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# Greyscale Morphology by the Umbra Method

by Dennis W. McGuire

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13. ABSTRACT (Maximum 200 words) <p>Matheron, Sternberg, Haralick, and others have shown that a greyscale-image morphology can be developed by means of the <i>umbra method</i>, that is, by the application of set morphology to the umbrae of the graphs of greyscale images. A twofold extension of the greyscale theory that issues from this method is here obtained. These extensions are achieved by means of a rigorous and detailed development of both the topological and algebraic aspects of the method.</p> <p>The umbra method represents greyscale images by the bounded nonnegative functions in the set <math>\mathcal{U}</math> of extended real valued (ERV) upper semicontinuous (USC) functions of <math>n</math> real variables. The set <math>\mathcal{U}</math> can be identified with the subspace of umbral members of the space of closed subsets of either <math>\mathfrak{R}^n \times [-\infty, \infty]</math> or <math>\mathfrak{R}^{n+1}</math> (<math>\mathfrak{R}</math> = the real continuum). It is shown here that in either case the identification leads to the same natural</p> <p style="text-align: right;">(continued on reverse)</p>				
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### 13. Abstract (cont'd)

hit-miss topology and complete lattice structure on  $\mathcal{U}$ . Relative to these structures, the set  $\mathcal{M}(\mathcal{U})$  of translationally invariant (TI) USC transformations of  $\mathcal{U}$  can be taken to define the *morphological transforms* of  $\mathcal{U}$ . One can, moreover, give  $\mathcal{M}(\mathcal{U})$  a natural hit-miss topology and complete lattice structure by taking advantage of the kernel theories of Matheron and Maragos; in fact, it is shown here that the kernels of the transforms in  $\mathcal{M}(\mathcal{U})$  are completely characterized by two properties: namely,  $\mathcal{K}$  is the kernel of a transform in  $\mathcal{M}(\mathcal{U})$  if and only if  $\mathcal{K}$  is a closed subset of  $\mathcal{U}$  that is also *up-closed* (i.e.,  $f$  is a member of  $\mathcal{K}$  if and only if  $f + t$  is likewise a member for all positive  $t$ ). This is the first extension obtained.

For the second extension, the lattice space of morphological (i.e., TI and USC) mappings of  $\mathcal{U}$  to the closed subsets of  $\mathfrak{R}^{n+1}$  is first introduced and developed. It is then shown that this space isomorphically includes and considerably generalizes  $\mathcal{M}(\mathcal{U})$ , and is also a proper setting in which to generalize the representation theorems of Banon and Barrera (which apply to TI set mappings) to apply to morphological mappings of the functions in  $\mathcal{U}$ .

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# 1 Introduction

A number of image-processing methodologies are available for the development of algorithms to perform the functions required for automatic/aided target recognition (ATR). Each is supported by a mathematical theory whose extent and coherency limit or define the rational application of that methodology in developing algorithms. This is not to say that a given methodology cannot be profitably used in intuitively motivated ways that transcend the boundaries of its established theory. But a result arrived at intuitively will have general practical utility only when it becomes clear how the result issues from the principles of application of the method. In a recent report,<sup>1</sup> I reviewed and amplified on the theory that supports the use of closed-set or euclidean morphology in processing binary images. In both that report and this, my main objective is to provide (to the extent now possible) a coherent and rigorous treatment of the principles of application of mathematical morphology to the image-processing tasks required to achieve ATR. I have pursued this objective by first reviewing the relevant concepts in the existing mathematical morphology literature, and then attempting to use these concepts to form a coherent body of principles for the applications sought. In this attempt, I have had to further develop some of the existing theory.

This report addresses the morphological processing of greyscale images. This case is of singular importance because the synthetic aperture radar and infrared imagery of greatest interest for ATR is generally greyscale. The first step in the development of a greyscale-image morphology is the choice of an appropriate set of mathematical functions to represent images. The most general and satisfactory form of greyscale theory is based on the *umbra method* that evolved in the work of Matheron,<sup>2</sup> Sternberg,<sup>3</sup> and Haralick *et al.*<sup>4</sup> For the representational role, this method chooses the bounded nonnegative members of the set  $\mathcal{U} = \mathcal{U}(\mathbb{R}^n)$  of extended real valued (ERV) upper semicontinuous (USC) functions of  $n$  (a positive integer) real variables. The rationale for this choice is based on various considerations.

First of all, the greyscale images typically encountered in practice can

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<sup>1</sup>D. W. McGuire, *The morphological processing of binary images*, Army Research Laboratory, ARL-TR-28 (1993).

<sup>2</sup>G. Matheron, *Théorie des ensembles aléatoires*, Ecole des Mines de Paris (1969).

<sup>3</sup>S. R. Sternberg, *Parallel architecture for image processing*, Proc. Third International IEEE Compsac, Chicago, IL (1979).

<sup>4</sup>R. M. Haralick, S. R. Sternberg, and X. Zhuang, *Image analysis using mathematical morphology*, IEEE PAMI-9, 4, 532-550 (1987).

be described as bounded, nonnegative, real valued functions of two variables that show occasional abrupt jumps but are in the main continuous. There is, however, no compelling reason to limit the dimension of the image field to two, and there are important technical reasons for allowing the functions at issue to assume the values  $\infty$  and  $-\infty$ . Since the mathematical concept of semicontinuity adequately captures the range of discontinuities exhibited, the choice made is at least reasonable. There is more justification than this, however.

As indicated in my earlier report,<sup>1</sup> closed-set morphology can be applied to the processing of binary images (and thereby yield a theory of their morphological processing) because we can choose to represent binary images as topologically closed subsets of the image field. If we allowed the set of points where the image has a nonzero intensity to be arbitrary, only the algebraic part of set morphology would be available, and we would therefore have no concept of a morphological image metric, a concept that seems indispensable. If the nonzero-intensity set is required to be closed, however, then the morphological image topology (i.e., the hit-miss topology) of Matheron<sup>5</sup> becomes an integral and highly useful part of the theory. Now, if a closed-set binary image is regarded as a function, say  $f$ , then  $f$  is a real valued USC function whose only values are zero and one. The more general ERV USC functions  $f$  are precisely the ones whose *cross sections* or *threshold sets*  $X_t(f) = \{x : f(x) \geq t\}$  are closed; equivalently, they are precisely the ones whose *umbrae* (loosely speaking, the set of points on or under the graph of  $f$ ) are also closed. It is indeed the closedness of the cross sections of image functions that makes the well-known *threshold decomposition method*<sup>6,7</sup> effective in defining certain "morphological" transforms of greyscale images, and it is precisely the closedness of the umbrae of image functions that makes it possible to apply closed-set morphology directly to greyscale images.

Having chosen the bounded nonnegative functions in  $\mathcal{U}$  to represent greyscale images, the next theoretical task is to determine what can be usefully meant by a *morphological transform*—what I will call an *M-transform*—of a greyscale image. The umbra method achieves this end by defining the *M-transforms* of greyscale images analogously to the corresponding transformations (the translationally invariant USC transformations) of closed sets and by more generally defining these transforms on  $\mathcal{U}$ .

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<sup>5</sup>G. Matheron, *Random Sets and Integral Geometry*, John Wiley & Sons (1975).

<sup>6</sup>J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press (1982).

<sup>7</sup>P. Maragos, *A representation theory for morphological image and signal processing*, IEEE PAMI-9, 4, 532-550 (1987).

The essence of the umbra method is as follows. The *umbra* of a function is, as mentioned, the set of points on or under the function's graph. Since the graphs of USC functions are generally not closed sets, and their umbrae invariably are, the idea is to use umbrae to represent such functions by closed subsets of the space  $S$  in which the graphs reside. In this way, one obtains a class of umbrae that (1) forms a subspace of the space of closed subsets of  $S$  (the field of operations of closed-set morphology) and (2) is in one-to-one correspondence with  $\mathcal{U}$ . Closed-set morphology theory can then be applied directly to the umbra subspace, and by means of the one-to-one correspondence thence to  $\mathcal{U}$ . The technically definitive umbra method emerges in the detailed pursuit of this roughly sketched program. This pursuit, carried out in rigorous detail in sections 2 and 3, forms about two-thirds of the substance of this report.

Specifically, the umbra and USC-function spaces that form the basis of the umbra method are treated in detail in section 2. Here the relevant hit-miss and myopic topologies are carefully defined and the various algebraic operations (the lattice operations and the Minkowski sum and difference) are incorporated into the appropriate topological framework. The morphological transforms of  $\mathcal{U}$  (the  $\mathcal{M}$ -transforms) are defined in section 3. Here I extend the existing theory by using the kernel theory of Matheron<sup>5</sup> and Maragos<sup>7</sup> to establish the morphological transform space  $\mathcal{M}(\mathcal{U})$  of  $\mathcal{U}$  by determining its natural hit-miss topology and complete lattice structure.

In section 4, I introduce and develop the space  $\mathcal{H}(\mathcal{U})$  of morphological (i.e., translationally invariant and USC) mappings of  $\mathcal{U}$  to the space  $\mathbf{F}(\mathbb{R}^{n+1})$  of closed subsets of  $\mathbb{R}^{n+1}$ . Here it is shown that  $\mathcal{H}(\mathcal{U})$  isomorphically includes and considerably generalizes  $\mathcal{M}(\mathcal{U})$ . This more general space has not, to my knowledge, been considered by other morphology theorists; moreover, it turns out to be a proper setting in which to generalize the representation theorems of Banon and Barrera (sect. 1.4) to the realm of greyscale morphology.<sup>8</sup> Section 4 concludes with the details of this generalization.

The remainder of this introduction is a terse (despite its length in pages) review of the needed concepts and results from my earlier report.<sup>1</sup> This review is intended only as a ready reference; the reader should consult the earlier report for further details and depth of understanding.

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<sup>8</sup>G.J.F. Banon and J. Barrera, *Minimal representations for translational-invariant set mappings by mathematical morphology*, SIAM J. Appl. Math., 51, No. 6, pp 1782-1789 (1991).

## 1.1 Ordered Topology

§ 1.1 Let  $(X, \preceq)$  be a poset. Then the set  $\{(x, y) \in X \times X : x \preceq y\}$  is called the graph of  $\preceq$  on  $X$ . If  $X$  has a topology  $\tau$ , and if the graph of  $\preceq$  on  $X$  is a closed subset of the product space  $X \times X$ , then  $\preceq$  is called a closed order in  $X$ , and  $(X, \tau, \preceq)$  is called a topological ordered space. A topological ordered space is called a (locally) compact ordered space if its topology is (locally) compact.

§ 1.2 If  $(X, \preceq)$  is a poset and  $A \subset X$ , then  $A$  is called a decreasing (increasing) set if  $x \in A$  and  $y \preceq x$  ( $x \preceq y$ )  $\implies y \in A$ .

§ 1.3 Every topological ordered space is Hausdorff.

§ 1.4 Let  $(X, \wedge, \vee)$  be a lattice and let  $\preceq$  denote its induced ordering (i.e., the ordering defined by  $x \preceq y \iff x \wedge y = x$ ). Then if  $(X, \tau, \preceq)$  is a topological ordered space, we call  $(X, \tau, \wedge, \vee)$  a closed-order lattice. A closed-order lattice whose topology is (locally) compact is called a (locally) compact closed-order lattice.

§ 1.5 A one-to-one mapping of a topological ordered space (closed-order lattice) onto another is called a topological-ordered-space (closed-order-lattice) isomorphism if the mapping is both a poset (lattice) isomorphism and a homeomorphism. Similar definitions apply to (locally) compact ordered spaces and (locally) compact closed-order lattices.

§ 1.6 If  $(X, \tau, \preceq)$  is a topological ordered space, then the class  $\tau_u$  of open decreasing subsets of  $X$  and the class  $\tau_l$  of open increasing subsets of  $X$  are topologies on  $X$  called its decreasing and increasing topologies, respectively. If  $(X, \tau, \preceq)$  is a compact ordered space, then  $\tau_u \cup \tau_l$  is a subbase for  $\tau$ .

§ 1.7 A topological ordered space  $(X, \tau, \preceq)$  whose increasing and decreasing topologies together form a subbase for  $\tau$  is called order resolvable. Compact ordered spaces and their subspaces are order resolvable.

§ 1.8 If  $(X, \tau, \preceq)$  is order resolvable, then two topologies  $\mu$  and  $\lambda$  on  $X$  such that  $\mu \cup \lambda$  is a subbase for  $\tau$ ,  $\mu \subset \tau_u$ , and  $\lambda \subset \tau_l$  are called upper and lower topologies for  $(X, \tau, \preceq)$ .

§ 1.9 Let  $\Omega$  be a topological space, let  $X$  be an order resolvable space, let  $\mu$  and  $\lambda$  be upper and lower topologies for  $X$ , let  $\omega$  be a point in  $\Omega$ , and let  $\Lambda$  map  $\Omega$  to  $X$ . Then (where LSC means lower semicontinuous)

1.  $\Lambda$  is called  $\mu$ -USC ( $\lambda$ -LSC) [at  $\omega$ ] if  $\Lambda$  is continuous [at  $\omega$ ] with respect to  $\mu$  ( $\lambda$ ).
2.  $\Lambda$  is  $\mu$ -USC ( $\lambda$ -LSC)  $\iff \Lambda$  is  $\mu$ -USC ( $\lambda$ -LSC) at every  $\omega$ .
3.  $\Lambda$  is continuous [at  $\omega$ ]  $\iff \Lambda$  is both  $\mu$ -USC and  $\lambda$ -LSC [at  $\omega$ ].

