Computer Science

The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory

J. Todd Wilson

May 1994

CMU-CS-94-186

Carnegie Mellon

The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory

J. Todd Wilson

May 1994

CMU-CS-94-186

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Thesis Committee:
Dana S. Scott, Chair
Stephen Brookes
Frank Pfenning
Richard Statman
Aleš Pultr (Charles Univ., Prague)

This research has been partially supported by the CMU School of Computer Science, RISC-Linz, Johannes Kepler Universität Linz, and the Medlar Project.

The views and conclusions contained in this document are those of the author and should not be interpreted as representing the official policies, either expressed or implied, of any of the sponsoring organizations.

Carnegie Mellon

School of Computer Science

DOCTORAL THESIS
in the field of
Pure and Applied Logic

The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory

J. TODD WILSON

Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

ACCEPTED:

Dana Scott	8-18-94
THESIS COMMITTEE CHAIR	DATE
[illegible]	8-18-94
DEPARTMENT HEAD	DATE

APPROVED:

[illegible]	11-22-94
DEAN	DATE

Abstract

This thesis studies the framework arising in the algebraic and categorical description of general (or "point-set") topology. Classically, a topological space is a set with structure, the structure being its collection of "open" sets, which taken together determine an abstract notion of proximity. The collection of all such open sets forms a special kind of complete lattice, and it is a class of complete lattices ("frames") motivated by these examples that is the focus of algebraic study—in short, one dispenses with the points and studies the algebra of open sets. This method has had successes not only in general topology, but has also found application in such diverse areas as logic, topos theory, and even computer science.

It is not these specific areas of application, however, with which the thesis is primarily concerned; rather, it is that part of the theory which they all share: the category of frames. This category has as a subcategory the category of complete Boolean algebras, and these two catgories stand in much the same relation as do the categories of topological spaces and sets. As with sets and spaces, complete Boolean algebras are in some ways better behaved categorically than frames, and so the former provides a potential source of information about the latter. For the purpose of obtaining this information, a construction for frames, called the "assembly tower" and present already at the beginnings of the subject, is studied sytematically and in this way found to be a key tool for uncovering both structural and algebraic properties of frames.

In addition to the above categorical approach to studying frames, the thesis also develops an algebraic approach using the new notion of extensional operator and the further notion of regular operator. The theory of these operators is again studied systematically and is shown to be closely related to, and indeed to provide a framework for better understanding, the well-known theory of "nuclei," which has been a fundamental part of frame theory for some time. A link is also established between regular operators and the assembly tower, thus connecting both of the approaches considered in the thesis.

As applications of these two theories, several open problems in the frame theory literature (most of a technical nature) are settled and several natural questions concerning the structure of the category of frames are answered.

Preface

This thesis studies the framework arising in the algebraic and categorical description of general (or "point-set") topology In this framework—variously called frame theory, locale theory, point-free topology, or even pointless topology—the focus is on the algebraic (lattice-theoretic) properties of the open sets of spaces rather than the points; thus "spaces" are defined by certain lattices of "open sets", and points become a derived notion. This shift in viewpoint often produces a uniformity not present with classical topological spaces, and moreover enables. through the use of topos theory, an enlargement of the scope of application of the general theory.

My interest in this subject, when the thesis was begun, was more algebraic and categorical and less topological, hence the topic and organization of this work. Thus. although I have gained through this work an appreciation for the topological aspects of frame theory, the reader coming to this work for insights into topology and topological methods will, I'm afraid, be disappointed. On the other hand. I hope that the work makes for interesting (universal) algebra and category theory.

As indicated in the title, the thesis is organized around the *assembly tower* construction It should be said, however, that this organization was more an act of hindsight than a conscious guiding principle present from the beginning of the work. The results presented here were developed in a mostly haphazard manner, sometimes forming connections that were more systematically explored. In fact, the very first result discovered was the equation for the fixedpoint set of a prenucleus (see 9.22.1, which is a bit more general), and the reader may enjoy contemplating how the rest of the thesis may have arisen from this single point.

As for physical organization, the thesis is divided into nine chapters, each with an introduction describing its contents, and 30 sections, each containing several subsections. The sections are numbered consecutively from the beginning of the thesis, and chapters contain varying numbers of sections For each subsection, there is at most one result, and references to such results are in the form 98.6, for the result of the 6th subsection of section 98. Occasionally, an additional level of numbering is used to label certain results or other objects, for example 98.6.1. Equations (displays) are numbered individually within each subsection and are referred to as in the notation 98.6(2).

As each chapter begins with an introduction, I won't go into any detailed description of individual chapters here. However, the reader will like to know that in Chapter 1, after an introductory section (Section 1) describing frame theory in broad context and introducing the assembly tower, the reader can find, in Section 2, a summary of the results of the thesis, including all of the main results (often with the ideas of the proofs), and with pointers to the subsections containing them. Thus Section 2 is much like a very detailed table of contents. The bibliography at the end of the thesis only contains those works actually cited in the text. For more comprehensive bibliographies, see those of [25] and [22]

The reader will also find "Exercises" scattered throughout the text. I do *not* believe that exercises, as they appear for example in a textbook, are appropriate material for a PhD thesis, and so I should explain that these are here merely to give me a way of including interesting but tangential material into the thesis, without having to devote to them the space necessary for a full development Thus, none of the results of the exercises are used in the main text, though they may be used in other exercises. (If, having heard this, however, the reader would like to try some of the "exercises" for himself or herself, I would certainly not discourage it!)

Finally, I would like to thank several people who have contributed to this thesis in one way or another First of all, there is my advisor, Dana Scott. Although I ventured out into the wide world of frame theory largely on my own and sometimes against his advice he nevertheless showed remarkable patience and support; even from a distance, I was able to learn many things, not the least of which was a deep sense of what research and teaching are all about He also made it possible for me to join him on leave to the Research Institute for Symbolic Computation (RISC) in Linz, Austria, which, in addition to being my first stay outside of North America, provided me with numerous opportunities to make valuable contacts in Europe. Also at Carnegie Mellon, I would like to thank my committee—Steve Brookes, Frank Pfenning, and Rick Statman—for their confidence in me, and several members of the Computer Science Department staff, especially Lydia Defilippo (now retired) and Becky Clark.

It is also my great honor to have Aleš Pultr, of Charles University in Prague, as my co-advisor It was he who awakened my interest in the topological aspects of frame theory and, through several long discussions, made the work meaningful for me. He also gave me several valuable suggestions and criticisms, which helped me to write a better thesis. In connection with this, I would also like thank JiříAdámek for making possible, through the TEMPUS project, several very productive visits to Prague.

At RISC, I would especially like to thank Jochen Pfalzgraf, who, on the one hand, organized a weekly category/topos theory seminar that encouraged work such as mine, and, on the other hand, made numerous arrangements, through the generous support of the MEDLAR project, for me to travel to conferences and meet with other researchers. I would like to thank RISC itself and its director, Bruno Buchberger, for (at times, hard-earned) financial support and Karoly Erdei for his help and understanding. Also at RISC, I would like to thank my officemates, Kim Wagner (another student of Dana) and Karel Stokkermans—especially Kim, who lived through many of the ups and downs of this work with me.

Of my external contacts, I would like to thank Pino Rosolini, who, through much correspondence and many discussions, was able to help me through to the end, and Harold Simmons, whose own work in frame theory provided a lasting motivation for me. Most of all, I would like to acknowledge my substantial debt to Peter Johnstone, whose generosity, from his initial encouragement to his long letters to me on my research, has meant a great deal to me as a student and researcher. The breadth and quality of his work has been a constant inspiration, and I can only hope that, in this thesis, I am able to demonstrate some of that influence.

On the production side, I would like to thank the Cornell Computer Science Department for the computing facilities with which the final stages of this thesis were completed, and I would like to thank Richard Zippel and Robert Constable for their patience while I was finishing up. This document was typeset with LaTeX, using the same style of organization as found in the book *Lecture Notes on Topoi and Quasitopoi*, by Oswald Wyler, and I would like to thank Prof Wyler for interesting discussions on this style of layout. The diagrams were typset with the macro package Xy-pic, developed by Kristoffer Rose. and I would like to thank Kris for his quick responses to several urgent questions that I, as a first-time user, had while preparing the final version of the manuscript

On a more personal note, I would like to thank Dr. Henderson Yeung, who was responsible, through amazingly tireless encouragement over the course of 12 years, for motivating me to pursue higher education at all and, once I was committed to getting a PhD, making sure I made it there

And lastly, and most importantly. I would like to thank my wife, Mary, without whose constant support, understanding, and at times greater devotion to the completion of my research and writing than I myself could muster, this thesis would never have been completed I hereby dedicate this thesis to her, with love.

CONTENTS

Chapter 1
INTRODUCTION AND SUMMARY OF RESULTS

This introductory chapter begins, in Section 1.1, with a look at the background and motivations of frame theory. Then, in Sections 1.2–1.5, we discuss the connections between frame theory and universal algebra, topos theory, logic, and the semantics of computation. Since our approach to the subject is somewhat untraditional, we take some space, starting from Section 1.6, to discuss other approaches. Then, in Section 1.11, we introduce the focus of our research, the assembly tower, and in Section 1.12 give some justification for this choice.

The second section of the chapter is a detailed summary of the results we obtain. All of the major results are included, with pointers to the rest of the thesis. In many cases, the ideas of proofs are outlined. Thus, the reader looking for a convenient entry point into the thesis, or wanting merely to scan the results, should go to Section 2

1. Introduction

1.1. Point-free topology. Frame theory arose from the observation that many properties and constructions of topological spaces can be described entirely in terms of the lattice of open sets of the space, without reference to the points. The open covering formulation of compactness is a good example. Other examples include connectedness, normality, regularity, compactifications, and (Vietoris) hyperspaces. In many of these examples, the classical formulations mention points (and closed sets), but equivalent, "point-free" formulations in terms of open sets are possible. Morphisms of topological spaces—the continuous functions—also have a point-free aspect, since the inverse image f^{-1} of a continuous function $f : X \to Y$ by definition takes open sets of Y to open sets of X These observations lead naturally to the replacement of the category of topological spaces by the category of *locales*, which I now describe.

A *frame* A is a complete lattice satisfying a strong distributive law,

$$a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}, \tag{1}$$

for every element a and subset S of A, and a frame morphism $f : A \to B$ is a function that preserves finite meets and arbitrary joins. These constitute the category **Frm** of frames. This is, of course, what we have with topological spaces: if X is a space, then the collection $\Omega(X)$ of open sets of X forms a frame, and for every continuous function $f : X \to Y$ between spaces, the inverse image restricted to open sets, $f^{-1} : \Omega(Y) \to \Omega(X)$, is a frame morphism. Notice, however, that the continuous function and the associated frame morphism go in opposite directions. For this reason, it is the *opposite* or *dual* category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ of *locales* that serves as a substitute for the category of spaces. The step from frames to locales is formally trivial, as it amounts to just "turning the arrows around", but, as Johnstone has emphasised [25][1], it is conceptually very important

1.2. Universal algebra. Frames may be described as algebraic objects, i.e., as sets with operations that satisfy equations However, they are infinitary algebras in the strongest sense· one needs proper classes of operations and equations for such a description. Nevertheless, frames share with finitary universal algebras a characteristic feature: the existence of free objects on any set of generators. The above facts, summarized by saying that **Frm** is *monadic* over **Set**, already imply a considerable amount about **Frm** (and thus, by duality, **Loc**). For example, **Frm** is complete and co-complete, and it inherits both limits and pullback-stable (regular epi, mono)-factorizations from **Set**

One source of interest in frames from an algebraic point of view is that they are a "borderline" case although **Frm** is monadic over **Set**, certain small modifications of frames no longer possess free objects. For example, if we substitute countable meets for finite meets in the basic operations of a frame, but still keep the same distributivity law, then free objects over countably infinite sets no longer exist (see the paper of Garcia and Nelson [10] for this and other examples). Perhaps the most basic example of this nature, and the first to be discovered (independently by Gaifman [9] and Hales [14]; see Solovay [47] for a simplified proof), is the case of complete Boolean algebras (cBa's) The category **cBa**, like the examples in [10], is a limit-closed full subcategory that, because of the non-existence of free algebras, is not reflective An obvious problem here, called the *reflection problem*, is to characterize the frames having reflections into **cBa**. This problem was considered by Simmons in a series of papers [42, 43, 44, 45, 40] but, despite several advances, has remained unsolved in general.

1.3. Topos theory. Toposes are categories that were introduced by Grothendieck (and others) as a generalization of the category of sheaves on a topological space to support powerful cohomology theories for use in algebraic geometry. The same categories were models of an elementary axiomatization of sheaf categories given by Lawvere and Tierney, who were interested instead in their "set-like" behavior. The first line of development culminated in Deligne's proof of the hardest of the Weil conjectures (specifically, an analog for finite fields of the Riemann Hypothesis), and the second, producing a topos-theoretic proof of the independence of the Continuum Hypothesis by Lawvere and Tierney, models of various intuitionistic theories, and other notable results in logic since then, continues to unfold (See the introduction to Johnstone [20] for more historical information, and the epilog in Mac Lane and Moerdijk [30] for an overview of the literature of topos theory and its many connections to other areas of mathematics.)

Since sheaves on a topological space are defined in terms of the open sets of the space, it is a simple matter to generalize the sheaf construction to locales, where it however still appears as a special case of Grothendiek's construction of sheaves on a

[1] This paper is an excellent survey of locale (and frame) theory and should be considered mandatory reading by anyone interested in learning about the subject; I will make reference to it often. In particular, the reader should consult it for an account of the relation between locale theory and topology and why the category of locales should be considered a good substitute for (and generalization of) the category of spaces.

site. Such *localic toposes*, apart from their motivational importance, are basic to the structure theory of toposes. For example, a theorem of Diaconescu and a theorem of Barr state that, respectively, every (Grothendieck) topos is an open quotient of a localic topos and an ordinary quotient of a Boolean localic topos (i.e., the sheaves over a cBa); a theorem of Joyal and Tierney states that every topos is the topos of G-equivariant sheaves for some groupoid G in the category of locales.

1.4. Logic. In addition to their more elaborate uses in the form of localic toposes as models of intuitionistic set theories, frames are themselves, at a more basic level, the appropriate models of first-order intuitionistic logic. Every frame has, along with the constants 0 (false) and 1 (true) and the binary operations $\wedge$ (and) and $\vee$ (or), a binary operation $\rightarrow$ (implies) that makes it into a Heyting algebra—where we have the familiar adjointness relationship,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c, \tag{2}$$

for all elements a, b, c of the frame—and thus a model of propositional intuitionistic logic. The basic infinitary operation $\bigvee$ and "derived" operation $\bigwedge$ give the structure necessary to interpret existential and universal quantification. In this respect, frames can again be compared to cBa's, which are the natural models of first-order classical logic.

1.5. Semantics of computation. The use of topology in the semantics of computation is one of the cornerstones of the subject, where it embodies the concept of informational approximation and where continuity provides a useful substitute for computability. Smyth [46] has linked this to the notion of a semidecidable property, and Abramsky [1] has explained it in terms of a logic of *finitely observable properties*, a view that is expanded in the book of Vickers [50], where it is called a logic of *affirmable assertions*. The idea is that to observe (or affirm) a conjunction of properties (or assertions) it is necessary to observe them all, whereas to observe a disjunction of properties it is only necessary to observe one of them; thus, with finite resources, one can observe finite conjunctions and arbitrary disjunctions. And, like infinite conjunctions, implications are not finitely affirmable, because they always have the chance of being refuted by an observation beyond the finite number of observations made. An example of the fruitfulness of this view is the paper of Abramsky and Vickers [2], where process semantics is treated in this framework—or, actually, in the more general but quite similar framework of *quantales* [38], where conjunction is replaced by a non-commutative operation more appropriate for modelling observations that introduce side-effects.

1.6. The topological approach. As locales are intended to be generalized spaces, the most common approach to their study is in terms of their relationship to the category of spaces, **Top**. For this, the notion of a *point* of an arbitrary locale is fundamental. If $\star$ is the one-point space, then a point of a space X is the same thing as a continuous map $p : \star \rightarrow X$, which gives a *frame morphism* $p^{-1} : \Omega(X) \rightarrow \Omega(\star)$. We therefore define a point of a frame A to be a frame morphism $A \rightarrow \mathbf{2}$, where $\mathbf{2} = \Omega(\star)$ is the two-element cBa. This is analogous to the situation in the spectral theory of distributive lattices, except that frames in general don't have enough points, in the sense that two different elements of a frame (thought of as open sets) may contain the same points If we identify such pairs of elements, however, we get a quotient frame that is a topology, and this process gives the object part of a functor pt : **Loc** $\to$ **Top** that is right adoint to the functor Ω : **Top** $\to$ **Loc**. The frames for which this quotient morphism (the counit of the adjunction) is an isomorphism are called *spatial*, and the spaces for which the unit is a homeomorphism are called *sober*—a "separation axiom" between T_0 and T_2 and independent of T_1. The sober spaces are thus a reflective full subcategory of all spaces, and locales are, strictly speaking, therefore only a generalization of sober spaces (though this isn't much of a restriction, see [25])

1.7. The categorical approach. As mentioned in 1.1, many topological notions are available, via suitable reformulation in terms of open sets, for locales. Another method of importing topological notions into locale theory is via category theory; the definition of "point" above is an example of this. Another example is localic products. since **Loc** is complete, it has arbitrary products, corresponding to coproducts in **Frm**. Localic products *don't* agree in general with space products, even on sober spaces[2]—but, notably, it is the locale products that are usually better behaved As one example of many, used here to emphasize also the constructive nature of locale theory, consider Tychonoff's theorem that the product of compact spaces is compact. For spaces, this theorem requires some form of choice (in general, it is equivalent to the Axiom of Choice (AC), for Hausdorff spaces it is equivalent to the Prime Ideal Theorem (PIT)); for locales, it not only is choice-free but, as Johnstone says, "viewed from the right perspective it becomes a triviality" [25, p 87]. It is only if one wants to show, in the Hausdorff case, that the resulting locale product has enough points (and hence recover the classical Tychonoff Theorem for Hausdorff spaces) that one needs PIT. Such constructiveness is not just an aesthetically pleasing feature of locale theory; it can be crucial in applications of the theory to "non-classical" settings—including, by a change-of-base result, classical fiberwise topology (see [25] and especially [23] for more discussion on this point)

1.8. The universal algebra approach. Although studying frames as generalized spaces may seem the most natural approach, studying them as universal algebras can also be profitable—and, in many cases, conceptually more simple. As Madden [32, p.109] points out, this is reminiscent of commutative algebra: "Even though [it] has been developed in large measure to support algebraic geometry, most expositions make scant reference to the geometric picture. This is simply a matter of efficiency" I will look here at just one example, the fundamental construction for universal algebras of the complete lattice of congruences. For frames the situation with respect to this construction is especially nice, because the congruences on a frame A, besides forming a frame themselves, have three useful descriptions other than the usual one as equivalence relations on A that are simultaneously subframes of $A \times A$. For the first two, the main observation is that since a congruence θ respects joins, every equivalence class of θ has a largest member, and θ is recoverable both from the operation taking an element to the largest member of its class and from the set of such largest members

[2] Note, however, that the product space is always the spatial (co)reflection of the localic product; thus they agree precisely when the localic product is spatial—which is the case, for example, with products of compact Hausdorff spaces or finite products of locally compact spaces

itself. In the first case, the operations j associated to congruences are exactly those satisfying

$$a \leq ja = j(ja) \qquad \text{and} \qquad j(a \wedge b) = ja \wedge jb \qquad (a, b \in A); \tag{3}$$

such an operation is called a *nucleus*. In the second case, the sets S associated to congruences are exactly those satisfying the closure conditions

$$T \subset S \text{ implies } \bigwedge T \in S \qquad \text{and} \qquad a \in A, s \in S \text{ imply } a \to s \in S, \tag{4}$$

which I call *maxsets* Furthermore it is clear (from its description as the set of the largest representatives of each θ-class) that S with the order inherited from A is isormophic to the quotient A/θ and that j provides the quotient morphism In this way, the frame of congruences on A is isomorphic to the set of nuclei on A, ordered pointwise and denoted $\mathsf{N}A$, as well as to the set of maxsets on A, when ordered by *reverse* inclusion. The frame $\mathsf{N}A$ is called the *assembly* of A.

The third description of the frame of congruences of A, or equivalently now of $\mathsf{N}A$, is as the universal solution in **Frm** to the problem of complementing the elements of A. To explain this, consider for any $a \in A$ the nucleus $c(a)$ defined by $c(a)x = a \vee x$; this nucleus is associated with the maxset $\{x : a \leq x\}$ and with the smallest congruence identifying a and 0 The assignment $a \mapsto c(a)$ is a frame morphism $c_A : A \to \mathsf{N}A$, which is both mono and epi in **Frm**, but is an isomorphism iff A is a cBa. For every $a \in A$, $c(a)$ is a *complemented*, or *Boolean* element of $\mathsf{N}A$ (meaning, of course, that there exists a necessarily unique $j \in \mathsf{N}A$ with $c(a) \vee j = 1$ and $c(a) \wedge j = 0$; in fact, j is the nucleus $u(a)$ defined by $u(a)x = a \to x$). With these preliminaries, the precise universal property of $\mathsf{N}A$ can now be stated: if $f : A \to B$ is a frame morphism such that $f(a)$ is Boolean for every $a \in A$, then there exists a unique frame morphism $\bar{f} : \mathsf{N}A \to B$ extending f (i.e., such that $f = \bar{f} \circ c_A$).

Finally, let me point out that the assignment $A \mapsto \mathsf{N}A$ can be extended (uniquely, by the universal property) to an endofunctor on **Frm** so that the morphisms c_A : $A \to \mathsf{N}A$ become components of a natural transformation $c : I \to \mathsf{N}$ from the identity functor. In terms of congruences, the morphism $\mathsf{N}f : \mathsf{N}A \to \mathsf{N}B$ takes a congruence on A to the congruence generated by its image under f. Every frame morphism $h : A \to B$, since it preserves arbitrary joins, has a right (order-)adjoint, $h_* : B \to A$; for $\mathsf{N}f$ this right adjoint, as is the case for universal algebras in general, is the function $(\mathsf{N}f)_* : \mathsf{N}B \to \mathsf{N}A$ that takes a congruence on B to its inverse image under f. The same function, in terms of maxsets, takes a maxset of B to its image under f_*.

1.9. The categorical structure of Frm. Category theory, in addition to being useful in the formulation of topological notions for locales, can also be applied directly to the study of the category **Frm** itself. For this, the "topological" adjunction **Loc** $\rightleftarrows$ **Top** and the "algebraic" adjunction **Set** $\rightleftarrows$ **Frm** already provide a great deal of information Further information can be gotten by factoring the latter adjunction through various intermediate categories, as explained in [25] Thus, by factoring through the category **CSlat** of complete join-semilattices (which arises by "forgetting" the finite meets), one can deduce the following property of **Frm**. given a (small) directed diagram, all of whose morphisms are mono, the morphisms of the colimiting cone of the diagram are then also mono. Another property deducible using this factorization is that coproducts (and more generally pushouts) distribute over products, in the sense that the canonical morphism

$$B \otimes_A \prod_{i \in I} C_i \longrightarrow \prod_{i \in I} B \otimes_A C_i$$

is an isomorphism, where I have written $\otimes_A$ for the coproduct under A (i.e., the pushout—a kind of tensor product), and where I have supressed mention of the morphisms. See the monograph of Joyal and Tierney [27] for these and other facts relating **CSlat** and **Frm**, especially the analogies developed there between frames and rings, between complete semilattices and Abelian groups, and between frame morphisms and modules, and an explanation of my use of the tensor $\otimes_A$ above. And, it is by factoring the same adjunction this time through the category of *preframes* (which arises by forgetting the finite joins, just keeping the directed ones) that the localic Tychonoff Theorem becomes a "triviality" (see the Pre^3 paper of Johnstone and Vickers [26]).

Yet, despite these sources of information about the categorical structure of **Frm**, many questions remain unanswered. For example, although pushouts preserve products, they do not preserve all limits; in particular, they do not preserve monomorphisms, corresponding to the fact that pullbacks in **Loc** do not preserve surjections. This can be seen as a defect of the category of locales—compared to the situation in spaces, where surjections are pullback-stable—and it thus becomes important to understand this discrepency. Although several classes of pushout-stable monos (here called *universal*) in **Frm** are known, one cannot even say, for example, whether regular monos, or more generally *equationally closed* monos (see the paper of Pultr and Tozzi [37]—these represent well, under some additional separation conditions, topological *quotients*) are universal. Universal monos are also closely connected to the reflection problem, since a frame is reflective iff it has a universal embedding into a cBa. Understanding the class of universal monos has been one of the main motivations of this work.

1.10. The topos-theoretic connection. I would like to mention one other possible approach to studying frames, besides those of topology, universal algebra, and category theory, suggested by the connections between frame theory and topos theory outlined in 1.3. Every locale A has associated to it its category of sheaves, $\mathbf{Sh}(A)$, which is a topos. Now it turns out that certain properties of the locale are reflected as categorical (or logical) properties of the topos and can be studied as such. As an algebraic example, a frame A is a cBa (assuming AC) iff every epimorphism of $\mathbf{Sh}(A)$ splits—a condition that, for the category of sets, is itself equivalent to AC. As a topological example, if a space A is *completely regular, first-countable*, and has no isolated points, then, in the (intuitionistic) set theory determined by $\mathbf{Sh}(A)$, every function $f : \mathbb{R} \to \mathbb{R}$ on the reals is continuous (see [48] for this and some references to similar results going back to the paper of Scott [39])

1.11. The assembly tower. I can now explain the assembly tower, a construction already appearing at the beginning of locale theory, a 1972 paper of Isbell [15], and around which my work is centered. Since N is an endofunctor on **Frm** naturally extending the identity, it can be iterated transfinitely (using pointwise colimits at limit ordinals) to produce an ordinal-indexed family of functors $\{\mathsf{N}^\alpha\}$ and natural transformations $\{c^{\beta,\alpha} : \mathsf{N}^\beta \to \mathsf{N}^\alpha\}_{\beta\leq\alpha}$. Thus. for every frame A there is a diagram

$$A \xrightarrow{c} \mathsf{N}A \xrightarrow{c} \mathsf{N}^2A \xrightarrow{c} \cdots \mathsf{N}^\alpha A \xrightarrow{c} \mathsf{N}^{\alpha+1}A \xrightarrow{c} \cdots, \tag{5}$$

where for each α I have written c for $c_{\mathsf{N}^\alpha} = c_A^{\alpha,\alpha+1}$ $\mathsf{N}^\alpha A \to \mathsf{N}^{\alpha+1}A$ and where, for each limit ordinal λ, $\mathsf{N}^\lambda A$ is the colimit of the diagram consisting of the morphisms $c_A^{\beta,\alpha} : \mathsf{N}^\beta A \to \mathsf{N}^\alpha A$, for all $\beta \leq \alpha < \lambda$. This diagram is called the *assembly tower* of A. I will write c^α for the morphism $c_A^{0,\alpha}$ and, in general, leave off subscripts whenvever possible. As each of the morphisms c in (5) is mono, it follows (from the general fact about colimits of monos mentioned before) that the morphisms c^α are also mono. And as the morphisms c are also epi, it follows easily by induction that the c^α are epi as well.

The first thing to note about the assembly tower is that it gives a "solution" to the reflection problem, in the following sense. Since every element of a cBa is complemented, given a frame morphism $f . A \to B$ to a cBa, there is by induction and the universal property of N a unique morphism $f^\alpha \cdot \mathsf{N}^\alpha A \to B$ extending f for every α. If any of these $\mathsf{N}^\alpha A$ is Boolean, then it clearly is the reflection of A in **cBa**. On the other hand, it can be shown that if A has a reflection in **cBa**, then it must be $\mathsf{N}^\alpha A$ for some α Thus. the reflection problem is reduced to finding conditions on A that correspond to the existence of such an α. One interesting aspect of this problem is that for all known examples of reflective frames A, N^2A is already Boolean and that, moreover, for certain classes of frames (for example the class of *coherent* frames), this holds in general—i.e , for any frame A in the class, A is reflective iff N^2A is Boolean.

The construction $A \xrightarrow{c} \mathsf{N}A$ also has a topological aspect For example, in [25] the locale extensionally the same as the frame $\mathsf{N}A$ is called the *dissolution* of A and denoted A_{d}; it is likened (except for the fact that it is not idempotent) to the discrete modification of A. Despite this, the assembly tower construction as a whole is really orthogonal to the spatial aspects of a locale, in a sense made clear by the previous paragraph. Namely, since 2 is Boolean, it follows that the frames A and $\mathsf{N}^\alpha A$ have the same points As one consequence of this, it turns out that the spatial reflection of $\mathsf{N}^\alpha A$, for any $\alpha \geq 2$, is always the same discrete space (which suggests, perhaps, that A_{dd} is closer to being the discrete modification of A than A_{d}, whose spatial reflection can fail to be discrete). On the other hand, the assembly tower can provide examples of "large" frames with "few" points [3]

1.12. Some philosophy. Although the main reason for focusing on the assembly tower in this thesis is that it underlies most of the results obtained, I can say something here about why the construction is likely to be of some use. In any algebraic category the calculation of limits is trivial. in the sense that it simply involves putting the natural structure on the limit of the underlying sets; it *is* really in the colimits that the algebraicity of the category makes itself known. For example, free algebras (given the fact that the "forgetful" functor is representable) and quotients by congruences are

[3] But by no means the best examples: for any sober space X, there exist arbitrarily large frames A with X as spatial reflection: Isbell et al. [17] construct examples of pointless localic groups, in the sense that they have no points other than the identity. Since such a locale G has exactly one point, it follows that for any index set I, $X \times G^I$, where the product and power are the localic ones, has X as its spatial part.

easily described as colimits, conversely, all colimits can be described naturally in terms of generators and relations Also, in **Frm**, the two results about colimits of monos and pushouts of products and the questions about universal monos, directly concern colimits To put it into a slogan—an apparently common aspiration among category theorists "Colimits are the Essence of Algebra"

Where the assembly tower fits in is this Suppose, for a moment, that we extend the category of frames to include a colimit for the assembly tower of each frame. Since each of these diagrams is *small-directed*, in the sense that every *set* of arrows has an upper bound, this colimit could be constructed as the union of all the frames in the diagram; it would have a proper class of elements in general and would have finite meets and joins of all sub*sets* of the underlying class (but would not have joins of all sub*classes*). Moreover, being the union of the tower, it would be Boolean, since every element appearing at level α of the tower becomes complemented at level $\alpha + 1$ In this extended world, the Boolean objects thus obtained would provide reflections for all frames

Categories similar to the one suggested by the above have been considered before, under the name κ-frames, by Madden [32] A κ-frame, for a regular cardinal κ, is a partially-ordered set that has joins for all sets of cardinality $< \kappa$ and has finite meets that distribute, as in (1) over all such joins The main difference between κ-frames and ordinary frames is that κ-frames are described by only a set of operations and equations, it follows from this that the category of κ-cBa's (the Boolean κ-frames) *is* a reflective subcategory of the category of κ-frames, because the analog of the assembly tower for κ-frames terminates at the κth level. Consideration of the categories of κ-cBa's goes back to at least the 1960's, and much is known about them In particular, they have nicely behaved colimits For example, they have the *strong amalgamation property* and the *congruence extension property* (in particular, pushouts of all monos are mono) and, consequently, all epis are surjective, a property that fails badly for **Frm** (just look at the assembly tower).

The hypothetical extension of **Frm** with which I started can now be seen as part of the category of κ-frames for κ equal to the cardinality of the universe V, in some extension of **Set** where V becomes a set, as is the case with Grothendieck universes. Or, we can assume the existence of an inaccessible cardinal κ and redefine **Frm** to be the category of all *small* frames—those of cardinality $< \kappa$, which are then identical to κ-frames Or, finally, we can adopt an approach similar to the "monster model" approach of model theorists working in Classification Theory: in whatever your particular application of frames, let κ be a regular cardinal larger than the cardinality of any frame you use Then all of your frames are actually κ-frames, and you might as well be working in this category This last is related to reflection principles proposed for use in category theory by Feferman [8] (following a suggestion of Kreisel).

The point is, since the Boolean subcategory is reflective and the reflection functor, being a left adjoint, preserves colimits, information about the colimits in the larger category can be gotten from the better-behaved colimits in the subcategory through the use of the reflection, i e , the assembly tower. This is essentially the technique Madden and Molitor use in [33] to characterize frame epimorphisms. Further use of this idea will described in 2 6.

2. Summary of results

Although the results obtained are all related in some way to the assembly tower, they divide naturally into several topics, each of which is treated seperately below, in the order that they appear in the thesis. To avoid interruption in these discussions, We will gather together here most of the prerequisite material we need. Throught this section, decimal numbers in parentheses refer to where the actual results may be found in thesis.

2.1. Some preliminaries. Recall first that every frame A has an operation $\rightarrow$ defined so as to satify the adjointness relation (2) with $\wedge$, and this $\rightarrow$ makes A into a complete Heyting algebra (cHa for the rest of this paragraph). Conversely every cHa satisfies the distributive law (1), and so is a frame. The cHa morphisms are by definition required to preserve all meets, all joins, and arrow; thus, every cHa morphism is a frame morphism but (it turns out) not conversely. For $a \in A$, the element $a \rightarrow 0$ is often written $\neg a$ and called the *negation* of a, since it corresponds to logical negation in a Heyting algebra, where neither $a \vee \neg a = 1$ nor $\neg\neg a = a$ holds in general.

For any nucleus $j \in NA$, we write A_j for the quotient of A by the congruence associated to j. Meets in the frame NA are computed *pointwise*, that is, for any $J \subset NA$ and $a \in A$,

$$(\bigwedge J)a = \bigwedge\{ja : j \in J\}.$$

However, neither joins nor arrow in NA is computed pointwise

In addition to the previously mentioned nuclei $c(a)$ and $u(a)$, called, respectively, *closed* and *open* because the quotients by these nuclei correspond to the closed and open subspaces of a topological space, there is for every $a \in A$ the *quasi-closed* nucleus $q(a)$ defined by

$$q(a)x = (x \rightarrow a) \rightarrow a. \qquad (x \in A).$$

The quotient $A_{q(a)}$ is a cBa, and, conversely, every cBa quotient of A has this form. The maxset corresponding to $q(a)$ is the set $\{x \rightarrow a : x \in A\}$. Thus, by the second closure condition of (4), this maxset is the smallest one containing the element a; it follows easily from this that every nucleus j can be written

$$j = \bigwedge\{q(a) : ja = a\}.$$

Since frame morphisms preserve complements, every frame quotient of a cBa is a cBa. And since, just as with universal algebras in general, the congruence lattice of a quotient A_j of A is isomorphic to the *interval*

$$[j, 1] = \{k \in NA \;\; j \le k \le 1\}$$

of the congruence lattice NA of A, it follows that $Q = \{q(a) \,.\, a \in A\}$ is an up-closed subset of NA An important quasi-closed nucleus is the double negation, $q(0)$, so called because $q(0)x = \neg\neg x$. An element $a \in A$ is called *regular* if it is a fixedpoint of $q(0)$, i.e., if $\neg\neg a = a$. In the frame $\Omega(X)$, these are exactly the regular open subsets of X.

2.2. Extensional operators. By an *operator* on a Heyting algebra A we will mean any function $l: A \to A$. Various kinds operators on frames, and their associated fixedpoint sets have been studied before. Nuclei, arising as a special case of (Joyal-Tierney) topologies on a topos, are the most important and widely studied. Other examples are the *prenuclei* of Banaschewski [3] and the *derivatives* of Golan and Simmons [12].

Prenuclei are like nuclei, except that they may not be idempotent (a prenucleus is idempotent just in case it's a nucleus), but for every prenucleus there is a unique nucleus with the same fixedpoints. It is for this reason prenuclei often arise: natural constructions involving nuclei frequently result in operators that are prenuclei but are not idempotent; one then only needs to find the associated nucleus, which is often accomplished by transfinitely iterating the prenucleus until it "converges" (13.8). As an illustration, the join of two nuclei j and k can be computed by iterating their composite $j \circ k$, which is only a prenucleus. Or, again, it was Johnstone's original choice-free proof [21] of the localic Tychonoff theorem—essentially involving the transfinite iteration of a particular prenucleus—that eventually led to Banaschewski's paper [3] and the isolation of the notions of prenucleus and preframe.

We introduce a general class of operators on Heyting algebras, called *extensional*, which have nuclei and prenuclei as examples. By definition, an operator l is extensional if it satisfies

$$a \leftrightarrow b \leq la \leftrightarrow lb \qquad (a, b \in A),$$

and, from the point of view of frames as models of first-order intuitionistic logic, these are perhaps the most natural class of operators to consider, since, logically, the definition can be interpreted to mean that l "preserves (degree of) equality": the results of applying l are as equal as the arguments to which it is applied. The extensional operators on A are also exactly the operators that are compatible with all Heyting-algebra congruences on A, in the sense that for any such congruence θ, $a\,\theta\,b$ implies $la\,\theta\,lb$, and they can be characterized in several other ways, as well (9.4). Extensional operators may also be seen to arise (at least when A is complete) from the topos-theoretic connection, where the extensional operators on A are in 1-1 correspondence with the morphisms $\Omega \to \Omega$ in $\mathrm{Sh}(A)$, and this helps to explain why many properties of extensional operators are equivalent to stronger, "uniform" (or "internalized") versions of these properties (as mentioned in 9.24).

Because of the relation to congruences, extensional operators provide a convenient means of doing calculations in Heyting algebras (9.6). We introduce and study several classes of extensional operators (see 9.12 for a graphical summary). For example, we find that the prenuclei of Banaschewski are precisely the inflationary and monotone extensional operators (9.10), and therefore that nuclei are precisely the extensional closure operators (9.11). We find, for every class of extensional operators defined, what we call the upper and lower *classifiers* for the class (9.13–18), and indicate how these lead to structure theorems for operators (9.21).

Finally, we find a simple formula for the set of fixedpoints of an inflationary extensional operator l (9.22.1):

$$\operatorname{fix} l = \{(la \to a) \to a : a \in A\} \tag{6}$$

This formula has potentially useful applications to prenuclei, since it allows us to get at the fixedpoints of the prenucleus without having to go through a transfinite iteration, which may be valuable in situations where issues of constructiveness preclude the use of ordinal iterations (whose length may not be bounded in advance)

2.3. Frame morphism adjoints and κ-frames. Recall that **Loc** was defined as the formal dual of **Frm**, that is, locale morphisms are just "turned around" versions of frame morphisms. Despite the abstractness of this definition, **Loc** is isomorphic to a concrete category in the following way. Every frame morphism $f : A \to B$ has a right adjoint $f_* : B \to A$, and, moreover, these right adjoints satisfy the laws $\mathrm{id}_* = \mathrm{id}$ and $(g \circ f)_* = f_* \circ g_*$. Thus, **Loc** is isomorphic to the category with objects those of **Frm** and with morphisms the functions f_* for frame morphisms f and composition ordinary function composition.

In 1975, Dowker and Strauss [7] gave a characterization of the functions $g : B \to A$ that appear as right adjoints of frame morphisms $A \to B$ This characterization (essentially the one in 10 7) explicitly involved the left adjoint, however, and, as pointed out by Johnstone [22, p 40], there still wasn't a description in terms of g alone We give such a description here (10.6): right adjoints to frame morphisms $A \to B$ can be characterized independently as those functions $g : B \to A$ satisfying

- $g(\bigwedge S) = \bigwedge\{g(s) : s \in S\}$ for every $S \subset B$,
- $g(b) = 1$ implies $b = 1$ for every $b \in B$, and
- for every $b \in B$ and $a_1, a_2 \in A$ with $a_1 \wedge a_2 \leq g(b)$, there exist $b_1, b_2 \in B$ such that $a_1 \leq g(b_1)$, $a_2 \leq g(b_2)$, and $b_1 \wedge b_2 \leq b$.

This is a special case of a more general result (10.5).

A κ-frame is defined as with a frame, except that arbitrary joins are replaced by joins of cardinality strictly less than κ, which is assumed to be a regular cardinal. (Let us use the terms κ-set, κ-family, κ-product, etc., to refer to objects similarly bounded in cardinality by κ.) These were introduced and studied in the paper of Madden [32]. Among the results proved there are a construction of the free functor F^λ_κ from λ-frames to κ-frames, where $\lambda \leq \kappa$ We prove, additionally, that F^λ_κ preserves all λ-products of λ-frames (11.6) and, when $\lambda > \omega$, equalisers of λ-frame morphisms as well (11.8). Thus F^λ_κ preserves all limits involving fewer than λ many morphisms. We also give examples to show that these results are the best possible (11 7 and 11.9). We furthermore show that all congruences on a κ-product of κ-frames are products of congruences on the factors (11.10) and conclude that all congruences on products of frames are products of congruences (11.11).

By looking at the construction of colimits in frames and κ-frames, we observe that every colimit of frames becomes a colimit of κ-frames when κ is chosen large enough (12.4). This forms the basis of one of the main results of Chapter 8 (28.5–6) Finally, we formulate the basic result about directed colimits of frames (12.7) and use this to sketch a proof in the exercises that, in **Frm**, directed colimits commute with arbitrary products (in fact with a larger class of limits, called $*$-compatible; see 12.9.1).

2.4. Regular operators. We define a *logical* operator to be an extensional operator l that is inflationary: $a \leq la$ for all $a \in A$. Regular operators are, by definition, the logical operators r satisfying $\neg\neg r = r$. They can be seen to arise from logical operators by considerations involving fixedpoint sets Thus, one feature of double negation of logical operators is that its associated congruence is the same as the equivalence determined by equality of fixedpoint sets. That is, for all logical operators l and m, $\operatorname{fix} l = \operatorname{fix} m$ iff $\neg\neg l = \neg\neg m$ (9.22). Thus, regular operators are completely determined by their fixedpoint sets, and every logical operator has the same fixedpoint set as a unique regular operator, moreover, regular operators are idempotent (see below), so their ranges and fixedpoint sets coincide, and this is the origin of the formula (6) for fixedpoints of logical operators.

Just like any $\neg\neg$-quotient of a frame, the collection of all regular operators on A, denoted $\mathsf{R}A$, is a cBa Meets and arrow in $\mathsf{R}A$ are computed pointwise Joins in a cBa are definable from meets and negation (by the de Morgan laws), and this leads to a simple formula for the join in $\mathsf{R}A$, which is also pointwise in a sense explained below (along with other properties of regular operators) once all of the necessary notions have been introduced

In order to discuss some of the properties of regular operators (including the pointwise description of joins in $\mathsf{R}A$ just mentioned) and give a characterization of their fixedpoint sets, we need to introduce the *regularity order* and the notion of *stability* Given elements $a, b \in A$, we say that b is *regular over* a, and write $b \trianglerighteq a$ or $a \trianglelefteq b$, if b is a regular element in the interval $[a, 1]$. This is easily seen to be equivalent to the equation $(b \to a) \to a = b$, and to the inequality $q(a) \leq q(b)$ of quasi-closed nuclei (20.1). The latter implies that $\trianglelefteq$ is a partial order; other elementary properties of $\trianglelefteq$ are, for every $a, b, c \in A$ (20 2, 20 8)

- $a \trianglelefteq b$ iff $b = x \to a$ for some $x \in A$.
- $a \trianglelefteq b$ imples $a \leq b$,
- $a \trianglelefteq c$ and $a \leq b \leq c$ imply $b \trianglelefteq c$;
- $a \trianglelefteq a \vee b$ implies $a \wedge b \trianglelefteq b$ (but not conversely);
- $a \trianglelefteq c$ and $b \trianglelefteq c$ imply $a \wedge b \trianglelefteq c$

We call a subset $S \subset A$ *stable* if it has a lower bound in the regularity ordering. It then has a greatest lower bound, namely its meet (20.5). As an example $\{a, b\}$ is stable iff $(a \to b) \to b = (b \to a) \to a$ (20 6). Since the partial order $(A, \trianglelefteq)$ is essentially the up-closed subset $Q = \{q(a) . a \in A\}$ of $\mathsf{N}A$ with the induced order, it follows that the regularity ordering also has joins for all nonempty sets S, which we denote by $\bigtriangledown S$. For each $a \in A$, the set $\{b : a \trianglelefteq b\}$, being exactly $\operatorname{fix} q(a)$, is a cBa under the ordering $\trianglelefteq$, and the join $\bigtriangledown$, when restricted to $\operatorname{fix} q(a)$, coincides with the cBa join (20.13)

Regular operators r are "regular" in another way, namely, in view of $(ra \to a) \to a = ra$, they are exactly those extensional operators that are inflationary in the regularity ordering. They can also be very usefully characterized as those operators r satisfying $r(a \to b) = a \to rb$ for all $a, b \in A$ (21.1) We can prove the following properties of aribtrary regular operators $r, s \in \mathsf{R}A$ for all $a, b \in A$ and $S \subset A$ (21 2.1-6, 21.3):

- $r(ra) = ra$ (they are idempotent),
- $a \trianglelefteq rb$ iff $ra \trianglelefteq rb$ (in particular they are $\trianglelefteq$-monotone: $a \trianglelefteq b$ implies $ra \trianglelefteq rb$),

- $r(sa) = (r \vee s)a = s(ra)$ (any two regular operators commute, and binary join is composition);
- if S is stable, then $r \bigwedge S = \bigwedge\{rs : s \in S\}$ (they preserve stable meets);
- $ra \wedge rb \leq r(a \wedge b)$,
- if S is nonempty, then $r \nabla S = \nabla\{ra : a \in S\}$ (they preserve regular joins);
- if $R \subset \mathsf{R}A$ is nonempty, then $(\bigvee R)a = \nabla\{ra : r \in R\}$ (nonempty joins in $\mathsf{R}A$ are pointwise with respect to ∇)

Fixedpoint sets of regular operators, which as you recall are the same as the fixedpoint sets of logical operators in general, can now be characterized (22.6) as the sets $S \subset A$ satisfying.

$$T \subset S,\ T \text{ stable implies } \bigwedge T \in S \quad \text{and} \quad a \in A, s \in S \text{ imply } a \to s \in S$$

Thus, they differ from maxsets (cf. (4)) only in that they are closed under stable meets, as opposed to all meets Equivalently, they are characterized in terms of the partial order $(A, \trianglelefteq)$ as the subsets both up-closed and closed under all existing meets; in short, they are the *complete filters* in $(A, \trianglelefteq)$

Now, finite meets and arbitrary joins of both maxsets and complete filters (in their reverse orderings) are computed the same way, and so the obvious inclusion of maxsets into complete filters induces a frame embedding $\mathsf{N}A \to \mathsf{R}A$ (23 1). In terms of operators. this embedding takes a nucleus j to the regular operator $\neg\neg j$, which I have been writing $\underline{\bar{\jmath}}$ in this context. Thus the composite embedding $A \to \mathsf{N}A \to \mathsf{R}A$ is given by $a \mapsto \underline{c(a)}$, which, if we rewrite it $\bar{c} : A \to \mathsf{R}A$ and simplify slightly. is

$$\bar{c}(a)x = (a \to x) \to x \qquad (x \in A).$$

The right adjoint to the inclusion $\mathsf{N}A \to \mathsf{R}A$, in terms of fixedpoint sets, closes a complete filter up under all meets to obtain a maxset. In terms of operators, the right adjoint, denoted $r \mapsto r^\circ$, is readily seen (23.2) to be given by

$$r^\circ a = \bigwedge\{rb : b \geq a\}. \tag{7}$$

As an application, we get a formula for the join of an arbitrary set J of nuclei (23.4(1)),

$$\begin{aligned} (\bigvee_{j \in J} j)a &= \bigwedge_{b \geq a} ((\bigvee_{j \in J} jb) \to b) \to b \\ &= \bigwedge_{b \geq a} (\bigwedge_{j \in J} jb \to b) \to b \end{aligned} \qquad (a \in A). \tag{8}$$

This formula is derived in a way reminiscent of that described at the end of 1.12 for **Frm**, by lifting J from $\mathsf{N}A$ to $\mathsf{R}A$ by the embedding (which preserves joins), computing the join in $\mathsf{R}A$ by the pointwise-∇ formula, and returning to $\mathsf{N}A$ by (7). Moreover, it solves a problem that, even for binary joins of nuclei, seemed "quite difficult" [41, p.242] An application of (8) is given below in 2.5. Finally, it should be mentioned that the formula for the arrow operation of $\mathsf{N}A$ appearing in the proof of

II.2.5 in Johnstone [22] can also be seen to arise in the same way from the pointwise arrow operation of $\mathsf{R}A$ (23.4(2))

Now, to complete the picture, we shift from the regular operators themselves to the categorical properties of $\mathsf{R}A$ as a frame, in particular its relation to the assembly tower. By the universal property of N, the embedding $\mathsf{N}A \to \mathsf{R}A$ can be extended to a morphism $\mathsf{N}^2A \to \mathsf{R}A$. This morphism, it turns out, is exactly the $\neg\neg$-quotient (23.6) Thus, the elements of $\mathsf{R}A$ can also be seen as the regular nuclei on $\mathsf{N}A$ (making that a third way in which they are "regular"!). As $Q = \{q(a) : a \in A\}$ is an up-closed subset of $\mathsf{N}A$, every nucleus $J \in \mathsf{N}^2A$, being inflationary, takes Q into itself and thus defines an operator J^* on A by the equation

$$Jq(a) = q(J^*a) \qquad (a \in A)$$

This operator is regular and the assignment $J \mapsto J^*$ gives, in terms of operators, the $\neg\neg$-quotient $\mathsf{N}^2A \to \mathsf{R}A$ (23.8) It also implies that for any two nuclei $J, K \in \mathsf{N}^2A$, $\neg\neg J = \neg\neg K$ iff J and K are equal when restricted to Q. A third implication is that $\neg\neg$ on N^2A preserves all meets, because meets are computed pointwise in both $\mathsf{R}A$ (applied to an element $a \in A$) and N^2A (applied the corresponding element $q(a) \in Q$) And, since $\neg\neg$ preserves arrow in any Heyting algebra, it follows that $\neg\neg$ is a complete Heyting algebra morphism (23.9), also called an *open* frame morphism since these correspond topologically to open continuous maps

The frame $\mathsf{R}A$ is also the limit of the Boolean quotients of A (24.1).

2.5. Free meets. Frame morphisms are required by definition to preserve only finite meets, but there are non-trivial situations in which meets of infinite sets are preserved as well As an example of this, consider meets of open sets of a space, easily seen to be given by the interior of their intersection These meets aren't normally preseved by the inverse images of continuous maps, but if the intersection is itself open, then it *is* preserved (since inverse image always preserves intersections). As an algebraic example, if every element $a \in S$ of a subset of frame has a complement $\neg a$ and, furthermore, the join $\bigvee\{\neg a : a \in S\}$ is itself complemented, then $\bigwedge S = \neg\bigvee\{\neg a : a \in S\}$ and so this meet is preserved by every frame homomorphism

Let us say that a subset S of a frame A *has a free meet* if $f(\bigwedge S) = \bigwedge\{f(s) : s \in S\}$ for every frame morphism $f : A \to B$, and that A has *free meets* if every subset of A has a free meet, i.e., every frame morphism out of A preserves all meets.

The equality $\bigwedge S = \neg\bigvee\{\neg a : a \in S\}$ in the algebraic example above suggests a way of looking for subsets of A with free meets: Since the embedding $A \to \mathsf{N}A$ freely complements the elements of A, and the embedding $\mathsf{N}A \to \mathsf{N}^2A$ freely complements all the joins of those complements, we might expect to find free meets for subsets of A by looking at their images in N^2A. This is the starting point of the first result of this section (25.3, 25.4): For any subset S of a frame A, the following four statements are equivalent:

1. S has a free meet in A
2. $c^2(\bigwedge S) = \bigwedge_{s \in S} c^2(s)$, where $c^2 : A \to \mathsf{N}^2A$ is the canonical injection.
3. For some $a \in A$, $\bigvee_{s \in S} u(s) = u(a)$ in $\mathsf{N}A$ (in fact, necessarily $a = \bigwedge S$)

4. Every $a \in A$ greater than $\bigwedge S$ can be written as the meet of a stable subset of the up-closure of S (see 2.4 for the definition of stability)

Notice that conditions 2–4 are conditions on, repsectively, $\mathsf{N}^2 A$, $\mathsf{N}A$, and A itself. The proof of the equivalence of condition 4 with the others relies heavily on formula (8) of 2 4 (applied to the join of open nuclei in Condition 3).

