivity analysis.

Some current R & D users of PEGASYS include: Washington University in St. Louis, The University of Maryland at College Park, The University of Iowa, McDonnell Douglas Aerospace Co., Ford Motor Co., Kelly AFB, The University of Technology in Espoo, Finland. PEGASYS is being used by NASA LBJ Space Center in Houston, and is being evaluated for use at Lockheed Fort Worth, McDonnell Aircraft Company and Ford Motor Company.

Thus, earlier research into the p-version which was supported by AFOSR, has rapidly been transferred into modern technology.

1.2 Hierarchic Modeling and Analysis of Structural Connections

In the last few years there has been a great deal of progress in the development of methods for controlling the errors of discretization in finite element analysis. It has been established that, with properly designed finite element meshes and p-extensions, it is possible to achieve *exponential* convergence rates for very large classes of problems and, with superconvergent extraction procedures, engineering data can be computed from finite element solutions with accuracies comparable to the accuracy of the strain energy. The stopping criterion is that *the data of interest must be substantially independent of the discretization.*

Control of errors of idealization is based on the idea that any particular model, for example a model based on the linear theory of elasticity, is embedded in a set of more general models, for example, ones that account for geometric and/or material nonlinearities. Which model is appropriate for a given task, depends on the goals of computation and the required accuracy. In general one starts with a simple linear model. Once a reasonably accurate approximate solution is available, then it is possible to judge whether the solution

violates the assumptions (i.e., restrictions) incorporated in the model. The key question is, whether removal of those restrictions would have a significant effect on the conclusions drawn for the model. If that is the case then the model has to be "upgraded" and a new solution obtained which, once again, has to be examined for consistency with respect to the restrictions incorporated in the model. The stopping criterion is that the data of interest must be *substantially independent of any restrictions imposed on the model.* The construction of hierarchic models for structural plates and shells is described in reference [2]. An investigation of hierarchic modelling techniques applied to fastened structural connections is presented in reference [3].

Fasteners are complicated structural assemblies, full stress analysis of which would require consideration of three-dimensional problems involving friction, contact, nonlinear material properties, and other factors, such as mode of installation, etc. A number of idealizing decisions are necessary in order to make the problem tractable by numerical methods. This generally means that the problem has to be reduced to a two-dimensional one and nonlinearities must either be ignored, or only mild forms of nonlinearities considered. The objectives are to compute (a) the total force acting on fastener groups; (b) the forces acting on individual fasteners; (c) critical combinations of stresses or strains in the neighborhood of the most heavily loaded fasteners; (d) the stress intensity factors for cracks in the vicinity of fasteners.

There is renewed interest in this problem, particularly in the aerospace industry, for two reasons: First, concern over the safety of aging aircraft requires re-evaluation of structural connections that remain in service much longer than the original design life. Second, previously established design procedures are based on assumptions of substantial ductility in the fasteners and the connected parts. These assumptions do not hold, in general, for composite materials. For this reason, determination of the distribution of forces in fastener groups subjected to critical loads and determination of the stress distributions in the neighborhood of fasteners, are important. Given the very low ductility of composite materials, use of the linear theory of elasticity is sufficient in a very large number of practical problems.

There may be considerable interest on the part of the Air Force in analyzing structural connections because there is a potential for substantial improvements in the reliability and performance of mathematical models for structural connections and their repairs. Each repair being a separate design problem, improvements in this area are of obvious importance to the Air Force.

The representation of fasteners in current modeling practice is a source of serious errors: In current modeling practice fasteners are usually represented by multipoint constraints, that is nodes of the finite element mesh are positioned at fastener locations and the nodes are connected by rigid or flexible bars. *This modeling practice is conceptually wrong and the computed forces in the fasteners are entirely discretization-dependent [3, 4, 5, 6, 7, 8].*

1.3 References

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2 Summary of Research Accomplishments

Need for iterative solvers and parallelization in the p-version.

Direct methods are fully satisfactory for the solution of the assembled system of linear equations in the p-version for small size problems or large sparse systems. But large size problems (with more than 30,000 degrees of freedom) which are today common in finite element analysis, are inefficient both in terms of CPU time and storage [1]. Iterative methods are a much better choice for problems with a large number of elements, and also do not suffer from fill-in. Iterative methods can be implemented on parallel computers and achieve computational load balance, that is, the computational load can be balanced among processors. Classical iterative methods: Jacobi, Gauss-Seidel, SOR, SSOR, Chebyshev methods, have been studied thoroughly and solve well-conditioned systems efficiently. But these methods converge very slowly for ill-conditioned systems.

Also, they do not exploit the special structure of the global stiffness matrix. Iterative methods, specially designed for the p-version, can lead to dramatic increases in the rate of convergence, to implementation on parallel computers and, hence, to impressive savings in cost. These new methods will be used for complicated problems involving hundreds of thousands of degrees of freedom. Such problems are now becoming common and very realistic.

2.1 Parallel Implementations based on the p-version

An iterative method based on the textured decomposition method has been developed in order to solve the systems of linear equations arising in the p-version of the finite element method. The iteration was used to implement the p-version in parallel on an MIMD computer NCUBE/six, the objectives are two-fold: to achieve high computational efficiency (that is computational load should be balanced among the processors) and simultaneously to achieve rapid convergence.

A superelement, consisting of four adjacent rectangular finite elements, is constructed for two dimensional problems. Based on the structural property of the shape functions, each superelement is partitioned into three blocks in two different ways, and a two-leaf textured decomposition (TD) is used. Computations for a superelement associated with each leaf are assigned to two processors and are performed in parallel. A new preconditioner is introduced to accelerate convergence in a preconditioned textured decomposition (PTD). A special local communication strategy is used to avoid global assembly and global communication.