§ 1.10 Let  $\mathfrak{R}$  denote the set of real numbers, let

$$\mu = \{\mathfrak{R}, \emptyset, (-\infty, t) : t \in \mathfrak{R}\},$$

and let  $\lambda = \{\mathfrak{R}, \emptyset, (t, \infty) : t \in \mathfrak{R}\}$ . Then  $\mu$  and  $\lambda$  are topologies on  $\mathfrak{R}$ ,  $\mu \cup \lambda$  is a subbase for the usual topology  $\tau$  of  $\mathfrak{R}$ ,  $(\mathfrak{R}, \tau, \leq)$  is a locally compact ordered space, and  $\mu$  and  $\lambda$  are its maximal upper and lower topologies. Let  $\mathfrak{R}^{(e)}$  denote the set of extended real numbers, let

$$\mu_e = \{\mathfrak{R}^{(e)}, \emptyset, [-\infty, t) : t \in \mathfrak{R}^{(e)}\},$$

and let  $\lambda_e = \{\mathfrak{R}^{(e)}, \emptyset, (t, \infty] : t \in \mathfrak{R}^{(e)}\}$ . Then  $\mu_e$  and  $\lambda_e$  are topologies on  $\mathfrak{R}^{(e)}$ ,  $\mu_e \cup \lambda_e$  is a subbase for the usual topology  $\tau_e$  of  $\mathfrak{R}^{(e)}$ ,  $(\mathfrak{R}^{(e)}, \tau_e, \leq)$  is a compact ordered space, and  $\mu_e$  and  $\lambda_e$  are its maximal upper and lower topologies.  $[(\mathfrak{R}, \tau, \inf, \sup)] (\mathfrak{R}^{(e)}, \tau_e, \inf, \sup)$  is a [conditionally] complete, [locally] compact closed-order lattice.

§ 1.11 If  $f$  is an ERV function on a topological space  $X$ , then

1.  $f$  is USC  $\iff \{x \in X : f(x) < t\}$  is open in  $X$  for all  $t \in \mathfrak{R}^{(e)}$ .
2.  $f$  is LSC  $\iff \{x \in X : f(x) > t\}$  is open in  $X$  for all  $t \in \mathfrak{R}^{(e)}$ .

§ 1.12 If  $t \in \mathfrak{R}^{(e)}$  and  $f$  is ERV on  $X$ , then the horizontal cross section of  $f$  at  $t$  is defined as  $X_t(f) = \{x \in X : f(x) \geq t\}$ . Also,

$$X_t^-(f) \equiv \{x \in X : f(x) > t\}.$$

§ 1.13 If  $f$  is ERV on  $X$ , then

- (1)  $f$  is USC if and only if  $X_t(f)$  is closed in  $X$  for all  $t \in \mathfrak{R}^{(e)}$ .
- (2)  $f$  is LSC if and only if  $X_t^-(f)$  is open in  $X$  for all  $t \in \mathfrak{R}^{(e)}$ .

If  $X$  is a first countable Hausdorff space, then

- (3)  $f$  is USC  $\iff f(x) \geq \limsup f(x_i) \forall x \in X$  and  $\forall \{x_i\}$  in  $X$  with limit  $x$ .
- (4)  $f$  is LSC  $\iff f(x) \leq \liminf f(x_i) \forall x \in X$  and  $\forall \{x_i\}$  in  $X$  with limit  $x$ .

Conditions (3) and (4) come up again and again in different guises and settings as criteria for semicontinuity. I refer to them and their relatives as the *usual semicontinuity criteria*.

Birkhoff,<sup>9</sup> Kelley,<sup>10</sup> and Nachbin<sup>11</sup> are general references for the material of this section.

## 1.2 Closed-Set Morphology

§ 1.14 Let  $S$  be a locally compact, second countable Hausdorff (LCS) space and let  $\mathbf{F}(S)$ ,  $\mathbf{G}(S)$ , and  $\mathbf{K}(S)$  respectively denote the classes of closed, open, and compact subsets of  $S$ . Then  $(\mathbf{F}(S), \subset)$  is a poset and  $(\mathbf{F}(S), \cap, \cup)$  is a complete distributive lattice with induced ordering  $\subset$ .

§ 1.15 Let  $\mathbf{F}^K = \{F \in \mathbf{F}(S) : F \cap K = \emptyset\}$  and let

$$\mathbf{F}_G = \{F \in \mathbf{F}(S) : F \cap G \neq \emptyset\}.$$

Then the hit-miss topology  $\tau$  of  $\mathbf{F}(S)$  is generated by

$$\{\mathbf{F}^K : K \in \mathbf{K}(S)\} \cup \{\mathbf{F}_G : G \in \mathbf{G}(S)\}.$$

$\tau$  is compact, second countable, and Hausdorff.  $\subset$  is a closed order in  $\mathbf{F}(S)$ ,  $(\mathbf{F}(S), \tau, \subset)$  is a compact ordered space,  $(\mathbf{F}(S), \tau, \cap, \cup)$  is a compact closed-order lattice,  $\{\mathbf{F}^K : K \in \mathbf{K}(S)\} \cup \emptyset$  is a base for an upper topology of  $(\mathbf{F}(S), \tau, \subset)$ , and  $\{\mathbf{F}_G : G \in \mathbf{G}(S)\} \cup \mathbf{F}(S)$  is a subbase for a companion lower topology.

§ 1.16 Let  $\mathbf{K}^F = \{K \in \mathbf{K}(S) : K \cap F = \emptyset\}$  and let

$$\mathbf{K}_G = \{K \in \mathbf{K}(S) : K \cap G \neq \emptyset\}.$$

The topology  $\nu$  generated on  $\mathbf{K}(S)$  by

$$\{\mathbf{K}^F : F \in \mathbf{F}(S)\} \cup \{\mathbf{K}_G : G \in \mathbf{G}(S)\}$$

is called its myopic topology.

The following are basic results about  $\nu$  and its relation to  $\tau$ .

<sup>9</sup>G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publications (1948).

<sup>10</sup>J. L. Kelley, *General Topology*, D. Van Nostrand (1955).

<sup>11</sup>L. Nachbin, *Topology and Order*, Van Nostrand Math. Studies No. 4, D. Van Nostrand (1965).

1.  $(\mathbf{K}(S), \mathbf{v})$  is an LCS space.
2. If  $S$  is not a compact space, then neither is  $(\mathbf{K}(S), \mathbf{v})$ , and  $\mathbf{v}$  is strictly stronger than the relative hit-miss topology of  $\mathbf{K}(S)$ .
3. If  $\mathcal{K}$  is a  $\mathbf{v}$ -compact subset of  $\mathbf{K}(S)$ , then the relative hit-miss topology of  $\mathcal{K}$  and the relative myopic topology of  $\mathcal{K}$  coincide.
4. A subset  $\mathcal{K}$  of  $\mathbf{K}(S)$  is  $\mathbf{v}$ -compact if and only if  $\mathcal{K}$  is closed in  $\mathbf{F}(S)$  and there exists a  $K_0 \in \mathbf{K}(S)$  such that  $K_0 \supset K$  for all  $K \in \mathcal{K}$ .
5.  $(\mathbf{K}(S), \mathbf{v}, \subset)$  is an order resolvable locally compact ordered space and  $(\mathbf{K}(S), \mathbf{v}, \cap, \cup)$  is a locally compact closed-order lattice.
6.  $(\mathbf{K}(S), \mathbf{v}, \cap, \cup)$  is distributive and has the universal lower bound  $\emptyset \subset S$ . It has no universal upper bound, unless  $S$  is compact.
7. If  $\mathcal{K} \subset \mathbf{K}(S)$  is not empty, then  $\inf \mathcal{K}$  exists in  $\mathbf{K}(S)$  but  $\sup \mathcal{K}$  need not, unless  $S$  is compact.

For the time being, the  $S$  will be dropped in  $\mathbf{F}(S)$ ,  $\mathbf{K}(S)$ , etc. With regard to convergence in  $\mathbf{F}$  and  $\mathbf{K}$ , the following theorems hold.

**Theorem 1.1** *A sequence  $\{F_i\}$  in  $\mathbf{F}$  converges to  $F \in \mathbf{F}$  if and only if (1)  $G \subset S$  is open and  $G \cap F \neq \emptyset \implies G \cap F_i \neq \emptyset \forall$  but at most finitely many  $F_i$  and (2)  $K \subset S$  is compact and  $K \cap F = \emptyset \implies K \cap F_i = \emptyset \forall$  but at most finitely many  $F_i$ .*

**Theorem 1.2** *A sequence  $\{F_i\}$  in  $\mathbf{F}$  converges to  $F \in \mathbf{F}$  if and only if (a) for each  $x \in F$  there exist  $x_i \in F_i$  for all but at most finitely many  $i$  such that  $x_i \rightarrow x$  and (b) if  $\{F_{i_k}\}$  is a subsequence of  $\{F_i\}$ , then every convergent sequence  $x_{i_k} \in F_{i_k}$  has its limit in  $F$ . In addition, (a) and (b) are respectively equivalent to (1) and (2) of Theorem 1.1.*

I refer to (a) and (b) of this theorem as *Matheron's convergence criteria*.

**Theorem 1.3** *A sequence  $\{K_i\}$  in  $\mathbf{K}$  converges in the myopic topology to  $K \in \mathbf{K}$  if and only if  $K_i \rightarrow K$  in the hit-miss topology of  $\mathbf{F}$  and there exists a  $K_0 \in \mathbf{K}$  such that  $K_0 \supset K_i$  for all  $i$ .*

§ 1.17 *Let  $\{F_i\}$  be a sequence in  $\mathbf{F}$  and let  $\mathcal{L}(\{F_i\})$  denote its set of limit points. Then the lower and upper limits of  $\{F_i\}$  are defined by*

1.  $\underline{\text{Lim}} F_i = \cap \{F : F \in \mathcal{L}(\{F_i\})\}$ .
2.  $\overline{\text{Lim}} F_i = \cup \{F : F \in \mathcal{L}(\{F_i\})\}$ .

**Theorem 1.4** (Upper-lower limit theorem) If  $\{F_i\}$  is a sequence in  $\mathbf{F}$ , then (a)  $\underline{\text{Lim}} F_i$  is the largest  $F \in \mathbf{F}$  that satisfies (a) of Theorem 1.2, (b)  $\overline{\text{Lim}} F_i$  is the smallest  $F \in \mathbf{F}$  that satisfies (b) of Theorem 1.2, and (c)  $F_i \rightarrow F \iff \underline{\text{Lim}} F_i = \overline{\text{Lim}} F_i = F$ .

The lower limit lies in  $\mathbf{F}$  by definition and the upper limit lies in  $\mathbf{F}$  by (b) of this theorem. The following is another instance of the usual semicontinuity criteria.

**Theorem 1.5** If  $\Psi : X \rightarrow \mathbf{F}$  and  $X$  is a first countable Hausdorff space, then  $\Psi$  is USC at  $x \in X \iff \Psi(x) \supset \overline{\text{Lim}} \Psi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$  and  $\Psi$  is LSC at  $x \in X \iff \Psi(x) \subset \underline{\text{Lim}} \Psi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ .

§ 1.18 If  $A$  and  $B$  are any subsets of  $\mathbb{R}^n$ , then their Minkowski sum is defined by  $A \oplus B = \{x : x = y + z, y \in A, z \in B\}$ ; if  $A$  or  $B$  is empty, then  $A \oplus B \equiv \emptyset$ . If  $A \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ , then  $A + x \equiv A \oplus \{x\}$ . The symmetric  $\check{A}$  of  $A \subset \mathbb{R}^n$  is defined as  $\{x : -x \in A\}$ .

§ 1.19 If  $A$  and  $B$  are subsets of  $\mathbb{R}^n$ , then

1.  $A, B \in \mathbf{K}(\mathbb{R}^n) \implies A \oplus B \in \mathbf{K}(\mathbb{R}^n)$ .
2.  $A \in \mathbf{F}(\mathbb{R}^n)$  and  $B \in \mathbf{K}(\mathbb{R}^n) \implies A \oplus B \in \mathbf{F}(\mathbb{R}^n)$ .
3.  $A, B \in \mathbf{F}(\mathbb{R}^n) \not\implies A \oplus B \in \mathbf{F}(\mathbb{R}^n)$ .

§ 1.20 The Minkowski difference  $A \ominus B$  of two arbitrary subsets of  $\mathbb{R}^n$  is defined by  $A \ominus B = (A^c \oplus B)^c$  where the  $c$  denotes complementation.

§ 1.21 Regardless of the topological character of  $B$ , the following hold.

1. If  $A$  is closed, then  $A \ominus B$  is closed.
2. If  $A$  is compact, then  $A \ominus B$  is compact.

The following also hold.

1.  $\oplus$  is continuous on  $\mathbf{F} \times \mathbf{K}$  and  $\mathbf{K} \times \mathbf{K}$  to  $\mathbf{F}$  and  $\mathbf{K}$ , respectively.
2.  $\ominus$  is only a USC mapping of  $\mathbf{F} \times \mathbf{K}$ ,  $\mathbf{K} \times \mathbf{K}$  and  $\mathbf{F} \times \mathbf{F}$  into  $\mathbf{F}$ ,  $\mathbf{K}$  and  $\mathbf{F}$ , respectively.
3.  $\cup$  is a continuous operation in  $\mathbf{F}(S)$  and  $\mathbf{K}(S)$ , but  $\cap$  is only USC.
4. The mapping  $F \mapsto \overline{F^c}$  of  $\mathbf{F}(S)$  to itself is LSC.

5. If  $S$  is locally connected, then the mapping  $F \mapsto \partial F$  (the boundary of  $F$ ) of  $\mathbf{F}(S)$  to itself is LSC and  $K \mapsto \partial K$  is an LSC mapping of  $\mathbf{K}(S)$  to itself.