This result also gives a solution to a problem first considered by Macnab in his thesis [31] and more recently by Niefield and Rosenthal [36] (see also Section 4.5 of [38]), namely to characterize those sets $S \subset A$ such that $S = j^{-1}(1)$ for some nucleus $j \in \mathsf{N}A$ or, equivalently, such that $S = f^{-1}(1)$ for some frame morphism $f : A \to B$. It is an easy consequence of the above that these sets are precisely the "free filters" (filters closed under free meets, see 25 5)

Finally, the results of this section can be combined with a result of Beazer and Macnab [5] to obtain the following (26.1): A frame A has free meets iff both of the following (independent—see 26.2) conditions hold:

- A is a biframe, i.e., its opposite is also a frame.
- $\mathsf{N}A$ is a cBa

2.6. Universal monos. A universal mono is a morphism $u : A \to B$ such that for any morphism $f : A \to D$, the pushout of u along f is mono. Clearly such a u is itself mono, since u is the pushout of u along the identity. Here are some other basic properties of universal monos (27.2), valid in any category with pushouts (where $f : A \to B$ and $g : B \to C$ are morphisms):

1. If f and g are universal monos, then so is $g \circ f$
2. If $g \circ f$ is a universal mono, then so is f.
3. The pushout of a universal mono along any morphism is universal
4. If $g \circ f$ is a universal mono and f is epi, then g is a universal mono.

These tell us how to get new universal monos from old ones; we also need some examples to start with. A first class of examples are the open monos; recall that a frame morphism is open when it is a complete Heyting algebra morphism. i.e., when it also preserves arrow and all meets—see [27] for more on open morphisms. A second class, recently investigated by Vermeulen [49] are the (localic) proper surjections; without the surjectivity restriction, such locale morphisms $p : A \to B$ correspond, by the change-of-base result mentioned in conjunction with the constructivity of locale theory in 1.7, to compact locales in $\mathbf{Sh}(B)$. A third class, also universal in any category with pushouts, are the components of natural monomorphisms from the identity functor (27 3(iv)). For $\mathbf{Frm}$, this class includes all of the morphisms $c^{\alpha} : A \to \mathsf{N}^{\alpha} A$. Combining this with condition 2 above, we see that first factors of the components of natural monomorphisms from the identity are also universal Interestingly, this last class includes *all* universal monos in $\mathbf{Frm}$: given any such $u : A \to B$, a natural monomorphism from the identity can be constructed that has u as a first factor of its component at A (Exercise 27.4 2)

To explain the main result of this section, we need a couple of definitions. Let α be an ordinal. We call a morphism $f : A \to B$ *α-mono* if $\mathsf{N}^\alpha f$ is mono and *α-epi* if the image of $\mathsf{N}^\alpha f$ contains the image of $c^\alpha : B \to \mathsf{N}^\alpha B$. Thus, for example, 0-mono means mono and 0-epi means onto. It is easy to see (28.2) that if f is α-mono then it is β-mono for all $\beta \leq \alpha$ and that if f is α-epi then it is γ-epi for all $\gamma \geq \alpha$. Note also that both of the notions of α-mono and α-epi are different for different α: if f is the composite $\neg\neg \circ c^{\alpha} : A \to (\mathsf{N}^\alpha A)_{\neg\neg}$ for a non-reflective frame A (such as the free frame on ω), then f is β-mono for all $\beta < \alpha$ but it is not α-mono; and f is α-epi but not β-epi for any $\beta < \alpha$ (28.4).

Now Madden and Molitor [33] have shown that f is an epimorphism iff it is α-epi for some α. Their proof can be simply explained using the idea, put forward at the end of 1.12, of an extension of **Frm** where the assembly tower of every frame A has a colimit, call it $\mathsf{N}^\infty A$, which is the union of the frames in the tower and gives a reflection of A into the subcategory of Boolean objects. Since N^∞ is a faithful left adjoint it both preserves and reflects epis, i.e., f is epi iff $\mathsf{N}^\infty f$ is. But in the category of Boolean objects epis are surjections, and therefore, since $N^\infty B$ is the union of the frames in the tower, there must be a stage α at which all of the elements of B have appeared in the range of $\mathsf{N}^\alpha f$, completing the proof.

The main results of this section are (28.5, 28.8)

- f is a universal mono iff it is an α-mono for every ordinal α.
- Conversely, f is an $(\alpha+1)$-mono iff the pushout of f along every α-epi is mono.

That a universal mono is α-mono for every α follows easily from basic properties of the assembly tower and universal monos. That a morphism f which is α-mono for every α is universal again uses the idea above. In detail, since $\mathsf{N}^\infty f$ is the union of the $\mathsf{N}^\alpha f$, which are all mono, $\mathsf{N}^\infty f$ is mono as well. And if g is a pushout of f, then since N^∞ preserves colimits, $\mathsf{N}^\infty g$ is a pushout of $\mathsf{N}^\infty f$. But in the category of Boolean objects, all monos are universal, so $\mathsf{N}^\infty g$ is mono and thus so is g. The second part of the result uses similar ideas, along with a positive answer to the (somewhat technical) question asked at the end of [33] (Lemma 28.6).

2.7. Combinatorial morphisms. Because of the way coproducts are computed in **Frm**, it turns out that the free extension $A[X]$ of a frame A by a set X is the subframe of the catesian power $A^{\mathsf{P}_f X}$ consisting of all the monotone functions, where $\mathsf{P}_f X$ is the set of finite subsets of X, ordered by reverse inclusion (29.1). And since every X-generated extension of A is a quotient of $A[X]$, the idea is that we can study frame morphisms out of A by studying congruences on $A[X]$. This generalizes the description of singly-generated frame extensions given by Banashewski [4], and many results of this section are generalizations of the results obtained there.

The easiest congruences to work with are the restrictions to $A[X]$ of congruences on $A^{\mathsf{P}_f X}$, which by a result mentioned in 2.3 must be products of $(\mathsf{P}_f X)$-indexed families of congruences on A. We call such congruences *standard*, and a frame morphism isomorphic to one of the form $A \to A[X]/\prod_i \theta_i$ *combinatorial*.

It is easy to see that every congruence on $A[X]$ has a least standard congruence greater than it, and this congruence is given by a simple formula (29.4). The process of standarization of congruences is preserved by pushout along an arbitrary morphism

(29.7) A combinatorial morphism $f : A \to A[X]/\prod_i \theta_i$ is mono iff $\bigwedge_i \theta_i = 0$, and so if this family of congruences has a free meet (equal to 0) in $\operatorname{Con} A$, then f is a universal mono (29.9), establishing a connection between free meets and universal monos. It also shows that every finitely generated combinatorial extension is universal (Corollary 29.9), since every finite set of congruences has a free meet.

In the case that X is finite, the theory becomes quite manageable: every congruence on $A[X]$ is standard (30.3). To prove this, we use a lemma about finite subsets of distributive lattices (30.2) that generalizes the familiar result that $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply $x = y$ for any three elements a, x, y. As another example of the manageability of the theory, because every congruence is standard, the process of standardization is trivial, and so pushouts of combinatorial morphisms have an especially simple description (30.4). This allows us to characterize finitely generated epis and finitely generated regular monos in terms of the family of congruences (30.5): a finitely generated combinatorial morphism $f : A \to A[X]/\prod_i \theta_i$ is epi iff $\theta_i \vee \theta_j = 1$ whenever $i \neq j$, and is a regular mono iff it satisfies a "patching" condition reminiscent of the definition of a sheaf.

Chapter 2
PRELIMINARIES

The reader of this thesis is assumed to be familiar with the basic notions and results of set theory (sets, ordinals, cardinals), lattice theory, universal algebra, and category theory. Of these four subjects, much of what we will need from lattice theory and category theory can already be found in Chapter I of Johnstone [22], another book of Johnstone ([24]) contains much of what we will need from set theory (and logic). In any case, to establish our notations, terminology, and background results, we review these four subjects in separate (and rather dense) sections below. General references are [19] for set theory, [13] for lattice theory [6] and [35] for universal algebra, and [29] and [18] for category theory (The reader may also wish to consult [34], which covers universal algebra from the point of view of category theory and, in the first chapter, discusses infinitary universal algebra a subject very important to us here but not mentioned in the other references on universal algebra)

3. Sets

3.1. Axioms and notations. Although it won't be necessary to specify precisely with which theory of sets we will work, the reader desiring such a commitment may take Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) for this purpose In addition to sets, we will also make use of *classes* (such as the class V of all sets, as well as more specific classes), but we will consider a class to be a linguistic objectification of a formula (with one free variable) rather than a fundamental entity. Thus, if C is a class represented by the formula $\phi(x)$, then $y \in C$ simply means $\phi(y)$ For some considerations it will be convenient—though, as we will see, not at all necessary—to assume the existence of inaccessible cardinals, or to likewise adopt other devices that allow us to distinguish between "small" and "large" sets. This is explained in more detail in Section 6.6 below

As for specific notation, ours is basically standard Examples: set membership ($a \in A$), subset ($A \subset B$, note that this includes the possibility that $A = B$; we will not need a notation for *proper* subsets), sets formed by comprehension or separation ($\{a \in A : \phi(a)\}$), the empty set ($\emptyset$), finite sets ($\{1, 2, 3\}$), union ($A \cup B$, $\bigcup\{A_i : i \in I\}$, $\bigcup_{i \in I} A_i$), intersection ($A \cap B$, $\bigcap\{A_i : i \in I\}$, $\bigcap_{i \in I} A_i$), set difference ($A - B$), power set ($\mathsf{P}A = \{X : X \subset A\}$), ordered n-tuples ($\langle x_1, \ldots, x_n \rangle$), binary cartesian products ($A \times B$), disjoint union ($\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$), relations *between* sets ($\theta \subset A \times B$, $a \;\theta\; b$ iff $\langle a, b \rangle \in \theta$) and relations *on* a set ($\rho \subset A \times A$), the domain and range of a relation ($\operatorname{dom} R = \{a : \exists b \; a \, R \, b\}$, $\operatorname{rng} R = \{b : \exists a \; a \, R \, b\}$), the converse or *inverse* of a relation ($R^{-1} = \{\langle b, a \rangle : \langle a, b \rangle \in R\}$), functions ($f : A \to B$, $A \xrightarrow{f} B$, $f(a)$, $(g \circ f)(a) = g(f(a))$), arbitrary cartesian products ($\prod_{i \in I} A_i$; elements of this set are functions $\sigma : I \to \bigcup_{i \in I} A_i$ such that $\sigma(i) \in A_i$ for all $i \in I$), and cartesian powers (A^I, elements are functions $\sigma : I \to A$)

3.2. Notations for functions. A function $f : A \to B$ (also called a *map*) is *injective* (or *one-to-one*, or *1-1*) if $f(a) = f(b)$ implies $a = b$ for all $a, b \in A$; f is *surjective* (or *onto*) if $\operatorname{rng} f = B$; f is *bijective* if f is both 1-1 and onto. Note that if $g \circ f$ is 1-1, then so is f, and if $g \circ f$ is onto, then so is g. If $S \subset A$, the *restriction* of f to S is $f \cap (S \times B)$; this function $S \to B$ is denoted $f|_S$. Functions $A^I \to A$ are called *I-ary operations* on A, when I has 0, 1, or 2 elements, these operations are called *nullary*, *unary*, or *binary*, respectively. We identify nullary operations on A with elements of A (i.e., *constants*), unary operations with functions $A \to A$, binary operations with functions $A \times A \to A$, and so on. Unary functions are also called *operators*, and the action of an operator $l : A \to A$ on an element $a \in A$ is written using juxtaposition: la

Functions defined "syntactically" need not always be given a name. we adopt the useful convention that if $p(x)$ represents an expression involving a variable x, and if for every $a \in A$ the result, $p(a)$, of "evaluating" the expression with x replaced by a yields an element of B, then the resulting function $A \to B$ is denoted by $x \mapsto p(x)$ or just $p(-)$. Example the (unary) squaring function on the integers might be denoted $n \mapsto n^2$ or $(-)^2$.

A *partial* function $f : A \to B$ is a *single-valued* relation $f \subset A \times B$, i.e., such that $a\ f\ b_1$ and $a\ f\ b_2$ imply $b_1 = b_2$ for all $a \in A$ and $b_1, b_2 \in B$. We say that $f(a)$ is *defined* if $a \in \operatorname{dom} f$. Finally, a *relation-class* is a class R such that $x \in R$ implies that x is an ordered pair, and a *function-class* is a relation-class that is single-valued, in the sense defined above Much of the notation and terminology for relations and (partial) functions also applies to relation- and function-classes.

3.3. Posets. A *partial order* on a set (or class) A is a binary relation $\leq$ on A that is *reflexive* ($a \leq a$ for all $a \in A$), *transitive* ($a \leq b$ and $b \leq c$ imply $a \leq c$), and *antisymmetric* ($a \leq b$ and $b \leq a$ imply $a = b$). A *linear order* additionally has, for every $a, b \in A$, either $a \leq b$ or $b \leq a$. A set A equipped with a partial order is called a *poset* (partially-ordered set); a linearly-ordered set is often called a *chain*. When the order is understood, a poset will often be denoted simply by naming its underlying set We also use the notation $b \geq a$ for $a \leq b$. We write $a < b$ for the conjunction of $a \leq b$ and $a \neq b$ (and similarly with $a > b$), and $a \leq b \leq c$ for the conjunction of $a \leq b$ and $b \leq c$. A simple but useful result about posets is the following:

Proposition (Yoneda Lemma for Posets). *If $(A, \leq)$ is a poset, then for every $a, b \in A$,*

(a) *$a \leq b$ if and only if for every $x \in A$, $x \leq a$ implies $x \leq b$*
(b) *$a = b$ if and only if for every $x \in A$. $x \leq a$ and $x \leq b$ are equivalent.*

Proof. (a) If $a \leq b$ and $x \leq a$, then $x \leq b$ by transitivity. Conversely, if $x \leq a$ implies $x \leq b$ for every $x \in A$, then since $a \leq a$ by reflexivity, putting $x = a$ we have $a \leq b$, proving (a). Part (b) follows from (a) and antisymmetry.

3.4. Associated constructions. If $(A, \leq)$ is a poset, then $(A, \leq^{\mathrm{op}})$, where $a \leq^{\mathrm{op}} b$ iff $b \leq a$, is also a poset, called the *dual* of A and denoted A^{op}. As a consequence, we have a duality for posets: every statement about posets has a dual statement, formed by replacing the order with the dual order, and a statement about a poset is true iff the dual statement is true about the dual poset. Hence, if a statement is true of all posets, then its dual is also true of all posets.

A map $f: A \to B$ between posets $\langle A, \leq\rangle$ and $\langle B, \leq'\rangle$ is *monotone* (or *order-preserving*) if $f(a) \leq' f(b)$ whenever $a \leq b$, and *antimonotone* (or *order-reversing*) if $f(a) \geq' f(b)$ whenever $a \leq b$. Note that $f: A \to B$ is antimonotone iff $f: A \to B^{\mathrm{op}}$, or equivalently $f: A^{\mathrm{op}} \to B$, is monotone. A map $f: A \to B$ is an *(order-)isomorphism* if it is a monotone bijection.

Starting with given posets we can construct new ones by restricting to a subposet or taking a product. If $\langle A, \leq\rangle$ is a poset and $S \subset A$, then the order on S *induced* by A is $\leq \cap (S \times S)$. Examples of posets with the induced order are *interval subposets*: if $a, b \in A$, then $[a, b] = \{x \in A : a \leq x \leq b\}$, with the induced order. If I is a set and, for each $i \in I$, $\langle A_i, \leq_i\rangle$ is a poset, then the cartesian product $\prod_{i\in I} A_i$ becomes a poset with order $\leq$ defined *pointwise*, i.e., for $\sigma, \tau \in \prod_{i\in I} A_i$, $\sigma \leq \tau$ iff $\sigma(i) \leq_i \tau(i)$ for all $i \in I$.

Given a poset $\langle A, \leq\rangle$, we can associate to each subset $S \subset A$ four other subsets:

(i) the *up-closure* of S: $\mathrm{upcl}\, S = \{a \in A : a \geq s \text{ for some } s \in S\}$,

(ii) the *down-closure* of S: $\mathrm{downcl}\, S = \{a \in A : a \leq s \text{ for some } s \in S\}$,

(iii) the *upper bounds* of S: $\mathrm{ub}\, S = \{a \in A : a \geq s \text{ for all } s \in S\}$, and

(iv) the *lower bounds* of S: $\mathrm{lb}\, S = \{a \in A : a \leq s \text{ for all } s \in S\}$.

S is *up-closed* or *down-closed* if $\mathrm{upcl}\, S = S$ or $\mathrm{downcl}\, S = S$. As a special case, we define $\uparrow a = \mathrm{upcl}\{a\} = \mathrm{ub}\{a\}$ and $\downarrow a = \mathrm{downcl}\{a\} = \mathrm{lb}\{a\}$, called respectively the *principal filter* and *principal ideal* generated by a. An element $a \in A$ is the *join* (or *least upper bound*, or *supremum*) of S if a is the least element of $\mathrm{ub}\, S$, i.e., $a \in \mathrm{ub}\, S$ and, for all $b \in \mathrm{ub}\, S$, $a \leq b$. Dually, a is the *meet* (or *greatest lower bound*, or *infimum*) of S if a is the greatest element of $\mathrm{lb}\, S$. Note that $\mathrm{ub}\, \emptyset = \mathrm{lb}\, \emptyset = A$, so that the join and meet of the emptyset, if they exist, are respectively the least element of A, denoted 0, and the largest element of A, denoted 1. We view join ($\bigvee$) and meet ($\bigwedge$) as partial functions $\bigvee, \bigwedge : \mathrm{P}A \to A$.

3.5. Ordinals. Informally, the class of ordinals is the (linear) order freely generated by the constant 0, the unary successor operation s, and the join operation, subject only to the condition that $x \leq s(x)$. This description takes transfinite recursion (and induction) as basic. Formally, ordinals can be identified with sets that are transitive and well-ordered by $\in$. A set α is *transitive* if $x \in \alpha$ and $y \in x$ imply $y \in \alpha$, or, more succinctly, $\bigcup\alpha \subset \alpha$ (or $\alpha \subset \mathrm{P}\alpha$). A poset $\langle A, \leq\rangle$ is a *well-ordering* if every non-empty subset of A has a least element. The informal description is then realized by taking $0 = \emptyset$, $s(x) = x \cup \{x\}$ and $\bigvee = \bigcup$. The class of all ordinals is denoted $\mathbb{O}$. As we have defined them, ordinals satisfy $\alpha = \{\beta \in \mathbb{O} : \beta < \alpha\}$ and $\beta < \alpha$ iff $\beta \in \alpha$. The finite ordinals are denoted $0, 1, 2, 3, \ldots$, as usual. The first infinite ordinal (= the set of all finite ordinals) is denoted ω.

An *ordinal sequence* $\langle a_\beta : \beta < \alpha\rangle$ is a function with domain α for which $\beta \mapsto a_\beta$ for all $\beta < \alpha$. Transfinite induction says that if X is a class of ordinals such that $0 \in X$, $s(x) \in X$ whenever $x \in X$, and $\bigcup S \in X$ whenever $S \subset X$, then $X = \mathbb{O}$. Transfinite recursion says that if G is a function-class defined at least on all ordinal sequences, then there is a unique function-class F such that $F(\alpha) = G(\langle F(\beta) : \beta < \alpha\rangle)$.

Every well-ordering is isomorphic to a unique ordinal. If α and β are ordinals, then the ordinal $\alpha+\beta$ is defined to be the unique ordinal isomorphic to the well-ordering that puts β "at the end of" α, i.e., to the poset $\langle A, \leq \rangle$, where $A = (\{0\} \times \alpha) \cup (\{1\} \times \beta)$, and where $\langle i, \gamma_1 \rangle \leq \langle j, \gamma_2 \rangle$ iff either $i < j$ or both $i = j$ and $\gamma_1 \leq \gamma_2$. Note that addition of ordinals is associative (but not commutative) and that whenever $\alpha \leq \beta$ there exists a unique γ such that $\alpha + \gamma = \beta$.

An ordinal α is either 0, a *successor* ordinal ($\alpha = s(\beta) = \beta + 1$ for some ordinal β), or a *limit* ordinal ($\alpha = \bigcup \alpha$). Transfinite induction and recursion can be restated using this classification of ordinals. As an example, we construct the *cumulative hierarchy*, $\{V_\alpha : \alpha \in \mathrm{O}\}$. We define $V_0 = \emptyset$, $V_{\alpha+1} = PV_\alpha$, and, if λ is a limit ordinal, $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$. One can prove, using transfinite induction, that each V_α is transitive, that the hierarchy is indeed cumulative ($\beta < \alpha$ implies $V_\beta \subset V_\alpha$), and that for every ordinal α, $\alpha \subset V_\alpha$ (and hence $\alpha \in V_{\alpha+1}$). As a consequence of the Axiom of Regularity (every set has an $\in$-minimal element), every set belongs to some set in the cumulative hierarchy; we define the *rank* of a set x, $\mathrm{rank}\, x$, to be the least ordinal α such that $x \in V_{\alpha+1}$. Note that $\mathrm{rank}\, \alpha = \alpha$ and $V_\alpha = \{x : \mathrm{rank}\, x < \alpha\}$.

3.6. Cardinals. Just as ordinals provide representatives of isomorphism types of well-orderings, cardinals represent isomorphism types of sets (where "isomorphism" means "bijection" in this case). We can achieve this by defining a *cardinal* to be an ordinal that is not isomorphic to any smaller ordinal. By the Axiom of Choice, every set X can be well-ordered and therefore admits a bijection to a unique cardinal, called its *cardinality* and denoted $|X|$. Addition, multiplication, and exponentiation of cardinals are defined by

$$\kappa + \lambda = |\kappa + \lambda|, \quad \kappa \cdot \lambda = |\kappa \times \lambda|, \quad \kappa^\lambda = |\kappa^\lambda|,$$

where the cardinalities are, respectively, of the ordinal sum, cartesian product, and cartesian power of κ and λ. We note that $|PX| = 2^{|X|} > |X|$. The least cardinal larger than κ is denoted κ^+.

A cardinal is *regular* if it is not the union of a smaller set of smaller cardinals; more precisely, κ is regular if, whenever $X \subset \kappa$ and $\kappa = \bigcup X$, then $|X| = \kappa$. Using the Axiom of Choice, one can show that κ^+ is regular for every cardinal κ.

4. Lattices

4.1. Semilattices and lattices. A *meet-semilattice* is a poset in which every finite set has a meet. Equivalently, a meet-semilattice $\langle A, \leq \rangle$ has a greatest element 1 (the empty meet) and a meet $a \wedge b$ for every two elements $a, b \in A$. This binary operation is associative ($a \wedge (b \wedge c) = (a \wedge b) \wedge c$), commutative ($a \wedge b = b \wedge a$), and idempotent ($a \wedge a = a$), and 1 is a unit for the operation ($a \wedge 1 = a$). The order relation is recovered from the meet operation by the equivalence $a \leq b$ iff $a \wedge b = a$, and the descriptions by order ($\leq$) and by operations ($1, \wedge$) are equivalent.

A function $f : A \to B$ between meet-semilattices that preserves finite meets (i.e., $f(\bigwedge S) = \bigwedge\{f(s) : s \in S\}$ for every finite set $S \subset A$) is called a *meet-semilattice (homo)morphism*. Every meet-semilattice morphism is monotone, and injective meet-semilattice morphisms *reflect order*: $f(a) \leq f(b)$ implies $a \leq b$ (proof. $f(a) \leq f(b)$ iff $f(a) = f(a) \wedge f(b) = f(a \wedge b)$ iff $a = a \wedge b$ iff $a \leq b$). For every $a \in A$, the unary operation $a \wedge -$ on A is monotone but is not a meet-semilattice morphism (unless $a = 1$), since it doesn't preserve 1.

Dually, a *join-semilattice* is a poset in which every finite set has a join; equivalently, a join-semilattice has a least element 0 and a join $a \vee b$ for every two of its elements Algebraically join-semilattices are the same as meet-semilattices (and may as well just be called *semilattices*)—they consist of one constant and one binary operation satisfying the same equations The difference is in the relation between the order and the binary operation $a \leq b$ iff $a \vee b = b$ Join-semilattice morphisms preserve finite joins, and the operations $a \vee -$ are monotone but are not join-semilattice morphisms (unless $a = 0$)

A *lattice* is a poset that is both a meet- and join-semilattice Equivalently it has a greatest element 1, a least element 0, and a meet $a \wedge b$ and join $a \vee b$ for every two of its elements. In addition to the semilattice equations satisfied by the pair 1 and $\wedge$ and by the pair 0 and $\vee$ the operations $\wedge$ and $\vee$ are related to each other by the *absorption laws*: $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ Lattice morphisms preserve finite meets and finite joins Note that the dual of a meet-semilattice is a join-semilattice and vice-versa, and that the dual of a lattice is a lattice To form the dual of a statement about lattices one interchanges 0 and 1 and interchanges $\wedge$ and $\vee$

4.2. Distributive and Boolean lattices. A lattice A is *distributive* if it satisfies $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$. One can then show that it also satisfies the dual law, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ Given elements a, b, c of a lattice A with $a \leq b \leq c$, $d \in A$ is called a *relative complement* of b in $[a, c]$ if $b \wedge d = a$ and $b \vee d = c$ (d is therefore necessarily also in the interval $[a, c]$). In general an element may have many relative complements in a given interval, but in a distributive lattice relative complements are unique when they exist (in fact, distributive lattices are characterized among lattices by this property) Relative complements in the interval $[0, 1]$ are called *(absolute) complements* It is clear from the definition that complements are preserved by lattice morphisms

A Boolean lattice is a distributive lattice in which every element has a complement. The operation taking an element to its complement is denoted $\neg$; thus $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ for every element a of a Boolean lattice Boolean lattices additionally satisfy $\neg\neg a = a$ for every a, as well as the *De Morgan Laws*: $\neg(a \wedge b) = (\neg a) \vee (\neg b)$ and $\neg(a \vee b) = (\neg a) \wedge (\neg b)$.

4.3. Complete lattices, closure operators, and adjunctions. Complete meet-semilattices, complete join-semilattices and complete lattices, and the morphisms between them, are defined as are their non-complete counterparts, except that joins and meets are required to exist for all subsets (not just finite subsets), and morphisms are required to preserve them Although the resulting three types of morphisms are different, the three types of posets are the same. any poset having all meets also has all joins, and vice-versa, since $\bigvee S = \bigwedge \mathrm{ub}\, S$ and $\bigwedge S = \bigvee \mathrm{lb}\, S$. The basic example of a complete lattice is the collection of all subsets of a set X, the order being $\subset$, meets and joins are given by intersection and union

A *closure operator* on a complete lattice A is an operator (function) $C : A \to A$ that is *inflationary* ($a \leq Ca$), *idempotent* ($CCa = Ca$), and monotone ($a \leq b$ implies $Ca \leq Cb$) Dually, a *co-closure operator* is *deflationary* ($Ca \leq a$), idempotent, and montone An element $a \in A$ is a *fixedpoint* of an operator C if $Ca = a$, the set of fixedpoints of C is denoted $\mathrm{fix}\, C$. Note that, for a closure or co-closure operator C, the set $\mathrm{fix}\,C$ is the same as the range of C. A *meet-closed* subset of A is simply a subset $S \subset A$ such that if $T \subset S$, then $\bigwedge T \in S$; a *join-closed* subset is defined dually. A meet-closed subset of a complete lattice is itself a complete lattice in the induced order, since is has all meets (and thus all joins); similarly with a join-closed subset.

Closure and co-closure operators and meet- and join-closed subsets can be partially ordered. we order the operators pointwise ($C_1 \leq C_2$ iff $C_1 a \leq C_2 a$ for all $a \in A$), and. for reasons that will become clear shortly, we order the join-closed subsets by inclusion ($S_1 \leq S_2$ iff $S_1 \subset S_2$) and the meet-closed subsets by reverse inclusion ($S_1 \leq S_2$ iff $S_2 \subset S_1$). Given two complete lattices A and B, a pair of monotone functions $l : A \to B$ and $r : B \to A$ are said to be *adjoint*, with l the *left adjoint* and r the *right adoint*, when $l(a) \leq b$ iff $a \leq r(b)$ for every $a \in A$ and $b \in B$. This situation is called an *adjunction* and denoted $l \dashv r$.

The relations between the above notions are spelled out in the following proposition

Proposition. *Suppose that A and B are complete lattices*

(a) *If C is a closure operator on A, then $\mathrm{fix}\,C$ is meet-closed (and thus a complete lattice). If $S \subset A$ is meet-closed, then the operator C on A defined by $Ca = \bigwedge\{b \in S : a \leq b\}$ is a closure operator. If $i : \mathrm{fix}\,C \to A$ is the inclusion, then $C \dashv i$.*

(b) *Dually, if C is a co-closure operator on A, then $\mathrm{fix}\,C$ is join-closed, and if $S \subset A$ is join-closed, then $Ca = \bigvee\{b \in S : b \leq a\}$ defines a co-closure operator. Moreover $i \dashv C$, where $i : \mathrm{fix}\,C \to A$ is the inclusion*

(c) *The correspondences in (a) and (b) between closure operators and meet-closed subsets and between co-closure operators and join-closed subsets are isomorphisms of posets, when the sets are ordered as in the previous paragraph.*

(d) *If $l : A \to B$ and $r : B \to A$ satisfy $l \dashv r$, then $r \circ l$ is a closure operator on A, and $l \circ r$ is a co-closure operator on B. In fact $l \circ r \circ l = l$ and $r \circ l \circ r = r$, and $l|_{\mathrm{fix}\, r\circ l}$ $(\mathrm{fix}\, r \circ l) \to (\mathrm{fix}\, l \circ r)$ is an isomorphism of posets, with inverse $r|_{\mathrm{fix}\, l\circ r}$. The operation l preserves all joins, and r preserves all meets. Conversely, any join-preserving function between complete lattices has a right adjoint, and, dually, any meet-preserving function has a left adjoint. Any pair of adjoints $l \dashv r$ satisfy*

$$l(a) = \bigwedge\{b : r(b) \geq a\} \qquad \text{and} \qquad r(b) = \bigvee\{a : l(a) \leq b\},$$

so that each of r, l determines the other. l is 1-1 iff r is onto, and l is onto iff r is 1-1.

4.4. Examples. A common source of adjunctions (in fact the *only* source of adjunctions between powersets) is described by the following proposition:

Proposition. *Suppose X and Y are sets, $\mathsf{P}X$ and $\mathsf{P}Y$ are the associated complete lattices of subsets (ordered by inclusion), and $R \subset X \times Y$ is any relation. Then $l_\exists : \mathsf{P}X \to \mathsf{P}Y$, $r_\exists : \mathsf{P}Y \to \mathsf{P}X$, $l_\forall : \mathsf{P}X \to (\mathsf{P}Y)^{\mathrm{op}}$, and $r_\forall : (\mathsf{P}Y)^{\mathrm{op}} \to \mathsf{P}X$, defined, for $S \subset X$, $T \subset Y$, and $Q = \exists, \forall$, by*

$$l_Q(S) = \{y \in Y \ \ Qs \in S \ \ s\,R\,y\} \qquad \text{and} \qquad r_Q(T) = \{x \in X : Qt \in T \ \ x\,R\,t\},$$

satisfy $l_Q \dashv r_Q$. Moreover, every adjunction $l_Q \dashv r_Q$ between $\mathsf{P}X$ and $\mathsf{P}Y$ (if $Q = \exists$) or $\mathsf{P}X$ and $(\mathsf{P}Y)^{\mathrm{op}}$ (if $Q = \forall$) arises in this way from a unique relation R defined by

$$x\,R\,y \quad \text{iff} \quad y \in l_Q(\{x\}) \quad \text{iff} \quad x \in r_Q(\{y\}).$$

PROOF (Sketch) For the first part, the reader may check that both $l_\exists(S) \subset T$ and $S \subset r_\exists(T)$ are equivalent to $\forall x \forall y(x \in S$ and $x\, R\, y$ imply $y \in T)$, and that both $l_\forall(S) \supset T$ and $S \subset r_\forall(T)$ are equivalent to $\forall x \forall y(x \in S$ and $y \in T$ imply $x\, R\, y)$ For the second part, note that l_Q, as a left adjoint, preserves joins and therefore is completely determined by its values on the singleton subsets $\{x\}$ of X (and these values as sets, are themselves determined by the elements they contain)

As an example of an "existential" adjunction, if $R = f : X \to Y$ is a function, then $l_\exists$ and $r_\exists$ are just the direct and inverse image, and the adjunction $l_\exists \dashv r_\exists$ is the familiar relation $f(S) \subset T$ iff $S \subset f^{-1}(T)$ for all $S \subset X$ and $T \subset Y$ As an example of a "universal" adjunction, consider a poset $\langle A, \leq\rangle$ Then the adjunction on $\mathrm{P}A$ induced by the binary relation $\leq$ is given by $l_\forall(S) = \mathrm{ub}\, S$ and $r_\forall(T) = \mathrm{lb}\, T$ The complete lattice of fixedpoints of $r_\forall \circ l_\forall$ is the so-called *Dedekind-MacNeille completion* of A, and the mapping $a \mapsto \downarrow a$ is an emedding of A into the completion that moreover preserves whatever meets and joins happen to exist in A. We will see several other examples of "universal" adjunctions later

Finally, we observe that the identity function, $i : A \to A$, is adjoint to itself $(i \dashv i)$ and that adjunctions may be composed if, in addition to $l \dashv r$ as above, $l' : B \to C$ and $r' : C \to B$ have $l' \dashv r'$, then $l' \circ l \dashv r \circ r'$

5. Universal algebra

5.1. Basic notions. In order to encompass all of the examples with which we will be dealing, it will be convenient to use a quite general definition of algebra. A *similarity type* is a (possibly proper) class Ω of *operation symbols* along with a function-class $\iota : \Omega \to V$ that assigns to each operator symbol $\omega \in \Omega$ an index set $\iota(\omega)$ called its *arity* An *algebra* of similarity type Ω (ι is often left implicit) is then a set A along with a function-class assigning to each $\omega \in \Omega$ an operation $\omega_A : A^{\iota(\omega)} \to A$.[1] As with partial orders, algebras are often denoted simply by naming their underlying set. If A and B are two algebras of type Ω, a function $f : A \to B$ is called an Ω-*(homo)morphism* if, for every function symbol $\omega \in \Omega$, f *preserves* ω; i.e., for every element $\sigma \in A^{\iota(\omega)}$, we have $f(\omega_A(\sigma)) = \omega_B(f \circ \sigma)$ Every identity function is a homomorphism, and the composite of two homomorphisms is a homomorphism A bijective homomorphism is called an *isomorphism*

As an example, we can take the similarity type of lattices to be $\Omega = \{\wedge, \vee, 1, 0\}$, with $\iota(\wedge) = \iota(\vee) = 2$ and $\iota(1) = \iota(0) = 0$, and display a typical lattice as $\langle A, \wedge, \vee, 1, 0\rangle$, where we have used the operation symbols themselves (effectively leaving off the subscript A) to denote the operations $\wedge, \vee : A \times A \to A$ and constants $1, 0 \in A$ Then Ω-morphisms correspond to lattice morphisms as we defined them in 4.1 As an example of an infinitary similarity type, we have κ-*complete semilattices*, where κ is an infinite cardinal These are posets in which every set of cardinality strictly less

[1] The reason for this definition, as stated above, is its *uniformity* it enables us to treat as algebras of a fixed similarity type structures (such as complete lattices) that are traditionally not able to be treated as such. In all cases that we consider in this thesis, however, the class of operations of an algebra will be derived from a single set associated to the algebra (such as a partial order), and so there will be no difficulty speaking of sets, or even classes, of algebras.

than κ has a join, and a homomorphism of κ-complete semilattices is required to preserve these joins. For the similarity type, we can take as operation symbols all of the cardinals λ with $\lambda < \kappa$ (and define $\iota(\lambda) = \lambda$), the operation λ on a κ-complete semilattice would then be interpreted by λ-indexed join (explicitly, it would take $\sigma \in A^\lambda$ to $\bigvee_{i<\lambda} \sigma(i)$) Finally, as an example where we need a proper class of operations, we consider complete semilattices Here, we can take as the class of operation symbols all of V (again with $\iota(I) = I$ for every $I \in V$) and associate with the operation symbol I the operation giving I-indexed joins

5.2. Subalgebras, products, and quotients. If A is a algebra of type Ω and $S \subset A$, then S is called a *subuniverse* of A if every operation ω_A of A restricts to S, i.e., for every $\sigma \in S^{\iota(\omega)}$, $\omega_A(\sigma) \in S$. The restricted operations then make S into an Ω-algebra, called a *subalgebra* of A, and the inclusion $S \to A$ becomes a homomorphism. Arbitrary intersections of subuniverses are clearly subuniverses; thus, for every $X \subset A$ there is a least subuniverse of A containing X. The associated subalgebra is called the subalgebra of A *generated* by X If I is a set and, for every $i \in I$, A_i is an Ω-algebra, then the cartesian product $A = \prod_{i\in I} A_i$ becomes an Ω-algebra when we define the operations pointwise: if $\sigma \in A^{\iota(\omega)}$, then $\omega_A(\sigma)$ is the function $i \mapsto \omega_{A_i}(\sigma_i)$, where, for every $j \in \iota(\omega)$, $\sigma_i(j) = \sigma(j)(i)$. For every $i \in I$, the ith projection, $\pi_i : A \to A_i$ given by $\pi_i(\sigma) = \sigma(i)$, is an onto homomorphism

A binary relation $\theta \subset A \times A$ is an *equivalence relation* on A if it is reflexive, symmetric, and transitive. For $a \in A$, the set $a/\theta = \{b \in A : a\ \theta\ b\}$ is called the *equivalence class* of θ containing a (which it does by reflexivity). The set $A/\theta = \{a/\theta : a \in A\}$ of equivalence classes of θ form a *partition* of A—a set of disjoint sets whose union is A. The (onto) function $A \to A/\theta$ which maps $a \mapsto a/\theta$ is called the *quotient*, or *natural*, map and is denoted $\natural_\theta$. An equivalence relation θ on an Ω-algebra A is called an Ω-*congruence* if, further, it is a subuniverse of the product algebra $A \times A$. Explicitly, this means that θ is *compatible* with every $\omega \in \Omega$: for every pair $\sigma, \tau \in A^{\iota(\omega)}$, if $\sigma(i)\ \theta\ \tau(i)$ for all $i \in \iota(\omega)$, then $\omega_A(\sigma)\ \theta\ \omega_A(\tau)$. The set of equivalence classes then becomes an Ω-algebra, where $\omega_{A/\theta}(\natural_\theta \circ \sigma) = \natural_\theta(\omega_A(\sigma))$ for every $\sigma \in A^{\iota(\omega)}$, and the natural map becomes a homomorphism.

It is easy to verify that the intersection of any set of congruences on A is again a congruence. As a consequence of Proposition 4.3, there is a closure operator Θ on $A \times A$ that takes $X \subset A \times A$ to the smallest congruence containing X (called the congruence *generated* by X), and the fixedpoints of this operator are exactly the congruences on A, which form a complete lattice under inclusion that we denote by $\mathrm{Con}\, A$. A congruence generated by a single ordered pair, such as $\Theta((a, b))$, or written more simply as $\Theta(a, b)$, is called a *principal* congruence. It has the property that, for any congruence θ, $\Theta(a, b) \leq \theta$ iff $a\ \theta\ b$. It is therefore easily seen that every congruence is a join of principal congruences: $\theta = \bigvee\{\Theta(a, b) : a\ \theta\ b\}$.

5.3. Homomorphism theorems. If $f : A \to B$ is a homomorphism, then the *kernel* of f is the relation $\ker f$ on A defined by $a\ (\ker f)\ a'$ iff $f(a) = f(a')$, which is easily seen to be a congruence. The function $g : A/\ker f \to B$ given by $g(a/\ker f) = f(a)$ is well-defined and is a 1-1 homomorphism, and so $f = g \circ \natural_{\ker f}$ is a *factorization* of f into an onto homomorphism followed by a 1-1 homomorphism If f is already onto, then g is an isomorphism; thus, onto homomorphisms and quotients by congruences amount to the same thing. In the same way, if $\theta \leq \ker f$, then f can be factored uniquely through $A \to A/\theta$ via g $A/\theta \to B$ defined by $g(a/\theta) = f(a)$ (though g will be neither 1-1 or onto in general) Conversely, note that for any composable homomorphisms $A \xrightarrow{h} C \xrightarrow{g} B$, we have $\ker h \leq \ker(g \circ h)$

The composite of two onto homomorphisms is onto, in terms of congruences, we get the following: if $\theta \leq \psi$ in $\mathrm{Con}\, A$, and we write ψ/θ for the congruence on A/θ consisting of the pairs $\{(a/\theta, b/\theta) : a\, \psi\, b\}$, then the map $A/\psi \to (A/\theta)/(\psi/\theta)$ given by $a/\psi \mapsto (a/\theta)/(\psi/\theta)$ is an isomorphism Also, if $[\theta, 1]$ is the interval subposet of $\mathrm{Con}\, A$, then the map $[\theta, 1] \to \mathrm{Con}\, A/\theta$ given by $\psi \mapsto \psi/\theta$ is an isomorphism. If f $A \to B$ is a homomorphism, and $f \times f : A \times A \to B \times B$ is the function defined by $(f \times f)((a, a')) = (f(a), f(a'))$, then for every $\psi \in \mathrm{Con}\, B$, $(f \times f)^{-1}(\psi)$ is a congruence on A It follows from this that the adjunction $\Theta \dashv \imath$ between $\mathsf{P}(B \times B)$ and $\mathrm{Con}\, B$ arising from the closure operator Θ on $\mathsf{P}(B \times B)$ ($\imath$ is the inclusion) can be composed with the adjunction $f \times f \dashv (f \times f)^{-1}$ between $\mathsf{P}(A \times A)$ and $\mathsf{P}(B \times B)$ to yield (by restriction) an adjunction $\Theta \circ (f \times f) \dashv (f \times f)^{-1}$ between $\mathrm{Con}\, A$ and $\mathrm{Con}\, B$

5.4. Lattice congruences. Before continuing with the general survey of Universal Algebra, we now look at some special properties of congruence relations on lattices and, in particular, distributive lattices. These are collected in the following proposition, whose straightforward proof we omit (though the reader with less familiarity with distributive lattices will find it a rewarding exercise)

Proposition. *Suppose A is a distributive lattice (i.e., with operations $\wedge, \vee, 0, 1$ that satisfy the lattice equations plus distributivity), $\theta \in \mathrm{Con}\, A$, and $a\, b, c, d \in A$ Then the following statements hold:*

(a) *$a\, \theta\, b$ iff $(a \wedge b)\, \theta\, (a \vee b)$*
(b) *$b, d \in a/\theta$ and $b \leq c \leq d$ imply $c \in a/\theta$.*
(c) *$b, d \in a/\theta$ imply $b \wedge d, b \vee d \in a/\theta$*
(d) *$a \wedge b\, \theta\, b$ iff $a\, \theta\, a \vee b$.*
(e) *If $c \leq d$, then $a\, \Theta(c,d)\, b$ iff $a \wedge c = b \wedge c$ and $a \vee d = b \vee d$*
(f) *If $a \leq b$ and $c \leq d$, then $\Theta(a,b) \wedge \Theta(c,d) = \Theta(a \vee c, b \wedge d)$.*
(g) *If M_1, M_2 are θ-congruence classes, then $M_1 \leq M_2$ (in A/θ) iff there exists $a \in M_1$ and $b \in M_2$ with $a \leq b$.*

Notes In fact, only parts (e) and (f) require distributivity. Part (a) implies that the congruence θ is determined by the pairs $(a, b) \in \theta$ with $a \leq b$. Parts (b) and (c) say that congruence classes are *convex* and closed under meets and joins (in fact, they are closed under any non-empty meets and joins that are compatible with θ) Given any two $a, b \in A$, we say that the interval $[a, a \vee b]$ *projects down* to the interval $[a \wedge b, b]$ and that the latter interval *projects up* to the former Part (d) then says that the intervals collapsed by a congruence are closed under projections Note also that, in a distributive lattice, projective intervals are isomorphic Part (e) is a characterization of principal congruences, and part (f) says that principal congruences are closed under finite meets (note that the empty meet, or largest congruence on A, is principal: $\Theta(0, 1)$).

5.5. Equational classes and free algebras. For the rest of this Section, it will be convenient to treat "algebras" on (possibly proper) classes along side algebras on sets, as we have been doing with function-classes and functions. Since the operations on such an "algebra" are function-classes, however, we must insist that they be *uniform*: thus, given a similarity type Ω, we define an *algebra-class* of type Ω to be a class A and a formula $\phi(x, y)$ with two free variables such that, for every $\omega \in \Omega$, the formula $\varphi(\omega, y)$ (with one free variable) represents a function-class whose domain includes the class of all functions $\sigma : \iota(\omega) \to A$, and whose range is contained in A (Note that every algebra is an algebra-class.)

Now, fix a similarity type Ω and a class of *variables* X. We define the class $T_\Omega(X)$ of Ω-*terms* over X (and we will leave off the subscript Ω where there can be no confusion) Because Ω (as well as X) can be a proper class, this requires some care. (And note further that, for reasons of effective "coding", each $x \in X$ must be $<_\Omega$-minimal in the sense defined below.) We proceed by analogy with the construction of ordinals We first define an order $<_\Omega$ on V by putting $x <_\Omega y$ if $y = \langle \omega, f \rangle$, $\omega \in \Omega$, $\operatorname{dom} f = \iota(\omega)$, and $x \in \operatorname{rng} f$. A set y is $<_\Omega$-*minimal* if there does not exist x such that $x <_\Omega y$. Given a set T, $x \in T$ is $<_\Omega$-*maximal in* T if there does not exist $y \in T$ such that $x <_\Omega y$. Let Ω_0 be the class of constants of Ω ($\omega \in \Omega_0$ iff $\iota(\omega) = \emptyset$) We call a set T Ω-*grounded in* X if, for every $<_\Omega$-minimal element x of T, either $x \in X$ or $x = \langle \omega, \emptyset \rangle$ for some $\omega \in \Omega_0$. We call T Ω-*transitive* if $y \in T$ and $x <_\Omega y$ imply $x \in T$. Finally, we define an Ω-*term* in X to be a set T such that

(1) T is Ω-grounded in X,
(2) T is Ω-transitive,
(3) $<_\Omega$ is a well-ordering on T (every $S \subset T$ has a $<_\Omega$-minimal element), and
(4) there is a unique $<_\Omega$-maximal element in T, called the *head* of T.

Analogously to ordinals, we can prove a *structural induction* and a *structural recursion* theorem. We define the *support* of a term T by $\operatorname{spt} T = T \cap X$, in view of the transitivity of terms (and structural induction), this is the set of variables "occurring" in the term T. As an example of structural recursion, we define evaluation of terms. Let A be an Ω-algebra-class, T be a term, and $\rho : X' \to A$ be a function, where $\operatorname{spt} T \subset X' \subset X$. Then the *value* $T[\rho]$ of T at ρ is defined by recursion as follows: if $T = x \in X$, then $T[\rho] = \rho(x)$; otherwise, if $\langle \omega, f \rangle$ is the head of T and (by recursion) $\sigma : \iota(\omega) \to A$ is the function defined by $\sigma(i) = f(i)[\rho]$ (or the empty function if $\iota(\omega) = \emptyset$), then $T[\rho] = \omega_A(\sigma)$. Operations on A of the form $\rho \mapsto T[\rho]$ are called *term functions*.

An *equation* in X is simply a pair $\langle L, R \rangle$ with $L, R \in T(X)$, which we write more suggestively as $L \approx R$. An equation $L \approx R$ is *satisfied* (or *holds*, or *is valid*) in an algebra-class A if, for every function $\rho : \operatorname{spt} L \cup \operatorname{spt} R \to A$, we have $L[\rho] = R[\rho]$. If E is a class of equations, then the class of all Ω-algebras satisfying each of the equations in E is denoted $\operatorname{Mod}(\Omega, E)$;[2] classes of algebras of the form $\operatorname{Mod}(\Omega, E)$ are called *equational classes*. As an example, we give a class of equations for complete join-semilattices, making it an equational class. Recall that the similarity type has an operation symbol for each set I, denoted here in the form $\bigvee_{i \in I}$, whose interpretation in the algebra is given by I-indexed join. The class of equations consists of the single equation $\bigvee_1 x \approx x$, along with for every set I, family of sets $\{J_i : i \in I\}$, and onto function $g : K \to \coprod_{i \in I} J_i$, the equation $\bigvee_{i \in I}(\bigvee_{j \in J_i} x_{i,j}) \approx \bigvee_{k \in K}(x_{i_k, j_k})$, where for every $k \in K$, $g(k) = \langle i_k, j_k \rangle$ and $j_k \in J_{i_k}$ (recall the definition of disjoint union).

[2] Again, just a reminder that, in all cases we consider, Ω and E will be such that each algebra in $\operatorname{Mod}(\Omega, E)$ is encodable by a set

The reader may enjoy verifying that these equations suffice to characterize complete join-semilattices (hint the equations amount to a generalized associative law, and the presence of the function g allows for change of index set, including permutations and repetitions, so it is a generalized idempotent and commutative law as well). The same idea would work for κ-complete semilattices, except that the sets I, J_i, and K would have cardinality less than κ note that, given any such I and J_i, in order to insure the existence of an appropriate K, it is necessary that κ be a *regular* cardinal, for then $|\bigsqcup_{i\in I} J_i| < \kappa$

The class $T(X)$ becomes an Ω-algebra-class when, for $\omega \in \Omega$ and σ $\iota(\omega) \rightarrow T(X)$, we set $\omega_{T(X)}(\sigma) = \{\langle\omega\ \sigma\rangle\} \cup \operatorname{rng}\sigma$ As such, it is *free* over X with repsect to the class of all Ω-algebras if A is any Ω-algebra, and $f . X \rightarrow A$ is any function-class, then there is a unique function-class $\overline{f} \cdot T(X) \rightarrow A$ such that $\overline{f}(x) = f(x)$ for all $x \in X$ and such that $\overline{f}$ is an Ω-homomorphism.

If E is a class of equations in X, let E' be the class of those equations $L \approx R$ that hold in every algebra satisfying every equation in E (we say that $L \approx R$ is *semantically entailed* by E). Then E', as a class of ordered pairs, is a congruence relation on $T(X)$, the class of equivalence classes $T(X)/E'$ inherits an Ω-algebra structure from $T(X)$ and, moreover, satisfies all the equations of E.[3] If we write $\overline{x}$ for the equivalence class x/E', then the algebra-class $T(X)/E'$ is free over $\{\overline{x} \cdot x \in X\}$ with respect to the class $\mathbf{Mod}(\Omega, E)$

For the last kind of freeness we will discuss, we need the notions of reduct and diagram. Suppose that Ω' is a subclass of the class of operation symbols Ω (with each operation in Ω' having the same arity as it does in Ω). Then every Ω-algebra is naturally also an Ω'-algebra when we forget about the extra operations. we say that the Ω'-algebra is a *reduct* of the Ω-algebra For an Ω'-algebra A, we define a new set of variables $X_A = \{x_a : a \in A\}$, and let Δ_A (called the *diagram* of A) be the class of all pairs $\langle L, R\rangle$, where $L, R \in T_{\Omega'}(X_A)$ and $L[\rho] = R[\rho]$ for the function $\rho \cdot X_A \rightarrow A$ defined by $\rho(x_a) = a$. Now, given any class E of Ω-equations, we have, using the previous construction of free algebra-classes, a free Ω-algebra-class $F = T_\Omega(X_A)/(E \cup \Delta_A)'$ (on generators X_A and with equations those semantically entailed by the equations in E and the diagram of A), which has the following freeness property: for any algebra $B \in \mathbf{Mod}(\Omega, E)$, whose reduct to Ω' we denote B', and any Ω'-homomorphism $f . A \rightarrow B'$, there is a unique function-class $\overline{f} . F \rightarrow B$ such that $\overline{f}$ is an Ω-homomorphism with $f = \eta_A \circ \overline{f}$, where $\eta_A : A \rightarrow F$ takes $a \in A$ to the equivalence class containing x_a (Note, incidentally, that in the special case that $\Omega = \Omega'$ and $A \in \mathbf{Mod}(\Omega, E)$, we have that $T_\Omega(X_A)/(E \cup \Delta_A)'$ is isomorphic to A; hence every algebra is a quotient of a free algebra.)

