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The Tensor Equation $AX + XA = \Phi(A, H)$ With Applications to Kinematics of Continua

Michael J. Scheidler

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Michael J. Scheidler

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Army Research Laboratory
ATTN: AMSRL-WT-TD
Aberdeen Proving Ground, MD 21005-5066

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The (second-order) tensor equation $AX + XA = \Phi(A, H)$ is studied for certain isotropic functions $\Phi(A, H)$ which are linear in H. Qualitative properties of the solution X and relations between the solutions for various forms of Φ are established for an inner product space of arbitrary dimension. These results, together with Rivlin's identities for tensor polynomials in two variables, are applied in three dimensions to obtain new explicit formulas for X in direct tensor notation as well as new derivations of previously known formulas. Several applications to the kinematics of continua are considered.

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1 Introduction

A variety of problems in continuum mechanics require the solution $\mathbf{X}$ of a linear algebraic equation of the form

$$\mathbf{AX} + \mathbf{XA} = \mathbf{\Phi}(\mathbf{A}, \mathbf{H}) . \tag{1.1}$$

Here $\mathbf{A}$, $\mathbf{X}$, and $\mathbf{H}$ are second-order tensors (i.e., linear transformations) on a two- or three-dimensional inner product space $\mathcal{V}$, and $\mathbf{\Phi}(\mathbf{A}, \mathbf{H})$ is an isotropic function of $\mathbf{A}$ and $\mathbf{H}$ which is linear in $\mathbf{H}$.

For example, consider a smooth motion with *deformation gradient* $\mathbf{F}$.[1] By the polar decomposition theorem, $\mathbf{F}$ has the unique multiplicative decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} , \tag{1.2}$$

where the proper orthogonal tensor $\mathbf{R}$ is the *rotation tensor*, and the symmetric, positive-definite tensors $\mathbf{U}$ and $\mathbf{V}$ are the *right* and *left stretch tensors*. Let $\mathbf{C}$ and $\mathbf{B}$ denote the *right* and *left Cauchy-Green tensors*:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2 \quad \text{and} \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2 . \tag{1.3}$$

Let the *stretching tensor* $\mathbf{D}$ and the *spin tensor* $\mathbf{W}$ denote the symmetric and skew parts of the *velocity gradient* $\mathbf{L}$:

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = \operatorname{sym}\mathbf{L} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}) , \quad \mathbf{W} = \operatorname{skw}\mathbf{L} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}) . \tag{1.4}$$

For any tensor field $\mathbf{A}$, let $\dot{\mathbf{A}}$ denote the *material time derivative* of $\mathbf{A}$, and let $\mathring{\mathbf{A}}$ denote the *Jaumann rate* of $\mathbf{A}$:

$$\mathring{\mathbf{A}} := \dot{\mathbf{A}} + \mathbf{AW} - \mathbf{WA} . \tag{1.5}$$

Then the material time derivatives of the stretch tensors are related to the material time derivatives of the Cauchy-Green tensors by the equations

$$\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} = \dot{\mathbf{C}} \quad \text{and} \quad \mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} = \dot{\mathbf{B}} . \tag{1.6}$$

The material time derivative of the left stretch tensor is related to the velocity gradient by the equation

$$\mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} = \mathbf{V}^2\mathbf{L}^{\mathrm{T}} + \mathbf{L}\mathbf{V}^2 , \tag{1.7}$$

and the Jaumann rate of the left stretch tensor is related to the stretching tensor by the equation

$$\mathbf{V}\mathring{\mathbf{V}} + \mathring{\mathbf{V}}\mathbf{V} = \mathbf{V}^2\mathbf{D} + \mathbf{D}\mathbf{V}^2 . \tag{1.8}$$

[1] We use the notation and terminology of Truesdell and Noll [1]; cf. also Wang and Truesdell [2], Gurtin [3], and Truesdell [4].

The material time derivative of the right stretch tensor is related to the tensor $\mathbf{D_R} = \mathbf{R}^T\mathbf{D}\mathbf{R}$ by the equivalent equations

$$\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} = 2\mathbf{U}\mathbf{D_R}\mathbf{U} \quad \text{and} \quad \mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1} = 2\mathbf{D_R}\,. \tag{1.9}$$

The stretching tensor is related to the Jaumann rate of the left Cauchy-Green tensor or its inverse by the equivalent equations

$$\mathbf{B}\mathbf{D} + \mathbf{D}\mathbf{B} = \mathring{\mathbf{B}} \quad \text{and} \quad \mathbf{B}^{-1}\mathbf{D} + \mathbf{D}\mathbf{B}^{-1} = -(\mathbf{B}^{-1})^{\circ}\,. \tag{1.10}$$

The skew tensor $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ is related to the velocity gradient by the equation

$$\mathbf{V}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{V} = \mathbf{L}\mathbf{V} - \mathbf{V}\mathbf{L}^T\,, \tag{1.11}$$

and the difference of $\mathbf{W}$ and $\mathbf{\Omega}$ is related to the stretching tensor by the equation

$$\mathbf{V}(\mathbf{W} - \mathbf{\Omega}) + (\mathbf{W} - \mathbf{\Omega})\mathbf{V} = \mathbf{V}\mathbf{D} - \mathbf{D}\mathbf{V}\,. \tag{1.12}$$

The tensor equations (1.6)–(1.12) have been studied by various authors; cf. Leonov [5], Sidoroff [6], Dienes [7], Guo [8], Hoger and Carlson [9], Hoger [10], Mehrabadi and Nemat-Nasser [11], Stickforth and Wegener [12]. and Guo, Lehmann and Liang [13]. These equations are of the general form (1.1) with $\mathbf{A} = \mathbf{V}$, $\mathbf{U}$, $\mathbf{U}^{-1}$, $\mathbf{B}$, or $\mathbf{B}^{-1}$, and with $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ of the form

$$\mathbf{H},\ \mathbf{A}^2\mathbf{H}^T + \mathbf{H}\mathbf{A}^2,\ \mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2,\ \mathbf{A}\mathbf{H}\mathbf{A},\ \mathbf{H}\mathbf{A} - \mathbf{A}\mathbf{H}^T,\ \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}\,. \tag{1.13}$$

In particular, for the kinematics applications discussed above, the coefficient tensor $\mathbf{A}$ in (1.1) is symmetric and positive-definite. These restrictions on $\mathbf{A}$ will be assumed for the present discussion only. They guarantee that a solution $\mathbf{X}$ exists and is unique. Indeed, relative to any principal basis $\{\mathbf{e}_i\}$ for $\mathbf{A}$, the components of $\mathbf{X}$ are given by the simple formula

$$X_{ij} = \frac{\Phi_{ij}}{a_i + a_j}\,, \tag{1.14}$$

where a_i is the (necessarily positive) eigenvalue of $\mathbf{A}$ corresponding to $\mathbf{e}_i$, and Φ_{ij} are the components of $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ relative to $\{\mathbf{e}_i\}$. Observe that $\mathbf{X}$ is symmetric (resp. skew) iff $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ is symmetric (resp. skew). Of course, to actually compute $\mathbf{X}$ by means of (1.14) we must first determine the eigenvalues and eigenvectors of $\mathbf{A}$.

For problems in which the eigenvalues and eigenvectors of $\mathbf{A}$ are not of primary interest, it may be more useful to express $\mathbf{X}$ directly in terms of the tensors $\mathbf{A}$ and $\mathbf{H}$. Explicit solutions of the this type have been derived by the authors cited above. For example, Sidoroff [6] and Guo [8] obtained the following solution of the tensor equation

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{H} \tag{1.15}$$

for the case where $\mathbf{H}$ (and hence $\mathbf{X}$) is skew and $\dim \mathcal{V} = 3$:

$$(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})\mathbf{X} = \left(I_{\mathbf{A}}^2 - II_{\mathbf{A}}\right)\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2) \,. \tag{1.16}$$

Here $I_{\mathbf{A}}$, $II_{\mathbf{A}}$, and $III_{\mathbf{A}}$ denote the principal invariants of $\mathbf{A}$, and the requirement that $\mathbf{A}$ be positive-definite guarantees that $I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}}$ is positive. Sidoroff and Guo arrived at this solution by first deriving a formula for the axial vector of $\mathbf{X}$ in terms of the axial vector of $\mathbf{H}$ and then converting this intermediate result to its equivalent tensor form (1.16). Hoger and Carlson [9] obtained the following solution of (1.15) for arbitrary $\mathbf{H}$ when $\dim \mathcal{V} = 3$:

$$\begin{aligned} 2III_{\mathbf{A}}(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})\mathbf{X} \;=\; & I_{\mathbf{A}}\mathbf{A}^2\mathbf{H}\mathbf{A}^2 - I_{\mathbf{A}}^2(\mathbf{A}^2\mathbf{H}\mathbf{A} + \mathbf{A}\mathbf{H}\mathbf{A}^2) \\ & + (I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})(\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2) \\ & + \left(I_{\mathbf{A}}^3 + III_{\mathbf{A}}\right)\mathbf{A}\mathbf{H}\mathbf{A} - I_{\mathbf{A}}^2 II_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) \\ & + \left[I_{\mathbf{A}}^2 III_{\mathbf{A}} + II_{\mathbf{A}}(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})\right]\mathbf{H}\,. \end{aligned} \tag{1.17}$$

Since equations of the form (1.16) and (1.17) are said to be displayed in direct notation, we will refer to such equations as *direct formulas* for $\mathbf{X}$ or *direct solutions* of (1.15). By a *general* direct solution of (1.15) we mean a solution, such as (1.17), which is valid for any tensor $\mathbf{H}$.

Although the component formula (1.14) is easily derived and is independent of the dimension of $\mathcal{V}$, the derivation of direct formulas for $\mathbf{X}$ is nontrivial, and the complexity of these formulas increases rapidly with the dimension of $\mathcal{V}$.[2] For example, when $\dim \mathcal{V} = 2$ the solution of (1.15) for skew $\mathbf{H}$ is[3] $I_{\mathbf{A}}\mathbf{X} = \mathbf{H}$, which is substantially simpler than its three-dimensional counterpart (1.16). Also, observe that there is no apparent simplification of the direct formula (1.17) when $\mathbf{H}$ is skew; in particular, it is by no means obvious that (1.17) and (1.16) are equivalent for skew $\mathbf{H}$. By utilizing Rivlin's [17] identities for tensor polynomials in two variables, Hoger and Carlson [9] were able to convert (1.17) to a form which does indeed collapse to (1.16) when $\mathbf{H}$ is skew.

This paper is devoted to the derivation and applications of direct solutions of the tensor equation (1.1) in three dimensions. Clearly, for any function $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ we can obtain a direct solution of (1.1) by replacing $\mathbf{H}$ with $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ in (1.17) or in any other general direct solution of (1.15). The resulting formulas will typically be more complicated than the direct solution of (1.15) from which they were obtained,

[2]Cf. the direct solutions of (1.15) obtained by Smith [14], Jameson [15], and Müller [16]. Their formulas are valid for arbitrary dimensions, but the complexity of these formulas is such that they would seem to be useful only for $\dim \mathcal{V} \leq 3$ or 4. Also, compare Hoger and Carlson's [9] solutions of (1.15) in two and three dimensions, and Mehrabadi and Nemat-Nasser's [11] solutions of (1.19) in two and three dimensions.

[3]This solution is a special case of the second of two general direct solutions of (1.15) obtained by Hoger and Carlson [9] in two dimensions.

although subsequent applications of the Cayley-Hamilton theorem or Rivlin's [17] identities may yield substantial simplifications in some cases. One of the goals of this paper is to develop methods which yield these simpler formulas more directly. Another goal is to derive the skew solution (1.16) and other simple solutions of (1.1) for skew $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ without resorting to intermediate results in terms of axial vectors or to the more complicated general direct solutions.

The paper is organized as follows. In Section 2 we study the fourth-order tensors $\mathsf{L}_{\mathbf{A}}$, $\mathsf{M}_{\mathbf{A}}$, and $\mathsf{N}_{\mathbf{A}}$ characterized by the conditions

$$\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{H}] \iff \mathbf{AX}+\mathbf{XA} = \mathbf{H}\,, \tag{1.18}$$

$$\mathbf{X} = \mathsf{M}_{\mathbf{A}}[\mathbf{H}] \iff \mathbf{AX}+\mathbf{XA} = \mathbf{AH}-\mathbf{HA}\,, \tag{1.19}$$

$$\mathbf{X} = \mathsf{N}_{\mathbf{A}}[\mathbf{H}] \iff \mathbf{AX}+\mathbf{XA} = \mathbf{A}^2\mathbf{H}-2\mathbf{AHA}+\mathbf{HA}^2\,. \tag{1.20}$$

Then $\mathbf{X}$ is the solution of the tensor equation (1.1) iff $\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{\Phi}(\mathbf{A},\mathbf{H})]$. In particular, $\mathsf{M}_{\mathbf{A}}[\mathbf{H}] = \mathsf{L}_{\mathbf{A}}[\mathbf{AH}-\mathbf{HA}]$ and $\mathsf{N}_{\mathbf{A}}[\mathbf{H}] = \mathsf{L}_{\mathbf{A}}[\mathbf{A}^2\mathbf{H}-2\mathbf{AHA}+\mathbf{HA}^2]$. Conversely, when $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ has one of the forms in (1.13), we show that there are simple relations for $\mathsf{L}_{\mathbf{A}}[\mathbf{\Phi}(\mathbf{A},\mathbf{H})]$ in terms of $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$, $\mathsf{M}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}]$, or $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$. The utility of these relations is due to the fact that direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ and $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ are simpler and easier to derive than general direct formulas for $\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ such as (1.17). The results in Section 2 are independent of the dimension of the inner product space $\mathcal{V}$. Furthermore, unlike the component formula (1.14), these results are valid for any tensor $\mathbf{A}$ with the property that (1.15) has a unique solution $\mathbf{X}$ for any given $\mathbf{H}$. Such a tensor $\mathbf{A}$ is necessarily nonsingular but need not be symmetric or definite.

In Sections 3–7 we assume that $\dim\mathcal{V} = 3$. Section 3 contains various tensor identities which will be utilized in the sequel. These include Rivlin's [17] identities for tensor polynomials in two variables as well as some new identities which follow from Rivlin's. In Section 4 we consider (1.15) with $\mathbf{X}$ and $\mathbf{H}$ restricted to the set $\mathcal{T}(\mathbf{A})$ of all tensors $\mathbf{K}$ such that $\operatorname{tr}(\mathbf{A}^n\mathbf{K}) = 0$ $(n = 0,1,2)$. We obtain necessary and sufficient conditions for the existence of a unique solution in $\mathcal{T}(\mathbf{A})$ (the possibility of other solutions outside $\mathcal{T}(\mathbf{A})$ is not excluded here), and we derive direct formulas for this solution. When $\mathbf{A}$ is symmetric these formulas are valid for any skew tensor $\mathbf{H}$; in particular, we recover the formula (1.16) of Sidoroff and Guo. These results do not require that $\mathbf{A}$ be nonsingular; some applications for which $\mathbf{A}$ might be singular are discussed below. Section 4 concludes with the derivation of necessary and sufficient conditions for the existence of a unique solution $\mathbf{X}$ of (1.15) with $\mathbf{H}$ unrestricted. The proof utilizes the results for the special case where $\mathbf{X}$ and $\mathbf{H}$ belong to $\mathcal{T}(\mathbf{A})$. In Section 5 we use the results in Sections 3 and 4 to derive direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ and $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ for arbitrary $\mathbf{H}$; these formulas, together with the relations for $\mathsf{L}_{\mathbf{A}}$ in terms of $\mathsf{M}_{\mathbf{A}}$ or $\mathsf{N}_{\mathbf{A}}$ obtained in Section 2, are in turn used to derive direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ which are valid for arbitrary $\mathbf{H}$. Then direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{\Phi}(\mathbf{A},\mathbf{H})]$ with $\mathbf{\Phi}$ as in (1.13) follow from these results and the identities in Section 2. In Section 6 we derive equations (1.6)–(1.12) and apply our results to the solution of these and related equations arising in the kinematics of continua. In Section 7 we discuss some additional kinematic formulas which can be obtained from various transformations of the results in Section 6. Although some of the algebraic and kinematic formulas derived in this paper have been obtained previously by other authors, the derivations given here are new, and we derive many new formulas as well.

Additional Applications

The general results in Sections 2–5 should prove useful for other problems in mechanics which lead to tensor equations of the form (1.1). Some of these problems are listed below.

1. Direct formulas for the derivatives of the stretch and rotation tensors with respect to the deformation gradient: Here $\mathbf{A} = \mathbf{U}$ or $\mathbf{V}$, and $\mathbf{\Phi}(\mathbf{A}, \mathbf{H})$ has some of the forms listed in (1.13) as well as

$$\mathbf{AH} + \mathbf{H}^{\mathrm{T}}\mathbf{A}, \quad \mathbf{HA} + \mathbf{AH}^{\mathrm{T}}, \quad \mathbf{AH} - \mathbf{H}^{\mathrm{T}}\mathbf{A}\,. \tag{1.21}$$

This problem is the subject of a follow-up paper [18]; cf. also Wheeler [19] and Chen and Wheeler [20] for a different approach to this problem. For a hyperelastic material, the results in [18]-[20] also yield direct formulas for the first Piola-Kirchhoff stress tensor in terms of the derivative of the strain energy function with respect to the right stretch tensor $\mathbf{U}$; cf. also Hoger [21], where this problem has been solved using Hoger and Carlson's [9] formula (1.17).

2. Direct formulas for one work-conjugate stress tensor in terms of another, or for a work-conjugate stress tensor in terms of the Cauchy or first Piola-Kirchhoff stress tensors (cf. Guo and Man [22]): Here $\mathbf{A} = \mathbf{U}$ or $\mathbf{V}$, and $\mathbf{\Phi}(\mathbf{A}, \mathbf{H})$ has some of the forms listed in (1.13) as well as

$$\mathbf{AH}, \quad \mathbf{HA}, \quad \sum_{r=1}^{m} \mathbf{A}^{m-r}\mathbf{H}\mathbf{A}^{r-1}\,. \tag{1.22}$$

3. The kinematics and dynamics of rigid bodies (cf. Truesdell [4, §I.10, I.13] and Scheidler [23]) and pseudo-rigid bodies (cf. Cohen and Muncaster [24]): Here the symmetric tensor $\mathbf{A}$ is either the current or the referential *Euler tensor.* If the mass is not confined to a single plane then $\mathbf{A}$ is positive-definite (Segner's Theorem). However, the results in Theorem 4.1 also apply when the mass is confined to a single plane, in which case $\mathbf{A}$ is singular.

4. Traction boundary value problems in finite elasticity: For $\mathbf{A}$ symmetric and $\mathbf{H}$ and $\mathbf{X}$ skew, the tensor equation (1.15) arises in connection with Signorini's expansion and Stopelli's theorems; cf. Wang and Truesdell [2, §7.2, 7.4]. For these applications the *astatic load* tensor $\mathbf{A}$ may be singular; the results in Section 4 are applicable in this case provided that the load system does not possess an axis of equilibrium.

5. Stability analysis of systems of ordinary differential equations: The tensor equation $\mathbf{A}^{\mathrm{T}}\mathbf{Y} + \mathbf{YA} = \mathbf{G}$ arises in the construction of quadratic Liapunov functions;

cf. Hahn [25, Ch. 4] and Gantmacher [26, §5.5]. At the end of Section 2 we show how general direct solutions of this equation can be obtained from general direct solutions of $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$.

2 The fourth-order tensors $\mathsf{L}_\mathbf{A}$, $\mathsf{M}_\mathbf{A}$, and $\mathsf{N}_\mathbf{A}$

Let Lin denote the set of all linear transformations (or second-order tensors) on the finite-dimensional real inner product space $\mathcal{V}$. Sym and Skw denote the subspaces of Lin consisting of all symmetric and skew tensors, respectively. Psym denotes the set of all symmetric and positive-definite tensors. The identity tensor is denoted by $\mathbf{I}$, and $\mathbf{A}^0 := \mathbf{I}$ for any tensor $\mathbf{A}$. Unless specified otherwise, $\mathbf{A}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{X}$, and $\mathbf{Y}$ denote arbitrary tensors. We assume that Lin is equipped with the inner product "$\cdot$" defined in terms of the trace function by $\mathbf{H} \cdot \mathbf{G} := \mathrm{tr}\,(\mathbf{H}^\mathrm{T}\mathbf{G})$, where $\mathbf{H}^\mathrm{T}$ denotes the transpose of $\mathbf{H}$.

By a *fourth-order tensor* we mean a linear transformation from Lin into Lin. The image of $\mathbf{H} \in$ Lin under a fourth-order tensor K is denoted by $\mathsf{K}[\mathbf{H}]$. In this section we study the properties of the fourth-order tensors $\mathsf{L}_\mathbf{A}$, $\mathsf{M}_\mathbf{A}$, and $\mathsf{N}_\mathbf{A}$ discussed in the Introduction. To facilitate the statement and proof of some of these properties, we introduce the fourth-order tensors $\mathsf{B}_\mathbf{A}$ and $\mathsf{C}_\mathbf{A}$:

$$\mathsf{B}_\mathbf{A}[\mathbf{X}] := \mathbf{AX} + \mathbf{XA} \quad \text{and} \quad \mathsf{C}_\mathbf{A}[\mathbf{X}] := \mathbf{AX} - \mathbf{XA}\,. \tag{2.1}$$

It is easily verified that $\mathsf{B}_\mathbf{A}$ and $\mathsf{C}_\mathbf{A}$ commute; indeed,

$$\mathsf{C}_\mathbf{A}\mathsf{B}_\mathbf{A} = \mathsf{B}_\mathbf{A}\mathsf{C}_\mathbf{A} = \mathsf{C}_{\mathbf{A}^2}\,. \tag{2.2}$$

Clearly, $\mathsf{C}_\mathbf{A}$ is singular for every tensor $\mathbf{A}$. Let Lin* denote the set of all $\mathbf{A} \in$ Lin such that $\mathsf{B}_\mathbf{A}$ is nonsingular, or, equivalently, such that the equation $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ has a unique solution $\mathbf{X}$ for any given $\mathbf{H}$. From the discussion in the Introduction we know that Psym $\subset$ Lin*. Necessary and sufficient conditions for $\mathbf{A} \in$ Lin* in terms of the principal invariants of $\mathbf{A}$ or in terms of the characteristic roots of $\mathbf{A}$ are discussed in Section 4. For this section we need only the following elementary results.