Matheron<sup>5</sup> and Serra<sup>6</sup> are general references for the material of this and the next section.

### 1.3 Transformation Space Theory

A mapping  $\Psi$  of  $\mathbf{F} = \mathbf{F}(\mathbb{R}^n)$  to itself is called a *transformation* of (or on)  $\mathbf{F}$ . The transformation type to be focussed on I call an  $\mathcal{M}$ -*transformation*. The *order preserving (increasing, isotone)* members of this class are more commonly known as *morphological filters*. In what follows, the shorthand  $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$  is used for the power set of  $\mathbb{R}^n$ .

§ 1.22 A mapping  $\Psi : \mathbf{F} \rightarrow \mathcal{P}$  is called *translationally invariant (TI)* if  $\Psi(F + x) = \Psi(F) + x$  for all  $F \in \mathbf{F}$  and  $x \in \mathbb{R}^n$ . A TI mapping is called an  $\mathcal{M}$ -*transformation* if it is into  $\mathbf{F}$  and USC. The kernel of a TI mapping  $\Psi : \mathbf{F} \rightarrow \mathcal{P}$  is the set  $\ker(\Psi) = \{F \in \mathbf{F} : \mathbf{0} \in \Psi(F)\}$ .

§ 1.23 If  $\Psi$  is a TI mapping of  $\mathbf{F}$  to  $\mathcal{P}$  and  $F \in \mathbf{F}$ , then

$$\Psi(F) = \{x \in \mathbb{R}^n : F - x \in \ker(\Psi)\}.$$

If  $\mathcal{K}$  is any subset of  $\mathbf{F}$ , then  $F \mapsto \{x \in \mathbb{R}^n : F - x \in \mathcal{K}\}$  defines a TI mapping of  $\mathbf{F}$  to  $\mathcal{P}$  whose kernel is  $\mathcal{K}$ .

**Theorem 1.6** (Matheron's closed kernel theorem) A TI mapping  $\Psi$  of  $\mathbf{F}$  to  $\mathcal{P}$  is into  $\mathbf{F}$  and USC if and only if  $\ker(\Psi)$  is closed in  $\mathbf{F}$ .

There is thus a one-to-one correspondence between the TI mappings of  $\mathbf{F}$  and  $\mathcal{P}(\mathbf{F})$ , and there is also a one-to-one correspondence between the class of  $\mathcal{M}$ -transformations of  $\mathbf{F}$  and the class of closed subsets of  $\mathbf{F}$ , i.e.,  $\mathbf{F}(\mathbf{F})$ . Since  $\mathbf{F}$  is an LCS space, the class of  $\mathcal{M}$ -transformations can be identified with  $\mathbf{F}(\mathbf{F})$  (understood to have its hit-miss topology), thus giving rise to the space  $\mathcal{M}(\mathbf{F})$  of  $\mathcal{M}$ -transformations.

§ 1.24 The transformation space  $\mathcal{M}(\mathbf{F})$  has a natural lattice structure that it acquires from  $\mathbf{F}(\mathbf{F})$  through the correspondence  $\Psi \leftrightarrow \ker(\Psi)$ . If  $\Psi$  and  $\Psi'$  are transformations in  $\mathcal{M}(\mathbf{F})$ , then the transformations  $\Psi \cap \Psi'$  and  $\Psi \cup \Psi'$  are defined in terms of their kernels by

$$\ker(\Psi \cap \Psi') = \ker(\Psi) \cap \ker(\Psi') \text{ and } \ker(\Psi \cup \Psi') = \ker(\Psi) \cup \ker(\Psi').$$

For all  $\Psi, \Psi' \in \mathcal{M}(\mathbf{F})$ , it is clear that  $\Psi \cap \Psi', \Psi \cup \Psi' \in \mathcal{M}(\mathbf{F})$ . Indeed it follows that  $(\mathcal{M}(\mathbf{F}), \cap, \cup)$  is a lattice and that  $\Psi \mapsto \ker(\Psi)$  is a lattice isomorphism of  $(\mathcal{M}(\mathbf{F}), \cap, \cup)$  onto  $(\mathbf{F}(\mathbf{F}), \cap, \cup)$ . Consequently,  $(\mathcal{M}(\mathbf{F}), \cap, \cup)$  is complete and distributive.

§ 1.25 The ordering  $\subseteq$  induced in  $\mathcal{M}(\mathbf{F})$  by its lattice operations can be characterized as follows: If  $\Psi$  and  $\Psi'$  are transformations in  $\mathcal{M}(\mathbf{F})$ , then  $\Psi \subseteq \Psi' \iff \Psi(F) \subset \Psi'(F)$  for all  $F \in \mathbf{F} \iff \ker(\Psi) \subset \ker(\Psi')$ . In addition, if  $\{\Psi_\alpha\}$  is any set of transformations in  $\mathcal{M}(\mathbf{F})$ , then

1.  $\inf\{\Psi_\alpha\} \equiv \cap_\alpha \Psi_\alpha$  has the kernel  $\cap_\alpha \ker(\Psi_\alpha)$  and

$$(\cap_\alpha \Psi_\alpha)(F) = \cap_\alpha \Psi_\alpha(F) \quad \forall F \in \mathbf{F}.$$

2.  $\sup\{\Psi_\alpha\} \equiv \cup_\alpha \Psi_\alpha$  has the kernel  $\overline{\cup_\alpha \ker(\Psi_\alpha)}$  and is the least  $\mathcal{M}$ -transformation such that  $(\cup_\alpha \Psi_\alpha)(F) \supset \overline{\cup_\alpha \Psi_\alpha(F)}$  for all  $F \in \mathbf{F}$ .

3. If  $\{\Psi_\alpha\} = \{\Psi_k\}$  is a finite set of  $\mathcal{M}$ -transformations, then

$$(\cup_k \Psi_k)(F) = \cup_k \Psi_k(F) \quad \text{for all } F \in \mathbf{F}.$$

§ 1.26 In the following, the hit-miss topologies of  $\mathbf{F}(\mathbf{F})$  and  $\mathcal{M}(\mathbf{F})$  are both denoted  $\nu$ . Under the correspondence  $\leftrightarrow$ :

1.  $(\mathcal{M}(\mathbf{F}), \subseteq)$  is poset isomorphic to  $(\mathbf{F}(\mathbf{F}), \subset)$ .

2.  $(\mathcal{M}(\mathbf{F}), \nu, \subseteq)$  is compact-ordered-space isomorphic to the compact ordered space  $(\mathbf{F}(\mathbf{F}), \nu, \subset)$ .

3.  $(\mathcal{M}(\mathbf{F}), \nu, \cap, \cup)$  is isomorphic as a compact closed-order lattice to  $(\mathbf{F}(\mathbf{F}), \nu, \cap, \cup)$ .

Therefore,  $\cup$  is a continuous operation in  $\mathcal{M}(\mathbf{F})$ , but  $\cap$  is only USC.

§ 1.27 If  $\Psi$  maps  $\mathbf{F}$  to  $\mathcal{P}$ , then  $\Psi$  is called increasing (decreasing) if  $\Psi(E) \supset \Psi(F)$  ( $\Psi(E) \subset \Psi(F)$ ) whenever  $E, F \in \mathbf{F}$  and  $E \supset F$ . Order preserving and order reversing are synonyms for increasing and decreasing. The mappings  $\Psi_\emptyset : F \mapsto \emptyset$  and  $\Psi_{\mathbb{R}^n} : F \mapsto \mathbb{R}^n$  (each for all  $F \in \mathbf{F}$ ) are  $\mathcal{M}$ -transformations that are both increasing and decreasing. They are called the trivial transformations.

§ 1.28 If  $\Psi$  is a TI mapping of  $\mathbf{F}$  to  $\mathcal{P}$ , then  $\Psi(\emptyset)$  is either  $\emptyset$  or  $\mathbb{R}^n$ , and likewise for  $\Psi(\mathbb{R}^n)$ . If  $\Psi$  is nontrivial and increasing (decreasing), then  $\Psi(\emptyset) = \emptyset$  ( $\mathbb{R}^n$ ) and  $\Psi(\mathbb{R}^n) = \mathbb{R}^n$  ( $\emptyset$ ).  $\Psi \in \mathcal{M}(\mathbf{F})$  is increasing (decreasing) if and only if  $\ker(\Psi)$  is an increasing (decreasing) set. The subspace  $\mathcal{M}_\uparrow(\mathbf{F})$  ( $\mathcal{M}_\downarrow(\mathbf{F})$ ) of increasing (decreasing)  $\Psi \in \mathcal{M}(\mathbf{F})$  is closed in  $\mathcal{M}(\mathbf{F})$ .

§ 1.29 If  $E \in \mathbf{F}$  is fixed, then the following are  $\mathcal{M}$ -transformations:  $F \mapsto \check{F}$ ,  $F \mapsto F \ominus E$ ,  $F \mapsto F \ominus \check{E}$ ,  $F \mapsto E \ominus F$ , and  $F \mapsto \check{E} \ominus F$ . The first three are increasing and the last two are decreasing. The transformations  $F \mapsto F \ominus \check{E}$  and  $F \mapsto \check{E} \ominus F$  are written  $\mathcal{E}_E$  and  $\tilde{\mathcal{D}}_E$ , respectively, and are called erosion and antidilation by  $E$ .

## 1.4 Supremum Representations

Matheron's<sup>5</sup> well-known supremum representation theorem for the increasing  $\Psi \in \mathcal{M}(\mathbf{F})$  can be stated as follows.

**Theorem 1.7** If  $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$ , then  $\Psi = \bigcup_{E \in \ker(\Psi)} \mathcal{E}_E$ ; that is,  $\Psi$  is the supremum of the set of erosions  $\{\mathcal{E}_E : E \in \ker(\Psi)\}$ . Moreover, if  $F \in \mathbf{F}$ , then  $\Psi(F) = \bigcup \{\mathcal{E}_E(F) : E \in \ker(\Psi)\}$ .

If  $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$  and is not trivial, then Maragos<sup>7</sup> defines the *minimal basis kernel*  $\mathcal{K}_{\min}(\Psi)$  of  $\Psi$  as the collection of minimal elements of  $\ker(\Psi)$  relative to  $\subset$ . This concept leads to Maragos' minimal representation version of Matheron's result, namely:

**Theorem 1.8** If  $\Psi \in \mathcal{M}_\uparrow(\mathbf{F})$ , then  $\Psi = \bigcup_{E \in \mathcal{K}_{\min}(\Psi)} \mathcal{E}_E$ . Moreover, if  $F \in \mathbf{F}$ , then  $\Psi(F) = \bigcup \{\mathcal{E}_E(F) : E \in \mathcal{K}_{\min}(\Psi)\}$ .

Banon and Barrera<sup>8</sup> have developed versions of the foregoing theorems for arbitrary  $\Psi \in \mathcal{M}(\mathbf{F})$ . If  $E, H \in \mathbf{F}$  and  $E \subset H$ , then  $[E, H] \equiv \{F \in \mathbf{F} : E \subset F \subset H\}$  and is called a closed interval of  $\mathbf{F}$ ; moreover, it follows that all closed intervals of  $\mathbf{F}$  are closed subsets of  $\mathbf{F}$ . In terms of this concept, one of the general representation theorems of Banon and Barrera can be expressed as follows.

**Theorem 1.9** If  $\Psi \in \mathcal{M}(\mathbf{F})$ , then  $\Psi = \bigcup_{[E, H] \subset \ker(\Psi)} \mathcal{E}_E \cap \tilde{\mathcal{D}}_H$ . Moreover, if  $F \in \mathbf{F}$ , then  $\Psi(F) = \bigcup \{\mathcal{E}_E(F) \cap \tilde{\mathcal{D}}_H(F) : [E, H] \subset \ker(\Psi)\}$ .

Banon and Barrera<sup>8</sup> define the *basis*  $\mathbf{B}_\Psi$  of a general  $\Psi \in \mathcal{M}(\mathbf{F})$  as the collection of maximal closed intervals of  $\mathbf{F}$  contained in  $\ker(\Psi)$ . This concept leads to the minimal representation theorem of Banon and Barrera that generalizes Theorem 1.8, namely:

**Theorem 1.10** If  $\Psi \in \mathcal{M}(\mathbf{F})$ , then  $\Psi = \bigcup_{[E, H] \in \mathbf{B}_\Psi} \mathcal{E}_E \cap \tilde{\mathcal{D}}_H$ . Moreover, if  $F \in \mathbf{F}$ , then  $\Psi(F) = \bigcup \{\mathcal{E}_E(F) \cap \tilde{\mathcal{D}}_H(F) : [E, H] \in \mathbf{B}_\Psi\}$ .

## 2 Umbra and USC-Function Spaces

There are two candidates for the resident space of the umbrae of the  $f \in \mathcal{U}$ , namely,  $\mathfrak{R}^n \times \mathfrak{R} = \mathfrak{R}^{n+1}$  and  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$  (where  $\mathfrak{R}^{(e)}$  denotes the set of extended real numbers). Although  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$  is the resident space of the graphs of the  $f \in \mathcal{U}$ , one is at liberty to define the umbrae of these  $f$  as subsets of either of the foregoing product spaces. The two choices, moreover, give rise to apparently different theories that are, in fact, equivalent. In the exposition that follows, I develop both theoretical lines in parallel so that they can be compared in their details at various points. I begin with the choice  $\mathfrak{R}^{n+1}$ .