When are these free algebra-classes free *algebras*? If Ω and X are both sets, it follows that $T(X)/E$ is a set.[4] On the other hand, for some classes of equations, $T(X)$ is a proper class while $T(X)/E$ is a set Of course this is true if E contains $x \approx y$, for example, but a more interesting example is that of complete semilattices.

[3] To construct the class $T(X)/E'$, we use an idea of Dana Scott: we take as equivalence classes not *all* the terms equivalent to a given one (which may be a proper class and thus not "collectable" into a class), but only the *set* of those with minimal rank

[4] To prove this, recursively assign an ordinal rank to every term, as we did with sets, and then show by induction (and *some* cardinal arithmetic) that for a sufficiently large cardinal κ we have $|T_\alpha(X)| < \kappa$ for all ordinals α

One can easily check that the free complete semilattice on a set X, with respect to the similarity type and equations given before, is isomorphic to the algebra $(\mathcal{P}X, \bigcup, \emptyset)$ (here $\bigcup$ represents the class of union operations. one for each index set). In other cases. such as for complete lattices, one can show that $T(X)/E$ is a proper class

6. Category theory

6.1. Basic notions. The theory of categories may be developed in several different ways, depending on the foundation used. For our purposes it will be sufficient to base it on the set theory introduced in Section 3. Thus, we define a category $\mathbf{A}$ to consist of a class of *objects* $A, B, C, \ldots$ and a class of *morphisms* (or *arrows*) $f, g, h, \ldots$ such that (1) every morphism f has a *domain* $\mathrm{dom}\, f$ and a *codomain* $\mathrm{cod}\, f$, which are objects; we write $f : A \to B$ to assert that $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$, and require that for any objects A and B the class $\hom_{\mathbf{A}}(A, B)$ of all morphisms $f : A \to B$ be a set; (2) for every object $A \in \mathbf{A}$ (where we are using $\in$ to indicate "is an object of"), there is an *identity* morphism $\mathrm{id}_A : A \to A$ (we often leave off the subscript), and for every two morphisms f, g with $\mathrm{cod}\, f = \mathrm{dom}\, g$, say $f : A \to B$ and $g : B \to C$, there is a *composite* morphism $g \circ f : A \to C$; and (3) identity morphisms are identities under composition, and composition is associative ($\mathrm{id} \circ f = f \circ \mathrm{id} = f$ and $h \circ (g \circ f) = (h \circ g) \circ f$, assuming all the morphisms are composable) A *functor* $F : \mathbf{A} \to \mathbf{C}$ between two categories is a function-class that assigns to every object A of $\mathbf{A}$ an object FA of $\mathbf{C}$ and to every morphism f of $\mathbf{A}$ a morphism Ff of $\mathbf{C}$ in such a way that domains, codomains, identities, and composition are preserved; i.e., if $f : A \to B$ then $Ff : FA \to FB$, and we have $F\mathrm{id}_A = \mathrm{id}_{FA}$ and $F(g \circ f) = Fg \circ Ff$ whenever $g \circ f$ is defined A *natural transformation* $\tau : F \to G$ between two functors $F, G : \mathbf{A} \to \mathbf{C}$ is a function-class assigning to every object A of $\mathbf{A}$ a morphism $\tau A : FA \to GA$ of $\mathbf{C}$ (called the *component* at A), such that $Gf \circ \tau A = \tau B \circ Ff$ whenever $f : A \to B$ is a morphism of $\mathbf{A}$.

Almost of all the categories we will be working with are *concrete*, that is to say, categories whose objects are sets with some additional structure and whose morphisms are certain functions between these sets (usually preserving the structure). Examples of concrete categories are **Set** (objects: sets; arrows: functions), **Top** (objects: topological spaces; arrows: continuous maps), **Pos** (objects: posets; arrows: monotone functions), and $\mathbf{Mod}(\Omega, E)$ (objects: algebras; arrows: homomorphisms). For every poset P, there is a category with an object for every element of P and an arrow from p to q whenever $p \leq q$ (and no other arrows). Functors between such (non-concrete) *poset-categories* are just monotone maps. For every category $\mathbf{A}$, there is the identity functor $\mathrm{id}_{\mathbf{A}}$ on $\mathbf{A}$, and the composition of two functors is a functor. A simple but important class of functors are the *forgetful* (or *reduct* functors) $\mathbf{Mod}(\Omega, E) \to \mathbf{Mod}(\Omega', E')$, where $\Omega' \subset \Omega$ and $E' \subset E$, which take an Ω-algebra to its reduct to Ω'. A category is *small* if its class of objects (and hence also its class of morphisms) forms a set.

Having indicated how our theory of categories is to be based on set theory, and given our main examples, we now proceed to outline the rest of the category-theoretic preliminaries quickly and ask the reader to consult the references for more information.

The *dual* of a category $\mathbf{A}$ is denoted $\mathbf{A}^{\mathrm{op}}$; as with posets, statements about categories have duals. and if a statement is true of all categories, then so is its dual.

A *subcategory* $\mathbf{C}$ of $\mathbf{A}$ has as objects and morphisms subclasses of the objects and morphisms of $\mathbf{A}$, in such a way that the inclusion $\mathbf{C} \to \mathbf{A}$ is a functor

Given a functor $G : \mathbf{C} \to \mathbf{A}$ and object $A \in \mathbf{A}$, a pair $\langle C, u\rangle$ consisting of an object $C \in \mathbf{C}$ and a morphism $u : A \to GC$ is called *universal* from A to G if for every other such pair $\langle C', u'\rangle$ there is a unique morphism $h : C \to C'$ with $u' = Gh \circ u$ If G is the inclusion of a subcategory, then a universal pair from A to G is called a *reflection* of A into the subcategory Dually, we speak of universal pairs from G to A, and a *coreflection* when G is an inclusion.

Two functors $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{A}$ are *adjoint*, written $F \dashv G$, when there is a family of bijections $\hom_{\mathbf{C}}(FA, C) \simeq \hom_{\mathbf{A}}(A, GC)$, natural in A and C We denote the *unit* of the adjunction by $\eta : \mathrm{id}_{\mathbf{A}} \to G \circ F$ and the *counit* by $\varepsilon : F \circ G \to \mathrm{id}_{\mathbf{C}}$; the components of these correspond under the bijection to appropriate identity morphisms If $F \dashv G$, then the pair $\langle FA, \eta A\rangle$ is universal from A to G for every $A \in \mathbf{A}$, and, conversely, if a universal pair $\langle FA, \eta A\rangle$ from A to G is given for each $A \in \mathbf{A}$, then F can be (uniquely) made into a functor such that $F \dashv G$ For us, the main examples of adjoint functors are $F \dashv G$, where G is a forgetful functor between equational classes, and F is the corresponding free functor arising from the universal property of free algebras. In case $\mathbf{C}$ is a subcategory of $\mathbf{A}$ and G is the inclusion, then F is called a *reflection* If, furthermore, G is a *full* inclusion (meaning that every morphism between objects in the subcategory is also in the subcategory), then the counit is a *natural isomorphism* (i.e., every component εC of the transformation is an *isomorphism*). Dually, a functor right adjoint to an inclusion is called a *coreflection* In case both the unit and counit are natural isomorphisms, the adjunction is called an *equivalence*, and $\mathbf{A}$ and $\mathbf{C}$ are said to be *equivalent* categories

If $\mathbf{J}$ is a small category, functors $D : \mathbf{J} \to \mathbf{A}$ are the objects of a category $\mathbf{A}^{\mathbf{J}}$, whose morphisms are natural transformations (composed coordinatewise) The *diagonal* functor $\Delta : \mathbf{A} \to \mathbf{A}^{\mathbf{J}}$ takes an object A to the functor which is A on all objects of $\mathbf{J}$ and id_A on all morphisms The category $\mathbf{A}$ is said to have *limits of type* $\mathbf{J}$ if Δ has a right adjoint, $\lim_{\mathbf{J}} : \mathbf{A}^{\mathbf{J}} \to \mathbf{A}$, and, dually, *colimits* of type $\mathbf{J}$ if Δ has a left adjoint, $\mathrm{colim}_{\mathbf{J}} : \mathbf{A}^{\mathbf{J}} \to \mathbf{A}$ Given a *diagram* $D : \mathbf{J} \to \mathbf{A}$ of type $\mathbf{J}$, the value $\lim D$ is called the *limit* of the diagram, and the counit $\varepsilon D : \Delta \lim D \to D$ is the *limit cone* (dual: *colimits* and *colimit cones*) More generally, a (as opposed to "the") limit of D is a universal pair from Δ to D (called a limit cone), and a colimit is a universal pair from D to Δ (a colimit cone) Given a morphism $\Delta A \to D$ in $\mathbf{A}^{\mathbf{J}}$, the corresponding morphism $A \to \lim D$ (or, more generally, $A \to L$ for a given limit L) is called the *mediating* morphism, and similarly with colimits

A functor $F : \mathbf{A} \to \mathbf{C}$ is said to *preserve* limits of type $\mathbf{J}$ if for every diagram $D : \mathbf{J} \to \mathbf{A}$, F takes limit cones for D to limit cones for $F \circ D$. F is said to create limits of type $\mathbf{J}$ if every limit on the image of F has a unique preimage in $\mathbf{A}$: i e , if whenever $D : \mathbf{J} \to \mathbf{A}$ is a diagram in $\mathbf{A}$ and $\mu : \Delta L \to F \circ D$ is a limit cone in $\mathbf{C}$, then there is a unique $\mu' : \Delta L' \to D$ such that $F \circ \mu' = \mu$, and moreover μ' is a limit cone Limits are preserved by right-adjoint functors (dually, colimits are preserved by left-adjoint functors). Limits and colimits in poset-categories are meets and joins, and the preservation of (co)limits by adjoint functors generalizes the same fact for adjunctions between posets (see 4 3) A category is *complete* if it posseses limits of type $\mathbf{J}$ for every small category $\mathbf{J}$ (dual *cocomplete*) It can be shown that a category is complete iff it has equalizers and all products (dual cocomplete = coproducts + coequalizers).

6.2. Particulars. We now look at special cases of limits (and colimits) in a category $\mathbf{A}$. Limits of discrete diagrams D (when every morphism of $\mathbf{J}$ is an identity morphism) are called *products* and are denoted, for a family $\{A_i : i \in I\}$, by $\prod_{i\in I} A_i$. The *projections* (making up the limit cone) are denoted $\pi_i : \prod_{i\in I} A_i \to A_i$. If, for every $i \in I$, there is a morphism $f_i : B \to A_i$ (so that the family $\{f_i : i \in I\}$ determines a natural transformation $\Delta B \to D$), the mediating morphism to the product is denoted $(f_i)_{i\in I} : B \to \prod_{i\in I} A_i$. (Dual: *coproducts*, $\coprod_{i\in I} A_i$, with *injections* $\nu_i : A_i \to \coprod_{i\in I} A_i$.) The limit of a pair $f,g : A \to B$ of morphisms is essentially given by a morphism $e : E \to A$ such that $f \circ e = g \circ e$ and such that any other such morphism factors uniquely through e (i.e., if $f \circ e' = g \circ e'$, then $e' = e \circ h$ for some (unique) h), it is called an *equalizer* of f and g (dual: *coequalizer*). The limit of a diagram $A \xrightarrow{f} B \xleftarrow{g} C$ is essentially an object P and two morphisms $A \xleftarrow{g'} P \xrightarrow{f'} C$, called a *pullback* of the diagram, such that $f \circ g' = g \circ f'$ and such any other such object and two morphisms factor through P in the obvious way. We say that f' is the pullback of f along g and that g' is the pullback of g along f. (Dual: *pushout*.)

A morphism $f : A \to B$ is a *section* if there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ (dual: *retraction*). Thus an isomorphism f is both a section and retraction (by the same g, which is called the *inverse* of f and denoted $g = f^{-1}$). f is a *monomorphism* (or is *mono*) if, $f \circ h = f \circ k$ implies $h = k$ (dual: *epimorphism*, or *epi*). If g is mono, then $g \circ f$ is mono iff f is mono. The equalizer of two morphisms is always mono; such monos are called *regular*. Given a regular mono m, the two morphisms resulting from pushing out m along itself have m as an equaliser. f is an *extremal mono* if, whenever $f = h \circ k$ and k is epi, then k is an isomorphism (dual: extremal epi). Sections are regular monos, and regular monos are extremal, but the implications cannot be reversed in general

Finally, a word on "the" versus "a". In most categories, limits, colimits, and so on, are unique only up to isomorphism. Thus, there is a difficulty speaking about "the" product of A and B. However, in the algebraic categories that we deal with here, there is a *canonical* choice of limit (and colimit), and so we will be justified in speaking about "the" limit or colimit, by which we will always mean the canonical one.

6.3. The category Set. What do all these concepts mean in the category **Set**? Any one-element set is a terminal object. Products are given by cartesian products with their projection functions: the mediating morphism for a family $\{f_i : B \to A_i\}$ takes an element $b \in B$ to the function σ_b defined by $\sigma_b(i) = f_i(b)$. The equalizer of two functions $f, g : A \to B$ can be given by the inclusion of the set $\{a \in A : f(a) = g(a)\}$ into A. The pullback of $f : A \to B$ and $g : C \to B$ is the set $\{(a,c) \in A \times C : f(a) = g(c)\}$; the functions g' and f' are, respectively, the projections onto the first and second coordinates. More generally, limits in **Set** can be computed as follows: given any diagram $D : \mathbf{J} \to \mathbf{Set}$, the limit of D can be taken to be the set consisting of those $\sigma \in \prod_{j\in \mathbf{J}} Dj$ such that $(Df)(\sigma(j)) = \sigma(k)$ for all morphisms $f : j \to k$ of $\mathbf{J}$. The limit cone consists of the projections onto each coordinate, and mediating morphisms are just as with the product. As for colimits, coproducts are disjoint unions, and the coequalizer of $f, g : X \to Y$ is the natural map $\natural_\theta : Y \to Y/R$, where R is the equivalence relation generated by the pairs $\{(f(x), g(x)) : x \in X\}$. In general, the colimit of a diagram $D : \mathbf{J} \to \mathbf{Set}$ is $(\coprod_{j\in \mathbf{J}} Dj)/R$, where $\langle i,x\rangle\ R\ \langle j,y\rangle$ (for $i,j \in \mathbf{J}$, $x \in Di$, $y \in Dj$) iff there exists morphisms $g' : i \to k$ and $f' : j \to k$ such that $(Dg')x = (Df')y$ In **Set**, sections, regular monos, extremal monos, and monos all correspond to injective functions Also, epis, extremal epis, and regular epis all correspond to surjective functions. the Axiom of Choice is equivalent to the statement that all surjective functions are retractions

6.4. The Pushout Lemma. The following Lemma lists the properties of pushouts that we will need in Chapter 8.

Lemma. *Pushouts of epis (resp , regular epis, retractions) are epi (resp , regular epis, retractions) Consider the commutative diagram*

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "f"'] & \bullet \arrow[r, "h"] \arrow[d, "f'"'] & \bullet \arrow[d, "f''"'] \\
\bullet \arrow[r, "g'"'] & \bullet \arrow[r, "h'"'] & \bullet
\end{tikzcd}$$

(a) *If both squares are pushouts, then the outer rectangle is a pushout (i.e., $h' \circ g'$ is a pushout of $h \circ g$ along f)*

(b) *If the outer rectangle is a pushout, and the pair $\langle f', g' \rangle$ is jointly-epi (i.e., $k_1 \circ f' = k_2 \circ f'$ and $k_1 \circ g' = k_2 \circ g'$ together imply $k_1 = k_2$; note that this is the case if the left square is a pushout), then the right square is a pushout*

(c) *If f is epi and f' is an isomorphism, then the left square is a pushout.*

Proof. See [18, p.183] for a proof of the first part of the Lemma. Parts (a) and (b) are standard ([29, p 72], [18, p.180]). For (c), suppose k_1 and k_2 are such that $k_1 \circ f = k_2 \circ g$. Let $m = k_2 \circ (f')^{-1}$. Then $m \circ f' = k_2$ and $m \circ g' \circ f = m \circ f' \circ g = k_2 \circ g = k_1 \circ f$ Since f is epi, $m \circ g' = k_1$. Thus m is the required mediating morphism (which is obviously unique).

6.5. Categorical properties of algebraic categories. A category $\mathbf{A}$ equivalent to a category of the form $\mathbf{Mod}(\Omega, E)$ will be called *algebraic*; if Ω can be taken to be a set, then $\mathbf{A}$ is called *monadic*.[5] An algebraic category is monadic iff it possesses free algebras over any set. This subsection is devoted to giving the categorical properties of an algebraic (or monadic) category $\mathbf{A} = \mathbf{Mod}(\Omega, E)$.

6.5.1. *$\mathbf{A}$ is complete, and limits in $\mathbf{A}$ are computed as limits of the underlying sets, with operations defined pointwise (the forgetful functor to* **Set** *creates limits)*

6.5.2. *In $\mathbf{A}$, regular epi = extremal epi = surjective, and mono = injective.*

6.5.3. *$\mathbf{A}$ has coequalizers; if $\mathbf{A}$ is monadic, then it also has coproducts and thus all colimits* The coequalizer of two morphisms $f, g : A \to B$ is given by the natural map $\natural_\theta\ B \to B/\theta$, where $\theta = \Theta(\{(f(a), g(a)) : a \in A\})$. Assuming that free algebras over any set exist, the coproduct of a family $\{A_i : i \in I\}$ of algebras (which we assume for convenience are disjoint) is given by

$$T_\Omega\Big(\bigcup_{i \in I} X_{A_i}\Big) \Big/ \Big(E \cup \bigcup_{i \in I} \Delta_{A_i}\Big)'$$

[5] The source of this term is the theory of monads; although this theory underlies much of what we discuss, and the knowledgable reader will see that our presentation is guided by it, I've avoided introducing it explicitly, because much of the technical baggage is unnecessary for our results

(X_A and Δ_A and the operation $(-)'$ on equations are defined in 5.5).

6.5.4. **A** *has both (extremal-epi, mono)-factorizations and (epi, extremal-mono)-factorizations.* The first just refers (in view of 6.5.2) to the usual fact that every morphism f can be factored as $f = h \circ g$, where g is extremal-epi (= onto) and h is mono (= 1-1), and that any other such factorization $f = h' \circ g'$ is equivalent in the sense that $h' = m \circ h'$ and $k' = k \circ m$ for a unique isomorphism m. The morphism g can be taken to be $\natural_{\ker f}$. The second says the same, except that g is epi and h is extremal-mono (but note that epis need not be onto and not all monos need be extremal).

6.5.5. Finally, we point out that if every $\omega \in \Omega$ has $|\iota(\omega)| < \kappa$ for some fixed regular cardinal κ (and, hence, **A** is monadic), then certain colimit constructions on **A** become simplified (technically, **A** is *locally κ-presentable*). For example, we say that **J** is a *κ-filtered* category if every subcategory **J**$'$ of **J** with less than κ morphisms has a cone over it (i.e., there exists a natural transformation $\mathrm{id}_{\mathbf{J}'} \to \Delta j$ between functors $\mathbf{J}' \to \mathbf{J}'$, for some $j \in \mathbf{J}$). Then, if **J** is a κ-filtered category, the colimit of any diagram of type **J** is calculated as in **Set**; i.e., the forgetful functor to **Set** creates limits of type **J**. In particular, if **J** is a (poset-category that is a) chain such that every $S \subset \mathbf{J}$ with $|S| < \kappa$ (a "κ-subset") has an upper bound, and $D : \mathbf{J} \to \mathbf{A}$ is a diagram such that for every $j \leq k$, $D(j \to k)$ is an inclusion, then $\operatorname{colim} D$ is just the union of the algeras Dj, $j \in \mathbf{J}$. Similarly, if $\{\theta_j : j \in \mathbf{J}\}$ is a chain of congruences in an algebra A with an upper bound for every κ-subset, then $\bigvee_{j\in \mathbf{J}} \theta_j = \bigcup_{j \in \mathbf{J}} \theta_j$.

6.6. Foundations. The properties of κ-filtered colimits in a monadic category with arities bounded by κ, as in 6.5.5, turn out to be very useful.[6] We now describe a few methods, some more "philosophical" than technical, for treating non-monadic algebraic categories as if they were bounded.

An *inaccessible* cardinal, which we will always write as ∞, is an uncountable, regular, strong-limit cardinal (uncountable, of course, means $|\infty| > \omega$, and κ is a strong-limit cardinal if $\lambda < \kappa$ implies $2^\lambda < \kappa$). Let it be said immediately that the existence of inaccessible cardinals cannot be proved (nor even proved consistent) in ZFC, however, they have a plausibility similar in nature to the existence of infinite sets (which are also "inaccessible" without the Axiom of Infinity). Given a fixed inaccessible cardinal ∞, let us call a set X *small* if $|X| < \infty$ and *large* otherwise. In addition to being "inaccesible" by unions (regularity) and powersets ($\lambda \mapsto 2^\lambda$), one can show that *any* set-theoretic construction involving small sets will result in a small set.

Assuming the existence of an inaccessible cardinal ∞, we may decide to restrict our attention to small sets, and hence small algebras, etc , since all of the sets we deal with in normal mathematical work are small. By accepting this restriction, we then make all of our algebraic categories bounded by ∞, and hence amenable to the use of the results mentioned in 6.5.5. If this helps us to prove something about small algebras, then it is worthwhile.

An extension of this approach posits an unlimited number of inaccessible cardinals (specifically, for every cardinal there is a larger inaccessible cardinal). This is equivalent to Grothendieck's method of *universes*: a universe is a set closed under all set-theoretic

[6] And they are one good reason why most universal algebra texts only treat the finitary case, where $\kappa = \omega$ (another, to be sure, is notational simplicity). For example, the famous Birkhoff Subdirect Representation Theorem is true only in the finitary case.

constructions, and one supposes that every set belongs to some universe The purpose of this is to recover what was lost with a single inaccesible cardinal, namely, there were some sets (the large ones) that couldn't be talked about, and with unlimited inaccessibles, every set becomes small at some point Unfortunately, to really exploit this extra generality involves some technical difficulties, since one is constantly needing to shift between universes

A better approach, developed by Feferman [8], is to adjoin a predicate S to the language of set theory, with the idea that $S(x)$ asserts that "x is small", and then add axioms which say that small sets satisfy the same formulas (in the language without S) as all sets. This solves the problem that universes were created for—to regain the universality lost by a single inaccessible—but without the associated technical disadvantages Furthermore, and quite importantly, the extension of ZFC to include S is *conservative*: no new theorems of ZFC can be proven. This differs from the other approaches, which require extra assumptions. The idea is roughly that, as with the Reflection Principle of set theory (see [19, p.89]), though an inaccessible cardinal offers absolute inaccessibility, the approach with S offers innaccesibility for any *finite* number of set-theoretic operations, which is all that can appear in a proof anyway.

Chapter 3

HEYTING ALGEBRAS AND EXTENSIONAL OPERATORS

Every frame is a Heyting algebra; thus, we begin our investigation of frames by looking at the properties of Heyting algebras After some basic facts in Section 7, we look in Section 8 at the relationship between Heyting algebra congruences and filters. In this respect, Heyting algebras are less like lattices and distributive lattices, and more like groups and rings: every congruence is determined by one of its equivalence classes. Although this material is well-known, we go into detail in order to bring out some additional information (for example Proposition 8.8) and to prepare for the results of the following section. In the last (and by far the largest) section, Section 9, we introduce and study the concept of extensional operator The first part of Section 9 introduces the method of calculation we use in Heyting algebras ("replacement principles"), and the rest of the section is devoted to several classes of extensional operators and their various properties.

7. Definitions and basic properties

7.1. Definitions. A *Heyting algebra* is an algebra $A = (A, \wedge, \vee, \rightarrow, 0, 1)$ such that

(a) $(A, \wedge, \vee, 0, 1)$ is a lattice, and
(b) $\rightarrow$ is a binary operation that satisfies

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for all $a, b, c \in A$, where $\leq$ is the lattice order

A *Heyting lattice* is a lattice that is the reduct of a Heyting algebra.

In any Heyting algebra, it will be convenient to define two other "arrow" operations, $\leftrightarrow$ ("bi-arrow") and $\twoheadrightarrow$ ("double-arrow"), by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \quad \text{and}$$
$$a \twoheadrightarrow b = (a \rightarrow b) \rightarrow b.$$

For notational convenience, we extend the Heyting operations, $\wedge$, $\vee$, and $\rightarrow$, to sets, by stipulating that these operations "distribute over" sets producing sets, thus, for example, if $a, b \in A$ and $S \subset A$, then

$$a \wedge (b \rightarrow S) = a \wedge \{b \rightarrow s : s \in S\} = \{a \wedge (b \rightarrow s) \cdot s \in S\}.$$

Similarly, if $f : A \rightarrow B$ is any function and $S \subset A$, then

$$f(S) = \{f(s) \,.\, s \in S\} \subset B.$$

Finally, recall that in any lattice, we consider ("arbitrary" or "infinite") joins and meets as partial functions $\bigwedge, \bigvee : PA \rightarrow A$. An expression involving such a meet or join may therefore be undefined, given two such expressions E and E', we use the notation

$$E \simeq E'$$

to indicate *directed equality*; i e , if E is defined then so is E' and $E = E'$ If both $E \simeq E'$ and $E' \simeq E$, then we write $E \asymp E'$

7.2. Here are some facts about Heyting algebras and Heyting lattices that we will regard as basic and use freely in calculations.

Proposition. *Suppose that A is a Heyting algebra, $a, b, c \in A$, and $S \subset A$ Then the following statements hold·*

(a) $a \rightarrow a = 1$
(b) $(a \rightarrow b) \wedge b = b$ *equivalently,* $b \leq a \rightarrow b$
(c) $a \wedge (a \rightarrow b) = a \wedge b$.
(d) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
(e) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$
(f) $1 \rightarrow a = a$.
(g) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *(and thus* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$*)*
(h) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$.
(i) *The operation* $a \rightarrow -$ *is monotone; the operation* $- \rightarrow a$ *is anti-monotone*
(j) $a \leq b$ *if and only if* $a \rightarrow b = 1$.
(k) $a \leq a \rightarrow b$
(l) $(a \rightarrow b) \rightarrow b = a \rightarrow b$
(m) $a \wedge \bigvee S \simeq \bigvee a \wedge S$
(n) $a \rightarrow \bigwedge S \simeq \bigwedge a \rightarrow S$
(o) $(\bigvee S) \rightarrow a \simeq \bigwedge S \rightarrow a$

Moreover, a lattice L is a Heyting lattice iff a (necessarily unique) binary operation $\rightarrow$ on L can be defined so that (a)–(d) *hold for all $a, b, c \in L$*

PROOF. For (a) and (b), use 7.1(1) on the inequalities $1 \wedge a \leq a$ and $b \wedge a \leq b$ Use it in the other direction on $a \rightarrow b \leq a \rightarrow b$ to get

$$a \wedge (a \rightarrow b) \leq b \tag{1}$$

and hence $a \wedge (a \rightarrow b) \leq a \wedge b$; the reverse inequality follows from (b) and the monotonicity of $a \wedge -$ Continuing with 7.1(1), and using the Yoneda lemma for posets (3.3), an arbitrary $x \in X$ is less than either side of (d) precisely when $x \wedge a \leq b \wedge c$, either side of (e) when $x \wedge a \wedge b \leq c$, and either side of (f) when $x \leq a$, proving (d), (e), and (f). Statements (g) and (h) are special cases of (m) and (o), since binary joins exist in A. Monotonicity, (i), follows from (d) and (h); see 9.3 for details (j) is trivial. The inequality (k) follows from (1), and, therefore, so does half of (l) For the other half, if $x \leq a \rightarrow b$, then $(a \rightarrow b) \rightarrow b \leq x \rightarrow b$ by antimonotonicity of $- \rightarrow b$, and so $x \leq (a \rightarrow b) \rightarrow b$ by two applications of 7.1(1).

Next, suppose that $\bigvee S$ exists. Then

$$\begin{aligned} \forall s \in S\ a \wedge s \leq x \quad &\text{iff} \quad \forall s \in S\ s \leq a \rightarrow x \\ &\text{iff} \quad \bigvee S \leq a \rightarrow x \\ &\text{iff} \quad a \wedge \bigvee S \leq x \end{aligned}$$

Thus $a \wedge \bigvee S$ is the least upper bound of $a \wedge S$, proving (m); the proofs of (n) and (o) are analogous

Finally, a proof that (a)–(d) constitute (along with the lattice equations) an equational axiomatization of Heyting algebras can be found in [22, I 1.10]

7.3. Regular and Boolean elements. If a is an element of a Heyting algebra A, then the element $a \to 0$ is often denoted by $\neg a$ and called the *negation* (or *pseudocomplement*) of a Note that $a \wedge \neg a = 0$; in fact, $\neg a$ is the largest element x such that $a \wedge x = 0$. We say that a is

(a) *regular* if $\neg\neg a = a$, and
(b) *Boolean* if $a \vee \neg a = 1$

Trivially, an element is Boolean iff it has a complement (sometimes we call Boolean elements *complemented*) Let $b \in A$ be Boolean. Since 7.2(i) implies that $\neg\neg b \geq b$, we have

$$\neg\neg b \vee \neg b \geq b \vee \neg b = 1,$$

and so $\neg\neg b \vee \neg b = 1$ But $\neg\neg b \wedge \neg b = 0$, showing that $\neg\neg b$ is also a complement of b. By the uniqueness of complements in a distributive lattice, $\neg\neg b = b$ Hence, every Boolean element is regular

7.4. Our main interest in Boolean elements is that they behave nicely with respect to the Heyting algebra operations (and, as we shall see later, certain operators on Heyting algebras).

Proposition. *Suppose A is a Heyting algebra, $b \in A$ is Boolean, and $S \subset A$ Then the following statements hold for all $a, c \in A$:*

(a) $a \wedge b \leq c$ *if and only if* $a \leq \neg b \vee c$
(b) $b \to c = \neg b \vee c$.
(c) $a \to b = \neg a \vee b$.
(d) $b \vee \bigwedge S = \bigwedge b \vee S$

PROOF. (a) Suppose $a \wedge b \leq c$. Then

$$\neg b \vee c \geq \neg b \vee (a \wedge b) = (\neg b \vee a) \wedge (\neg b \vee b) = \neg b \vee a \geq a$$

so that $a \leq \neg b \vee c$ Conversely, suppose that $a \leq \neg b \vee c$. Then

$$a \wedge b \leq (\neg b \vee c) \wedge b = (\neg b \wedge b) \vee (c \wedge b) = c \wedge b \leq c.$$

(b) For any $x \in A$, 7.1(1) and (a) give

$$x \leq b \to c \quad \text{iff} \quad x \wedge b \leq c \quad \text{iff} \quad x \leq \neg b \vee c$$

(c) Since $a \to b \geq a \to 0$ and $a \to b \geq b$, one direction is clear. The other direction is equivalent by (a) to

$$(a \to b) \wedge \neg b \leq a \to 0,$$

which is true iff $a \wedge (a \to b) \wedge \neg b \leq 0$. But this is clear, since $a \wedge (a \to b) \leq b$ and $b \wedge \neg b = 0$.

(d) Using $\neg\neg b = b$, (b), and 7.2(n), we have

$$b \vee \bigwedge S \asymp \neg b \to \bigwedge S \asymp \bigwedge \neg b \to S \asymp \bigwedge b \vee S$$

7.5. Boolean lattices. Recall that a Boolean lattice is a distributive lattice in which every element has a complement (which is necessarily unique). The following proposition gives an alternate description of Boolean lattices.

Proposition. *If A is a Heyting lattice in which every element is regular, then A is a Boolean lattice.*

Proof. Suppose every element of A is regular and let $a \in A$. It is enough to show that $a \vee \neg a = 1$. Now,

$$\begin{aligned} a \vee \neg a &= \neg\neg(a \vee \neg a) \\ &= \neg(\neg a \wedge \neg\neg a) \\ &= \neg 0 = 1 \end{aligned}$$

7.6. Here are some facts about Boolean lattices, generalizing the De Morgan laws of 4.2.

Proposition. *Suppose A is a Boolean lattice and $S \subset A$. Then*
(a) *The map $x \mapsto \neg x$ is an order isomorphism $A \to A^{op}$*
(b) $\neg \bigvee S \asymp \bigwedge \neg S$
(c) $\neg \bigwedge S \asymp \bigvee \neg S$

Proof. Since every element of A is regular, the map $a \mapsto \neg a$ is onto ($a \in A$ has preimage $\neg a$), moreover,

$$\neg a \leq \neg b \quad \text{iff} \quad \neg a \wedge b \leq 0 \quad \text{iff} \quad b \leq \neg\neg a \quad \text{iff} \quad b \leq a$$

for every $a, b \in A$, showing that it is also an order-reversing embedding, proving (a). Parts (b) and (c) follow immediately from (a), since $\neg$, by reversing the order, swaps meets and joins.

8. Filters and congruences

8.1. Filters. A *filter* in a meet-semilattice A is a subset $F \subset A$ satisfying the following three conditions

8.1.1. $1 \in F$.
8.1.2. $a \in F$ and $a \leq a'$ imply $a' \in F$ $\quad (a, a' \in A)$
8.1.3. $a \in F$ and $b \in F$ imply $a \wedge b \in F$ $\quad (a, b \in A)$

8.2. Filters from Congruences. Every meet-semilattice congruence gives an example of a filter:

Proposition. *Suppose θ is a congruence on a meet-semilattice A. Then $1/\theta$ is a filter in A*

PROOF. We check 8 1 1–8.1 3 for $1/\theta$ Since $1 \in 1/\theta$, 8.1.1 is clear. Next suppose $a \in 1/\theta$ and $a \leq a'$ Then $a\,\theta\,1$ and $a'\,\theta\,a'$, so $a = a \wedge a'\,\theta\,1 \wedge a' = a'$ Thus, $a\,\theta\,a'$ and so $a' \in 1/\theta$, verifying 8 1 2. Finally, suppose $a, b \in 1/\theta$ Then $a\,\theta\,1$ and $b\,\theta\,1$ so that $a \wedge b\,\theta\,1$. Thus, $a \wedge b \in 1/\theta$, verifying 8 1.3.

8.3. The equivalence of filters and congruences. In the rest of this section we show that for Heyting algebras the filters arising from congruences are typical· associated to each filter F in a Heyting algebra A is a congruence $\Theta(F)$ on A such that the operations

$$F \mapsto \Theta(F) \quad \text{and} \quad \theta \mapsto 1/\theta$$

are mutually inverse isomorphisms between the set of filters in A, ordered by inclusion, and $\mathrm{Con}\,A$.

We start with the following proposition.

Proposition. *Let A be a Heyting algebra Then for every $a, x\ y \in A$ the following statements are equivalent·*

(a) $a \rightarrow x = a \rightarrow y$
(b) $a \wedge x = a \wedge y$
(c) $a \leq x \rightarrow y$
(d1) $y \in [a \wedge x, a \rightarrow x]$
(d2) $x \in [a \wedge y, a \rightarrow y]$.

PROOF Since $a \wedge (a \rightarrow b) = a \wedge b$ and $a \rightarrow (a \wedge b) = (a \rightarrow a) \wedge (a \rightarrow b) = a \rightarrow b$ for every $a, b \in A$. the two operations $a \wedge -$ and $a \rightarrow -$ are related to each other by the equations

$$\begin{aligned} a \wedge - &= a \wedge (a \rightarrow -), \\ a \rightarrow - &= a \rightarrow (a \wedge -) \end{aligned} \tag{1}$$

By applying these two operations to the equations (a) and (b), respectively, it follows that (a)$\Leftrightarrow$(b) for every $a, x, y \in A$ Now consider the conjunction

$$a \wedge x \leq y \quad \text{and} \quad a \wedge y \leq x. \tag{2}$$

Clearly, (2) is equivalent to the conjuction

$$a \wedge x \leq a \wedge y \quad \text{and} \quad a \wedge y \leq a \wedge x$$

and thus equivalent to (b) Alternatively, (2) is equivalent to

$$a \leq x \rightarrow y \quad \text{and} \quad a \leq y \rightarrow x$$

and thus to (c). Alternatively again, (2) is equivalent to

$$a \wedge x \leq y \quad \text{and} \quad y \leq a \rightarrow x$$

and thus to (d1); the case of (d2) is similar.

8.4. Notational convention. We will exploit the equivalence of the equations

$$a \rightarrow x = a \rightarrow y \quad \text{and} \quad a \wedge x = a \wedge y,$$

and denote them both by

$$a * x = a * y$$

More formally (and more generally) the symbols $*$, $*'$, etc, appearing in a formula may denote either $\rightarrow$ or $\wedge$ but must denote the same operation in all of their occurences This notation will only be used in a formula if all resulting instances of the formula are equivalent Thus $1 * a = a$ denotes the two (valid) equations $1 \rightarrow a = a$ and $1 \wedge a = a$, and the reader may verify the truth (and hence equivalence) of all four instances of the equation $a * (a *' b) = a * b$ (to be used later in fact)

8.5. Proposition. *If F is a filter in a Heyting algebra A, then for every pair of elements $x, y \in A$, the following two statements are equivalent*

(a) $a * x = a * y$ *for some* $a \in F$

(b) $x \leftrightarrow y \in F$

Moreover if we define the relation $\Theta(F)$ to hold between x and y just in case these statements are true, then $\Theta(F)$ is a congruence

Proof. The equivalence of (a) and (b) follows directly from 8 3 if (a) holds, then $a \leq x \leftrightarrow y$, and thus using 8 1.2 so does (b), conversely, if (b), then $x \leftrightarrow y \leq x \leftrightarrow y$ gives (a) with $a = x \rightarrow y$

We now show that $\Theta(F)$ is a congruence Since $1 \in F$ (8 1.1), it follows that $\Theta(F)$ is reflexive, $\Theta(F)$ is obviously symmetric; for transitivity, suppose $x\,\Theta(F)\,y\,\Theta(F)\,z$ Choose $a, b \in F$ such that $a \wedge x = a \wedge y$ and $b \wedge y = b \wedge z$ Then $a \wedge b \in F$ by 8 1 3, and

$$(a \wedge b) \wedge x = b \wedge (a \wedge x) = b \wedge (a \wedge y) = a \wedge (b \wedge y) = a \wedge (b \wedge z) = (a \wedge b) \wedge z,$$

so that $x\,\Theta(F)\,z$. We have thus shown that $\Theta(F)$ is an equivalence relation

To complete the proof we need to show that $\Theta(F)$ repects the basic operations of A. Suppose $x\,\Theta(F)\,x'$ and $y\,\Theta(F)\,y'$, and choose $a, b \in F$ such that

$$a \wedge x = a \wedge x' \qquad \text{and} \qquad b * y = b * y'$$

As before, $a \wedge b \in F$ Now,

$$(a \wedge b) \wedge (x \wedge y) = (a \wedge x) \wedge (b \wedge y) = (a \wedge x') \wedge (b \wedge y') = (a \wedge b) \wedge (x' \wedge y'),$$

thus, $x \wedge y\,\Theta(F)\,x' \wedge y'$ Similarly, using distributivity,

$$\begin{aligned}(a \wedge b) \wedge (x \vee y) &= (a \wedge b \wedge x) \vee (a \wedge b \wedge y)\\ &= (a \wedge b \wedge x') \vee (a \wedge b \wedge y') = (a \wedge b) \wedge (x' \vee y'),\end{aligned}$$

and so $x \vee y\,\Theta(F)\,x' \vee y'$ Finally,

$$\begin{aligned}(a \wedge b) \rightarrow (x \rightarrow y) &= (a \wedge x) \rightarrow (b \rightarrow y)\\ &= (a \wedge x') \rightarrow (b \rightarrow y') = (a \wedge b) \rightarrow (x' \rightarrow y'),\end{aligned}$$

showing that $x \rightarrow y\,\Theta(F)\,x' \rightarrow y'$ This completes the proof

8.6. Proposition. *Suppose A is a Heyting algebra. Then*
(a) *$F \subset F'$ if and only if $\Theta(F) \subset \Theta(F')$, for all filters F, F' in A, and*
(b) *$\Theta(1/\theta) = \theta$, for all $\theta \in \mathrm{Con}\, A$.*

PROOF. To prove (a), assume $F \subset F'$. Then $x\, \Theta(F)\, y$ implies $x \leftrightarrow y \in F$, so that $x \leftrightarrow y \in F'$ and $x\, \Theta(F')\, y$, showing that $\Theta(F) \subset \Theta(F')$. Conversely, assume $\Theta(F) \subset \Theta(F')$. then, $a = a \leftrightarrow 1 \in F$ implies $a\, \Theta(F)\, 1$, so that $a\, \Theta(F')\, 1$ and $a \in F'$, showing that $F \subset F'$.

To prove (b) suppose $x\, \Theta(1/\theta)\, y$. Then $a * x = a * y$ for some $a \in 1/\theta$. Hence,

$$x = 1 * x\ \theta\ a * x = a * y\ \theta\ 1 * y = y$$

Thus, $x\ \theta\ y$, and we have shown that $\Theta(1/\theta) \subset \theta$. For the converse, suppose $x\ \theta\ y$. Then $1 = x \rightarrow x\ \theta\ x \rightarrow y$, and, similarly, $1\ \theta\ y \rightarrow x$. Thus, $1\ \theta\ x \leftrightarrow y$. So $x \leftrightarrow y \in 1/\theta$ and $x\, \Theta(1/\theta)\, y$, which shows $\theta \subset \Theta(1/\theta)$.

8.7. Theorem. *Let A be a Heyting algebra. Then Θ, as defined in 8.5, is an isomorphism between the set of filters on A, ordered by inclusion, and $\mathrm{Con}\, A$, its inverse is given by $\theta \mapsto 1/\theta$.*

PROOF. Θ maps filters to congruences by 8.5. It is order-preserving and 1-1 by 8.6(a), and it is onto by 8.6(b). Thus Θ is an isomorphism. Finally, 8.6(b) shows that Θ^{-1} is given by $\theta \mapsto 1/\theta$.

8.8. Proposition. *Suppose A is a Heyting algebra. Then the principal congruences of A are exactly those associated with principal filters in A. If F is the principal filter $\uparrow a$, then the equivalence class of $\Theta(F)$ containing x is the interval $[a \wedge x, a \rightarrow x]$. In symbols, $x/\Theta(\uparrow a) = [a \wedge x, a \rightarrow x]$.*

PROOF. It is clear from the preceding results that $\Theta(a, b) = \Theta(\uparrow a \leftrightarrow b)$ and $\Theta(\uparrow c) = \Theta(c, 1)$, establishing the first part of the proposition. The second part follows from 8.5 and 8.3:

$$x\, \Theta(\uparrow a)\, y \quad \text{iff} \quad a * x = a * y \quad \text{iff} \quad y \in [a \wedge x, a \rightarrow x]$$

9. Extensional operators on Heyting Algebras

9.1. Operators. Recall that an *operator* on a poset A is simply a function $l : A \rightarrow A$, that composition of operators l and m is denoted, as usual, by $m \circ l$ (so that $(m \circ l)a = mla$), and that a closure operator is inflationary, monotone, and idempotent.

If A is a lattice or Heyting algebra, then the collection of all operators on A is likewise a lattice or Heyting algebra, where the order on operators is pointwise

$$l \leq m \quad \text{if and only if} \quad \forall a \in A\ la \leq ma.$$

The operations $l \wedge m$, $l \vee m$, and $l \rightarrow m$ and constants 0 and 1 are therefore also pointwise: $(l \wedge m)a = la \wedge ma$, $0a = 0$, etc. (Of course, this is just another presentation of the product $A^{|A|}$.)

9.2. Here are two simple results illustrating the above notions

Proposition. *Suppose that p and q are operators on a lattice A Then*
(a) *if p and q are inflationary and monotone, then $p \vee q \leq (p \circ q) \wedge (q \circ p)$.*
(b) *If p and p' are monotone and q is idempotent, then $p \leq q$ and $p' \leq q$ imply $p \circ p' \leq q$*

Proof Let p and q be as in (a), and fix $a \in A$ Since q is inflationary, we have both $a \leq qa$ and $pa \leq qpa$ Using the monotonicity of p on the former, $pa \leq pqa$, and hence $pa \leq pqa \wedge qpa$ Similarly, using that p is inflationary and that q is monotone, we have $qa \leq pqa \wedge qpa$ and hence $pa \vee qa \leq pqa \wedge qpa$ Rewriting, $(p \vee q)a \leq ((p \circ q) \wedge (q \circ p))a$, and since a was arbitrary, (a) follows

Now suppose p, p' and q are as in (b), and fix $a \in A$ If $p \leq q$ and $p' \leq q$, then applications of the assumptions yield

$$pp'a \leq pqa \leq qqa = qa$$

and so (b) follows

9.3. Proposition. *Suppose that p is an operator on a lattice A. Then the following three statements are equivalent·*
(a) *p is monotone*
(b) *$p(a \wedge b) \leq pa \wedge pb$ for every $a, b \in A$*
(c) *$pa \vee pb \leq p(a \vee b)$ for every $a, b \in A$*

Proof By symmetry, (b) is equivalent to $p(a \wedge b) \leq pa$ over all $a, b \in A$. But the pairs (x, y) with $x \leq y$ and the pairs $(a \wedge b, b)$ with a and b arbitrary are coextensive, since $x \leq y$ iff $x = x \wedge y$ Thus $(a) \Leftrightarrow (b)$ The proof of $(a) \Leftrightarrow (c)$ is dual.

9.4. Extensional operators. The main property of the operators we will be studying is *extensionality*, which is introduced by the following proposition.

Proposition. *Suppose l is an operator on the Heyting algebra A Then the following statements are equivalent*
(a) *$x \,\theta\, y$ implies $lx \,\theta\, ly$ for all $x, y \in A$ and $\theta \in \mathrm{Con}\, A$.*
(b) *$a \wedge lx \leq l(a * x) \leq a \rightarrow lx$ for all $a, x \in A$.*
(c) *$a * lx = a * l(a *' x)$ for all $a, x \in A$.*
(d) *$a * x = a * y$ implies $a * lx = a * ly$ for all $a, x, y \in A$*
(e) *$x \leftrightarrow y \leq lx \leftrightarrow ly$ for all $x, y \in A$*

An operator satisfying these conditions is called *extensional*

Proof By Proposition 8 8, (b) is equivalent to $l(a * x)\ \Theta(\uparrow a)\ lx$, which is the case if l satisfies (a), since, taking $\theta = \Theta(\uparrow a)$, we have $a \,\theta\, 1$ and therefore $a * x \,\theta\, 1 * x = x$. Thus (a) implies (b)

Next, since $a * (a *' x) = a * x$, applying the monotone operator $a * -$ to (b) yields

$$a * lx \leq a * l(a *' x) \leq a * lx.$$

Thus $a * lx = a * l(a *' x)$, and (b) implies (c)

Assume (c) and that $a *' x = a *' y$. Then

$$a * lx = a * l(a *' x) = a * l(a *' y) = a * ly,$$

proving that (c) implies (d).

Now (d) is equivalent by 8 3 to the statement

$$a \leq x \leftrightarrow y \quad \text{implies} \quad a \leq lx \leftrightarrow ly \qquad (a, x, y \in A), \tag{1}$$

which is in turn equivalent to (e) by Yoneda (3 3)

Finally, assume (e), let $\theta \in \operatorname{Con} A$. and suppose $x\,\theta\,y$. Then $x\,\Theta(1/\theta)\,y$ by 8 6(b), and so $x \leftrightarrow y \in 1/\theta$. But $1/\theta$ is up-closed and so $lx \leftrightarrow ly \in 1/\theta$ It follows that $lx\,\theta\,ly$. Hence (e) implies (a), and the proof is complete

9.5. Discussion. Condition (a) of the proposition says that l is compatible with every congruence of A Clearly, therefore, every *polynomial function* on A is extensional (where a polynomial function is defined analogously to a term function, except that "constants" from A may appear in the term). More generally, for any extensional operators l and m, the operators $l \wedge m$, $l \vee m$, $l \rightarrow m$, and $l \circ m$, and the constant operators 0 and 1, are extensional In other words, the extensional operators form a sub-Heyting algebra of the algebra of all operators. Notice that the "free" occurences of $*$ and $*'$ in (b) and (c) each give rise to two distinct conditions whose equivalence does not follow from 8 3 but rather is established in the course of the proof above. The word "extensional" used to describe these operators is derived from condition (e), which can be interpreted to mean that l "preserves (degree of) equality" the results of applying l are as equal as the arguments to which it is applied, in a logic with A as the truth values.

9.6. Replacement principles. The approach to calculation in Heyting algebras we will be using is based on *replacement principles*, i e., rules that allow replacement of subexpressions by other expressions These naturally involve Heyting algebra congruences and, thus, extensional operators.

Proposition. *Let l be an extensional operator on a Heyting algebra A, and let $a, b \in A$. Then,*

(a) $(a \leftrightarrow b) * la = (a \leftrightarrow b) * lb$.
(b) $a * la = a * l1$.
(c) $(b \rightarrow a) * lb = (b \rightarrow a) * l(a \wedge b)$.
(d) $(b \rightarrow a) * lb = (b \rightarrow a) * la$, *if* $a \leq b$.

Proof First of all, (a) is just a restatement, using 8.3, of condition (e) of 9.4 Next, replacing b with 1 in (a) results in (b), since $a \leftrightarrow 1 = 1$; likewise, replacing a with $a \wedge b$ in (a) results in (c), since

$$b \leftrightarrow (a \wedge b) = (b \rightarrow (a \wedge b)) \wedge ((a \wedge b) \rightarrow b) = (b \rightarrow a) \wedge 1 = b \rightarrow a$$

Finally, (d) is just a special case of (c).

9.7. Some useful equalities. As an example of the use of 9.6, and for future reference, we now prove some Heyting algebra equations. Further use of these principles will be made throughout the rest of this section and in Chapter 6 on regular operators The equations below are proved valid in any Heyting algebra A, and free variables are assumed to vary over A

9.7.1. $a \to (b \to c) = b \to (a \to c)$

PROOF. We have

$$\begin{aligned}(a \to (b \to c)) \to (b \to c) &= (b \wedge (a \to (b \to c))) \to c \\ &= (b \wedge (a \to c)) \to c && \text{by 9.6(b)} \\ &= b \to ((a \to c) \to c)\end{aligned}$$

9.7.2. $a \to (a \wedge b) = (a \to b) \to a$.

PROOF We have

$$\begin{aligned}(a \to (a \wedge b)) \to (a \wedge b) &= (a \to (a \wedge b)) \to a && \text{by 9.6(d)} \\ &= (a \to b) \to a && \text{by 9.6(b)}\end{aligned}$$

9.7.3. *If $c \geq a$, then $(b \to c) \to a = (b \to a) \wedge (c \to a)$*

PROOF Assume $c \geq a$ Since $- \to a$ is anti-monotone, $(b \to c) \to a \leq c \to a$; thus,

$$\begin{aligned}(b \to c) \to a &= ((b \to c) \to a) \wedge (c \to a) \\ &= ((b \to a) \to a) \wedge (c \to a) && \text{by 9.6(d)}\end{aligned}$$

9.7.4. $(b \wedge c) \to a = (b \to a) \wedge (c \to a)$

PROOF We have

$$\begin{aligned}((b \wedge c) \to a) \to a &= (b \to (c \to a)) \to a \\ &= (b \to a) \wedge ((c \to a) \to a) && \text{by 9.7.3.}\end{aligned}$$

9.7.5. $(b \to a) \to b = (b \to a) \wedge (a \to b)$.