Proposition 2.1 *The following conditions are equivalent:*

(1) $\mathbf{A} \in$ Lin*;
(2) $\mathbf{A}^\mathrm{T} \in$ Lin*;
(3) $\mathbf{QAQ}^{-1} \in$ Lin* *for every nonsingular tensor* $\mathbf{Q}$;
(4) $\mathbf{A}$ *is nonsingular and* $\mathbf{A}^{-1} \in$ Lin*.

Proof: The equivalence of (1) and (2) follows from the equivalence of the equations $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ and $\mathbf{A}^\mathrm{T}\mathbf{X}^\mathrm{T} + \mathbf{X}^\mathrm{T}\mathbf{A}^\mathrm{T} = \mathbf{H}^\mathrm{T}$. Similarly, the equivalence of (1) and (3) follows from the equivalence of the equations $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ and

$$(\mathbf{QAQ}^{-1})(\mathbf{QXQ}^{-1}) + (\mathbf{QXQ}^{-1})(\mathbf{QAQ}^{-1}) = \mathbf{QHQ}^{-1}.$$

Now suppose that $\mathbf{A}$, and hence $\mathbf{A}^T$, is singular. Then there are nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{Au} = \mathbf{0}$ and $\mathbf{A}^T\mathbf{v} = \mathbf{0}$. Therefore,

$$\mathsf{B}_\mathbf{A}[\mathbf{u} \otimes \mathbf{v}] = (\mathbf{Au}) \otimes \mathbf{v} + \mathbf{u} \otimes (\mathbf{A}^T\mathbf{v}) = \mathbf{0}.$$

Since $\mathbf{u} \otimes \mathbf{v} \neq \mathbf{0}$, $\mathsf{B}_\mathbf{A}$ is singular and hence $\mathbf{A} \notin \mathrm{Lin}^*$. Thus $\mathbf{A} \in \mathrm{Lin}^* \Rightarrow \mathbf{A}$ is nonsingular. That $\mathbf{A} \in \mathrm{Lin}^* \Rightarrow \mathbf{A}^{-1} \in \mathrm{Lin}^*$ follows from the equivalence of the equations $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ and $\mathbf{A}^{-1}\mathbf{X} + \mathbf{XA}^{-1} = \mathbf{A}^{-1}\mathbf{HA}^{-1}$. Hence, (1) implies (4). Conversely, if $\mathbf{A}^{-1} \in \mathrm{Lin}^*$ then $\mathbf{A} = (\mathbf{A}^{-1})^{-1} \in \mathrm{Lin}^*$. □

Proposition 2.1 shows that nonsingularity of $\mathbf{A}$ is necessary for $\mathbf{A} \in \mathrm{Lin}^*$; as we will see in Section 4, it is not sufficient. *For the remainder of this section we assume that* $\mathbf{A} \in \mathrm{Lin}^*$. Unless specified otherwise, no additional restrictions are imposed on $\mathbf{A}$. We denote the inverse of $\mathsf{B}_\mathbf{A}$ by $\mathsf{L}_\mathbf{A}$:

$$\mathsf{L}_\mathbf{A} := (\mathsf{B}_\mathbf{A})^{-1}. \tag{2.3}$$

Then $\mathsf{B}_\mathbf{A}[\mathbf{X}] = \mathbf{H}$ iff $\mathbf{X} = \mathsf{L}_\mathbf{A}[\mathbf{H}]$, which is equivalent to the statement (1.18). If I denotes the fourth-order identity tensor, then

$$\mathsf{L}_\mathbf{A}\mathsf{B}_\mathbf{A} = \mathsf{B}_\mathbf{A}\mathsf{L}_\mathbf{A} = \mathsf{I}, \tag{2.4}$$

which is equivalent to the relations

$$\mathsf{L}_\mathbf{A}[\mathbf{AH} + \mathbf{HA}] = \mathbf{A}\mathsf{L}_\mathbf{A}[\mathbf{H}] + \mathsf{L}_\mathbf{A}[\mathbf{H}]\mathbf{A} = \mathbf{H}. \tag{2.5}$$

If $\mathbf{A}$ is symmetric, then from $(2.1)_1$ it follows that $\mathsf{B}_\mathbf{A}$, and hence its inverse $\mathsf{L}_\mathbf{A}$, maps symmetric tensors to symmetric tensors and skew tensors to skew tensors.

The fourth-order tensor $\mathsf{M}_\mathbf{A}$ is defined by

$$\mathsf{M}_\mathbf{A} := \mathsf{L}_\mathbf{A}\mathsf{C}_\mathbf{A} = \mathsf{C}_\mathbf{A}\mathsf{L}_\mathbf{A}, \tag{2.6}$$

where $(2.6)_2$ follows by multiplying $(2.2)_1$ on the left and right by $\mathsf{L}_\mathbf{A}$ and then using (2.4). From $(2.1)_2$ it follows that (2.6) is equivalent to the relations

$$\mathsf{M}_\mathbf{A}[\mathbf{H}] = \mathsf{L}_\mathbf{A}[\mathbf{AH} - \mathbf{HA}] = \mathbf{A}\mathsf{L}_\mathbf{A}[\mathbf{H}] - \mathsf{L}_\mathbf{A}[\mathbf{H}]\mathbf{A}. \tag{2.7}$$

By replacing $\mathbf{H}$ with $\mathbf{AH} - \mathbf{HA}$ in (1.18), we see that $(2.7)_1$ is equivalent to the statement (1.19). From (2.6), (2.2), and (2.4), we obtain the relations

$$\mathsf{M}_\mathbf{A}\mathsf{B}_\mathbf{A} = \mathsf{B}_\mathbf{A}\mathsf{M}_\mathbf{A} = \mathsf{C}_\mathbf{A} = \mathsf{L}_\mathbf{A}\mathsf{C}_{\mathbf{A}^2} = \mathsf{C}_{\mathbf{A}^2}\mathsf{L}_\mathbf{A}. \tag{2.8}$$

By (2.1) we see that (2.8) is equivalent to the relations

$$\begin{aligned} \mathsf{M}_\mathbf{A}[\mathbf{AH} + \mathbf{HA}] &= \mathbf{A}\mathsf{M}_\mathbf{A}[\mathbf{H}] + \mathsf{M}_\mathbf{A}[\mathbf{H}]\mathbf{A} = \mathbf{AH} - \mathbf{HA} \\ &= \mathsf{L}_\mathbf{A}[\mathbf{A}^2\mathbf{H} - \mathbf{HA}^2] = \mathbf{A}^2\mathsf{L}_\mathbf{A}[\mathbf{H}] - \mathsf{L}_\mathbf{A}[\mathbf{H}]\mathbf{A}^2. \end{aligned} \tag{2.9}$$

If $\mathbf{A}$ is symmetric, then $\mathbf{AH} - \mathbf{HA}$ is symmetric (resp. skew) if $\mathbf{H}$ is skew (resp. symmetric). Hence, by $(2.7)_1$ and the comment following (2.5), we see that if $\mathbf{A}$ is symmetric then $\mathsf{M}_\mathbf{A}$ maps symmetric tensors to skew tensors and skew tensors to symmetric tensors.

The fourth-order tensor $\mathsf{N}_\mathbf{A}$ is defined by

$$\mathsf{N}_\mathbf{A} := \mathsf{M}_\mathbf{A}\mathsf{C}_\mathbf{A} = \mathsf{C}_\mathbf{A}\mathsf{M}_\mathbf{A} = \mathsf{L}_\mathbf{A}(\mathsf{C}_\mathbf{A})^2 = (\mathsf{C}_\mathbf{A})^2\mathsf{L}_\mathbf{A}, \tag{2.10}$$

where $(2.10)_{2-4}$ follow from (2.6). Since

$$(\mathsf{C}_\mathbf{A})^2[\mathbf{H}] = \mathbf{A}^2\mathbf{H} - 2\mathbf{AHA} + \mathbf{HA}^2, \tag{2.11}$$

(2.10) is equivalent to the relations

$$\begin{aligned} \mathsf{N}_\mathbf{A}[\mathbf{H}] &= \mathsf{M}_\mathbf{A}[\mathbf{AH} - \mathbf{HA}] = \mathbf{A}\mathsf{M}_\mathbf{A}[\mathbf{H}] - \mathsf{M}_\mathbf{A}[\mathbf{H}]\mathbf{A} \\ &= \mathsf{L}_\mathbf{A}[\mathbf{A}^2\mathbf{H} - 2\mathbf{AHA} + \mathbf{HA}^2] \\ &= \mathbf{A}^2\mathsf{L}_\mathbf{A}[\mathbf{H}] + \mathsf{L}_\mathbf{A}[\mathbf{H}]\mathbf{A}^2 - 2\mathbf{A}\mathsf{L}_\mathbf{A}[\mathbf{H}]\mathbf{A}. \end{aligned} \tag{2.12}$$

By replacing $\mathbf{H}$ with $\mathbf{A}^2\mathbf{H} - 2\mathbf{AHA} + \mathbf{HA}^2$ in (1.18), we see that $(2.12)_3$ is equivalent to the statement (1.20).

Proposition 2.2 *Let $\mathcal{S}_\mathbf{A}$ denote the set consisting of the fourth-order tensors introduced above:*

$$\mathcal{S}_\mathbf{A} := \{\mathsf{B}_\mathbf{A}, \mathsf{C}_\mathbf{A}, \mathsf{L}_\mathbf{A}, \mathsf{M}_\mathbf{A}, \mathsf{N}_\mathbf{A}\}. \tag{2.13}$$

Then any two tensors in $\mathcal{S}_\mathbf{A}$ commute, and each tensor $\mathsf{K}_\mathbf{A} \in \mathcal{S}_\mathbf{A}$ has the following properties:

$$\begin{aligned} \mathsf{K}_\mathbf{A}[\mathbf{A}_1\mathbf{H}\mathbf{A}_2] &= \mathbf{A}_1\mathsf{K}_\mathbf{A}[\mathbf{H}]\mathbf{A}_2, \ \textit{if } \mathbf{A}_1 \textit{ and } \mathbf{A}_2 \textit{ commute with } \mathbf{A}, & (2.14) \\ \mathsf{K}_{\mathbf{QAQ}^{-1}}[\mathbf{QHQ}^{-1}] &= \mathbf{Q}\mathsf{K}_\mathbf{A}[\mathbf{H}]\mathbf{Q}^{-1}, \ \textit{for any nonsingular tensor } \mathbf{Q}. & (2.15) \end{aligned}$$

In particular, $\mathsf{K}_\mathbf{A}[\mathbf{H}]$ is an isotropic function of $\mathbf{A}$ and $\mathbf{H}$ which is linear in $\mathbf{H}$, and

$$\mathsf{K}_\mathbf{A}[\mathbf{A}^m\mathbf{H}\mathbf{A}^n] = \mathbf{A}^m\mathsf{K}_\mathbf{A}[\mathbf{H}]\mathbf{A}^n, \ \textit{for any integers } m \textit{ and } n. \tag{2.16}$$

Proof: That $\mathcal{S}_\mathbf{A}$ is commutative follows from (2.2), (2.4), (2.6), (2.8), and (2.10). The easiest way to establish the other properties is to first prove them for the tensors $\mathsf{B}_\mathbf{A}$ and $\mathsf{C}_\mathbf{A}$, and then use the definitions (2.3), (2.6), and (2.10) to prove the corresponding results for $\mathsf{L}_\mathbf{A}$, $\mathsf{M}_\mathbf{A}$, and $\mathsf{N}_\mathbf{A}$. We prove (2.14) and leave the proof of (2.15) to the reader. It is easily seen that (2.14) holds for $\mathsf{K}_\mathbf{A} = \mathsf{B}_\mathbf{A}$ or $\mathsf{C}_\mathbf{A}$. To prove (2.14) for $\mathsf{K}_\mathbf{A} = \mathsf{L}_\mathbf{A}$, apply $\mathsf{L}_\mathbf{A}$ to $\mathsf{B}_\mathbf{A}[\mathbf{A}_1\mathbf{X}\mathbf{A}_2] = \mathbf{A}_1\mathsf{B}_\mathbf{A}[\mathbf{X}]\mathbf{A}_2$ to obtain $\mathbf{A}_1\mathbf{X}\mathbf{A}_2 = \mathsf{L}_\mathbf{A}[\mathbf{A}_1\mathsf{B}_\mathbf{A}[\mathbf{X}]\mathbf{A}_2]$, and then set $\mathbf{X} = \mathsf{L}_\mathbf{A}[\mathbf{H}]$. To prove (2.14) for $\mathsf{K}_\mathbf{A} = \mathsf{M}_\mathbf{A}$, use the fact that (2.14) holds for $\mathsf{K}_\mathbf{A} = \mathsf{C}_\mathbf{A}$ and $\mathsf{L}_\mathbf{A}$:

$$\begin{aligned} \mathsf{M}_\mathbf{A}[\mathbf{A}_1\mathbf{H}\mathbf{A}_2] &= \mathsf{L}_\mathbf{A}[\mathsf{C}_\mathbf{A}[\mathbf{A}_1\mathbf{H}\mathbf{A}_2]] = \mathsf{L}_\mathbf{A}[\mathbf{A}_1\mathsf{C}_\mathbf{A}[\mathbf{H}]\mathbf{A}_2] \\ &= \mathbf{A}_1\mathsf{L}_\mathbf{A}[\mathsf{C}_\mathbf{A}[\mathbf{H}]]\mathbf{A}_2 = \mathbf{A}_1\mathsf{M}_\mathbf{A}[\mathbf{H}]\mathbf{A}_2. \end{aligned}$$

The proof of (2.14) for $\mathsf{K}_{\mathbf{A}} = \mathsf{N}_{\mathbf{A}}$ is similar. □

The equations (2.7) give two expressions for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ in terms of $\mathsf{L}_{\mathbf{A}}$, $\mathbf{A}$, and $\mathbf{H}$. Alternate expressions are

$$\begin{aligned} \mathsf{M}_{\mathbf{A}}[\mathbf{H}] &= \mathbf{H} - 2\mathsf{L}_{\mathbf{A}}[\mathbf{HA}] = \mathbf{H} - 2\mathsf{L}_{\mathbf{A}}[\mathbf{H}]\mathbf{A} \\ &= 2\mathsf{L}_{\mathbf{A}}[\mathbf{AH}] - \mathbf{H} = 2\mathbf{A}\mathsf{L}_{\mathbf{A}}[\mathbf{H}] - \mathbf{H}\,. \end{aligned} \tag{2.17}$$

The relations $(2.17)_{1,3}$ follow from $(2.7)_1$ and (2.5); for example,

$$\begin{aligned} \mathsf{M}_{\mathbf{A}}[\mathbf{H}] &= \mathsf{L}_{\mathbf{A}}[\mathbf{AH} - \mathbf{HA}] = \mathsf{L}_{\mathbf{A}}[\mathbf{AH} + \mathbf{HA} - 2\mathbf{HA}] \\ &= \mathsf{L}_{\mathbf{A}}[\mathbf{AH} + \mathbf{HA}] - 2\mathsf{L}_{\mathbf{A}}[\mathbf{HA}] = \mathbf{H} - 2\mathsf{L}_{\mathbf{A}}[\mathbf{HA}]\,. \end{aligned}$$

Then $(2.17)_{2,4}$ follow from $(2.17)_{1,3}$ and (2.16) with $\mathsf{K}_{\mathbf{A}} = \mathsf{L}_{\mathbf{A}}$.

Proposition 2.3 *The fourth-order tensor* $\mathsf{L}_{\mathbf{A}}$ *can be expressed in terms of the fourth-order tensor* $\mathsf{M}_{\mathbf{A}}$ *by the formulas*

$$\mathsf{L}_{\mathbf{A}}[\mathbf{H}] = \tfrac{1}{2}(\mathbf{H} - \mathsf{M}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}^{-1} = \tfrac{1}{2}\mathbf{A}^{-1}(\mathbf{H} + \mathsf{M}_{\mathbf{A}}[\mathbf{H}])\,. \tag{2.18}$$

Similarly, $\mathsf{L}_{\mathbf{A}}$ *can be expressed in terms of* $\mathsf{N}_{\mathbf{A}}$ *by the formulas*

$$\begin{aligned} \mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \tfrac{1}{4}(\mathbf{A}^{-1}\mathbf{H} + \mathbf{HA}^{-1} - \mathbf{A}^{-1}\mathsf{N}_{\mathbf{A}}[\mathbf{H}]\mathbf{A}^{-1}) \\ &= \tfrac{1}{4}\mathbf{A}^{-1}(\mathbf{AH} + \mathbf{HA} - \mathsf{N}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}^{-1}\,. \end{aligned} \tag{2.19}$$

Proof: (2.18) follows from (2.17). Then from (2.18) we have

$$\mathsf{L}_{\mathbf{A}}[\mathbf{H}] = \tfrac{1}{4}(\mathbf{A}^{-1}\mathbf{H} + \mathbf{HA}^{-1} + \mathbf{A}^{-1}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] - \mathsf{M}_{\mathbf{A}}[\mathbf{H}]\mathbf{A}^{-1}),$$

and from (2.12) we have

$$\begin{aligned} \mathbf{A}^{-1}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] - \mathsf{M}_{\mathbf{A}}[\mathbf{H}]\mathbf{A}^{-1} &= \mathbf{A}^{-1}(\mathsf{M}_{\mathbf{A}}[\mathbf{H}]\mathbf{A} - \mathbf{A}\mathsf{M}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}^{-1} \\ &= \mathbf{A}^{-1}(-\mathsf{N}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}^{-1}\,. \ \square \end{aligned}$$

Now suppose that by some means we have obtained a direct solution of the tensor equation (1.19), or, equivalently, a direct formula for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$, which is valid for any tensor $\mathbf{H}$. As we will see in Section 5, such formulas are relatively simple and easily derived when $\dim \mathcal{V} = 3$. Then direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ which are valid for any tensor $\mathbf{H}$ follow from (2.18).[4] Alternatively, we can use the relations $(2.12)_{1,2}$ and the direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ to obtain direct formulas for $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$, and then use (2.19) to obtain direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ which are valid for any $\mathbf{H}$. In any of these formulas we can, of course, replace the $\mathbf{A}^{-1}$ terms by a polynomial in $\mathbf{A}$ via the

[4]For the tensor equation $\mathbf{A}^T\mathbf{X} + \mathbf{XA} = \mathbf{H}$ ($\mathbf{H}$ symmetric), Barnett and Storey [27] obtained a relation analogous to $(2.18)_1$; cf. their equations (1.2), (2.1), and (2.2). Their relation is equivalent to $(2.18)_1$ when both $\mathbf{H}$ and $\mathbf{A}$ are symmetric. They did not obtain direct solutions.

Cayley-Hamilton theorem. These techniques will be used in Section 5 to generate direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in three dimensions.

By (1.18) with $\mathbf{H} \to \mathbf{\Phi}(\mathbf{A},\mathbf{H})$, the unique solution $\mathbf{X}$ of the tensor equation $\mathbf{AX} + \mathbf{XA} = \mathbf{\Phi}(\mathbf{A},\mathbf{H})$ is given by $\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{\Phi}(\mathbf{A},\mathbf{H})]$. If $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ is an isotropic function of $\mathbf{A}$ and $\mathbf{H}$ which is linear in $\mathbf{H}$ (in particular, if $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ has one of the forms in (1.13), (1.21), or (1.22)), then by (2.15) it follows that $\mathbf{X}$ is also an isotropic function of $\mathbf{A}$ and $\mathbf{H}$ which is linear in $\mathbf{H}$. When $\mathbf{\Phi}(\mathbf{A},\mathbf{H}) = \mathbf{AH} - \mathbf{HA}$, this solution can also be written as $\mathbf{X} = \mathsf{M}_{\mathbf{A}}[\mathbf{H}]$. As we will see in Section 5, for arbitrary $\mathbf{H}$ the direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ are much simpler than the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$, which should not be too surprising in view of the method described above for generating the latter formulas. For other functions $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ in the list (1.13), the existence of relatively simple direct formulas for $\mathbf{X}$ is due to the fact that there are simple expressions for $\mathsf{L}_{\mathbf{A}}[\mathbf{\Phi}(\mathbf{A},\mathbf{H})]$ in terms of $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$, $\mathsf{M}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}]$, or $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$. We derive these identities below. Similar results hold for $\mathbf{\Phi}(\mathbf{A},\mathbf{H})$ of the form (1.21) and (1.22); cf. Scheidler [18].