Three model problems: A Poisson equation on a rectangular domain with a smooth true solution, a Laplace equation on a rectangular domain with a near singular solution, and a Poisson equation on L-shaped domain, are solved. The conjugate gradient method, the textured decomposition method, the recursive textured decomposition method, both with and without preconditioning; and the classical iterative methods (Jacobi, Gauss-Seidel, SOR), are used to solve the three model problems. Load balance, speedup ratio, and spectral radii of the various iterations are studied. The test results indicate that recursive PTD with a local communication strategy gives at least a 30% improvement in computational time over the other methods.

This method has been presented at papers at the SIAM National Meetings and will appear in the SIAM J. on Scientific Computing [2].

2.2 Multi-p Methods

A natural analogy to the multigrid method, which is used in connection with the finite difference method or the h-version of the finite element method, is the multi-p method which is used in connection with the p-version of the finite element method and hierarchical shape functions. Each method, multigrid or multi-p, is based on a fundamental iterative scheme, e.g., Gauss-Seidel or SOR. We have studied multi-p methods based on general linear stationary iteration schemes. Various V-cycle algorithms for the multi-p methods are formulated including standard multi-p V-cycle (SMPV), modified multi-p V-cycle (MMPV) and varying multi-p V-cycle (VMPV). Convergence results for each of the V-cycle algorithms have been provided. We have shown that, using a general linear stationary iterative scheme as a smoother, the standard multi- V-cycle algorithm has a linear convergence rate, but this rate is faster than that of the smoother. The modified multi-p V-cycle algorithm also has a linear convergence rate, but again this rate is faster than the rate of the standard multi-p V-cycle. The convergence of the varying multi-p V-cycle algorithm is a consequence of the convergence of both standard and modified V-cycle methods. Numerical experiments on representative problems have been conducted, and the numerical results agree and support our theoretical analysis [3].

In addition, we have studied nested multi-p methods. An error estimate has been derived for the nested multi-p methods. By comparing the nested multi-p methods with the multi-p V-cycle methods, we found that, at low accuracy, the nested multi-p methods are more efficient, but, at high accuracy, the multi-p V-cycle methods. This leads to the so-called accelerated multi-p V-cycle methods which are a combination of the nested multi-p methods and the V-cycle methods. Convergence of the accelerated multi-p V-cycle methods is proved. Numerical results indicate that the accelerated multi-p V-cycle methods are 80% more efficient than the underlying iteration. Some of the numerical results are shown in the figure 1.