PROOF Since $- \to b$ is anti-monotone, $(b \to a) \to b \leq a \to b$; thus,

$$\begin{aligned}(b \to a) \to b &= ((b \to a) \to b) \wedge (a \to b) \\ &= ((b \to a) \to a) \wedge (a \to b) && \text{by 9.6(a).}\end{aligned}$$

(Note that the b we replaced with a was "within the scope" of both $a \to b$ and $b \to a$, justifying the replacement by 9.6(a))

9.8. Beyond extensional operators. In the rest of this section, we take a look at some of the properties of extensional operators on a Heyting algebra A that moreover are inflationary, monotone, and/or idempotent, and some examples of such We begin with the inflationary and the monotone extensional operators.

Proposition.
(a) *An extensional operator l is inflationary iff $l1 = 1$*
(b) *An arbitrary operator l is extensional and monotone if and only if either of the following two equivalent conditions hold.*

$$a \to b \le la \to lb \qquad (a, b \in A), \tag{1}$$
$$(a \to b) * la \le (a \to b) * lb \qquad (a, b \in A). \tag{2}$$

PROOF. (a) If l is inflationary, $1 \le l1$. Conversely, if $l1 = 1$, then for any $a \in A$, 9 6(b) gives $a \wedge la = a \wedge l1 = a \wedge 1 = a$, and so $a \le la$.
(b) If l is extensional and monotone, then, by 9 6(c),

$$(a \to b) \wedge la = (a \to b) \wedge l(a \wedge b) \le l(a \wedge b) \le lb.$$

and so $a \to b \le la \to lb$. Conversely, if l satisfies (1), then for any $x, y \in A$. both $x \to y \le lx \to ly$ and $y \to x \le ly \to lx$, and so, taking the meet of these inequalities, $x \leftrightarrow y \le lx \leftrightarrow ly$. Hence, l is extensional by 9.4(e). Moreover, since $x \le y$ iff $x \to y = 1$, monotinicity of l follows directly from (1).

Thus it remains to show the equivalence of (1) and (2). Now (1) is equivalent to $(a \to b) \wedge la \le lb$, and thus to $(a \to b) \wedge la \le (a \to b) \wedge lb$, which is one of the instances of (2). Applying $(a \to b) \to -$ to this last inequality yields the other.

9.9. Logical operators and quasinuclei. It will be convenient to have names for the classes of operators described in the previous proposition. Thus, an inflationary extensional operator will be called *logical*, and a monotone extensional operator will be called a *quasinucleus* (A diagram showing the relations between the classes of operators introduced in this section can be found in 9.12.)

In view of (1) above, quasinuclei might alternatively be called "uniformly monotone". In fact, part (b) of the Proposition is actually a special case of a quite general phenomenon occuring with extensional operators—namely, in a sense that can be made precise (see the remarks at the end of this section), an extesional operator satisfies an ordinary property (such as monotonicity) iff it satisfies the corresponding "uniform version" of the property.

9.10. Prenuclei. Next, we look at the extensional operators that are both inflationary and monotone—i.e., the monotone logical operators, or, if you prefer, the inflationary quasinuclei. Fortunately, we don't have to make up our minds what to call them, since these operators have been studied before. In [3], Banaschewski introduces the notion of a *prenucleus*, which is an inflationary, monotone operator p satisfying the condition

$$a \wedge pb \le p(a \wedge b) \qquad (a, b \in A). \tag{1}$$

Proposition. *Prenuclei are precisely the monotone logical operators (or inflationary quasinuclei).*

PROOF We show that (1) is equivalent to the extensionality of p assuming that p is inflationary and montone First, (1) is clearly equivalent to $a \wedge pb \le a \wedge p(a \wedge b)$ and,

ADA289360

since the reverse inequality holds by the monotonicity of p, thus to $a \wedge pb = a \wedge p(a \wedge b)$. But this equation is equivalent of the extensionality of p by 9.4(c).

9.11. Nuclei. Finally, we identify the extensional closure operators. Recall that a *nucleus* is an operator j satisfying

$$a \leq ja = jja \qquad \text{and} \qquad j(a \wedge b) = ja \wedge jb \qquad (a, b \in A) \tag{1}$$

Proposition. *Nuclei are exactly the idempotent prenuclei. Thus, they are exactly the extensional closure operators. Moreover, a operator j on A is a nucleus iff it satisfies*

$$a \to jb = ja \to jb \qquad (a, b \in A) \tag{2}$$

Proof. Both prenuclei and nuclei are inflationary and monotone (the monotonicity of nuclei following because they preserve binary meets: cf. 9.3). If p is an idempotent prenucleus, then, by two applications of 9.10(1),

$$pa \wedge pb \leq p(pa \wedge b) \leq pp(a \wedge b) = p(a \wedge b)$$

for any $a, b \in A$. The reverse inquality is again 9.3. Thus p is a nucleus. Conversely, if j is a nucleus, then

$$a \wedge jb \leq ja \wedge jb = j(a \wedge b)$$

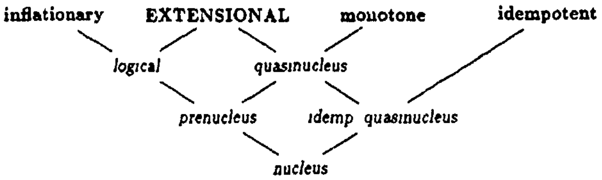
for every $a, b \in A$, and so j is an idempotent prenucleus.

For the second part of the Proposition, note first that any j satisfying (2) is inflationary (take $b = a$) and idempotent (take $a = jb$), and therefore since $a \to b \leq a \to jb = ja \to jb$, j is monotone and extensional by 9.8(1). Thus j is a nucleus. Conversely, if j is a nucleus, then $ja \to jb \leq a \to jb$ (since j is inflationary and $- \to jb$ is antimonotone) and, using 9.6(c),

$$ja \wedge (a \to jb) = j(a \wedge jb) \wedge (a \to jb) \leq j(a \wedge jb) \leq jjb = jb,$$

so that $a \to jb \leq ja \to jb$, proving (2).

9.12. Summary of classes of operators. The definitions and relations between the operators we have introduced in this section are summarized by the following diagram.



The properties are on the top row, and the lines are implications in the upward direction.

9.13. Classifying extensional operators. We now look at examples of the operators described above In fact, for each type of operator. we consider a pair of operators of that type (each defined by a Heyting polynomial) that "classify" the action of all the operators of that type

Proposition. *For any elements a and b of a Heyting algebra A, the operators $e_\uparrow(a\ b)$ and $e_\downarrow(a,b)$ defined for every $x \in X$ by*

$$e_\uparrow(a,b)x = (x \leftrightarrow a) \to b \qquad \text{and} \qquad e_\downarrow(a,b)x = a \wedge (b \leftrightarrow x) \tag{1}$$

are extensional. Furthermore, the two families of operators $e_\uparrow(a,b)$ and $e_\downarrow(a,b)$, for $a, b \in A$, classify (the action of) extensional operators, in the sense that, for every extensional operator l.

$$l \le e_\uparrow(a,b) \quad \text{iff} \quad la \le b, \tag{2}$$

$$e_\downarrow(a.b) \le l \quad \text{iff} \quad a \le lb. \tag{3}$$

PROOF. Since $e_\uparrow(a.b)$ and $e_\downarrow(a,b)$ are polynomial functions they are clearly extensional If $l \le e_\uparrow(a,b)$. then $la \le e_\uparrow(a,b)a = (a \leftrightarrow a) \to b = 1 \leftrightarrow b = b$ Conversely, if $la \le b$, then. using 9 6(a) and the monotonicity of $(x \leftrightarrow a) \to -$ for every $x \in A$,

$$lx \le (x \leftrightarrow a) \to lx = (x \leftrightarrow a) \to la \le (x \leftrightarrow a) \to b = e_\uparrow(a,b)x. \tag{4}$$

Thus $l \le e_\uparrow(a,b)$. Next, if $e_\downarrow(a,b) \le l$, then $a = a \wedge (b \leftrightarrow b) = e_\downarrow(a,b)b \le lb$. Conversely, if $a \le lb$, then, for similar reasons as above,

$$e_\downarrow(a,b)x = a \wedge (b \leftrightarrow x) \le lb \wedge (b \leftrightarrow x) = lx \wedge (b \leftrightarrow x) \le lx. \tag{5}$$

Thus $e_\downarrow(a,b) \le l$

9.14. Proposition. *The operator $e_\uparrow(a,b)$ is logical iff $a \le b$. Define $l_\uparrow(a,b) = e_\uparrow(a.b)$ when $a \le b$ and*

$$l_\downarrow(a,b)x = x \vee e_\downarrow(a,b)x \qquad (x \in A) \tag{1}$$

for every $a.b \in A$. Then $l_\downarrow(a,b)$ is also logical, and (a,b) and $l_\downarrow(a,b)$ classify logical operators in the sense of Proposition 9 13.

PROOF Since $e_\uparrow(a,b)1 = (a \leftrightarrow 1) \to b = a \to b$, the first part of the Proposition follows from 9.8(a) For any logical operator l, $la \le b$ implies $a \le b$ (since l is inflationary). Hence the $e_\uparrow(a,b)$, for $a \le b$, are the appropriate "upper" classifiers for logical operators The operators $l_\downarrow(a,b)$ are clearly inflationary, hence logical, and if l is any logical operator. then $e_\downarrow(a,b)x \le lx = x \vee lx$ iff $x \vee e_\downarrow(a,b)x \le lx$, and therefore the $l_\downarrow(a,b)$ are the appropriate "lower" classifiers.

9.15. Proposition. *The operators $q_\uparrow(a.b)$ and $q_\downarrow(a.b)$ defined by*

$$q_\uparrow(a,b)x = (x \to a) \to b \qquad \text{and} \qquad q_\downarrow(a,b)x = a \wedge (b \to x) \tag{1}$$

are quasinuclei and, furthermore, classify quasinuclei in the sense of Proposition 9 13

Proof Since $q_\uparrow(a,b)$ and $q_\downarrow(a,b)$ are polynomial functions, they are extensional. Both the operations $- \to b$ and $- \to a$ are anti-monotone, so their composition is monotone, thus $q_\uparrow(a,b)$ is a quasinucleus. The operations $a \wedge -$ and $b \to -$ are both monotone, and so $q_\downarrow(a,b)$ is also a quasinucleus.

If l is a quasinucleus then l satisfies $(x \to y) * lx \leq (x \to y) * ly$ by 9.8(2). Hence, the same argument used for (5) and (6) in 9.13 with $\to$ instead of $\to$, and with the nontrivial equality replaced by $\leq$ can be used here to show that $q_\uparrow(a,b)$ and $q_\downarrow(a,b)$ classify quasinuclei.

9.16. Corresponding directly to 9.14 we have the following result for prenuclei, the proof of which we omit, since it is an obvious translation of the proof of 9.14.

Proposition. *The operator $q_\uparrow(a,b)$ is a prenucleus iff $a \leq b$. Define $p_\uparrow(a,b) = q_\uparrow(a,b)$ when $a \leq b$ and*

$$p_\downarrow(a,b)x = x \vee q_\downarrow(a,b)x \qquad (x \in A) \tag{1}$$

for every $a,b \in A$. Then $p_\downarrow(a,b)$ is also a prenucleus, and $p_\uparrow(a,b)$ and $p_\downarrow(a,b)$ classify prenuclei in the sense of Proposition 9.13.

9.17. Evaluation adjunctions. In all of the preceeding propositions, the "classifying" properties of the operators involved are adjointness relationships. Consider extensional operators, for example: for each $a \in A$ there is the evaluation mapping $e \mapsto ea$ from the poset of extensional operators on A to A, which is monotone by the definition of order on operators. The classifying properties of $e_\uparrow(a,b)$ and $e_\downarrow(a,b)$ then say that the mappings $e_\uparrow(a,-)$ and $e_\downarrow(a,-)$ are respectively left and right adjoints to evaluation at a. The case of quasinuclei is similar. For logical operators and prenuclei we get an adjunction instead between the posets of operators on A and the interval $[a,1]$. A consequence of these adjunctions is that evaluation at a preserves whatever meets and joins exist in the algebra of operators (remembering, of course, that for logical operators and prenuclei, the codomain is the interval $[a,1]$). In other words (with the noted restriction) meets and joins, when they exist, are pointwise.

9.18. Classifying nuclei. Pointwise joins of idempotent operators are rarely idempotent. This is in particular the case with nuclei, and so, by the comments of the previous paragraph, we can't expect nuclei to have "upper" classifiers. They do have a restricted form of upper classifier, however (which, despite the restriction, are still "complete" in the sense mentioned in 15.4), and the lower classifiers for prenuclei are in fact already idempotent, and so are lower classifiers for nuclei. This is spelled out in the following proposition.

Proposition. *For every $a,b \in A$, the operators $q_\uparrow(a,a)$ and $p_\downarrow(a,b)$ are nuclei. Hence, for every nucleus j, we have*

$$j \leq q_\uparrow(a,a) \quad \textit{iff} \quad ja = a. \tag{1}$$

$$p_\downarrow(a,b) \leq j \quad \textit{iff} \quad a \leq jb \tag{2}$$

Proof $q_\uparrow(a,a)$ is a prenucleus by 9.16, and is idempotent since $(x \to a) \to a = x \to a$ by 7.2(1). The upper classifying property of $q_\uparrow(a,a)$ as a nucleus follows from that of the prenucleus since $ja \le a$ iff $ja = a$, as j is inflationary. Similarly, it will be enough for the lower classifying property of $p_\downarrow(a,b)$ to show that $p_\downarrow(a,b)$ is idempotent. By definition, $p_\downarrow(a,b)x = x \vee q_\downarrow(a,b)$. Now

$$\begin{aligned} q_\downarrow(a,b)p_\downarrow(a,b)x &= q_\downarrow(a,b)(x \vee q_\downarrow(a,b)x) \\ &= a \wedge (b \to (x \vee (a \wedge (b \to x)))) \\ &= a \wedge (b \to (x \vee (1 \wedge (1 \to x)))) \\ &= a \wedge (b \to x) = q_\downarrow(a,b)x, \end{aligned}$$

and so

$$p_\downarrow(a,b)p_\downarrow(a,b) = p_\downarrow(a,b) \vee q_\downarrow(a,b)p_\downarrow(a,b) = p_\downarrow(a,b) \vee q_\downarrow(a,b) = p_\downarrow(a,b).$$

9.19. Discussion. The preceeding proposition shows that meets of nuclei, when they exist, are pointwise. In Chapter 5, when we see that nuclei on complete Heyting algebras correspond exactly to frame congruences (i.e., equivalence relations compatible with finite meets and arbitrary joins) we will see that—at least in the typical case that $a \le b$—the nuclei $p_\downarrow(a,b)$ correspond to the principal congruences $\Theta(a,b)$.

9.20. Extensional operators on complete Heyting algebras. We now show that when A is complete, all of the operator algebras considered above are likewise complete.

Proposition. *If E is a class of operators on A of one of the types considered above (i.e., extensional, logical, quasinucleus, prenucleus, or nucleus), then the pointwise meet of E is again of the same type.*

PROOF. If each operator in E is extensional, then for any $a, x, y \in A$ with $a \to ex = a \to ey$ for all $e \in E$, we have

$$\begin{aligned} a \to (\textstyle\bigwedge E)x &= a \to \textstyle\bigwedge Ex = \textstyle\bigwedge a \to Ex \\ &= \textstyle\bigwedge a \to Ey = a \to \textstyle\bigwedge Ey = a \to (\textstyle\bigwedge E)y. \end{aligned}$$

Thus $\bigwedge E$ is extensional by 9.4(d). Clearly meets of inflationary operators are again inflationary and similarly with monotone operators. Thus, to complete the proof, we only need to show that pointwise meets of nuclei are idempotent. Suppose J is a class of nuclei and $a \in A$. Then, by monotonicity and idempotence of the elements of J,

$$(\textstyle\bigwedge J)(\textstyle\bigwedge J)a = \textstyle\bigwedge J(\textstyle\bigwedge Ja) \le \displaystyle\bigwedge_{j \in J} j(ja) = \textstyle\bigwedge Ja.$$

9.21. Structure theorems for operators. So far, we have only used the classifying operators of various types to show that the operations on the corresponding operator algebras are pointwise, which can anyhow be established directly. The more important use of classifying operators is to prove structure theorems to the effect that every operator of a given type is a join or meet of special operators.

Theorem. *Suppose A is a complete Heyting algebra. Then every operator of the classes considered (extensional, logical, quasinucleus, prenucleus, or nucleus) is both a join of lower classifiers and a meet of upper classifiers of the same type (except that nuclei only have lower classifiers).*

PROOF. We prove the theorem for joins of lower classifiers of extensional operators, the other cases are similar. So let e be an extensional operator. We show that, in the complete lattice of extensional operators

$$e = \bigvee\{e_1(a,b) : a \leq eb\}$$

Now, for every $a, b \in A$ with $a \leq eb$, we have $e_1(a,b) \leq e$ by 9.13(3). Thus e is bigger than the join. Conversely, let $x \in A$ be arbitrary. Then $e_1(ex,x)$ is included in the join, since $ex \leq ex$. But $e_1(ex,x)x = ex \wedge (x \to x) = ex$, and so the value of the join at x is at least ex. Thus e is less than the join, completing the proof.

9.22. (Pre)fixedpoints of extensional operators. Recall that a fixedpoint of an operator $l : A \to A$ is an element $a \in A$ such that $la = a$. We call a a *prefixedpoint* of l if $la \leq a$, and write $\mathrm{prefix}\, l$ for the set of such elements. Since the identity is the smallest logical operator, and $\to$ on logical operators is pointwise, it follows that $(\neg\neg l)a = (la \to a) \to a$ for all $a \in A$. By abuse of notation, we also write $\neg\neg l$ for the same operation when l is only an extensional operator (the least extensional operator is the constant 0 and not the identity).

Proposition. *Suppose l and m are extensional operators on A. Then the following hold.*

(a) $\mathrm{prefix}\, l = \mathrm{fix}\, \neg\neg l = \mathrm{rng}\, \neg\neg l$.

(b) $\mathrm{prefix}\, l = \mathrm{prefix}\, m$ *iff* $\neg\neg l = \neg\neg m$.

If l and m are logical, then prefix *may be replaced by* fix.

PROOF. If l is a logical operator, then since l is inflationary, it follows that $\mathrm{prefix}\, l = \mathrm{fix}\, l$, explaining the last part of the Proposition. Also note that for any extensional operator l, $\neg\neg l$ is always logical, and so

$$\mathrm{prefix}\, \neg\neg l = \mathrm{fix}\, \neg\neg l \tag{1}$$

Let us first prove that

$$\mathrm{rng}\, \neg\neg l \subset \mathrm{prefix}\, m \quad \text{iff} \quad m \leq \neg\neg l \tag{2}$$

This follows from the following calculation for an arbitrary $a \in A$:

$$\begin{aligned}
(\neg\neg l)a \in \mathrm{prefix}\, m \quad &\text{iff} \quad m((la \to a) \to a) \leq (la \to a) \to a \\
&\text{iff} \quad (la \to a) \wedge m((la \to a) \to a) \leq a \\
&\text{iff} \quad (la \to a) \wedge ma \leq a \qquad \text{by extensionality} \\
&\text{iff} \quad ma \leq (la \to a) \to a
\end{aligned}$$

Next, to prove (a), let $la \leq a$. Then $(la \to a) \to a = 1 \to a = a$, so $(\neg\neg l)a = a$, proving $\mathrm{prefix}\, l \subset \mathrm{fix}\, \neg\neg l$. The inclusion $\mathrm{fix}\, \neg\neg l \subset \mathrm{rng}\, \neg\neg l$ is trivial. Finally, putting $m = l$ in (2), we conclude $\mathrm{rng}\, \neg\neg l \subset \mathrm{prefix}\, l$, completing the proof of (a).

For (b), replace m in (2) by $\neg\neg m$, and use the equalities of (a) and (1) to get

$$\text{prefix}\, l \subset \text{prefix}\, m \quad \text{iff} \quad \neg\neg m \leq \neg\neg l$$

By symmetry, (b) follows.

9.22.1. Let us just restate the first part of the Proposition in the form in which we will most often use it:

Corollary. *If l is a logical operator, then*

$$\text{fix}\, l = \{la \twoheadrightarrow a : a \in A\}.$$

9.23. Discussion. A logical operator, by definition, is regular (in the Heyting algebra of logical operators) if $\neg\neg l = l$. The Proposition then says that every logical operator l has the same fixedpoint set as a unique regular operator (namely $\neg\neg l$). Regular operators are the subject of Chapter 6, where, among many other things, a characterization of the fixedpoint sets of regular operators (and thus, by the Proposition, the fixedpoint sets of logical operators, and the prefixedpoint sets of extensional operators) is given (22.7).

Also, notice that for extensional operators l and m, we have $\neg\neg l = \neg\neg m$ iff $la \rightarrow a = ma \rightarrow a$ for all $a \in A$. Since, by the Proposition, this is just in case l and m have the same prefixedpoints, the Proposition is another example of the uniformity of extensional operators: we have $\forall a \in A\ \ la \rightarrow a = ma \rightarrow a$ iff $\forall a \in A\ \ la \leq a \Leftrightarrow ma \leq a$.

9.24. Final Remarks. The theory of extensional operators on a (complete) Heyting algebra A, as developed in this section, is the beginning of a more comprehensive theory, which includes all of the results on regular operators in Chapter 6, as well as possessing some "semantic" connections to topos theory by way of the sheaf topos $\mathbf{Sh}(A)$. For example (assuming that A is complete), the elememts of A correspond to the global elements of Ω in $\mathbf{Sh}(A)$, i.e., to the morphisms $1 \rightarrow \Omega$. Every morphism $f : \Omega \rightarrow \Omega$ therefore induces by composition an operator on A. It can be shown that these operators are all extensional and that every extensional operator on A arises in this way from a unique morphism.

This connection suffices to explain the relation between properties of extensional operators and their corresponding "uniform properties", as mentioned in 9.9 and 9.23. This relation can be seen to arise from an "Ω-rule" for $\mathbf{Sh}(A)$. Namely, morphisms with domain Ω in $\mathbf{Sh}(A)$ are completely determined by their effect on global elements of Ω, and hence for any formula $\phi(\overline{x})$ involving arbitrary extensional operators (represented by their associated morphisms $\Omega \rightarrow \Omega$), where $\overline{x}$ is a sequence of n variables of type Ω, we have that if the associated morphism $f_\phi : \Omega^n \rightarrow \Omega$ is such that $f_\phi \circ \overline{a} = \text{true}$ for all $\overline{a} : 1 \rightarrow \Omega^n$, then $f_\phi = \text{true}$—which, unwound, amounts to the "uniform" version of ϕ.

There are other, more topos-theoretic connections, but these are beyond the scope of the present investigation. Various additional properties of the operators considered in this section are given in the exercises.

ADA289360

9.25. Exercises. In the following exercises, all classifying operators are *upper* classifiers, i.e., $q(a,b)$ means $q_{\uparrow}(a,b)$, etc. It is an additional exercise to formulate and prove corresponding properties for the lower classifiers

9.25.1. Show that

(a) $\epsilon(a_1,b_1) = \epsilon(a_2,b_2)$ iff $a_1 \leftrightarrow a_2 \leq b_1 \leftrightarrow b_2$

(b) $q(a_1,b_1) = q(a_2,b_2)$ iff $b_1 = b_2$ and (writing $b = b_1 = b_2$) $(a_1 \leftrightarrow a_2) \to b = b$.

9.25.2. Show that

(a) $q(a_1,b_1) \circ q(a_2,b_2) = q((a_1 \to b_1) \to a_2, (b_1 \to a_2) \to b_2)$.

(b) $q(a,b)^3 = q(a,b)^2$ (exponents denote iteration)

(c) $q(a,b)$ is idempotent iff $(b \to a) \to b = b$.

(d) $q(a,b)$ is a nucleus iff $(b \to a) \to a = b$ and then $q(a,b) = q(b,b)$

Chapter 4

FRAMES AND κ-FRAMES

Frames are the main objects of study in this thesis. However, as was pointed out by Madden [32], the auxilliary notion of κ-frame. since the category of such has a full, reflective subcategory of Boolean objects, is potentially useful for obtaining results about frames. The paper of Madden and Molitor [33] has just such a result, and we obtain another such result in Chapter 8 Thus, this chapter is concerned with both frames and κ-frames.

We introduce frames in Section 10, give the standard construction of free frames, and prove a new characterization of the right adjoints of frame morphisms Then, in Section 11 we introduce κ-frames (the definitions and basic results are taken from [32]), and follow this with a series of new results characterizing the types of limits preserved by the free functor from the category of λ-frames to the category of κ-frames, where $\lambda < \kappa$. We also prove a result a result about congruences on products of κ-frames, and we deduce from this a similar result for frames Finally, in Section 12, we look at explicit descriptions of various colimits of frames and κ-frames and, in an Exercise, sketch a proof that products and directed colimits commute in the category of frames

10. Frames

10.1. Frames and locales. A *frame* is, by definition, a complete lattice A that satisfies the following infinite distributive law:

$$a \wedge \bigvee S = \bigvee a \wedge S \qquad (a \in A,\ S \subset A). \tag{1}$$

We refer to this law as "frame distributivity". If A and B are frames, then a function $f : A \to B$ is a *frame (homo)morphism* if f preserves finite meets and arbitrary joins (including the empty meet and join: $f(1) = 1$ and $f(0) = 0$) Frames and frame morphisms clearly form a category, which we denote **Frm**. The opposite category is denoted **Loc**; thus $\mathbf{Loc} = \mathbf{Frm}^{op}$. Objects of this category (which are of course just the same as objects of **Frm**) are called *locales*, and morphisms are called *continuous maps*, in keeping with the topological terminology.

10.2. Proposition. *Suppose A is a complete lattice. Then A is a frame if and only if A is a Heyting algebra. In this case, the frame and Heyting algebra structures on A are related by the equation*

$$a \to b = \bigvee\{x \in A : a \wedge x \le b\} \qquad (a, b \in A). \tag{1}$$

PROOF. In a complete Heyting algebra, joins of all subsets exist by definition, so that frame distibutivity is just 7 2(m). Conversely, suppose A is a frame, and take (1) as the definition of $\to$ Then,

$$a \wedge (a \to b) = a \wedge \bigvee\{x \in A . a \wedge x \le b\} = \bigvee\{a \wedge x : a \wedge x \le b\} \le b$$

On the other hand, if $a \wedge x \leq b$, then x is one of the elements in the join on the right of (1), and so $x \leq a \rightarrow b$ Thus $x \wedge a \leq b$ iff $x \leq a \rightarrow b$, and so $\rightarrow$ makes A into a Heyting algebra Since a lattice can have only one Heyting algebra structure, it must be defined in A by (1), completing the proof

10.3. Open morphisms and cBa's. Although, by the Proposition, frames and complete Heyting algebras determine the same class of lattices, they are distinguished as algebras by the operations we take as fundamental For frames, we take finite meets and arbitrary joins as is reflected in our definition of frame morphism For complete Heyting algebras, we take all meets and joins as well as the arrow operation Thus, a complete Heyting algebra morphism is simultaneously a Heyting algebra morphism and a complete lattice morphism We denote the category of complete Heyting algebras and morphisms by **cHa**, and note that it is a *non-full* subcategory of **Frm**. A frame morphism that preserves arrow (as given by (1)) and all meets (and thus is a complete Heyting algebra morphism between the associated complete Heyting algebras) is called *open*, since these correspond (as continuous locale maps) to open continuous functions between topological spaces

Similarly to frames and complete Heyting algebras, Boolean frames, that is, frames in which every element is Boolean, are the same thing as complete Boolean algebras (cBa's), that is, Boolean lattices that are complete. Morphisms between cBa's, by definition, preserve all meets and joins, as well as negation $\neg$, and we denote the category of cBa's and cBa-morphisms by **cBa** and treat it as a subcategory of **Frm**. Because frame morphisms preserve complements, every frame morphism between cBa's is a cBa morphism (since it then also preserves meets by the De Morgan laws 7 6), and so, unlike **cHa** and **Frm** **cBa** is a *full* subcategory of **Frm**. Also note that cBa-morphisms preserve arrow (definable from join and negation), and so **cBa** can also be thought of as a full subcategory of **cHa**

10.4. Frames are monadic. It is clear how the equational presentation of complete join-semilattices given in Section 5 can be expanded to include the operations of finite meets and the equations for frame distributivity and thus give an equational presentation of frames. The following proposition gives a concrete description of the free frame on a set of generators It follows that **Frm** is monadic and therefore has the properties 6.5.1–4

Proposition. *Let X be a set, and let $F_0(X)$ denote the poset of all finite subsets of X ordered by reverse inclusion (i e, $s \leq t$ if and only if $t \subset s$) Then the set $F_\infty(X)$ of all down-closed subsets of $F_0(X)$, ordered by inclusion, is (a presentation of) the free frame on generators X. Joins and meets in $F_\infty(X)$ are given by union and intersection, the insertion of generators is $x \mapsto \downarrow x$, and the unique extension $\bar{f} : F_\infty(X) \rightarrow A$ of a mapping f $X \rightarrow A$ is given by*

$$\bar{f}(S) = \bigvee_{s \in S} \bigwedge f(s) \qquad (1)$$

We remark that $F_0(X)$, with the operation of $\cup$ and unit element $\emptyset$, is the free semilattice on X The reason for the terminology $F_0(X)$ and $F_\infty(X)$ will become clear in the next section

Proof See [22], II.1.2

10.5. Right adjoints of frame morphisms. Suppose that $f : A \to B$ is a frame morphism. Recall from Proposition 4.3 that, since f preserves arbitrary joins, f has a right adjoint, $f_* : B \to A$, and f and f_* are related to each other by the equations

$$f_*b = \bigvee\{a \in A : fa \leq b\} \qquad (b \in B); \tag{1}$$

$$fa = \bigwedge\{b \in B : a \leq f_*b\} \qquad (a \in A) \tag{2}$$

Moreover, these right adjoints to frame morphisms satisfy the laws $\mathrm{id}_* = \mathrm{id}$ and $(gf)_* = f_*g_*$, showing that the operation $f \mapsto f_*$ is a functor $(-)_* : \mathbf{Loc} \to \mathbf{Set}$. Also, f_* is 1-1 iff f is onto, and f_* is onto iff f is 1-1, and (therefore) f is an isomorphism iff f_* is a bijection.

The remarks above imply that the functor $(-)_*$ gives a faithful representation of the category **Loc** in the category **Set**. When is a function $g : B \to A$ the right adjoint of a frame morphism? Preserving meets is a necessary condition, and this will insure the existence of a left adjoint f, given by (2) above. The question of what conditions on g insure that f preserves finite meets is answered more generally for any class of meets by the following result.

Theorem. *Suppose that A and B are frames and that $g : B \to A$ is a monotone mapping with left adjoint $f : A \to B$. Then, for any set I, the following are equivalent*

(a) *f preserves all I-indexed meets*

(b) *For every $b \in B$ and I-indexed family $\{a_i\}$ of elements of A with $\bigwedge_i a_i \leq g(b)$, there exists an I-indexed family $\{b_i\}$ of elements of B such that $a_i \leq g(b_i)$ for all $i \in I$ and $\bigwedge_i b_i \leq b$.*

Proof Suppose that f preserves I-indexed meets, $b \in B$, and $\{a_i\}$ is a family with $\bigwedge_i a_i \leq g(b)$. By adjointness, $f(\bigwedge_i a_i) \leq b$, and so, by assumption on f, $\bigwedge_i f(a_i) \leq b$. Thus, if we define

$$b_i = f(a_i) \qquad (i \in I), \tag{3}$$

we have $g(b_i) = g(f(a_i)) \geq a_i$ for all $i \in I$ by adjointness, and so the family $\{b_i\}$ has the required properties, showing that (a) implies (b).

In the other direction, suppose that $g : B \to A$ is a monotone map for which (b) holds, and let $\{a_i\}$ be an I-indexed family of elements of A. Since f (being a left adjoint) is monotone, to show that f preserves the meet of $\{a_i\}$, it will be enough to show that $\bigwedge_i f(a_i) \leq f(\bigwedge_i a_i)$. Now, by equation (2) above,

$$f(\bigwedge_i a_i) = \bigwedge\{b \in B : \bigwedge_i a_i \leq g(b)\} \tag{4}$$

and

$$\bigwedge_i f(a_i) = \bigwedge_i \bigwedge\{b_i \in B : a_i \leq g(b_i)\} \tag{5}$$

So suppose $b \in B$ is such that $\bigwedge_i a_i \le g(b)$, as in the meet in (4). Then by assumption there exists a family $\{b_i\}$ as in (b). But $a_i \le g(b_i)$ for all $i \in I$ implies that each b_i is a member of the ith set of the meet in (5), and so

$$\bigwedge_i f(a_i) \le \bigwedge_i b_i \le b$$

By the choice of b, it therefore follows that $\bigwedge_i f(a_i) \le f(\bigwedge_i a_i)$, and the proof is complete.

10.6. Corollary. *If A and B and frames, then a mapping $g: B \to A$ is the right adjoint of a frame morphism $A \to B$ if and only if the following three conditions hold.*

(a) $g(\bigwedge S) = \bigwedge g(S)$ *for every* $S \subset B$,
(b) $g(b) = 1$ *implies* $b = 1$ *for every* $b \in B$, *and*
(c) *for every* $b \in B$ *and* $a_1, a_2 \in A$ *with* $a_1 \wedge a_2 \le g(b)$, *there exist* $b_1, b_2 \in B$ *such that* $a_1 \le g(b_1)$, $a_2 \le g(b_2)$, *and* $b_1 \wedge b_2 \le b$

Proof. Since a function preserves finite meets iff it preserves the empty meet (i.e., 1) and binary meets, the only part that needs to be checked is that condition (b) of the theorem reduces to condition (b) of the Corollary when I is empty. But this is clear, since the hypothesis of the former then reduces to $1 \le g(b)$, while the conclusion reduces to $1 \le b$

10.7. Remarks. Note that if I is finite, then by replacing (3) in the proof of the theorem with

$$b_i = b \vee fa_i \qquad (i \in I) \tag{3'}$$

we can conclude by the finite dual distributive law that the inequality $b_1 \wedge b_2 \le b$ in part (c) of the Corolllary can be replaced with an equality

It is shown in [11, IV 1 26] that right adjoints of frame morphisms g can be characterized by (a) and the non-first-order condition that the extension of g to ideals preserves primeness (i e., if P is a prime ideal of B, then the ideal of A generated by the image $g(P)$ is prime) The proof uses the Prime Ideal Theorem

We also note the following:

Proposition. *Suppose $f: A \to B$ is a map between frames with right adjoint f_*. Then f preserves binary meets if and only if*

$$f_*(fa \to b) = a \to f_*b \qquad (a \in A,\ b \in B). \tag{1}$$

Proof. Suppose $f \dashv f_*$ and let $a, b, c \in A$ be arbitrary. Then

$$c \le f_*(fa \to b) \quad \text{iff} \quad fc \le fa \to b \quad \text{iff} \quad fa \wedge fc \le b \tag{2}$$

and

$$c \le a \to f_*b \quad \text{iff} \quad a \wedge c \le f_*b \quad \text{iff} \quad f(a \wedge c) \le b \tag{3}$$

Now, by Yoneda (3.3), (1) holds just in case $c \le f_*(fa \to b)$ iff $c \le a \to f_*b$ for all $a, b, c \in A$. By (2) and (3), this is just in case $fa \wedge fc \le b$ iff $f(a \wedge c) \le b$ for all $a, b, c \in A$, i.e., just in case f preserves binary meets

11. κ-frames

11.1. Definitions. Let κ be a regular cardinal. By a *κ-set*, we mean a set whose cardinality is strictly less than κ Other κ-notions are defined in the obvious way: a κ-family is a family indexed by a κ-set, a κ-product is a product of fewer than κ objects. and so on

A *κ-frame* is a poset A such that

(a) every finite subset of A has a meet,

(b) every κ-subset of A has a join ("A has κ-joins"), and

(c) $a \wedge \bigvee S = \bigvee a \wedge S$ for all $a \in A$ and κ-sets $S \subset A$ (the "κ-distributive law")

A κ-frame morphism preserves finite meets and κ-joins. We denote the category of κ-frames and κ-morphisms by **κ-Frm**.

As with frames. it is clear how expand the equational presentation for κ-complete semilattices, given in 5 1, to an equational presentation of κ-frames that moreover uses only a set of operations, all of whose arities are bounded by κ. Thus the category **κ-Frm** is locally κ-presentable and enjoys all of the properties listed in 6.5.

Note that **0-Frm** (no joins at all) is essentially the category of meet-semilattices, and **ω-Frm** (finite joins) is essentially the category of distributive lattices For any regular cardinals $\lambda \leq \kappa$, there is an obvious forgetful functor U_κ^λ : **κ-Frm** $\rightarrow$ **λ-Frm**. Similarly, for every regular κ, there is a forgetful functor U_∞^κ : **Frm** $\rightarrow$ **κ-Frm**. Since all of these categories are monadic, these forgetful functors have left adjoints F_κ^λ : **λ-Frm** $\rightarrow$ **κ-Frm** and F_∞^κ : **κ-Frm** $\rightarrow$ **Frm**. Explicit descriptions for these left adjoints, generalizing 10.4, will be recalled below.

For uniformity in the treatment of frames and κ-frames, we would like to have **Frm** equal to **κ-Frm** for some κ. This can be partially achieved by the devices explained in 6.6., and this issue will be taken up again in 18.4, after we have explained the relationship between **κ-Frm** and the category **κ-cBa** of κ-complete Boolean algebras (i.e., Boolean κ-frames)

11.2. Free functors. For the rest of this section λ and κ will be regular cardinals with $\lambda \leq \kappa$.

The free functor F_κ^λ : **λ-Frm** $\rightarrow$ **κ-Frm** left adjoint to the forgetful functor U_κ^λ has a description generalizing that of the functor F_∞ (10.4). Let A be a λ-frame. A subset of A is called a *λ-ideal* if it is down-closed and closed under λ-joins If $S \subset A$, then the λ-ideal generated by S is

$$\{a \in A : a \leq \bigvee T, T \subset S, |T| < \lambda\} \tag{1}$$

A λ-ideal is *κ-generated* if it is generated by a κ-set.

Proposition. *Suppose that A is a λ-frame Then $F_\kappa^\lambda(A)$ can be taken to be the κ-frame of all κ-generated λ-ideals on A, ordered by inclusion. Finite meets are given by intersection and the join of a κ-set of λ-ideals is the λ-ideal generated by their union. The insertion of generators $A \rightarrow F_\kappa^\lambda(A)$ is given by $a \mapsto \downarrow a$. If f $A \rightarrow B$ is a λ-morphism to a κ-frame B and J is the λ-ideal generated by the κ-set $S \subset J$, then $\bar{f} \cdot F_\kappa^\lambda(A) \rightarrow B$ has $\bar{f}(J) = \bigvee f(S)$ In particular, if $g : A \rightarrow A'$ is a λ-morphism between λ-frames, then $F_\kappa^\lambda(g)(J) = \mathrm{downcl}\, g(J)$.*

Proof See [32, Prop 1 2] We just remark here that if I and J are λ-ideals κ-generated by G and H, then $I \cap J$ is generated by the κ-set $G \wedge H$, and that we need the regularity of κ to show that κ-joins of κ-generated ideals are κ-generated (as well as to show that (1) is a λ-ideal).

11.3. Preservation properties of F_κ^λ. Being a left adjoint, F_κ^λ preserves all colimits. What about limits? We first make the simple observation that F_κ^λ preserves monomorphisms

Lemma. *If $f\ A \to B$ is a monomorphism of λ-frames, then $F_\kappa^\lambda f$ is a monomorphism of κ-frames*

Proof Recall that monomorphisms in both categories are just the 1-1 morphisms. Suppose that $I, J \in F_\kappa^\lambda A$ are such that $\operatorname{downcl} f(I) = \operatorname{downcl} f(J)$. Then for every $i \in I$, there exists $j \in J$ such that $f(i) \leq f(j)$, and for every $j \in J$, there exists $i \in I$ such that $f(j) \leq f(i)$ But semilattice monomorphisms reflect order: $f(i) \leq f(j)$ implies $i \leq j$ and vice-versa Since I and J are down-closed, it follows that $I = J$

11.4. Lemma. *Let I be a κ-set, and for each $i \in I$, let A_i be a κ-frame Define, for $i \in I$ and $a_i \in A_i$, the element $\delta_i(a_i) \in \prod_i A_i$ by*

$$\delta_i(a_i)(j) = \begin{cases} a_i, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then the map $A_i \to \prod_i A_i$ given by

$$a_i \mapsto \delta_i(a_i) \tag{2}$$

preserves all joins and all non-empty meets existing in A_i, and every $\sigma \in \prod_i A_i$ is a κ-join of elements of the form $\delta_i(a_i)$:

$$\sigma = \bigvee_{i \in I} \delta_i(\sigma(i)). \tag{3}$$

Proof Easy

11.5. Lemma. *Suppose that I is a λ-set and A_i is a λ-frame for all $i \in I$. If J_i is a κ-generated λ-ideal on A_i for all $i \in I$, then $\prod_i J_i$ is a κ-generated λ-ideal on $\prod_i A_i$ Conversely, every κ-generated λ-ideal on $\prod_i A_i$ has this form*

Proof. That products of λ-ideals are λ-ideals is trivial, since the order and operations in a product are pointwise If each of the J_i is κ-generated, say by $S_i \subset J_i$, then every $a_i \in J_i$ is smaller than a λ-join of the S_i, and so (using 11 4(3)), every $\sigma \in \prod_i J_i$ is less than a join of $\lambda\ \lambda = \lambda$ elements of the set $\bigcup_{i \in I}\{\delta_i(s_i)\ .\ s_i \in S_i\}$, which has fewer than $\lambda \cdot \kappa = \kappa$ elements. This proves the first part of the Lemma.

For the second part, let J be a κ-generated λ-ideal on $\prod_i A_i$, and for every $i \in I$ let $J_i = \{a_i \in A_i \cdot \delta_i(a_i) \in J\}$ Since J is down-closed, we have

$$\sigma \in J \text{ implies } \delta_i(\sigma(i)) \in J \text{ for every } i \in I \tag{1}$$

Fix an $i \in I$ Then for every $a_i \in A_i$, $\exists\sigma \in J$ $\sigma(i) = a_i$ is equivalent to $\delta_i(a_i) \in J$. Thus J_i is the direct image of J under the projection $\pi_i : \prod_i A_i \to A_i$, and hence J_i is a κ-generated λ-ideal on A_i. Now, if $\sigma \in J$, then by (1) $\sigma(i) \in J_i$ for all $i \in I$. Conversely, if $\sigma(i) \in J_i$, or equivalently $\delta_i(\sigma(i)) \in J$, for every $i \in I$, then since J is a λ-ideal, $\sigma \in J$ by 11 4(3) Thus $J = \prod_i J_i$, and the proof is complete.

11.6. Theorem. *F_κ^λ preserves λ-products*

PROOF Suppose that $\{A_i \ i \in I\}$ is a λ-family of λ-frames. By Proposition 11.2, $F_\kappa^\lambda \prod_i A_i$ is the κ-frame of all κ-generated λ-ideals on $\prod_i A_i$. But it follows easily from Lemma 11.5 (and Proposition 11.2) that this κ-frame is isomorphic to $\prod_i F_\kappa^\lambda A_i$. Furthermore, the map $F_\kappa^\lambda \pi_i$, being direct image under projection, is the ith projection on $\prod_i F_\kappa^\lambda A_i$, as argued in the proof of 11 5, and so λ-product diagrams are preserved by F_κ^λ, as required

11.7. Example. We give an example to show that products of λ many factors need not be preserved by F_κ^λ when $\lambda < \kappa$. For this, consider the product $\mathbf{2}^\lambda$, where $\mathbf{2}$ is the two-element frame (of course, $\mathbf{2}^\lambda$ is isomorphic to the power-set $\mathrm{P}\lambda$ under the inclusion ordering). Since every ideal on $\mathbf{2}$ is principal, so is every λ-indexed product of ideals on $\mathbf{2}$ But the set J of all λ-subsets of λ is a λ-ideal by the regularity of λ, is λ^+-generated (by the set of all singleton subsets of λ), and is not principal. It follows that the product $\mathbf{2}^\lambda$ is not preserved by $F_{\lambda^+}^\lambda$.

11.8. Theorem. *If $\lambda > \omega$, then F_κ^λ preserves equalizers.*

Corollary. *If $\lambda > \omega$, then F_κ^λ preserves all λ-limits (i.e., limits where the indexing category has a λ-set of morphisms)*

PROOF Suppose $\lambda > \omega$, and let $f, g : A \to C$ be two λ-frame morphisms with equalizer $E = \{a \in A \,.\, f(a) = g(a)\}$ and inclusion $i \cdot E \to A$. Since F_κ^λ is a functor, we have $F_\kappa^\lambda f \circ F_\kappa^\lambda i = F_\kappa^\lambda g \circ F_\kappa^\lambda i$; and by Lemma 11.3, $F_\kappa^\lambda i$ is injective Therefore, to establish the Theorem, it will suffice to show, by Proposition 11.2, that any κ-generated λ-ideal J on A with $\mathsf{downcl}\, f(J) = \mathsf{downcl}\, g(J)$ has $J = \mathsf{downcl}\, J'$ for some κ-generated λ-ideal J' on E.

So, suppose $\mathsf{downcl}\, f(J) = \mathsf{downcl}\, g(J)$. Then

$$\forall j \in J\ \exists k \in J\ \ f(j) \le g(k) \qquad \text{and} \qquad \forall k \in J\ \exists j \in J\ \ g(k) \le f(j) \tag{1}$$

Now, let $j \in J$ be arbitrary, and define a sequence $\{a_n : n < \omega\}$ of elements on J by induction as follows Start by putting $a_0 = j$. Next, if $n+1$ is odd and a_n is defined, then choose $k \in J$ such that $f(a_n) \le g(k)$, which we can do by (1), and put $a_{n+1} = a_n \vee k$. Then $a_n \le a_{n+1} \in J$ and $f(a_n) \le g(a_{n+1})$ (by monotonicity of g). On the other hand, if $n+1$ is even and a_n is defined, then choose $j \in J$ such that $g(a_n) \le f(j)$, again possible by (1), and put $a_{n+1} = a_n \vee j$ Then $a_n \le a_{n+1} \in J$ and $g(a_n) \le f(a_{n+1})$ (by monotonicity of f).

Now, as J is closed under countable joins and $\{a_n : n < \omega\}$ is increasing, we have

$$\bigvee_{n<\omega} a_{2n} = \bigvee_{n<\omega} a_{2n+1} = \bigvee_{n<\omega} a_{2n+2} = a \in J. \tag{2}$$

But, since f and g preserve countable joins, we have, by (2),

$$f(a) = f(\bigvee_n a_{2n}) = \bigvee_n f(a_{2n}) \leq \bigvee_n g(a_{2n+1}) = g(\bigvee_n a_{2n+1}) = g(a)$$

and

$$g(a) = g(\bigvee_n a_{2n+1}) = \bigvee_n g(a_{2n+1}) \leq \bigvee_n f(a_{2n+2}) = f(\bigvee_n a_{2n+2}) = f(a).$$

Thus, $f(a) = g(a)$, and so $a \in E$. Summarizing, we have found for every $j \in J$ an element $a \in E \cap J$ with $j \leq a$. Choosing one such a for each generator of J, and letting J' be the λ-ideal on E (κ-)generated by these elements, we have $J = \text{downcl}\, J'$, as required. Thus the proof of the Theorem is complete.

The Corollary follows immediately from the present Theorem and Theorem 11.6, by the construction of limits from products and equalizers (see [29, V 2, Theorem 1] for details of this construction).

11.9. Example. We give an example to show that if $\lambda = \omega$, then equalizers need not be preserved by F_κ^λ when $\lambda < \kappa$. Let A and C both be the frame $\langle \omega + 1, \leq \rangle$ (i.e., with $0 < 1 < 2 < \cdots < \omega$), let $f: A \to C$ be the function with $f(n) = 2n$ ($n < \omega$) and $f(\omega) = \omega$, and let $g: A \to C$ be the function with $g(0) = 0$, $g(n) = 2n - 1$ ($0 < n < \omega$), and $g(\omega) = \omega$. Then f and g are clearly ω-frame (in fact frame) morphisms, and the equalizer E of f and g is the two-element frame $\{0, \omega\}$. Now, $J = \{a \in A : a < \omega\}$ is an ω^+-generated ω-ideal on A that is not generated by any ω-ideal of E. However, $\text{downcl}\, f(J) = J = \text{downcl}\, g(J)$, and so J is in the equalizer of $F_{\omega^+}^\omega f$ and $F_{\omega^+}^\omega g$. Thus, the equalizer is not preserved by $F_{\omega^+}^\omega$.

11.10. Products of congruences. We now prove a result about congruences on products of κ-frames and frames, whose proof is quite similar to that of Lemma 11.5.

Theorem. *Suppose that I is a κ-set and A_i is a κ-frame for all $i \in I$. If θ_i is a congruence on A_i for all $i \in I$, then $\prod_i \theta_i$ is a congruence on $\prod_i A_i$. Conversely, every congruence on $\prod_i A_i$ has this form.*

Proof. The first part of the Theorem is trivial, since all operations on the product are pointwise.

For the second part, let θ be a congruence on $\prod_i A_i$, and for every $i \in I$, define the relation θ_i on A_i by $a_i \,\theta_i\, b_i$ iff $\delta_i(a_i) \,\theta\, \delta_i(b_i)$ (δ_i was defined in Lemma 11.4). Clearly each relation θ_i is a congruence on A_i, since it is the inverse image of the congruence θ under the non-empty-meet- and join-preserving mapping 11.4(2). But now $\sigma \,\theta\, \tau$ implies

$$\delta_i(\sigma(i)) = \delta_i(1) \wedge \sigma \;\theta\; \delta_i(1) \wedge \tau = \delta_i(\tau(i)),$$

and thus $\sigma(i) \,\theta_i\, \tau(i)$ for all $i \in I$.

Conversely, if $\sigma(i) \,\theta_i\, \tau(i)$, or equivalently $\delta_i(\sigma(i)) \,\theta\, \delta_i(\tau(i))$, for all $i \in I$, then since θ is a κ-frame congruence, we use Lemma 11.4(3) to get $\sigma \,\theta\, \tau$. Thus $\theta = \prod_i \theta_i$ and the proof is complete.

11.11. Corollary. *All congruences on arbitrary frame products are products of congruences on the factors.*

PROOF We can either prove this directly, by redoing the proof of Theorem 11.10 without the cardinality restrictions or (to anticipate 12.4 below), we can choose a large enough regular cardinal κ th· frame congruences on the product are the same as κ-frame congruences or ' ·ct. and then invoke 11.10 directly.

11.12. Example. The ·xample 11.7, which shows that products of λ many factors need not be preserved by $F_{\lambda^+}^{\lambda}$, can b· modified to produce a congruence on a product of λ many λ-frames that is not a product of congruences. Thus, let θ be the relation on the frame $\mathbf{2}^{\lambda}$ (or equivalently, $\langle \mathrm{P}\lambda, \subset \rangle$) defined for $S, T \subset \lambda$ by $S\ \theta\ T$ iff $|S \Delta T| < \lambda$, where Δ is the *symmetric difference* operation $S \Delta T = (S - T) \cup (T - S)$. One can check, for any two I-indexed families $\{S_i\}$ and $\{T_i\}$ of subsets of λ, that $(\bigcup_i S_i) \Delta (\bigcup_i T_i) \subset \bigcup_i (S_i \Delta T_i)$, and so κ-joins are compatible with θ by the regularity of κ. Similarly, one can check, for any subsets S_1, S_2, T_1, T_2, that $(S_1 \cap S_2) \Delta (T_1 \cap T_2) \subset (S_1 \Delta T_1) \cup (S_2 \Delta T_2)$, and therefore finite meets repect θ as well, making θ a κ-frame congruence. But, $0/\theta$ is just the ideal J of Example 11.7 and is not a product of ideals. Thus θ cannot be a product of congruences.