Consider the case $\mathbf{\Phi}(\mathbf{A},\mathbf{H}) = \mathbf{AHA}$, i.e., the tensor equation

$$\mathbf{AX} + \mathbf{XA} = \mathbf{AHA}. \tag{2.20}$$

Alternate expressions for the solution $\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{AHA}]$ are given by the following identities:

$$\begin{aligned} \mathsf{L}_{\mathbf{A}}[\mathbf{AHA}] &= \mathbf{A}\mathsf{L}_{\mathbf{A}}[\mathbf{H}]\mathbf{A} = \mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}] \\ &= \tfrac{1}{2}\mathbf{A}(\mathbf{H} - \mathsf{M}_{\mathbf{A}}[\mathbf{H}]) = \tfrac{1}{2}(\mathbf{H} + \mathsf{M}_{\mathbf{A}}[\mathbf{H}])\mathbf{A} \\ &= \tfrac{1}{4}(\mathbf{AH} + \mathbf{HA} - \mathsf{N}_{\mathbf{A}}[\mathbf{H}]). \end{aligned} \tag{2.21}$$

$(2.21)_1$ follows from (2.16); then $(2.21)_{3,4,5}$ follow from (2.18) and $(2.19)_2$. To obtain $(2.21)_2$, observe that (2.20) is equivalent to the tensor equation

$$\mathbf{A}^{-1}\mathbf{X} + \mathbf{X}\mathbf{A}^{-1} = \mathbf{H}, \tag{2.22}$$

and that the solution of (2.22) is $\mathbf{X} = \mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}]$. For $\mathbf{A} \in \mathrm{Psym}$ and $\mathbf{H} \in \mathrm{Sym}$, relations equivalent to some of those in (2.21) were observed by Mehrabadi and Nemat-Nasser [11][5] and Cohen and Muncaster [24, Ch. 6].[6] Our derivations above and in

[5] In their analysis of the tensor equation $(1.9)_2$ for $\dot{\mathbf{U}}$, Mehrabadi and Nemat-Nasser obtained relations which, for symmetric $\mathbf{A}$ and $\mathbf{H}$, are equivalent to the relations $\mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}] = \frac{1}{2}(\mathbf{H}+\mathsf{M}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}$ and $\mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}] = \frac{1}{4}(\mathbf{AH}+\mathbf{HA} - \mathsf{N}_{\mathbf{A}}[\mathbf{H}])$; cf. equations (8.8), (8.12), (8.13), and (8.16) in their paper. They also derived a direct formula for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ in three dimensions and used this formula, together with the latter of the two relations above, to obtain a direct solution of $(1.9)_2$; cf. $(6.12)_1$ in this paper, which we will obtain by essentially the same technique. However, our derivation of direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ in Section 5 differs substantially from the method used in [11].

[6] Cohen and Muncaster considered a tensor equation of the form $\mathbf{A}^{-1}\mathbf{X} + \mathbf{X}\mathbf{A}^{-1} + c(\operatorname{tr}\mathbf{X})\mathbf{A}^{-1} = \mathbf{G}$, where $\mathbf{A} \in \mathrm{Psym}$ is the referential Euler tensor and $\mathbf{G}$ is symmetric. This equation arises in

the next paragraph differ from theirs; in particular, we do not rely on the symmetry of $\mathbf{A}$ or $\mathbf{H}$.

Compared with $(2.21)_2$, the formula for $\mathsf{M}_{\mathbf{A}^{-1}}$ in terms of $\mathsf{M}_{\mathbf{A}}$ is much simpler:

$$\mathsf{M}_{\mathbf{A}^{-1}} = -\mathsf{M}_{\mathbf{A}} . \tag{2.23}$$

Indeed, from (1.19) we see that $\mathbf{X} = \mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ iff $\mathbf{A}^{-1}\mathbf{X}+\mathbf{X}\mathbf{A}^{-1} = \mathbf{A}^{-1}(-\mathbf{H})-(-\mathbf{H})\mathbf{A}^{-1}$ iff $\mathbf{X} = \mathsf{M}_{\mathbf{A}^{-1}}[-\mathbf{H}]$. Note that some of the identities in (2.21) can also be obtained by replacing $\mathbf{A}$ with $\mathbf{A}^{-1}$ in (2.18) and (2.19) and then using (2.23) and (2.12).

For the case $\mathbf{\Phi}(\mathbf{A},\mathbf{H}) = \mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2$, we have the identities

$$\begin{aligned}
\mathsf{L}_{\mathbf{A}}[\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2] &= \mathbf{A}^2\mathsf{L}_{\mathbf{A}}[\mathbf{H}] + \mathsf{L}_{\mathbf{A}}[\mathbf{H}]\mathbf{A}^2 \\
&= 2\mathsf{L}_{\mathbf{A}}[\mathbf{H}]\mathbf{A}^2 + \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A} = 2\mathbf{A}^2\mathsf{L}_{\mathbf{A}}[\mathbf{H}] - \mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A} \\
&= \mathbf{A}\mathbf{H} - \mathsf{M}_{\mathbf{A}}[\mathbf{H}]\mathbf{A} = \mathbf{H}\mathbf{A} + \mathbf{A}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] \\
&= \tfrac{1}{2}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A} + \mathsf{N}_{\mathbf{A}}[\mathbf{H}]) .
\end{aligned} \tag{2.24}$$

$(2.24)_1$ follows from (2.16); then $(2.24)_{2,3}$ follow from (2.9). $(2.24)_{4,5}$ follow from $(2.24)_{2,3}$ and (2.13). Finally, $(2.24)_6$ follows from $(2.12)_4$ and $(2.21)_5$, or from $(2.24)_{4,5}$ and (2.12). Next, consider the case $\mathbf{\Phi}(\mathbf{A},\mathbf{H}) = \mathbf{A}^2\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{A}^2$. By applying $\mathsf{L}_{\mathbf{A}}$ to the identity

$$\mathbf{A}^2\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{A}^2 = \mathbf{A}^2(\operatorname{sym}\mathbf{H}) + (\operatorname{sym}\mathbf{H})\mathbf{A}^2 + \mathbf{A}^2(-\operatorname{skw}\mathbf{H}) - (-\operatorname{skw}\mathbf{H})\mathbf{A}^2$$

and then using (2.9) with $\mathbf{H} \to -\operatorname{skw}\mathbf{H}$, we obtain the identity

$$\begin{aligned}
\mathsf{L}_{\mathbf{A}}[\mathbf{A}^2\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{A}^2] &= \mathsf{L}_{\mathbf{A}}[\mathbf{A}^2(\operatorname{sym}\mathbf{H}) + (\operatorname{sym}\mathbf{H})\mathbf{A}^2] \\
&\quad + (\operatorname{skw}\mathbf{H})\mathbf{A} - \mathbf{A}(\operatorname{skw}\mathbf{H}) .
\end{aligned} \tag{2.25}$$

Alternate expressions for $\mathsf{L}_{\mathbf{A}}[\mathbf{A}^2\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{A}^2]$ follow from this and (2.24) with $\mathbf{H} \to \operatorname{sym}\mathbf{H}$. In particular, we have

$$\begin{aligned}
&\mathsf{L}_{\mathbf{A}}[\mathbf{A}^2\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{A}^2] + \mathbf{A}(\operatorname{skw}\mathbf{H}) - (\operatorname{skw}\mathbf{H})\mathbf{A} \\
&\quad = \mathbf{A}(\operatorname{sym}\mathbf{H}) - \mathsf{M}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}]\mathbf{A} = (\operatorname{sym}\mathbf{H})\mathbf{A} + \mathbf{A}\mathsf{M}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}] \\
&\quad = \tfrac{1}{2}(\mathbf{A}(\operatorname{sym}\mathbf{H}) + (\operatorname{sym}\mathbf{H})\mathbf{A} + \mathsf{N}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}]) .
\end{aligned} \tag{2.26}$$

Finally, for the case $\mathbf{\Phi}(\mathbf{A},\mathbf{H}) = \mathbf{H}\mathbf{A} - \mathbf{A}\mathbf{H}^{\mathrm{T}}$, we have the identity

$$\mathsf{L}_{\mathbf{A}}[\mathbf{H}\mathbf{A} - \mathbf{A}\mathbf{H}^{\mathrm{T}}] = \operatorname{skw}\mathbf{H} - \mathsf{M}_{\mathbf{A}}[\operatorname{sym}\mathbf{H}] . \tag{2.27}$$

---

the analysis of gyroscopic motions of pseudo-rigid elastic bodies. On multiplying this equation by $\mathbf{A}$ and taking the trace of the result we may solve for $\operatorname{tr}\mathbf{X}$ and reduce the original equation to the form (2.22) with $\mathbf{H} = \mathbf{G} - c(2+3c)^{-1}\operatorname{tr}(\mathbf{G}\mathbf{A})$. Then their equations (6.3.12), (6.3.14), and (6.3.16) are equivalent to our relation $\mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}] = \frac{1}{2}(\mathbf{H} + \mathsf{M}_{\mathbf{A}}[\mathbf{H}])\mathbf{A}$. They did not obtain direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ or $\mathsf{L}_{\mathbf{A}^{-1}}[\mathbf{H}]$.

This follows from the identity

$$\mathbf{HA} - \mathbf{AH}^{\mathrm{T}} = -\mathbf{A}(\operatorname{sym}\mathbf{H}) + (\operatorname{sym}\mathbf{H})\mathbf{A} + \mathbf{A}(\operatorname{skw}\mathbf{H}) + (\operatorname{skw}\mathbf{H})\mathbf{A}\,,$$

(2.5) with $\mathbf{H} \to \operatorname{skw}\mathbf{H}$, and $(2.7)_1$ with $\mathbf{H} \to \operatorname{sym}\mathbf{H}$.

We conclude this section with a result mentioned in the introduction in connection with Liapunov functions for systems of differential equations.

Proposition 2.4 *The tensor equation* $\mathbf{A}^{\mathrm{T}}\mathbf{Y} + \mathbf{YA} = \mathbf{G}$ *has a unique solution* $\mathbf{Y}$ *for any given* $\mathbf{G}$ *iff* $\mathbf{A} \in \mathrm{Lin}^*$. *If* $\mathbf{A} \in \mathrm{Lin}^*$, *then* $\mathbf{X} = \sum_{m,n} c_{m,n}\mathbf{A}^m\mathbf{H}\mathbf{A}^n$ *is a general direct solution of* $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ *iff* $\mathbf{Y} = \sum_{m,n} c_{m,n}(\mathbf{A}^{\mathrm{T}})^m\mathbf{G}\mathbf{A}^n$ *is a general direct solution of* $\mathbf{A}^{\mathrm{T}}\mathbf{Y} + \mathbf{YA} = \mathbf{G}$. *Here* m *and* n *are integers, the sums are assumed to be finite, and the coefficients* $c_{m,n}$ *may depend on* $\mathbf{A}$ *but not on* $\mathbf{H}$ *or* $\mathbf{G}$.

Proof: We use the well-known fact that $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}$ are similar, i.e., there is a nonsingular tensor $\mathbf{S}$ such that $\mathbf{A}^{\mathrm{T}} = \mathbf{SAS}^{-1}$. Let $\mathbf{Y} = \mathbf{SX}$ and $\mathbf{G} = \mathbf{SH}$. Then the equations $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{Y} + \mathbf{YA} = \mathbf{G}$ are equivalent, and the results of the proposition follow. □

Since $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ iff $\mathbf{X} = \mathsf{L}_{\mathbf{A}}[\mathbf{H}]$, Proposition 2.4 can be used to transform the general direct formulas (5.6) and (5.17)–(5.20) for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in three dimensions into general direct solutions $\mathbf{Y}$ of $\mathbf{A}^{\mathrm{T}}\mathbf{Y} + \mathbf{YA} = \mathbf{G}$. This proposition cannot be applied to the general direct formulas (5.22), (5.28), and (5.33) since some of the coefficients in these formulas depend on $\mathbf{H}$.[7]

3 Some tensor identities in three dimensions

The derivations of the results in Sections 4–7 utilize various identities involving one or two second-order tensors and their principal invariants. For convenience we have collected most of these identities in the present section. With the exception of some comments at the end of Section 4, *for the remainder of this paper we assume that the underlying inner product space* $\mathcal{V}$ *is three-dimensional.* Unless specified otherwise, the tensors $\mathbf{A}$ and $\mathbf{H}$ are arbitrary.

The *principal invariants* of $\mathbf{A}$ are denoted by $I_{\mathbf{A}}$, $II_{\mathbf{A}}$, and $III_{\mathbf{A}}$, and its *characteristic roots* (in the complex field) are denoted by a_1, a_2, a_3. Then

$$\det(x\mathbf{I} - \mathbf{A}) = x^3 - I_{\mathbf{A}}x^2 + II_{\mathbf{A}}x - III_{\mathbf{A}} = \prod_{i=1}^{3}(x - a_i)\,, \tag{3.1}$$

[7] If in Proposition 2.4 we allowed the coefficients $c_{m,n}$ in the formula for $\mathbf{X}$ to depend on $\mathbf{H}$, say $c_{m,n} = \hat{c}_{m,n}(\mathbf{A},\mathbf{H})$, then since $\mathbf{H} = \mathbf{S}^{-1}\mathbf{G}$ it follows that the coefficients in the corresponding formula for $\mathbf{Y}$ would depend not only on $\mathbf{A}$ and $\mathbf{G}$ but also on the tensor $\mathbf{S}$ in the similarity transformation $\mathbf{A}^{\mathrm{T}} = \mathbf{SAS}^{-1}$.

where $(3.1)_1$ holds for any real number x, $(3.1)_2$ holds for any complex x, and

$$
\begin{aligned}
I_{\mathbf{A}} &= \operatorname{tr}\mathbf{A} = a_1 + a_2 + a_3\,, && (3.2)\\
II_{\mathbf{A}} &= \tfrac{1}{2}\left(I_{\mathbf{A}}^2 - I_{\mathbf{A}^2}\right) = a_1a_2 + a_2a_3 + a_3a_1\,, && (3.3)\\
III_{\mathbf{A}} &= \det\mathbf{A} = a_1a_2a_3\,. && (3.4)
\end{aligned}
$$

The characteristic root a_i is real iff a_i is an eigenvalue of $\mathbf{A}$. The *second* and *third moments*[8] of $\mathbf{A}$ are

$$
\begin{aligned}
I_{\mathbf{A}^2} &= I_{\mathbf{A}}^2 - 2II_{\mathbf{A}} = a_1^2 + a_2^2 + a_3^2\,, && (3.5)\\
I_{\mathbf{A}^3} &= I_{\mathbf{A}}^3 - 3I_{\mathbf{A}}II_{\mathbf{A}} + 3III_{\mathbf{A}} = a_1^3 + a_2^3 + a_3^3\,. && (3.6)
\end{aligned}
$$

The Cayley-Hamilton theorem implies that

$$\mathbf{A}^3 = I_{\mathbf{A}}\mathbf{A}^2 - II_{\mathbf{A}}\mathbf{A} + III_{\mathbf{A}}\mathbf{I}\,. \tag{3.7}$$

The *adjoint* of $\mathbf{A}$ is the tensor

$$\operatorname{adj}\mathbf{A} := \mathbf{A}^2 - I_{\mathbf{A}}\mathbf{A} + II_{\mathbf{A}}\mathbf{I} = III_{\mathbf{A}}\mathbf{A}^{-1}\,, \tag{3.8}$$

where the expression on the right is valid only when $\mathbf{A}$ is nonsingular.

Since the expression $I_{\mathbf{A}}\mathbf{I} - \mathbf{A}$ occurs frequently in the sequel, we introduce a special symbol for it:

$$\tilde{\mathbf{A}} := I_{\mathbf{A}}\mathbf{I} - \mathbf{A}\,. \tag{3.9}$$

Then

$$I_{\tilde{\mathbf{A}}} = 2I_{\mathbf{A}} \quad \text{and} \quad \mathbf{A} = \tfrac{1}{2}I_{\tilde{\mathbf{A}}}\mathbf{I} - \tilde{\mathbf{A}}\,. \tag{3.10}$$

Expressions for the determinant of $\tilde{\mathbf{A}}$ in terms of $\mathbf{A}$ are

$$
\begin{aligned}
III_{\tilde{\mathbf{A}}} &= I_{\mathbf{A}}II_{\mathbf{A}} - III_{\mathbf{A}} = \tfrac{1}{3}\left(I_{\mathbf{A}}^3 - I_{\mathbf{A}^3}\right)\\
&= (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)\,. && (3.11)
\end{aligned}
$$

$(3.11)_{1,3}$ follow from (3.1) with $x = I_{\mathbf{A}}$, and $(3.11)_2$ follows from $(3.6)_1$. $(3.11)_3$ also follows from the fact that the characteristic roots of $\tilde{\mathbf{A}}$ are[9] $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_1$. If $\mathbf{A}$ is nonsingular, then from $(3.8)_2$, $(3.3)_1$, and $(3.11)_1$, we obtain[10]

$$II_{\mathbf{A}} = III_{\mathbf{A}}I_{\mathbf{A}^{-1}} \quad \text{and} \quad III_{\tilde{\mathbf{A}}} = III_{\mathbf{A}}(I_{\mathbf{A}}I_{\mathbf{A}^{-1}} - 1)\,. \tag{3.12}$$

[8] Cf. Ericksen [28, §38].

[9] For any scalar polynomial $p(x)$, the characteristic roots of the corresponding tensor polynomial $p(\mathbf{A})$ are $p(a_i)$; cf. Theorem 3, p. 84, in [29]. By choosing $p(x) = I_{\mathbf{A}} - x$ it follows that the characteristic roots of $\tilde{\mathbf{A}}$ are $I_{\mathbf{A}} - a_i = a_j + a_k$ for distinct i, j, k.

[10] Cf. Stickforth [30].

Expressions for the second principal invariant of $\tilde{\mathbf{A}}$ in terms of $\mathbf{A}$ are

$$\begin{aligned} II_{\tilde{\mathbf{A}}} &= I_{\mathbf{A}}^2 + II_{\mathbf{A}} = \tfrac{1}{2}\left(3I_{\mathbf{A}}^2 - I_{\mathbf{A}^2}\right) = I_{\mathbf{A}^2} + 3II_{\mathbf{A}} \\ &= (a_1+a_2)(a_2+a_3) + (a_2+a_3)(a_3+a_1) + (a_3+a_1)(a_1+a_2)\,. \end{aligned} \tag{3.13}$$

By replacing $\mathbf{A}$ with $\tilde{\mathbf{A}}$ in (3.8) and using (3.9), $(3.10)_1$, and $(3.13)_1$, we obtain the following expressions for the adjoint of $\tilde{\mathbf{A}}$:

$$\operatorname{adj}\tilde{\mathbf{A}} = \mathbf{A}^2 + II_{\mathbf{A}}\mathbf{I} = III_{\tilde{\mathbf{A}}}\tilde{\mathbf{A}}^{-1}\,, \tag{3.14}$$

where the expression on the right is valid only when $\tilde{\mathbf{A}}$ is nonsingular.[11] Let

$$\begin{aligned} \Delta_{\mathbf{A}} &:= I_{\mathbf{A}}^2 - II_{\mathbf{A}} = I_{\mathbf{A}^2} + II_{\mathbf{A}} = \tfrac{1}{2}\left(I_{\mathbf{A}}^2 + I_{\mathbf{A}^2}\right) = \tfrac{1}{2}I_{\tilde{\mathbf{A}}^2} \\ &= \tfrac{1}{2}[(a_1+a_2)^2 + (a_2+a_3)^2 + (a_3+a_1)^2]\,. \end{aligned} \tag{3.15}$$

This invariant appears in the direct solution (1.16) of Sidoroff and Guo and also in several other formulas in the sequel. Observe that $2\Delta_{\mathbf{A}}$ is the second moment of $\tilde{\mathbf{A}}$.

The following identities are due to Rivlin [17]:

$$\begin{aligned} \mathbf{A}^2\mathbf{H}\mathbf{A}^2 &= II_{\mathbf{A}}\mathbf{A}\mathbf{H}\mathbf{A} - III_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) \\ &\quad + I_{\mathbf{A}^2\mathbf{H}}\mathbf{A}^2 + \alpha^{(3)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + III_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}}\mathbf{I}\,, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \mathbf{A}^2\mathbf{H}\mathbf{A} + \mathbf{A}\mathbf{H}\mathbf{A}^2 &= I_{\mathbf{A}}\mathbf{A}\mathbf{H}\mathbf{A} - III_{\mathbf{A}}\mathbf{H} \\ &\quad + I_{\mathbf{A}\mathbf{H}}\mathbf{A}^2 + \alpha^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + III_{\mathbf{A}}I_{\mathbf{H}}\mathbf{I}\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2 + \mathbf{A}\mathbf{H}\mathbf{A} &= I_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) - II_{\mathbf{A}}\mathbf{H} \\ &\quad + I_{\mathbf{H}}\mathbf{A}^2 + \alpha^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \alpha^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\,, \end{aligned} \tag{3.18}$$

where

$$\alpha^{(3)}_{\mathbf{A},\mathbf{H}} = III_{\mathbf{A}}I_{\mathbf{H}} - II_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}} = -I_{\tilde{\mathbf{A}}\mathbf{A}^2\mathbf{H}}\,, \tag{3.19}$$

$$\alpha^{(2)}_{\mathbf{A},\mathbf{H}} = I_{\mathbf{A}^2\mathbf{H}} - I_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}} = -I_{\tilde{\mathbf{A}}\mathbf{A}\mathbf{H}}\,, \tag{3.20}$$

$$\alpha^{(1)}_{\mathbf{A},\mathbf{H}} = I_{\mathbf{A}\mathbf{H}} - I_{\mathbf{A}}I_{\mathbf{H}} = -I_{\tilde{\mathbf{A}}\mathbf{H}}\,, \tag{3.21}$$

$$\alpha^{(0)}_{\mathbf{A},\mathbf{H}} = I_{\mathbf{A}^2\mathbf{H}} - I_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}} + II_{\mathbf{A}}I_{\mathbf{H}} = I_{(\operatorname{adj}\mathbf{A})\mathbf{H}}\,. \tag{3.22}$$

The first expressions in (3.19)-(3.22) are the ones given by Rivlin [17]; the second expressions follow easily from these and (3.7)-(3.9). From (3.9), $(3.15)_1$, and (3.18), we obtain the identities

$$\begin{aligned} \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} &= \mathbf{A}\mathbf{H}\mathbf{A} - I_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) + I_{\mathbf{A}}^2\mathbf{H}\,, \\ &= -(\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2) + \Delta_{\mathbf{A}}\mathbf{H} + I_{\mathbf{H}}\mathbf{A}^2 + \alpha^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \alpha^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\,. \end{aligned} \tag{3.23}$$

[11]The identities $(3.10)_1$ $(3.11)_1$, $(3.13)_1$, and $(3.14)_2$ were observed by Guo [8]. Various authors have observed one or both of the identities $(3.11)_{1,3}$, often under the assumption that $\mathbf{A} \in$ Psym.