### 2.1 The Umbra Space $U \subset F(\mathfrak{R}^{n+1})$

Distinguish a  $t$ -axis in  $\mathfrak{R}^{n+1}$  by denoting its points

$$\{(x, t) : x \in \mathfrak{R}^n, t \in \mathfrak{R}\}.$$

**Definition 2.1** *If  $A \subset \mathfrak{R}^{n+1}$  is nonempty, then the umbra of  $A$ , denoted  $\Upsilon(A)$ , is the subset of  $\mathfrak{R}^{n+1}$  given by  $\{(x, t) : (x, t') \in A, t \leq t'\}$ ; if  $A = \emptyset$ , let  $\Upsilon(A) = \emptyset$ . A subset of  $\mathfrak{R}^{n+1}$  that contains  $(x, t)$  whenever it contains  $(x, t')$  with  $t' > t$  will be called *umbral*.*

$\Upsilon$  maps subsets of  $\mathfrak{R}^{n+1}$  to umbral subsets of  $\mathfrak{R}^{n+1}$ . Note that if  $A$  is umbral, then  $\Upsilon(A) = A$ .  $\Upsilon$  can be regarded as operating on subsets of  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$  if the above definition is extended as follows.

**Definition 2.2** *If  $A$  is a nonempty subset of  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$ , then*

$$\Upsilon(A) \equiv \{(x, t) \in \mathfrak{R}^{n+1} : (x, t') \in A, t \leq t'\}.$$

Note that the more general  $\Upsilon$  continues to produce subsets of  $\mathfrak{R}^{n+1}$  exclusively. The more general understanding of Definition 2.2 is needed to define the umbra of the graph of an  $f \in \mathcal{U}$ .

**Definition 2.3** *If  $f \in \mathcal{U}$ , then  $G_f \equiv \{(x, f(x)) \in \mathfrak{R}^n \times \mathfrak{R}^{(e)} : x \in \mathfrak{R}^n\}$  is called the *graph* of  $f$ .*

The umbra of the graph of  $f$  is accordingly

$$\Upsilon(G_f) = \{(x, t) \in \mathfrak{R}^{n+1} : t \leq f(x)\}.$$

**Definition 2.4** Let  $\mathbf{F}(\mathfrak{R}^{n+1})$  denote the space of closed subsets of  $\mathfrak{R}^{n+1}$ , where  $\mathbf{F}(\mathfrak{R}^{n+1})$  is assumed to be carrying its hit-miss topology  $\tau$ , and let  $\mathbf{U}$  denote the set of umbrae in  $\mathbf{F}(\mathfrak{R}^{n+1})$ . Then  $\mathbf{U}$  inherits a topology  $\omega$  from  $(\mathbf{F}(\mathfrak{R}^{n+1}), \tau)$  that we will call the hit-miss topology of  $\mathbf{U}$ .

**Definition 2.5** If  $A$  is a subset of  $\mathfrak{R}^{n+1}$ , then let

$$\mathbf{U}^A = \{U \in \mathbf{U} : U \cap A = \emptyset\} \text{ and } \mathbf{U}_A = \{U \in \mathbf{U} : U \cap A \neq \emptyset\}.$$

**Proposition 2.1**  $\omega$  is generated subbasically on  $\mathbf{U}$  by

$$\{\mathbf{U}^K : K \in \mathbf{K}(\mathfrak{R}^{n+1})\} \cup \{\mathbf{U}_G : G \in \mathbf{G}(\mathfrak{R}^{n+1})\}$$

where  $\mathbf{K}(\mathfrak{R}^{n+1})$  and  $\mathbf{G}(\mathfrak{R}^{n+1})$  are, respectively, the classes of compact and open subsets of  $\mathfrak{R}^{n+1}$ .

**Proof** An upper topology for  $\mathbf{F}(\mathfrak{R}^{n+1})$  has the base

$$\{\mathbf{F}^K : K \in \mathbf{K}(\mathfrak{R}^{n+1})\} \cup \emptyset$$

and a companion lower topology has the subbase

$$\{\mathbf{F}_G : G \in \mathbf{G}(\mathfrak{R}^{n+1})\} \cup \mathbf{F}(\mathfrak{R}^{n+1}).$$

The relative upper topology of  $\mathbf{U}$  in  $\mathbf{F}(\mathfrak{R}^{n+1})$  therefore has the base  $\{\mathbf{U}^K : K \in \mathbf{K}(\mathfrak{R}^{n+1})\} \cup \emptyset$ , where  $\emptyset$  now refers to the empty collection of subsets of  $\mathbf{U}$ . Similarly, the relative lower topology of  $\mathbf{U}$  in  $\mathbf{F}(\mathfrak{R}^{n+1})$  has the subbase  $\{\mathbf{U}_G : G \in \mathbf{G}(\mathfrak{R}^{n+1})\} \cup \mathbf{U}$ .

**Theorem 2.1** (Matheron)  $(\mathbf{U}, \omega)$  is a compact LCS space closed under arbitrary intersections and finite unions,  $(\mathbf{U}, \cap, \cup)$  is a complete distributive lattice with induced ordering  $\subset$ , and  $\Upsilon(G_f) \in \mathbf{U} \forall f \in \mathcal{U}$ .

**Proof** Since  $(\mathbf{U}, \omega)$  is a subspace of a second countable Hausdorff space, it follows that  $\omega$  is second countable and Hausdorff. To prove compactness, moreover, it is enough to show that  $\mathbf{U}$  is a  $\tau$ -closed subset of  $\mathbf{F}(\mathfrak{R}^{n+1})$ . Accordingly, let  $U_i \rightarrow F$  in  $\mathbf{F}(\mathfrak{R}^{n+1})$  where each  $U_i \in \mathbf{U}$ . We show that  $F \in \mathbf{U}$ , i.e.,  $(x, t) \in F$  whenever  $(x, t') \in F$  with  $t' > t$ . Put  $\delta = t' - t$ . There are  $(x_i, t'_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t'_i) \rightarrow (x, t')$ . Thus  $(x_i, t'_i - \delta) \in U_i$  for all but at most finitely many  $i$  and  $(x_i, t'_i - \delta) \rightarrow (x, t' - \delta) = (x, t)$ . Hence  $(x, t) \in F$  and it follows that  $F$  is a closed umbra.  $(\mathbf{U}, \omega)$  is therefore a compact LCS space.

Because arbitrary intersections and finite unions of umbrae (closed sets) are umbrae (closed sets), it follows that  $\mathbf{U}$  is closed under arbitrary intersections and finite unions. Moreover  $(\mathbf{U}, \cap, \cup)$  is clearly distributive and obviously has the induced ordering  $\subset$ . Now a lattice with a universal upper bound in which every subset has an infimum is necessarily complete.<sup>9</sup> Since  $\mathbf{U}$  is closed under arbitrary intersections, it follows that every subset of  $\mathbf{U}$  has an infimum. Completeness thus follows from the fact that  $(\mathbf{U}, \cap, \cup)$  has the universal bounds  $\mathfrak{R}^{n+1}$  and  $\emptyset$ . To prove that  $\Upsilon(G_f)$  is a closed subset of  $\mathbf{F}(\mathfrak{R}^{n+1})$  for all  $f \in \mathcal{U}$ , let  $\{(x_i, t_i)\}$  be an  $\mathfrak{R}^{n+1}$ -convergent sequence in  $\Upsilon(G_f)$  with limit  $(x, t)$ . We easily see that  $f(x) \geq t$  because  $f$  is USC and  $f(x_i) \geq t_i$  for all  $i$ , so that  $f(x) \geq \limsup f(x_i) \geq t$ . This completes the proof.

**Corollary 2.1** *Hence we have the following:*

1.  $(\mathbf{U}, \omega, \subset)$  is a compact ordered space.
2.  $(\mathbf{U}, \omega, \cap, \cup)$  is a compact closed-order lattice.

**Remark 2.1** *Since  $\mathbf{U} \subset \mathbf{F}(\mathfrak{R}^{n+1})$  is closed, we have the following characterization of convergence in  $\mathbf{U}$ :*

1.  $U_i \in \mathbf{U}$  and  $U_i \rightarrow U$  in  $\mathbf{F}(\mathfrak{R}^{n+1}) \implies U \in \mathbf{U}$  and  $U_i \rightarrow U$  in  $\mathbf{U}$ .
2. Conversely,  $U_i \rightarrow U$  in  $\mathbf{U} \implies U_i \rightarrow U$  in  $\mathbf{F}(\mathfrak{R}^{n+1})$ .

*Matheron's convergence criteria* for sequences in  $\mathbf{U}$  now follow readily:

**Theorem 2.2** *A sequence  $\{U_i\}$  in  $\mathbf{U}$  converges to  $U \in \mathbf{U}$  if and only if (a) for each  $(x, t) \in U$  there exist  $(x_i, t_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$  and (b) if  $\{U_{i_k}\}$  is a subsequence of  $\{U_i\}$ , then every convergent sequence  $(x_{i_k}, t_{i_k}) \in U_{i_k}$  has its limit in  $U$ .*

**Remark 2.2** *If  $\{U_i\}$  is a sequence in  $\mathbf{U}$  (a  $\mathbf{U}$ -sequence), then its limit points relative to  $\mathbf{F}(\mathfrak{R}^{n+1})$  coincide with its limit points relative to  $\mathbf{U}$ .*

Lower and upper limits may therefore be defined as follows.

**Definition 2.6** *Let  $\{U_i\}$  be a sequence in  $\mathbf{U}$  and let  $\mathcal{L}(\{U_i\})$  denote its set of limit points. Then we define  $\underline{\text{Lim}} U_i = \cap\{U : U \in \mathcal{L}(\{U_i\})\}$  and  $\overline{\text{Lim}} U_i = \cup\{U : U \in \mathcal{L}(\{U_i\})\}$ .*

**Remark 2.3** The upper (lower) limit in  $\mathbf{F}(\mathfrak{R}^{n+1})$  of a  $\mathbf{U}$ -sequence is equal to its upper (lower) limit in  $\mathbf{U}$ .

With these results we can readily establish the *upper-lower limit theorem* and the *usual semicontinuity criteria* (Thm. 2.4) for  $\mathbf{U}$ .

**Theorem 2.3** (Upper-lower limit theorem for  $\mathbf{U}$ ) *If  $\{U_i\}$  is a sequence in  $\mathbf{U}$ , then (a)  $\underline{\text{Lim}} U_i$  is the largest  $U \in \mathbf{U}$  that satisfies condition (a) of Theorem 2.2, (b)  $\overline{\text{Lim}} U_i$  is the smallest  $U \in \mathbf{U}$  that satisfies condition (b) of Theorem 2.2, and (c)  $U_i \rightarrow U$  in  $\mathbf{U}$  if and only if*

$$\underline{\text{Lim}} U_i = \overline{\text{Lim}} U_i = U.$$

**Theorem 2.4** *If  $\Phi : X \rightarrow \mathbf{U}$  and  $X$  is a first countable Hausdorff space, then  $\Phi$  is USC at  $x \in X \iff \Phi(x) \supset \overline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ , and  $\Phi$  is LSC at  $x \in X \iff \Phi(x) \subset \underline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ .*

The following result gives technically useful criteria for verifying the semicontinuities of a mapping into  $\mathbf{U}$ ; it follows from Theorems 2.2, 2.3, and 2.4.

**Remark 2.4** *If  $X$  is a first countable Hausdorff space, then*

1.  $\Phi : X \rightarrow \mathbf{U}$  is USC  $\iff x_i \rightarrow x$  in  $X$ ,  $(\xi_k, \tau_k) \in \Phi(x_{i_k}) \forall k$ , and  $(\xi_k, \tau_k) \rightarrow (\xi, \tau) \implies (\xi, \tau) \in \Phi(x)$ .
2.  $\Phi : X \rightarrow \mathbf{U}$  is LSC  $\iff x_i \rightarrow x$  in  $X$  and  $(\xi, \tau) \in \Phi(x) \implies$  there exist  $(\xi_i, \tau_i) \in \Phi(x_i)$  for all but at most finitely many  $i$  such that  $(\xi_i, \tau_i) \rightarrow (\xi, \tau)$ .

It is not generally true that  $\Upsilon$  maps closed subsets of  $\mathfrak{R}^{n+1}$  to closed umbrae. Consider, for instance, the closed subset

$$F = \{(x, \tan x) : x \in [0, \pi/2)\}$$

of  $\mathfrak{R}^2$ . There is no point in  $\Upsilon(F)$  with the  $x$ -coordinate  $\frac{\pi}{2}$ , but for all real  $t$  there are sequences in  $\Upsilon(F)$  that converge to  $(\frac{\pi}{2}, t)$ . We therefore have the following.

**Remark 2.5**  $\Upsilon$  maps  $\mathbf{F}(\mathfrak{R}^{n+1})$  onto a proper subset of  $\mathbf{U}$ .

## 2.2 The Umbra Space $\mathbf{V} \subset \mathbf{F}(\mathfrak{R}^n \times [-\infty, \infty])$

I now turn to the candidate  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$ . Let  $\mathfrak{R}^{(e)}$  have its usual compact topology and let  $\mathcal{E}$  denote the product space  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$ . Denote the points of  $\mathcal{E}$  by  $(x, t)$ , where  $x \in \mathfrak{R}^n$  and  $t \in \mathfrak{R}^{(e)}$ .

**Definition 2.7** *If  $A$  is a nonempty subset of  $\mathcal{E}$ , then the umbra of  $A$ , denoted  $\Upsilon_{\mathcal{E}}(A)$ , is the subset of  $\mathcal{E}$  given by  $\{(x, t) : (x, t') \in A, t \leq t'\}$ ; if  $A = \emptyset$ , let  $\Upsilon_{\mathcal{E}}(A) = \emptyset$ . A subset of  $\mathcal{E}$  that contains  $(x, t)$  whenever it contains  $(x, t')$  with  $t' > t$  is called *umbral*.*

$\Upsilon_{\mathcal{E}}$  maps subsets of  $\mathcal{E}$  to umbral subsets of  $\mathcal{E}$ . Note again that if  $A$  is umbral, then  $\Upsilon_{\mathcal{E}}(A) = A$  and that if  $f \in \mathcal{U}$ , then

$$\Upsilon_{\mathcal{E}}(G_f) = \{(x, t) \in \mathcal{E} : t \leq f(x)\},$$

which is generally not the same as  $\Upsilon(G_f)$ .