12. Limits and colimits

12.1. Remarks. By 6.5.1, limits of frames and κ-frames are computed as in **Set**. Coequalizers are given as in 6.5.3, but for frames a more concrete description is possible (see 13.7). As for coproducts, the infinite case isn't any harder than the binary case, and so we start by looking at that. (See Johnstone [22, II.2.12] for a development that includes the infinite case.)

12.2. Coproducts of frames. Given frames A and B, we call a subset $S \subset A \times B$ a *bi-ideal* on $A \times B$ if it is down-closed (in the product order) and, for every set I, has $\langle \bigvee_i a_i, b \rangle \in S$ whenever $\langle a_i, b \rangle \in S$ for all $i \in I$, and has $\langle a, \bigvee_i b_i \rangle \in S$ whenever $\langle a, b_i \rangle \in S$ for all $i \in I$.

Proposition. *Suppose A and B are frames. Then the coproduct of A and B can be represented by the set $A \oplus B$ of all bi-ideals on $A \times B$, ordered by inclusion. The injections $\nu_A : A \to A \oplus B$ and $\nu_B : B \to A \oplus B$ are given by $\nu_A(a) = \downarrow\langle a, 1 \rangle$ and $\nu_B(b) = \downarrow\langle 1, b \rangle$, and if $f : A \to C$ and $g : B \to C$ are frame morphisms, the mediating morphism $m : A \oplus B \to C$ is given by*

$$m(S) = \bigvee\{f(a) \wedge g(b) : \langle a, b \rangle \in S\}.$$

12.3. Coproducts of κ-frames, and infinitary coproducts. We record the obvious modification for κ-frames. A subset $S \subset A \times B$ is a *κ-bi-ideal* on $A \times B$ if it is down-closed, has same closure with respect to joins as above (when I is a κ-set), and is κ-generated (with respect to these closure conditions). The injections and the mediating morphism are the same (except that we define $m(S)$ in terms of a κ-generating set, instead of S itself). And, as with free κ-frames, the regularity of κ is used here to conclude that the join of a κ-set of κ-generated ideals is still κ-generated

Also for the record we mention that in the infinite case, one requires a (κ-)bi-ideal to be closed under (κ-)joins in each coordinate while the others remain fixed and that instead of taking subsets of the whole cartesian product, one takes subsets only of those elements that have 1 in all but finitely many coordinates

12.4. Colimits in κ-Frm vs. Frm. The basic relation we will be exploiting between κ-frames and frames is that "locally" the categories **Frm** and κ-**Frm** "look the same" First we have that if A is a poset with $|A| < \kappa$, then A is a frame iff A is a κ-frame Also, for such an A, frame congruences and κ-frame congruences on A coincide Finally, in the case where $\{A_i : i \in I\}$ is a family of frames with $|\prod_i A_i| < \kappa$, the notions of bi-ideals and κ-bi-ideals on (the appropriate subset of) the product agree, and therefore the coproducts, as constructed above, agree as well Putting together these observations, we get the following result:

Theorem. *If $D : \mathbf{J} \to$ **Frm** is a diagram of frames, with colimit cone $\nu : D \to \Delta C$, then there exists a regular cardinal κ (in fact we can choose $\kappa = |\prod_{j \to j'} Di|^+$, where the product is over all morphisms of $\mathbf{J}$) such that $U^\kappa_\infty \circ \nu : U^\kappa_\infty \circ D \to U^\kappa_\infty \circ \Delta C$ is a colimit cone in κ-**Frm***

Proof Let κ be chosen as above Then κ is a regular cardinal larger than any frame that appears in the construction of colim D from coproducts and coequalizers By the previous observations, it follows that the colimit, whether calculated in **Frm** or in κ-**Frm**, is the same poset, from which the Proposition follows

12.5. Operator description of frame coproducts. Given a bi-ideal $I \subset A \times B$ and an element $a \in A$, there exists a largest element $\phi(a) \in B$ such that $(a, \varphi(a)) \in I$ (namely $\phi(a) = \bigvee\{b : (a,b) \in I\}$, which is in I by the join-closure property on the second coordinate) Similarly, for every $b \in B$, there is a largest $\psi(b) \in A$ with $(\psi(b), b) \in I$. Either operation determines I, since

$$I = \{(a,b) : b \leq \phi(a)\} = \{(a,b) : a \leq \psi(b)\}, \tag{1}$$

and, considered as operations $\phi : A \to B^{\mathrm{op}}$ and $\psi : B^{\mathrm{op}} \to A$, they are adjoint ($\phi \dashv \psi$). Moreover it is clear from (1) that every such adjoint pair determines a unique bi-ideal, and so the frame coproduct of A and B can just as well be presented in terms of adjoint pairs

Wigner [51] gives such a description in terms of just the operations ϕ, which we reproduce here without proof

Theorem. *Suppose that A and B are frames Let $A \otimes B$ be the set of all operations $\phi : A \to B$ satisfying $\phi(\bigvee S) = \bigwedge \varphi(S)$ for all $S \subset A$ Then $A \otimes B$, ordered pointwise, is a frame coproduct of A and B The injections $\nu_A : A \to A \otimes B$ and $\nu_B : B \to A \otimes B$ are given by*

$$\nu_A(a_0)(a) = \begin{cases} 1, & \text{if } a \leq a_0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_B(b_0)(a) = \begin{cases} b_0, & \text{if } a \neq 0 \\ 1, & \text{if } a = 0. \end{cases}$$

If $f : A \to C$ and $g : B \to C$ are frame morphisms, then the mediating morphism $m : A \otimes B \to C$ and its right adjoint $m_* : C \to A \otimes B$ are given by

$$m(\phi) = \bigvee_{a \in A} f(a) \wedge g(\phi(a)) \quad \text{and} \quad m_*(c)(a) = g_*(f(a) \to c).$$

12.6. Pushouts of frames. All of the descriptions of binary coproducts we have given can be easily modified to give descriptions of the pushout of two morphisms $f : C \to A$ and $g : C \to B$. We remark on the necessary modifications and leave the verifications to the reader.

In Proposition 12.2, the coproduct of A and B was given as the set $A \oplus B$ of all bi-ideals on $A \times B$, ordered by inclusion. For the pushout of $f : C \to A$ and $g : C \to B$, one takes instead the set $A \oplus_C B$ of all bi-ideals S that additionally satisfy

$$\langle a \wedge f(c), b\rangle \in S \quad \text{if and only if} \quad \langle a, g(c) \wedge b\rangle \in S \qquad (a \in A, b \in B, c \in C), \qquad (1)$$

again ordered by inclusion. The description of the injections has to be modified (for $\nu_A(a)$, for example, one takes the smallest bi-ideal satisfying (1) and containing $\langle a, 1\rangle$), but the description of the mediating morphism does not.

In Theorem 12.5 the coproduct was given as the set $A \otimes B$ of all operations $\phi : A \to B$ satisfying $\phi(\bigvee S) = \bigwedge \phi(S)$ for all $S \subset A$, ordered pointwise. For the pushout, one takes the set $A \otimes_C B$ of those operations that additionally satify

$$\phi(f(c) \wedge a) = g(c) \to \phi(a) \qquad (a \in A, c \in C), \qquad (2)$$

again ordered pointwise. Unfortunately, there doesn't appear to be any useful formula for the injection into the pushout, as there is for the coproduct, but again the formulas for the mediating morphism and its adjoint remain valid.

12.7. Directed colimits of frames. Several authors ([51], [16], and, at least implicitly, [27]) have observed that the directed colimit of diagram in **Frm** is computed by taking the directed limit of the underlying sets and right adjoint maps, and have used this to prove that if all of the morphisms in the diagram are mono, then the canonical injections into the colimit are mono as well.

We state these results here and refer the reader to the papers above for proofs.

Theorem. *The functor* $(-)_* : \mathbf{Loc} \to \mathbf{Set}$ *creates filtered limits.*

More explicitly, suppose $D : \mathbf{J} \to \mathbf{Frm}$ is a diagram on an ω-filtered category (6.5.5). Then the colimit of D may be taken to be the set of all those $\sigma \in \prod_{j \in J} Dj$ such that for every morphism $f : i \to j$ of $\mathbf{J}$, $(Df)_*(\sigma(j)) = \sigma(i)$, ordered pointwise. (Meets in this frame are also pointwise.) For each $j \in \mathbf{J}$, the canonical injection $\nu_j : Dj \to \operatorname{colim} D$ is the left adjoint to the projection on the jth coordinate. Given a cone $\tau : D \to \Delta A$, the mediating morphism $\operatorname{colim} D \to A$ is the left adjoint to the function that takes an element $a \in A$ to the function $j \mapsto (\tau j)_*(a)$.

12.8. Corollary. If in a filtered diagram $D : \mathbf{J} \to \mathbf{Frm}$ of frames, Df is mono for every morphism f of $\mathbf{J}$, then the canonical injections ν_j are all mono.

12.9. Exercises.

12.9.1. Products commute with directed limits in **Frm**.

(a) Suppose that $G, H : \mathbf{J} \to \mathbf{Frm}$ are two diagrams in **Frm**, and $\tau : G \to H$ is a natural transformation that satisfies $Gf \circ (\tau i)_* = (\tau j)_* \circ Hf$ (as functions) for all morphisms $f : i \to j$ in $\mathbf{J}$.

Let $\mu : \lim_J G \to G$ and $\nu : \lim_J H \to H$ be limit cones let $\lim \tau : \lim G \to \lim H$ be the unique frame morphism such that $\tau j \circ \mu j = \nu j \circ \lim \tau$ for all $j \in \mathbf{J}$ and let $\lim \tau_* : \lim H \to \lim G$ be the unique function such that $(\tau j)_* \circ \nu j = \mu j \circ \lim \tau_*$ (as functions) for all $j \in \mathbf{J}$ Show that $(\lim \tau)_* = \lim \tau_*$.

(b) Call $F : \mathbf{I} \times \mathbf{J} \to \mathbf{Frm}$ *$*$-compatible* if, whenever $f : i \to i'$ and $g : j \to j'$ are morphisms of $\mathbf{I}$ and $\mathbf{J}$, we have $F(f, j) \circ F(i, g)_* = F(i', g)_* \circ F(f, j')$ Use (a) to prove that if F is a $*$-compatible diagram such that, for every $i \in \mathbf{I}$, $F(i, -) : \mathbf{J} \to \mathbf{Frm}$ is filtered, then the canonical morphism $\operatorname{colim}_J \lim_I F \to \lim_I \operatorname{colim}_J F$ is an isomorphism of frames.

(c) Show that if $F : \mathbf{I} \times \mathbf{J} \to \mathbf{Frm}$ is such that, for every $i \in \mathbf{I}$, $F(i, -)$ is a *discrete* diagram, then F is $*$-compatible Conclude that in $\mathbf{Frm}$, directed colimits commute with arbitrary products

12.9.2. Let κ be a regular cardinal Use the fact that the free functor F^{κ}_{∞} preserves colimits and κ-products to derive from Exercise 12 9 1 that κ-products commute with directed colimits of κ-frames

Chapter 5
THE ASSEMBLY TOWER

Associated to every frame A is another frame $\mathsf{N}A$, called its assembly, and this process can be iterated transfinitely to produce the assembly tower of A. The assembly tower is the main tool used for obtaining information about the "meet-structure" of a frame in Chapter 7 and about pushout-stable monomorphisms in Chapter 8. The assembly tower is also related to the extensional operators of Chapter 3, with the regular operators of Chapter 6 as intermediary Thus, the material of this chapter forms the core of the thesis

We begin in Section 13 with the equivalence of the complete lattices of congruences, nuclei, and maxsets on a frame establishing the definition of the assembly. Since congruences are naturally associated to quotients, these quotients have descriptions in terms of nuclei and maxsets, as well, which are looked at next The section ends with a look at how elements of the assembly can be generated by incomplete data—congruences generated by sets of pairs, nuclei generated by prenuclei, and maxsets generated by certain subsets Section 14 is devoted to showing that the assembly $\mathsf{N}A$ is a frame, this is accomplished by establishing a formula for meets of maxsets Section 15 then looks at special elements of $\mathsf{N}A$ (and $\mathsf{N}(\mathsf{N}A)$) and their rules of calculation In Section 16, the universal property of the assembly functor N is established, and the action of N on morphisms is looked at from three different angles. Some preservation properties of N are also proved Section 17 gives the construction of the assembly tower and its basic properties, and finally Section 18 discusses the similar construction for κ-frames

13. The assembly: congruences, nuclei, maxsets, quotients

13.1. Frame congruences. Recall that a congruence θ on a frame A is an equivalence relation on A that is compatible with finite meets and arbitrary joins. It follows that every equivalence class of θ has a largest member (namely its join). Thus, there is associated to each congruence θ on A both an operation $j_\theta : A \to A$, defined by

$$j_\theta a = \bigvee a/\theta, \tag{1}$$

and a subset $M_\theta \subset A$ defined by

$$M_\theta = \{a \in A \cdot a \geq b \text{ whenever } a\,\theta\,b\} \tag{2}$$

Thus j_θ takes an element to the largest element θ-related to it, and M_θ is the set of all such largest elements

13.2. Nuclei and Maxsets. The properties of the operations j_θ and subsets M_θ in (1) and (2) above can be sumarized as follows.

Proposition. *Let A be a frame*

(a) *An operation $j\ A \to A$ is equal to j_θ for some $\theta \in \mathrm{Con}\,A$ if and only if j is a nucleus* i e ,

$$a \leq ja = jja \quad \text{and} \quad j(a \wedge b) = ja \wedge jb \quad (a, b \in A). \tag{1}$$

and then $\theta = \{(a,b) : ja = jb\}$.

(b) *A subset $M \subset A$ is equal to M_θ for some $\theta \in \mathrm{Con}\,A$ if and only if*

$$S \subset M \text{ implies } \bigwedge S \in M \quad \text{and} \quad a \in A, m \in M \text{ imply } a \to m \in M \tag{2}$$

and then $\theta = \{(a,b)\ \forall m \in M\ a \leq m \Leftrightarrow b \leq m\}$

(c) *For a frame congruence θ the associated j_θ and M_θ are related to each other by*

$$M_\theta = \mathrm{fix}\, j_\theta = j_\theta A \quad \text{and} \quad j_\theta a = \bigwedge\{m \in M_\theta\ \ a \leq m\} \tag{3}$$

Nuclei, as extensional closure operators, were studied in Chapter 3. A subset M satsifying (2) will be called a *maxset* We call a subset $M \subset A$ satisfying the first part of (2) *meet-closed*, and one satifying the second part an *arrow-ideal*

Proof. The proof is standard see, for example. [22], II 2 2–3 [27], III 4, Prop 2, or [50], Thm 6.2.9. In fact (cf. [27]), the proof can be split into two stages, corresponding to the compatibility of θ with joins and finite meets; the first stage is the equivalence of the following (cf. Proposition 4 3).

(i) there is a complete join-semilattice congruence θ on A such that for all $a \in A$, $ja = \bigvee\{b\ \ a\,\theta\, b\}$.

(ii) j is a closure operator.

(iii) $a \leq jb$ iff $ja \leq jb$ for all $a\ b \in A$

(iv) there is $M \subset A$, closed under all meets, with $ja = \bigwedge\{m \in M : a \leq m\}$ for all $a \in A$.

One can moreover extract from these proofs the result that if R is the relation defined between pairs $(a,b) \in A \times A$ and $s \in A$ by $(a.b)\ R\ s$ iff $a \to s = b \to s$ and if $l_\forall \dashv r_\forall$ is the associated "universal" adjunction, then $r_\forall \circ l_\forall = \Theta$ (the closure operator taking a set of pairs to the smallest congruence containing them) and $l_\forall \circ r_\forall$ is the closure operator that takes a subset of A to the smallest maxset containing it See 13 7 and 13 9 for explicit descriptions of these closure operators

13.3. Frame quotients. Associated to every congruence θ on a frame A there is also, of course, the frame quotient A/θ and natural map $A \to A/\theta$, $a \mapsto a/\theta$ How does this relate to j_θ and M_θ?

Corollary. *Suppose A is a frame and $\theta \in \mathrm{Con}\,A$. Then M_θ, with the order induced by A, is order isomorphic to the quotient A/θ (by the correspondence $m \sim m/\theta$), and, up to this isomorphism, $j_\theta\ \ A \to M_\theta$ is the quotient map The right adjoint $(j_\theta)_*\ .\ M_\theta \to A$ to j_θ is the inclusion (j is a "reflection")*

Proof The order on frame congruence classes is easily seen to be the same, by 5.4(g) and (iii) in the proof above, as the order on their maximal elements, and (*iii*) moreover states that j_θ is left adjoint to the inclusion. The rest is trivial

ADA289360

13.4. Notational remark. It will be convenient to have a notation for the quotient of a frame A by the congruence associated to a nucleus j. Following Johnstone [22], we denote this A_j and, for concreteness, take the elements of A_j to be the fixedpoints of j and the order on A_j to be that induced by A as in the Corollary

13.5. Quotient operations. Finite meets and arbitrary joins of equivalence classes in the quotient are naturally given in terms of representatives; it is less clear what the infinite meet and arrow operations are For quotients given by maxsets (or fixedpoints of nuclei), the reverse is true:

Proposition. *Let A be a frame and j a nucleus on A If we denote the meet, join, and arrow operations of A_j by $\bigwedge^j$, $\bigvee^j$, and $\rightarrow^j$ then for every $S \subset A_j$ and $a, b \in A_j$, we have*

(a) $\bigwedge^j S = \bigwedge S$.
(b) $a \rightarrow^j b = a \rightarrow b$, *and*
(c) $\bigvee^j S = j \bigvee S$

PROOF As A_j is a meet-closed subset of A with j as reflection, (a) and (c) are clear In particular, finite meets in A_j agree with those of A and since A_j is closed under $\rightarrow$ by the second part of 13 2(2), (b) follows as well.

13.6. The assembly. We have seen that the complete lattice of congruences on a frame A can be described in three equivalent ways by congruences, nuclei, and maxsets In the sequel we make frequent use of all three, and, in an (admittedly weak) effort to remain neutral as to representation, we will call this complete lattice the *assembly* of A. (We will also make an effort when introducing a new concept associated with the assembly to describe it in terms of all three representations.) When the representation of the assembly is important, we will write $\mathsf{Con}\,A$, as usual, for the lattice of congruences on A (ordered by inclusion), $\mathsf{N}A$ for the lattice of nuclei on A (ordered pointwise), and $\mathsf{Max}\,A$ for the lattice of maxsets on A (ordered by reverse inclusion). In general, however, we will stick to the notation $\mathsf{N}A$, since it has become standard usage

13.7. Congruences generated by pairs. In this and the following paragraphs, we look at how elements of the assembly can be generated by various data The first such result, generation of congruences by pairs, follows directly from the observation made in the proof of 13 2

Proposition. *Suppose that A is a frame, and let θ be the congruence on A generated by $\{(a_i, b_i) : i \in I\}$. Then $x\ \theta\ y$ if and only if $x \rightarrow s = y \rightarrow s$ whenever $a_i \rightarrow s = b_i \rightarrow s$ for all $i \in I$.*

13.8. Nuclei generated from prenuclei. Recall that a prenucleus is an extensional operator that is inflationary and monotone (9.9) and that nuclei are precisely the idempotent prenuclei (9.10).

Proposition. *Suppose j is a prenucleus on a frame A. and define by transfinite* *recursion the sequence* $\{j^\alpha \mid \alpha \in \mathbb{C},\ \alpha > 0\}$ *of operators on* A *by*

$$j^1(a) = ja$$
$$j^{\alpha+1}(a) = j(j^\alpha(a)), \text{ and}$$
$$j^\lambda(a) = \bigvee_{\alpha<\lambda} j^\alpha(a) \quad \text{if } \lambda \text{ is a limit}$$

Then for some ordinal γ, j^γ *is a nucleus, and* j^γ *is the least nucleus pointwise greater than* j. *Furthermore,* $\operatorname{fix} j^\gamma = \operatorname{fix} j$.

PROOF. Since $j^1 = j$ is a prenucleus, compositions of prenuclei are larger prenuclei (by a simple argument from the definition), and (nonempty) pointwise joins of prenuclei are prenuclei (9.17), it follows by induction that each j^α is a prenucleus and that $j^\beta \le j^\alpha$ whenever $\beta \le \alpha$. If k is a nucleus with $j \le k$, then again by induction it is easy to verify (using 9.2(b)) that $j^\alpha \le k$ for all α. Thus, j^γ, if it is idempotent, is the least nucleus greater than j. But, since there are only a set of prenuclei on A, and the sequence $\{j^\alpha\}$ is increasing, there must be a γ such that $j \circ j^\gamma = j^\gamma$, from which it follows (by induction up to γ) that j^γ is idempotent.

It remains to verify that $\operatorname{fix} j^\gamma = \operatorname{fix} j$. But for any $a \in A$, $ja \le a$ implies $j^\alpha a \le a$ again by induction on α, and the converse implication is trivial, since $\{j^\alpha\}$ is increasing. This completes the proof.

13.9. Proposition. *Suppose* A *is a frame,* $T \subset A$, S *is an arrow-ideal of* A, *and* M *is a meet-closed subset of* A. *Then,*

(a) *The smallest maxset containing* T *is* $\{\bigwedge_i a_i \to t_i : a_i \in A, t_i \in T\}$;
(b) *The smallest maxset containing* S *is* $\{\bigwedge R : R \subset S\}$; *and*
(c) *The largest maxset contained in* M *is* $\{m \in M : a \to m \in M \text{ for all } a \in A\}$.

We call the set in (c) the *core of* M, written $\operatorname{core} M$.

PROOF. (a) Let T' be the set listed. Clearly, any maxset containing T must contain T', thus it is enough to show that T' is a maxset. It is closed under meets, because we can write a meet of meets as a single meet. And it is an arrow-ideal because of the laws

$$a \to \bigwedge X = \bigwedge a \to X \qquad \text{and} \qquad a \to (a' \to x) = (a \wedge a') \to x,$$

for $X \subset A$ and $x \in A$.

(b) Let S' be the set listed, which is meet-closed and contained in any maxset containing S. If $a \in A$ and $\bigwedge R \in S'$, then $a \to \bigwedge R = \bigwedge a \to R \in S'$, since for every $r \in R$, $a \to r \in S$, as S is an arrow-ideal.

(c) Let $M' = \operatorname{core} M$, which is an arrow-ideal by definition. If $R \subset M'$, then $\bigwedge R \in M$, since M is meet-closed, and for every $a \in A$, $a \to \bigwedge R = \bigwedge a \to R \in M$, since $a \to R \subset M$ by the definition of M'. Thus M' is a maxset. But any other maxset contained in M, since it is an arrow-ideal, must be contained in M', showing that M' is the largest maxset contained in M.

14. The assembly as a frame

14.1. Lattice operations in the assembly. The assembly of a frame A is a complete lattice that, by 13.2, we may take to consist of congruences, nuclei or maxsets. We now look at the lattice operations in the assembly in terms of these three perspectives.

Since arbitrary intersections of congruences are congruences, the meet operation in $\mathrm{Con}\, A$ is just intersection. The join of a set of congruences is, of course, the congruence generated by their union, but this won't have a simple description, in general. The meet operation in $\mathsf{N}A$ is given by pointwise meet, as was proved in Proposition 9.20: if $J \subset \mathsf{N}A$, then

$$(\bigwedge J)a = \bigwedge Ja \qquad (a \in A) \tag{1}$$

A formula for joins of nuclei will be given in 23.4 as an application of results developed in Chapter 6. Finally, since intersections of maxsets are maxsets, the join operation of $\mathrm{Max}\, A$ (remember that the order is reversed) is given by intersection. This is valuable in that we have at least one representation for which joins are easy to compute; see Theorem 14.4 below for an application.

14.2. Meets of maxsets. The meet operation of $\mathrm{Max}\, A$ is readily seen to be given by meet-closure of the union, since unions of arrow-ideals are arrow-ideals, and so 13.9(b) applies. We give another approach to this result, which carries with it some additional formulas for meets of maxsets.

Suppose $\mathcal{M} \subset \mathrm{Max}\, M$ is a collection of maxsets on A, each $M \in \mathcal{M}$ with its associated nucleus j_M, as determined in 13.2(3). Then $\bigwedge \mathcal{M}$ is associated to the nucleus $\bigwedge_M j_M$, and since this association is by fixedpoints, we have, by 14.1(1),

$$\bigwedge \mathcal{M} = \{a \in A : \bigwedge_M j_M a = a\}. \tag{1}$$

The following Lemma gives two other formulas for $\bigwedge \mathcal{M}$.

Lemma. *With the notation as above, we have*

$$\bigwedge \mathcal{M} = \{\bigwedge_M a_M : a_M \in M \text{ for each } M \in \mathcal{M}\} \tag{2}$$

Alternatively, the meet of a set of maxsets is the meet-closure of their union:

$$\bigwedge \mathcal{M} = \{\bigwedge T : T \subset \bigcup \mathcal{M}\}. \tag{3}$$

Proof. Let S_1, S_2, and S_3 be the sets on the right sides of (1), (2), and (3). We show that $S_1 \subset S_2 \subset S_3 \subset S_1$, and hence that all of these sets are equal.

The first two of these inclusions, $S_1 \subset S_2$ and $S_2 \subset S_3$, are trivial, so we only need to argue for the last, $S_3 \subset S_1$. Suppose $a \in S_3$, and pick $T \subset \bigcup \mathcal{M}$ with $a = \bigwedge T$. For each $M \in \mathcal{M}$, define

$$T_M = T \cap M \qquad \text{and} \qquad t_M = \bigwedge T_M .$$

Then $T = \bigcup_M T_M$, and for every $M \in \mathcal{M}$ we have $a \le t_M \in M$, since M is meet-closed and $T \supset T_M$ implies

$$a = \bigwedge T \le \bigwedge T_M = t_M$$

Therefore $j_M a \le j_M t_M = t_M$ for all $M \in \mathcal{M}$, and thus we have

$$a \le \bigwedge_M j_M a \le \bigwedge_M t_M = \bigwedge\{\bigwedge T_M \quad M \in \mathcal{M}\} = \bigwedge T = a$$

Therefore, $a = \bigwedge_M j_M a$, and so $a \in S_1$

14.2.1. Corollary. *If $M, N \in \mathsf{Max}\, A$, then*

$$M \wedge N = \{m \wedge n \quad m \in M \text{ and } n \in N\} \tag{4}$$
$$= \{a \in A \quad j_M a \wedge j_N a = a\} \tag{5}$$

PROOF If we put $\mathcal{M} = \{M, N\}$, then (4) is just equation (2) of the Lemma, while (5) is just (1)

14.3. Remark. Statement (4) of the Corollary above was apparently first noticed by Dana Scott, who used this description of binary meet of maxsets to give a proof of the following theorem Our proof is essentially the same, except that we have used (5) instead of (4) (so that certain choices made in the proof are canonical)

14.4. Theorem. *NA is a frame*

PROOF. We verify the frame distributive law for $\mathsf{Max}\, A$. Suppose $M \in \mathsf{Max}\, A$ and, for all $i \in I$ $N_i \in \mathsf{Max}\, A$ We only have to show that

$$M \wedge \bigvee_{i \in I} N_i \le \bigvee_{i \in I} M \wedge N_i, \tag{1}$$

since the reverse inequality holds in any complete lattice Recall that $\le$ and $\bigvee$ in (1) are reverse inclusion and set intersection, and let $x \in \bigcap_{i \in I} M \wedge N_i$ Then, by 14 2 1(5), $x = j_M x \wedge j_{N_i} x$ for all $i \in I$ Now for any particular $i \in I$,

$$j_M x \to x = j_M x \to (j_M x \wedge j_{N_i} x) = j_M x \to j_{N_i} x \in N_i,$$

since $j_{N_i} \in N_i$ and N_i is an arrow-ideal Thus, $j_M x \to x \in \bigvee N_i$ But then, since $j_M x \ge x$, we have $x = j_M x \wedge (j_M x \to x)$, which implies, by 14.2.1(4), that x is a member of the left side of (1), and we are done

15. Special elements and their properties

15.1. Principal congruences. We assume throughout this section that A is a frame Every congruence $\theta \in \mathrm{Con}\,A$ is a join of principal congruences, and furthermore, for each principal congruence $\Theta(a\ b)$, we have

$$\Theta(a\ b) = \Theta(0,a) \wedge \Theta(b,1) \tag{1}$$

by 5 4(f) Thus $\mathrm{Con}\,A$ is generated as a frame by

$$\{\Theta(0\ a) \cdot a \in A\} \cup \{\Theta(b,1) : b \in A\}. \tag{2}$$

We therefore begin our investigation of special elements of the assembly by looking the elements in (2).

Proposition.
(a) *$\Theta(0.a)$ is associated to the nucleus $c(a) = a \vee -$ and to the maxset $\uparrow a$*
(b) *$\Theta(b.1)$ is associated to the nucleus $u(b) = b \to -$, and $\mathrm{fix}\,u(b)$ is order-isomorphic to $\downarrow b$ by the operations $b \wedge -$ and $b \to -$.*
(c) *For any $j \in \mathrm{N}A$ and $a.b \in A$ with $a \leq b$, $c(b) \wedge u(a) \leq j$ iff $ja = jb$ iff $b \leq ja$. In particular, $c(b) \leq j$ iff $b \leq j0$, and $u(a) \leq j$ iff $ja = 1$.*
(d) *For any $a \in A$, $c(a)$ and $u(a)$ are complementary elements of $\mathrm{N}A$. Moreover, a and b are complements in A iff $c(a) = u(b)$ iff $c(b) = u(a)$*
(e) *The map $c\ \ A \to \mathrm{N}A$ given by $a \mapsto c(a)$ is a frame morphism It is both mono and epi and is an isomorphism iff A is Boolean.*
(f) *The map $u \cdot A \to \mathrm{N}A$ is an "anti-frame morphism": $u(\bigwedge F) = \bigvee u(F)$ for every finite set $F \subset A$. and $u(\bigvee S) = \bigwedge u(S)$ for every set $S \subset A$*

Proof (a) That the operation $c(a)$ is a nucleus follows from basic properties of a distributive lattice By 5.4(e), $(c,d) \in \Theta(0,a)$ iff $a \vee c = a \vee d$, and so $c(a)$ is associated to $\Theta(0,a)$ by 13.2(a) We have $c(a)x = x$ iff $a \vee x = x$ iff $a \leq x$, and so $\mathrm{fix}\,c(a) = \uparrow a$.
(b) That $u(b)$ is a nucleus follows from basic properties of a Heyting algebra By 5 4(e). $(c,d) \in \Theta(b,1)$ iff $b \wedge c = b \wedge d$, and this is equivalent, by 8 3, to $b \to c = b \to d$, so $u(b)$ is associated to $\Theta(b,1)$ If $x \leq b$, then $b \wedge (b \to x) = b \wedge x = x$, and if $b \to x = x$, then $b \to (b \wedge x) = b \to x = x$ Thus, since both $b \wedge -$ and $b \to -$ are monotone, they constitute the required order-isomorphism
(c) Suppose that $a \leq b$ Then by (a), (b), and (1), the nucleus $c(b) \wedge u(a)$ is associated to the principal congruence $\Theta(a,b)$, and so for any $\theta \in \mathrm{Con}\,A$, $j_\theta a = j_\theta b$ iff $(a,b) \in \theta$ iff $\Theta(a,b) \leq \theta$ iff $c(b) \wedge u(a) \leq j_\theta$ Since every $j \in \mathrm{N}A$ is j_θ for some $\theta \in \mathrm{Con}\,A$, this proves the first equivalence. Since $a \leq b$, the second equivalence follows from 13.2(iii). Finally, since $c(b)$ and $u(a)$ correspond to $\Theta(0,a)$ and $\Theta(b,1)$, the last part of (c) follows from the first
We prove (d) and (e) together. Starting with (e), if $S \subset A$, then the pointwise join of the nuclei $c(S)$ is clearly given by $c(\bigvee S)$, which. since it is a nucleus, must be the join of $c(S)$ in NA. Similary, if $F \subset A$ is finite, then the (pointwise) meet of $c(F)$, by the dual distributive law, is $c(\bigwedge F)$. Thus c is a frame morphism. It is obviously mono since, for instance, $c(a)0 = c(b)0$ iff $a = b$.
Now back to (d), we have $\Theta(0,a) \wedge \Theta(a.1) = \Theta(a,a)$ by 5.4(f), and so $c(a) \wedge u(a) = 0$; and we have $(0,1) \in \Theta(0,a) \circ \Theta(a,1) \subset \Theta(0,a) \vee \Theta(a,1)$, and so $c(a) \vee u(a) = 1$.

Thus $c(a)$ and $u(a)$ are complementary. Since c is a frame monomorphism a and b are complementary in A iff $c(a)$ and $c(b)$ are complementary in NA. But complements are unique and so this is iff $c(b) = u(a)$, or $c(a) = u(b)$, completing the proof of (d).

To finish off (e), we observe that any two frame morphisms out of NA that agree on $c(A)$ must also agree on $u(A)$, since by (d) these are the complements of the elements $c(A)$. Therefore, they must agree on all of NA, since this frame is generated by $c(A) \cup u(A)$. Thus $c: A \to NA$ is epi. If c is an isomorphism in particular onto, then for every $a \in A$, $u(a) = c(b)$ for some $b \in A$, and so a is complemented by (d). Hence A is Boolean. Conversely, if every element of A is complemented, then $u(A) = c(A)$ and so NA is generated by $c(A)$ alone. But $c(A)$ is already a subframe of NA, and so $c(A) = NA$ and c is an isomorphism.

(f) By (c), $u(a) \leq j$ iff $j = 1$. But j preserves finite meets, so for any finite set F, $\bigvee u(F) \leq j$ iff $\forall a \in F$ $u(a) \leq j$ iff $\forall a \in F$ $ja = 1$ iff $j(\bigwedge F) = 1$ iff $u(\bigwedge F) \leq j$. Thus $\bigvee u(F) = u(\bigwedge F)$ by Yoneda (3.3). For the other part, we simply recall that meets of nuclei are pointwise and apply 7.2(o).

15.2. Discussion. The nuclei $c(a)$ are called *closed*, and the $u(a)$ are called *open*, because they correspond, through the duality $\mathbf{Loc} = \mathbf{Frm}^{op}$, to open and closed sublocales of the locale A. Note that, since

$$a \to (x \to y) = (a \wedge x) \to y = (a \wedge (a \to x)) \to y = (a \to x) \to (a \to y),$$

the nucleus $u(a)$ preserves arrow, we already know (7.2(n)) that it preserves meets. Thus, since arrow and meets are the same in $A_{u(a)}$ and A (13.5), the quotient map $u(a) : A \to A_{u(a)}$ is open in the sense of 10.3. Conversely, it can be shown that if $j : A \to A_j$ is open, then $j = u(a)$ for some $a \in A$ (in fact, $a = \bigwedge\{x \in A : jx = 1\}$).

By (a) and (b), the intervals $[b, 1]$ and $[0, a]$ are quotients of A, with quotient maps $x \mapsto b \vee x$ and $x \mapsto a \wedge x$. By (c), the fact that every congruence is a join of principal congruences can be expressed with the formula

$$j = \bigvee\{c(ja) \wedge u(a) : a \in A\} \tag{1}$$

15.3. Quasiclosed nuclei. By 15.2(1), every nucleus is generated from "below" by open and closed nuclei. We now look at nuclei that generate from "above". Recall from Proposition 9.18 that the operator $q_1(a, a)$, where $q_1(a, a)(x) = x \to a$, is a nucleus. We denote this nucleus more simply by $q(a)$, and, following Johnstone [25], we call such nuclei *quasi-closed*. The basic properties of quasi-closed nuclei are given in the following result.

Proposition. *Let j be a nucleus on A and $a \in A$.*
(a) $\operatorname{fix} q(a) = \{x \to a : x \in A\}$.
(b) $j \leq q(a)$ *iff* $ja = a$.
(c) $j = \bigwedge\{q(a) : ja = a\}$.
(d) A_j *is Boolean iff* $j = q(j0)$.

Proof. Every element $x \to a$ is a fixedpoint of $q(a)$ by 7.2(l). Conversely, if $a \to b = b$, then $b = x \to a$ for $x = a \to b$, proving (a). Part (b) is just (1) of Proposition 9.18. Since nuclei are determined by their fixedpoint sets (cf., 13.2), a simple argument derives (c) from (b) For (d), we recall that a frame is Boolean iff every one of its elements is regular (7 5) Thus, by 13.5, A_j is Boolean iff $(ja \to j0) \to j0 = ja$ for all $a \in A$ But $ja \to j0 = a \to j0$ by 9 11, and so (c) follows

15.4. Discussion. It follows from part (a) of the Proposition that $\operatorname{fix} q(a)$ is the smallest maxset containing A and therefore (b), read in terms of fixedpoints, says something obvious. $a \in \operatorname{fix} j$ iff $\operatorname{fix} q(a) \subset \operatorname{fix} j$. Similarly, (c) becomes an obvious statement about (unions of) fixedpoint sets. Also. note that (c) is the "completeness" result for nuclei, analogous to the results of 9 21, that was alluded to in 9.18 Part (d) says that the Boolean quotients of A are precisely the quotients $A_{q(a)}$, for $a \in A$. since $q(a)0 = a$ And since every frame quotient of a Boolean frame is Boolean (as frame morphisms preserve complements), it therefore follows that the set

$$Q = \{q(a) \quad a \in A\} \tag{1}$$

is an up-closed subset of $\mathsf{N}A$

A special quasi-closed nucleus is *double-negation*, $q(0)$, so-called because $q(0)a = \neg\neg a$ We often write $A_{\neg\neg}$ for $A_{q(0)}$ Note that, since $A_{c(a)} \cong \uparrow a$, it follows from 13.5 that $(A_{c(a)})_{\neg\neg} = A_{q(a)}$ (apparently explaining the term "quasi-closed") A nucleus j is called *dense* if $j0 = 0$. It follows from part (b) that j is dense iff $j \leq q(0)$

15.5. Calculation with special nuclei. We now collect together the rules we will be using for calculations involving closed, open, and quasi-closed nuclei

Proposition. *Let $j \in \mathsf{N}A$ and $a \in A$.*
(a) $j \vee c(a) = j \circ c(a)$
(b) $u(a) \vee j = u(a) \circ j$.
(c) $j \to q(a) = q(ja \to a)$.
(d) $j \vee q(a) = q(ja \to a)$.

Proof. For any nuclei j_1, j_2, and k, if $j_1 \leq k$ and $j_2 \leq k$. then $j_1 \circ j_2 \leq k$ by 9.2(b). Thus, if one can show that $j_1 \circ j_2$ is a nucleus (in fact just idempotent), then $j_1 \circ j_2 = j_1 \vee j_2$

Now, since $a \vee j(a \vee x) \leq j(a \vee x)$, we have $j(a \vee j(a \vee x)) \leq j(a \vee x)$ by 13.2(iii), and so $j \circ c(a)$ is idempotent. proving (a) And since, by extensionality of j, $a \to j(a \to jx) = a \to jjx = a \to jx$, $u(a) \circ j$ is idempotent, proving (b).

To prove (c), let $k \in \mathsf{N}A$ be arbitrary Then $k \leq j \to q(a)$ iff $k \wedge j \leq q(a)$ iff $(k \wedge j)a \leq a$, by 15 3(b) But

$$(k \wedge j)a = ka \wedge ja = k(ja \to a) \wedge ja,$$

by extensionality of k, and so, continuing the chain of equivalences, $(k \wedge j)a \leq a$ iff $k(ja \to a) \leq ja \to a$ iff $k \leq q(ja \to a)$, again by 15.3(b). Thus (c) follows from Yoneda (3.3).

For (d), we observe that, since $j \vee q(a) \geq q(a)$ and Q is up-closed (15.4), $j \vee q(a) = q(x)$ for *some* x; *we must only show that* $x = ja \to a$. First, since $ja \to a \in \operatorname{fix} j$ by 9 22.1, $j \leq q(ja \to a)$. And since $(ja \to a) \to a = ja \to a$, $q(a) \leq q(ja \to a)$ by 15.3(b), and thus $j \vee q(a) \leq q(ja \to a)$ Conversely, we need to show that if $j \vee q(a) \leq q(x)$, then $ja \to a \leq x$, this will suffice since $x \leq y$ whenever $q(x) \leq q(y)$ (which means

$y \twoheadrightarrow x = y$ by 15 3(b)) So suppose x is such that $jx = x$ and $x \twoheadrightarrow a = x$ Then since $a \leq x$ and both j and $- \twoheadrightarrow a$ are monotone, we have

$$ja \twoheadrightarrow a \leq jx \twoheadrightarrow a = x \twoheadrightarrow a = x,$$

as required

15.6. Some higher identities. Some of the nucleic identities we will need later involve the *second assembly* of A, namely $\mathsf{N}^2A = \mathsf{N}(\mathsf{N}A)$. Elements of this frame are nuclei on $\mathsf{N}A$, and we denote them with capital letters $J, K, \ldots$ First we prove a Lemma, which should be thought of as a continuation of 7.4

Lemma. *Let j be a nucleus on a frame A, and let $b \in A$ be Boolean Then $jb = b \vee j0$*

PROOF We clearly have $b \leq jb$ and $j0 \leq jb$, and so $b \vee j0 \leq jb$ Conversely by 7 4(a), $jb \leq b \vee j0$ iff $(b \to 0) \wedge jb \leq j0$ But this follows from the extensionality of j, since $(b \to 0) \wedge jb = (b \to 0) \wedge j0 \leq j0$.

15.7. Propostion. *Let $a \in A$ and $J \in \mathsf{N}^2A$. Then the following hold*

(a) $(J0)a = (Jc(a))0$
(b) $c(c(a)) = u(u(a))$
(c) $c(u(a)) = u(c(a))$
(d) $q(q(a)) = c(q(a))$

PROOF (a) Since $c(a)$ is a Boolean element of $\mathsf{N}A$ by 15 1(d), we can use Lemma 15 6 and 15 5(a) to get

$$(Jc(a))0 = (J0 \vee c(a))0 = (J0 \circ c(a))0 = (J0)(c(a)0) = (J0)a$$

(b,c) Since $c(a)$ and $u(a)$ are complements of one another by the first part of 15.1(d), both (b) and (c) follow directly from the second part of 15 1(d)

(d) For any $j \in J$, we have, by 15 5(c),

$$q(q(a))j = (j \to q(a)) \to q(a) = q(ja \to a) \to q(a) = q((a \twoheadrightarrow (ja \to a)) \to a)$$

Now, by 9 7.1, $(a \twoheadrightarrow (ja \to a)) \to a = (ja \to (a \twoheadrightarrow a)) \to a = (ja \to a) \to a$ But, by 15.5(d), $q(ja \twoheadrightarrow a) = j \vee q(a) = c(q(a))j$, completing the proof

16. The functor N and its universal property

16.1. Recall from 5 3 that for every frame morphism $f \colon A \to B$, there is an adjunction $\Theta \circ (f \times f) \dashv (f \times f)^{-1}$ between $\mathrm{Con}\,A$ and $\mathrm{Con}\,B$, which we will write here as $\mathsf{N}f \dashv (\mathsf{N}f)_*$, the left adjoint of which takes a congruence on A to the congruence generated by its image under f in B, and the right adjoint of which takes a congruence on B to its inverse image under f We also write $\mathsf{N}f \dashv (\mathsf{N}f)_*$ for the associated adjunction between $\mathsf{N}A$ and $\mathsf{N}B$ and between $\mathrm{Max}\,A$ and $\mathrm{Max}\,B$

Proposition. *For every frame morphism $f : A \to B$, the map $\mathsf{N}f : \mathsf{N}A \to \mathsf{N}B$ is also a frame morphism, and in this way N becomes a functor* **Frm** $\to$ **Frm**. *For every $a \in A$ we have*

$$(\mathsf{N}f)(c(a)) = c(f(a)) \quad \text{and} \quad (\mathsf{N}f)(u(a)) = u(f(a)). \tag{1}$$

In particular, the morphisms $c_A : A \to \mathsf{N}A$ are components of a natural transformation from the identity functor to N.

PROOF Since $(\mathsf{N}f)(\Theta(a,b)) = \Theta(f(a), f(b))$, and since f preserves 0 and 1, the equations of (1) are clear by 15.1(a,b). Also, since f preserves binary meets and joins, the formula 5.4(f) shows that $\mathsf{N}f$ preserves binary meets of principal congruences. But every congruence is a join of principal congruences, and so the meet of any two congruences can be reduced, via frame distributivity, to a join of a set of binary meets of principal congruences. Now $\mathsf{N}f$ preserves joins, since it is a left adjoint, and it preserves binary meets of principal congruences, as observed above; thus it follows that $\mathsf{N}f$ preserves all finite meets, and hence is a frame morphism. It is straightforward to verify that $f \mapsto \mathsf{N}f$ is a functor.

The statement about c being a natural transformation follows from the first equation of (1), which is exactly the naturality condition.

16.2. The universal property. The frame $\mathsf{N}A$ and the embedding $c_A : A \to \mathsf{N}A$ can be characterized independently of representation by the ("universal") property that every element of A becomes *freely complemented* in $\mathsf{N}A$. Since it doesn't involve any extra effort to prove a more general theorem, we construct the universal complementation of an arbitrary subset $S \subset A$; the stated result about $\mathsf{N}A$ then follows by taking $S = A$.

Theorem. *Suppose A is a frame and $S \subset A$. Let $\mathsf{N}_S A$ be the subframe of $\mathsf{N}A$ generated by $c(A) \cup u(S)$, and note that $c_A : A \to \mathsf{N}_S A$. Then, given a frame morphism $f : A \to B$ such that $f(s)$ is complemented for every $s \in S$, there exists a unique frame morphism $\overline{f} : \mathsf{N}_S A \to B$ such that $f = \overline{f} \circ c_A$.*

PROOF Since every element of $\mathsf{N}_S A$ is generated by images of elements of A and (some of) their complements, the morphism $c_A : A \to \mathsf{N}_S A$ is epi, as in 15.1(e). Thus, the required morphism $\overline{f}$, if it exists, is unique. To show existence, consider the morphism $\mathsf{N}f : \mathsf{N}A \to \mathsf{N}B$. By 16.1(1), we have $(\mathsf{N}f)(c(a)) = c(f(a))$ for all $a \in A$; and since every element $f(s)$ is complemented in B, with complement $\neg f(s)$, we have $(\mathsf{N}f)(u(s)) = u(f(s)) = c(\neg f(s))$ by 15.1(d). Thus, the image of the generators of $\mathsf{N}_S A$ under $\mathsf{N}f$ is contained in $c_B(B)$, which is a subframe of $\mathsf{N}B$ (and is isomorphic to B, since c_B is mono). Hence $(\mathsf{N}f)(\mathsf{N}_S A) \subset c_B(B)$, and we can take $\overline{f} = c_B^{-1} \circ (\mathsf{N}f)|_{\mathsf{N}_S A}$.

16.3. Although we have defined $\mathsf{N}f$ and $(\mathsf{N}f)_*$ in terms of congruences, these operations have nice descriptions in terms of nuclei and maxsets, as well. Recall (13.9) that core M, for a meet-closed subset $M \subset A$, is the largest maxset contained in M.

Proposition. *Suppose $f : A \to B$ is a frame morphism. Then,*
(a) *$(\mathsf{N}f)(j)$, for $j \in \mathsf{N}A$, is the smallest nucleus k on B for which $f(ja) \le kf(a)$*

for all $a \in A$.

(b) $(\mathrm{N}f)_*(k) = [k]$ *for every* $k \in \mathrm{N}B$, *where* $[k]$ *is the nucleus on* A *defined by*

$$[k]a = f_*(kf(a)) \qquad (a \in A); \tag{1}$$

(c) $(\mathrm{N}f)(M) = \mathrm{core}\,(f_*)^{-1}(M)$ *for every* $M \in \mathrm{Max}\,A$ *and*

(d) $(\mathrm{N}f)_*(N) = f_*(N)$ *for every* $N \in \mathrm{Max}\,B$

PROOF (a) Given $j \in \mathrm{N}A$, consider $\mathrm{N}f$ on the congruence $\theta \in \mathrm{Con}\,A$ associated to j. Clearly, $\theta = \Theta(\{(a, ja) \mid a \in A\})$, and so $(\mathrm{N}f)(\theta) = \Theta(\{(f(a), f(ja)) \mid a \in A\})$. But this is easily seen to imply (a)

(b) Given $k \in \mathrm{N}B$ we again consider the associated congruence $\psi \in \mathrm{Con}\,B$. The nucleus $(\mathrm{N}f)_*(k)$ takes $a \in A$ to the largest $b \in A$ such that $(a, b) \in (\mathrm{N}f)_*(\psi) = (f \times f)^{-1}(\psi)$, or equivalently $(f(a), f(b)) \in \psi$. But if $a \leq b$, then $(f(a), f(b)) \in \psi$ exactly when $f(b) \leq kf(a)$ or by adjointness, $b \leq f_*(kf(a))$. The largest such b is, of course, $f_*(kf(a))$, proving (b) (Note that it follows from this line of argument that $[k]$ actually *is* a nucleus on A.)

We prove (d) before (c) Note that (d) will follow from (b) if we can show that, for every $k \in \mathrm{N}B$, $f_*(\mathrm{fix}\,k) = \mathrm{fix}\,[k]$. Suppose $kb = b$. Then, since f_* and k are montone and $f(f_*(b)) \leq b$ by adjointness

$$[k]f_*(b) = f_*(kf(f_*(b))) \leq f_*(kb) = f_*(b),$$

showing that $f_*(b) \in \mathrm{fix}\,k$. Conversely, suppose that $[k]a = a$, i e, $f_*(kf(a)) = a$. But $kf(a) \in \mathrm{fix}\,k$, and so $a \in f_*(\mathrm{fix}\,k)$, as required (Again, it follows from this line of argument that $f_*(N)$ is a maxset on A. For a direct proof, use 10 7.)