We define the subspace $\mathcal{T}(\mathbf{A})$ of Lin as follows:

$$\mathcal{T}(\mathbf{A}) := \{\mathbf{H} \in \text{Lin} : \text{tr}\,\mathbf{H} = \text{tr}\,(\mathbf{A}\mathbf{H}) = \text{tr}\,(\mathbf{A}^2\mathbf{H}) = 0\}\,. \tag{3.24}$$

In particular, every tensor $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ is deviatoric. It is easily verified that $\mathcal{T}(\tilde{\mathbf{A}}) = \mathcal{T}(\mathbf{A})$. By the Cayley-Hamilton theorem, $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff $\text{tr}\,(\mathbf{A}^k\mathbf{H}) = 0$ for every nonnegative integer k; if $\mathbf{A}$ is nonsingular then $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff $\text{tr}\,(\mathbf{A}^k\mathbf{H}) = 0$ for every integer k. For any tensor $\mathbf{L}$, let $\mathcal{P}(\mathbf{L})$ denote the subspace of Lin consisting of all polynomials in $\mathbf{L}$. Then $\mathcal{T}(\mathbf{A}) = \mathcal{P}(\mathbf{A}^T)^\perp$, the orthogonal complement of the subspace of all polynomials in $\mathbf{A}^T$. Since $\text{tr}\,(\mathbf{S}\mathbf{H}) = 0$ for every symmetric tensor $\mathbf{S}$ and skew tensor $\mathbf{H}$, it follows that

$$\text{Skw} \subset \mathcal{T}(\mathbf{A}) = \mathcal{P}(\mathbf{A})^\perp\,, \quad \forall\,\mathbf{A} \in \text{Sym}\,. \tag{3.25}$$

More generally, suppose that there is a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathcal{V}$ consisting of eigenvectors of $\mathbf{A}$: $\mathbf{A}\mathbf{e}_i = a_i\mathbf{e}_i$ $(i = 1,2,3)$. Equivalently, $\mathbf{A} = \sum_{i=1}^3 a_i\mathbf{e}_i \otimes \mathbf{e}^i$, where $\{\mathbf{e}^i\}$ is the reciprocal basis of $\{\mathbf{e}_i\}$. Let $H^i{}_j = \mathbf{e}^i \cdot \mathbf{H}\mathbf{e}_j$ denote the components of the tensor $\mathbf{H}$ relative to $\{\mathbf{e}_i\}$. If the eigenvalues of $\mathbf{A}$ are distinct, then $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff $H^1{}_1 = H^2{}_2 = H^3{}_3 = 0$. If $a_1 \neq a_2 = a_3$, then $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff $H^1{}_1 = 0$ and $H^2{}_2 + H^3{}_3 = 0$. Finally, $a_1 = a_2 = a_3 =: a$ iff $\mathbf{A} = a\mathbf{I}$ iff $\mathcal{T}(\mathbf{A})$ is the set of all deviatoric tensors.

When $\mathbf{H} \in \mathcal{T}(\mathbf{A})$, the identities (3.16)–(3.18) and $(3.23)_2$ simplify substantially and can be used to obtain other useful identities. The results are summarized in

Proposition 3.1 *The following identities hold for any tensor* $\mathbf{A}$ *and any* $\mathbf{H} \in \mathcal{T}(\mathbf{A})$:

$$\begin{aligned}
\mathbf{A}^2\mathbf{H}\mathbf{A}^2 &= II_{\mathbf{A}}\mathbf{A}\mathbf{H}\mathbf{A} - III_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A})\,, && (3.26)\\
\mathbf{A}^2\mathbf{H}\mathbf{A} + \mathbf{A}\mathbf{H}\mathbf{A}^2 &= I_{\mathbf{A}}\mathbf{A}\mathbf{H}\mathbf{A} - III_{\mathbf{A}}\mathbf{H}\,, && (3.27)\\
\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2 + \mathbf{A}\mathbf{H}\mathbf{A} &= I_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) - II_{\mathbf{A}}\mathbf{H}\,, && (3.28)\\
(\text{adj}\,\mathbf{A})\mathbf{H}(\text{adj}\,\mathbf{A}) &= III_{\mathbf{A}}(I_{\mathbf{A}}\mathbf{H} - \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A})\,, && (3.29)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} &= \Delta_{\mathbf{A}}\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2)\,, && (3.30)\\
\tilde{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A})\tilde{\mathbf{A}} &= III_{\tilde{\mathbf{A}}}\mathbf{H}\,, && (3.31)\\
(\text{adj}\,\tilde{\mathbf{A}})\mathbf{H}(\text{adj}\,\tilde{\mathbf{A}}) &= III_{\tilde{\mathbf{A}}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A})\,. && (3.32)
\end{aligned}$$

If $\tilde{\mathbf{A}}$ *is nonsingular then (3.31) and (3.32) are equivalent. If* $\mathbf{A}$ *is nonsingular then (3.29) is equivalent to the identity*

$$III_{\mathbf{A}}\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1} = I_{\mathbf{A}}\mathbf{H} - \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}\,. \tag{3.33}$$

If $\mathbf{A}$ *is symmetric then (3.26)–(3.32) hold for any skew tensor* $\mathbf{H}$. *If* $\mathbf{A}$ *is symmetric and nonsingular then (3.33) holds for any skew tensor* $\mathbf{H}$.

Proof: (3.26)-(3.28) and (3.30) follow immediately from (3.16)-(3.23) and the definition of $\mathcal{T}(\mathbf{A})$. Then (3.8) and (3.26)-(3.28) yield (3.29) and (3.33). To prove (3.31), replace $\mathbf{H}$ with $\mathbf{AH} + \mathbf{HA}$ in $(3.23)_1$ and use (3.27), (3.28), and $(3.11)_1$. Similarly, (3.32) follows from $(3.14)_1$, (3.26), (3.28), and $(3.11)_1$. Since $\mathcal{T}(\mathbf{A}) = \mathcal{T}(\tilde{\mathbf{A}})$, (3.32) can also be obtained by replacing $\mathbf{A}$ with $\tilde{\mathbf{A}}$ in (3.29) and using (3.9) and $(3.10)_1$. If $\tilde{\mathbf{A}}$ is nonsingular then $\operatorname{adj}\tilde{\mathbf{A}} = III_{\tilde{\mathbf{A}}}\tilde{\mathbf{A}}^{-1}$, in which case (3.31) and (3.32) are easily seen to be equivalent. The statements for symmetric $\mathbf{A}$ follow from these results and (3.25). □

The identity (3.31) is the key to our construction of simple direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ when $\mathbf{H} \in \mathcal{T}(\mathbf{A})$.[12] Note that since $\operatorname{tr}(\mathbf{A}^k(\mathbf{A}^m\mathbf{H}\mathbf{A}^n)) = \operatorname{tr}(\mathbf{A}^{k+m+n}\mathbf{H})$, for any nonnegative integers m and n we have

$$\mathbf{A}^m\mathbf{H}\mathbf{A}^n - \mathbf{A}^n\mathbf{H}\mathbf{A}^m \in \mathcal{T}(\mathbf{A}), \quad \forall\, \mathbf{A},\mathbf{H} \in \mathrm{Lin}\,; \tag{3.34}$$

and if $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ then any linear combination of terms of the form $\mathbf{A}^m\mathbf{H}\mathbf{A}^n$ belongs to $\mathcal{T}(\mathbf{A})$. In particular, for $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ each of the expressions in Proposition 3.1 belongs to $\mathcal{T}(\mathbf{A})$. We conclude this section with a result which will be utilized in the proof of Theorem 4.1.

Proposition 3.2 $\mathbf{AHA} = \mathbf{0}$ *for each* $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ *iff* $\mathbf{A} = \mathbf{u} \otimes \mathbf{v}$ *for some vectors* $\mathbf{u}$ *and* $\mathbf{v}$. *Similarly,* $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = \mathbf{0}$ *for each* $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ *iff* $\mathbf{A} = \frac{1}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{I} - \mathbf{u} \otimes \mathbf{v}$ *for some vectors* $\mathbf{u}$ *and* $\mathbf{v}$. *In both cases,* $\mathcal{T}(\mathbf{A})$ *is the set of all tensors* $\mathbf{H}$ *such that* $\operatorname{tr}\mathbf{H} = \mathbf{v}\cdot\mathbf{H}\mathbf{u} = 0$.

Proof: The "if" part of the first result is straightforward. Conversely, suppose that

$$\mathbf{AHA} = \mathbf{0}, \quad \forall \mathbf{H} \in \mathcal{T}(\mathbf{A}). \tag{$*$}$$

Then $\mathbf{A}$ is singular. Now in general, $\mathbf{A}$ has rank zero or one iff $\mathbf{A} = \mathbf{u} \otimes \mathbf{v}$ for some vectors $\mathbf{u}$ and $\mathbf{v}$. Hence, it suffices to show that if $\mathbf{A}$ has rank two then $(*)$ leads to a contradiction. Since $\mathbf{A}(\mathbf{b}\otimes\mathbf{c}) - (\mathbf{b}\otimes\mathbf{c})\mathbf{A} \in \mathcal{T}(\mathbf{A})$ for any vectors $\mathbf{b}$ and $\mathbf{c}$ (cf. (3.34)), $(*)$ implies that

$$\mathbf{A}^2\mathbf{b} \otimes \mathbf{A}^{\mathrm{T}}\mathbf{c} = \mathbf{A}\mathbf{b} \otimes (\mathbf{A}^{\mathrm{T}})^2\mathbf{c}, \quad \forall\, \mathbf{b},\mathbf{c} \in \mathcal{V}. \tag{$\ddagger$}$$

If $\mathbf{A}$ has rank two, we may choose $\mathbf{c}$ so that $\mathbf{A}^{\mathrm{T}}\mathbf{c} \neq \mathbf{0}$ and $(\mathbf{A}^{\mathrm{T}})^2\mathbf{c} \neq \mathbf{0}$. Then $(\ddagger)$ implies that for every vector $\mathbf{b}$ there is an $a \neq 0$ such that $\mathbf{A}(\mathbf{A}\mathbf{b}) = \mathbf{A}^2\mathbf{b} = a\mathbf{A}\mathbf{b}$. Then a is an eigenvalue of $\mathbf{A}$, and it is not hard to show that a is independent of $\mathbf{b}$, so that $\mathbf{A}^2 = a\mathbf{A}$. If $\mathbf{P} := \frac{1}{a}\mathbf{A}$ then $\mathbf{P}^2 = \mathbf{P}$. Hence, $\mathbf{A} = a\mathbf{P}$ for some projection

[12] The identities (3.30)-(3.32) were derived in Scheidler [23] by the methods used here but under the assumption that $\mathbf{A}$ is symmetric. A major special case of (3.31) was obtained independently and by a different method by Chen and Wheeler [20]. For symmetric $\mathbf{A}$, they showed that (3.31) holds for any $\mathbf{H}$ such that $\mathbf{e}\cdot\mathbf{He} = 0$ for every eigenvector of $\mathbf{A}$. The set of all such $\mathbf{H}$ coincides with $\mathcal{T}(\mathbf{A})$ when $\mathbf{A}$ has three distinct eigenvalues but otherwise is properly included in $\mathcal{T}(\mathbf{A})$.

$\mathbf{P}$ of rank two. Then there is a basis $\{\mathbf{e}_i\}$ of $\mathcal{V}$ with reciprocal basis $\{\mathbf{e}^i\}$ such that $\mathbf{P} = \mathbf{e}_2 \otimes \mathbf{e}^2 + \mathbf{e}_3 \otimes \mathbf{e}^3$, in which case $(*)$ implies that $H^2{}_2 = H^3{}_3 = H^2{}_3 = H^3{}_2 = 0$ for each $\mathbf{H} \in \mathcal{T}(\mathbf{A})$. But from the comments preceding Proposition 3.1, we see that $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ if $H^1{}_1 = 0$ and $H^2{}_2 = -H^3{}_3 \neq 0$, which is a contradiction. Finally, note that since $\mathcal{T}(\tilde{\mathbf{A}}) = \mathcal{T}(\mathbf{A})$ for any tensor $\mathbf{A}$, we have $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = \mathbf{0}$ for each $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = \mathbf{0}$ for each $\mathbf{H} \in \mathcal{T}(\tilde{\mathbf{A}})$ iff $\tilde{\mathbf{A}} = \mathbf{u} \otimes \mathbf{v}$ iff (cf. (3.10)) $\mathbf{A} = \frac{1}{2}(\mathbf{u} \cdot \mathbf{v})\mathbf{I} - \mathbf{u} \otimes \mathbf{v}$. □

4 Existence and uniqueness of solutions. Direct solutions in $\mathcal{T}(\mathbf{A})$

Here, as in the previous section, we do not require that $\mathbf{A} \in \mathrm{Lin}^*$, i.e., that the equation $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ has a unique solution $\mathbf{X}$ for *every* $\mathbf{H} \in \mathrm{Lin}$. Instead, in Theorem 4.1 and Proposition 4.2 we determine necessary and sufficient conditions for the existence of a unique solution as well as simple direct formulas for this solution when $\mathbf{X}$ and $\mathbf{H}$ are restricted to the subspace $\mathcal{T}(\mathbf{A})$. These results are then used in the proof of Theorem 4.2, which gives necessary and sufficient conditions for $\mathbf{A} \in \mathrm{Lin}^*$. We begin with the following result which is utilized in the proofs.

Proposition 4.1 *If* $\mathbf{X} \in \mathcal{T}(\mathbf{A})$ *then* $\mathbf{AX} + \mathbf{XA} \in \mathcal{T}(\mathbf{A})$; *i.e.* $\mathbb{B}_{\mathbf{A}}$ *maps* $\mathcal{T}(\mathbf{A})$ *into itself. Conversely, if* $\mathbf{A}$ *is nonsingular and* $\mathbf{AX} + \mathbf{XA} \in \mathcal{T}(\mathbf{A})$, *then* $\mathbf{X} \in \mathcal{T}(\mathbf{A})$. *In particular,* $\ker \mathbb{B}_{\mathbf{A}} \subset \mathcal{T}(\mathbf{A})$ *if* $\mathbf{A}$ *is nonsingular.*

Proof: The first part is just a special case of the result stated after (3.34). Conversely, suppose that $\mathbf{A}$ is nonsingular and $\mathbf{AX} + \mathbf{XA} \in \mathcal{T}(\mathbf{A})$. Then $2\,\mathrm{tr}\,(\mathbf{A}^{k+1}\mathbf{X}) = \mathrm{tr}\,(\mathbf{A}^k(\mathbf{AX} + \mathbf{XA})) = 0$ for any integer k, so that $\mathbf{X} \in \mathcal{T}(\mathbf{A})$.[13] Finally, $\mathbf{X} \in \ker \mathbb{B}_{\mathbf{A}}$ iff $\mathbf{AX} + \mathbf{XA} = \mathbf{0} \in \mathcal{T}(\mathbf{A})$. □

Theorem 4.1 *The following conditions are equivalent:*

(1) $III_{\tilde{\mathbf{A}}} \neq 0$;
(2) *The restriction of* $\mathbb{B}_{\mathbf{A}}$ *to the subspace* $\mathcal{T}(\mathbf{A})$ *is nonsingular;*
(3) *For each* $\mathbf{H} \in \mathcal{T}(\mathbf{A})$, *the equation* $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$ *has exactly one solution* $\mathbf{X} \in \mathcal{T}(\mathbf{A})$ *(the possibility of other solutions outside of the subspace* $\mathcal{T}(\mathbf{A})$ *is not excluded).*

When $III_{\tilde{\mathbf{A}}} \neq 0$, *direct formulas for the solution* $\mathbf{X} \in \mathcal{T}(\mathbf{A})$ *are*

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathbf{X} &= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = \mathbf{AHA} - I_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) + I_{\mathbf{A}}{}^2\mathbf{H} \\ &= \Delta_{\mathbf{A}}\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2). \end{aligned} \tag{4.1}$$

If $\mathbf{A}$ *is symmetric with* $III_{\tilde{\mathbf{A}}} \neq 0$ *and if* $\mathbf{H}$ *is skew, then (4.1) is the only skew solution of* $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$.

[13] If we dropped the assumption that $\mathbf{A}$ is nonsingular then we could only conclude that $\mathrm{tr}\,(\mathbf{A}^{k+1}\mathbf{X}) = 0$ holds for nonnegative k, which does not imply $\mathrm{tr}\,\mathbf{X} = 0$.

Proof: From the first part of Proposition 4.1, we see that (2)$\Leftrightarrow$(3). From (3.31) with $\mathbf{H} \to \mathbf{X}$, we have

$$\tilde{\mathbf{A}}\mathbf{B}_{\mathbf{A}}[\mathbf{X}]\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{AX} + \mathbf{XA})\tilde{\mathbf{A}} = III_{\tilde{\mathbf{A}}}\mathbf{X}, \quad \forall \mathbf{X} \in \mathcal{T}(\mathbf{A}). \tag{$*$}$$

If $III_{\tilde{\mathbf{A}}} \neq 0$, then by $(*)$ it follows that the conditions $\mathbf{X} \in \mathcal{T}(\mathbf{A})$ and $\mathbf{B}_{\mathbf{A}}[\mathbf{X}] = \mathbf{0}$ imply $\mathbf{X} = \mathbf{0}$. Hence, (1)$\Rightarrow$(2)$\Leftrightarrow$(3); and (3), together with $(*)$, implies $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = III_{\tilde{\mathbf{A}}}\mathbf{X}$, i.e., $(4.1)_1$. Then $(4.1)_{2,3}$ follow from $(3.23)_1$ and (3.30). If $\mathbf{A}$ is symmetric and $\mathbf{H}$ is skew then $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}$ is skew. Since Skw $\subset \mathcal{T}(\mathbf{A})$, and since (4.1) is the only solution in $\mathcal{T}(\mathbf{A})$, it follows that (4.1) is the only solution in Skw. It remains to show that (3) implies (1). Suppose that (3) holds and that $III_{\tilde{\mathbf{A}}} = 0$. Then by $(*)$, it follows that $\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} = \mathbf{0}$ for each $\mathbf{H} \in \mathcal{T}(\mathbf{A})$. Hence, by Proposition 3.2, we must have $\mathbf{A} = \frac{1}{2}(\mathbf{u}\cdot\mathbf{v})\mathbf{I} - \mathbf{u}\otimes\mathbf{v}$ for some vectors $\mathbf{u}$ and $\mathbf{v}$, in which case $\mathbf{H} \in \mathcal{T}(\mathbf{A})$ iff tr $\mathbf{H} = \mathbf{v}\cdot\mathbf{Hu} = \mathbf{0}$. But then for any vector $\mathbf{w}$ orthogonal to $\mathbf{u}$, the tensor $\mathbf{H} := \mathbf{u}\otimes\mathbf{w}$ belongs to $\mathcal{T}(\mathbf{A})$ and satisfies $\mathbf{AH} + \mathbf{HA} = \mathbf{0}$, which contradicts (3). Hence $III_{\tilde{\mathbf{A}}}$ must be nonzero. □

Recalling the definition (3.15) of $\Delta_{\mathbf{A}}$, we see that $(4.1)_3$ is the direct solution (1.16) obtained by Sidoroff [6] and Guo [8] under the assumptions $\mathbf{A} \in$ Psym and $\mathbf{H} \in$ Skw. Various necessary and sufficient conditions for $III_{\tilde{\mathbf{A}}} \neq 0$ follow from the results in the previous section and are summarized in

Proposition 4.2 *The following conditions are equivalent:*

(1) $III_{\tilde{\mathbf{A}}} \neq 0$;
(2) $\tilde{\mathbf{A}}$ *is nonsingular;*
(3) $I_{\mathbf{A}}II_{\mathbf{A}} \neq III_{\mathbf{A}}$;
(4) $I_{\mathbf{A}}^3 \neq I_{\mathbf{A}^3}$;
(5) $I_{\mathbf{A}}$ *is not an eigenvalue of* $\mathbf{A}$;
(6) $a_i + a_j \neq 0, \quad \forall i \neq j \in \{1,2,3\}$.

Since any nonreal characteristic roots of $\mathbf{A}$ occur in a complex conjugate pair, from (6) we see that if $\mathbf{A}$ has a characteristic root with nonzero real and imaginary parts then $III_{\tilde{\mathbf{A}}} \neq 0$. Also note that $\tilde{\mathbf{A}}$ may be nonsingular even if $\mathbf{A}$ is singular. Indeed, from (6) we see that the conditions

$$a_1 = 0, \quad a_2 \neq 0, \quad a_3 \neq 0, \quad a_2 + a_3 \neq 0 \tag{4.2}$$

are sufficient for $III_{\tilde{\mathbf{A}}} \neq 0$, and that if $a_1 = 0$ then the other conditions in (4.2) are necessary for $III_{\tilde{\mathbf{A}}} \neq 0$. It follows that if $\tilde{\mathbf{A}}$ is nonsingular then the null space of $\mathbf{A}$ has dimension at most one.

By combining the above results, we obtain

Theorem 4.2 *The following conditions are equivalent:*

(1) $\mathbf{A} \in \mathrm{Lin}^*$;
(2) $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *are nonsingular;*
(3) $III_{\mathbf{A}} \neq 0$ *and* $III_{\tilde{\mathbf{A}}} \neq 0$;
(4) $I_{\mathbf{A}}II_{\mathbf{A}} \neq III_{\mathbf{A}} \neq 0$;
(5) $III_{\mathbf{A}} \neq 0$ *and* $I_{\mathbf{A}}I_{\mathbf{A}^{-1}} \neq 1$;
(6) $a_i + a_j \neq 0, \quad \forall i,j \in \{1,2,3\}$;
(7) *neither* 0 *nor* $I_{\mathbf{A}}$ *is an eigenvalue of* $\mathbf{A}$.