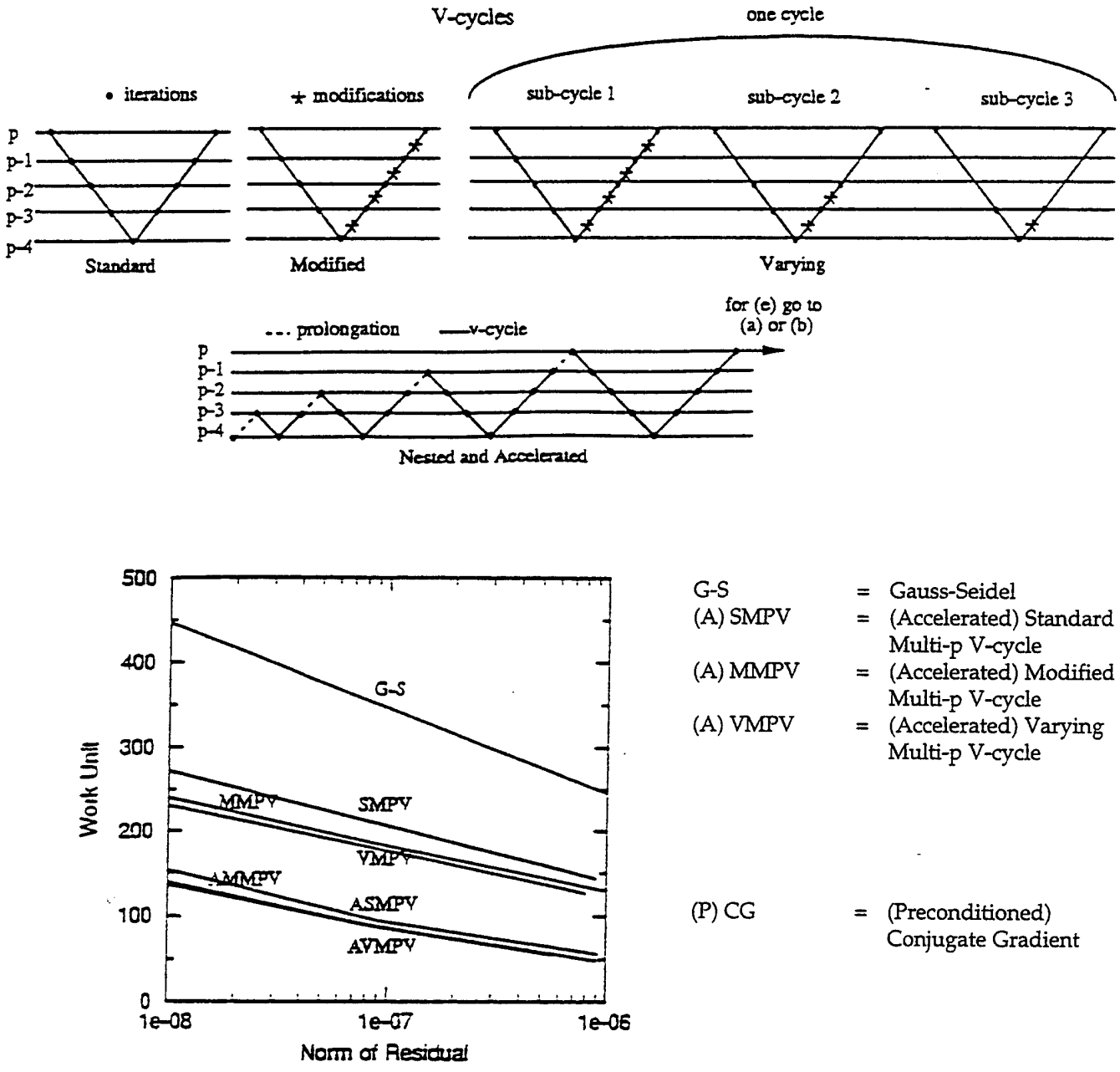

Figure 1. Comparison of the multi-p V-cycles.

2.3 Multi-p Preconditioning

In general, preconditioned conjugate gradient methods are regarded as very promising iterative methods for solving linear systems of equations. In particular, conjugate gradient methods preconditioned by first condensing the finite elements and then using linear elements to construct a preconditioner have been studied by some researchers [4].

An algebraic theory for multi-p methods has been presented and analyzed. Convergence and symmetric properties are proved under suitable conditions. It is then shown how these multi-p methods can be used as preconditioners for the conjugate gradient method (CG). In particular, it is shown that given any preconditioner M_p to CG, a multi-p preconditioner B_p based on M_p can be constructed, which leads to a smaller condition number (and hence faster convergence). When B_p is applied as a preconditioner to condensed finite elements, the condition number is shown to grow slower than $C(1+\log^2 p)$, the best currently known result for 2-D problems in the p-version of the finite element analysis [3].

Numerical experiments on representative problems indicate that the condition numbers after multi-p preconditionings are, in fact, independent of p. The numerical results also show greater efficiency for PCG with the multi-p preconditioners in terms of number of iterations and CPU time when compared with two sophisticated linear equation solvers: (1) a direct frontal solver specially designed for the p-version of the finite element analysis; (2) a highly tuned preconditioned CG code in ITPACK. Preliminary comparisons of the number of iterations are also made with ROCKITS [5], a new commercial code used for the p-version.

The methods presented are intended for use in the p-version of the finite element analysis, but are general in nature and can be applied to a wide variety of problems.

The following three dimensional elastostatic problem on a brick domain illustrates our results.

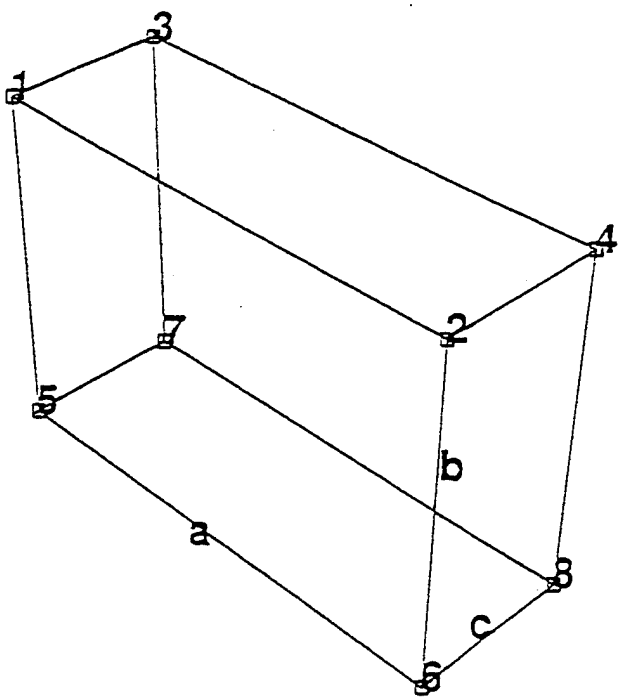

Figure 2. A 3-D elastostatic problem on a brick domain.

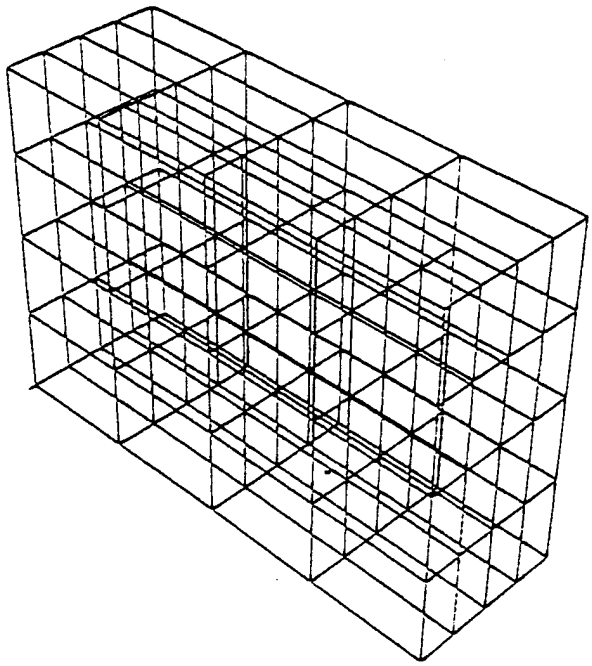

Figure 3. Uniform mesh with 64 elements.