For (c), let $M \in \mathrm{Max}\,A$, and observe that $(\mathrm{N}f)(M)$ is the least $N \in \mathrm{Max}\,B$ with $M \leq (\mathrm{N}f)_*(N)$, i.e., using (d), $M \supset (\mathrm{N}f)_*(N) = f_*(N)$. This is equivalent to the condition $N \subset (f_*)^{-1}(M)$, and since M is meet-closed and f_* preserves meets, it follows that $(f_*)^{-1}(M)$ is meet-closed, and so the required N, by 13 9(c), is $\mathrm{core}\,(f_*)^{-1}(M)$

16.4. We finish this section with a few preservation properties of the functor N. Recall that a nucleus $j \in \mathrm{N}A$ is dense if $j0 = 0$. We say that a frame morphism $h : A \to B$ is *dense* if $h_*(0) = 0$, or equivalently, $h(a) = 0$ implies $a = 0$. Thus h is dense just in case the nucleus associated to $\ker h$ is dense

Proposition. *Suppose that* $f : A \to B$ *is a frame morphism. Then,*

(a) *If* f *is onto, then* $\mathrm{N}f$ *is onto,*

(b) *If* f *is epi, then* $\mathrm{N}f$ *is epi, and*

(c) f *is mono if and only if* $\mathrm{N}f$ *is dense*

Also, the functor $\mathrm{N} : \mathbf{Frm} \to \mathbf{Frm}$ *preserves products*

PROOF (a) If f is onto, then by 16 1(1), $(\mathrm{N}f)(c(A)) = c(B)$ and $(\mathrm{N}f)(u(A)) = u(B)$. Since $\mathrm{N}B$ is generated by $c(B) \cup u(B)$, this proves that $\mathrm{N}f$ is onto

(b) The preservation of epis is elementary category theory: if f is epi, then since c_B is epi, we have that $c_B \circ f = \mathrm{N}f \circ c_A$ is epi, and so $\mathrm{N}f$ is epi

(c) If $\Delta = 0$ is the identity congruence on B, then obviously

$$(\mathrm{N}f)_*(\Delta) = (f \times f)^{-1}(\Delta) = \ker f.$$

which is 0 iff f is mono

Finally, by 11.11, congruences on frame products are products of congruences on the factors In addition to this, we need only observe that the projection from a product of congruences to one of the factors ($\theta \mapsto \theta_i$ in the proof of 11 10) is the same as the direct image of the congruence under the projection mapping (i e , $(\mathsf{N}\pi_i)(\theta) = \theta_i$)

17. The assembly tower of a frame

17.1. The construction. Let A be a frame We construct an ordinal sequence $\{\mathsf{N}^\alpha A : \alpha \in \mathbb{O}\}$ of frames and a doubly-indexed ordinal sequence $\{c^\alpha_\beta : \mathsf{N}^\beta A \to \mathsf{N}^\alpha A \mid \beta, \alpha \in \mathbb{O}, \beta \leq \alpha\}$ of frame morphisms, together called the *assembly tower* of A, by simultaneous ordinal recursion For the frames, we set

$$\begin{aligned} \mathsf{N}^0 A &= A, \\ \mathsf{N}^{\alpha+1} A &= \mathsf{N}(\mathsf{N}^\alpha A), \quad \text{and} \\ \mathsf{N}^\lambda &= \operatorname*{colim}_{\alpha<\lambda} \mathsf{N}^\alpha A, \quad \text{if } \lambda \text{ is a limit,} \end{aligned}$$

where more explicitly, the diagram D over which the colimit is taken is indexed by the ordinal λ and has $D\alpha = \mathsf{N}^\alpha A$ for $\alpha < \lambda$ and $D(\beta \to \alpha) = c^\alpha_\beta$ for $\beta \leq \alpha \leq \lambda$. For the morphisms, we set

$$\begin{aligned} c^\alpha_\alpha &= \mathrm{id}_{\mathsf{N}^\alpha A}, \\ c^{\alpha+1}_\beta &= c_{\mathsf{N}^\alpha A} \circ c^\alpha_\beta, \quad \beta \leq \alpha, \\ c^\lambda_\beta &= \text{the canonical injection } \mathsf{N}^\beta A \to \mathrm{colim}_{\alpha<\lambda} \mathsf{N}^\alpha A, \\ &\quad \text{when } \lambda > \beta \text{ is a limit, and} \\ c^\alpha_\lambda &= \text{the mediating morphism determined by } \{c^\alpha_\beta \ \ \beta < \lambda\}, \\ &\quad \text{when } \lambda < \alpha \text{ is a limit.} \end{aligned}$$

By induction, note that, for all $\gamma \leq \beta \leq \alpha$, $c^\alpha_\gamma = c^\alpha_\beta \circ c^\beta_\gamma$. When more than one frame is being discussed, it may be necessary to add an extra subscript, as in $(c_A)^\alpha_\beta$ and $(c_B)^\alpha_\beta$, to distinguish between the morphisms. On the other hand, we write c^α for c^α_0 and (as usual) c for c^1_0 whenever possible

If $f : A \to B$ is a frame morphism, then we define $\mathsf{N}^\alpha f : \mathsf{N}^\alpha A \to \mathsf{N}^\alpha B$ by recursion as follows: $\mathsf{N}^0 f = f$, $\mathsf{N}^{\alpha+1} f = \mathsf{N}(\mathsf{N}^\alpha f)$, and if λ is a limit, then $\mathsf{N}^\lambda f$ is the mediating morphism determined by $\{(c_B)^\lambda_\alpha \circ \mathsf{N}^\alpha f \ \ \alpha < \lambda\}$. It is easy to see by induction (using the universal property of the colimit at limit stages) that each N^α becomes a functor $\mathbf{Frm} \to \mathbf{Frm}$. We can again show by induction that if $\beta \leq \alpha$, then

$$\mathsf{N}^\alpha f \circ (c_A)^\alpha_\beta = (c_B)^\alpha_\beta \circ \mathsf{N}^\beta f, \tag{1}$$

and so c^α_β becomes a natural transformation $\mathsf{N}^\beta \to \mathsf{N}^\alpha$. We also note that

$$(c_{\mathsf{N}^\gamma A})^\alpha_\beta = (c_A)^{\gamma+\alpha}_{\gamma+\beta} \qquad \text{and} \qquad \mathsf{N}^\gamma((c_A)^\alpha_\beta) = (c_A)^{\alpha+\gamma}_{\beta+\gamma} \tag{2}$$

17.2. Here are a few more facts about the assembly tower of a frame, also, of course, established by induction

Proposition. *Suppose A is a frame and the assembly tower of A is constructed as above Then the following hold·*

(a) *For all ordinals α, β with $\beta \leq \alpha$, the morphism c^α_β is both mono and epi*

(b) *For every morphism $f \colon A \to B$, where B is a Boolean frame, and every ordinal α, there exists a unique frame morphism $\overline{f} \colon N^\alpha A \to B$ such that $f = \overline{f} \circ c^\alpha$*

PROOF (a) By the first equation of (2) above, we can assume that $\beta = 0$ For any A, c_A is mono and epi by 15 1(e), and compositions of monos or epis are likewise mono or epi Thus (a) will follow by induction if we can show that limit-ordinal-indexed colimits of monos and epis are mono and epi For monos this is just Corollary 12 8 For epis, this follows from the universal property of the colimit in detail, if λ is a limit ordinal and $D \colon \lambda \to \mathbf{Frm}$ is a diagram with colimit cone $\nu \colon D \to \Delta\,\mathrm{colim}_\lambda D$, and if $f, g \colon \mathrm{colim}_\lambda D \to A$ are morphisms such that $f \circ \nu 0 = g \circ \nu 0$, then for every ordinal $\alpha < \lambda$, we have

$$f \circ \nu\alpha \circ D(0 \to \alpha) = f \circ \nu 0 = g \circ \nu 0 = g \circ \nu\alpha \circ D(0 \to \alpha),$$

and thus, since $D(0 \to \alpha)$ is epi, $f \circ \nu\alpha = g \circ \nu\alpha$ Setting $\tau\alpha \colon D\alpha \to A$ to be this common value results in a cone $\tau \colon D \to \Delta A$, and since both f and g are mediating morphisms with respect to τ, we have $f = g$ by the universal property of the colimit

(b) Each c^α is epi by (a), so any such $\overline{f}$ will be unique Morphisms $\overline{f}_\alpha \colon N^\alpha A \to B$, $\alpha \in \mathbf{O}$, can be constructed by recursion on α by setting $\overline{f}_0 = f$, using the universal property of N (16 2) for all successor ordinals (since every element of B is complemented), and using the mediating morphism from the colimit for all limit ordinals Uniqueness is guaranteed by the universal properties of N and colimits.

17.3. The reflection problem. Since **cBa** is a subcategory of **Frm**, 17.2(b) implies that, for a frame A, if $N^\alpha A$ is Boolean for some α, then $N^\alpha A$ (along with the morphism c^α) is a *reflection of A into* **cBa** In this case, we say simply that A "has a reflection" Gaifman [9] and Hales [14] showed independently that (in our terminology) the free complete Boolean algebra-class (with unary negation and joins of every arity) on a countably infinite set is a proper class. Thus, since the reflection of $F_\infty(\omega)$, if it existed, would be the free cBa on ω, the frame $F_\infty(\omega)$ has no reflection, and its assembly tower grows arbitrarily large The *reflection problem* is to characterize those frames with reflections

17.4. Exercises.

17.4.1. Use 16 4 and Exercise 12 9.1 to show that, for every ordinal α, the functor N^α preserves products

18. The assembly tower for κ-frames

18.1. The κ-assembly. Throughout this section, we let κ be a fixed regular cardinal

The universal property of N on **Frm** can be easily generalized to κ-**Frm** Thus, for a κ-frame A, we let d_A $A \to \mathsf{B}A$ be the result in κ-**Frm** of freely complementing the elements of A This can be constructed as a quotient of a free extension of A by new elements $\{a'$ $a \in A\}$, where we divide by (the congruence determined by) equations saying that a' is the complement of a A concrete description of this κ-frame is given by the following result, which moreover shows the close relation B has to N.

Proposition. *Suppose A is a κ-frame. Then* Con A *is a frame and* $\mathsf{B}A$ *is the sub-κ-frame of* Con A *generated by the principal congruences The morphism* d_A $A \to \mathsf{B}A$ *is given by* $d_A(a) = \Theta(0\ a)$ *and for any morphism* f $A \to B$, $\mathsf{B}f$ *is the restiction to* $\mathsf{B}A$ *of the (join-preserving) function* Con $A \to$ Con B *taking a congruence on A to the congruence generated by its image (under $f \times f$) on B. The morphisms* $d_A : A \to \mathsf{B}A$ *are mono and epi and are the components of a natural transformation from the identity functor to* B

Proof See [32], 5 1 and 5.2. The statements not proved there may be proved entirely analogously to the corresponding statements about N.

18.2. The κ-assembly tower. Just as with the assembly tower for frames, we can iterate the functor B to produce an ordinal sequence $\{\mathsf{B}^\alpha \cdot \alpha \in \mathbb{O}\}$ of functors and a doubly-indexed sequence $\{d^\alpha_\beta$ $\mathsf{B}^\beta \to \mathsf{B}^\alpha \mid \beta, \alpha \in \mathbb{O}, \beta \leq \alpha\}$ of natural transformations, and this sequence will have properties analogous to those of 17.1(1,2) and 17.2. However, when it comes to the reflection problem, there is an important difference: since the ordinal κ (by regularity) is κ-filtered as a category and κ-**Frm** is locally κ-presentable, the colimit used to construct $\mathsf{B}^\kappa A$ is just the union of the $\mathsf{B}^\alpha A$ for $\alpha < \kappa$ (assuming that we identify each κ-frame in the tower with its image under d) Thus the result is a Boolean κ-frame, and it follows that the full subcategory κ-**cBa** of κ-complete Boolean algebras is reflective, with reflection functor B^κ.

We mention one additional connection with the functor N.

Proposition. *Let A be a frame. Then for every ordinal α there is a unique κ-frame morphism* $e^\alpha_A : \mathsf{B}^\alpha A \to U^\kappa_\infty \mathsf{N}^\alpha A$ *such that* $U^\kappa_\infty c^\alpha_A = e^\alpha_A \circ d^\alpha_A$.

We remark that e^α_A is proved to be a monomorphism in 28.6.

Proof. The morphism e^α_A is constructed by recursion, as follows We start with $e^0_A = \mathrm{id}_A$. Next, if e^α_A $\mathsf{B}^\alpha A \to U^\kappa_\infty \mathsf{N}^\alpha A$ is defined, then every element in $\mathsf{B}^\alpha A$ becomes complemented in $U^\kappa_\infty \mathsf{N}^{\alpha+1} A$ via the morphism $U^\kappa_\infty c^{\alpha+1}_\alpha \circ e^\alpha_A$, and so we take $e^{\alpha+1}_A$ to be the unique morphism guaranteed by the universal property of B. Finally, for limit λ, we define e^λ_A to be the mediating morphism corresponding to the cone $\{U^\kappa_\infty c^\lambda_\alpha \circ e^\alpha_A \cdot \alpha < \lambda\}$

18.3. The category κ-cBa. We have just seen that κ-**cBa** is a reflective subcategory of κ-**Frm** The advantage of this stems from the following result of Lagrange [28].

Theorem. *In the category* κ-**cBa**,
(a) *the pushout of a monomorphism along any morphism is mono, and*
(b) *every epi is surjective*

18.4. Foundations: ∞-Frm and ∞-cBa. We recall from 6 6 that we may assume the existence of an inaccessible cardinal ∞ (equivalently, we may assume the existence of a single universe U, and let $\infty = |U|$) and restrict our attention to small frames, i e , those of cardinality less that ∞, thereby making **Frm** a full subcategory of ∞-**Frm** In addition to making possible a uniform treatment of frames and κ-frames, we also have the fact that ∞-**Frm** contains ∞-**cBa** as a reflective subcategory (whereas **cBa** is not reflective in **Frm**), and this allows us to bypass an excursion through κ-**Frm**, and the choice of a sufficiently large regular cardinal, to give a more straightforward proof of Theorem 28 5 in Chapter 8

However, since the only properties of U (or ∞) we use are of the kind axiomatizable using the predicate S, the work of Feferman shows that anything we prove about frames using these assumptions and restrictions has a proof without them Thus, the use of ∞-**Frm** is really a matter of convenience, and, to illustrate this point, we give in 28 6 an alternate proof of Theorem 28 5 without any extra assumptions or restrictions

Chapter 6
REGULAR OPERATORS

This chapter builds on the material in Chapter 3 on extensional operators using the information on frames and the assembly tower of a frame, as developed in Chapters 4 and 5. After the basic definitions and examples in Section 19, and the observation that the set RA of regular operators on a frame A is a cBa, Section 20 introduces the regularity ordering and stable sets, which are the key to understanding the properties of regular operators and developing their applications These notions are thoroughly studied in this section and then, in Section 21, used to establish the many properties of regular operators In Section 22, fixpoints sets of regular operators are characterized as being the complete filters in the regularity ordering, and this characterization, in addition to being a culmination of work on fixedpoint sets of logical operators started in Chapter 3, is also the base for establishing the relation between regular operators and the assembly tower, which is the subject of Section 23. This section begins by establishing a formula for joins of nuclei, and then proceeds to show that RA is isomorphic to the double-negation quotient of the second assembly N^2A. In this way, a canonical Booleanization of A is given a concrete description by operators on A The rest of the section is devoted to giving a formula for double negation on N^2A in terms of operators, and then using this formula to show that double negation is an open quotient Finally, in Section 24, it is shown how RA is a limit of a diagram consisting of all of the Boolean quotients of A

19. Extensional, logical, and regular operators on frames

19.1. Definitions. Throughout this chapter, A will denote a fixed frame.

Recall that an *extensional operator* on A is a function $l : A \to A$ satisfying the equivalent conditions of Proposition 9.4, and that a logical operator is an inflationary extensional operator As explained in 9.17, it follows from 9.13 and 9.20 that the class of extentional operators on A, ordered pointwise, has pointwise meets, joins, and arrow, and hence is a complete Heyting subalgebra of the cartesian power A^A. By definition, an extensional operator l is logical if $\mathrm{id}_A \leq l$, and therefore the logical operators also form a frame, being the closed quotient $[\mathrm{id}_A, 1]$ (cf., 15.2) of the frame of extensional operators Meets and arrow in the frame of logical operators are the same as those of the frame of extensional opearators by 13.5, i.e., pointwise. (Non-empty joins are also pointwise, since it is a *closed* quotient.)

We define a *regular operator* on A to be a regular element of the frame of logical operators, and let RA be the set of regular operators on A, ordered pointwise Thus, a regular operator r satisfies $(r \to \mathrm{id}_A) \to \mathrm{id}_A = r$, or

$$ra = (ra \to a) \to a \qquad (a \in A) \tag{1}$$

It follows immediately from 15.3(d) that RA is a cBa, and (again from 13.5) the meet and arrow operations on RA are pointwise.

19.2. Examples of regular operators. Since $(x \rightarrow a) \twoheadrightarrow a = x \rightarrow a$ for any elements x and a of a frame A, the following is an easy consequence of the definitions

Proposition. *Let l be an operator on A, and define operators l' and $\bar{l}$ by*

$$l'(x) = lx \rightarrow x \qquad (x \in A)$$

and

$$\bar{l}(x) = l''(x) = lx \twoheadrightarrow x \qquad (x \in A)$$

If l is extensional, then l' and $\bar{l}$ are regular operators

We note that the converse is also true, because every regular operator r has the forms l_1' and $\bar{l}_2$ for the extensional operators $l_1 = r'$ and $l_2 = r$, since $(r')' = \bar{r} = r$ by definition

19.3. Corollary. *For any $a \in A$, the following are regular operators*
(a) $u(a)x = a \rightarrow x$
(b) $\bar{c}(a)x = (a \vee x) \twoheadrightarrow x = a \twoheadrightarrow x$.
(c) $q'(a)x = (x \rightarrow a) \rightarrow x$.
(d) $\bar{q}(a)x = (x \rightarrow a) \twoheadrightarrow x$
Note that we have written $\bar{c}(a)$, $q'(a)$, and $\bar{q}(a)$ instead of the more correct $\overline{c(a)}$, $q(a)'$, and $\overline{q(a)}$ Later, we will be treating $\bar{c}$, q', and $\bar{q}$ as functions $A \rightarrow RA$, and so this notation is more appropriate

20. The regularity ordering

20.1. Regularity. Recall that an element b of a frame A is called *regular* if $b \twoheadrightarrow 0 = b$ Our main notion is a generalization of this

Proposition. *If a and b are elements of a frame A, then the following statements are equivalent*
(a) *b is regular in $A_{c(a)} = [a, 1]$*
(b) $b \twoheadrightarrow a = b$
(c) $q(a) \leq q(b)$.

If these statements are true of a and b, then we say that b is *regular over* a, and write $b \trianglerighteq a$ or $a \trianglelefteq b$

Proof The frame $A_{c(a)} = [a\ 1]$ has a as its least element, and its arrow operation agrees with that of A by 13 5 It follows that (a) and (b) are equivalent
Next $b \twoheadrightarrow a = b$ means that b is a fixedpoint of $q(a)$, and this is equivalent to $q(a) \leq q(b)$ by 15.4(b), thus, (b) is equivalent to (c), and the proof is complete.

20.2. Here are a few simple facts concerning the relation $\trianglelefteq$

Proposition. *Suppose A is a frame and $a, b, c \in A$. Then*
(a) *$\langle A, \trianglelefteq\rangle$ is a partial order (which we denote $A^{\trianglelefteq}$) that is isomorphic to the subset $Q = \{q(a) \cdot a \in A\}$ of NA with the induced order*
(b) *$a \trianglelefteq b$ implies $a \leq b$*

(c) $a \trianglelefteq b$ *implies* $(b \to a) \to b = b$
(d) $a \trianglelefteq c$ *and* $a \leq b \leq c$ *imply* $b \trianglelefteq c$.

PROOF. Since $q(a)0 = a$ for all $a \in A$, the map $q : A \to NA$ is 1-1, so (a) follows from 20.1(c).

Suppose that $a \trianglelefteq b$ Then $b = b \twoheadrightarrow a \geq a$, and so $b \geq a$, which proves (b) Furthermore, since $a \to b = 1$, we have, by 9.7.5,

$$(b \twoheadrightarrow a) \to b = (b \twoheadrightarrow a) \wedge (a \to b) = b \twoheadrightarrow a = b,$$

which proves (c)

Finally, suppose that $a \trianglelefteq c$ and $a \leq b \leq c$. Then by the (anti-)monotonicity properties of $\twoheadrightarrow$ it follows that

$$(c \twoheadrightarrow b) \to b \leq (c \twoheadrightarrow b) \to c \leq (c \twoheadrightarrow a) \to c$$

But $(c \twoheadrightarrow a) \to c = c$ by (c), so that $c \twoheadrightarrow b \leq c$ and thus $b \trianglelefteq c$, proving (d).

20.3. Upper sets in $A^{\trianglelefteq}$. If $a \in A$, the *up-closure* of a in $A^{\trianglelefteq}$, which we denote $\Uparrow a$, is defined as usual by

$$\Uparrow a = \{b : b \trianglerighteq a\}.$$

By 15.3(a), it is clear from the definitions that $\Uparrow a$ is none other than the maxset fix $q(a)$ of A, and thus, by 15.3(d) is a cBa. Being a maxset of A, it also follows that $\Uparrow a$ is meet-closed:

$$s \trianglerighteq a \text{ for all } s \in S \quad \text{implies} \quad \bigwedge S \trianglerighteq a. \tag{1}$$

20.4. Stable sets and elements. We will call a subset $S \subset A$ *stable* if it has a lower bound in $A^{\trianglelefteq}$, i e, if there exists $a \in A$ such that $a \trianglelefteq s$ for all $s \in S$. Notice that by this definition all one-element subsets, as well as the empty subset, are stable. Given two elements $a, b \in A$, we indicate that $\{a, b\}$ is stable with the notation $a \smile b$. Finally, a subset $S \subset A$ is *pairwise stable* if $a \smile b$ for all $a, b \in S$, and *finitely stable* if every finite subset of S is stable (these two notions will be shown equivalent in 20.10).

20.5. Propostion. *Suppose $S \subset A$ is stable. Then the meet of S in $A^{\trianglelefteq}$ exists and is equal to $\bigwedge S$*

PROOF. Let s_0 be such that $s_0 \trianglelefteq s$ for all $s \in S$. Then

$$s_0 \trianglelefteq \bigwedge S \tag{1}$$

by 20.3(1) Thus it is enough that $\bigwedge S \trianglelefteq s$ for all $s \in S$. But this follows from 20.1(d) and the assumption $\forall s\ \ s_0 \trianglelefteq s$, because for any $s \in S$ we get

$$s_0 \leq \bigwedge S \leq s$$

by applying 20.2(b) to (1)

20.6. We can characterize pairwise stability as follows

Proposition. *The following statements are equivalent for elements a and b of a frame A*

(a) $a \smile b$
(b) $(a \rightarrow b) \rightarrow a = a$ *and* $(b \rightarrow a) \rightarrow b = b$
(c) $a \twoheadrightarrow b = b \twoheadrightarrow a$

PROOF By 20 5 (a) is equivalent to the conjunction of $a \trianglerighteq a \wedge b$ and $b \trianglerighteq a \wedge b$, i.e , to

$$a \twoheadrightarrow (a \wedge b) = a \quad \text{and} \quad b \twoheadrightarrow (a \wedge b) = b. \tag{1}$$

But (1) is equivalent by 9 7 2. to (b)

The equivalence of (b) and (c) is shown by the following chain of equivalences where we have used 9 7 5 between the second and third lines

$$(a \rightarrow b) \rightarrow a = a \quad \text{and} \quad (b \rightarrow a) \rightarrow b = b$$
$$(a \rightarrow b) \rightarrow a \leq a \quad \text{and} \quad (b \rightarrow a) \rightarrow b \leq b$$
$$(a \twoheadrightarrow b) \wedge (b \rightarrow a) \leq a \quad \text{and} \quad (b \twoheadrightarrow a) \wedge (a \rightarrow b) \leq b$$
$$a \twoheadrightarrow b \leq b \twoheadrightarrow a \quad \text{and} \quad b \twoheadrightarrow a \leq a \twoheadrightarrow b$$
$$a \twoheadrightarrow b = b \twoheadrightarrow a$$

20.7. The relation between $\trianglelefteq$ and $\wedge$. We formulate a general relation between $\wedge$ and $\trianglelefteq$, from which we can derive several others by instantiation

Proposition. *Suppose A is a frame and $a, b, c, d \in A$ are such that*

$$a \trianglerighteq b \quad \text{and} \quad c \trianglerighteq d. \tag{1}$$

Then

$$a \wedge c \trianglerighteq b \wedge d \quad \text{iff} \quad ((a \rightarrow d) \rightarrow a) \wedge ((c \rightarrow b) \rightarrow c) \leq a \wedge c$$

PROOF Assume (1), and note that $a \geq b \wedge d$ and $c \geq b \wedge d$ Then the result follows easily from this calculation

$$\begin{aligned}
(a \wedge c) \twoheadrightarrow (b \wedge d) &= (a \twoheadrightarrow (b \wedge d)) \wedge (c \twoheadrightarrow (b \wedge d)) && \text{by } 9\,7\,4 \\
&= ((a \rightarrow (b \wedge d)) \rightarrow a) \wedge ((c \rightarrow (b \wedge d)) \rightarrow c) && \text{by } 9.6(d) \\
&= (((a \rightarrow b) \wedge (a \rightarrow d)) \rightarrow a) \wedge (((c \rightarrow b) \wedge (c \rightarrow d)) \rightarrow c) \\
&= ((a \rightarrow d) \rightarrow ((a \rightarrow b) \rightarrow a)) \wedge ((c \rightarrow b) \rightarrow ((c \rightarrow d) \rightarrow c)) \\
&= ((a \rightarrow d) \rightarrow a) \wedge ((c \rightarrow b) \rightarrow c) && \text{by } 20\,2(c)
\end{aligned}$$

20.8. Corollaries.
(a) *If $a \trianglerighteq b$, then $a \wedge c \trianglerighteq b \wedge c$ iff $(c \rightarrow b) \rightarrow c \leq a \rightarrow c$.*
(b) *If $a \vee b \trianglerighteq a$, then $b \trianglerighteq a \wedge b$*
(c) *If $a \trianglerighteq b$ and $a \trianglerighteq c$, then $a \trianglerighteq b \wedge c$*

PROOF (a) Suppose $a \unrhd b$. Since $c \unrhd c$, we can use the Proposition to get

$$a \wedge c \unrhd b \wedge c \quad \text{iff} \quad ((a \to c) \to a) \wedge ((c \to b) \to c) \leq a \wedge c$$

The latter is equivalent by adjointness to

$$\begin{aligned}(c \to b) \to c &\leq ((a \to c) \to a) \to (a \wedge c) \\ &= (a \to (a \wedge c)) \to (a \wedge c) \qquad \text{by 9.7.2} \\ &= a \to (a \wedge c) = a \to c\end{aligned}$$

(b) We can use (a) on $a \vee b \unrhd a$ and b to obtain $b \unrhd a \wedge b$, if we can show that $(b \to a) \to b \leq (a \vee b) \to b = a \to b$ But this follows trivially from the anti-monotonicity of $- \to b$

(c) We can apply the Propostion to $a \unrhd b$ and $a \unrhd c$, to obtain $a \unrhd b \wedge c$. if we can show that

$$((a \to c) \to a) \wedge ((a \to b) \to a) \leq a$$

But both expressions on the left are equal to a by 20.2(c), so this is trivial.

20.9. Discussion. If we call an interval $[a, b]$ *regular* if $a \unlhd b$, then 20.8(b) says that the set of regular intervals is closed under downward transposition It is not, however, closed under upward transposition. let **3** be the three-element chain $(0 < 1 < 2)$ and $A = (\mathbf{3} \times \mathbf{3}) - \{(0,2)\}$ have the induced ordering. Then the interval $[(0,0),(0,1)]$ is regular, but this projects up to $[(1,0).(1,1)]$, which is not regular.

If, in analogy with 20 3, we define

$$\Downarrow a = \{c \,.\, a \unrhd c\},$$

then 20.8(c) and 20.2(d) say that $\Downarrow a$ is a filter in $[0, a]$

(a) $a \in \Downarrow a$,
(b) $c \in \Downarrow a$ and $c \leq b \leq a$ imply $b \in \Downarrow a$; and
(c) $c, d \in \Downarrow a$ imply $c \wedge d \in \Downarrow a$

20.10. The stability of a set is not in general determined by the stability of its finite subsets. For example, consider the subframe of $(\mathsf{P}\omega, \subset)$ consisting of $\emptyset$ along with all the cofinite subsets of ω (i e , those X such that $\omega - X$ is finite). Then every finite subset of $S = \mathsf{P}\omega - \{\emptyset\}$ is stable (over its intersection, which is cofinite), but S is not itself stable (since no element of S is stable over $\emptyset$)

However, using 20.8(c), we can show that finite stability and pairwise stability are the same

Propostion. *A subset of frame is pairwise stable iff it is finitely stable.*

PROOF. That finite stability implies pairwise stability is trivial: we prove the converse. Suppose A is a frame, $S \subset A$ is pairwise stable, and $s_1, \ldots, s_n \in S$ By pairwise stability and 20.5, $s_i \unrhd s_i \wedge s_j$ for every $1 \leq i, j \leq n$ It follows by repeated applications of 20 8(c) that

$$s_i \unrhd s_1 \wedge \cdots \wedge s_n \qquad (1 \leq i \leq n). \tag{1}$$

Thus $\{s_1, \ldots, s_n\}$ is stable

20.11. Joins and arrow in $A^{\trianglelefteq}$. The set Q of 20.2(a) is up-closed, as observed in 15 4. Thus, every non-empty subset of $A^{\trianglelefteq}$ has a join If $a, b \in A$ and $S \subset A$ is nonempty, then we denote the join of a and b and the join of S in $A^{\trianglelefteq}$ by, respectively,

$$a \nabla b \quad \text{and} \quad \nabla S$$

Another consequence of Q being up-closed is that it is closed under the arrow operation of $\mathsf{N}A$ This induces an operation on $A^{\trianglelefteq}$ which, for $a, b \in A$, we denote

$$a \mathrel{-\!\!\triangleright} b$$

Proposition. *Suppose $a, b \in A$ Then*

$$a \nabla b = (a \twoheadrightarrow b) \twoheadrightarrow a = (b \twoheadrightarrow a) \twoheadrightarrow b = (a \leftrightarrow b) \to x, \tag{1}$$

where x is any element of $\{a, b, a \wedge b, a \vee b\}$, and

$$a \mathrel{-\!\!\triangleright} b = (b \twoheadrightarrow a) \to b \tag{2}$$

Proof The first two equations in (1) follow from 15 5(d)

$$q(a) \vee q(b) = q(q(a)b \twoheadrightarrow b) = q((b \twoheadrightarrow a) \twoheadrightarrow b),$$
$$q(b) \vee q(a) = q(q(b)a \twoheadrightarrow a) = q((a \twoheadrightarrow b) \twoheadrightarrow a)$$

We get the last equation in (1) by calculating

$$\begin{aligned} ((a \twoheadrightarrow b) \to a) \to a &= ((a \twoheadrightarrow b) \to a) \to (a \twoheadrightarrow b) && \text{by } 9.6(\text{d}) \\ &= ((a \to b) \wedge ((a \twoheadrightarrow b) \to a)) \to b \\ &= ((a \to b) \wedge (b \to a)) \twoheadrightarrow b && \text{by } 9\ 6(\text{b}) \\ &= (a \leftrightarrow b) \to x. && \text{by } 9\ 6(\text{a}) \end{aligned}$$

We get (2) from 15 5(c).

$$q(a) \to q(b) = q(q(a)b \to b) = q((b \twoheadrightarrow a) \to b).$$

20.12. Here are some basic facts about ∇ and $\mathrel{-\!\!\triangleright}$.

Proposition. *Suppose a $b \in A$. Then,*

(a) $a \trianglelefteq b$ *iff* $a \mathrel{-\!\!\triangleright} b = 1$
(b) *If* $a \geq b$, *or if* $a \smile b$, *then* $a \mathrel{-\!\!\triangleright} b = a \to b$ *and* $a \nabla b = a \twoheadrightarrow b$
(c) *If* $a \smile b$, *then* $a \leq b$ *iff* $a \trianglelefteq b$
(d) $(a \nabla b) \mathrel{-\!\!\triangleright} a = (a \nabla b) \twoheadrightarrow a = b \mathrel{-\!\!\triangleright} a$.
(e) $(a \wedge b) \to c = (a \to c) \nabla (b \to c)$.

Note: It will follow as a special case of 21 2.8 that both $\to$ and $\mathrel{-\!\!\triangleright}$ distribute over ∇

Proof Since $1 = a \rightarrowtriangle b = (b \twoheadrightarrow a) \to b$ iff $b \twoheadrightarrow a \leq b$, (a) is clear.

Next, note that, by 20 11,

$$a \triangledown b = (a \rightarrowtriangle b) \to b \tag{1}$$

If $a \geq b$, then $b \to a = 1$, so that $b \twoheadrightarrow a = a$ and thus

$$a \rightarrowtriangle b = (b \twoheadrightarrow a) \to b = a \to b;$$

if $a \smile b$, then $b \twoheadrightarrow a = a \twoheadrightarrow b$ by 20.6, so that

$$a \rightarrowtriangle b = (b \twoheadrightarrow a) \to b = (a \twoheadrightarrow b) \to b = a \to b$$

In either case, $a \rightarrowtriangle b = a \to b$, thus, we also have, by (1),

$$a \triangledown b = (a \rightarrowtriangle b) \to b = (a \to b) \to b$$

Therefore (b) holds

If $a \smile b$, then by (b) $a \rightarrowtriangle b = a \to b$ Since $a \leq b$ iff $a \to b = 1$ and $a \trianglelefteq b$ iff $a \rightarrowtriangle b = 1$ by (a), this proves (c)

The first equation of (d) follows from (b), the second follows from 20.11 since

$$(a \triangledown b) \to a = ((a \twoheadrightarrow b) \twoheadrightarrow a) \to a = (a \twoheadrightarrow b) \to a = b \rightarrowtriangle a$$

Finally, $a \to c \smile b \to c$, since both are regular over c; thus

$$\begin{aligned}(a \to c) \triangledown (b \to c) &= (a \to c) \twoheadrightarrow (b \to c) && \text{by (b)}\\ &= b \to ((a \to c) \twoheadrightarrow c) && \text{by 9 7.1}\\ &= b \to (a \to c) = (a \wedge b) \to c\end{aligned}$$

Therefore (e) holds, and the proof is complete.

20.13. Infinite and relative joins in $A^{\trianglelefteq}$. We've seen that ∇S exists only when S is nonempty A closely related operation, which is defined for all $S \subset A$, is the relative join: if $a \in A$, we define the *relative join* of S over a, denoted $\nabla^a S$, by

$$\nabla^a S = (\bigvee S) \twoheadrightarrow a = (\text{[illegible]} a) \to a \tag{1}$$

Notice that, by 13 5(c), ∇^a is just the join operation in the frame $A_{\Uparrow(a)}$ The relation between ∇ and ∇^a is spelled out in the following propostion.

Propostion. *Suppose that A is a frame, $a \in A$, and $S \subset {\Uparrow}a$ is nonempty. Then*

$$\nabla S = \nabla^a S \tag{2}$$

More generally, for any $S \subset A$, the operations ∇ and ∇^a are related by the equations

$$\nabla S = \nabla^a a \triangledown S, \quad \text{if } a \in S, \tag{3}$$
$$\nabla^a S = \nabla \{a\} \cup (S \twoheadrightarrow a). \tag{4}$$

PROOF Observe that for any a and x in A, $q(a)x = x \twoheadrightarrow a$ is the least element greater than x which is regular over a It follows that $\bigvee^a S$ is the least element regular over a and greater than each $s \in S$. Thus, if $S \subset \Uparrow a$ is nonempty, then since $\bigvee S \trianglerighteq s \trianglerighteq a$ for each $s \in S$ $\bigvee S \geq \bigvee^a S$ But, since $\trianglelefteq$ and $\leq$ coincide on $\Uparrow a$ by 20 12(c), it follows that $\bigvee^a S$ is regular over every $s \in S$ Thus $\bigvee S = \bigvee^a S$

Now suppose $S \subset A$ is arbitrary If $a \in S$, then $a \vee S$ is nonempty, and so

$$\bigvee S = \bigvee a \vee S = \bigvee^a a \vee S$$

by (2), since $a \vee S \subset \Uparrow a$ This proves (3) Finally, since

$$a \in \{a\} \cup (S \twoheadrightarrow a) \subset \Uparrow a,$$

we can use (2) again to obtain

$$\bigvee \{a\} \cup (S \twoheadrightarrow a) = \bigvee^a \{a\} \cup (S \twoheadrightarrow a) \tag{5}$$

But the second equation of (1) shows that to compute $\bigvee^a T$, each $a \in T$ can be ignored and each $s \twoheadrightarrow a \in T$ can be replaced by s. Thus the right-hand side of (5) is equal to $\bigvee^a S$, and we have proved (4)

21. Properties of regular operators

21.1. A characterization of regular operators. As with the equation 9.11(2) for nuclei, we can give a single equation characterizing regular operators

Proposition. *The following statements are equivalent for an arbitrary operator r on a frame A:*

(a) *r is a regular operator*

(b) *$r(a \to b) = a \to rb$ for all a $b \in A$*

PROOF Suppose that r is a regular operator, and $a, b \in A$ Then, since

$$b \trianglelefteq a \to b \trianglelefteq r(a \to b), \tag{1}$$

we have

$$\begin{aligned} r(a \to b) &= r(a \to b) \twoheadrightarrow (a \to b) && \text{by (1)} \\ &= a \to (r(a \to b) \twoheadrightarrow b) && \text{by 9.7 1} \\ &= a \to r(a \to b) && \text{by (1)} \\ &= a \to rb && \text{by extensionality} \end{aligned}$$

Thus (a) implies (b)

Suppose r satisfies (b) Let $a, x, y \in A$ and suppose that $a \to x = a \to y$ Then

$$a \to rx = r(a \to x) = r(a \to y) = a \to ry,$$

thus, r is extensional by 9 4(d) It remains to show that $ra \twoheadrightarrow a \leq ra$ for all $a \in A$ But,

$$\begin{aligned}(ra \twoheadrightarrow a) \to ra &= r((ra \twoheadrightarrow a) \to a) \\ &= r(ra \to a) \\ &= ra \to ra = 1.\end{aligned}$$

Thus, r is a regular operator. and (b) implies (a).

21.2. Basic properties. We now prove a series of results about regular operators. Looking over the examples in 19 3, we see that

$$a \to -, \quad a \twoheadrightarrow -, \quad a \mathrel{-\!\!\triangleright} -, \quad \text{and} \quad a \nabla - \tag{1}$$

are all regular operators, for every $a \in A$. It will be useful to keep these examples in mind as we develop the properties of regular operators below (Some of these properties. when applied to the operators in (1), are even somewhat surprising.)

In the following, A is a frame, l is an extensional operator on A, and r is a regular operator on A.

21.2.1. (r is idempotent) $r(ra) = ra$.

Proof. We have

$$\begin{aligned}r(ra) &= r((ra \to a) \to a) \\ &= (ra \to a) \to ra \\ &= (ra \to a) \to a && \text{by 9.6(d), since } rc \geq a \\ &= ra.\end{aligned}$$

21.2.2. (fix r is upward-closed) *If* $ra = a$ *and* $a \trianglelefteq b$, *then* $rb = b$.

Proof. We have

$$\begin{aligned}rb &= r((b \to a) \to a) \\ &= (b \to a) \to ra \\ &= (b \to a) \to a = b\end{aligned}$$

21.2.3. $l(ra) \to ra = l(a) \to ra$.
This is another replacement principle we shall use frequently.

Proof We have

$$\begin{aligned}l(ra) \to ra &= l(ra) \to (ra \twoheadrightarrow a) \\ &= ((ra \to a) \wedge l(ra)) \to a \\ &= ((ra \to a) \wedge l(a)) \to a && \text{by 9.6(d)} \\ &= l(a) \to (ra \twoheadrightarrow a) \\ &= l(a) \to ra.\end{aligned}$$

21.2.4. (r is $\trianglelefteq$-monotone)
(a) $a \trianglelefteq rb$ *iff* $ra \trianglelefteq rb$
(b) $a \trianglelefteq b$ *implies* $ra \trianglelefteq rb$.

PROOF (a) Since $a \trianglelefteq ra$, the "if" direction is just transitivity of $\trianglelefteq$ For the "only if" direction, assume $a \trianglelefteq rb$ Then,

$$\begin{aligned}(rb \to ra) \to ra &= (rb \to a) \to ra && \text{by 21 2 3}\\ &= r((rb \to a) \to a) \\ &= r(rb) && \text{by assumption}\\ &= rb && \text{by 21 2 1}\end{aligned}$$

Thus $ra \trianglelefteq rb$.
(b) If $a \trianglelefteq b$, then $a \trianglelefteq rb$, and (b) follows from (a)

21.2.5. (Regular operators commute) *Suppose* r_1 *and* r_2 *are any two regular operators Then*

$$r_1(r_2a) = r_1a \triangledown r_2a = r_2(r_1a)$$

PROOF. We prove the first equation. the second following by symmetry

$$\begin{aligned}r_1a \triangledown r_2a &= ((r_1a \to r_2a) \to r_1a) \to r_1a \\ &= ((a \to r_2a) \to a) \to r_1a && \text{by 21 2 3}\\ &= (r_2a \to a) \to r_1a && \text{because } a \leq r_2a\\ &= r_1((r_2a \to a) \to a) = r_1(r_2a).\end{aligned}$$

21.2.6. (r preserves stable meets) *Suppose* $S \subset A$ *is stable Then*

$$r \bigwedge S = \bigwedge rS$$

PROOF Suppose that $a \trianglelefteq s$ for every $s \in S$ Then,

$$\begin{aligned}r \bigwedge S &= r(\bigwedge(S \to a) \to a) \\ &= r((\bigvee S \to a) \to a) \\ &= (\bigvee S \to a) \to ra \\ &= \bigwedge(S \to a) \to ra = \bigwedge r((S \to a) \to a) = \bigwedge rS\end{aligned}$$

21.2.7.
(a) $r(a \wedge b) = ((a \to b) \to ra) \wedge ((b \to a) \to rb)$
(b) $r(a \wedge b) \geq ra \wedge rb$

PROOF. Since. by 9.7 5,

$$(a \to b) \to a = a \to (a \wedge b) \quad \text{and} \quad (b \to a) \to b = b \to (a \wedge b), \tag{1}$$

it follows that $(a \to b) \to a \smile (b \to a) \to b$ Also.

$$\begin{aligned} a \wedge b &= (a \wedge b) \twoheadrightarrow (a \wedge b) \\ &= (a \twoheadrightarrow (a \wedge b)) \wedge (b \twoheadrightarrow (a \wedge b)) && \text{by 9 7.4} \\ &= ((a \to b) \to a) \wedge ((b \to a) \to b). && \text{by (1)} \end{aligned}$$

Thus. by 21.2.6, we have.

$$\begin{aligned} r(a \wedge b) &= r\big(((a \to b) \to a) \wedge ((b \to a) \to b)\big) \\ &= r((a \to b) \to a) \wedge r((b \to a) \to b) \\ &= ((a \to b) \to ra) \wedge ((b \to a) \to rb) \end{aligned}$$

This proves (a), (b) is a trivial consequence of (a)

21.2.8. (r sub-preserves ∇)
(a) $r \nabla S \trianglerighteq \nabla rS \quad (S \subset A)$.
(b) $a \triangledown rb = ra \triangledown rb = ra \triangledown b$

Note that (a) and (b) can be combined into the following: *If $S = S_1 \cup S_2$, and both S_1 and S_2 are nonempty, then*

$$r \nabla S = (\nabla S_1) \triangledown (\nabla rS_2).$$

PROOF. (a) Suppose $S \subset A$ has $s_0 \in S$, and so $rs_0 \in rS$. For any $s \in S$, $\nabla S \trianglerighteq s$, so by 21.2.4(b), $r \nabla S \trianglerighteq rs$. Thus

$$r \nabla S \trianglerighteq \nabla rS$$

Conversely, for every $s \in S$, $s \trianglelefteq rs \trianglelefteq \nabla rS$. Thus,

$$\nabla S \trianglelefteq \nabla rS. \tag{1}$$

Now, rs_0 is a fixedpoint of r by 21.2.1, and $rs_0 \trianglelefteq \nabla rS$ Thus, ∇rS is also a fixedpoint of r by 21.2.2, and so we can use 21.2.4(a) on (1) to obtain

$$r \nabla S \trianglelefteq \nabla rS.$$

(b) We prove the first equation, the second follows by symmetry. Since rb is a fixedpoint of r, so is $a \triangledown rb$ by 21.2.2. Thus, using (a),

$$a \triangledown rb = r(a \triangledown rb) = ra \triangledown r(rb) = ra \triangledown rb.$$

21.3. Joins in $\mathsf{R}A$. Suppose $R \subset \mathsf{R}A$ Since $\mathsf{R}A$ is Boolean. the De Morgan laws (7.6) hold. Thus, using the fact that meets and arrow in $\mathsf{R}A$ are pointwise, and the formula 20.13(1), we get, for every $a \in A$,

$$(\bigvee R)a = (\neg \bigwedge \neg R)a = (\bigwedge Ra \to a) \to a = \nabla^{a} Ra$$

Thus, we have the formula

$$(\bigvee R)a = \nabla^a Ra \qquad (a \in A) \tag{1}$$

If R is nonempty, then (1) reduces by Proposition 20 13 to

$$(\bigvee R)a = \nabla Ra \qquad (a \in A), \tag{2}$$

since then Ra is nonempty and each of its elements is regular over a. Thus, just as with meets and arrow, joins in $\mathsf{R}A$ are calculated pointwise (provided we use ∇) We also note the following

Proposition. *For every* $r_1, r_2 \in \mathsf{R}A$.

$$r_1 \vee r_2 = r_1 \circ r_2$$

Proof This follows from (2), with $R = \{r_1, r_2\}$, and 21.2.5.

22. Fixedpoint sets of regular operators

22.1. Complete filters. We define a *complete filter* in a partial order $(P, \leq)$ to be a subset $F \subset P$ that is up-closed and closed under whatever meets exist in P. Note that if P has a largest element 1, then $1 \in F$ for every complete filter F, since it is the empty meet We denote the set of complete filters in P by $\mathsf{CompFilt}\,P$ and order it by *reverse* inclusion. In this way $\mathsf{CompFilt}\,P$ becomes a complete lattice (since arbitrary intersections of complete filters are clearly complete filters)

We continue our assumption that A is a frame

22.2. Proposition. *If r is a regular operator on A, then $\mathsf{fix}\,r$ is a complete filter in $A^{\trianglelefteq}$ Moreover if r' is another regular operator, then*

$$r \leq r' \quad \textit{implies} \quad \mathsf{fix}\,r \supset \mathsf{fix}\,r'$$

Proof That $\mathsf{fix}\,r$ is up-closed is just 21.2.2. Suppose $S \subset \mathsf{fix}\,r$ and $\bigwedge S$ exists in $A^{\trianglelefteq}$, i.e. S is stable Then, since r preserves stable meets by 21 2 6,

$$r(\bigwedge S) = \bigwedge rS = \bigwedge S,$$

and so $\bigwedge S \in \mathsf{fix}\,r$ Thus $\mathsf{fix}\,r$ is a complete filter

For the second part, we simply note that $r'a = a$ and $r \leq r'$ implies

$$a \leq ra \leq r'a = a,$$

so that $ra = a$

22.3. Regular operators from complete filters. The previous proposition is one direction in an equivalence between regular operators on A and complete filters in

$A^{\trianglelefteq}$, analogous to the equivalence between nuclei and maxsets. For the other direction, suppose F is a complete filter in $A^{\trianglelefteq}$. For $a \in A$, put

$$F_a = F \cap \Uparrow a = \{b \in F \,.\, a \trianglelefteq b\}$$

Then we define the operator r_F on A by

$$r_F a = \bigwedge F_a. \tag{1}$$

Proposition. *If F is a complete filter in $A^{\trianglelefteq}$, then r_F, as defined above, is a regular operator. Moreover, if F' is another complete filter in $A^{\trianglelefteq}$, then*

$$F \subset F' \quad \textit{implies} \quad r_F \geq r_{F'}.$$

PROOF Assume $a, x, y \in A$ are such that $a * x = a * y$, we then show that $a * r_F x = a * r_F y$, establishing that r_F is extensional by 9.4(d). Suppose $b \in F_x$. If we put $b' = b \triangledown y$, then $b' \trianglerighteq b \in F$ implies $b' \in F$ since F is up-closed, and so $b' \in F_y$. But

$$\begin{aligned} a * b &= a * (b \triangledown x) && \text{since } b \trianglerighteq x \\ &= a * (b \triangledown y) && \text{by replacement} \\ &= a * b'. \end{aligned}$$

Thus, we have shown that for every $b \in F_x$ there exists $b' \in F_y$ such that $a * b = a * b'$. It follows that $a * r_F x \geq a * r_F y$. A similar argument shows that $a * r_F x \leq a * r_F y$. Thus r_F is extensional.

For regularity, we simply note that, since each element of F_a is regular over a, it follows from 20.3(1) that $r_F a$ is also regular over a.

For the second part of the Proposition, suppose $F' \in \mathsf{CompFilt}\, A^{\trianglelefteq}$ and $F \subset F'$. Then, for any $a \in A$, $F_a \subset F'_a$, and so $r_F a \geq r_{F'} a$. Since a was arbitrary, $r_F \geq r_{F'}$.

22.4. Proposition. *Suppose A is a frame. Then,*
(a) *$\mathrm{fix}\, r_F = F$, for all $F \in \mathsf{CompFilt}\, A^{\trianglelefteq}$, and*
(b) *$r_{\mathrm{fix}_r} = r$, for all $r \in \mathsf{R}A$.*

PROOF If $a \in A$ is a fixedpoint of r_F, then since F_a is stable and F is closed under stable meets, it follows that $a = r_F a = \bigwedge F_a \in F$. Thus $\mathrm{fix}\, r_F \subset F$. Conversely, if $a \in F$, then $a \in F_a$, and so $r_F a = a$ and $a \in \mathrm{fix}\, r_F$. Thus, $F \subset \mathrm{fix}\, r_F$, proving (a).

Next, suppose r is a regular operator on A, and choose $a \in A$. Since r is idempotent (21.2 1), $ra \in \mathrm{fix}\, r$. Thus, since $ra \trianglerighteq a$, $ra \in (\mathrm{fix}\, r)_a$. If also $b \in (\mathrm{fix}\, r)_a$, then $rb = b \trianglerighteq a$, so that $b \trianglerighteq ra$ by 22.2.4(a). Thus, ra is the least element of the set $(\mathrm{fix}\, r)_a$; hence, $r_{\mathrm{fix}_r} a = ra$. Since a was arbitrary, $r_{\mathrm{fix}_r} = r$, proving (b).

22.5. Theorem. *The two operations*

$$r \mapsto \mathrm{fix}\, r \qquad \textit{and} \qquad F \mapsto r_F$$

are inverse isomorphisms between $\mathsf{R}A$ and $\mathsf{CompFilt}\, A^{\trianglelefteq}$

Proof Put together 22 2 22.3, and 22 4.

22.6. Finally, we state for the record the consequence of 22 5, as it was pointed out in 9 22

Corollary. *The sets that can arise as the fixedpoint sets of logical operators, or equivalently as the prefixedpoint sets of extensional operators, on a frame A are precisely the complete filters in $A^{\trianglelefteq}$.*

22.7. Meets in $\mathsf{CompFilt}\,A$. We can easily adapt the results for meets in the lattice of maxsets (14 2) to meets in the lattice of complete filters

Proposition. *Suppose that $\mathcal{F} \subset \mathsf{CompFilt}\,A^{\trianglelefteq}$ Then*

$$\begin{aligned}\bigwedge \mathcal{F} &= \{a \in A \quad \bigwedge_{F\in\mathcal{F}} r_F a = a\}\\ &= \{\bigwedge_{F\in\mathcal{F}} a_S \quad a_F \in \mathcal{F} \text{ for each } F \in \mathcal{F}, \text{ and } \{a_F \quad F \in \mathcal{F}\} \text{ is stable}\}\\ &= \{\bigwedge T \quad T \subset \bigcup \mathcal{F} \text{ and } T \text{ is stable}\}\end{aligned}$$

Proof Analogous to 14 2, using the results of this section and the properties of stable sets

22.7.1. Corollary. *If $S, T \in \mathsf{CompFilt}\,A^{\trianglelefteq}$, then*

$$S \wedge T = \{s \wedge t \,.\, s \in S \text{ and } t \in T\}$$

Proof For all $s \in S$ and $t \in T$ we have (see the proof of 21 2 7)

$$s \wedge t = ((s \to t) \to s) \wedge ((t \to s) \to t) \in S \wedge T,$$

since $(s \to t) \to s \smile (t \to s) \to t$ The Corollary follows

23. $\mathsf{R}A$ and the assembly tower

23.1. The embedding of $\mathsf{N}A$ into $\mathsf{R}A$. Recall that a maxset of a frame A is an arrow-ideal of A closed under all meets, while a complete filter in $A^{\trianglelefteq}$ is an arrow-ideal of A closed under all stable meets. Thus, $\mathsf{Max}\,A \subset \mathsf{CompFilt}\,A^{\trianglelefteq}$ Moreover, in both lattices, arbitrary joins are given by intersection, binary meets are given by "pairwise" meets (14.2 1(4) and 22.7.1), and the top elements are the same (namely, the set $\{1\}$), hence, $\mathsf{Max}\,A$ is really a subframe of $\mathsf{CompFilt}\,A^{\trianglelefteq}$

Because of the isomorphisms

$$\mathsf{Max}\,A \simeq \mathsf{N}A \qquad \text{and} \qquad \mathsf{CompFilt}\,A^{\trianglelefteq} \cong \mathsf{R}A,$$

the subframe inclusion of the previous paragraph induces a frame embedding $\mathsf{N}A \to \mathsf{R}A$ This embedding is given by $j \mapsto \bar{j}$, where (recall 19 2)

$$\bar{j}a = (ja \to a) \to a \qquad (a \in A), \tag{1}$$

since (1) gives a regular operator with $\operatorname{fix}\bar{\jmath} = \operatorname{fix}\jmath$, by 9.22(a)

23.2. The right adjoint. The operation $\jmath \mapsto \bar{\jmath}$, being a frame homomorphism, has a right adjoint, which we denote $r \mapsto r^\circ$.