**Definition 2.8** *Let  $\mathbf{V}$  denote the set of umbrae in  $\mathbf{F}(\mathcal{E})$  and let  $\epsilon$  denote the hit-miss topology of  $\mathbf{F}(\mathcal{E})$ . Then  $\mathbf{V}$  inherits a topology  $\sigma$  from  $(\mathbf{F}(\mathcal{E}), \epsilon)$  that we will call the *hit-miss topology of  $\mathbf{V}$* .*

$\sigma$  has a subbasic structure similar to that of  $\omega$ .

**Definition 2.9** *If  $A$  is a subset of  $\mathcal{E}$ , then let*

$$\mathbf{V}^A = \{U \in \mathbf{V} : U \cap A = \emptyset\} \text{ and } \mathbf{V}_A = \{U \in \mathbf{V} : U \cap A \neq \emptyset\}.$$

**Remark 2.6**  $\sigma$  is generated subbasically on  $\mathbf{V}$  by

$$\{\mathbf{V}^K : K \in \mathbf{K}(\mathcal{E})\} \cup \{\mathbf{V}_G : G \in \mathbf{G}(\mathcal{E})\}.$$

With a bit more work, we can also obtain the following counterpart of Theorem 2.1 and its corollary.

**Theorem 2.5 (Matheron)**  *$(\mathbf{V}, \sigma)$  is a compact LCS space closed under arbitrary intersections and finite unions,  $(\mathbf{V}, \cap, \cup)$  is a complete distributive lattice with induced ordering  $\subset$ , and  $\Upsilon_{\mathcal{E}}(G_f) \in \mathbf{V} \forall f \in \mathcal{U}$ .*

**Proof** Since  $(\mathbf{V}, \sigma)$  is a subspace of a second countable Hausdorff space, it follows that  $\sigma$  is second countable and Hausdorff. To prove compactness we show that  $\mathbf{V}$  is an  $\epsilon$ -closed subset of  $\mathbf{F}(\mathcal{E})$ . Accordingly, let  $U_i \rightarrow F$  in  $\mathbf{F}(\mathcal{E})$  where each  $U_i \in \mathbf{V}$ . We show that  $F \in \mathbf{V}$ , i.e.,  $(x, t) \in F$  whenever  $(x, t') \in F$  with  $t' > t$ . First assume that  $t'$  and  $t$  are finite and put  $\delta = t' - t$ . There are  $(x_i, t'_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t'_i) \rightarrow (x, t')$ . Thus  $(x_i, t'_i - \delta) \in U_i$  for all but at most finitely many  $i$  and  $(x_i, t'_i - \delta) \rightarrow (x, t' - \delta) = (x, t)$ . Hence  $(x, t) \in F$ . If  $t = -\infty$ , we note that  $(x_i, -\infty) \in U_i$  for all but at most finitely many  $i$  and that  $(x_i, -\infty) \rightarrow (x, -\infty)$ . Hence  $(x, -\infty) \in F$ . If  $t' = \infty$ , there are still  $(x_i, t'_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t'_i) \rightarrow (x, t' = \infty)$ , and we have the two cases  $t \in \mathfrak{R}$  and  $t = -\infty$  to consider. If  $t \in \mathfrak{R}$ , there is clearly a sequence  $\{\tau_i\}$  such that  $\tau_i \leq t'_i$  and  $\tau_i \rightarrow t$ . Thus  $(x_i, \tau_i) \in U_i$  for all but at most finitely many  $i$  and  $(x_i, \tau_i) \rightarrow (x, t) \in F$ . Likewise,  $(x_i, -\infty) \in U_i$  for all but at most finitely many  $i$  and  $(x_i, -\infty) \rightarrow (x, -\infty) \in F$ . We have thus proved that  $F$  is a closed umbra, and this shows that  $\mathbf{V}$  is a closed and therefore compact subset of  $\mathbf{F}(\mathcal{E})$ . Hence  $(\mathbf{V}, \sigma)$  is a compact LCS space.

Because arbitrary intersections and finite unions of umbrae (closed sets) are umbrae (closed sets), it follows that  $\mathbf{V}$  is closed under arbitrary intersections and finite unions. Moreover  $(\mathbf{V}, \cap, \cup)$  is clearly distributive and obviously has the induced ordering  $\subset$ . Recall that a lattice with a universal upper bound in which every subset has an infimum is necessarily complete. Since  $\mathbf{V}$  is closed under arbitrary intersections, it follows that every subset of  $\mathbf{V}$  has an infimum. Completeness thus follows from the evident fact that  $(\mathbf{V}, \cap, \cup)$  has the universal bounds  $\mathcal{E}$  and  $\emptyset$ . To prove that  $\Upsilon_{\mathcal{E}}(G_f)$  is a closed subset of  $\mathbf{F}(\mathcal{E})$  for all  $f \in \mathcal{U}$ , let  $\{(x_i, t_i)\}$  be an  $\mathcal{E}$ -convergent sequence in  $\Upsilon_{\mathcal{E}}(G_f)$  with limit  $(x, t)$ . Since  $f$  is USC and  $f(x_i) \geq t_i$  for all  $i$ , we see that  $f(x) \geq \limsup f(x_i) \geq t$ .

**Corollary 2.2** *Hence we have the following:*

1.  $(\mathbf{V}, \sigma, \subset)$  is a compact ordered space.
2.  $(\mathbf{V}, \sigma, \cap, \cup)$  is a compact closed-order lattice.

**Remark 2.7** Since  $\mathbf{V} \subset \mathbf{F}(\mathcal{E})$  is closed, we have the following characterization of convergence in  $\mathbf{V}$ .

1. If  $U_i \in \mathbf{V}$  and  $U_i \rightarrow U$  in  $\mathbf{F}(\mathcal{E})$ , then  $U \in \mathbf{V}$  and  $U_i \rightarrow U$  in  $\mathbf{V}$ .
2. Conversely, if  $U_i \rightarrow U$  in  $\mathbf{V}$ , then  $U_i \rightarrow U$  in  $\mathbf{F}(\mathcal{E})$ .

Matheron's convergence criteria for sequences in  $\mathbf{V}$  again follow readily from this characterization:

**Theorem 2.6** A sequence  $\{U_i\}$  in  $\mathbf{V}$  converges to  $U \in \mathbf{V}$  if and only if (a) for each  $(x, t) \in U$  there exist  $(x_i, t_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$  and (b) if  $\{U_{i_k}\}$  is a subsequence of  $\{U_i\}$ , then every convergent sequence  $(x_{i_k}, t_{i_k}) \in U_{i_k}$  has its limit in  $U$ .

**Remark 2.8** If  $\{U_i\}$  is a sequence in  $\mathbf{V}$  (a  $\mathbf{V}$ -sequence), then its limit points relative to  $\mathbf{F}(\mathcal{E})$  coincide with its limit points relative to  $\mathbf{V}$ .

**Definition 2.10** Let  $\{U_i\}$  be a sequence in  $\mathbf{V}$  and let  $\mathcal{L}(\{U_i\})$  denote its set of limit points. Then we define  $\underline{\text{Lim}} U_i = \bigcap \{U : U \in \mathcal{L}(\{U_i\})\}$  and  $\overline{\text{Lim}} U_i = \bigcup \{U : U \in \mathcal{L}(\{U_i\})\}$ .

**Remark 2.9** The upper (lower) limit in  $\mathbf{F}(\mathcal{E})$  of a  $\mathbf{V}$ -sequence is equal to its upper (lower) limit in  $\mathbf{V}$ .

With these results we can readily establish the upper-lower limit theorem and the usual semicontinuity criteria (Thm. 2.8) for  $\mathbf{V}$ .

**Theorem 2.7** (Upper-lower limit theorem for  $\mathbf{V}$ ) If  $\{U_i\}$  is a sequence in  $\mathbf{V}$ , then (a)  $\underline{\text{Lim}} U_i$  is the largest  $U \in \mathbf{V}$  that satisfies condition (a) of Theorem 2.6, (b)  $\overline{\text{Lim}} U_i$  is the smallest  $U \in \mathbf{V}$  that satisfies condition (b) of Theorem 2.6, and (c)  $U_i \rightarrow U$  in  $\mathbf{V}$  if and only if

$$\underline{\text{Lim}} U_i = \overline{\text{Lim}} U_i = U.$$

**Theorem 2.8** If  $\Phi : X \rightarrow \mathbf{V}$  and  $X$  is a first countable Hausdorff space, then  $\Phi$  is USC at  $x \in X \iff \Phi(x) \supset \overline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ , and  $\Phi$  is LSC at  $x \in X \iff \Phi(x) \subset \underline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ .

We therefore obtain the following counterpart of Remark 2.4.

**Remark 2.10** *If  $X$  is a first countable Hausdorff space, then*

1.  $\Phi : X \rightarrow V$  is USC  $\iff x_i \rightarrow x$  in  $X$ ,  $(\xi_k, \tau_k) \in \Phi(x_{i_k}) \forall k$ , and  $(\xi_k, \tau_k) \rightarrow (\xi, \tau) \implies (\xi, \tau) \in \Phi(x)$ .
2.  $\Phi : X \rightarrow V$  is LSC  $\iff x_i \rightarrow x$  in  $X$  and  $(\xi, \tau) \in \Phi(x) \implies$  there exist  $(\xi_i, \tau_i) \in \Phi(x_i)$  for all but at most finitely many  $i$  such that  $(\xi_i, \tau_i) \rightarrow (\xi, \tau)$ .

Here the development deviates from that of Section 2.1. Indeed  $\Upsilon_{\mathcal{E}}$  has a more elegant topological relation to the spaces  $F(\mathcal{E})$  and  $V$  than  $\Upsilon$  does to  $F(\mathfrak{R}^{n+1})$  and  $U$ . Compare the following with Remark 2.5.

**Proposition 2.2**  $\Upsilon_{\mathcal{E}}$  is a continuous closed mapping of  $F(\mathcal{E})$  onto  $V$ .

**Proof** First we show that  $\Upsilon_{\mathcal{E}}$  maps  $F(\mathcal{E})$  onto  $V$ . For this it is sufficient to prove that  $\Upsilon_{\mathcal{E}}(F) \in V$  for all  $F \in F(\mathcal{E})$ . Let  $\{(x_i, t_i)\}$  be a convergent sequence in  $\Upsilon_{\mathcal{E}}(F)$  with limit  $(x, t)$ . We show that  $(x, t) \in \Upsilon_{\mathcal{E}}(F)$  and hence that  $\Upsilon_{\mathcal{E}}(F)$  is a closed umbra. Since each  $(x_i, t_i)$  is in  $\Upsilon_{\mathcal{E}}(F)$ , there are  $\tau_i \geq t_i$  such that  $(x_i, \tau_i) \in F$  for all  $i$ . Since  $\{\tau_i\}$  is a sequence in  $\mathfrak{R}^{(e)}$ , it follows that there is a subsequence  $\{\tau_{i_k}\}$  such that  $\tau_{i_k} \rightarrow \tau \geq t$ . Thus  $(x_{i_k}, \tau_{i_k}) \rightarrow (x, \tau)$  and since  $F$  is closed, it follows that  $(x, \tau) \in F$ . Since  $\tau \geq t$ , we may therefore conclude that  $(x, t) \in \Upsilon_{\mathcal{E}}(F)$ . Thus  $\Upsilon_{\mathcal{E}}$  maps  $F(\mathcal{E})$  onto  $V$ . To prove that this mapping is continuous, we let  $\{F_i\}$  be a convergent sequence in  $F(\mathcal{E})$  with limit  $F$ , and demonstrate that  $\Upsilon_{\mathcal{E}}(F_i) \rightarrow \Upsilon_{\mathcal{E}}(F)$  in  $V$  by appeal to Theorem 2.6.

Suppose that  $(x_{i_k}, t_{i_k}) \in \Upsilon_{\mathcal{E}}(F_{i_k})$  for all  $k$  and  $(x_{i_k}, t_{i_k}) \rightarrow (x, t)$ . By showing that  $(x, t) \in \Upsilon_{\mathcal{E}}(F)$ , we establish condition (b) of Theorem 2.6. There are  $\tau_{i_k} \geq t_{i_k}$  such that  $(x_{i_k}, \tau_{i_k}) \in F_{i_k}$  for all  $k$ . Since  $\{\tau_{i_k}\}$  has a convergent subsequence with limit  $\tau \geq t$ , and since  $F_i \rightarrow F$  in  $F(\mathcal{E})$ , it follows that  $(x, \tau) \in F$  and hence that  $(x, t) \in \Upsilon_{\mathcal{E}}(F)$ . To establish condition (a) of Theorem 2.6, suppose that  $(x, t) \in \Upsilon_{\mathcal{E}}(F)$ . Then  $(x, \tau) \in F$  for some  $\tau \geq t$ , and it follows that there are  $(x_i, \tau_i) \in F_i$  for all but at most finitely many  $i$  such that  $(x_i, \tau_i) \rightarrow (x, \tau)$ . If  $\tau = t$ , we are done. If  $t = -\infty$ , then  $(x_i, -\infty) \in \Upsilon_{\mathcal{E}}(F_i)$  for all but at most finitely many  $i$  and  $(x_i, -\infty) \rightarrow (x, -\infty) = (x, t)$  and we are again done. If  $\tau = \infty > t > -\infty$ , it is clear that  $(x_i, t) \in \Upsilon_{\mathcal{E}}(F_i)$  for all but at most finitely many  $i$  and  $(x_i, t) \rightarrow (x, t)$ . The remaining case requiring consideration is  $\infty > \tau > t > -\infty$ . Let  $\tau - t = \delta > 0$ . Then  $(x_i, \tau_i - \delta) \in \Upsilon_{\mathcal{E}}(F_i)$  for all but at most finitely many  $i$  and  $(x_i, \tau_i - \delta) \rightarrow (x, t)$ . Hence  $\Upsilon_{\mathcal{E}}$  maps  $F(\mathcal{E})$  continuously onto  $V$ .