A 3-D elastostatic problem on a brick domain–CPU time (sec.)

p	1	2	3	4	5	6	7	8
DOF	369	1269	2169	3789	6129	9381	13737	19389
Direct Solver	3	10	25	62	148	331	703	1487
PCG by Multi-p	4	8	15	32	64	123	237	735
PCG by SSOR	11	60	153	472	1186	2111	-	-

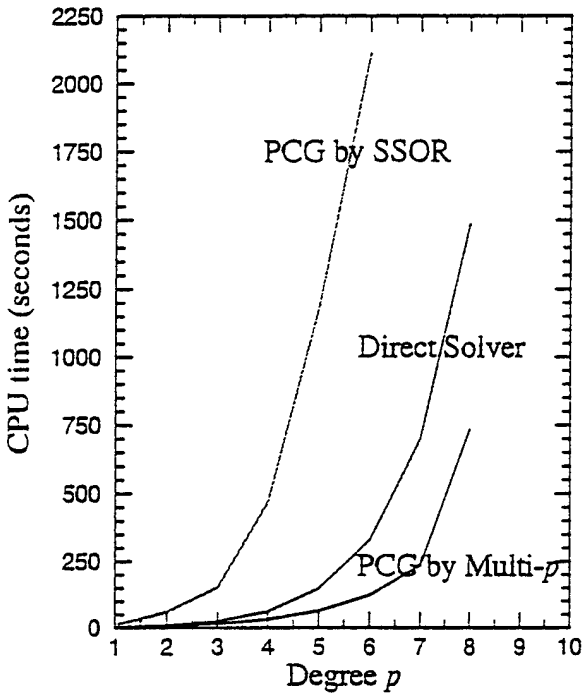

Figure 4. Relationship between CPU time and degree p.

These results have been reported at SIAM National Meetings and will appear in SIAM J. on Scientific Computing [3, 6].

2.4 Conditioning of Global Stiffness Matrices

In order to properly design preconditioners and to understand the rate of convergence of iterative schemes, it is important to know the condition number of global stiffness matrices appearing in the p-version.

We have presented a theory for bounding the minimum eigenvalues, maximum eigenvalues and condition numbers of stiffness matrices arising from the p-version of finite element analysis. Both lower and upper bounds are derived for the minimum eigenvalues, maximum eigenvalues and the condition numbers, which are valid for stiffness matrices based on a set of general basis functions that can be used in the p-version. Although the theory is specifically developed for the p-version and general basis functions, it is shown that the methodology used can be applied easily to both the h-version and the $h-p$ version of the finite element methods. For a class of hierarchical basis functions that has been popularly used in the p-version, explicit bounds are derived for the minimum eigenvalues, maximum eigenvalues and condition numbers of stiffness matrices. Our results show that the condition numbers of the stiffness matrices grow at most as $p^{4(d-1)}$, where d is the number of dimensions. This growth is quite slow compared with the general bounds, which provides new theoretical support for using this class of basis functions. Numerical results are also presented which indicate that our theoretical bounds are quite sharp.

2.5 The Problem of Model Selection

This phase of the work is motivated by the recognition that proper model selection is an essential prerequisite for reliable numerical simulation of physical systems. Specifically, work on model selection is focussed on two areas: One is the development of proper modelling techniques for fastened structural connections, the other is model selection for structural plates and shells made of laminated composites. Both areas are of substantial practical importance: In computations involving durability and damage tolerance, reliable and accurate estimation of the load distribution within aeronautical structures is essential. Loads are typically transmitted through lugs, actuators and fasteners. Increasingly, aircraft components are fabricated of composite materials. Reliable analytical procedures for structural and strength analyses of these important structural elements are not currently available.

The problem of fastened structural connections is approached by the use of space enrichment techniques: The finite element space is enlarged through the introduction of the fundamental solution, multiplied by a cutoff function, such that the nearly singular character of the fastener interacting with the plate is well represented. This method is expected to lead to convenient and accurate representation of the structural action of large numbers of fasteners.

In the case of laminated plates the stress distributions near boundaries, discontinuities and at interfaces are generally very different from the stress distribution in the interior regions. Boundary layer effects are normally present, and the problem in those regions is essentially three-dimensional. Hierarchic models for laminated plates make it possible to approximate the three-dimensional problem through the solution of two-dimensional problems without the expense of a fully three-dimensional analysis.

The proper choice of model depends on the problem description and the data of interest. For this reason, the model definition itself must be adaptive. In order to make this possible, a hierarchic sequence of models has been defined. The essential property of hierarchic plate models is that the exact solutions corresponding to the sequence of models converge to the exact solution of the fully three-dimensional problem:

$$\lim_{i \to 0} \| u_{EX}^{(3D)} - u_{EX}^{(HM|i)} \|_E = 0$$

where $u_{EX}^{(3D)}$ is the exact solution of the fully three-dimensional problem, $u_{EX}^{(HM|i)}$ is the exact solution of the ith hierarchic model. Subsequently the following desirable properties were stated and the conditions for achieving them clarified:

1. (a) With respect to the thickness (t) approaching zero, the exact solution of each model should converge to the exact solution of the fully three-dimensional problem in energy norm:

$$\lim_{t \to 0} \frac{\| u_{EX}^{(3D)} - u_{EX}^{(HM|i)} \| E}{\| u_{EX}^{(3D)} \|_E} = 0 \quad i = 1, 2, \ldots$$

   In the case of plates the limiting solution is the Kirchhoff plate model. The first model in the hierarchy is the Reissner-Mindlin model.