Proof: The equivalence of (2)–(7) follows from Proposition 4.2 and $(3.12)_2$. That (1) implies (2) follows from Proposition 2.1 and Theorem 4.1. Conversely, if (2) holds then by Proposition 4.1 and Theorem 4.1 we have $\ker \mathbf{B_A} \subset \mathcal{T}(\mathbf{A})$ and $\ker \mathbf{B_A} \cap \mathcal{T}(\mathbf{A}) = \{\mathbf{0}\}$, respectively. Hence, $\ker \mathbf{B_A} = \{\mathbf{0}\}$. □

From (4) of Theorem 4.2 we see that $\mathbf{A} \in \mathrm{Lin}^*$ if $\mathbf{A}$ is nonsingular and deviatoric. From (6) we see that $\mathbf{A} \in \mathrm{Lin}^*$ if the characteristic roots of $\mathbf{A}$ have positive real parts (e.g., $\mathbf{A} \in \mathrm{Psym}$), or if the characteristic roots of $\mathbf{A}$ have negative real parts (e.g., $-\mathbf{A} \in \mathrm{Psym}$), or if $\mathbf{A}$ has a nonzero eigenvalue and a characteristic root with nonzero real and imaginary parts.

The only part of Theorem 4.2 that carries over (with obvious modifications) to arbitrary dimensions is the equivalence of (1) and (6). This result is a special case of

Theorem 4.3 [14] *Let $\mathcal{V}$ be a real or complex inner product space of dimension N. Let $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ have characteristic roots a_i and b_i $(i = 1, \ldots, N)$. Then the tensor equation $\mathbf{BX} + \mathbf{XA} = \mathbf{H}$ has a unique solution $\mathbf{X}$ for any given $\mathbf{H}$ iff $b_i + a_j \neq 0$ for each $i, j \in \{1, \ldots, N\}$.*

When $\mathcal{V}$ is a real N-dimensional inner product space and $\mathbf{B} = \mathbf{A}$ or $\mathbf{A}^{\mathrm{T}}$, the condition $b_i + a_j \neq 0$ reduces to $a_i + a_j \neq 0$. As in the three-dimensional case, this condition can also be expressed in terms of the N principal invariants of $\mathbf{A}$, which we denote by $I_{\mathbf{A}}^{(1)} = \mathrm{tr}\,\mathbf{A}$, $I_{\mathbf{A}}^{(2)}, \ldots, I_{\mathbf{A}}^{(N)} = \det \mathbf{A}$. Let $I_{\mathbf{A}}^{(0)} = 1$, let $I_{\mathbf{A}}^{(k)} = 0$ if $k < 0$ or $k > N$, and let $A_{\mathcal{H}}$ denote the $N \times N$ matrix whose element in the ith row and jth column is $(-1)^j I_{\mathbf{A}}^{(2i-j)}$. The matrix $A_{\mathcal{H}}$ (or its transpose) is known as the *Hurwitz matrix* associated with $\mathbf{A}$. Observe that $\det A_{\mathcal{H}}$ is a polynomial in the principal invariants of $\mathbf{A}$; in particular, $\det A_{\mathcal{H}} = III_{\mathbf{A}}(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}})$ when $N = 3$. For arbitrary N, $a_i + a_j \neq 0$ for each $i, j \in \{1, \ldots, N\}$ iff $\det A_{\mathcal{H}} \neq 0$. This follows immediately from the identities

$$\begin{aligned} \det A_{\mathcal{H}} &= (-1)^{N(N+1)/2}\, 2^{-N} \prod_{i \leq j} (a_i + a_j) \\ &= (-1)^{N(N+1)/2} (\det \mathbf{A}) \prod_{i<j} (a_i + a_j)\,; \end{aligned} \tag{4.3}$$

cf. Hahn [25, §2.6,2.7] and Gantmacher [26, §5.6,5.7].

[14] Most of the proofs in the literature deal only with the special case where $\mathcal{V}$ is the vector space of N-tuples of complex numbers and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{X}$ are complex $N \times N$ matrices; cf. Gantmacher [29, §8.3], Bellman [31, §12.13], Jacob and Polak [32], and Feintuch and Rubin [33] for four different proofs. These proofs utilize the fact that every characteristic root of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$; hence, they do not carry over directly to a real vector space. However, the corresponding theorem for real matrix equations and, consequently, the general theorem stated above, can be obtained from the result for complex matrix equations.

5 Direct formulas for $\mathsf{L}_{\mathbf{A}}$, $\mathsf{M}_{\mathbf{A}}$, and $\mathsf{N}_{\mathbf{A}}$ in three dimensions

In this section we assume that $\mathbf{A} \in \mathrm{Lin}^*$ *and* $\mathcal{V}$ *is three-dimensional.* Then from Theorem 4.1 and (1.18) it follows that *for each* $\mathbf{H} \in \mathcal{T}(\mathbf{A})$,

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} \\ &= \mathbf{A}\mathbf{H}\mathbf{A} - I_{\mathbf{A}}(\mathbf{A}\mathbf{H} + \mathbf{H}\mathbf{A}) + I_{\mathbf{A}}^2\mathbf{H} \\ &= \Delta_{\mathbf{A}}\mathbf{H} - (\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2); \end{aligned} \tag{5.1}$$

in particular, these formulas hold whenever $\pm\mathbf{A} \in \mathrm{Psym}$ *and* $\mathbf{H} \in \mathrm{Skw}$, *in which case* $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ *is skew.* For arbitrary $\mathbf{H}$, all of the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$, $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$, and $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ derived below will be obtained from $(5.1)_{1,2}$, the results in Sections 2 and 3, and the fact that $\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A} \in \mathcal{T}(\mathbf{A})$ for each $\mathbf{A}, \mathbf{H} \in \mathrm{Lin}$, which is just a special case of (3.34). This last fact allows us to replace $\mathbf{H}$ with $\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}$ in (5.1).

By replacing $\mathbf{H}$ with $\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}$ in $(5.1)_1$ and using $\mathsf{M}_{\mathbf{A}}[\mathbf{H}] = \mathsf{L}_{\mathbf{A}}[\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}]$ and $\tilde{\mathbf{A}} = I_{\mathbf{A}}\mathbf{I} - \mathbf{A}$, we obtain

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] &= \tilde{\mathbf{A}}(\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A})\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{H}\tilde{\mathbf{A}} - \tilde{\mathbf{A}}\mathbf{H})\tilde{\mathbf{A}} \\ &= \mathbf{A}(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}) - (\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}})\mathbf{A} = (\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}})\tilde{\mathbf{A}} - \tilde{\mathbf{A}}(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}) \\ &= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}^2 - \tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}} \end{aligned} \tag{5.2}$$

for any tensor $\mathbf{H}$. Similarly, from $(5.1)_2$ with $\mathbf{H} \to \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}$, we obtain

$$III_{\tilde{\mathbf{A}}}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] = \mathbf{A}^2\mathbf{H}\mathbf{A} - \mathbf{A}\mathbf{H}\mathbf{A}^2 + I_{\mathbf{A}}(\mathbf{H}\mathbf{A}^2 - \mathbf{A}^2\mathbf{H}) + I_{\mathbf{A}}^2(\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}). \tag{5.3}$$

A lengthier derivation of this result follows by replacing $\mathbf{H}$ with $\mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}$ in $(5.1)_3$; this yields some $\mathbf{A}^3$ terms which can be reduced by means of the Cayley-Hamilton theorem (3.7). Mehrabadi and Nemat-Nasser [11] obtained (5.3) by repeated applications of the Cayley-Hamilton theorem.

In view of (1.19), the direct formulas (5.2) and (5.3) yield a variety of direct solutions $\mathbf{X} = \mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ of the tensor equation $\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{H} - \mathbf{H}\mathbf{A}$, $\mathbf{H}$ arbitrary. In particular, if $\mathbf{A}$ and $\mathbf{H}$ are symmetric then $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ is skew, and we have the alternate formulas

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] &= 2\,\mathrm{skw}\,(\mathbf{A}\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}) = 2\,\mathrm{skw}\,(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}^2) \\ &= 2\,\mathrm{skw}\left(\mathbf{A}^2\mathbf{H}\mathbf{A} - I_{\mathbf{A}}\mathbf{A}^2\mathbf{H} + I_{\mathbf{A}}^2\mathbf{A}\mathbf{H}\right) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} -III_{\tilde{\mathbf{A}}}\mathsf{M}_{\mathbf{A}}[\mathbf{H}] &= 2\,\mathrm{skw}\,(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}\mathbf{A}) = 2\,\mathrm{skw}\,(\tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}}) \\ &= 2\,\mathrm{skw}\left(\mathbf{A}\mathbf{H}\mathbf{A}^2 - I_{\mathbf{A}}\mathbf{H}\mathbf{A}^2 + I_{\mathbf{A}}^2\mathbf{H}\mathbf{A}\right). \end{aligned} \tag{5.5}$$

Next, we derive direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ and $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ for arbitrary $\mathbf{H}$. From (2.18) and $(5.2)_1$, we obtain

$$\begin{aligned}
2\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \left[\mathbf{H} + \frac{1}{III_{\tilde{\mathbf{A}}}}\tilde{\mathbf{A}}(\mathbf{HA} - \mathbf{AH})\tilde{\mathbf{A}}\right]\mathbf{A}^{-1} \\
&= \mathbf{HA}^{-1} + \frac{1}{III_{\tilde{\mathbf{A}}}}\tilde{\mathbf{A}}(\mathbf{H} - \mathbf{AHA}^{-1})\tilde{\mathbf{A}} \\
&= \mathbf{A}^{-1}\left[\mathbf{H} + \frac{1}{III_{\tilde{\mathbf{A}}}}\tilde{\mathbf{A}}(\mathbf{AH} - \mathbf{HA})\tilde{\mathbf{A}}\right] \\
&= \mathbf{A}^{-1}\mathbf{H} + \frac{1}{III_{\tilde{\mathbf{A}}}}\tilde{\mathbf{A}}(\mathbf{H} - \mathbf{A}^{-1}\mathbf{HA})\tilde{\mathbf{A}}\,.
\end{aligned} \tag{5.6}$$

Other formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ follow from (2.18), $(5.2)_{2-5}$, and (5.3)–(5.5). In particular, if $\mathbf{A}$ and $\mathbf{H}$ are symmetric then $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ is symmetric, and

$$\begin{aligned}
\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \left[\frac{1}{2}\mathbf{H} + \frac{1}{III_{\tilde{\mathbf{A}}}}\operatorname{skw}(\tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}})\right]\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1}\left[\frac{1}{2}\mathbf{H} + \frac{1}{III_{\tilde{\mathbf{A}}}}\operatorname{skw}(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}^2)\right].
\end{aligned} \tag{5.7}$$

Alternate expressions for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ follow by substituting $(3.8)_2$ for $\mathbf{A}^{-1}$.

Since $\mathsf{N}_{\mathbf{A}}[\mathbf{H}] = \mathsf{M}_{\mathbf{A}}[\mathbf{AH} - \mathbf{HA}] = \mathsf{M}_{\mathbf{A}}[\mathbf{H}\tilde{\mathbf{A}} - \tilde{\mathbf{A}}\mathbf{H}]$, by replacing $\mathbf{H}$ with $\mathbf{AH} - \mathbf{HA}$ in $(5.2)_1$ and replacing $\mathbf{H}$ with $\mathbf{H}\tilde{\mathbf{A}} - \tilde{\mathbf{A}}\mathbf{H}$ in $(5.2)_2$, we obtain

$$\begin{aligned}
III_{\tilde{\mathbf{A}}}\mathsf{N}_{\mathbf{A}}[\mathbf{H}] &= \tilde{\mathbf{A}}(\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2 - 2\mathbf{AHA})\tilde{\mathbf{A}} \\
&= \mathbf{A}^2(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}) + (\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}})\mathbf{A}^2 - 2\mathbf{A}(\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}})\mathbf{A} \\
&= \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^2\mathbf{H} + \mathbf{H}\tilde{\mathbf{A}}^2 - 2\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}})\tilde{\mathbf{A}} \\
&= \tilde{\mathbf{A}}^3\mathbf{H}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}^3 - 2\tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}}^2\,.
\end{aligned} \tag{5.8}$$

Another expression for $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ can be obtained by using $\tilde{\mathbf{A}} = I_{\mathbf{A}}\mathbf{I} - \mathbf{A}$ in $(5.8)_1$, expanding, and then using (3.7) to reduce the $\mathbf{A}^3$ terms. Similarly, we can reduce the $\tilde{\mathbf{A}}^3$ terms in $(5.8)_4$ by (3.7) with $\mathbf{A} \to \tilde{\mathbf{A}}$. The results are

$$\begin{aligned}
III_{\tilde{\mathbf{A}}}\mathsf{N}_{\mathbf{A}}[\mathbf{H}] &= -2\mathbf{A}^2\mathbf{HA}^2 + 2I_{\mathbf{A}}(\mathbf{A}^2\mathbf{HA} + \mathbf{AHA}^2) - 2II_{\tilde{\mathbf{A}}}\mathbf{AHA} \\
&\quad + \Gamma_{\mathbf{A}}(\mathbf{AH} + \mathbf{HA}) - 2I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{H} \\
&= -2\tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}}^2 + I_{\tilde{\mathbf{A}}}(\tilde{\mathbf{A}}^2\mathbf{H}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}}^2) - 2II_{\tilde{\mathbf{A}}}\tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} \\
&\quad + III_{\tilde{\mathbf{A}}}(\tilde{\mathbf{A}}\mathbf{H} + \mathbf{H}\tilde{\mathbf{A}})\,,
\end{aligned} \tag{5.9}$$

where alternate expressions for $II_{\tilde{\mathbf{A}}}$ are given by (3.13), and

$$\Gamma_{\mathbf{A}} = I_{\mathbf{A}}II_{\mathbf{A}} + III_{\mathbf{A}} = III_{\tilde{\mathbf{A}}} + 2III_{\mathbf{A}}\,. \tag{5.10}$$

By using Rivlin's identities (3.16) and (3.17) in $(5.9)_1$, we obtain

$$
\begin{aligned}
III_{\tilde{\mathbf{A}}}\mathsf{N}_{\mathbf{A}}[\mathbf{H}] &= -4II_{\mathbf{A}}\mathbf{AHA} + (I_{\mathbf{A}}II_{\mathbf{A}} + 3III_{\mathbf{A}})(\mathbf{AH}+\mathbf{HA}) - 4I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{H} \\
&\quad - 2\left(\alpha^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A}^2 + \eta^{(3)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + III_{\mathbf{A}}\alpha^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\right), \qquad (5.11)
\end{aligned}
$$

where $\alpha^{(2)}_{\mathbf{A},\mathbf{H}}$ and $\alpha^{(1)}_{\mathbf{A},\mathbf{H}}$ are given by (3.20) and (3.21), and

$$
\eta^{(3)}_{\mathbf{A},\mathbf{H}} = -I_{\mathbf{A}}I_{\mathbf{A}^2\mathbf{H}} + \Delta_{\mathbf{A}}I_{\mathbf{AH}} + III_{\mathbf{A}}I_{\mathbf{H}}. \qquad (5.12)
$$

Finally, by using Rivlin's identity (3.18) in (5.11), we obtain

$$
\begin{aligned}
III_{\tilde{\mathbf{A}}}\mathsf{N}_{\mathbf{A}}[\mathbf{H}] &= 4II_{\mathbf{A}}(\mathbf{A}^2\mathbf{H}+\mathbf{HA}^2) - 3III_{\tilde{\mathbf{A}}}(\mathbf{AH}+\mathbf{HA}) + 4\left(II_{\mathbf{A}}^2 - I_{\mathbf{A}}III_{\mathbf{A}}\right)\mathbf{H} \\
&\quad - 2\left(\eta^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A}^2 + \eta^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \eta^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\right), \qquad (5.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta^{(2)}_{\mathbf{A},\mathbf{H}} &= I_{\mathbf{A}^2\mathbf{H}} - I_{\mathbf{A}}I_{\mathbf{AH}} + 2II_{\mathbf{A}}I_{\mathbf{H}}, \qquad (5.14)\\
\eta^{(1)}_{\mathbf{A},\mathbf{H}} &= -I_{\mathbf{A}}I_{\mathbf{A}^2\mathbf{H}} + II_{\tilde{\mathbf{A}}}I_{\mathbf{AH}} + (III_{\mathbf{A}} - 2I_{\mathbf{A}}II_{\mathbf{A}})I_{\mathbf{H}}, \qquad (5.15)\\
\eta^{(0)}_{\mathbf{A},\mathbf{H}} &= 2II_{\mathbf{A}}I_{\mathbf{A}^2\mathbf{H}} + (III_{\mathbf{A}} - 2I_{\mathbf{A}}II_{\mathbf{A}})I_{\mathbf{AH}} + \left(2II_{\mathbf{A}}^2 - I_{\mathbf{A}}III_{\mathbf{A}}\right)I_{\mathbf{H}}. \qquad (5.16)
\end{aligned}
$$

In the remainder of this section we derive some additional formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ for arbitrary $\mathbf{H}$. By substituting the formula $(5.9)_1$ for $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ into $(2.19)_1$ and then using $(3.23)_1$, we obtain

$$
\begin{aligned}
2III_{\tilde{\mathbf{A}}}\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \mathbf{AHA} - I_{\mathbf{A}}(\mathbf{AH}+\mathbf{HA}) + II_{\tilde{\mathbf{A}}}\mathbf{H} \\
&\quad + I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{A}^{-1}\mathbf{HA}^{-1} - III_{\mathbf{A}}(\mathbf{A}^{-1}\mathbf{H}+\mathbf{HA}^{-1}) \\
&= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} + II_{\mathbf{A}}\mathbf{H} \\
&\quad + I_{\mathbf{A}}III_{\mathbf{A}}\mathbf{A}^{-1}\mathbf{HA}^{-1} - III_{\mathbf{A}}(\mathbf{A}^{-1}\mathbf{H}+\mathbf{HA}^{-1}). \qquad (5.17)
\end{aligned}
$$

The direct formula $(5.17)_1$ (with $III_{\tilde{\mathbf{A}}}$ and $II_{\tilde{\mathbf{A}}}$ replaced by the equivalent expressions $(3.11)_1$ and $(3.13)_1$) was stated without proof by Leonov [5] and Stickforth and Wegener [12] for the case $\mathbf{A} \in$ Psym and $\mathbf{H} \in$ Sym.[15] If we use the relation $III_{\mathbf{A}}\mathbf{A}^{-1} = \operatorname{adj}\mathbf{A}$ in (5.17), we obtain

$$
\begin{aligned}
2III_{\tilde{\mathbf{A}}}\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \mathbf{AHA} - I_{\mathbf{A}}(\mathbf{AH}+\mathbf{HA}) + II_{\tilde{\mathbf{A}}}\mathbf{H} \\
&\quad + \frac{I_{\mathbf{A}}}{III_{\mathbf{A}}}(\operatorname{adj}\mathbf{A})\mathbf{H}(\operatorname{adj}\mathbf{A}) - (\operatorname{adj}\mathbf{A})\mathbf{H} - \mathbf{H}(\operatorname{adj}\mathbf{A}) \\
&= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} + II_{\mathbf{A}}\mathbf{H} \\
&\quad + \frac{I_{\mathbf{A}}}{III_{\mathbf{A}}}(\operatorname{adj}\mathbf{A})\mathbf{H}(\operatorname{adj}\mathbf{A}) - (\operatorname{adj}\mathbf{A})\mathbf{H} - \mathbf{H}(\operatorname{adj}\mathbf{A}). \qquad (5.18)
\end{aligned}
$$

[15]Leonov attributes the result to L. M. Zubov. Stickforth and Wegener refer the reader to some lecture notes by Stickforth.