Proposition. *Suppose r is a regular operator on A. Then,*
(a) $\operatorname{fix} r^\circ$ *is the meet-closure of* $\operatorname{fix} r$.
(b) $r^\circ a = \bigwedge\{rb \;\; b \geq a\}$

PROOF. For (a), note that, by adjointness, $\operatorname{fix} r^\circ$ must be the smallest maxset of A that contains $\operatorname{fix} r$ But, since $\operatorname{fix} r$ is an arrow-ideal, this maxset is just the meet-closure of $\operatorname{fix} r$, by 13.9(b).

To prove (b), recall from 13 2(c) that every nucleus j satisfies

$$ja = \bigwedge\{b \in \operatorname{fix}\jmath : b \geq a\} \tag{1}$$

To show that the meet in (1), with j replaced by r°, is equal to the meet in (b), we therefore need to show that the two sets

$$\{b \in \operatorname{fix} r^\circ \;\; b \geq a\} \qquad \text{and} \qquad \{rb : b \geq a\} \tag{2}$$

are *cofinal* (i.e., have the same meet) Going right to left, since $\operatorname{fix} r \subset \operatorname{fix} r^\circ$, we have $a \leq rb \in \operatorname{fix} r^\circ$ whenever $b \geq a$ Conversely, suppose that $b \in \operatorname{fix} r^\circ, b \geq a$ By (a), b is a meet of fixedpoints of r, each of which (since r is idempotent and inflationary) is of the form rc for some $c \geq a$ This completes the proof

23.3. Lemma. *Suppose $J \subset \mathsf{N}A$ and $j_1, j_2 \in \mathsf{N}A$. Then,*
(a) $\bigvee J = (\bigvee \bar{J})^\circ$.
(b) $j_1 \to j_2 = (\bar{\jmath}_1 \to \bar{\jmath}_2)^\circ$.

PROOF. Since $\jmath \mapsto \bar{\jmath}$ is 1-1 we have $(\bar{\jmath})^\circ = \jmath$ for all $j \in \mathsf{N}A$; therefore, since $\jmath \mapsto \bar{\jmath}$ is a frame morphism,

$$(\bigvee \bar{J})^\circ = (\overline{\bigvee J})^\circ = \bigvee J.$$

Also, using Proposition 10.7,

$$(\bar{\jmath}_1 \to \bar{\jmath}_2)^\circ = j_1 \to (\bar{\jmath}_2)^\circ = \jmath_1 \to \jmath_2$$

23.4. Joins and arrow of nuclei. Using the Lemma, we can now take advantage of the pointwise formula for joins of regular operators (21.3), and the pointwise arrow operation, to get formulas for joins and arrow of nuclei:

Theorem. *Under the hypotheses of the Lemma,*

$$(\bigvee J)a = \bigwedge_{b \geq a} (\bigwedge Jb \to b) \to b = \bigwedge_{b \geq a} ((\bigvee Jb) \to b) \to b \tag{1}$$

and

$$(j_1 \to \jmath_2)a = \bigwedge_{b \geq a} j_1 b \to j_2 b \tag{2}$$

PROOF. By the Lemma and previous proposition (as well as 7 6 and 21 3),

$$(\bigvee J)a = (\bigvee \overline{J})^{\circ}a = (\neg \bigwedge \neg \overline{J})^{\circ}a = \bigwedge_{b \geq a} (\neg \bigwedge \neg \overline{J})b = \bigwedge_{b \geq a} (\bigwedge Jb \to b) \to b.$$

The second equation of (1) is clear

For (2) we have, again by the Lemma and previous proposition,

$$(j_1 \to j_2)a = (\overline{j_1} \to \overline{j_2})^{\circ}a = \bigwedge_{b \geq a} \overline{j_1}b \to \overline{j_2}b$$

But

$$\overline{j_1}b \to \overline{j_2}b = \overline{j_2}(\overline{j_1}b \to b) = \overline{j_2}(j_1 b \to b) = j_1 b \to \overline{j_2}b,$$

thus, to establish (2) it suffices to show that the sets

$$\{j_1 b \to \overline{j_2}b \;\; b \geq a\} \qquad \text{and} \qquad \{j_1 b \to j_2 b \,.\, b \geq a\}$$

are cofinal. Now, since $\overline{j_2}b \geq j_2 b$ for all b, each element on the left is greater than one on the right. For the converse, we note that, since $j_2 b \in \mathrm{fix}\,\overline{j_2}$,

$$j_1 b \to j_2 b \geq j_1(j_2 b) \to j_2 b = j_1(j_2 b) \to \overline{j_2}(j_2 b)$$

23.5. Lemma. *Suppose A is a frame. Then, for any $r \in \mathrm{R}A$ and $a \in A$, we have*

$$r \vee \overline{q}(a) = \overline{q}(ra).$$

PROOF. For any $x \in A$ we have

$$\begin{aligned}
(r \vee \overline{q}(a))x &= r(\overline{q}(a)x) && \text{by Proposition 21.3}\\
&= r(a \nabla x)\\
&= ra \nabla x && \text{by 21 2 8}\\
&= \overline{q}(ra)x
\end{aligned}$$

23.6. Theorem. $\mathrm{R}A \simeq (\mathrm{N}^2 A)_{\neg\neg}$.

PROOF Let f $\mathrm{N}^2 A \to \mathrm{R}A$ be the unique extension (guaranteed to exist by 16 2 since every element of $\mathrm{R}A$ is Boolean) of the frame embedding $\mathrm{N}A \to \mathrm{R}A$ of 23.1 Since the latter is mono, f is dense by 16.4(c) As double negation is the only nucleus that is at the same time dense and quasi-closed, it will be enough to show that f is surjective—equivalently, that f_* is 1-1 By 16 3(b) and $\mathrm{NR}A \simeq \mathrm{R}A$, $f_* r = [r]$, where

$$[r]j = (r \vee \overline{j})^{\circ}.$$

So, suppose r_1 and r_2 are such that $[r_1] = [r_2]$ Then, for any $a \in A$, evaluating at the nucleus $q(a)$ gives, by the Lemma,

$$[r_1]q(a) = (r_1 \vee \overline{q}(a))^{\circ} = (\overline{q}(r_1 a))^{\circ} = q(r_1 a).$$

and, similarly, $[r_2]q(a) = q(r_2 a)$. But q is 1-1, so $r_1 a = r_2 a$. Since a was arbitrary, $r_1 = r_2$, proving that f_* is 1-1, as desired.

23.7. We now aim to get a description of the map $\neg\neg : \mathsf{N}^2A \to \mathsf{R}A$ in terms of operators. We do this by going through $\mathsf{RN}A$, with the following extra result.

Proposition. *The map $d : \mathsf{RN}A \to \mathsf{R}A$ defined for $R \in \mathsf{RN}A$ by $dR = r$, where*

$$ra = b \quad \text{iff} \quad Rq(a) = q(b) \qquad (a, b \in A), \tag{1}$$

is a cBa morphism.

PROOF. First of all note that r, as given in (1), is well-defined, since the regular operator R is inflationary and the set of nuclei of the form $q(a)$ for $a \in A$ is up-closed (15.4). Next, we check that r is a regular operator, using 21.1 and that q is 1-1: for every $a, b \in A$,

$$\begin{aligned} q(r(a \to b)) &= Rq(a \to b) && \text{by (1)} \\ &= R(c(a) \to q(b)) && \text{by 15.5(c), since } (a \vee b) \to b = a \to b \\ &= c(a) \to Rq(b) \\ &= c(a) \to q(rb) && \text{by (1)} \\ &= q(a \to rb). \end{aligned}$$

Next, we check that d preserves meets. Suppose $\mathcal{Q}$ is a subset of $\mathsf{RN}A$ and put $Q = d\mathcal{Q}$. Then, since meets in $\mathsf{RN}A$ are pointwise, we have, for every $a \in A$,

$$(\bigwedge \mathcal{Q})q(a) = \bigwedge \mathcal{Q}q(a) = \bigwedge q(Qa).$$

Transposing to $(\mathsf{N}A)^{\trianglelefteq}$ by 20.2(a), we see that, since the set Qa is stable, we can use 20.5 to conclude that $\bigwedge q(Qa) = q(\bigwedge Qa)$. Thus, since meets in $\mathsf{R}A$ are also pointwise,

$$(\bigwedge \mathcal{Q})q(a) = q(\bigwedge Qa) = q((\bigwedge Q)a).$$

Hence $d(\bigwedge \mathcal{Q}) = \bigwedge d\mathcal{Q}$, proving that d preserves meets. We complete the proof by checking that d preserves $\neg$. Suppose $R \in \mathsf{RN}A$ and $a \in A$, and put $r = dR$. Then,

$$(\neg R)q(a) = Rq(a) \to q(a) = q(ra) \to q(a) = q(ra \twoheadrightarrow a),$$

by 20.11. Now $ra \geq a$, so that $ra \twoheadrightarrow a = ra \to a$ by 20.12(b). Thus,

$$(\neg R)q(a) = q(ra \to a) = q((\neg r)a),$$

and so $d(\neg R) = \neg dR$.

23.8. **Proposition.** *The morphism $f : \mathsf{N}^2A \to \mathsf{R}A$ given by the composite*

$$\mathsf{N}^2A \xrightarrow{c} \mathsf{N}^3A \xrightarrow{\neg\neg} \mathsf{RN}A \xrightarrow{d} \mathsf{R}A \tag{1}$$

satisfies $fJ = r$, where

$$ra = b \quad \text{iff} \quad Jq(a) = q(b) \qquad (a, b \in A). \tag{2}$$

Moreover, $\neg\neg : \mathsf{N}^2A \to \mathsf{R}A$ *is given by the same formula: thus* $f = \neg\neg$

PROOF. By 23.1, the map $\mathsf{N}^2A \to \mathsf{RN}A$ is given by $J \mapsto \overline{J}$. Thus, the first part of the Proposition will follow if we can show that for every $a \in A$, $Jq(a) = \overline{J}q(a)$ Indeed, suppose that $J \in \mathsf{N}^2A$, and let r be defined as in (2) Notice that since

$$q(a) \leq Jq(a) = q(ra),$$

we have $ra \trianglerighteq a$ for all $a \in A$. But now for any $a \in A$,

$$\begin{aligned}\overline{J}q(a) &= (Jq(a) \to q(a)) \to q(a)\\ &= (q(ra) \to q(a)) \to q(a)\\ &= q(ra \to a) \to q(a)\\ &= q((ra \to a) \to a) = q(ra) = Jq(a)\end{aligned}$$

For the second part of the proposition, note that we now have two morphisms $\mathsf{N}A \to \mathsf{R}A$

$$\mathsf{N}A \xrightarrow{c} \mathsf{N}^2A \underset{\neg\neg}{\overset{f}{\rightrightarrows}} \mathsf{R}A$$

The bottom composite is given by $j \mapsto \overline{j}$ (as we know from 23 1). But the calculation

$$\begin{aligned}q((fc(j))a) &= c(j)(q(a)) && \text{by (2)}\\ &= j \vee q(a)\\ &= q(ja \to a) && \text{by 15.5(d)}\\ &= q(\overline{j}a),\end{aligned}$$

for all $j \in \mathsf{N}A$ *and* $a \in A$, *shows that the top composite is given by the same formula* Therefore, as $c : \mathsf{N}A \to \mathsf{N}^2A$ is epi by 15.1(e), we have $f = \neg\neg$

23.9. Corollary. *The morphism* $\neg\neg\ \mathsf{NN}A \to \mathsf{R}A$ *is open.*

PROOF In any Heyting algebra double negation preserves $\to$:

$$\begin{aligned}((a \to b) \to 0) \to 0 &= (((a \to 0) \to 0) \wedge (b \to 0)) \to 0 && \text{by 9.7 3}\\ &= ((a \to 0) \to 0) \to ((b \to 0) \to 0)\end{aligned}$$

The proof that $\neg\neg\ \mathsf{N}^2A \to \mathsf{R}A$ preserves meets is identical to the proof that the map $d : \mathsf{RN}A \to \mathsf{R}A$ of 23 7 preserves meets

24. $\mathsf{R}A$ as a limit

24.1. Given a frame A, we can construct a diagram of cBa's as follows. The vertices of the diagram are the frames $A_{q(a)}$ for each $a \in A$, and there is an arrow $A_{q(a)} \to A_{q(b)}$ iff $q(a) \leq q(b)$, in which case the morphism is the natural quotient morphism. Let us call this diagram $D(A)$ In terms of maxsets, it's easy to see that what we have are the maxsets $\Uparrow a$ for each $a \in A$, and a morphism $\Uparrow a \to \Uparrow b$ whenever $a \trianglelefteq b$, which is given for all $x \in \Uparrow a$, by $x \mapsto b \nabla x$.

We now describe a cone from RA to $D(A)$. For every $a \in A$, we have the map $e_a : RA \to \Uparrow a$, 'evaluate at a', given by $e_a r = ra$. Since $a \trianglelefteq ra$ for each $r \in RA$, this indeed maps RA to $\Uparrow a$; it preserves all meets, since meets in RA are pointwise and meets in $\Uparrow a$ are the same as in A, and it preserves complements, since $(r \to 0)a = ra \to a$. Thus, the evaluation maps are cBa morphisms. They determine a cone since, for each $a \trianglelefteq b$ and each $r \in RA$, we have, by 21.2.8(b),

$$b \,\triangledown\, ra = r(b \,\triangledown\, a) = rb$$

Theorem. *With notation as above, the cone $(e_a : RA \to \Uparrow a)_{a \in A}$ is a limit of the diagram $D(A)$.*

Proof. Suppose $(f_a : C \to \Uparrow a)_{a \in A}$ is another cone to $D(A)$. This means in particular that, for all $c \in C$,

$$a \trianglelefteq f_a c \qquad (a \in A) \tag{1}$$

and

$$b \,\triangledown\, f_a c = f_b c \qquad (a, b \in A,\ a \trianglelefteq b) \tag{2}$$

The goal is to define a morphism $f : C \to RA$ by which this cone factors through the cone $(e_a)_{a \in A}$. Such a morphism, were it to exist, would thus have to satisfy

$$(e_a \circ f)c = f(c)a = f_a c \qquad (c \in C)$$

for every $a \in A$. Taking the second equation as the definition of f, it remains to show that

(a) $f(c)$ is a regular operator for all $c \in C$, and

(b) f is a frame morphism.

For (a), it suffices by 21.1 to check that

$$f_{a \to b} c = a \to f_b c$$

for all $a, b \in A$. But, as $b \trianglelefteq a \to b$, we have

$$\begin{aligned} f_{a \to b} c &= (a \to b) \,\triangledown\, f_b c && \text{by (2)} \\ &= a \to (b \,\triangledown\, f_b c) && \text{by 21.2.5} \\ &= a \to f_b c. && \text{by (1)} \end{aligned}$$

For (b), recall (20.13) that joins in the maxset $\Uparrow a$ of A are given by ∇^a; then, for every $a \in A$ and $S \subset C$,

$$f(\bigvee S)a = f_a \bigvee S = \nabla^a f_a S = (\bigvee f(S))a,$$

by 21.3(1), so that f preserves joins. Similarly, given $a \in A$ and a finite set $T \subset C$,

$$f(\bigwedge T)a = f_a(\bigwedge T) = \bigwedge f_a T = (\bigwedge f(T))a,$$

proving that f preserves finite meets. This completes the proof.

Chapter 7
FREE MEETS

This short chapter answers the "local" question of which meets in a frame A are preserved by every frame morphism out of A and the corresponding "global" question of which frames A have the property that every frame morphism out of A preserves all meets The answer to the first question is given in Section 25, and this is used to give a characterization of the dual kernels of frame morphisms. The answer to the second question, which is simply that A is a biframe (i e., its dual is also a frame) and NA is Boolean, is given in Section 26 Two examples follow, showing that these two conditions on A are independent

25. The characterization

25.1. Definitions. A subset S of a frame A has a *free meet* if $f(\bigwedge S) = \bigwedge fS$ for every frame morphism $f \quad A \to B$ The frame A itself *has free meets* if every subset $S \subset A$ has a free meet

Note that since frame morphisms preserve finite meets (and are monotone) a subset S of A has a free meet just in case the filter generated by S has a free meet We could therefore restrict attention to filters without any loss of generality, although we won't have any reason to do so (except, perhaps, in Corollary 25 4, where a restriction to up-closed sets would make the statement of the result simpler)

25.2. Lemma. *Suppose A is a frame and $S \subset A$. Then $\bigvee u(S) = u(a)$ implies $a = \bigwedge S$ and similarly $\bigwedge c^2(S) = c^2(a)$ implies $a = \bigwedge S$.*

PROOF. Suppose $\bigvee u(S) = u(a)$ Then negating both sides and using 15.1(d), we get

$$\bigwedge c(S) = c(a), \tag{1}$$

and evaluating both sides of this at 0, we get $\bigwedge S = a$ For the second part, we evaluate both sides of $\bigwedge c^2(S) = c^2(a)$ at $0 \in NA$ to get (1), and then evaluate again at 0, as before, to get $\bigwedge S = a$.

25.3. Theorem. *Suppose that S is a subset of a frame A Then the following five statements are equivalent·*

(a) *S has a free meet in A*
(b) *$\bigwedge c^2(S) = c^2(\bigwedge S)$ in N^2A*
(c) *$\bigvee u(S) = u(\bigwedge S)$ in NA*
(d1) *$\bigwedge(S \to a) \to a = ((\bigwedge S) \to a) \to a$ in A, for every $a \in A$*
(d2) *$\bigwedge(S \to a) \to a = a$ in A, for every $a \geq \bigwedge S$*

Note that condition (b) can be replaced by "$\bigwedge c^2(S) = c^2(a)$ for some $a \in A$" and that condition (c) can be replaced by "$\bigvee u(S) = u(a)$ for some $a \in A$", for then $a = \bigwedge S$ in both cases by 25.2

PROOF. We prove $(a)\Rightarrow(b)\Rightarrow(c)\Rightarrow(a)$ and $(a)\Rightarrow(d1)\Rightarrow(d2)\Rightarrow(c)$

First, (a) trivially implies (b), since $c^2 : A \to \mathrm{N}^2A$ is a frame morphism. Because $u(\bigvee u(S)) = \bigwedge u^2(S)$ by 15.1(f) and $u^2 = c^2$ by 15.7(b), (b) is the result of applying u to both sides of (c); since u is 1-1, this shows that (b) implies (c). Now assume (c), and let $f : A \to B$ be a frame morphism. Then, again using 15.1(f), and the second part of 16.1(1),

$$\begin{aligned} u(f\bigwedge S) &= (\mathrm{N}f)(u(\bigwedge S)) \\ &= (\mathrm{N}f)(\bigvee u(S)) = \bigvee(\mathrm{N}f)(u(S)) \\ &= \bigvee u(fS), \end{aligned}$$

and so $f\bigwedge S = \bigwedge fS$ follows from 25.2. Thus (c) implies (a).

Next, considering for every $a \in A$ the frame morphism $q(a) : A \to A_{q(a)}$, and the fact (13.5(a)) that meets in both frames are the same, (a) clearly implies (d1). Condition (d2) is a special case of (d1), since $((\bigwedge S) \to a) \to a = a$ whenever $a \geq \bigwedge S$. Finally, since $\bigvee u(S) \leq u(\bigwedge S)$ always, (c) is equivalent to $u(\bigwedge S) \leq \bigvee u(S)$ and thus, by 15.1(c), to $(\bigvee u(S))(\bigwedge S) = 1$. An application of the formula 23.4(1) for joins in $\mathrm{N}A$ (and expansion of the definition of u) reduces this to

$$\bigwedge_{a \geq \wedge S} [\bigwedge(S \to a) \to a] \to a = 1.$$

As the expression in square brackets is $\geq a$, the equivalence of (c) and (d2) follows, completing the proof.

25.4. Recall the definition of stability from 20.4. The following corollary gives another way of expressing condition (d2) above.

Corollary. *A subset S of a frame A has a free meet iff every element $a \geq \bigwedge S$ can be written as the meet of a stable subset of the upward closure of S.*

PROOF. Since $(S \to a) \to a$ is a stable subset of the upward closure of S, necessity follows trivially from the implication $(a)\Rightarrow(d2)$ of the Theorem. Conversely, if T is a stable subset of the upward closure of S with meet a, then

$$a \leq \bigwedge(S \to a) \to a \leq \bigwedge(T \to a) \to a = \bigwedge T = a,$$

and so $a = \bigwedge(S \to a) \to a$, verifying condition (d2) of the Theorem.

25.5. As an application of Theorem 25.3, we can derive a characterization of the "dual kernels" of frame morphisms, i.e., the sets $f^{-1}(1) \subset A$ for frame morphisms $f : A \to B$.

Theorem. *A subset F of a frame A is equal to $f^{-1}(1)$ for some frame morphism $f : A \to B$ iff F is a free filter, i.e.,*

(a) *F is upward closed, and*

(b) *if $S \subset F$ and $\bigwedge(S \to a) \to a = a$ for every $a \geq \bigwedge S$, then $\bigwedge S \in F$.*

Note that, by the previous theorem, (b) is equivalent to saying that F is closed under free meets.

PROOF Considering the above note, sets of the form $f^{-1}(1)$ are clearly free filters Conversely, supposing that $F \subset A$ is a free filter, consider the nucleus $j = \bigvee u(F)$ If $a \in F$, then $ja \geq u(a)a = 1$, and so $ja = 1$ And if $ja = 1$, then $u(a) \leq \bigvee u(F)$ by 15 1(c), and so $u(a) = u(a) \wedge \bigvee u(F) = \bigvee u(a \vee F)$, using 15 1(f) Thus $a \vee F$ has a free meet and $\bigwedge a \vee F = a$ But $a \vee F \subset F$ by (a), and therefore $a \in F$ by (b). We have therefore shown that $a \in F$ iff $ja = 1$, and so $F = j^{-1}1$ for the frame morphism $j : A \to A_j$

26. Frames with free meets

26.1. We now turn to the question of which frames have free meets. Using the previous theorem and a result of Beazer and Macnab [5] this question can be answered as follows·

Theorem. *A frame A has free meets iff both of the following conditions hold*
(a) *A is a biframe*
(b) *$\mathsf{N}A$ is Boolean.*

PROOF First of all observe that A is a biframe iff c $A \to \mathsf{N}A$ preserves meets, for $c(\bigwedge S) = \bigwedge c(S)$ is equivalent to $a \vee \bigwedge S = \bigwedge a \vee S$ for every $a \in A$ Suppose now that A has free meets By the preceeding comment, A is a biframe Also, by the previous theorem,

$$\bigwedge(S \to a) \to a = ((\bigwedge S) \to a) \to a,$$

and so for every $a \in A$, there is a smallest element $d(a)$ with $(d(a) \to a) \to a = 1$ Therefore, according to the result of Beazer and Macnab, $\mathsf{N}A$ is Boolean

Conversely, suppose A is a biframe and $\mathsf{N}A$ is Boolean Then, as c $A \to \mathsf{N}A$ preserves all meets, and c $\mathsf{N}A \to \mathsf{N}^2A$, being an isomorphism by 15 1(e), also preserves all meets, the composite $c^2 . A \to \mathsf{N}^2A$ preserves all meets Hence, A has free meets by the previous theorem

26.2. Examples. The conditions (a) and (b) above are independent, as the following two examples show·

(i) Let $A = (\omega + 1)^{op}$ (i e , the chain $\omega < \ldots < 2 < 1 < 0$) Then A is complete and therefore a biframe (in fact, even a completely distributive frame) But n is a dense element of A for every $n \neq \omega$, in the sense that $n \to \omega = \omega$, and $\bigwedge_{n \neq \omega} n = \omega$, showing that n doesn't have a least dense cover $d(n)$, and therefore that $\mathsf{N}A$ is not Boolean. Notice that ω may be replaced here by any limit ordinal and that together these complete chains form a class of canonical examples of frames whose assemblies are not Boolean, in the sense that the failure of a frame to have a Boolean assembly may be traced to the existence in the frame of such a complete chain with every element dense over the meet

(ii) The same A, with N applied, yields an example of a non-biframe with a Boolean assembly. In detail, using the isomorphism of $\mathsf{N}A$ with the poset of maxsets of A ordered by reverse inclusion, and the degeneracy of $\to$ and $\bigwedge$ in A, $\mathsf{N}A$ can be identified with the set of all subsets of A (ordered by reverse inclusion) of the following two types:

(a) finite sets containing 0, and
(b) infinite sets containg both 0 and ω.

Join is given by intersection, and meet is given by union, with ω added if the result is infinite. $\mathsf{N}A$ is not a biframe, because a counterexample to

$$S \vee \bigwedge\{T_n : n < \omega\} = \bigwedge\{S \vee T_n : n < \omega\} \tag{1}$$

can be found by setting $S = \{0, \omega\}$ and $T_n = \{0, n\}$, for then ω is an element of the left side of (1) but not the right.

On the other hand, it is easy to see that $\{0\}$ and $\{0, \omega\}$ are the least dense elements over sets of type (a) and (b), respectively, and thus that N^2A is Boolean.

26.3. Remark. The independence of conditions (a) and (b) above can also be used to show the independence of the statements "f preserves meets" and "$\mathsf{N}f$ preserves meets." If $\mathsf{N}A$ is Boolean and A is not a biframe, then $f = c : A \to \mathsf{N}A$ doesn't preserve meets while $\mathsf{N}f$ (being an isomorphism) does. And if A is a biframe with $\mathsf{N}A$ not Boolean, then the same f preserves meets, but $\mathsf{N}f$ doesn't (otherwise $\mathsf{N}A$ would be a biframe and hence Boolean).

26.4. Exercises.

26.4.1. Modify the proof of Theorem 25.3 to obtain the equivalence of the following statements, for $S \subset A$ and $b \in A$.

(a) $\bigwedge fS \leq fb$, for every frame morphism $f : A \to B$
(b) $\bigwedge c^2(S) \leq c^2(b)$.
(c) $\bigvee u(S) \geq u(b)$
(d1) $\bigwedge(S \to a) \to a \leq (b \to a) \to a$, for every $a \in A$.
(d2) $\bigwedge(S \to a) \to a = a$, for every $a \geq b$.

26.4.2. Given a frame A, define $\mathsf{M}A$ to be the subframe of N^2A generated by all elements of the form $\bigwedge\{c^2(s) : s \in S\}$, where $S \subset A$. Show the following:

(a) If $f : A \to B$ is a frame morphism such that every subset of the image $f(A)$ of A has a free meet in B, then the morphism $\mathsf{N}^2 f$ restricted to $\mathsf{M}A$ factors through $c_B^2 : B \to \mathsf{N}^2B$, and therefore f extends to a morphism $\overline{f} : \mathsf{M}A \to B$ in a canonical way.

(b) M can be made into an endofunctor on **Frm** so that the embeddings $c^2 : A \to \mathsf{M}A$ are components of a natural transformation from the identity functor to M.

(c) A has free meets iff $c^2 : A \to \mathsf{M}A$ is an isomorphism.

(d) Is $c^2 : A \to \mathsf{M}A$ epi? (I don't know the answer.)

Chapter 8

UNIVERSAL MONOS AND COMBINATORIAL MORPHISMS

In Chapter 7 we used the second-level assembly N^2A of a frame A to characterize the free meets in A In the first half of this chapter, we make use of the whole assembly tower to characterize those monos whose pushouts are always mono These results make essential use of the theory developed in Chapters 4 and 5 and are a further example of the use of κ-frames to obtain results about frames. Section 27 introduces the class of universal monos proves some closure properties of the class and gives several families of examples In Section 28, the notions of α-mono and α-epi are introduced and their basic properties investigated. Then the main results are proved f is a universal mono iff it is α-mono for every α, and f is $(\alpha + 1)$-mono if the pushout of f along every α-epi is mono The first of these results is proved in two ways, once using the categories ∞-**Frm** and ∞-**cBa** introduced in Section 18 (which rely on inaccessible cardinals and entail a restriction of the frames considered), and a second time using an excursion through the category κ-frames where κ is chosen based on the size of the instance of the result to be proven, this second proof depends on a Lemma about the relation between the assembly tower of a frame and its κ-assembly tower when it is considered to be a κ-frame The section closes with a proof that α-epis are stable under pushout.

The second half of the chapter is concerned with combinatorial morphisms. These provide a means of treating categorical properties of a morphism $f : A \to B$ (like being an epi or regular mono) in terms of algebraic data on A (namely congruences). Section 29 treats the general theory, while Section 30 treats the finitely generated case Section 29 starts with a concrete description of the free extension $A[X]$ of a frame by a set X of "indeterminates" including the effect of the functor $A \mapsto A[X]$ on morphisms Then the notion of standardization of congruences is introduced and several results relating congruences and their standardizations are proved Finally, both pushouts and iterated extensions of combinatorial morphisms are considered, and a connection is established with free meets In Section 30, finitely generated combinatorial morphisms are examined since standardization no longer plays a role, the theory becomes considerably simpler in this case The results lead up to a characterisation, in terms of properties of congruences, of finitely generated epis and regular monos

27. Universal monos

27.1. Definition. A morphism $f : A \to B$ in a category with pushouts is a *universal mono* (or just *universal*) if the pushout of f along every morphism $h : A \to D$ is mono.

27.2. Here are some basic properties of universal monos in any category with pushouts

Proposition. *Every universal mono is mono, and every pushout of a universal mono is itself universal Suppose $f : A \to B$ and $g : B \to C$ are morphisms Then*

(a) *If f and g are universal, then so is $g \circ f$.*
(b) *If $g \circ f$ is universal, then so is f.*
(c) *If $g \circ f$ is universal and f is epi, then g is universal*
(d) *If $g \circ f$ is epi and g is universal, then f is epi*

We summarize (a) and (b) by saying that composites and first factors of universal monos are mono

PROOF. Every morphism is a pushout of itself along the identity, so universal monos are mono Part (a) of the Pushout Lemma (6 4) says that iterated pushouts are pushouts along the composite; it easily follows from this that pushouts of universal monos are universal Parts (a) and (b) also easily follow from the corresponding parts of the Pushout Lemma, since composites and first factors of monos are mono, as does part (c), since if f is epi then g is a pushout of $g \circ f$ along f and hence universal.

For part (d), suppose that $g \circ f$ is epi, g is universal, and $k_1, k_2 : B \to D$ are morphisms such that $k_1 \circ f = k_2 \circ f$. The idea will be to "push out" the pair k_1, k_2 along g and then use the fact that $g \circ f$ is epi. Thus, we construct the following diagram

$$\begin{tikzcd}
A \arrow[rr, two heads, "g\circ f"] \arrow[dr, "f"'] & & C \arrow[d, "k_1'"] \arrow[dr, "k_2'"] & \\
& B \arrow[ur, "g"] \arrow[d, shift right, "k_1"'] \arrow[d, shift left, "k_2"] & \bullet \arrow[dr, "g_2''"] & \bullet \arrow[d, "g_1''"] \\
& D \arrow[ur, "g_1'"] \arrow[urr, "g_2'"'] \arrow[rr, "l"'] & & \bullet,
\end{tikzcd}$$

where k_1' and k_2' are the pushouts of k_1 and k_2 along g (with g_1' and g_2' as the remaining morphisms of these pushout squares), and where g_1'' and g_2'' are the pushouts of g_1' and g_2' along each other, with $l = g_2'' \circ g_1' = g_1'' \circ g_2'$ as the "diagonal" of this pushout square Since g is universal, both g_1' and g_2' are universal, and so both g_2'' and g_1'' are mono, thus l is mono

The rest of the proof is a diagram chase showing that $l \circ k_1 = l \circ k_2$ follows from the assumptions that $k_1 \circ f = k_2 \circ f$ and that $g \circ f$ is epi. In detail (where I am omitting the symbol "$\circ$", and where parentheses indicate the expression(s) to be replaced in the next step), we have

$$g_2''(k_1'g)f = (g_2''g_1')(k_1f) = g_1''(g_2'k_2)f = g_1''k_2'gf,$$

and therefore, since gf is epi, $g_2''k_1' = g_1''k_2'$. Now, using this,

$$(l)k_1 = g_2''(g_1'k_1) = (g_2''k_1')g = g_1''(k_2'g) = (g_1''g_2')k_2 = lk_2.$$

Finally, since l is mono, we conclude that $k_1 = k_2$.

27.3. Examples. (1) Every isomorphism and, more generally, every *section* (i e, a morphism $f : A \to B$ for which there is $g : B \to A$ with $g \circ f = \mathrm{id}_A$) is universal, as can be easily shown (6.4).

(ii) Every *open* mono is universal (For a proof, see [27], V.4, Proposition 1.) Recall that a frame morphism is open iff it is a cHa morphism, i.e., iff it preserves arrow and arbitrary meets.

(iii) The monos corresponding to localic *proper* surjections are universal. This fact appears as Proposition 4.2 in the paper by Vermeulen [49], see that paper for a detailed account of proper maps. We recall the lattice-theoretic characterization: $f : A \to B$ is proper iff f_* preserves directed joins and $f_*(f(a) \vee b) = a \vee f_*(b)$ for all $a \in A$ and $b \in B$.

(iv) For any category $\mathbf{C}$ with pushouts, if $\tau : I \to F$ is a natural monomorphism from the identity functor $I : \mathbf{C} \to \mathbf{C}$ to an arbitrary functor $F : \mathbf{C} \to \mathbf{C}$, then every component τA of τ is universal. This is because, for every morphism $h : A \to D$, the monomorphism $\tau D : D \to FD$ factors, by naturality of τ, through the pushout of $\tau A : A \to FA$ along h, making the latter mono. In particular, all of the morphisms $c^\alpha : A \to \mathrm{N}^\alpha A$ (as defined in 17.1) are universal.

(v) Combining (iv) and 27.2(b), we conclude that first factors of components of natural monomorphisms from the identity are universal. As examples, we have the morphisms $c^2 : A \to \mathrm{M}A$ of Exercise 26.4.2. Conversely, as is sketched in Exercise 27.4.2 below, if directed colimits of monos are mono (as is the case with **Frm**: see 12.8), then every universal mono is the first factor of a component of an appropriately constructed natural monomorphism $\tau : I \to F$. Treating F as a kind of "free extension" functor, this lends some credence to the intuitive idea that universal monos represent free extensions.

27.4. Exercises.

27.4.1. Show that the construction used in the proof of Proposition 27.2(d) of pushing out a pair of morphisms along another morphism can be given a universal property similar to that of an *ordinary pushout*. Observe that if a category has pushouts, it also has pushouts of pairs and that pushouts of univeral monos along pairs are universal.

27.4.2. Let $\mathbf{C}$ be a cocomplete category. The phrase "directed colimits of monos are mono" in part (c) below means that whenever $\{f_i : A \to B_i\}$ is a *directed source of monos*—i.e., a family of monos with common domain such that for any pair of indices i, j, there is an index k such that f_k factors through both f_i and f_j—then the common map from A to the colimit of the family is mono.

(a) Fix a morphism $f : A \to B$ of $\mathbf{C}$. For every morphism $g : A \to D$, let u_g be the pushout of f along g, and for every object D of $\mathbf{C}$, let $\tau_f D : D \to F_f D$ be the morphism from D to the colimit $F_f D$ of the source $\{u_g : g \in \hom(A, D)\}$. Show that F_f is the object part of a functor $F_f : \mathbf{C} \to \mathbf{C}$ and that the morphisms $\tau_f D$ are the components of a natural transformation $\tau_f : I \to F_f$. Moreover, show that $\tau_f A$ factors through f.

(b) The identity functor I is an object of the functor category $[\mathbf{C}, \mathbf{C}]$ of all endofunctors on $\mathbf{C}$, and the comma category $(I, [\mathbf{C}, \mathbf{C}])$ has as objects all natural transformations $I \to G$. Show that for each object A of $\mathbf{C}$, the assignment

$$(\tau : I \to G) \mapsto (\tau A : A \to GA)$$

determines a functor $(I, [\mathbf{C}, \mathbf{C}]) \to (A, \mathbf{C})$, and that this functor is right adjoint to the functor determined by the assignment

$$(f\ A \to B) \mapsto (\tau_f \cdot I \to F_f)$$

(c) Show that if directed colimits of monos are mono in $\mathbf{C}$ and $f \cdot A \to B$ is a universal mono, then τ_f is a natural monomorphism Conclude that every universal mono is the first factor of a component of a natural monomorphism from the identity

28. Relation to the assembly tower

28.1. Definitions. Let $f\ A \to B$ be a frame morphism We say that f is *α-mono* if $\mathsf{N}^\alpha f$ is mono and that f is *α-epi* if the image of $c^\alpha\ B \to \mathsf{N}^\alpha B$ is contained in the image of $\mathsf{N}^\alpha f\ \ \mathsf{N}^\alpha A \to \mathsf{N}^\alpha B$ In symbols: $\mathrm{im}(c^\alpha_B) \subset \mathrm{im}(\mathsf{N}^\alpha f)$

This last condition can be expressed diagramatically in two ways:

$$\begin{tikzcd}
\mathsf{N}^\alpha A \arrow[rr, "\mathsf{N}^\alpha f"] \arrow[dr, two heads] & & \mathsf{N}^\alpha B \\
& \bullet \arrow[ur] & \\
A \arrow[uu, "c^\alpha_A"] \arrow[rr] & & B, \arrow[uu, dashed, "c^\alpha_B"'] \arrow[ul, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{N}^\alpha A \arrow[rr, "\mathsf{N}^\alpha f"] & & \mathsf{N}^\alpha B \\
& \bullet \arrow[ul] \arrow[dr, two heads] & \\
A \arrow[uu, "c^\alpha_A"] \arrow[rr] \arrow[ur, dashed] & & B. \arrow[uu, "c^\alpha_B"']
\end{tikzcd}$$

In the left diagram, the morphism $\mathsf{N}^\alpha f$ has been factored through its image, and c^α_B factors through this In in right diagram, the pullback of $\mathsf{N}^\alpha f$ along c^α_B is onto.

28.2. Here are a few simple facts concerning these notions.

Proposition.

(a) *Composites and first factors of α-monos are α-mono*

(b) *An α-mono is β-mono for all $\beta \le \alpha$.*

(c) *An α-mono composed with a β-mono is a* $\min(\alpha, \beta)$*-mono.*

(d) *Second factors of α-epis are α*-epi.

(e) *An α-epi is β-epi for all $\beta \ge \alpha$.*

PROOF. Part (a) is trivial, since general monos satisfy these conditions and N^α is a functor For part (b), suppose f is α-mono and $\beta \le \alpha$ Let γ be the unique ordinal for which $\alpha + \gamma = \beta$. Then, since $\mathsf{N}^\alpha f$ is mono, so is $(\mathsf{N}^\alpha f) \circ c^\gamma_{\mathsf{N}^\beta A} = c^\gamma_{\mathsf{N}^\beta B} \circ \mathsf{N}^\beta f$, and thus $\mathsf{N}^\beta f$ is mono as well. Part (c) is a simple corollary of parts (a) and (b). Part (d) follows from a simple property of images. namely, the image of a composite is contained in the image of the second factor. Finally, suppose f is α-epi (as in the left diagram above) and $\beta \ge \alpha$. Then, with γ such that $\alpha + \gamma = \beta$, applying N^γ to the image factorization of $\mathsf{N}^\alpha f$ and recalling that N^γ preserves surjections (by 16.4(a) and induction), we see that B also factors through the image of $\mathsf{N}^\beta f$ and thus is β-epi

28.3. Proposition. *If $f . A \to B$ is α-epi and $g . B \to C$ is β-epi, then $g \circ f$ is $(\alpha + \beta)$-epi*

Proof Let γ be such that $\beta + \gamma = \alpha + \beta$, and consider the following diagram.

$$\begin{tikzcd}
 & N^{\alpha+\beta}A \arrow[r, "N^{\alpha+\beta}f"] & N^{\alpha+\beta}B \arrow[r, "N^{\alpha+\beta}g"] & N^{\alpha+\beta}C & \\
N^{\alpha}A \arrow[ur, "c^{\beta}"] \arrow[r, "N^{\alpha}f"] & N^{\alpha}B \arrow[ur, "c^{\beta}"] & N^{\beta}A \arrow[ul, "c^{\gamma}_A"'] \arrow[r, "N^{\beta}f"] & N^{\beta}B \arrow[ul, "c^{\gamma}_B"'] \arrow[r, "N^{\beta}g"] & N^{\beta}C \arrow[ul, "c^{\gamma}_C"'] \\
 & A \arrow[ul, "c^{\alpha}"] \arrow[ur, "c^{\beta}"] \arrow[r, "f"'] & B \arrow[ul, "c^{\alpha}_B"'] \arrow[ur, "c^{\beta}"] \arrow[r, "g"'] & C \arrow[ur, "c^{\beta}_C"'] &
\end{tikzcd}$$

(Notice that I have simplified the subscripts on the arrows labelled c^{γ}.) Since g is β-epi, $\operatorname{im}(c^{\beta}_C) \subset \operatorname{im}(N^{\beta}g)$. Thus, by composing with c^{γ}_C, we have

$$\operatorname{im}(c^{\alpha+\beta}_C) \subset \operatorname{im}(c^{\gamma}_C \circ N^{\beta}g) = \operatorname{im}((N^{\alpha+\beta}g) \circ c^{\gamma}_B). \tag{1}$$

Now, since f is α-epi, $\operatorname{im}(c^{\alpha}_B) \subset \operatorname{im}(N^{\alpha}f)$. Applying the functor N^{β}, we get $\operatorname{im}(c^{\gamma}_B) \subset \operatorname{im}(N^{\alpha+\beta}f)$, so that composing with $N^{\alpha+\beta}g$ yields

$$\operatorname{im}((N^{\alpha+\beta}g) \circ c^{\gamma}_B) \subset \operatorname{im}((N^{\alpha+\beta}g) \circ N^{\alpha+\beta}f) = \operatorname{im}(N^{\alpha+\beta}(g \circ f)) \tag{2}$$

Putting together (1) and (2), we have $\operatorname{im}(c^{\alpha+\beta}_C) \subset \operatorname{im}(N^{\alpha+\beta}(g \circ f))$, as required.

28.4. Other properties, and distinctness. Informally, the notions of α-mono and α-epi give a "ranking" to arbitrary morphisms: if $\alpha < \beta$, then an β-mono is "more mono" than a α-mono, and an α-epi is "more epi" than a β-epi. (Formally, we could define the *mono-rank* of a morphism f to be the (possibly empty or total) initial segment of the ordinals consisting of those α for which f is α-mono, and the *epi-rank* of a morphism to be the smallest ordinal α for which f is α-epi (or ∞ if there aren't any), but this technicality won't be necessary.) At one extreme of the ranking are the 0-epis, which are just the surjections, and the monos that are α-mono for all ordinals α, which we will prove shortly, are precisely the universal monos. At the other extreme, 0-mono is synonymous with mono, and Madden and Molitor [33] show that (in our terminology) a morphism is epi iff it is α-epi for some, and therefore for any sufficiently large, ordinal α.

We have already seen what happens to the ranks of monos and epis under composition and, respectively, first and second factors. Although pullbacks of monos are always mono, there doesn't seem to be any connection between the rank of a mono and the rank of its pullback. On the other hand, we will show (Corollary 28.9) that ranks of epis are preserved under pushout.

Finally, we consider examples showing that the notions of α-mono and α-epi are different for each ordinal α; stated another way, each possible mono- and epi-rank is achieved by some morphism. Let A be any non-reflective frame (for example the free frame on ω), fix an ordinal α, and let B be the cBa $(N^{\alpha}A)_{\neg\neg}$. Consider the morphism $f_{\alpha} = \neg\neg \circ c^{\alpha}_A : A \to B$. Since B is a cBa, it follows that for all $\beta \leq \alpha$,

$N^\beta f_\alpha = \neg\neg \circ c^{\gamma}_{N^\beta A}$, where $\beta + \gamma = \alpha$. This is mono for all $\beta < \alpha$ but not mono when $\beta = \alpha$ (for then it is just $\neg\neg$, which is mono only if A is reflective). Thus f_α is β-mono for all $\beta < \alpha$, but it is not α-mono. By similar reasoning, $N^\beta f_\alpha$ is onto iff $\beta \geq \alpha$, and hence f_α is α-epi but not β-epi for any $\beta < \alpha$.

28.5. Characterization of universal monos. The next two theorems are the main results of the section. They characterize universal monos in terms of their mono-rank and, conversely, the mono-rank of a morphism in terms of its degree of "universality".

Theorem. *A morphism f is a universal mono if and only if it is α-mono for every ordinal α.*

PROOF. Suppose f is universal, let α be an ordinal, and consider the diagram

$$\begin{tikzcd}
N^\alpha A \arrow[rr, "N^\alpha f"] \arrow[dr, "f'"] & & N^\alpha B \\
& \bullet \arrow[ur, dashed] & \\
A \arrow[uu, dotted, "c^\alpha_A"] \arrow[rr, "f"'] & & B, \arrow[uu, "c^\alpha_B"'] \arrow[ul, "d"']
\end{tikzcd}$$

where the lower-left "square" is a pushout and the dotted morphism is the unique morphism from the pushout. Since f is universal, f' is mono. The morphism d is epi, since it is the pushout of an epi, and therefore, since c^α_B is universal, the dotted morphism is also mono (in fact universal) by 27.2(c). Thus the composite $N^\alpha f$ is mono, and hence f is α-mono.

Conversely, suppose f is α-mono for all ordinals α, let $h : A \to D$ be an arbitrary morphism, and consider the two diagrams

$$\begin{tikzcd}
D \arrow[r, "f'"] & P \\
A \arrow[u, "h"] \arrow[r, "f"'] & B \arrow[u, "h'"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
N^\infty D \arrow[r, "N^\infty f'"] & N^\infty P \\
N^\infty A \arrow[u, "N^\infty h"] \arrow[r, "N^\infty f"'] & N^\infty B, \arrow[u, "N^\infty h'"']
\end{tikzcd}
\qquad (1)$$

where the first diagram is a pushout in **Frm** and the second, living in the category ∞-**cBa** (and hence *also in* ∞-**Frm**), is the functor N^∞ applied to the first. (See Section 18.4 for information about ∞-**cBa**, ∞-**Frm**, and the functor N^∞.) Since N^∞ is a left adjoint, it preserves colimits, and so the second diagram is a pushout in ∞-**cBa** (but, incidentally, not necessarily in ∞-**Frm**). Also, since $N^\alpha f$ is mono for all α, $N^\infty f$ is mono in ∞-**Frm** (since by 6.5.5 it is a union of monos), and hence also in ∞-**cBa**. But all monos are universal in ∞-**cBa** by 18.3(a), and so $N^\infty f'$ is mono in ∞-**cBa**. Thus $N^\infty f'$ is mono in ∞-**Frm**, and hence f' is mono in **Frm**.

28.6. An alternate proof. The second part of the proof above uses the categories ∞-**cBa** and ∞-**Frm**, which depend (in a set-theoretical foundation for category theory) on an inaccessible cardinal or a Grothendieck universe, as well as implying a limitation on the "frames" considered, this is discussed in Section 6.6 (see also 18.4). As also mentioned there, these assumptions and limitations are not necessary, but merely convenient. For illustration, we give an alternate proof of 28.5 using only ZFC. The proof starts with the following lemma, which answers the question asked at the end of [33]: "Is the morphism e^κ_A (in the diagram below) always mono?".

Lemma. *Let κ be a regular cardinal. Then the morphism e^κ_A in the diagram*

$$\begin{tikzcd} A \arrow[rr, "c^\kappa_A"] \arrow[dr, "d^\kappa_A"'] & & \mathsf{N}^\kappa A \\ & B^\kappa A \arrow[ur, "e^\kappa_A"'] & \end{tikzcd}$$

of κ-frames, as constructed in 18.2, is (a universal) mono. (Note that we have implicitly applied the forgetful functor U^κ_∞ to $c^\kappa_A : A \to \mathsf{N}^\kappa A$.)

PROOF. The notation follows that of Section 18. We first extend the functor $\mathsf{N}^\kappa : \mathbf{Frm} \to \mathbf{Frm}$ to a functor $\mathsf{N}^\kappa : \kappa\text{-}\mathbf{Frm} \to \kappa\text{-}\mathbf{Frm}$ by precomposing with the free functor $F^\kappa_\infty : \kappa\text{-}\mathbf{Frm} \to \mathbf{Frm}$; that is, for every κ-frame A, we define $\mathsf{N}^\kappa A := \mathsf{N}^\kappa F^\kappa_\infty(A)$. Then, the morphisms $c^\kappa_A \circ \iota : A \to F^\kappa_\infty(A) \to \mathsf{N}^\kappa A$ are mono and, moreover, components of a natural transformation $I \to \mathsf{N}^\kappa$ from the identity functor on κ-$\mathbf{Frm}$. By Example (iv) of 27.3, these morphisms are therefore universal in κ-$\mathbf{Frm}$ (just as, when A is a frame, we already knew they were in $\mathbf{Frm}$). Since d^κ_A is epi and κ-$\mathbf{Frm}$ has pushouts, we can therefore apply 27.2(c) to conclude that e^κ_A is universal.

Theorem (bis). *A frame morphism f is a universal mono if and only if it is α-mono for every ordinal α.*

PROOF. The "only if" direction is just as before.

For the "if" direction, we start with the same pushout appearing as the first diagram of (1) above. As is explained in 12.4, this pushout becomes a pushout in κ-$\mathbf{Frm}$ when κ is large enough (for instance, $\kappa = |B \oplus_A C|^+$ suffices). Then, applying the Boolean reflector $\mathsf{B}^\kappa : \kappa\text{-}\mathbf{Frm} \to \kappa\text{-}\mathbf{cBa}$, we obtain a pushout in κ-$\mathbf{cBa}$. Thus, if we can show that $\mathsf{B}^\kappa f$ is mono, then the rest of the proof proceeds as before with ∞ replaced by κ. But to show this, we can form (in the category κ-$\mathbf{Frm}$) the diagram

$$\begin{tikzcd} \mathsf{N}^\kappa A \arrow[rrr, "\mathsf{N}^\kappa f"] & & & \mathsf{N}^\kappa B \\ & \mathsf{B}^\kappa A \arrow[ul, "e^\kappa_A"'] \arrow[r, "\mathsf{B}^\kappa f"] & \mathsf{B}^\kappa B \arrow[ur, "e^\kappa_B"] & \\ A \arrow[uu, "c^\kappa_A"] \arrow[ur, "d^\kappa_A"] \arrow[rrr, "f"'] & & & B, \arrow[uu, "c^\kappa_B"'] \arrow[ul, "d^\kappa_B"'] \end{tikzcd}$$

in which the morphism e^κ_A is mono by the Lemma. Thus, as $\mathsf{N}^\kappa f$ is mono by assumption, so is $(\mathsf{N}^\kappa f) \circ e^\kappa_A = e^\kappa_B \circ \mathsf{B}^\kappa f$, implying that $\mathsf{B}^\kappa f$ is also mono, as required.

28.7. The universality of α-monos. As the second main result of this section, we obtain a kind of converse to Theorem 28.5. We start with a Lemma that is actually the case $\alpha = 0$ of the theorem that follows it.

Lemma. *Suppose that f is a frame morphism Then $\mathsf{N}f$ is mono if and only if pushouts of f along surjections are mono*

PROOF The proof refers to the following diagram:

$$\begin{tikzcd}
B \arrow[r, "g'"] & \bullet \arrow[r, equal] & \bullet \\
A \arrow[u, "f"] \arrow[r, "g"'] & P \arrow[u, "f'"'] \arrow[r, "h"'] & Q \arrow[u, "f''"']
\end{tikzcd}$$

Suppose first that $\mathsf{N}f$ is mono, g is onto, and the left square above is a pushout: we show that f' is mono Factor f' as a surjection h followed by an injection f'', as shown in the right square Since h is epi, the right square is a pushout by part (c) of the Pushout Lemma (6 4), and hence the outer rectangle is a pushout by part (a) Therefore. $(\mathsf{N}f)(\ker g) = (\mathsf{N}f)(\ker(h \circ g))$ and. using that $\mathsf{N}f$ is mono, $\ker g = \ker(h \circ g)$ Thus h is an isomorphism and f' is mono.