2. (b) When $u_{EX}^{(3D)}$ is smooth then the hierarchic models yield optimal rates of convergence,

that is,

$$\lim_{t\to 0} \frac{\|u_{EX}^{(3D)} - u_{EX}^{(HM|i)}\|_E}{\|u_{EX}^{(3D)}\|_E} \leq Ct^{\alpha_i} \quad i = 1, 2, \ldots$$

where $\alpha_i >$ is the largest possible constant.

When the region of interest includes the boundaries of plates and shells then boundary layer effects must be taken into consideration. The exact solution of plate models may differ very substantially from the exact solution of the corresponding fully three-dimensional problem at the boundaries. Thus, when the data of interest depend on the solution at the boundaries, which is often the case in engineering applications, proper model selection through the use of hierarchic models is essential.

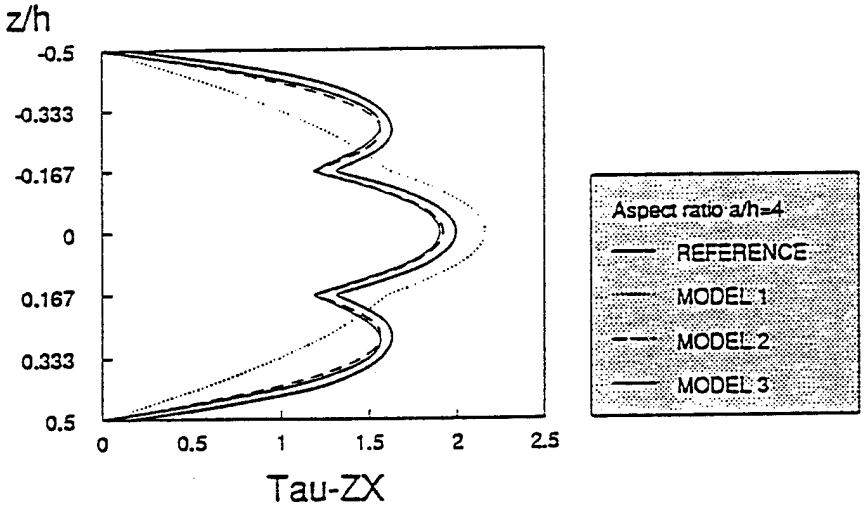

Figure 5. Typical through-thickness shear stress distribution computed for a 3-ply (-45 + 45 - 45) soft-simply supported square plate from a hierarchic sequence of models. The reference is the fully three-dimensional model.

2.6 Numerical Analysis of Material Interface Singularities in Two Dimensions

Eigenpairs

The solution for the linear elasticity problem in two dimensions (2-D) in the vicinity of a singular point can be expanded in the form [7]:

$$\vec{u} = \sum_{i=1}^{\infty} \sum_{m=0}^{M} C_{im} r^{\alpha_i} \ln^m r \, \vec{f}_{im}(\theta) \tag{1}$$

where C_{im} are the coefficients of the asymptotic expansion (called the generalized stress intensity factors - GSIFs), and α_i and $\vec{f}_{im}(\theta)$ are eigenpairs which depend on the boundary conditions.

At present, the possibility to evaluate these GSIFs, which determine failure initiation at singular points in a 2-D domain, especially when the singularity is caused by multi-material interfaces, is very limited.

A numerical method based on the Steklov problem for the computation of the eigenpairs resulting from singularities due to corners, abrupt changes in material properties and boundary conditions is presented (see [8]).

Numerical studies have indicated that the computed values converge strongly, are accurate and inexpensive from both the computational point of view and the point of view of human time needed for input preparation.

This method is very important from the practical point of view because it provides a rigorous quantitative basis for investigating failure events, such as delaminations of composite materials at corners, and failure in electronic devices.

The Steklov Weak Form.

Notation: We denote the two displacements (variables) in the x and y directions by u_x and u_y respectively. The normal and tangential displacements and tractions, will be denoted by T_n, T_t and u_n, u_t, respectively. In the vicinity of the corner we assume that no body forces are present.

Let us consider a domain Ω_R shown in the figure 6, where r, θ are the coordinates of a cylindrical coordinate system located in the singular point. On the boundaries Γ_1 and Γ_2 homogeneous boundary conditions are introduced.

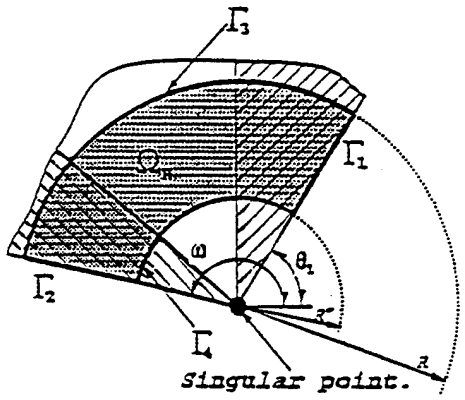

Figure 6. Domain and notations for the modified Steklov formulation.

In Ω_R, u_x and u_y may be represented as follows:

$$\vec{u} \stackrel{\Delta}{=} \left\{ \begin{matrix} u_x \\ u_y \end{matrix} \right\} = r^{\alpha} \left\{ \begin{matrix} f(\theta) \\ g(\theta) \end{matrix} \right\}. \tag{2}$$

Under special (exceptional) circumstances, $\vec{u}$ may also have additional terms $\ln r$ terms, however this case is not treated in the following.

Using (2), on Γ_3 we have:

$$(\partial \vec{u}/\partial \nu) = (\alpha/R)\vec{u} \quad , \quad (x, y) \in \Gamma_3, \tag{3}$$

and a similar condition on Γ_4.