We note that a mapping is called *closed* if the image of each closed domain set is closed in the range of the mapping. Now every closed subset of  $F(\mathcal{E})$  is compact because  $F(\mathcal{E})$  is a compact space. It is, moreover, generally true that a continuous image of a compact set is compact and that the compact subsets of a Hausdorff space are closed. Since  $V$  is a Hausdorff space, it is accordingly clear that  $\Upsilon_{\mathcal{E}}$  is a closed mapping, as stated. This completes the proof.

The continuous umbra mapping of  $F(\mathcal{E})$  onto  $V$  is an instance of the general situation that leads to the concept of a *quotient topology*. In fact,  $\sigma$  is the quotient topology of  $V$  relative to the mapping  $\Upsilon_{\mathcal{E}}$  and the topology  $\epsilon$  of its domain. This can be seen as follows.

**Definition 2.11** *Let  $(X, \tau)$  be a topological space, let  $Y$  be a set, and let  $\Lambda$  map  $X$  onto  $Y$ . Then the quotient topology of  $Y$  relative to  $\Lambda$  and  $\tau$  is the strongest topology on  $Y$  for which  $\Lambda$  is continuous.*

From general topology<sup>10</sup> we have the following result.

**Theorem 2.9** *If  $\Lambda$  is a continuous mapping of a topological space  $(X, \tau)$  onto a topological space  $(Y, \tau')$  such that  $\Lambda$  is either a closed or an open mapping, then  $\tau'$  is the quotient topology.*

**Corollary 2.3**  *$\sigma$  is the quotient topology of  $V$  relative to  $\Upsilon_{\mathcal{E}}$  and  $\epsilon$ .*

## 2.3 The Umbra Subspace $\widehat{V} \subset V$

**Definition 2.12** *The closed supports in  $\mathbb{R}^n$  of the members of  $U$ ,  $V$ , and  $\mathcal{U}$ , together with related notions, are defined as follows.*

1. *If  $U \in \mathcal{U}$ , then the support of  $U$ , denoted  $\Delta U$ , is the closed subset  $\overline{\{x : (x, t) \in U\}}$  of  $\mathbb{R}^n$ .*
2. *If  $U \in V$ , then the support of  $U$ , denoted  $\Delta U$ , is the closed subset  $\overline{\{x : (x, t) \in U, t > -\infty\}}$  of  $\mathbb{R}^n$ .*
3. *If  $f \in \mathcal{U}$ , then the support of  $f$ , denoted  $\Delta f$ , is the closed subset  $\overline{\{x : f(x) > -\infty\}}$  of  $\mathbb{R}^n$ .*
4. *If  $U \in V$ , then the subset  $\dot{U}$  of  $U$  defined by*

$$\dot{U} = \{(x, t) : (x, t) \in U, x \in \Delta U\}$$

*will be called the core of  $U$ , and the supset  $\ddot{U} = U \cup (\mathbb{R}^n, -\infty)$  of  $U$  will be called the full augmentation of  $U$ .*

5. When  $\Delta U \subset \mathfrak{R}^n$  is compact, we say that  $U$  has compact support.
6. If  $f \in \mathcal{U}$  and  $\Delta f \subset \mathfrak{R}^n$  is compact, we likewise say that  $f$  has compact support.

**Remark 2.11** If  $U$  and  $V$  are umbrae in  $\mathbf{V}$ , then

1.  $\dot{U}$  and  $U$  are in  $\mathbf{V}$ .
2.  $\dot{U} \subset V \subset \ddot{U} \implies \dot{V} = \dot{U}$  and  $\ddot{V} = \ddot{U}$ .
3.  $\dot{U} = \ddot{U} \iff \dot{U} = \ddot{U}$ .

**Definition 2.13** Let  $\widehat{\mathbf{V}}$  be the set of distinct full augmentations of the umbrae in  $\mathbf{V}$ , and let  $\widehat{\sigma}$  be the relative hit-miss topology of  $\widehat{\mathbf{V}}$  in  $\mathbf{V}$ . If  $A$  is a subset of  $\mathcal{E}$ , then let

$$\widehat{\mathbf{V}}^A = \{U \in \widehat{\mathbf{V}} : U \cap A = \emptyset\} \text{ and } \widehat{\mathbf{V}}_A = \{U \in \widehat{\mathbf{V}} : U \cap A \neq \emptyset\}.$$

**Remark 2.12** The topology  $\widehat{\sigma}$  is generated subbasically on  $\widehat{\mathbf{V}}$  by

$$\{\widehat{\mathbf{V}}^K : K \in \mathbf{K}(\mathcal{E})\} \cup \{\widehat{\mathbf{V}}_G : G \in \mathbf{G}(\mathcal{E})\}.$$

Now observe that  $\widehat{\mathbf{V}} \subset \mathbf{V} \subset \mathbf{F}(\mathcal{E})$  and that each space is a closed subspace of one(s) following it. We therefore obtain a theorem and corollary for the space  $\widehat{\mathbf{V}}$  that are virtually identical to Theorems 2.1 and 2.5 and their corollaries, namely:

**Theorem 2.10**  $(\widehat{\mathbf{V}}, \widehat{\sigma})$  is a compact LCS space closed under arbitrary intersections and finite unions,  $(\widehat{\mathbf{V}}, \cap, \cup)$  is a complete distributive lattice with induced ordering  $\subset$ , and  $\Upsilon_{\mathcal{E}}(G_f) \in \widehat{\mathbf{V}}$  for all  $f \in \mathcal{U}$ .

**Corollary 2.4** Hence we have the following:

1.  $(\widehat{\mathbf{V}}, \widehat{\sigma}, \subset)$  is a compact ordered space.
2.  $(\widehat{\mathbf{V}}, \widehat{\sigma}, \cap, \cup)$  is a compact closed-order lattice.

Indeed the rest of the material following Corollary 2.2 up to and including Remark 2.10 continues to be valid for the space  $(\widehat{\mathbf{V}}, \widehat{\sigma})$  relative to both  $\mathbf{F}(\mathcal{E})$  and  $\mathbf{V}$ . Now consider the following.

**Definition 2.14** Let  $\Lambda$  be the mapping of  $\mathbf{V}$  onto  $\widehat{\mathbf{V}}$  given by  $U \mapsto \dot{U}$ .

**Proposition 2.3**  $\Lambda$  is a continuous closed mapping of  $\mathbf{V}$  onto  $\widehat{\mathbf{V}}$ .

**Proof** To prove continuity, first suppose that  $(x, t) \in \bar{U}$ . If  $(x, t) \in U$ , then since  $U_i \rightarrow U$  in  $\mathbf{V}$ , there are  $(x_i, t_i) \in U_i \subset \bar{U}_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$ . If  $(x, t) \notin U$ , then  $(x, t) = (x, -\infty) \in \bar{U}_i$  for all  $i$ . On the other hand, if  $(x_{i_k}, t_{i_k}) \in \bar{U}_{i_k}$  and  $(x_{i_k}, t_{i_k}) \rightarrow (x, t)$ , then either infinitely many of the  $(x_{i_k}, t_{i_k})$  are in the corresponding  $U_{i_k}$ , in which case  $(x, t) \in U \subset \bar{U}$ , or all but finitely many of the  $(x_{i_k}, t_{i_k}) = (x_{i_k}, -\infty) \rightarrow (x, -\infty) \in \bar{U}$ . This proves continuity.

Because  $\widehat{\mathbf{V}}$  is a Hausdorff space, we again see that the closedness of  $\Lambda$  follows from the compactness of the domain space  $\mathbf{V}$  and the facts that (1) a continuous image of a compact set is compact and (2) the compact subsets of a Hausdorff space are closed.

**Corollary 2.5** *Hence we have the following:*

1.  $\hat{\sigma}$  is the quotient topology of  $\widehat{\mathbf{V}}$  relative to  $\Lambda$  and  $\sigma$ .
2.  $\Lambda \circ \Upsilon_{\mathcal{E}}$  is a continuous closed mapping of  $\mathbf{F}(\mathcal{E})$  onto  $\widehat{\mathbf{V}}$ .

## 2.4 Topological Equivalence of $\mathbf{U}$ and $\widehat{\mathbf{V}}$

If  $U \in \mathbf{U}$ , let  $\Omega(U) = \bar{U} \cup \Pi(-\infty)$ , where the overbar denotes closure in  $\mathcal{E}$ , and  $\Pi(-\infty)$  denotes the subset  $(\mathfrak{R}^n, -\infty)$  of  $\mathcal{E}$ .

**Remark 2.13**  $\Omega$  is one-to-one onto  $\widehat{\mathbf{V}}$ , and if  $V \in \widehat{\mathbf{V}}$ , then

$$\Omega^{-1}(V) = \{(x, t) \in V : t \in \mathfrak{R}\}.$$

**Lemma 2.1**  $\Omega^{-1}$  is continuous, and  $\Omega$  is USC.

**Proof** Let  $V_i \rightarrow V$  in the  $\hat{\sigma}$ -topology of  $\widehat{\mathbf{V}}$ . Then if  $t \in \mathfrak{R}$  and  $(x, t) \in V$ , it follows that there are  $(x_i, t_i) \in V_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$ . Moreover, since  $t$  is real, all but at most finitely many of the  $t_i$  must be real. Hence  $\{\Omega^{-1}(V_i)\}$  and  $\Omega^{-1}(V)$  satisfy condition (a) of Theorem 2.2. Suppose, then, that  $(x_k, t_k) \in \Omega^{-1}(V_{i_k})$  for all  $k$  and that  $(x_k, t_k) \rightarrow (x, t)$  in  $\mathfrak{R}^{n+1}$ . Then  $(x, t) \in V$  and  $t \in \mathfrak{R}$ , and it follows that  $\Omega^{-1}(V_i) \rightarrow \Omega^{-1}(V)$  in the  $\omega$ -topology of  $\mathbf{U}$ . Hence  $\Omega^{-1}$  is continuous.

For the second part, let  $U_i \rightarrow U$  in the  $\omega$ -topology of  $\mathbf{U}$ , and suppose that  $(x_k, t_k) \in \Omega(U_{i_k}) = \bar{U}_{i_k} \cup \Pi(-\infty)$  for all  $k$ . We show that if  $(x_k, t_k) \rightarrow (x, t)$  in  $\mathcal{E}$ , then  $(x, t) \in \Omega(U) = \bar{U} \cup \Pi(-\infty)$ . If  $t = -\infty$ , there is nothing to prove. If  $t \in \mathfrak{R}$ , then all but

finitely many of the  $t_k \in \mathfrak{R}$ , and it follows that  $(x, t) \in U \subset \Omega(U)$ . Suppose, finally, that  $t = \infty$ . Since either infinitely many of the  $t_k$  are real, or all but finitely many of them are  $\infty$ , we have without loss of generality the following two cases to consider: (1)  $t_k \in \mathfrak{R}$  for all  $k$  and (2)  $t_k = \infty$  for all  $k$ . In both cases, given any  $\tau \in \mathfrak{R}$ , there is a real sequence  $\{\tau_k\}$  such that  $(x_k, \tau_k) \in U_{i_k}$  and  $(x_k, \tau_k) \rightarrow (x, \tau)$ . Hence  $(x, \tau) \in U$  for all real  $\tau$ , and it follows that  $(x, \infty) \in \bar{U}$ . This completes the proof.

**Theorem 2.11**  $\Omega$  is a homeomorphism of  $U$  onto  $\widehat{V}$ .

**Proof** By the last lemma, it is sufficient to show that  $\Omega$  is LSC. Let  $U_i \rightarrow U$  in the  $\omega$ -topology of  $U$  and let  $(x, t) \in \Omega(U)$ . It is clear that if  $t \neq \infty$ , then there exist  $(x_i, t_i) \in \Omega(U_i)$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$  in  $\mathcal{E}$ . The rest of the proof is to show that the same implication holds when  $t = \infty$ . For this, the following is first established: *If  $B_\epsilon(x)$  is the open ball in  $\mathfrak{R}^n$  with radius  $\epsilon > 0$  and center at  $x$ , and if  $(x, \infty) \in \Omega(U)$ , then for each positive  $\epsilon$  and  $\lambda$ , it follows that  $B_\epsilon(x) \times (\lambda, \infty] \cap U_i \neq \emptyset$  for all but at most finitely many  $i$ .* To see this, note that  $(x, \infty) \in \bar{U}$  implies that  $(x, t) \in U$  for all  $t \in \mathfrak{R}$ . Hence the open set  $S_{\epsilon, \delta}(x, t) \equiv B_\epsilon(x) \times (t - \delta, t + \delta)$  hits all but at most finitely many of the  $U_i$  for all positive  $\epsilon$  and  $\delta$ , because  $U_i \rightarrow U$  in the  $\omega$ -topology of  $U$ . The italicized assertion now follows from the fact that  $B_\epsilon(x) \times (\lambda, \infty] \supset S_{\epsilon, \delta}(x, t)$  for some real  $t$  and positive  $\delta$ .