If the formula $(3.8)_2$ for $III_\mathbf{A}\mathbf{A}^{-1}$ is substituted into the last term in $(5.17)_1$, we obtain

$$2III_{\tilde{\mathbf{A}}}\mathsf{L}_\mathbf{A}[\mathbf{H}] = \mathbf{AHA} - (\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) + \Delta_\mathbf{A}\mathbf{H} + I_\mathbf{A} III_\mathbf{A}\mathbf{A}^{-1}\mathbf{HA}^{-1}. \tag{5.19}$$

The formulas $(5.18)_2$ and (5.19) are due to Müller [16] and Jameson [15], respectively.[16] By using (3.8) in (5.19) or $(5.18)_1$ and expanding, we obtain

$$\begin{aligned} 2III_\mathbf{A} III_{\tilde{\mathbf{A}}}\mathsf{L}_\mathbf{A}[\mathbf{H}] &= I_\mathbf{A}\mathbf{A}^2\mathbf{HA}^2 - I_\mathbf{A}^2(\mathbf{A}^2\mathbf{HA} + \mathbf{AHA}^2) + III_{\tilde{\mathbf{A}}}(\mathbf{A}^2\mathbf{H} + \mathbf{HA}^2) \\ &\quad + \left(I_\mathbf{A}^3 + III_\mathbf{A}\right)\mathbf{AHA} - I_\mathbf{A}^2 II_\mathbf{A}(\mathbf{AH} + \mathbf{HA}) + \alpha_\mathbf{A}\mathbf{H}, \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} \alpha_\mathbf{A} &= I_\mathbf{A}^2 III_\mathbf{A} + II_\mathbf{A} III_{\tilde{\mathbf{A}}} = I_\mathbf{A} II_\mathbf{A}^2 + III_\mathbf{A}\Delta_\mathbf{A} \\ &= I_\mathbf{A} II_\mathbf{A}^2 + I_\mathbf{A}^2 III_\mathbf{A} - II_\mathbf{A} III_\mathbf{A}. \end{aligned} \tag{5.21}$$

In view of (1.18) and $(3.11)_1$, this is the direct formula (1.17) of Hoger and Carlson [9];[17] cf. also Smith [14].[18]

Substitution of Rivlin's identities (3.16)-(3.18) into (5.20), together with $(3.23)_1$, yields

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathsf{L}_\mathbf{A}[\mathbf{H}] &= \mathbf{AHA} - I_\mathbf{A}(\mathbf{AH} + \mathbf{HA}) + I_\mathbf{A}^2\mathbf{H} \\ &\quad + \beta^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A}^2 + \beta^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \beta^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I} \\ &= \tilde{\mathbf{A}}\mathbf{H}\tilde{\mathbf{A}} + \beta^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A}^2 + \beta^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \beta^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} 2\beta^{(2)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_\mathbf{A}}\left(I_\mathbf{A} I_{\mathbf{A}^2\mathbf{H}} - I_\mathbf{A}^2 I_{\mathbf{AH}} + III_{\tilde{\mathbf{A}}} I_\mathbf{H}\right) = I_\mathbf{A} I_{\mathbf{A}^{-1}\mathbf{H}} - I_\mathbf{H} \\ &= \frac{1}{III_\mathbf{A}}(-I_\mathbf{A} I_{\tilde{\mathbf{A}}\mathbf{AH}} + III_{\tilde{\mathbf{A}}} I_\mathbf{H}) = I_{\tilde{\mathbf{A}}\mathbf{A}^{-1}\mathbf{H}}, \end{aligned} \tag{5.23}$$

[16] Müller and Jameson obtained direct solutions of the tensor equation $\mathbf{BX}+\mathbf{XA} = \mathbf{H}$ for arbitrary dimensions. Müller's formula is quite complicated. For the case $\mathbf{B} = \mathbf{A}^\mathrm{T}$, both authors simplified their general results, and Jameson listed the corresponding formulas in two and three dimensions. For $\mathbf{B} = \mathbf{A}^\mathrm{T}$ and $\dim \mathcal{V} = 3$, the formulas of Müller and Jameson are equivalent to those obtained from $(5.18)_2$ and (5.19), respectively, via Proposition 2.4; they reduce to $(5.18)_2$ and (5.19) when $\mathbf{A}$ is symmetric. Their general formulas (for $\mathbf{A}$ and $\mathbf{P}$ unrelated) also reduce to $(5.18)_2$ and (5.19) when $\mathbf{B} = \mathbf{A}$.

[17] Hoger and Carlson showed that (1.17) is a solution of $\mathbf{AX}+\mathbf{XA} = \mathbf{H}$ for any tensor $\mathbf{H}$ and any tensor $\mathbf{A}$ such that $III_\mathbf{A}(I_\mathbf{A} II_\mathbf{A} - III_\mathbf{A}) \neq 0$. They established uniqueness of this solution when $\mathbf{A}$ is also symmetric.

[18] For symmetric $\mathbf{H}$, Smith obtained a direct solution of the tensor equation $\mathbf{A}^\mathrm{T}\mathbf{X} + \mathbf{XA} = \mathbf{H}$ for arbitrary $N = \dim \mathcal{V}$. This solution has the form $(\det A_\mathcal{H})\mathbf{X} = \sum_{m,n=1}^{N} c_{m,n}(\mathbf{A}^\mathrm{T})^{m-1}\mathbf{HA}^{n-1}$, where $A_\mathcal{H}$ is the Hurwitz matrix associated with $\mathbf{A}$ (cf. Section 4), and the coefficients $c_{m,n}$ are complicated polynomials in the principal invariants of $\mathbf{A}$ involving cofactors of $A_\mathcal{H}$. Smith gave explicit expressions for these coefficients when $\dim \mathcal{V} = 2, 3, 4$. For $\dim \mathcal{V} = 3$, his formula is equivalent to the one obtained from (1.17) via Proposition 2.4; this formula reduces to (1.17) when $\mathbf{A}$ is symmetric.

$$
\begin{aligned}
2\beta^{(1)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left[-I_{\mathbf{A}}^2 I_{\mathbf{A}^2\mathbf{H}} + \left(I_{\mathbf{A}}^3 - III_{\mathbf{A}}\right) I_{\mathbf{A}\mathbf{H}} + \hat{\beta}_{\mathbf{A}} I_{\mathbf{H}}\right] \\
&= -I_{\mathbf{A}\mathbf{H}} + 2I_{\mathbf{A}} I_{\mathbf{H}} - I_{\mathbf{A}}^2 I_{\mathbf{A}^{-1}\mathbf{H}} = I_{\hat{\mathbf{A}}\mathbf{H}} - I_{\mathbf{A}} I_{\hat{\mathbf{A}}\mathbf{A}^{-1}\mathbf{H}}\,, \qquad (5.24) \\
2\beta^{(0)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left(III_{\hat{\mathbf{A}}} I_{\mathbf{A}^2\mathbf{H}} + \hat{\beta}_{\mathbf{A}} I_{\mathbf{A}\mathbf{H}} + \beta_{\mathbf{A}} I_{\mathbf{H}}\right) \\
&= -I_{\mathbf{A}} I_{\hat{\mathbf{A}}\mathbf{H}} + III_{\hat{\mathbf{A}}} I_{\mathbf{A}^{-1}\mathbf{H}} = III_{\mathbf{A}}(I_{\mathbf{A}} I_{\mathbf{A}^{-2}\mathbf{H}} - I_{\mathbf{A}^{-1}\mathbf{H}})\,, \qquad (5.25)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\beta}_{\mathbf{A}} &= I_{\mathbf{A}}(III_{\mathbf{A}} - III_{\hat{\mathbf{A}}}) = I_{\mathbf{A}}(2III_{\mathbf{A}} - I_{\mathbf{A}} II_{\mathbf{A}})\,, \qquad (5.26) \\
\beta_{\mathbf{A}} &= II_{\mathbf{A}} III_{\hat{\mathbf{A}}} - I_{\mathbf{A}}^2 III_{\mathbf{A}} = I_{\mathbf{A}} II_{\mathbf{A}}^2 - III_{\mathbf{A}} II_{\hat{\mathbf{A}}} \\
&= I_{\mathbf{A}} II_{\mathbf{A}}^2 - I_{\mathbf{A}}^2 III_{\mathbf{A}} - II_{\mathbf{A}} III_{\mathbf{A}}\,. \qquad (5.27)
\end{aligned}
$$

From (5.22) and $(3.23)_2$, we also have

$$
\begin{aligned}
III_{\hat{\mathbf{A}}} \mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= -(\mathbf{A}^2\mathbf{H} + \mathbf{H}\mathbf{A}^2) + \Delta_{\mathbf{A}}\mathbf{H} \\
&\quad + \gamma^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{A}^2 + \gamma^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \gamma^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\,, \qquad (5.28)
\end{aligned}
$$

where

$$
\begin{aligned}
2\gamma^{(2)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left(I_{\mathbf{A}} I_{\mathbf{A}^2\mathbf{H}} - I_{\mathbf{A}}^2 I_{\mathbf{A}\mathbf{H}} + \Gamma_{\mathbf{A}} I_{\mathbf{H}}\right) \\
&= \frac{1}{III_{\mathbf{A}}}(-I_{\mathbf{A}} I_{\hat{\mathbf{A}}\mathbf{A}\mathbf{H}} + \Gamma_{\mathbf{A}} I_{\mathbf{H}}) = I_{\mathbf{H}} + I_{\mathbf{A}} I_{\mathbf{A}^{-1}\mathbf{H}}\,, \qquad (5.29) \\
2\gamma^{(1)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left[-I_{\mathbf{A}}^2 I_{\mathbf{A}^2\mathbf{H}} + \left(I_{\mathbf{A}}^3 + III_{\mathbf{A}}\right) I_{\mathbf{A}\mathbf{H}} - I_{\mathbf{A}}^2 II_{\mathbf{A}} I_{\mathbf{H}}\right] \\
&= I_{\mathbf{A}\mathbf{H}} - I_{\mathbf{A}^2} I_{\mathbf{A}^{-1}\mathbf{H}} = -(I_{\hat{\mathbf{A}}\mathbf{H}} + I_{\mathbf{A}} I_{\hat{\mathbf{A}}\mathbf{A}^{-1}\mathbf{H}})\,, \qquad (5.30) \\
2\gamma^{(0)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left(\Gamma_{\mathbf{A}} I_{\mathbf{A}^2\mathbf{H}} - I_{\mathbf{A}}^2 II_{\mathbf{A}} I_{\mathbf{A}\mathbf{H}} + \gamma_{\mathbf{A}} I_{\mathbf{H}}\right) \\
&= -I_{\mathbf{A}} I_{\hat{\mathbf{A}}\mathbf{H}} + \Gamma_{\mathbf{A}} I_{\mathbf{A}^{-1}\mathbf{H}} = III_{\mathbf{A}}(I_{\mathbf{A}^{-1}\mathbf{H}} + I_{\mathbf{A}} I_{\mathbf{A}^{-2}\mathbf{H}})\,, \qquad (5.31)
\end{aligned}
$$

$\Gamma_{\mathbf{A}}$ is given by (5.10), and

$$
\begin{aligned}
\gamma_{\mathbf{A}} &= II_{\mathbf{A}}\Gamma_{\mathbf{A}} - I_{\mathbf{A}}^2 III_{\mathbf{A}} = I_{\mathbf{A}} II_{\mathbf{A}}^2 - III_{\mathbf{A}}\Delta_{\mathbf{A}} \\
&= I_{\mathbf{A}} II_{\mathbf{A}}^2 - I_{\mathbf{A}}^2 III_{\mathbf{A}} + II_{\mathbf{A}} III_{\mathbf{A}}\,. \qquad (5.32)
\end{aligned}
$$

The direct formula (5.28) with the expressions $(5.29)_3$, $(5.30)_2$, and $(5.31)_3$ for $\gamma^{(i)}_{\mathbf{A},\mathbf{H}}$ is due to Sidoroff [6].[19] The direct formula (5.28) with the expressions $(5.29)_1$, $(5.30)_1$, and $(5.31)_1$ for $\gamma^{(i)}_{\mathbf{A},\mathbf{H}}$ was obtained by Hoger and Carlson [9] by essentially the same method as used here, i.e., from the direct formula (5.20) and Rivlin's identities. Hoger

[19] For $\mathbf{A} \in$ Psym, Sidoroff derived $(5.1)_3$ for skew $\mathbf{H}$ and (5.28) for symmetric $\mathbf{H}$ and then combined these to obtain (5.28) for arbitrary $\mathbf{H}$. He noted that (5.28) yields a solution of $\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{H}$ for any tensor $\mathbf{A}$ such that $III_{\mathbf{A}}(I_{\mathbf{A}} II_{\mathbf{A}} - III_{\mathbf{A}}) \neq 0$, but did not establish uniqueness for this case.

and Carlson observed that (5.28) collapses to the formula $(5.1)_3$ of Sidoroff and Guo when $\mathbf{A} \in$ Psym and $\mathbf{H}$ is skew. Indeed, for any $\mathbf{A} \in \mathrm{Lin}^*$ and any $\mathbf{H} \in \mathcal{T}(\mathbf{A})$, we see that (5.22) and (5.28) collapse to $(5.1)_{1,2}$ and $(5.1)_3$, respectively. The direct formulas (5.6), (5.7), and (5.17)–(5.20) do not have this property.

From $(5.22)_2$ and the identity $\mathbf{A}^2 = \tilde{\mathbf{A}}^2 + 2I_{\mathbf{A}}\mathbf{A} - I_{\mathbf{A}}^2\mathbf{I}$, we obtain

$$\begin{aligned} III_{\tilde{\mathbf{A}}}\mathsf{L}_{\mathbf{A}}[\mathbf{H}] &= \tilde{\mathbf{A}}\left(\mathbf{H} + \beta^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\right)\tilde{\mathbf{A}} - \gamma^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{A} + \delta^{(1)}_{\mathbf{A},\mathbf{H}}\mathbf{I} \\ &= \tilde{\mathbf{A}}\left(\mathbf{H} + \beta^{(2)}_{\mathbf{A},\mathbf{H}}\mathbf{I}\right)\tilde{\mathbf{A}} + \gamma^{(1)}_{\mathbf{A},\mathbf{H}}\tilde{\mathbf{A}} + \delta^{(0)}_{\mathbf{A},\mathbf{H}}\mathbf{I}, \end{aligned} \tag{5.33}$$

where

$$2\delta^{(0)}_{\mathbf{A},\mathbf{H}} = \frac{III_{\tilde{\mathbf{A}}}}{III_{\mathbf{A}}}\alpha^{(0)}_{\mathbf{A},\mathbf{H}} = III_{\tilde{\mathbf{A}}}I_{\mathbf{A}^{-1}\mathbf{H}}, \tag{5.34}$$

$$\begin{aligned} 2\delta^{(1)}_{\mathbf{A},\mathbf{H}} &= \frac{1}{III_{\mathbf{A}}}\left(\delta_{\mathbf{A}}I_{\mathbf{A}^2\mathbf{H}} + \hat{\delta}_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}} + II_{\mathbf{A}}\delta_{\mathbf{A}}I_{\mathbf{H}}\right) \\ &= I_{\mathbf{A}}I_{\mathbf{A}\mathbf{H}} + \delta_{\mathbf{A}}I_{\mathbf{A}^{-1}\mathbf{H}}, \end{aligned} \tag{5.35}$$

and

$$\delta_{\mathbf{A}} = III_{\tilde{\mathbf{A}}} - I_{\mathbf{A}}^3 = -(I_{\mathbf{A}}\Delta_{\mathbf{A}} + III_{\mathbf{A}}) = -\tfrac{1}{3}\left(2I_{\mathbf{A}}^3 + I_{\mathbf{A}^3}\right), \tag{5.36}$$

$$\begin{aligned} \hat{\delta}_{\mathbf{A}} &= I_{\mathbf{A}}^4 + \hat{\beta}_{\mathbf{A}} = I_{\mathbf{A}}(III_{\mathbf{A}} - \delta_{\mathbf{A}}) = I_{\mathbf{A}}(I_{\mathbf{A}}\Delta_{\mathbf{A}} + 2III_{\mathbf{A}}) \\ &= I_{\mathbf{A}}(I_{\mathbf{A}}^3 - I_{\mathbf{A}}II_{\mathbf{A}} + 2III_{\mathbf{A}}). \end{aligned} \tag{5.37}$$

Also, by (5.23) and (5.30), we have

$$-\gamma^{(1)}_{\mathbf{A},\mathbf{H}} = I_{\mathbf{A}}\beta^{(2)}_{\mathbf{A},\mathbf{H}} + \tfrac{1}{2}I_{\tilde{\mathbf{A}}\mathbf{H}}. \tag{5.38}$$

We conclude this section by considering the special case where $\mathbf{A}$ and $\mathbf{H}$ are time-dependent tensors with $\mathbf{H}$ related to $\mathbf{A}$, its time rate of change $\dot{\mathbf{A}}$, and some time-dependent tensor $\mathbf{Z}$ as follows:

$$\mathbf{H} = \dot{\mathbf{A}} + \mathbf{A}\mathbf{Z} - \mathbf{Z}\mathbf{A}; \tag{5.39}$$

cf. (6.4) and (7.14). Then $\mathrm{tr}\,(\mathbf{p}(\mathbf{A})\mathbf{H}) = \mathrm{tr}\,(\mathbf{p}(\mathbf{A})\dot{\mathbf{A}})$, where $\mathbf{p}(\mathbf{A})$ is any polynomial in $\mathbf{A}$. In particular, we have

$$I_{\mathbf{H}} = I_{\dot{\mathbf{A}}} = \dot{I}_{\mathbf{A}}, \quad I_{\tilde{\mathbf{A}}\mathbf{H}} = \dot{II}_{\mathbf{A}}, \tag{5.40}$$

$$I_{\mathbf{A}\mathbf{H}} = I_{\mathbf{A}\dot{\mathbf{A}}} = \tfrac{1}{2}\dot{I}_{\mathbf{A}^2} = I_{\mathbf{A}}\dot{I}_{\mathbf{A}} - \dot{II}_{\mathbf{A}}, \tag{5.41}$$

$$I_{\mathbf{A}^2\mathbf{H}} = I_{\mathbf{A}^2\dot{\mathbf{A}}} = \tfrac{1}{3}\dot{I}_{\mathbf{A}^3} = I_{\mathbf{A}}^2\dot{I}_{\mathbf{A}} - \dot{III}_{\tilde{\mathbf{A}}}, \tag{5.42}$$

$$I_{(\mathrm{adj}\,\mathbf{A})\mathbf{H}} = I_{(\mathrm{adj}\,\mathbf{A})\dot{\mathbf{A}}} = \dot{III}_{\mathbf{A}}, \quad I_{\mathbf{A}^{-1}\mathbf{H}} = I_{\mathbf{A}^{-1}\dot{\mathbf{A}}} = \dot{III}_{\mathbf{A}}/III_{\mathbf{A}}. \tag{5.43}$$

These identities yield alternate expressions for the coefficients in the direct formulas (5.22), (5.28), and (5.33) for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ and for the coefficients in the identities (3.16)–(3.18) and $(3.23)_2$.

6 Applications to kinematics of continua

In this section we apply the preceding results to the derivation of some kinematic formulas for a material undergoing a smooth motion in three-dimensional space. By taking the material time derivative of the relations $\mathbf{U}^2 = \mathbf{C}$ and $\mathbf{V}^2 = \mathbf{B}$, we obtain the tensor equations (1.6) for $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$. Hence,

$$\dot{\mathbf{U}} = \mathsf{L}_{\mathbf{U}}[\dot{\mathbf{C}}] \quad \text{and} \quad \dot{\mathbf{V}} = \mathsf{L}_{\mathbf{V}}[\dot{\mathbf{B}}]. \tag{6.1}$$

A variety of direct formulas for $\dot{\mathbf{U}}$ in terms of $\dot{\mathbf{C}}$ and $\mathbf{U}$, and for $\dot{\mathbf{V}}$ in terms of $\dot{\mathbf{B}}$ and $\mathbf{V}$, follow from (6.1) and the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in Section 5. In particular, the formulas that follow from (5.20) and (5.28) are due to Hoger and Carlson [9].

By differentiating $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and using the relation between the velocity gradient and the material time derivative of the deformation gradient,

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \tag{6.2}$$

we obtain the following formulas for the material time derivatives of $\mathbf{B}$ and $\mathbf{B}^{-1}$:

$$\dot{\mathbf{B}} = \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^{\mathrm{T}} \quad \text{and} \quad -(\mathbf{B}^{-1})^{\cdot} = \mathbf{B}^{-1}\dot{\mathbf{B}}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{B}^{-1}. \tag{6.3}$$

Then (6.3), (1.4), and the definition (1.5) of the Jaumann rate yield the tensor equations (1.10) for $\mathbf{D}$. Hence,

$$\mathbf{D} = \mathsf{L}_{\mathbf{B}}[\mathring{\mathbf{B}}] = -\mathsf{L}_{\mathbf{B}^{-1}}[(\mathbf{B}^{-1})^{\circ}]. \tag{6.4}$$

A variety of direct formulas for $\mathbf{D}$ in terms of $\mathring{\mathbf{B}}$ and $\mathbf{B}$, or in terms of $(\mathbf{B}^{-1})^{\circ}$ and $\mathbf{B}^{-1}$, follow from (6.4) and the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in Section 5. In particular, the formulas which follow from (6.4) and $(5.17)_1$ are due to Leonov [5], and the formula which follows from $(6.4)_1$ and (5.28) is due to Sidoroff [6]. Also note that the identities (5.40)–(5.43) can be used in the expressions for the coefficients in the direct formulas for $\mathbf{D}$ which follow from (6.4), (5.22), (5.28), and (5.33).