Multiplying the equilibrium equation by $\vec{v} = \{v_x\ ,\ v_y\}^T \in H^1(\Omega_R) \times H^1(\Omega_R)$, we obtain after some mathematical manipulations the following weak form, called the modified Steklov weak form:

$$\begin{aligned} &\textbf{Seek} \qquad \alpha \in \mathcal{C}\ ,\ \vec{0} \neq \vec{u} \in H^1(\Omega_R) \times H^1(\Omega_R), \\ &\mathcal{B}(\vec{u},\vec{v}) + \textstyle\sum_{i=1}^{2} \mathcal{M}_i(\vec{u},\vec{v}) - (\mathcal{N}_R(\vec{u},\vec{v}) + \mathcal{N}_{R^*}(\vec{u},\vec{v})) = \\ &\alpha\left(\mathcal{M}_R(\vec{u},\vec{v}) + \mathcal{M}_{R^*}(\vec{u},\vec{v})\right), \quad \forall \vec{v} \in H^1(\Omega_R) \times H^1(\Omega_R) \end{aligned} \tag{4}$$

where:

$$\begin{aligned} \mathcal{B}(\vec{u},\vec{v}) &\stackrel{\Delta}{=} \textstyle\int\int_{\Omega_R} ([D]\vec{v})^T [E][D]\vec{u}\, d\Omega, \\ \textstyle\sum_{i=1}^{2} \mathcal{M}_i(\vec{u},\vec{v}) &= \textstyle\sum_{i=1}^{2} \int_{\Gamma_i} \vec{v}^T [A_1]^T [A_k]\vec{u}\, ds, \end{aligned}$$

$$\mathcal{M}_R(\vec{u},\vec{v}) = \int_{\theta} \left[\vec{v}^T [A_1]^T [A_3][E][A_5]\vec{u}\right]_{r=R} d\theta, \tag{5}$$

$$\mathcal{N}_R(\vec{u},\vec{v}) = \int_{\theta} \left[\vec{v}^T [A_1]^T [A_3][E][D^{(\theta)}]\vec{u}\right]_{r=R} d\theta\ , \tag{6}$$

where the various matrices are given in [9], and $[E]$ is the material matrix.

Remark 1 The domain Ω_R does not include singular points, hence no special refinements of the finite element mesh is required. Furthermore, Ω_R is very small in size.

The bilinear forms $\mathcal{N}_R$ and $\mathcal{N}_{R^*}$ are *non-symmetric* with respect to $\vec{u}$ and $\vec{v}$. As a consequence, the symmetric properties of the weak form are destroyed. This means that in general complex eigenvalues and eigenvectors exist.

Also note that the formulation of the weak form has not limited the domain Ω_R to be isotropic, and in fact can be applied to multi-material anisotropic interface, as will be demonstrated by a numerical example.

The expressions in (4) are reformulated in the framework of the p-version of the finite element method, where the solution of the eigenproblem are the desired eigenpairs.

Numerical Example

Consider two orthotropic materials, graphite and adhesive (epoxy), bonded together, with plane strain condition assumed. See figure 7.

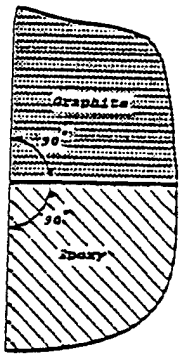

Figure 7. Orthotropic bonded materials test problem.

This problem was chosen to demonstrate the Steklov method for anisotropic multi-materials with a singular point. The material properties are listed in Table 1, where E is the modulus of elasticity, ν is the Poisson's ratio, and G is the modulus of rigidity.
The first three exact non-zero eigenvalues, obtained using the Lekhnitskii stress potentials, are given in the following; $(\alpha_1)_{EX} = 0.905 \pm 0.0000i$, $(\alpha_2)_{EX} = 1.000 \pm 0.0000i$, $(\alpha_3)_{EX} = 1.944 \pm 0.3051i$.

The mesh used for this example problem has the minimum possible number of finite elements, i.e. one element in each anisotropic subdomain. The exact eigenvalues are given with an accuracy of up to the third digit, so that the accuracy of the numerical results may be assessed up to about 0.01% relative error.

Convergence curves of the absolute relative error in the eigenvalues is shown in figure 8.

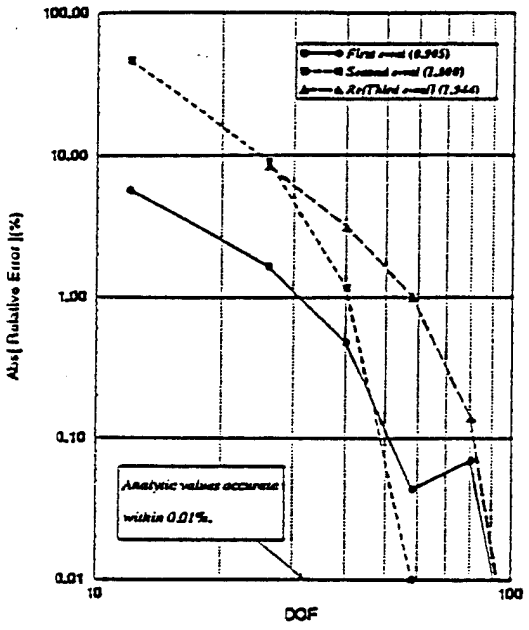

Figure 8. Convergence of first three eigenvalues.

The results demonstrate an excellent convergence rate for the coarsest mesh possible.