For each positive integer  $k$ , let  $\epsilon_k = k^{-1}$  and let  $\lambda_k = k$ . By what we have just shown, for each  $k$  there is a positive integer  $I_k$  such that  $U_i \cap B_{\epsilon_k}(x) \times (\lambda_k, \infty] \neq \emptyset$  for all  $i \geq I_k$ . For

$$p = I_1, I_1 + 1, \dots, I_2 - 1$$

let  $(x_p, t_p) \in U_p \cap B_{\epsilon_1}(x) \times (\lambda_1, \infty]$ . For

$$p = I_2, I_2 + 1, \dots, I_3 - 1$$

let  $(x_p, t_p) \in U_p \cap B_{\epsilon_2}(x) \times (\lambda_2, \infty]$ . Continuing in this way, we define a sequence  $\{(x_i, t_i)\}$ ,  $i \geq I_1$  such that

$$(x_p, t_p) \in U_p \cap B_{\epsilon_k}(x) \times (\lambda_k, \infty] \text{ for } p = I_k, I_k + 1, \dots, I_{k+1} - 1.$$

Since this sequence has  $\mathcal{E}$ -limit  $(x, \infty)$ , the proof is complete.

## 2.5 Hit-Miss Topology of $\mathcal{U}$

I now prove two very similar theorems (given by Serra,<sup>6</sup> but without distinguishing  $U$  from  $\widehat{V}$ ) that provide us with one-to-one mappings,  $\Gamma$  and  $\widehat{\Gamma}$ , of  $U$  and  $\widehat{V}$  onto  $\mathcal{U}$ . Each of these bijections induces upon  $\mathcal{U}$  the topology that makes the bijection a homeomorphism. In view of Theorem 2.11, the two topologies induced in this way must be identical, and this is how the *hit-miss topology of  $\mathcal{U}$*  is obtained. We begin with some preliminaries. We call the set  $\Pi(t) \equiv (\mathfrak{R}^n, t)$  the *horizontal plane* in  $\mathfrak{R}^{n+1}$  (respectively  $\mathcal{E}$ ) at *altitude  $t$* . If we let  $\Pi_n$  denote the projection mapping  $\Pi_n((x, t)) = x$  for all  $(x, t) \in \mathfrak{R}^{n+1}$  (respectively  $\mathcal{E}$ ), we then obtain the following result.

**Proposition 2.4**  $\Pi_n$  is a closed mapping of  $\mathcal{E}$  to  $\mathfrak{R}^n$ ; i.e., it maps closed subsets of  $\mathcal{E}$  to closed subsets of  $\mathfrak{R}^n$ .

**Proof** Let  $F \in \mathbf{F}(\mathcal{E})$  and let  $\{x_i\}$  be a convergent sequence in  $\Pi_n(F)$  with limit  $x$ . Then it follows that there are  $t_i \in \mathfrak{R}^{(e)}$  such that  $(x_i, t_i) \in F$  for all  $i$ . Since  $\{t_i\}$  is a sequence in a compact space, it follows that there is a subsequence  $\{i_k\}$  such that  $t_{i_k} \rightarrow t \in \mathfrak{R}^{(e)}$ . Therefore  $(x_{i_k}, t_{i_k}) \rightarrow (x, t) \in F$  because  $F$  is closed. Consequently  $x \in \Pi_n(F)$ , and it follows that  $\Pi_n(F)$  is closed in  $\mathfrak{R}^n$ .

Note, however, that  $\Pi_n$  is not a closed mapping of  $\mathfrak{R}^{n+1}$  to  $\mathfrak{R}^n$ .

**Theorem 2.12** Each  $U \in \mathcal{U}$  induces an  $f_U \in \mathcal{U}$  by means of the formula  $f_U(x) = \sup\{t : (x, t) \in U\}$  ( $x \in \mathfrak{R}^n$ ). Moreover,  $\Upsilon(G_{f_U}) = U$ ,  $\Delta f_U = \Delta U$ , and  $X_t(f_U) = \Pi_n(U \cap \Pi(t)) \forall t \in \mathfrak{R}$ .

**Proof** The supremum formula clearly defines an ERV function on  $\mathfrak{R}^n$ . We must show that  $f_U$  is USC. We do this by showing that the horizontal cross sections of  $f_U$  are closed in  $\mathfrak{R}^n$ . Note that  $X_\tau(f_U) = \{x : f_U(x) \geq \tau\}$ . Thus if  $\tau = -\infty$ , then  $X_\tau(f_U) = \mathfrak{R}^n$ , which is clearly closed. If  $\tau \in \mathfrak{R}$ , then

$$X_\tau(f_U) = \{x : \sup\{t : (x, t) \in U\} \geq \tau\} =$$

$$\{x : (x, t) \in U \text{ for some } t \geq \tau\} = \{x : (x, \tau) \in U\}.$$

Since  $\{x : (x, \tau) \in U\} = \Pi_n(U \cap \Pi(\tau))$ , we see that

$$X_t(f_U) = \Pi_n(U \cap \Pi(t)) \forall t \in \mathfrak{R}.$$

Let  $\{x_i\}$  be a convergent sequence in  $\mathfrak{R}^n$  such that  $(x_i, \tau) \in U$  for all  $i$  and let  $x_i \rightarrow x$ . Then since  $(x_i, \tau) \rightarrow (x, \tau)$  in  $\mathfrak{R}^{n+1}$  and  $U$

is closed, it follows that  $(x, \tau) \in U$ . Hence  $X_\tau(f_U)$  is also closed when  $\tau$  is real. If  $\tau = \infty$ , then  $X_\infty(f_U) = \{x : f_U(x) = \infty\}$ . Let  $\{x_i\}$  be a convergent sequence in  $\mathfrak{R}^n$  such that  $f_U(x_i) = \infty$  for all  $i$  and let  $x_i \rightarrow x$ . Given any real  $\tau$ , there are  $t_i$  such that  $(x_i, t_i) \in U$  for all  $i$  and  $(x_i, t_i) \rightarrow (x, \tau)$ . Thus  $(x, \tau) \in U$  for all  $\tau \in \mathfrak{R}$ , and it follows that  $f_U(x) = \infty$ ; hence  $X_\infty(f_U)$  is closed and the proof that  $f_U \in \mathcal{U}$  is complete.

By definition,

$$\begin{aligned} \Upsilon(G_{f_U}) &= \{(x, t) \in \mathfrak{R}^{n+1} : f_U(x) \geq t\} = \\ &= \{(x, t) \in \mathfrak{R}^{n+1} : \sup\{\tau : (x, \tau) \in U\} \geq t\}. \end{aligned}$$

It is therefore clear that  $\Upsilon(G_{f_U}) \supset U$ . On the other hand, if  $(x, t) \in \Upsilon(G_{f_U})$ , then  $\sup\{\tau : (x, \tau) \in U\} \geq t$ . If  $t$  is less than the supremum, then there is a  $\tau > t$  such that  $(x, \tau) \in U$ , and we see that  $(x, t) \in U$ . If  $t = \sup\{\tau : (x, \tau) \in U\}$ , then there is a sequence  $\{t_i\}$  such that  $(x, t_i) \in U$  for all  $i$  and  $t_i \rightarrow t$ . Hence  $(x, t) \in U$ , because  $U$  is closed, and  $\Upsilon(G_{f_U}) = U$ .

We must finally show that  $\Delta f_U = \Delta U$ . We have by definition that  $\Delta f_U = \overline{\{x : f_U(x) > -\infty\}}$  and  $\Delta U = \overline{\{x : (x, t) \in U\}}$ . It is clear that if  $f_U(x) > -\infty$ , then there is a real  $t$  such that  $(x, t) \in U$ . Thus  $\{x : f_U(x) > -\infty\} \subset \{x : (x, t) \in U\}$ . It is equally clear that  $(x, t) \in U \implies f_U(x) > -\infty$ , since  $t$  is real. Hence  $\{x : f_U(x) > -\infty\} = \{x : (x, t) \in U\}$ , and the proof is complete.

**Corollary 2.6** *The mapping  $\Gamma : U \mapsto f_U$  of  $\mathbf{U}$  is one-to-one onto  $\mathcal{U}$ .*

**Theorem 2.13** *Each  $U \in \mathbf{V}$  induces an  $f_U \in \mathcal{U}$  by means of the formula  $f_U(x) = \sup\{t : (x, t) \in U\}$  ( $x \in \mathfrak{R}^n$ ). Also,  $\Upsilon_{\mathcal{E}}(G_{f_U}) = \ddot{U}$ ,  $\Delta f_U = \Delta U$ , and  $X_t(f_U) = \Pi_n(\ddot{U} \cap \Pi(t)) \forall t \in \mathfrak{R}^{(e)}$ .*

**Proof** The supremum formula continues to define an ERV function on  $\mathfrak{R}^n$ . We show that  $f_U$  is USC by showing that the horizontal cross sections of  $f_U$  are closed in  $\mathfrak{R}^n$ . We again have that  $X_{-\infty}(f_U) = \mathfrak{R}^n$ , which is clearly closed. If  $\tau \in \mathfrak{R}$ , then  $X_\tau(f_U) = \Pi_n(U \cap \Pi(\tau))$  as before. Since  $U \cap \Pi(\tau)$  is closed in  $\mathcal{E}$ , it follows that  $\Pi_n(U \cap \Pi(\tau))$  is closed in  $\mathfrak{R}^n$ . If  $\tau = \infty$ , then again

$$X_\infty(f_U) = \{x : f_U(x) = \infty\} = \{x : (x, \infty) \in U\} = \Pi_n(U \cap \Pi(\infty)),$$

which is clearly closed. Hence  $f_U \in \mathcal{U}$ . It is now obvious that

$$X_t(f_U) = \Pi_n(\ddot{U} \cap \Pi(t)) \forall t \in \mathfrak{R}^{(e)}.$$

By definition,

$$\begin{aligned} \Upsilon_{\mathcal{E}}(G_{f_U}) &= \{(x, t) \in \mathcal{E} : f_U(x) \geq t\} = \\ &= \{(x, t) \in \mathcal{E} : \sup\{\tau : (x, \tau) \in U\} \geq t\}. \end{aligned}$$

Therefore  $\Upsilon_{\mathcal{E}}(G_{f_U}) \supset U$ , and, since  $\Upsilon_{\mathcal{E}}(G_{f_U}) \supset \Pi(-\infty)$ , we see that  $\Upsilon_{\mathcal{E}}(G_{f_U}) \supset \bar{U}$ . On the other hand, if  $(x, t) \in \Upsilon_{\mathcal{E}}(G_{f_U})$ , then  $\sup\{\tau : (x, \tau) \in U\} \geq t$ . If  $t = -\infty$ , it is clear that  $(x, t) \in \bar{U}$ . If  $t > -\infty$  and less than the supremum, then there is a  $\tau > t$  such that  $(x, \tau) \in U$ , and we see that  $(x, t) \in U$ . If  $t = \sup\{\tau : (x, \tau) \in U\}$ , then there is a sequence  $\{t_i\}$  such that  $(x, t_i) \in U$  for all  $i$  and  $t_i \rightarrow t$ . Hence  $(x, t) \in U$ , because  $U$  is closed, and it follows that  $\Upsilon_{\mathcal{E}}(G_{f_U}) = \bar{U}$ .

We must finally show that  $\Delta f_U = \Delta U$ . We have by definition that  $\Delta f_U = \{x : f_U(x) > -\infty\}$  and  $\Delta U = \{x : (x, t) \in U, t > -\infty\}$ . It is clear that if  $f_U(x) > -\infty$ , then there is a real  $t$  such that  $(x, t) \in U$ . Thus  $\{x : f_U(x) > -\infty\} \subset \{x : (x, t) \in U, t > -\infty\}$ . It is equally clear that  $t > -\infty$  and  $(x, t) \in U \implies f_U(x) > -\infty$ . Hence  $\{x : f_U(x) > -\infty\} = \{x : (x, t) \in U, t > -\infty\}$ .

**Corollary 2.7** *The mapping  $\hat{\Gamma} : U \mapsto f_U$  of  $\mathbf{V}$  restricted to  $\hat{\mathbf{V}}$  is one-to-one onto  $\mathcal{U}$ .*

We now have two ways to endow  $\mathcal{U}$  with a hit-miss topology and thereby obtain a setting in which the  $\mathcal{M}$ -transforms of the functions in  $\mathcal{U}$  can be defined similarly to the  $\mathcal{M}$ -transformations of  $\mathbf{F}(\mathbb{R}^n)$ . We may give  $\mathcal{U}$  the hit-miss topology of either  $U$  or  $\hat{\mathbf{V}}$  by declaring either  $\Gamma$  or  $\hat{\Gamma}$  a homeomorphism. But we have already seen that the topologies acquired by  $\mathcal{U}$  in this way are identical. Indeed we have the following.

**Proposition 2.5**  $\Omega = \hat{\Gamma}^{-1} \circ \Gamma$ .

**Proof** If  $U \in \mathcal{U}$ , then  $\hat{\Gamma}^{-1} \circ \Gamma(U) = \hat{\Gamma}^{-1}(f_U) \supset \bar{U} \cup \Pi(-\infty)$ . Moreover, for  $(x, t)$  to be in  $\hat{\Gamma}^{-1}(f_U)$ , it must be true that  $f_U(x) \geq t$ . If  $(x, t) \notin \bar{U} \supset U$  and  $t > -\infty$ , then it follows that

$$f_U(x) = \sup\{\tau : (x, \tau) \in U\} < t.$$

Thus  $(x, t) \notin \hat{\Gamma}^{-1}(f_U)$  and the proposition follows.