Let $\mathbf{L}_{\mathbf{R}}$ denote the *rotated velocity gradient*:

$$\mathbf{L}_{\mathbf{R}} := \mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R} = \mathbf{D}_{\mathbf{R}} + \mathbf{W}_{\mathbf{R}}. \tag{6.5}$$

Here $\mathbf{D}_{\mathbf{R}}$ and $\mathbf{W}_{\mathbf{R}}$ denote the *rotated stretching* and *spin tensors*:

$$\mathbf{D}_{\mathbf{R}} := \mathbf{R}^{\mathrm{T}}\mathbf{D}\mathbf{R} = \operatorname{sym}\mathbf{L}_{\mathbf{R}} \quad \text{and} \quad \mathbf{W}_{\mathbf{R}} := \mathbf{R}^{\mathrm{T}}\mathbf{W}\mathbf{R} = \operatorname{skw}\mathbf{L}_{\mathbf{R}}. \tag{6.6}$$

By differentiating $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and using (6.2), $(1.4)_2$, $\mathbf{F} = \mathbf{R}\mathbf{U}$, and $(6.6)_1$, we obtain

$$\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F} = 2\mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U}. \tag{6.7}$$

This, together with $(1.6)_1$, yields the tensor equations (1.9) for $\dot{\mathbf{U}}$. Hence,

$$\dot{\mathbf{U}} = 2\,\mathsf{L}_{\mathbf{U}}[\mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U}] = 2\,\mathsf{L}_{\mathbf{U}^{-1}}[\mathbf{D}_{\mathbf{R}}]\,. \tag{6.8}$$

Although direct formulas for $\dot{\mathbf{U}}$ in terms of $\mathbf{D}_{\mathbf{R}}$ and $\mathbf{U}$ follow from (6.8) and the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in Section 5, we can obtain simpler formulas by using the identities (2.21). Indeed, from (6.8) and (2.21) we obtain the following relations for the material time derivative of $\mathbf{U}$:

$$\begin{aligned} \dot{\mathbf{U}} &= \mathbf{U}(\mathbf{D}_{\mathbf{R}} - \mathsf{M}_{\mathbf{U}}[\mathbf{D}_{\mathbf{R}}]) = (\mathbf{D}_{\mathbf{R}} + \mathsf{M}_{\mathbf{U}}[\mathbf{D}_{\mathbf{R}}])\mathbf{U} \\ &= \tfrac{1}{2}(\mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U} - \mathsf{N}_{\mathbf{U}}[\mathbf{D}_{\mathbf{R}}])\,. \end{aligned} \tag{6.9}$$

Then a variety of direct formulas for $\dot{\mathbf{U}}$ follow from $(6.9)_{1,2}$ and the direct formulas (5.2)–(5.5) for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$. In particular, we have

$$\begin{aligned} \dot{\mathbf{U}} &= \left[\mathbf{D}_{\mathbf{R}} + \frac{1}{III_{\tilde{\mathbf{U}}}}\tilde{\mathbf{U}}(\mathbf{U}\mathbf{D}_{\mathbf{R}} - \mathbf{D}_{\mathbf{R}}\mathbf{U})\tilde{\mathbf{U}}\right]\mathbf{U} \\ &= \left[\mathbf{D}_{\mathbf{R}} + \frac{1}{III_{\tilde{\mathbf{U}}}}(\tilde{\mathbf{U}}\mathbf{D}_{\mathbf{R}}\tilde{\mathbf{U}}^2 - \tilde{\mathbf{U}}^2\mathbf{D}_{\mathbf{R}}\tilde{\mathbf{U}})\right]\mathbf{U} \\ &= \left[\mathbf{D}_{\mathbf{R}} + \frac{2}{III_{\tilde{\mathbf{U}}}}\operatorname{skw}(\tilde{\mathbf{U}}\mathbf{D}_{\mathbf{R}}\tilde{\mathbf{U}}^2)\right]\mathbf{U}\,. \end{aligned} \tag{6.10}$$

Some of the formulas which follow from $(6.9)_3$ and the direct formulas for $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ in Section 5 are

$$2\,\dot{\mathbf{U}} = \mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U} + \frac{1}{III_{\tilde{\mathbf{U}}}}\tilde{\mathbf{U}}(2\mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U} - \mathbf{U}^2\mathbf{D}_{\mathbf{R}} - \mathbf{D}_{\mathbf{R}}\mathbf{U}^2)\tilde{\mathbf{U}}\,, \tag{6.11}$$

and

$$\begin{aligned} III_{\tilde{\mathbf{U}}}\dot{\mathbf{U}} &= \mathbf{U}^2\mathbf{D}_{\mathbf{R}}\mathbf{U}^2 - I_{\mathbf{U}}(\mathbf{U}^2\mathbf{D}_{\mathbf{R}}\mathbf{U} + \mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U}^2) + II_{\tilde{\mathbf{U}}}\mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U} \\ &\quad - III_{\mathbf{U}}(\mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U}) + I_{\mathbf{U}}III_{\mathbf{U}}\mathbf{D}_{\mathbf{R}} \\ &= 2II_{\mathbf{U}}\mathbf{U}\mathbf{D}_{\mathbf{R}}\mathbf{U} - 2III_{\mathbf{U}}(\mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U}) + 2I_{\mathbf{U}}III_{\mathbf{U}}\mathbf{D}_{\mathbf{R}} \\ &\quad + \alpha^{(2)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{U}^2 + \eta^{(3)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{U} + III_{\mathbf{U}}\,\alpha^{(1)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{I} \\ &= -2II_{\mathbf{U}}(\mathbf{U}^2\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U}^2) + 2III_{\tilde{\mathbf{U}}}(\mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{D}_{\mathbf{R}}\mathbf{U}) \\ &\quad + 2(I_{\mathbf{U}}III_{\mathbf{U}} - II_{\mathbf{U}}^2)\mathbf{D}_{\mathbf{R}} + \eta^{(2)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{U}^2 + \eta^{(1)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{U} + \eta^{(0)}_{\mathbf{U},\mathbf{D}_{\mathbf{R}}}\mathbf{I}\,, \end{aligned} \tag{6.12}$$

where the $\alpha^{(i)}$ are defined by (3.19)–(3.22) and the $\eta^{(i)}$ are defined by (5.12) and (5.14)–(5.16). The direct formulas $(6.12)_{1,3}$ were obtained by Mehrabadi and Nemat-Nasser [11].[20] The formula $(6.12)_3$ was also derived by Guo, Lehmann, and Liang [13]. Now for any integers m and n,

$$\mathbf{U}^m = \mathbf{R}^T\mathbf{V}^m\mathbf{R} \quad \text{and} \quad \mathbf{U}^m\mathbf{D}_{\mathbf{R}}\mathbf{U}^n = \mathbf{R}^T(\mathbf{V}^m\mathbf{D}\mathbf{V}^n)\mathbf{R}\,. \tag{6.13}$$

[20] Their formula corresponding to $(6.12)_1$ contains a misprint; $II_{\mathbf{U}}$ should be replaced with $III_{\mathbf{U}}$ in the last term of their formula.

By use of (6.13) we may convert (6.10)–(6.12) to formulas for $\dot{\mathbf{U}}$ in terms of $\mathbf{D}$, $\mathbf{V}$, and $\mathbf{R}$. When this conversion is applied to $(6.12)_1$, we recover a formula obtained by Hoger [10].

From $(1.6)_2$, $(6.3)_1$, and $\mathbf{B} = \mathbf{V}^2$, we obtain the tensor equation (1.7) for $\dot{\mathbf{V}}$. Hence,

$$\dot{\mathbf{V}} = \mathsf{L}_{\mathbf{V}}[\mathbf{V}^2\mathbf{L}^T + \mathbf{L}\mathbf{V}^2]. \tag{6.14}$$

Direct formulas for $\dot{\mathbf{V}}$ in terms of $\mathbf{L}$ and $\mathbf{V}$ follow from (6.14) and the direct formulas for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$ in Section 5; in particular, the formula for $\dot{\mathbf{V}}$ obtained from $(5.17)_1$ is due to Stickforth and Wegener [12]. However, simpler formulas can be obtained by using the identities (2.26). Indeed, from (6.14), (2.26), (1.4), and (1.5), we obtain the following relations for Jaumann rate of $\mathbf{V}$:

$$\begin{aligned} \overset{\circ}{\mathbf{V}} &= \mathbf{V}\mathbf{D} - \mathsf{M}_{\mathbf{V}}[\mathbf{D}]\mathbf{V} = \mathbf{D}\mathbf{V} + \mathbf{V}\mathsf{M}_{\mathbf{V}}[\mathbf{D}] \\ &= \tfrac{1}{2}(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V} + \mathsf{N}_{\mathbf{V}}[\mathbf{D}]). \end{aligned} \tag{6.15}$$

These relations can also be derived as follows. From (1.7) and $(1.4)_1$ we have

$$\mathbf{V}\dot{\mathbf{V}} + \dot{\mathbf{V}}\mathbf{V} = \mathbf{V}^2\mathbf{D} + \mathbf{D}\mathbf{V}^2 - \mathbf{V}^2\mathbf{W} + \mathbf{W}\mathbf{V}^2. \tag{6.16}$$

But this is easily seen to be equivalent to the tensor equation (1.8) for $\overset{\circ}{\mathbf{V}}$. Hence, $\overset{\circ}{\mathbf{V}} = \mathsf{L}_{\mathbf{V}}[\mathbf{V}^2\mathbf{D} + \mathbf{D}\mathbf{V}^2]$, and the identities (2.24) yield (6.15). A variety of direct formulas for the Jaumann rate of $\mathbf{V}$ follow from $(6.15)_{1,2}$ and the direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ in Section 5. In particular, we have

$$\begin{aligned} III_{\tilde{\mathbf{V}}}(\overset{\circ}{\mathbf{V}} - \mathbf{D}\mathbf{V}) &= \mathbf{V}\tilde{\mathbf{V}}(\mathbf{V}\mathbf{D} - \mathbf{D}\mathbf{V})\tilde{\mathbf{V}} \\ &= \mathbf{V}(\tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}}^2 - \tilde{\mathbf{V}}^2\mathbf{D}\tilde{\mathbf{V}}) \\ &= \mathbf{V}\,\mathrm{skw}\,(\tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}}^2). \end{aligned} \tag{6.17}$$

Some of the formulas which follow from $(6.15)_3$ and the direct formulas for $\mathsf{N}_{\mathbf{A}}[\mathbf{H}]$ in Section 5 are

$$2\overset{\circ}{\mathbf{V}} = \mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V} + \frac{1}{III_{\tilde{\mathbf{V}}}}\tilde{\mathbf{V}}(\mathbf{V}^2\mathbf{D} + \mathbf{D}\mathbf{V}^2 - 2\mathbf{V}\mathbf{D}\mathbf{V})\tilde{\mathbf{V}}, \tag{6.18}$$

and

$$\begin{aligned} III_{\tilde{\mathbf{V}}}\overset{\circ}{\mathbf{V}} &= -\mathbf{V}^2\mathbf{D}\mathbf{V}^2 + I_{\mathbf{V}}(\mathbf{V}^2\mathbf{D}\mathbf{V} + \mathbf{V}\mathbf{D}\mathbf{V}^2) - II_{\tilde{\mathbf{V}}}\mathbf{V}\mathbf{D}\mathbf{V} \\ &\quad + I_{\mathbf{V}}II_{\mathbf{V}}(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V}) - I_{\mathbf{V}}III_{\mathbf{V}}\mathbf{D} \\ &= -\tilde{\mathbf{V}}^2\mathbf{D}\tilde{\mathbf{V}}^2 + I_{\mathbf{V}}(\tilde{\mathbf{V}}^2\mathbf{D}\tilde{\mathbf{V}} + \tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}}^2) - II_{\tilde{\mathbf{V}}}\tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}} + I_{\mathbf{V}}III_{\tilde{\mathbf{V}}}\mathbf{D} \\ &= -2II_{\mathbf{V}}\mathbf{V}\mathbf{D}\mathbf{V} + (I_{\mathbf{V}}II_{\mathbf{V}} + III_{\mathbf{V}})(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V}) - 2I_{\mathbf{V}}III_{\mathbf{V}}\mathbf{D} \\ &\quad - \alpha^{(2)}_{\mathbf{V},\mathbf{D}}\mathbf{V}^2 - \eta^{(3)}_{\mathbf{V},\mathbf{D}}\mathbf{V} - III_{\mathbf{V}}\,\alpha^{(1)}_{\mathbf{V},\mathbf{D}}\mathbf{I} \\ &= 2II_{\mathbf{V}}(\mathbf{V}^2\mathbf{D} + \mathbf{D}\mathbf{V}^2) - III_{\tilde{\mathbf{V}}}(\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V}) + 2(II_{\mathbf{V}}^2 - I_{\mathbf{V}}III_{\mathbf{V}})\mathbf{D} \\ &\quad - \eta^{(2)}_{\mathbf{V},\mathbf{D}}\mathbf{V}^2 - \eta^{(1)}_{\mathbf{V},\mathbf{D}}\mathbf{V} - \eta^{(0)}_{\mathbf{V},\mathbf{D}}\mathbf{I}. \end{aligned} \tag{6.19}$$

The formula $(6.19)_4$ was obtained by Guo, Lehmann, and Liang [13].

Next, we derive direct formulas for the skew tensors

$$\boldsymbol{\Omega} := \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = \mathbf{R}\boldsymbol{\Omega}_{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{6.20}$$

and

$$\boldsymbol{\Omega}_{\mathbf{R}} := \mathbf{R}^{\mathrm{T}}\dot{\mathbf{R}} = -\dot{\mathbf{R}}^{\mathrm{T}}\mathbf{R} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{R}\,. \tag{6.21}$$

Then direct formulas for $\dot{\mathbf{R}}$ follow from the relations

$$\dot{\mathbf{R}} = \boldsymbol{\Omega}\mathbf{R} = \mathbf{R}\boldsymbol{\Omega}_{\mathbf{R}}\,. \tag{6.22}$$

By differentiating the polar decompositions (1.2) and using (6.2), $(6.20)_1$, $(6.21)_1$, and the fact that $\dot{\mathbf{V}} = \dot{\mathbf{V}}^{\mathrm{T}}$ and $\dot{\mathbf{U}} = \dot{\mathbf{U}}^{\mathrm{T}}$, we obtain the formulas

$$\dot{\mathbf{V}} = \mathbf{L}\mathbf{V} - \mathbf{V}\boldsymbol{\Omega} = \mathbf{V}\mathbf{L}^{\mathrm{T}} + \boldsymbol{\Omega}\mathbf{V} \tag{6.23}$$

and

$$\dot{\mathbf{U}} = (\mathbf{L}_{\mathbf{R}} - \boldsymbol{\Omega}_{\mathbf{R}})\mathbf{U} = \mathbf{U}(\mathbf{L}_{\mathbf{R}}^{\mathrm{T}} + \boldsymbol{\Omega}_{\mathbf{R}})\,, \tag{6.24}$$

where

$$\mathbf{L}_{\mathbf{R}}^{\mathrm{T}} := (\mathbf{L}_{\mathbf{R}})^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{R} = \mathbf{D}_{\mathbf{R}} - \mathbf{W}_{\mathbf{R}}\,. \tag{6.25}$$

Observe that $(6.23)_2$ and $(6.24)_2$ are equivalent to the following equations for the skew tensors $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_{\mathbf{R}}$:

$$\mathbf{V}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{V} = \hat{\mathbf{L}} \quad \text{and} \quad \mathbf{U}\boldsymbol{\Omega}_{\mathbf{R}} + \boldsymbol{\Omega}_{\mathbf{R}}\mathbf{U} = \hat{\mathbf{L}}_{\mathbf{R}}\,, \tag{6.26}$$

where

$$\begin{aligned} \hat{\mathbf{L}} &:= \mathbf{L}\mathbf{V} - \mathbf{V}\mathbf{L}^{\mathrm{T}} = 2\,\mathrm{skw}\,(\mathbf{L}\mathbf{V}) \\ &= \mathbf{D}\mathbf{V} - \mathbf{V}\mathbf{D} + \mathbf{W}\mathbf{V} + \mathbf{V}\mathbf{W}\,, \end{aligned} \tag{6.27}$$

and

$$\begin{aligned} \hat{\mathbf{L}}_{\mathbf{R}} &:= \mathbf{L}_{\mathbf{R}}\mathbf{U} - \mathbf{U}\mathbf{L}_{\mathbf{R}}^{\mathrm{T}} = 2\,\mathrm{skw}\,(\mathbf{L}_{\mathbf{R}}\mathbf{U}) = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{L}}\mathbf{R} \\ &= \mathbf{D}_{\mathbf{R}}\mathbf{U} - \mathbf{U}\mathbf{D}_{\mathbf{R}} + \mathbf{W}_{\mathbf{R}}\mathbf{U} + \mathbf{U}\mathbf{W}_{\mathbf{R}}\,. \end{aligned} \tag{6.28}$$

In particular, $(6.26)_1$ with $\hat{\mathbf{L}}$ given by $(6.27)_1$ is the tensor equation (1.11) for $\boldsymbol{\Omega}$. From (6.26) it follows that

$$\boldsymbol{\Omega} = \mathsf{L}_{\mathbf{V}}[\hat{\mathbf{L}}] \quad \text{and} \quad \boldsymbol{\Omega}_{\mathbf{R}} = \mathsf{L}_{\mathbf{U}}[\hat{\mathbf{L}}_{\mathbf{R}}]\,. \tag{6.29}$$

Note that the tensor equation $(6.26)_2$ can also be obtained by multiplying the terms in $(6.26)_1$ on the left by $\mathbf{R}^{\mathrm{T}}$ and on the right by $\mathbf{R}$, and then using $\mathbf{U} = \mathbf{R}^{\mathrm{T}}\mathbf{V}\mathbf{R}$, (6.21), and (6.28). Similarly, $(6.29)_2$ also follows from $(6.29)_1$ by using (2.15) with $\mathsf{K} = \mathsf{L}$, $\mathbf{A} = \mathbf{V}$, and $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$. Since $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}_{\mathbf{R}}$ are skew, direct formulas for $\boldsymbol{\Omega}$ and

$\boldsymbol{\Omega}_{\mathbf{R}}$ follow from (6.29) and the direct formulas (5.1) for $\mathsf{L}_{\mathbf{A}}[\mathbf{H}]$. In particular, for $\boldsymbol{\Omega}$ we obtain

$$\begin{aligned} III_{\mathbf{V}}\boldsymbol{\Omega} &= \tilde{\mathbf{V}}\hat{\mathbf{L}}\tilde{\mathbf{V}} = \mathbf{V}\hat{\mathbf{L}}\mathbf{V} - I_{\mathbf{V}}(\mathbf{V}\hat{\mathbf{L}} + \hat{\mathbf{L}}\mathbf{V}) + I_{\mathbf{V}}^2\hat{\mathbf{L}} \\ &= \Delta_{\mathbf{V}}\hat{\mathbf{L}} - (\mathbf{V}^2\hat{\mathbf{L}} + \hat{\mathbf{L}}\mathbf{V}^2) = \Delta_{\mathbf{V}}\hat{\mathbf{L}} - (\mathbf{B}\hat{\mathbf{L}} + \hat{\mathbf{L}}\mathbf{B}). \end{aligned} \tag{6.30}$$

Various expressions for the invariant $\Delta_{\mathbf{V}}$ follow from (3.15); in particular, we have

$$\Delta_{\mathbf{V}} := I_{\mathbf{V}}^2 - II_{\mathbf{V}} = I_{\mathbf{B}} + II_{\mathbf{V}} = \tfrac{1}{2}(I_{\mathbf{V}}^2 + I_{\mathbf{B}}). \tag{6.31}$$

The direct formula $(6.30)_3$ for $\boldsymbol{\Omega}$ is due to Guo [8]. The formula $(6.30)_1$ was obtained recently by Chen and Wheeler [20].[21]

From (6.30) and (6.27) we see that $\boldsymbol{\Omega}$ depends on $\mathbf{L}$ and $\mathbf{V}$, or, equivalently, on $\mathbf{D}$, $\mathbf{W}$, and $\mathbf{V}$. Since $\mathbf{W}$ and $\boldsymbol{\Omega}$ are both tensor measures of material spin, one might expect the dependence of $\boldsymbol{\Omega}$ on $\mathbf{W}$ to be of a simpler form than its dependence on $\mathbf{D}$ and $\mathbf{V}$. This turns out to be the case, although it is not obvious from the formulas (6.30). Instead, we use the identity (2.27), together with $(6.29)_1$, $(6.27)_1$, (1.4), and (2.23), to obtain the relations

$$\boldsymbol{\Omega} = \mathbf{W} - \mathsf{M}_{\mathbf{V}}[\mathbf{D}] = \mathbf{W} + \mathsf{M}_{\mathbf{V}^{-1}}[\mathbf{D}]. \tag{6.32}$$

$(6.32)_1$ may also be obtained as follows. From the expression $(6.27)_3$ for $\hat{\mathbf{L}}$, we see that $(6.26)_1$ is equivalent to the tensor equation (1.12) for $\mathbf{W} - \boldsymbol{\Omega}$;[22] then by (1.19) we have $\mathbf{W} - \boldsymbol{\Omega} = \mathsf{M}_{\mathbf{V}}[\mathbf{D}]$. From $(6.32)_1$ and the direct formulas for $\mathsf{M}_{\mathbf{A}}[\mathbf{H}]$ in Section 5, it follows that

$$\begin{aligned} III_{\mathbf{V}}(\boldsymbol{\Omega} - \mathbf{W}) &= \tilde{\mathbf{V}}(\mathbf{DV} - \mathbf{VD})\tilde{\mathbf{V}} = 2\,\mathrm{skw}\,(\tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}}\mathbf{V}) \\ &= \tilde{\mathbf{V}}(\tilde{\mathbf{V}}\mathbf{D} - \mathbf{D}\tilde{\mathbf{V}})\tilde{\mathbf{V}} = \tilde{\mathbf{V}}^2\mathbf{D}\tilde{\mathbf{V}} - \tilde{\mathbf{V}}\mathbf{D}\tilde{\mathbf{V}}^2 = 2\,\mathrm{skw}\,(\tilde{\mathbf{V}}^2\mathbf{D}\tilde{\mathbf{V}}) \\ &= \mathbf{VDV}^2 - \mathbf{V}^2\mathbf{DV} + I_{\mathbf{V}}(\mathbf{V}^2\mathbf{D} - \mathbf{DV}^2) + I_{\mathbf{V}}^2(\mathbf{DV} - \mathbf{VD}) \\ &= 2\,\mathrm{skw}\,(\mathbf{VDV}^2 - I_{\mathbf{V}}\mathbf{DV}^2 + I_{\mathbf{V}}^2\mathbf{DV}). \end{aligned} \tag{6.33}$$

Since $\mathbf{W}$ is skew, we may also obtain $(6.33)_1$ from $(6.30)_1$, the expresssion $(6.27)_3$ for $\hat{\mathbf{L}}$, and the identity (3.31). From (6.27)–(6.29) we see that results analogous to

[21] I had also derived $(6.30)_1$ prior to seeing their paper. Chen and Wheeler [20] first derived a direct formula for $D_{\mathbf{F}}\mathbf{R}$, the derivative of the rotation tensor with respect to the deformation gradient, and then obtained $(6.30)_1$ from the relations $\dot{\mathbf{R}} = D_{\mathbf{F}}\mathbf{R}[\dot{\mathbf{F}}]$ and $\dot{\mathbf{F}} = \mathbf{LF}$. Although they did not solve any tensor equations in deriving their formula for $D_{\mathbf{F}}\mathbf{R}$, they did utilize a special case of the identity (3.31); cf. the footnote in Section 3.