2.7 Extraction of GSIFs Using the Principle of Minimum Complementary Energy

By utilizing the principle of minimum complementary energy in conjunction with the Steklov formulation and the p-version of the finite element method, the GSIFs for anisotropic as well as for isotropic domains, can be computed with high accuracy.

The following method is proposed. A small subdomain, Ω_R, is constructed around the singular point, by intersecting a circle of radius R centered in the singular point, and the domain Ω. The approximated pointwise finite element solution is applied on the boundaries of Ω_R.

The finite element method based on the principle of minimum complementary energy is used now over Ω_R, where the trial and test functions are the approximated eigenpairs with the GSIFs as unknowns. The finite element formulation chooses these coefficients such that the complementary energy in Ω_R is maximized. Solving the finite element system of equation over Ω_R, one obtains an approximation for the series coefficients.

Numerical experiments for the scalar elliptic problem (for both isotropic and anisotropic materials) showed that the GSIFs converge to the exact values as fast as the error in the strain energy, thus exhibit superconvergence. For a detail discussion and numerical examples see [10].

	Graphite	Adhesive
$E_{11} \times 10^6$psi	20.	1.4
E_{22}	2.	1.4
E_{33}	2.	1.4
ν_{12}	0.450	0.3
ν_{23}	0.040	0.3
ν_{31}	0.045	0.3
$G_{12} \times 10^6$psi	1.1	2.7

Table 1: Material properties.

Summary

It is expected that, much like crack extensions, failure initiation and propagation can be correlated with the GSIFs, but neither the computational procedures, nor the analytical understanding are available for the anisotropic multi-material interfaces.

A reliable numerical method for the computation of the eigenpairs and the generalized stress intensity factors (GSIFs) which characterize the solution in the neighborhood of singular points in anisotropic multi-material interfaces has been presented.

Numerical experiments and mathematical analysis indicate that the computed values converge strongly, are accurate and inexpensive from the points of view of human time needed for input data preparation, and required computer time.

This method is very important because it provides a rigorous quantitative basis for investigating failure events, such as delamination of composite materials, and failure in electronic devices.

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4 Papers Published and Presented Since the Start of the Project

4.1 Published Papers

1. "Hierarchical Finite Elements and Precomputed Arrays", by M.P. Rossow and I.N., Katz, Int. J. for Num. Method in Engr., Vol. 13, No. 6 (1978) pp. 977–999.

2. "Nodal Variables for Complete Conforming Finite Elements of Arbitrary Polynomial Order", by I.N. Katz, A.G. Peano, and M.P. Rossow, Computers and Mathematics with Applications, Vol. 4, No. 2, (1978), pp. 85–112.

3. "Hierarchic Solid Elements for the p-version of the Finite Element Method", by I.N. Katz, B.A. Szabo and A.G. Peano (in preparation).

4. "p-convergent Finite Element Approximations in Linear Elastic Fracture Mechanics", by A.K. Mehta (doctoral dissertation), Department of Civil Engineering, Washington University (1978).

5. "An Improved p-version Finite Element Algorithm and a Convergence Result for the p-version", by A.G. Kassos, Jr. (doctoral dissertation) Department of Systems Science and Mathematics, Washington University, (August, 1979).

6. "Hierarchic Families for the p-version of the Finite Element Method", I. Babuska, I.N. Katz and B.A. Szabo, invited paper presented at the Third IMACS International Symposium on Computer Methods for Partial Differential Equations, published in Advances in Computer Methods for Partial Differential Equations - III (1979) pp. 278–286.

7. "The p-version of the Finite Element Method", I. Babuska, B.A. Szabo and I.N. Katz, SIAM J. of Numerical Analysis, Vol. 18, No. 3, June 1981, pp. 515–545.

8. "Hierarchic Triangular Elements with one Curved Side for the p-version of the Finite Element Method", by I.N. Katz (in preparation).

9. "The p-version of the Finite Element Method for Problems Requiring C^1-Continuity", by D.W. Wang (doctoral dissertation), Department of Systems Science and Mathematics, Washington University, August 1982.

10. "Implementation of a C^1 Triangular Element based on the p-version of the Finite Element Method", by I.N. Katz, D.W. Wang and B. Szabo, Proceedings of the Symposium in Advances and Trends in Structural and Solid Mechanics, October 4-7, 1982, Washington, D.C.

11. "Implementation of a C^1-triangular Element based on the p-version of "The Finite Element Method", by Douglas W. Wang, I.N. Katz and B.A. Szabo, Computers and Structures, Vol. 19, No. 3, pp. 381-392, (1984).

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14. "A Reliable Root Solver for Automatic Computation with Application to Stress Analysis of a Composite Plane Wedge," by X-R. Ying (doctoral dissertation), Department of Systems Science and Mathematics, Washington University, December, 1986.

15. "Computation of Stress Field Parameters in Areas of Steep Stress Gradients", by B.A. Szabo, Communications in Applied Numerical Methods, Vol. 2, pp. 133–137, (1986).

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4.2 Presented Papers

1. "Hierarchical Approximation in Finite Element Analysis", by I.N. Katz, International Symposium on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23-27, 1977.

2. "Efficient Generation of Hierarchical Finite Elements Through the Use of Precomputed Arrays", by M.P. Rossow and I.N. Katz, Second Annual ASCE Engineering Mechanics Division Speciality Conference, North Carolina State University, Raleigh, NC, May 23-25, 1977.

3. "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I.N. Katz, SIAM 1977 National Meeting, Philadelphia, PA, June 13-15, 1977.