Conversely, suppose that pushouts of f along surjections are mono. Let θ_1 and θ_2 be congruences on A such that

$$(\mathsf{N}f)(\theta_1) = (\mathsf{N}f)(\theta_2) \tag{1}$$

Since $(\mathsf{N}f)(\theta_1) = (\mathsf{N}f)(\theta_2)$ iff $(\mathsf{N}f)(\theta_1 \wedge \theta_2) = (\mathsf{N}f)(\theta_1 \vee \theta_2)$, we may assume in what follows that $\theta_1 \leq \theta_2$. Put $P = A/\theta_1$ and $Q = A/\theta_2$ with natural maps as shown in the bottom row of the diagram, and assume that both squares are pushouts; by (1), the pushout of h along f' is an isomorphism. But now, by assumption on f, we have that f' is mono, and so, composing with the isomorphism and taking the first factor, h is mono as well. Thus $\theta_1 = \theta_2$, showing that $\mathsf{N}f$ is mono.

28.8. Theorem. *A frame morphism f is $(\alpha+1)$-mono if and only if the pushout of f along every α-epi is mono*

PROOF. Let $f : A \to B$ be such that pushouts of f along α-epis are mono Then, for any congruence θ on $\mathsf{N}^\alpha A$, we construct the following diagram:

$$\begin{tikzcd}
B \arrow[rr, "c_B^\alpha"] \arrow[dr, "c'"] & & \mathsf{N}^\alpha B \arrow[rr, "p''"] & & \bullet \\
& \bullet \arrow[ur, dashed, "u"] \arrow[rr, "p'"] & & \bullet \arrow[ur, dashed, "u'"] & \\
A \arrow[uu, "f"] \arrow[rr, "c_A^\alpha"'] & & \mathsf{N}^\alpha A \arrow[uu, "\mathsf{N}^\alpha f"'] \arrow[ul, "f'"] \arrow[rr, "p"'] & & (\mathsf{N}^\alpha A)/\theta. \arrow[ul, "f''"] \arrow[uu, "g"']
\end{tikzcd}$$

Here, the right square is a pushout, f' is the pushout of f along c_A^α, and f'' is the pushout of f' along p (and thus also a pushout of f along $p \circ c_A^\alpha$ by part (a) of the Pushout Lemma) The dotted morphism u is the unique morphism such that

$$u \circ c' = c_B^\alpha \quad \text{and} \quad u \circ f' = \mathsf{N}^\alpha f, \tag{1}$$

and u' is the unique morphism such that (with the help of the first part of (1))

$$u' \circ p' \circ c' = p'' \circ c^{\alpha}_{B} = p'' \circ u \circ c' \qquad \text{and} \qquad u' \circ f'' = g. \tag{2}$$

Since c' is the pushout of an epi, it is epi as well, and so we conclude from the first part of (2) that $u' \circ p' = p'' \circ u$. As in first part of the proof of 28.5, u is universal Now, both the right square and the lower-right "oblique" square (with f' and p as base) are pushouts, and both of the adjoining triangles commute by the second parts of (1) and (2) Thus, since the upper-right oblique square is commutative (as we pointed out above), it is also a pushout by part (b) of the Pushout Lemma. Hence u' is mono But, since $p \circ c^{\alpha}_{A}$, being the composite of an α-epi and a 0-epi, is α-epi by 28 3, f'' is therefore mono by the assumption on f, and so g is mono Since θ was arbitrary, we conclude by the Lemma that $\mathsf{N}^{\alpha+1} f$ is mono, and hence that f is $(\alpha+1)$-mono.

For the other direction, suppose $g\colon A \to C$ is an arbitrary α-epi, and consider the following diagram

$$\begin{tikzcd}
\mathsf{N}^{\alpha}B \arrow[rrrr, "\mathsf{N}^{\alpha}g'"] \arrow[drr, "q'"] & & & & \mathsf{N}^{\alpha}P \\
& & Q' \arrow[urr, dashed, "e"] \arrow[dr, dashed, "d"] & & \\
& B \arrow[ul, "c^{\alpha}_{B}"] \arrow[rr, "g'"] & & P \arrow[ur, "c^{\alpha}_{P}"'] & \\
& A \arrow[u, "f"] \arrow[rr, "g"] \arrow[dl, "c^{\alpha}_{A}"'] & & C \arrow[u, "f'"'] \arrow[dr, "c^{\alpha}_{C}"] & \\
& & Q \arrow[uuu, "v"] \arrow[ur, tail, "m"] \arrow[drr, "i"] & & \\
\mathsf{N}^{\alpha}A \arrow[uuuuu, "\mathsf{N}^{\alpha}f"] \arrow[urr, "q"] \arrow[rrrr, "\mathsf{N}^{\alpha}g"'] & & & & \mathsf{N}^{\alpha}C \arrow[uuuuu, "\mathsf{N}^{\alpha}f'"']
\end{tikzcd}$$

This diagram is constructed as follows. starting with f and g, we form the pushout, given by the *inner square* with vertex P. The outer square is N^{α} applied to the inner one, and we factor $\mathsf{N}^{\alpha}g$ as a surjection q followed by an injection i Since g is α-epi, c^{α}_{C} factors through Q by a morphism m, which is therefore mono, as shown. Next, the middle vertical arrow, labelled v, is the pushout of $\mathsf{N}^{\alpha}f$ along q. By the universal property of this pushout, we have the dotted morphism e such that

$$e \circ q' = \mathsf{N}^{\alpha}g' \qquad \text{and} \qquad e \circ v = \mathsf{N}^{\alpha}f' \circ i. \tag{3}$$

Finally, since

$$q' \circ c^{\alpha}_{B} \circ f = q' \circ (\mathsf{N}^{\alpha}f) \circ c^{\alpha}_{A} = v \circ q \circ c^{\alpha}_{A} = v \circ m \circ g,$$

the universal property of the inner pushout gives the dotted morphism d such that

$$d \circ g' = q' \circ c^{\alpha}_{B} \qquad \text{and} \qquad d \circ f' = v \circ m. \tag{4}$$

Now, having constructed this diagram, we can complete the proof of the theorem: If f is $(\alpha+1)$-mono, then, by the Lemma, pushouts of $\mathsf{N}^{\alpha}f$ along surjections are mono; in particular, v is mono But then $v \circ m$, which equals $d \circ f'$ (as in the second equation of (4)), is mono, and thus so is f'

28.9. Stability of epi-rank under pushout. In the second part of the proof above, we didn't use all of the information contained in the diagram constructed there. In particular, we didn't use (3) or the first equation of (4). Extracting this information, we get the following corollary.

Corollary. *The pushout of an α-epi along any morphism is α-epi.*

PROOF. The proof uses the diagram constructed above, starting with an α-epi $g : A \to C$ and a morphism $f : A \to B$, which we assume this time is arbitrary. As pointed out in 28.4, every α-epi is epi; thus g is epi and therefore so is g'. Now, using the first equations of (4) and (3),

$$e \circ d \circ g' = e \circ q' \circ c_B^\alpha = (\mathsf{N}^\alpha g') \circ c_B^\alpha = c_P^\alpha \circ g'$$

Since g' is epi, we conclude that $e \circ d = c_P^\alpha$. But then, since q' is onto it factors through the image of $\mathsf{N}^\alpha g'$, showing that c_P^α (via d) does as well. Hence, g' is α-epi, and the proof is complete.

29. Combinatorial Morphisms

29.1. Free extensions of frames. The theorem below gives a concrete description of the free extension of a frame A by a set X of "indeterminates". First, however, we look at how this description was derived. The free extension of A by generators X is isomorphic to the coproduct $A \oplus F(X)$, where $F(X)$ is the free frame on X, can easily be seen from the universal property involved. By 12.5, this frame is isomorphic to the frame of all functions $\phi : F(X) \to A$ such that $\phi(\bigvee S) = \bigwedge \phi(S)$ for all $S \subset F(X)$, ordered pointwise. Since $F(X)$ is the free $\bigvee$-semilattice on the $\wedge$-semilattice $\mathsf{P}_f X$ of finite subsets of X ordered by reverse inclusion, such operators ϕ are completely determined by their values on $\mathsf{P}_f X$, which can be arbitrarily chosen subject only to the restriction that ϕ turn into meets any joins of $F(X)$ under which $\mathsf{P}_f X$ is closed. But the only such joins are the trivial ones: $s \cap t = s$ for $s \subset t$, and so ϕ must be anti-monotone, but otherwise can be arbitrary. Finally, instead of considering *anti-monotone* functions on the finite subsets of X ordered by *reverse* inclusion, we prefer to turn things right side up.

Thus, with this motivation, we define $A[X]$ to be the set of all monotone functions $\mathsf{P}_f X \to A$, ordered pointwise, where $\mathsf{P}_f X$ is ordered by inclusion. We denote elements of $\mathsf{P}_f X$ by $s, t, \ldots$, and elements of $A[X]$ by $m, n, \ldots$. Just as with polynomials in X over a ring A, we may think of an element $m \in A[X]$ as assigning a "coefficient" $m(s)$ to each "monomial" $s \in \mathsf{P}_f X$ (be careful, though: these polynomials have been "saturated" to make them unique, and lattice arithmetic is different from ring arithmetic). For each $a \in A$ and $x \in X$, we define the following elements of $A[X]$ by their action on an arbitrary $s \in \mathsf{P}_f X$:

$$m_a(s) = a;$$

$$m_x(s) = \begin{cases} 1 & \text{if } x \in s, \\ 0 & \text{otherwise} \end{cases}$$

The functions m_a and m_x are clearly monotone. Finally, we define $\Delta : A \to A[X]$ by $\Delta(a) = m_a$ for every $a \in A$. We can now state the following result.

Theorem. *For any frame A and set X, $A[X]$ is a frame and $\Delta : A \to A[X]$ is a frame morphism making $A[X]$ freely generated over A with the mapping $x \mapsto m_x$ as the insertion of generators. Meets and joins in the frame $A[X]$ are pointwise, and the arrow operation is given by the formula*

$$(m \to n)(s) = \bigwedge_{s \subseteq t} m(t) \to n(t) \tag{1}$$

For any frame morphism $f : A \to B$ and mapping $\gamma : X \to B$, the unique extension $\overline{f} : A[X] \to B$ is given by

$$\overline{f}(m) = \bigvee_{s \in \mathrm{P}_f X} f(m(s)) \wedge \bigwedge \gamma(s) \tag{2}$$

PROOF. Arbitrary pointwise meets and joins of monotone functions are monotone, and so $A[X]$ is a subframe of the product $A^{\mathrm{P}_f X}$. (Elements of this product will be written, as usual, as functions $\sigma : \mathrm{P}_f X \to A$.) The function Δ is just the diagonal morphism, and hence is a frame morphism. I claim that the right adjoint $\imath_*$ to the inclusion $\imath : A[X] \to A^{\mathrm{P}_f X}$ is given by the formula

$$\imath_*(\sigma)(s) = \bigwedge_{s \subseteq t} \sigma(t) \tag{3}$$

Indeed, for any s, we have $m(s) = \bigwedge_{s \subseteq t} m(t)$ by monotonicity and so $m = \imath_*(\imath(m))$ for every $m \in A[X]$; conversely, for any s, we clearly have $\sigma(s) \geq \imath_*(\sigma)(s)$, and therefore $\sigma \geq \imath(\imath_*(\sigma))$ for every $\sigma \in A^{\mathrm{P}_f X}$. Since meets are computed pointwise in $A[X]$, it follows that the arrow in $A[X]$ is given by reflecting the pointwise arrow using (3). But this is just (1).

We now show the extension property. First, note that for any $a \in A$, $m_a \wedge \bigwedge_{x \in s} m_x$ is that monotone function which is a on all sets containing s, and 0 otherwise. It therefore follows easily by monotonicity that every $m \in A[X]$ can be written

$$m = \bigvee_{s \in P_f X} m_{m(s)} \wedge \bigwedge_{x \in s} m_x \tag{4}$$

Thus, $A[X]$ is generated (using finite meets and arbitrary joins) by $\{m_a : a \in A\} \cup \{m_x : x \in X\}$ and moreover, the formula (2) is defined in the only way possible, given (4). Thus, it remains to show that $\overline{f}$ as defined in (2) is a frame morphism. The preservation of joins is easy, since joins are pointwise in $A[X]$, are preserved by f, and commute with both binary meets and arbitrary joins in A. Since $\overline{f}$ is therefore monotone, the inequality $\overline{f}(m \wedge n) \leq \overline{f}(m) \wedge \overline{f}(n)$ is clear, and since $\bigwedge \gamma(\emptyset) = 1 = f(1)$, $\overline{f}(1) = \overline{f}(m_1) = 1$ is likewise clear. For the final inequality, we use the monotonicity of m, n, and f to get

$$\begin{aligned}
\overline{f}(m) \wedge \overline{f}(n) &= \Big(\bigvee_{s \in P_f X} f(m(s)) \wedge \bigwedge \gamma(s) \Big) \wedge \Big(\bigvee_{t \in P_f X} f(n(t)) \wedge \bigwedge \gamma(t) \Big) \\
&= \bigvee_{s,t \in P_f X} f(m(s) \wedge n(t)) \wedge \bigwedge \gamma(s \cup t) \\
&\leq \bigvee_{s,t \in P_f X} f(m(s \cup t) \wedge n(s \cup t)) \wedge \bigwedge \gamma(s \cup t) \\
&= \bigvee_{u \in P_f X} f((m \wedge n)(u)) \wedge \bigwedge \gamma(u) \\
&= \overline{f}(m \wedge n)
\end{aligned}$$

29.2. $A[X]$ **as a functor of** A. Let X be a set. For each frame morphism $f : A \to B$, there is by the universal property of $A[X]$ a unique morphism $f[X] : A[X] \to B[X]$ such that $f[X](m_x) = m_x$ for all $x \in X$ and such that the left square of

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "f"'] & A[X] \arrow[r, "i"] \arrow[d, "f[X]"] & A^{P_f X} \arrow[d, "f^{P_f X} = f \circ -"] \\
B \arrow[r, "\Delta"'] & B[X] \arrow[r, "i"'] & B^{P_f X}
\end{tikzcd}$$

commutes It is easily seen (e g , by using (4) above) that this morphism is given by $f[X](m) = f \circ m$. Thus the right square also commutes By the isomorphism $A[X] \simeq A \oplus F(X)$ and the fact that colimits commute with themselves (or just argue from the universal property), the left square above is a pushout. Since pushouts preserve products (see Section 1.9), the outer rectangle is also pushout, so that, by the Pushout Lemma (6 4), the right square is as well.

29.3. Congruences on $A[X]$. Just as any frame is a quotient of a free frame $F(X)$ for some set X, so any frame $A \to B$ over A is a quotient of some free extension $A[X]$ of A. We are thus led to consider congruences on $A[X]$. If $\{\theta_s\}$ is a family of congruences on A indexed by $P_f X$, then the relation $\prod_s \theta_s$ on $A[X]$, defined by

$$m \, (\textstyle\prod_s \theta_s) \, n \quad \text{iff} \quad \forall s \in P_f X \;\; m(s) \, \theta_s \, n(s) \tag{1}$$

is clearly a congruence on $A[X]$ In general, a congruence on any subframe of a product will be called *standard* if it is the restriction of a product of congruences, as in (1); by 11.11, all congruences on "full" frame products are standard. On $A[X]$, however, there can be non-standard congruences: see Exercise 30.6.1.

29.4. Standardization. If i is the inclusion $A[X] \to A^{P_f X}$, then

$$\mathsf{N}(A[X]) \xrightarrow{\mathsf{N}i} \mathsf{N}(A^{P_f X}) \xrightarrow{\simeq} (\mathsf{N}A)^{P_f X}$$

takes congruences on $A[X]$ to the congruences they generate on the product. and $(\mathsf{N}i)_*$ restricts congruences on the product to the subframe. It follows that the mapping

$\theta \mapsto (\mathsf{N}i).((\mathsf{N}i)\theta)$ gives the smallest standard congruence of $A[X]$ greater than θ, i e, its "standardization" It also follows from this that the standard congruences on $A[X]$ form a maxset in $\mathsf{N}(A[X])$

The following result gives an explicit description of standardization, to state it, we introduce some notation If $m\ \ \mathsf{P}_f X \to A$ is a function, $s \in \mathsf{P}_f X$, and $a \in A$, then we denote by $m\langle a/s\rangle$ that function which is identical to m except (possibly) at s, where its value is a If m is monotone, then we say that the *substitution* $\langle a/s\rangle$ is *appropriate* for m if $m\langle a/s\rangle$ is still monotone. Note that, because operations are pointwise, $m\langle a/s\rangle \wedge n\langle b/s\rangle = (m \wedge n)\langle a \wedge b/s\rangle$, etc

For every $a \in A$ and $s \in \mathsf{P}_f X$, we introduce the monotone "test function" $\delta_{c,s}$: $\mathsf{P}_f X \to A$, defined for every $t \in \mathsf{P}_f X$ by

$$\delta_{c,s}(t) = \begin{cases} 0, & \text{if } t \subset s \text{ and } t \neq s, \\ c, & \text{if } t = s, \\ 1 & \text{otherwise} \end{cases}$$

Clearly, every substitution $\langle a/s\rangle$ is appropriate for $\delta_{c,s}$, and we have

$$\delta_{c,s}\langle a/s\rangle = \delta_{a,s}. \tag{1}$$

The reader may verify the following important property of these test functions.

$$\delta_{m(s),s} = (m \wedge \bigwedge_{x \in s} m_x) \vee \bigvee_{x \in X \setminus s} m_x \qquad (m \in A[X], s \in \mathsf{P}_f X) \tag{2}$$

The importance of this property is that the "coefficient" of s in m can be extracted (with the test function as result) using finite meets and arbibrary joins with elements of $A[X]$ not depending on m

Theorem. *With notation as above, suppose that θ is a congruence on A Then for every $s \in \mathsf{P}_f X$ the following conditions on a pair of elements $a, b \in A$ are equivalent*

(a) $m\langle a/s\rangle\, \theta\, m\langle b/s\rangle$ *for every (appropriate)* $m \in A[X]$.

(b) $\delta_{a,s}\, \theta\, \delta_{b,s}$

(c) $m\langle a/s\rangle\, \theta\, m\langle b/s\rangle$ *for some* $m \in A[X]$.

These conditions define a congruence θ_s on A, and the mapping $\mathsf{N}(A[X]) \to \mathsf{N}(A^{\mathsf{P}_f X})$ given by $\theta \mapsto \prod_s \theta_s$ is the standardization morphism $\mathsf{N}i$, *which moreover is onto*

PROOF We first show the equivalence of (a)–(c) If (a) holds, then for any $c \in A$ we can take $m = \delta_{c,s}$ and use (1) to get (b). The implication $(b)\Rightarrow(c)$ is trivial. For $(c)\Rightarrow(a)$, let m be as in (c), and suppose that $n \in A[X]$ is appropriate for both $\langle a/s\rangle$ and $\langle b/s\rangle$. Then n is also appropriate for $\langle a \wedge b/s\rangle$ and $\langle a \vee b/s\rangle$, and so by reflexitivity of θ we have

$$n\langle a \wedge b/s\rangle\, \theta\, n\langle a \wedge b/s\rangle \qquad \text{and} \qquad n\langle a \vee b/s\rangle\, \theta\, n\langle a \vee b/s\rangle. \tag{3}$$

Now, we can join $m\langle a/s\rangle\, \theta\, m\langle b/s\rangle$ with the left side of (3) to obtain

$$(m \vee n)\langle a/s\rangle = m\langle a/s\rangle \vee n\langle a \wedge b/s\rangle\, \theta\, m\langle b/s\rangle \vee n\langle a \wedge b/s\rangle = (m \vee n)\langle b/s\rangle,$$

and thus $(m \vee n)\langle a/s\rangle\, \theta\, (m \vee n)\langle b/s\rangle$ and we can can meet this with the right side of (3) to obtain

$$n\langle a/s\rangle = ((m \vee n) \wedge n)\langle a \wedge (a \vee b)/s\rangle\, \theta\, ((m \vee n) \wedge n)\langle b \wedge (a \vee b)/s\rangle = n\langle b/s\rangle.$$

and thus $n(a/s)\ \theta\ n(b/s)$ Hence (c) implies (a).

Next. since $\delta_{\bigvee T,s} = \bigvee \delta_{T,s}$ and $\delta_{\bigwedge T,s} = \bigwedge \delta_{T,s}$ for every $T \subset A$, it is clear from (b) that θ_s is a congruence Optimistically denoting the mapping $\theta \mapsto \prod_s \theta_s$ by $\mathsf{N}i$, we now prove that $\mathsf{N}i \dashv (\mathsf{N}i)_*$. For every $s \in \mathsf{P}_f X$ and $a, b \in A$, using condition (b) we get

$$a\,(\textstyle\prod_s \theta_s)_s\, b \quad \text{iff} \quad \delta_{a,s}\,(\textstyle\prod_s \theta_s)\,\delta_{b,s} \quad \text{iff} \quad a\,\theta_s\,b.$$

and so $\prod_s (\prod_s \theta_s)_s = \prod_s \theta_s$, proving that $(\mathsf{N}i) \circ (\mathsf{N}i)_*$ is the identity. Thus it remains to show that $\theta \leq (\mathsf{N}i)_*((\mathsf{N}i)(\theta))$ for all congruences θ on $A[X]$. So, let θ be such and suppose that $m\ \theta\ n$. Then, writing

$$S = \bigwedge_{x \in s} m_x \qquad \text{and} \qquad S' = \bigvee_{x \in X \setminus s} m_x,$$

we can use (2) to get. for every $s \in \mathsf{P}_f X$,

$$\delta_{m(s),s} = (m \wedge S) \vee S'\ \theta\ (n \wedge S) \vee S' = \delta_{n(s),s}$$

and therefore $m(s)\ \theta_s\ n(s)$, by (b) Hence, $m(\prod_s \theta_s)n$

Finally, since $(\mathsf{N}i) \circ (\mathsf{N}i)_*$ is the identity, $\mathsf{N}i$ is onto, and the proof is complete

29.5. Extraction congruences. The notion of "extraction" of coefficients via meet and join with fixed elements. as described above, is naturally associated with a congruence.

Lemma. *Let A be a distributive lattice or a frame. Then,*

(a) *For every four elements $a, b. x, y \in A$, we have*

$$(x \wedge a) \vee b = (y \wedge a) \vee b \quad \textit{if and only if} \quad (x \vee b) \wedge a = (y \vee b) \wedge a. \tag{1}$$

(b) *Fixing $a. b \in A$, the relation $\hat\theta(a, b)$ between x and y defined by the equations in (1) is a congruence on A If A is a frame, $\hat\theta(a, b)$ is the congruence associated to the nucleus $u(a) \vee c(b)$.*

Proof. (a) Applying the operation $- \wedge a$ to the first equation and using the distributivity law twice yields the second equation. Similarly, applying $- \vee b$ to the second equation yields the first.

(b) Since the operation $(- \wedge a) \vee b$ preserves binary meets and all existing non-empty joins, its kernel is clearly is a congruence. When A is a frame, we have, by 15.5, $(u(a) \vee c(b))x = (u(a) \circ c(b))x = a \rightarrow (b \vee x)$, and so the equality of the associated congruence and $\theta'(a, b)$ follows from 8 3.

29.6. Test functions and extraction congruences. We extend the notation $\hat\theta(a, b)$ of the Lemma to sets $S, T \subset A$ by

$$\hat\theta(S\ T) = \hat\theta(\bigwedge S, \bigvee T).$$

The relation between test functions and extraction congruences is spelled out in the following result.

Proposition. *Suppose f $A \to B$ is a frame morphism, $\gamma : X \to B$ is a mapping, and $\overline{f}$ $A[X] \to B$ is the unique extension of f with $\overline{f}(m_x) = \gamma(x)$ for all $x \in X$. Let $s \in \mathsf{P}_f X$, and let $\hat{\theta}_s$ denote the congruence $\hat{\theta}(\gamma(s), \gamma(X \setminus s))$. Then*

(a) $\overline{f}(\delta_{a,s}) = \overline{f}(\delta_{b,s})$ *if and only if $f(a)\, \hat{\theta}_s\, f(b)$, for all $a, b \in A$ and*

(b) $f(m(s))\, \hat{\theta}_s\, f(n(s))$ *if and only if $\overline{f}(m)\, \hat{\theta}_s\, \overline{f}(n)$, for every m $n \in A[X]$*

Proof. Suppose $m, n \in A[X]$ We first show that

$$\overline{f}(\delta_{m(s),s}) = \overline{f}(\delta_{n(s),s}) \quad \text{iff} \quad \overline{f}(m)\, \hat{\theta}_s\, \overline{f}(n) \tag{1}$$

By the basic property 29 4(2) of test functions,

$$\overline{f}(\delta_{m(s),s}) = \overline{f}\Big((m \wedge \bigwedge_{x \in s} m_x) \vee \bigvee_{x \in X \setminus s} m_x\Big) \tag{2}$$

Since $\overline{f}$ preserves the meets and joins in (2) (s is finite), and since $\overline{f}(m_x) = \gamma(x)$ by assumption, (2) becomes

$$\overline{f}(\delta_{m(s),s}) = (\overline{f}m \wedge \bigwedge \gamma(s)) \vee \bigvee \gamma(X \setminus s) \tag{3}$$

We have a similar formula for $\overline{f}(\delta_{n(s),s})$, and so (1) holds by the definition of $\hat{\theta}_s$

Now to prove (a), we let $a, b \in A$ and apply (1) to $m = m_a$ and $n = m_b$ Since $\overline{f}(m_a) = f(a)$ and $\overline{f}(m_b) = f(b)$ by assumption, this proves (a) To prove (b), it suffices by (1) to prove that $\delta_{m(s),s} = \delta_{m_{m(s)}(s),s}$, and similarly for n. But this is clear, since $m_{m(s)}(s) = m(s)$.

29.7. Combinatorial morphisms. Let us call a morphism $A \to B$ *combinatorial* if it is isomorphic to one of the form $A \to A[X]/\prod_s \theta_s$, for some family $\{\theta_s\}$ of congruences on A indexed by $\mathsf{P}_f X$. Although we don't expect combinatorial morphisms to behave nicely with respect to pushouts, so that for example the right-hand square (and hence the whole rectangle) of the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A[X] & \longrightarrow & A[X]/\prod_s \theta_s \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[X]} & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & B[X] & \longrightarrow & B[X]/\prod_s \theta'_s, \end{array} \qquad \begin{array}{c} \theta'_s = (\mathsf{N}f)(\theta_s), \\ g(m/\prod_s \theta_s)(s) = f(m(s))/\prod_s \theta'_s, \end{array} \tag{1}$$

is always a pushout (for the same reason that we don't expect direct images of maxsets to be maxsets), pushouts do respect the *process* of standarization, as the following commutative diagram (which is just N applied to the right-hand square of the diagram in 29.2) shows·

$$\begin{array}{ccc} \mathsf{N}(A[X]) & \xrightarrow{\mathsf{N}i} & (\mathsf{N}A)^{\mathsf{P}_f X} \\ {\scriptstyle \mathsf{N}(f[X])}\downarrow & & \downarrow{\scriptstyle (\mathsf{N}f)^{\mathsf{P}_f X}} \\ \mathsf{N}(B[X]) & \xrightarrow[\mathsf{N}i]{} & (\mathsf{N}B)^{\mathsf{P}_f X}. \end{array}$$

Thus, even though the right-hand square in (1) is not always a pushout, it does always commute (which establishes the formula for g in (1), using 29.2), and could even be called a pushout "up to standardization".

29.8. Iterated extensions. Combinatorial morphisms can be composed in some cases. Specifically, if $A \to A[X]/\prod_s \theta_s$ is a combinatorial morphism, Y is a set disjoint from X, and for each $t \in \mathsf{P}_f Y$, ψ_t is a *standard* congruence on $A[X]/\prod_s \theta_s$, then, up to isomorphism, each ψ_t is a family $\{\psi_{ts}\}$ of congruences on A such that $\theta_s \le \psi_{ts}$ for all $s \in \mathsf{P}_f X$ and $t \in \mathsf{P}_f Y$, and so the composite is $A \to A[X \cup Y]/\prod_{ts} \theta_{ts}$. The relations between the appropriate (equivalence classes of) monotone functions can be deduced from the following proposition, whose straightforward but tedious proof we omit.

Proposition. *Suppose that $A \to A[X]/\prod_s \theta_s$ is a combinatorial morphism and Y is a set disjoint from X. Then there is an isomorphism*

$$(A[X]/\textstyle\prod_s \theta_s)[Y] \simeq A[X \cup Y]/\textstyle\prod_{ts} \theta_s,$$

which takes, going left to right, a monotone function $m : \mathsf{P}_f Y \to A[X]/\prod_s \theta_s$ to the equivalence class $\hat{m}/\prod_{ts} \theta_s$, where $\hat{m} : \mathsf{P}_f(X \cup Y) \to A$ is given by

$$\hat{m}(s \cup t) = m(t)(s), \qquad (s \in \mathsf{P}_f X,\, t \in \mathsf{P}_f Y)$$

and which takes an equivalence class $n/\prod_{ts} \theta_s$ of monotone functions $\mathsf{P}_f(X \cup Y) \to A$ to the monotone function $\mathsf{P}_f Y \to A[X]/\prod_s \theta_s$ given by $t \mapsto \check{n}(t)/\prod_s \theta_s$, where

$$\check{n}(t)(s) = n(s \cup t) \qquad (s \in \mathsf{P}_f X,\, t \in \mathsf{P}_f Y).$$

29.9. The connection with free meets. Using this notion of combinatorial morphism, we can now show a connection between free meets and universal monos, as studied in Chapter 7 and the first part of this chapter.

Theorem. *Let $g : A \to A[X]/\prod_s \theta_s$ be a combinatorial morphism. Then,*

(a) *g is mono if and only if $\bigwedge_s \theta_s = 0$, and*

(b) *if the family $\{\theta_s\}$ has a free meet in $\mathsf{N}(A[X])$ that is equal to 0, then g is a universal mono.*

Corollary. *Every finitely generated combinatorial extension is universal.*

In Theorem 30.3 below, we show that every finitely generated extension is combinatorial, hence, by this result, universal.

PROOF. (a) This follows easily from the equivalences

$$\begin{aligned} g(a) = g(b) \quad &\text{iff} \quad m_a\, (\textstyle\prod_s \theta_s)\, m_b \\ &\text{iff} \quad \forall s \quad m_a(s)\, \theta_s\, m_b(s) \\ &\text{iff} \quad \forall s \quad a\, \theta_s\, b \\ &\text{iff} \quad a\, (\textstyle\bigwedge_s \theta_s)\, b \end{aligned}$$

(b) Assume $\{\theta_s\}$ has free meet 0, let $f : A \to B$ be a frame morphism, and let $\theta = (\mathsf{N}(f[X]))(\prod_s \theta_s)$. By the remarks at the end of 29.7, the standardization of θ is $\prod_s \theta'_s$, where $\theta'_s = (\mathsf{N}(f[X]))(\theta_s)$ for each s. But, since $\{\theta_s\}$ has a free meet, we therefore have

$$\bigwedge_s \theta'_s = (\mathsf{N}(f[X]))(\bigwedge_s \theta_s) = (\mathsf{N}(f[X]))(0) = 0$$

Thus, by part (a), the morphism $B \to B[X]/\prod_s \theta'_s$ is mono. But since $\theta \le \prod_s \theta'_s$, this morphism factors through $B \to B[X]/\theta$, proving that the latter is mono, as required.

For the Corollary, we simply note that if X is finite, then $P_f X$ is finite, and every finite set has a free meet.

30. Finitely generated combinatorial morphisms

30.1. In this last section, we look at finitely generated extensions $A[X]$ (X finite), for which, as we prove below, all congruences are standard. This clears up the problem with pushouts (cf. 29.7) and makes it possible to characterize, for example, when finitely generated (combinatorial) morphisms are epi and when they are regular.

For definiteness, we fix a positive integer n and take $X = \bar{n} = \{0, 1, \ldots, n-1\}$. Thus the variables $s, t, \ldots$ now range over subsets of $\bar{n}$, etc.

30.2. Finite extraction congruences. Recall (4.2) that for any elements a, x, y of a distributive lattice,

$$a \wedge x = a \wedge y \quad \text{and} \quad a \vee x = a \vee y \quad \text{imply} \quad x = y \tag{1}$$

The next result, which implies (as is proved in the Theorem that follows it) that any finite polynomial may be recovered from its extracted coefficients, is a generalization of this.

Lemma. *Let G be any finite subset of a distributive lattice A. Then for any $x, y \in A$, if $x\ \hat{\theta}(s, G \setminus s)\ y$ for every $s \subset G$, then $x = y$.*

Note that we recover (1) as the special case $G = \{a\}$.

Proof. The proof is by induction on the size of $G \subset A$, using (1) for the induction step. First note that the case $G = \emptyset$ is trivial, since the hypothesis reduces to $x = y$ in this case.

For the induction step, suppose the Lemma is true for all sets of cardinality n, let G have $n+1$ elements, and suppose that x and y are such that $x\ \hat{\theta}(s, G \setminus s)\ y$ for every $s \subset G$. Choose an element $g \in G$ and write $G = G' \cup \{g\}$, so that G' has n elements. Then, an inspection of the definition of $\hat{\theta}(-, -)$ as given by the left equation of 29.5(1) reveals that for every $s' \subset G'$,

$$\langle x \wedge g, y \wedge g \rangle \in \hat{\theta}(s', G' \setminus s') \quad \text{iff} \quad \langle x, y \rangle \in \hat{\theta}(s' \cup \{g\}, G \setminus (s' \cup \{g\})) \tag{2}$$

Similarly, the right equation of 29.5(1) shows that for every $s' \subset G'$,

$$\langle x \vee g, y \vee g \rangle \in \hat{\theta}(s', G' \setminus s') \quad \text{iff} \quad \langle x, y \rangle \in \hat{\theta}(s', G \setminus s') \tag{3}$$

But the right sides of (2) and (3) are true by our assumption on x and y, and so the left sides of (2) and (3) hold for every $s' \subset G'$. By the induction hypothesis, therefore, $x \wedge g = y \wedge g$ and $x \vee g = y \vee g$. But then $x = y$ by (1), and so the theorem follows by induction.

30.3. Theorem. *Suppose that $f : A \to B$ is a frame morphism, $\gamma : \overline{n} \to B$ is a mapping, and $\overline{f} : A[\overline{n}] \to B$ is the unique extension of f with $\overline{f}(m_i) = \gamma(i)$ for all $i = 1, \ldots, n$. For every $s \in \mathrm{P}_f X$, let $\hat{\theta}_s$ denote the congruence $\hat{\theta}(\gamma(s), \gamma(X \setminus s))$. Then*

$$\ker \overline{f} = \prod_{s \subset \overline{n}} f^{-1}(\hat{\theta}_s). \tag{1}$$

Corollary. *Every congruence on $A[\overline{n}]$ is standard.*

Proof. For any $m, n \in A[\overline{n}]$, we have

$$\begin{aligned} \langle m, n \rangle \in \textstyle\prod_s f^{-1}(\hat{\theta}_s) \quad &\text{iff} \quad \forall s \quad m(s)\,(f^{-1}(\hat{\theta}_s))\,n(s) \\ &\text{iff} \quad \forall s \quad f(m(s))\,\hat{\theta}_s\,f(n(s)) \\ &\text{iff} \quad \forall s \quad \overline{f}(m)\,\hat{\theta}_s\,\overline{f}(n), \end{aligned} \tag{2}$$

the last equivalence following from Proposition 29.6(b). But the statement in (2) is equivalent by the Lemma to $\overline{f}(m) = \overline{f}(n)$ and thus to $\langle m, n \rangle \in \ker \overline{f}$, completing the proof of the Theorem.

To prove the Corollary, let θ be a congruence on $A[\overline{n}]$, let $\natural_\theta : A[\overline{n}] \to A[\overline{n}]/\theta$ be the natural map, and consider the morphism $f = \natural_\theta \circ \Delta : A \to A[\overline{n}]/\theta$. Then, the unique extension $\overline{f}$ of f with $\overline{f}(m_i) = m_i/\theta$ for all $i \in \overline{n}$ is just $\overline{f} = \natural_\theta$, with kernel θ. Therefore, by the Theorem, θ is standard—and, incidentally,

$$\theta_s = f^{-1}(\hat{\theta}_s) \qquad (s \subset \overline{n}) \tag{3}$$

30.4. Cokernel pairs. Since all congruences on finitely generated free extensions are standard, it follows that the outer rectangle of diagram 29.7(1) is a pushout when $X = \overline{n}$. We use this to give a description of the cokernel pair of a finitely generated (combinatorial) morphism.

Lemma. *Let $f : A \to A[\overline{n}]/\prod_s \theta_s$ be a combinatorial morphism. Then the following diagram is a pushout*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & A[\overline{n}]/\prod_s \theta_s \arrow[d, "g"] \\ A[\overline{n}]/\prod_s \theta_s \arrow[r, "h"'] & A[\overline{2n}]/\prod_{st}(\theta_s \vee \theta_t), \end{tikzcd}$$

where

$$\begin{aligned} g(m/\textstyle\prod_s \theta_s)(s \cup t) &= m(s)/(\theta_s \vee \theta_t), \\ h(m/\textstyle\prod_s \theta_s)(s \cup t) &= m(t)/(\theta_s \vee \theta_t) \end{aligned} \qquad (s \subset \overline{n},\ t \subset \overline{2n} \setminus \overline{n})$$

PROOF Let θ be a congruence on A with natural map $\natural_\theta$ $A \to A/\theta$ Then since pushouts of quotients of A correspond to joins of congruences, we have (up to isomorphism over A) $(N\natural_\theta)(\psi) = \theta \vee \psi$ for all congruences ψ on A We conclude therefore from 29 7(1) that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\natural_\theta"'] & A[\overline{n}]/\prod_s \theta_s \arrow[d, "k"] \\ A/\theta \arrow[r] & A[\overline{n}]/\prod_s(\theta \vee \theta_s), \end{tikzcd} \qquad k(m/\textstyle\prod_s \theta_s)(s) = m(s)/\prod_s(\theta \vee \theta_s),$$

is a pushout. Thus, $(Nf)(\theta) = \prod_s(\theta \vee \theta_s)$. It remains to establish the formulas for g and h By 29 7(1), g is given by

$$g(m/\textstyle\prod_s \theta_s)(s) = f(m(s))/\prod_s t(\theta_s \vee \theta_t)$$

But $f(m(s))$ is the equivalence class containing the constant function $m_{m(s)}$, and so, using the isomorphisms of Proposition 29 8, the formula for g follows The formula for h is similar.

30.5. Dominions, epis. and regular monos. The *dominion* of a morphism $f : A \to B$ is by definition the equalizer of its cokernel pair Thus, in our case, it may be taken to be the subframe D of B consisting of precisely those elements b such that $g(b) = h(b)$ for every two morphisms $g, h : B \to C$ that agree on A Thus, f is epi iff its dominion is all of B

Another way of defining the dominion of f is as the smallest subframe of D of B containing A such that the inclusion $D \to B$ is a regular mono. Thus, f is a regular mono iff its dominion is A

We are now in a position to state the following result, which uses the description of cokernel pairs in Lemma 30 4 to characterize finitely generated epis and regular monos in terms of their associated family of congruences

Theorem. *If f $A \to A[\overline{n}]/\prod_s \theta_s$ is a combinatorial morphism, then we have the following*

(a) *The dominion of f is equal to*

$$\{m/\textstyle\prod_s \theta_s : m(s)\ (\theta_s \vee \theta_t)\ m(t) \text{ for all } s,t \subset \overline{n}\}$$

(b) *f is epi if and only if $\theta_s \vee \theta_t = 1$ whenenver $s \neq t$*

(c) *f is a regular mono if and only if it satisfies the following "sheaf-like" condition· whenever $m \in A[X]$ is such that $m(s)\ (\theta_s \vee \theta_t)\ m(t)$ for all $s, t \subset \overline{n}$, then there exists a unique $a \in A$ such that $m(s)\ \theta_s\ a$ for all $s \subset \overline{n}$*

PROOF Part (a) is an obvious translation of the definition using Lemma 30.4 For (b), note that if $\theta_s \vee \theta_t = 1$ whenever $s \neq t$, then every $m \in A[X]$ satisfies the condition in part (a) for $m/\prod_s \theta_s$ to be in the dominion of f Hence, f is epi. Conversely, suppose that f is epi, and let s and t be different subsets of $\overline{n}$; we may assume that $s \not\subset t$ Then $m = \delta_1$, has $m(s) = 1$ and $m(t) = 0$, and so since f is epi, we conclude from part (a) that $1 = m(s)\ (\theta_s \vee \theta_t)\ m(t) = 0$. Thus $\theta_s \vee \theta_t = 1$ This proves (b) Finally, part (c) follows easily from part (a) using the equivalence of the statements "f is regular" and "the dominion of f is A".

30.6. Exercises.

30.6.1.

(a) Show that if θ is a regular element of $N(A[X])$ (i e., $\neg\neg\theta = \theta$), then θ is standard (Hint: the identity congruence is standard.)

(b) If B is a cBa and X is a set, show conversely that every standard congruence on $A[X]$ is regular and thus that standardization on $N(B[X])$ is given by double negation

(c) Use (b) to give examples of non-standard congruences on free extensions.

Chapter 9
CONCLUSION AND PROBLEMS

The goal of my research has been show how a single construction, the assembly tower construction, can be used to illuminate a variety of aspects of frame theory and lead to solutions of some interesting problems I don't want to claim that have achieved this goal in its entirety, but I do hope to have proven that such an investigation is both viable and worthwhile and to have laid a foundation for further investigation along these lines.

In this rest of this chapter, I want to discuss some possible continuations of the work described in this thesis I start by mentioning some general directions and then proceed to discuss specific problems arising from the research described in particular chapters.

Constructiveness. In describing the set-theoretic foundations for the work described in this thesis, I assumed a set theory containing the Axiom of Choice. This was necessary in order to use the method of κ-frames. For example, the result (12 4) that every frame colimit becomes a κ-frame colimit for large enough κ is an unavoidable application of the Axiom of Choice, since we must be able to choose, for any frame A, a (regular) cardinal κ with $\kappa > |A|$, and this implies that A can be well-ordered

However, since the conclusions I drew using this method—for example that f is a universal mono iff f is α-mono for all α (28 5)—seem to have a constructive meaning, it becomes interesting to ask whether some other methods would yield the same results constructively

Separation Conditions. One feature of the work described in this thesis is that it concerns *arbitrary* frames I have not, as is typical in topology, assumed any extra conditions on the "spaces" I have been studying. On the other hand, one would expect that extra conditions (regularity, compactness, etc) would lead to better theorems, and I have only begun investigating this possibility

Foundations. This work touches on some interesting foundational questions It clearly becomes more convenient when we can work inside an inaccessible cardinal, or use algebras with proper classes of elements, and although all of these devices proved in the end to be unnecessary, one would like to have a better understanding of just when such extra assumptions are harmless. For example, in collecting algebra-classes into classes, I needed to be sure that they were codable as sets (see footnotes 1 and 2 in Section 5), although this requirement didn't seem to have anything to do with the matter at hand.

Extensional Operators. There is still much to be said here about the relationship between extensional operators and topos theory (cf., 9 23) For example, what is the best way to formulate the "uniformity" property of extensional operators? Can one use extensional operators to construct toposes from Heyting algebras? Does our hard-won formula for joins of nuclei (23 4(1)) follow from a result about unions in toposes?

The Reflection Problem. The original motivation of the assembly tower construction was to get at Boolean reflections of frames, when they exist. Although a solution to the problem of characterizing the reflective frames would probably not be of much use in itself, it seems nevertheless to require a new insight into the category of frames One can hope therefore that further research on this particular problem will lead to results of a more wide-spread interest.

Universal Monos. Although I characterized universal monos $f : A \to B$ in terms of the assembly tower (28 5), the form of this characterization is quite similar to that of the "solution" to the reflection problem afforded by the assembly tower mentioned in 1.11 One would like, in other words, a more intrinsic description—one given in terms of A and B alone.

Another observation to be made here is that the main results for universal monos do not require many of the properties of the category of frames, mainly just the existence of a limit-closed subcategory like **cBa** (reflectiveness is not necessary, of course) and an assembly construction I already have some partial results in this direction, and it would be interesting to see how far they can be pushed

BIBLIOGRAPHY

[1] S Abramsky *Domain Theory and the Logic of Observable Properties* PhD thesis, Queen Mary College, University of London, 1987

[2] S. Abramsky and S Vickers Quantales observational logic, and process semantics. Technical Report DOC 90/1, Department of Computing, Imperial College, London, 1990.

[3] B. Banaschewski. Another look at the localic Tychonoff theorem *Comment. Math Univ. Carolinae* 29(4).647–656 1988

[4] B Banaschewski Singly generated frame extensions *J. Pure Appl Algebra*, 83:1–21, 1992.

[5] R Beazer and D S Macnab Modal extensions of Heyting algebras. *Colloq Math* , 41 1–12, 1979

[6] S. Burris and H. P Sankappanavar. *A Course in Universal Algebra* Graduate Texts in Mathematics 78. Springer-Verlag, 1981

[7] C. H Dowker and D P Strauss Products and sums in the category of frames In *Categorical Topology*, pages 208–219. Springer-Verlag, 1976 Lecture Notes in Mathematics 540

[8] S. Feferman Set-theoreitcal foundations of category theory In Saunders Mac Lane, editor. *Reports of the Midwest Category Seminar III*, pages 201–247. Springer-Verlag. 1969. Lecture Notes in Mathematics 106

[9] H Gaifman Infinite Boolean polynomials I. *Fund. Math* , 54.229–250, 1964

[10] O Garcia and E Nelson. On the non-existence of free complete distributive lattices Unpublished, Feb. 1984

[11] G Gierz, K. H. Hofmann, K Keimel, J. D. Lawson. M. W. Mislove, and D S Scott *A Compendium of Continuous Lattices* Springer-Verlag, 1980

[12] J. S. Golan and H Simmons. *Derivatives, Nuclei and Dimensions on the Frame of Torsion Theories* Pitman Research Notes in Mathematics 188 Longman Scientific & Technical. 1988.

[13] G. Gratzer *General Lattice Theory* Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften 52. Birkhäuser, Stuttgart, 1978.

[14] A W Hales On the non-existence of free complete Boolean algebras *Fund Math* , 54.45–66, 1964.

[15] J. R. Isbell. Atomless parts of spaces *Math Scand* 31·5–32, 1972

[16] J. R. Isbell. Direct limits of meet-continuous lattices *J. Pure Appl. Alg.*, 23.33–35, 1982

[17] J. R. Isbell, I. Kříž A Pultr. and J Rosický Remarks on localic groups In *Categorical Algebra and its Applications*. pages 154–172 Springer-Verlag, 1988. Lecture Notes in Mathematics 1348.

[18] S. Strecker J. Adámek. H. Herlich *Abstract and Concrete Categories· The Joy of Cats.* Pure and Applied Mathematics John Wiley & Sons, 1990

[19] T. Jech. *Set Theory* Pure and Applied Mathematics 79 Academic Press, 1978

[20] P. T. Johnstone *Topos Theory* L M.S. Monographs 10. Academic Press, 1977.

[21] P T Johnstone. Tychonoff's theorem without the Axiom of Choice *Fund. Math*, 113:21–35 1981

[22] P. T. Johnstone *Stone Spaces.* Cambridge Studies in Advanced Mathematics 3. Cambridge University Press, 1982.

[23] P. T Johnstone The point of pointless topology *Bull. Amer Math. Soc.*, 8(1) 41–53, 1983.

[24] P. T. Johnstone. *Notes on Logic and Set Theory.* Cambridge University Press, 1987

[25] P. T. Johnstone. The art of pointless thinking: A student's guide to the category of locales. In H. Herlich and H.-E. Porst, editors, *Category Theory at Work*, pages 85–107. Heldermann Verlag, Berlin, 1991.

[26] P T. Johnstone and S. Vickers. Preframe presentations present. In A Carboni, M C Pedicchio, and G. Rosolini, editors, *Category Theory*, pages 193–212. Springer-Verlag, 1990. Lecture Notes in Mathematics 1488.

[27] A. Joyal and M. Tierney An extension of the Galois theory of Grothendieck. Mem. Amer. Math. Soc 309, 1984.

[28] R. LaGrange. Amalgamation and epimorphism in m-complete boolean algebras *Algebra Univ.*, 4.277–279, 1974.

[29] S Mac Lane *Categories for the Working Mathematician.* Graduate Texts in Mathematics 5. Springer-Verlag, 1971.

[30] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic A First Introduction to Topos Theory* Universitext. Springer-Verlag, 1992.

[31] D. S. Macnab. *An Algebraic Study of Modal Operators on Heyting Algebras with Applications to Topology and Sheafification* PhD thesis, University of Aberdeen, Scotland, 1976.

[32] J. J. Madden. κ-frames *J Pure Appl. Algebra*, 70.107–127, 1991.

[33] J J Madden and A Molitor Epimorphisms of frames *J Pure Appl Algebra*. 70:129-132, 1991

[34] E. G. Manes *Algebraic Theories* Graduate Texts in Mathematics 26 Springer-Verlag. 1976

[35] R McKenzie G McNulty and W. Taylor. *Algebras, Lattices, Varieties*, volume 1 Brooks/Cole Pub Co, 1987.

[36] S Niefield and K Rosenthal Componental nuclei In *Categorical Algebra and its Applications* pages 299-306 Springer-Verlag, 1988. Lecture Notes in Mathematics 1348

[37] A. Pultr and A. Tozzi Equationally closed subframes and representation of quotient spaces *Cahiers de Top et Géom Diff. Cat.*, 34(3).167-183, 1993

[38] K I Rosenthal *Quantales and their Applications* Pitman Research Notes in Mathematics 234 Longman Scientific & Technical, 1990

[39] D Scott Extending the topological interpretation to intuitionistic analysis II In J. Myhill, A Kino, and R E. Vesley, editors, *Intuitionism and Proof Theory*, pages 235-255 North-Holland. Amsterdam, 1970.

[40] H. Simmons. Boolean reflections of frames and extended Cantor-Bendixson analysis. Unpublished

[41] H. Simmons A framework for topology In A Macintyre, L. Pacholski, and J. Paris, editors. *Logic Colloquium '77*, pages 239-251. North-Holland, 1978.

[42] H Simmons Spaces with Boolean assemblies. *Colloq Math*, 43.23-39, 1980.

[43] H. Simmons. An algebraic version of Cantor-Bendixson analysis. In *Categorical Aspects of Topology and Analysis* pages 310-323 Springer-Verlag, 1982. Lecture Notes in Mathematics 915.

[44] H. Simmons. Near-discreteness of modules and spaces as measured by Gabriel and Cantor. *J Pure Appl Algebra*. 56:119-162, 1989.

[45] H Simmons Separating the discrete from the continuous by iterating derivatives. *Bull Soc Math Belg.*, 41:417-463, 1989.

[46] M. Smyth. Powerdomains and predicate transformers: A topological view In J. Diaz. editor. *Automata, Languages and Programming*, pages 662-675 Springer-Verlag, 1983 Lecture Notes in Computer Science 154.

[47] R. Solovay A new proof of a theorem of Gaifman and Hales *Bull Amer. Math. Soc*. 72 282-284, 1966

[48] A S Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction. vol II* Studies in Logic and the Foundations of Mathematics 123. North-Holland, 1988

[49] J. J C. Vermeulen Proper maps of locales *J. Pure Appl Algebra*, 92:79–107, 1994.

[50] S Vickers *Topology via Logic* Cambridge Tracts in Theoretical Computer Science 5. Cambridge University Press. 1989

[51] D. Wigner Two notes on frames *J. Austr. Math. Soc. (Series A)*, 28:257–268, 1979.