[22] The tensor equation (1.12) was stated (without proof) by Green [34] and derived by Green and McInnis [35]. In both papers it was noted that $\det\tilde{\mathbf{V}} \neq 0$ and stated (without proof) that this condition implies that (1.12) may be solved for $\boldsymbol{\Omega} - \mathbf{W}$ in terms of $\mathbf{V}$ and $\mathbf{D}$, but no solutions were given. The tensor equations (1.7), (1.11), and (1.12) were derived by Dienes [7]. He obtained the solution $\boldsymbol{\omega} = \mathbf{w} + \tilde{\mathbf{V}}^{-1}\mathbf{z}$ of (1.12), where $\boldsymbol{\omega}$, $\mathbf{w}$ and $\mathbf{z}$ are the axial vectors of $\boldsymbol{\Omega}$, $\mathbf{W}$, and $\mathbf{DV} - \mathbf{VD}$, respectively, but did not obtain an explicit formula (cf. (3.14)) for $\tilde{\mathbf{V}}^{-1}$.

(6.30)–(6.33) and (1.12) also hold for $\mathbf{\Omega_R}$. They may be obtained from (6.30)–(6.33) and (1.12) by the replacements

$$\mathbf{V} \to \mathbf{U}, \quad \mathbf{B} \to \mathbf{C}, \quad \mathbf{\Omega} \to \mathbf{\Omega_R}, \quad \hat{\mathbf{L}} \to \hat{\mathbf{L}}_\mathbf{R}, \quad \mathbf{W} \to \mathbf{W_R}, \quad \mathbf{D} \to \mathbf{D_R}. \tag{6.34}$$

Hoger [10] derived $(6.33)_6$ by substituting her formula for $\dot{\mathbf{U}}$ into the identity

$$\mathbf{\Omega} - \mathbf{W} = \mathbf{R}\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{R}^T - \mathbf{D}, \tag{6.35}$$

which follows from $(6.24)_2$. Stickforth and Wegener [12] derived a formula equivalent to $(6.33)_7$ by substituting their formula for $\dot{\mathbf{V}}$ into the identity

$$\mathbf{\Omega} = \mathbf{V}^{-1}\mathbf{L}\mathbf{V} - \mathbf{V}^{-1}\dot{\mathbf{V}}, \tag{6.36}$$

which follows from $(6.23)_1$. Our derivation of $(6.33)_6$ is closer in spirit, though different in detail, to that of Mehrabadi and Nemat-Nasser [11]. They derived the analog of the tensor equation (1.12) for $\mathbf{\Omega_R}$ and solved this equation to obtain the analog of $(6.33)_6$ for $\mathbf{\Omega_R}$; they also noted the corresponding result for $\mathbf{\Omega}$.[23]

A variety of additional formulas for $\dot{\mathbf{V}}$ and $\dot{\mathbf{U}}$ can be obtained by substituting the formulas (6.30) for $\mathbf{\Omega}$ into (6.23), and by substituting the analog of (6.30) for $\mathbf{\Omega_R}$ into (6.24). In particular, the formulas for $\dot{\mathbf{V}}$ and $\dot{\mathbf{U}}$ which follow from $(6.30)_3$, $(6.23)_1$, and $(6.24)_1$ are due to Guo [8], and the formula for $\dot{\mathbf{V}}$ which follows from $(6.30)_1$ and $(6.23)_1$ was obtained by Chen and Wheeler [20]. From $(6.15)_{1,2}$ and $(6.32)_1$, or from (6.23), (1.4), and (1.5), we obtain the following simple expressions for the Jaumann rate of $\mathbf{V}$ in terms of $\mathbf{D}$, $\mathbf{\Omega} - \mathbf{W}$, and $\mathbf{V}$:

$$\begin{aligned} \overset{\circ}{\mathbf{V}} &= \mathbf{V}\mathbf{D} + (\mathbf{\Omega} - \mathbf{W})\mathbf{V} = \mathbf{D}\mathbf{V} - \mathbf{V}(\mathbf{\Omega} - \mathbf{W}) \\ &= \tfrac{1}{2}[\mathbf{V}\mathbf{D} + \mathbf{D}\mathbf{V} + (\mathbf{\Omega} - \mathbf{W})\mathbf{V} - \mathbf{V}(\mathbf{\Omega} - \mathbf{W})] \\ &= \operatorname{sym}[(\mathbf{D} + \mathbf{\Omega} - \mathbf{W})\mathbf{V}]. \end{aligned} \tag{6.37}$$

By substituting the formulas (6.33) into (6.37), we recover the formulas (6.17) and (6.18). Similarly, from $(6.9)_{1,2}$ and the analog of $(6.32)_1$ for $\mathbf{\Omega_R}$, or directly from (6.24) and (6.5), we see that

$$\begin{aligned} \dot{\mathbf{U}} &= [\mathbf{D_R} - (\mathbf{\Omega_R} - \mathbf{W_R})]\mathbf{U} = \mathbf{U}(\mathbf{D_R} + \mathbf{\Omega_R} - \mathbf{W_R}) \\ &= \tfrac{1}{2}[\mathbf{U}\mathbf{D_R} + \mathbf{D_R}\mathbf{U} + \mathbf{U}(\mathbf{\Omega_R} - \mathbf{W_R}) - (\mathbf{\Omega_R} - \mathbf{W_R})\mathbf{U}] \\ &= \operatorname{sym}[\mathbf{U}(\mathbf{D_R} + \mathbf{\Omega_R} - \mathbf{W_R})]. \end{aligned} \tag{6.38}$$

The formulas $(6.38)_{1,3}$ were obtained by Mehrabadi and Nemat-Nasser [11]. By substituting the analog of (6.33) for $\mathbf{\Omega_R} - \mathbf{W_R}$ into (6.38), we recover the formulas (6.10) and (6.11).

[23] Equations similar to (1.12) arise in the kinematics of elastic-plastic deformations based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts; cf. Nemat-Nasser [36, 37] and Obata *et al.* [38].

7 Additional kinematic formulas

If the deformation gradient $\mathbf{F}$ is known then the Cauchy-Green tensors $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are easy to calculate, whereas the stretch tensors $\mathbf{U} = \sqrt{\mathbf{C}}$ and $\mathbf{V} = \sqrt{\mathbf{B}}$ are generally more difficult to compute since they involve a tensor square root. Hence, for some applications it might be useful to have direct formulas similar to those in Section 6 but involving $\mathbf{C}$ or $\mathbf{B}$ instead of $\mathbf{U}$ or $\mathbf{V}$. To this end we utilize a formula for the square root of a symmetric, positive-definite tensor due to Ting [39] and Stickforth [30]:[24]

$$III_{\hat{\mathbf{V}}}\mathbf{V} = -\mathbf{B}^2 + \Delta_{\mathbf{V}}\mathbf{B} + I_{\mathbf{V}}III_{\mathbf{V}}\mathbf{I}, \tag{7.1}$$

with an analogous formula for $\mathbf{U}$ in terms of $\mathbf{C}$. Recall that $\Delta_{\mathbf{V}}$ is given by (6.31), and note that

$$II_{\mathbf{V}} = \tfrac{1}{2}(I_{\mathbf{V}}^2 - I_{\mathbf{B}}), \quad III_{\mathbf{V}} = \sqrt{III_{\mathbf{B}}}, \quad III_{\hat{\mathbf{V}}} = I_{\mathbf{V}}II_{\mathbf{V}} - III_{\mathbf{V}}\,. \tag{7.2}$$

Thus if we wish to express $\mathbf{V}$ solely in terms of $\mathbf{B}$ and its principal invariants, we need an explicit formula for $I_{\mathbf{V}}$ in terms of the principal invariants of $\mathbf{B}$. In three dimensions this formula is rather complicated.[25] Of course, given $\mathbf{B}$ one could also compute the eigenvalues b_i of $\mathbf{B}$ and then compute the coefficients in (7.1) from the eigenvalues $\sqrt{b_i}$ of $\mathbf{V}$. In any case, by substituting (7.1) or its analog for $\mathbf{U}$ into the direct formulas in Section 6, we obtain corresponding formulas in terms of $\mathbf{B}$ or $\mathbf{C}$. The scalar coefficients in these new formulas are more complicated and involve the principal invariants of $\mathbf{U}$ or $\mathbf{V}$; cf. Hoger and Carlson [9], where direct formulas were obtained for $\dot{\mathbf{U}}$ in terms of $\dot{\mathbf{C}}$ and $\mathbf{C}$, and for $\dot{\mathbf{V}}$ in terms of $\dot{\mathbf{B}}$ and $\mathbf{B}$. Here we list only the simplest of the results which can be derived by this procedure, namely the formula for $\mathbf{\Omega} - \mathbf{W}$ obtained from (7.1) and $(6.33)_6$:

$$\begin{aligned} III_{\hat{\mathbf{V}}}^2(\mathbf{\Omega} - \mathbf{W}) &= \mathbf{B}\mathbf{D}\mathbf{B}^2 - \mathbf{B}^2\mathbf{D}\mathbf{B} + I_{\mathbf{V}}^2(\mathbf{B}^2\mathbf{D} - \mathbf{D}\mathbf{B}^2) + \varepsilon_{\mathbf{V}}(\mathbf{D}\mathbf{B} - \mathbf{B}\mathbf{D}) \\ &= \hat{\mathbf{B}}(\mathbf{D}\mathbf{B} - \mathbf{B}\mathbf{D})\hat{\mathbf{B}} - 2I_{\mathbf{V}}III_{\hat{\mathbf{V}}}(\mathbf{D}\mathbf{B} - \mathbf{B}\mathbf{D}) \\ &= \hat{\mathbf{B}}(\hat{\mathbf{B}}\mathbf{D} - \mathbf{D}\hat{\mathbf{B}})\hat{\mathbf{B}} - 2I_{\mathbf{V}}III_{\hat{\mathbf{V}}}(\hat{\mathbf{B}}\mathbf{D} - \mathbf{D}\hat{\mathbf{B}}) \\ &= 2\,\mathrm{skw}\,(\hat{\mathbf{B}}^2\mathbf{D}\hat{\mathbf{B}} + 2I_{\mathbf{V}}III_{\hat{\mathbf{V}}}\mathbf{D}\hat{\mathbf{B}})\,, \end{aligned} \tag{7.3}$$

where

$$\varepsilon_{\mathbf{V}} = I_{\mathbf{V}}(I_{\mathbf{V}}^3 - 2III_{\hat{\mathbf{V}}}) = I_{\mathbf{V}}\left(I_{\mathbf{V}}I_{\mathbf{B}} + 2\sqrt{III_{\mathbf{B}}}\right)\,, \tag{7.4}$$

and

$$\hat{\mathbf{B}} = I_{\mathbf{V}}^2\mathbf{I} - \mathbf{B}\,. \tag{7.5}$$

[24] This result follows from the Cayley-Hamilton theorem. An equivalent formula, but with more complicated expressions for the coefficients, had been obtained by Hoger and Carlson [40].

[25] Cf. Hoger and Carlson [40], Sawyers [41], and Stickforth [30].

The formulas (7.1)–(7.5) also hold with the replacements (6.34).

In our derivation of (6.33) we utilized the relation $(6.32)_1$ for $\mathbf{\Omega}$. By using $(6.32)_2$ instead, we find that (6.33) and (7.1)–(7.5) hold under the replacements

$$\mathbf{\Omega} \leftrightarrow \mathbf{W}, \quad \mathbf{V} \rightarrow \mathbf{V}^{-1}, \quad \mathbf{B} \rightarrow \mathbf{B}^{-1}, \tag{7.6}$$

and that the analogs of (6.33) and (7.1)–(7.5) obtained via the replacements (6.34) hold under the replacements

$$\mathbf{\Omega_R} \leftrightarrow \mathbf{W_R}, \quad \mathbf{U} \rightarrow \mathbf{U}^{-1}, \quad \mathbf{C} \rightarrow \mathbf{C}^{-1}. \tag{7.7}$$

For a nonsingular tensor, say $\mathbf{A}$, let the corresponding letter in sans serif type denote the *distortional part* of $\mathbf{A}$:

$$\mathsf{A} := III_{\mathbf{A}}^{-1/3}\mathbf{A}; \tag{7.8}$$

then $\det \mathsf{A} = 1$. The distortional parts[26] F, U, V, B, and C of $\mathbf{F}$, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{B}$, and $\mathbf{C}$ are unaffected by the dilatational part of the deformation, that is, by the value of $\det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V}$. Note that the polar decompositions (1.2), the relations (1.3) for the Cauchy-Green tensors, and the formulas (7.1)–(7.2) and their analog for $\mathbf{U}$ and $\mathbf{C}$, also hold with the replacements

$$\mathbf{F} \rightarrow \mathsf{F}, \quad \mathbf{U} \rightarrow \mathsf{U}, \quad \mathbf{V} \rightarrow \mathsf{V}, \quad \mathbf{B} \rightarrow \mathsf{B}, \quad \mathbf{C} \rightarrow \mathsf{C}. \tag{7.9}$$

For any tensor $\mathbf{H}$ let

$$\mathbf{H}_0 := \mathbf{H} - \tfrac{1}{3} I_{\mathbf{H}} \mathbf{I} \tag{7.10}$$

denote the *deviatoric part* of $\mathbf{H}$, so that $I_{\mathbf{H}_0} = 0$. Since the rate of change of volume per unit volume, or rate of dilatation, is given by

$$(\det \mathbf{F})^{\cdot} / \det \mathbf{F} = I_{\mathbf{L}} = I_{\mathbf{D}}, \tag{7.11}$$

the deviatoric part $\mathbf{D}_0$ of the stretching tensor $\mathbf{D}$ is unaffected by the rate of dilatation; hence, $\mathbf{D}_0$ is a measure of the rate of change of shape or rate of distortion. Note that the relations (1.4), (6.5), (6.6), and (6.25) also hold with the replacements

$$\mathbf{D} \rightarrow \mathbf{D}_0, \quad \mathbf{L} \rightarrow \mathbf{L}_0, \quad \mathbf{L_R} \rightarrow (\mathbf{L_R})_0, \quad \mathbf{D_R} \rightarrow (\mathbf{D_R})_0. \tag{7.12}$$

Now it is intuitively obvious that $\dot{\mathbf{R}}$, and thus $\mathbf{\Omega}$, should be unaffected by the rate of dilatation. Indeed, from (6.33) we see that the spherical part of $\mathbf{D}$, $\frac{1}{3} I_{\mathbf{D}} \mathbf{I}$, cancels out, so that $\mathbf{D}$ can be replaced with $\mathbf{D}_0$. Likewise, we expect that $\mathbf{\Omega}$ should be unaffected by the dilatational part of the deformation. Indeed, on using $\mathbf{V} = III_{\mathbf{V}}^{1/3} \mathsf{V}$ in (6.33) we

[26]These tensors have found useful applications in the constitutive theory of hyperelastic materials (cf. Ogden [42, Ch. 7], Rubin [43], and Charrier *et al.* [44]) and elastic-plastic materials (cf. Willis [45]).

see that the $III_{\mathbf{V}}$ terms cancel out, so that $\mathbf{V}$ can be replaced with V. In the same way we find that the formulas (6.26)–(6.33), (7.1)–(7.5), (1.11), (1.12), and the formulas for $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_{\mathbf{R}}$ obtained via the replacements (6.34), (7.6), or (7.7), also hold with the replacements (7.9), or with the replacements (7.12), or with both replacements together. In general, the other formulas in Section 6 fail to hold with just one of the replacements (7.9) or (7.12). However, all of the formulas in Section 6 as well as the formulas (1.2)–(1.12) and (7.1)–(7.5) hold when the replacements (7.9) and (7.12) are made together.[27] There are essentially two ways to show this. We can either use (7.11) to show that $\dot{\mathsf{F}} = \mathbf{L}_0\mathsf{F}$, and then proceed as in the derivation of the original results, that is, by differentiating the polar decompositions $\mathsf{F} = \mathbf{R}\mathsf{U} = \mathsf{V}\mathbf{R}$. Or we can start from the original results and convert them by using (7.11), the relations

$$\dot{\mathsf{V}} = III_{\mathbf{V}}^{-1/3}\dot{\mathbf{V}} - \tfrac{1}{3}I_{\mathbf{D}}\mathsf{V} \quad \text{and} \quad \dot{\mathsf{B}} = III_{\mathbf{B}}^{-1/3}\dot{\mathbf{B}} - \tfrac{2}{3}I_{\mathbf{D}}\mathsf{B}\,, \tag{7.13}$$

and their analogs for $\dot{\mathsf{U}}$ and $\dot{\mathsf{C}}$, which follow from (7.8) and (7.11).

Since $I_{\mathbf{D}_0} = 0$ and $III_{\mathsf{U}} = III_{\mathsf{V}} = III_{\mathsf{B}} = III_{\mathsf{C}} = 1$, there is some simplification in some of the direct formulas which follow from the replacements (7.9) and (7.12);[28] if the motion is isochoric then these simplifications hold for the original formulas as well. We consider one example here. From (6.4) with the replacements (7.9) and (7.12), we have

$$\mathbf{D}_0 = \mathsf{L}_{\mathsf{B}}[\overset{\circ}{\mathsf{B}}] = -\mathsf{L}_{\mathsf{B}^{-1}}[(\mathsf{B}^{-1})^{\circ}]\,. \tag{7.14}$$

These relations and the direct formulas which follow from $(5.17)_1$ were observed by Leonov [5]. From $(7.14)_1$, the direct formula $(5.33)_2$, the identities (5.40)–(5.43), and the fact that $\dot{III}_{\mathsf{B}} = 0$, we obtain the following formulas for the deviatoric stretching tensor in terms of the Jaumann rate of the distortional part of $\mathbf{B}$:

$$III_{\tilde{\mathsf{B}}}\mathbf{D}_0 = \tilde{\mathsf{B}}(\overset{\circ}{\mathsf{B}} - \tfrac{1}{2}\beta_1\mathbf{I})\tilde{\mathsf{B}} + \tfrac{1}{2}\beta_2\tilde{\mathsf{B}}\,, \tag{7.15}$$

where

$$\beta_1 = \operatorname{tr}\overset{\circ}{\mathsf{B}} = \operatorname{tr}\dot{\mathsf{B}} = \dot{I}_{\mathsf{B}}\,, \quad \beta_2 = \operatorname{tr}(\mathsf{B}\overset{\circ}{\mathsf{B}}) = \operatorname{tr}(\mathsf{B}\dot{\mathsf{B}}) = I_{\mathsf{B}}\dot{I}_{\mathsf{B}} - \dot{II}_{\mathsf{B}}\,. \tag{7.16}$$

The *polar rate*[29] of a tensor field $\mathbf{A}$ is defined by

$$\begin{aligned} \overset{*}{\mathbf{A}} &:= \dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{A} = \mathbf{R}\,(\mathbf{R}^{\mathrm{T}}\mathbf{A}\mathbf{R})^{\cdot}\,\mathbf{R}^{\mathrm{T}} \\ &= \overset{\circ}{\mathbf{A}} + \mathbf{A}(\boldsymbol{\Omega} - \mathbf{W}) - (\boldsymbol{\Omega} - \mathbf{W})\mathbf{A}\,. \end{aligned} \tag{7.17}$$

[27] Several of these results have been noted by Mehrabadi and Nemat-Nasser [11].

[28] Cf. the formula for $\dot{\mathsf{U}}$ obtained by Mehrabadi and Nemat-Nasser [11, (8.18)].

[29] There are a variety of names used in the mechanics literature for this invariant rate. Here we follow Dienes [46], who was motivated by the fact that the rotation tensor $\mathbf{R}$ in the definition $\boldsymbol{\Omega} := \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ arises from the polar decomposition of the deformation gradient.

In particular,

$$\overset{*}{\mathbf{V}} = \mathbf{R}\dot{\mathbf{U}}\mathbf{R}^{\mathrm{T}} \quad \text{and} \quad \overset{*}{\mathbf{B}} = \mathbf{R}\dot{\mathbf{C}}\mathbf{R}^{\mathrm{T}}. \tag{7.18}$$

By setting $\mathbf{A} = \mathbf{V}$ in $(7.17)_3$ and using the formulas for $\overset{\circ}{\mathbf{V}}$ and $\mathbf{\Omega} - \mathbf{W}$ in Section 6, we can obtain a variety of direct formulas for $\overset{*}{\mathbf{V}}$. However, it is simpler to note that by $(7.18)_1$, (6.13), and (2.15), direct formulas for $\overset{*}{\mathbf{V}}$ in terms of $\mathbf{V}$ and one or more of the tensors $\mathbf{D}$, $\mathbf{W}$, $\mathbf{L}$, and $\mathbf{\Omega}$ are given by (6.8)–(6.12), (6.24), and (6.38) with the replacements

$$\dot{\mathbf{U}} \to \overset{*}{\mathbf{V}}, \quad \mathbf{U} \to \mathbf{V}, \quad \mathbf{D_R} \to \mathbf{D}, \quad \mathbf{L_R} \to \mathbf{L}, \quad \mathbf{W_R} \to \mathbf{W}, \quad \mathbf{\Omega_R} \to \mathbf{\Omega}. \tag{7.19}$$

Also, $(7.18)_2$ and (6.7) yield the the simple formula $\overset{*}{\mathbf{B}} = 2\mathbf{VDV}$, which was derived by different means by Dienes [7].

Finally, we note that direct formulas for the material time derivatives of $\mathbf{U}^{-1}$, $\mathbf{V}^{-1}$, V^{-1}, and U^{-1}, and for the Jaumann and polar rates of $\mathbf{V}^{-1}$ and V^{-1}, follow from the results in this and the previous section and the identities

$$(\mathbf{A}^{-1})^{\cdot} = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}, \quad (\mathbf{A}^{-1})^{\circ} = -\mathbf{A}^{-1}\overset{\circ}{\mathbf{A}}\mathbf{A}^{-1}, \quad (\mathbf{A}^{-1})^{*} = -\mathbf{A}^{-1}\overset{*}{\mathbf{A}}\mathbf{A}^{-1},$$

which hold for any tensor field $\mathbf{A}$.

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