4. "Hierarchical Complete Conforming Tetrahedral Elements of Arbitrary Polynomial Order", by I.N. Katz, presented at SIAM 1977 Fall Meeting, Albuquerque, NM, October 31–November 2, 1977.

5. "A Hierarchical Family of Complete Conforming Prismatic Finite Elements of Arbitrary Polynomial Order", by I.N. Katz, presented at SIAM 1978 National Meeting, Madison, WI, May 24-26, 1978.

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7. "Smooth Approximation to a Function in H^2(D) by Modified Bernstein Polynomials over Triangles", by A.G. Kassos, Jr. and I.N. Katz, presented at the SIAM 1979 Fall Meeting, Denver, Colorado, November 12-14, 1979.

8. "Triangles with one Curved Side for the p-version of the Finite Element Method", by I.N. Katz, presented at the SIAM 1980 Spring Meeting, Alexandra, VA, June 5-7, 1980.

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12. "Computer Implementation of a C^1-Triangular Element based on the p-version of the Finite Element Method," by D.W. Wang and I.N. Katz, SIAM 30th Anniversary Meeting, July 19-23, 1982, Stanford, California.

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14. "p-Convergent Polynomial Approximations in $H_0^2(\Omega)$," by D.W. Wang and I.N. Katz, Fourth Texas Symposium on Approximation Theory, Department of Mathematics, Texas A& M University, College Station, Texas 77843, January 17-21, 1983.

15. "Design Aspects of Adaptive Finite Element Codes," by D.W. Wang, I.N. Katz and M.Z. Qian, ASCE-EMD (American Society of Civil Engineers-Engineering Mechanics Division) Speciality Conference, Purdue University, May 25-28, 1983.

16. "Smoothing Stresses Computed Pointwise by the p-version of the Finite Element Method," by I.N. Katz and X-R. Ying, SIAM 1980 National Meeting, Denver, Colorado, June 6-8, 1983.

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19. "Stress Singularities at Angular Corners of Composite Plates," X-R. Ying and I.N. Katz, SIAM Spring Meeting, Pittsburgh, PA, June 1985.

20. "Computation of the Amplitude of Stress Singular Terms for Cracks and Reentrant Corners," B.A. Szabo and I. Babuska presented at the 19th National Symposium on Fracture Mechanics, San Antonio, Texas, June 30-July 2, 1986.

21. "A Global Algorithm for Finding all Roots of an Analytic Function in a Finite Domain," X-R. Ying and I.N. Katz, presented at the SIAM 1986 National Meeting, Boston, Massachusetts, July 21-25, 1986.

22. "A Reliable Argument Principle Algorithm to Find the Number of Zeros of an Analytic Function in a Bounded Domain," X-R. Ying and I.N. Katz, presented at the Symposium on the Impact of Mathematical Analysis on the Solution of Engineering Problems, University of Maryland, September 17-19, 1986.

23. "On Stress Analysis with Large Length Rations," by B.A. Szabo, presented at First World Congress on Computational Mechanics, Austin, Texas, September 24, 1986.

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25. "Computation of Eigenvalues and Eigenfunctions for the Stress Singularity in a Composite Wedge," X-R. Ying and I.N. Katz, presented at the Sixth International Conference on Mathematical Modelling, August 4-7, St. Louis, Missouri.

26. "Use of Dominated Functions to Find a Bounded Domain Containing all the Roots of a Nonlinear Functions," X-R. Ying and I.N. Katz, 1988 SIAM National Meeting, July 10-15, 1988, Minneapolis, Minnesota.

27. "A Reliable Root Solver for Automatic Computation", X-R Ying and I. N. Katz, 1989 SIAM National Meeting, July 17-21, 1989, San Diego California.

28. "A parallel Implementation of the p-version of the Finite Element Method", Y. Zhu and I. N. Katz, ICIAM '91 (July 1991) Boston Massachusetts.

29. "Models for Fastened Structural Connections", B.A. Szabo and J. Bortman, 1991 USAF Structural Integrity Program Conference, San Antonio, TX, December 2-5, 1991.

30. "Multi-p Methods Based on Textured Decomposition", N. Hu and I.N. Katz, SIAM 40th Anniversary Meeting, July 20-24, 1992, Los Angeles, CA.

31. "A Preconditioned Textured Decomposition for Parallel Implementation of the p-version of the Finite Element Method", Y. Zhu and I.N. Katz, SIAM 40th Anniversary Meeting, July 20-24, 1992, Los Angeles, CA.

32. "Mathematical Models: Theory and Practice", B.A. Szabo, Invited lecture, 1992 IBM Europe Institute, Oberlech, Austria, July 20-24, 1993.

33. "A Recursive Textured Decomposition for the Parallel Implementation of the p-version", Y. Zhu and I. N. Katz, 1993 SIAM Annual Meeting, July 12-16, Philadelphia, Pennsylvania.

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36. "Stress Intensity Factors for Multi-site Damage Panels", Ricardo L. Actis and Barna A. Szabó, 1992 USAF Structural Integrity Program Conference, San Antonio, Texas.

37. "Stresses Computed from Hierarchic Plates Models: The case of Laminated Plates", Ricardo L. Actis and Barna A. Szabó, MAFELAB April 27-30, 1993.

38. "Conjugate Gradient Methods Preconditioned by Multi-p Processes", N. Hu and I.N. Katz, 1994 SIAM Annual Meeting, July 25-29, San Diego, California.

39. "A Numerical Method for Solving the Hamilton-Jacobi PDE with Quadratic Cost", L. Chuang and I.N. Katz, 1994 SIAM Annual Meeting, July 25-29, San Diego, California.