**Corollary 2.8** *Let  $\gamma$  denote the topology induced on  $\mathcal{U}$  by declaring  $\Gamma$  a homeomorphism. If  $\mathcal{U}$  is carrying the  $\gamma$ -topology, then  $\hat{\Gamma}$  is a homeomorphism of  $\hat{\mathbf{V}}$  onto  $\mathcal{U}$ .*

With respect to  $\gamma$ , then,  $\Gamma$  and  $\widehat{\Gamma}$  are homeomorphisms of  $U$  and  $\widehat{V}$ , respectively, onto  $\mathcal{U}$ . Henceforth we call  $\gamma$  the *hit-miss topology of  $\mathcal{U}$* . To clarify the character of the space  $(\mathcal{U}, \gamma)$ , we will use the homeomorphism  $\widehat{\Gamma}$  to translate the theory of convergence and semicontinuity from  $(\widehat{V}, \widehat{\sigma})$  to  $(\mathcal{U}, \gamma)$ . In doing this we focus on establishing the  $(\mathcal{U}, \gamma)$ -versions of *Matheron's convergence criteria* (Cor. 2.9 and Thm. 2.15), the definition of upper and lower limits (Def. 2.15), the *upper-lower limit theorem* (Thm. 2.16), and the *usual semicontinuity criteria* (Thm. 2.17). The  $\gamma$ -convergence of a sequence  $\{f_i\}$  in  $\mathcal{U}$  (a  $\mathcal{U}$ -sequence) to an  $f \in \mathcal{U}$  will be denoted  $f_i \rightarrow f = \lim f_i$ . First I state, for reference, the  $\widehat{V}$ -version of Matheron's convergence criteria.

**Theorem 2.14** *A sequence  $\{U_i\}$  in  $\widehat{V}$  converges to  $U \in \widehat{V}$  if and only if (a) for each  $(x, t) \in U$  there exist  $(x_i, t_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$  and (b) if  $\{U_{i_k}\}$  is a subsequence of  $\{U_i\}$ , then every  $\mathcal{E}$ -convergent sequence  $(x_{i_k}, t_{i_k}) \in U_{i_k}$  has its limit in  $U$ .*

If we put  $f_i = \widehat{\Gamma}(U_i)$  and  $f = \widehat{\Gamma}(U)$ , then this result can be restated in terms of the  $f_i$  and  $f$  as follows.

**Corollary 2.9** *If  $\{f_i\}$  is a  $\mathcal{U}$ -sequence and  $f \in \mathcal{U}$ , then  $f_i \rightarrow f$  in  $\mathcal{U}$  if and only if both of the following hold:*

(i) *For each  $(x, t) \in \mathcal{E}$  such that  $t \leq f(x)$ , there exist  $(x_i, t_i) \in \mathcal{E}$  for all but at most finitely many  $i$  such that*

$$t_i \leq f_i(x_i) \text{ and } (x_i, t_i) \rightarrow (x, t) \text{ in } \mathcal{E}.$$

(ii) *If  $\{f_{i_k}\}$  is a subsequence of  $\{f_i\}$ , then  $x_{i_k} \rightarrow x \in \mathbb{R}^n$ ,  $t_{i_k} \leq f_{i_k}(x_{i_k}) \forall k$ , and  $t_{i_k} \rightarrow t$  in  $\mathbb{R}^{(e)}$  together imply that  $t \leq f(x)$ .*

I will call conditions (i) and (ii) *Matheron's first and second convergence criteria* for  $(\mathcal{U}, \gamma)$ . Serra<sup>6</sup> has indicated the somewhat different pair of convergence criteria given in the next theorem.

**Theorem 2.15** *If  $\{f_i\}$  is a  $\mathcal{U}$ -sequence and  $f \in \mathcal{U}$ , then  $f_i \rightarrow f$  if and only if (a) for each  $x \in \mathbb{R}^n$  there exist  $x_i \in \mathbb{R}^n$  such that  $x_i \rightarrow x$  and  $f_i(x_i) \rightarrow f(x)$  and (b) if  $\{f_{i_k}\}$  is a subsequence of  $\{f_i\}$ , then  $x_{i_k} \rightarrow x$  in  $\mathbb{R}^n$  implies that  $\limsup f_{i_k}(x_{i_k}) \leq f(x)$ .*

**Proof** By applying Corollary 2.9, we first show that (ii) and (b) are equivalent. For (ii)  $\implies$  (b), let  $f_{i_k}$  be a subsequence of  $f_i$ , let  $x_{i_k} \rightarrow x$ , let  $f_{i_k}(x_{i_k}) \rightarrow t$ , and put  $f_{i_{k_j}}(x_{i_{k_j}}) = t_{i_{k_j}}$ . It is now an immediate consequence of (ii) that  $\limsup f_{i_k}(x_{i_k}) \leq f(x)$ . Hence (ii)  $\implies$  (b). For (b)  $\implies$  (ii), let  $f_{i_k}$  be a subsequence of  $f_i$  and let  $(x_{i_k}, t_{i_k}) \rightarrow (x, t)$  with  $t_{i_k} \leq f_{i_k}(x_{i_k})$  for all  $k$ . Since by (b)  $f(x) \geq \limsup f_{i_k}(x_{i_k})$  and  $\limsup f_{i_k}(x_{i_k}) \geq \limsup t_{i_k} = t$ , it is clear that (b)  $\implies$  (ii).

We now show that (i) and (ii) imply (a). Let  $f_i \rightarrow f$  in  $\mathcal{U}$  and choose  $t = f(x)$  in (i) so that  $t_i \rightarrow f(x)$ . Since  $t_i \leq f_i(x_i)$ , we have that  $\limsup f_i(x_i) \geq \liminf f_i(x_i) \geq f(x)$ . If  $\limsup f_i(x_i) > f(x)$ , then  $x_i$  has a subsequence  $x_{i_k}$  such that  $f_{i_k}(x_{i_k}) \rightarrow \tau > f(x)$ . Since this contradicts (ii), we have

$$f(x) = \limsup f_i(x_i) \geq \liminf f_i(x_i) \geq f(x).$$

Hence  $f_i(x_i) \rightarrow f(x)$ .

We finally show that (a) implies (i). First suppose that  $t = f(x)$  in (i), let  $x_i$  be the sequence converging to  $x$  given by (a), and put  $t_i = f_i(x_i)$ . Thus (i) follows from (a) in this case. Continuing with the same  $x_i$ , suppose that  $t < f(x) < \infty$  in (i). Then we can choose (putting  $\delta = f(x) - t$ )  $t_i = f_i(x_i) - \delta \rightarrow f(x) - \delta = t$ . Thus (i) again follows from (a). Finally, if  $t < f(x) = \infty$  in (i), then there are  $t_i \leq f_i(x_i)$  such that  $t_i \rightarrow t$  because  $f_i(x_i)$  is tending to  $\infty$ . This completes the proof.

**Corollary 2.10** *The following are results of the proof just given.*

1. Condition (a)  $\implies$  Matheron's first convergence criterion.
2. Condition (b)  $\iff$  Matheron's second convergence criterion.

The  $\widehat{\mathcal{V}}$ -version of Definition 2.10 gives us the following definition of upper and lower limits. (Note that the (partial) ordering  $\leq$  defined in  $\mathcal{U}$  by  $f \leq g \iff f(x) \leq g(x) \forall x \in \mathfrak{R}^n$  makes  $(\mathcal{U}, \leq)$  a complete poset. If  $\{f_\alpha\}$  is any set of functions in  $\mathcal{U}$ , then  $\inf\{f_\alpha\}$  (the greatest lower bound of  $\{f_\alpha\}$  relative to  $\leq$ ) is given for each  $x \in \mathfrak{R}^n$  by  $\inf\{f_\alpha(x)\}$ , i.e., by the pointwise infimum. It is not generally true, however, that  $\sup\{f_\alpha\}$  (the least upper bound of  $\{f_\alpha\}$  relative to  $\leq$ ) is given by the pointwise supremum.)

**Definition 2.15** *Let  $\{f_i\}$  be a sequence in  $\mathcal{U}$ , and let  $\mathcal{L}(\{f_i\})$  denote its set of limit points. Then for each  $x \in \mathfrak{R}^n$  we define*

$$(\underline{\text{Lim}} f_i)(x) = \inf\{f(x) : f \in \mathcal{L}(\{f_i\})\}$$

and

$$(\overline{\text{Lim}} f_i)(x) = \sup\{f(x) : f \in \mathcal{L}(\{f_i\})\}.$$

$\underline{\text{Lim}} f_i \in \mathcal{U}$  is immediate and  $\overline{\text{Lim}} f_i \in \mathcal{U}$  follows from Theorem 2.16.

**Definition 2.16** Let  $\{f_i\}$  be a sequence in  $\mathcal{U}$  and let  $f \in \mathcal{U}$ .

1. If  $\{f_i\}$  and  $f$  satisfy Matheron's first convergence criterion, then we say that  $f_i$  lower semiconverges to the lower semi-limit  $f$ .
2. If  $\{f_i\}$  and  $f$  satisfy Matheron's second convergence criterion, then we say that  $f_i$  upper semiconverges to the upper semi-limit  $f$ .

This terminology is handy for stating the following  $(\mathcal{U}, \gamma)$ -version of the upper-lower limit theorem.

**Theorem 2.16** (Upper-lower limit theorem for  $(\mathcal{U}, \gamma)$ ) If  $\{f_i\}$  is a sequence in  $\mathcal{U}$ , then (a)  $\underline{\text{Lim}} f_i$  is the supremum of the lower semi-limits of  $f_i$ , (b)  $\overline{\text{Lim}} f_i$  is the infimum of the upper semi-limits of  $f_i$ , and (c)  $f_i \rightarrow f$  if and only if  $\underline{\text{Lim}} f_i = \overline{\text{Lim}} f_i = f$ .

This theorem follows from the  $\widehat{\mathcal{V}}$ -version of Theorem 2.7; moreover, note that the terms infimum and supremum are used relative to the poset  $(\mathcal{U}, \leq)$ . I conclude this section with the  $(\mathcal{U}, \gamma)$ -version of the usual semicontinuity criteria.

**Theorem 2.17** If  $\Phi : X \rightarrow \mathcal{U}$  and  $X$  is a first countable Hausdorff space, then  $\Phi$  is USC at  $x \in X \iff \Phi(x) \geq \overline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ , and  $\Phi$  is LSC at  $x \in X \iff \Phi(x) \leq \underline{\text{Lim}} \Phi(x_i) \forall \{x_i\}$  in  $X$  that converge to  $x$ .

We also have the following counterpart of Remarks 2.4 and 2.10.

**Remark 2.14** If  $X$  is a first countable Hausdorff space, then

1.  $\Phi : X \rightarrow \mathcal{U}$  is USC  $\iff x_i \rightarrow x$  in  $X$ ,  $(\xi_k, \tau_k) \rightarrow (\xi, \tau)$  in  $\mathcal{E}$ , and  $\Phi(x_{i_k})(\xi_k) \geq \tau_k$  for all  $k$  together imply that  $\Phi(x)(\xi) \geq \tau$ .
2.  $\Phi : X \rightarrow \mathcal{U}$  is LSC  $\iff x_i \rightarrow x$  in  $X$ ,  $(\xi, \tau) \in \mathcal{E}$ , and  $\Phi(x)(\xi) \geq \tau \implies$  there exist  $(\xi_i, \tau_i) \in \mathcal{E}$  such that  $\Phi(x_i)(\xi_i) \geq \tau_i$  for all but at most finitely many  $i$  and  $(\xi_i, \tau_i) \rightarrow (\xi, \tau)$  in  $\mathcal{E}$ .

## 2.6 Myopic Topology of the Spaces $V_c$ , $\widehat{V}_c$ , and $U_c$

Because of the compactness of the  $t$ -axis of  $\mathcal{E}$ , it follows that the core of every  $U \in V$  with compact support is a compact subset of  $\mathcal{E}$ . Conversely, every compact  $U \in V$  has compact support and consists of the core of  $U$  in union with a compact subset of  $\Pi(-\infty)$ .

**Definition 2.17** Let  $V_c$  be the set of compact  $U \in V$  (the set of umbrae in  $K(\mathcal{E})$ ), and let  $\rho$  be the relative myopic topology of  $V_c$  in  $K(\mathcal{E})$ .

Henceforth assume that  $V_c$  is carrying its  $\rho$ -topology.

**Remark 2.15** The following results are easily obtained:

1. If  $U_i \in V_c$  and  $U_i \rightarrow U$  myopically in  $K(\mathcal{E})$ , then  $U \in V_c$  and  $U_i \rightarrow U$  in  $V_c$ .
2. Conversely, if  $U_i \rightarrow U$  in  $V_c$ , then  $U_i \rightarrow U$  myopically in  $K(\mathcal{E})$ .
3.  $V_c$  is a myopically closed subspace of  $K(\mathcal{E})$ .
4. If  $\mathcal{K} \subset V_c$ , then  $\mathcal{K}$  is closed (compact) in  $V_c \iff \mathcal{K}$  is closed (compact) in  $K(\mathcal{E})$ .
5.  $V_c$  is an LCS space.
6.  $\Upsilon_{\mathcal{E}}$  maps  $K(\mathcal{E})$  onto  $V_c$ .

**Proposition 2.6** A subset  $\mathcal{K}$  of  $V_c$  is  $\rho$ -compact if and only if  $\mathcal{K}$  is  $\sigma$ -closed in  $V$  and  $\exists$  a  $V \in V_c$  such that  $U \subset V$  for all  $U \in \mathcal{K}$ .

**Proof** For  $\implies$  let  $\mathcal{K}$  be a  $\rho$ -compact subset of  $V_c$ . Since  $\mathcal{K}$  is then a  $\nu$ -compact subset of  $K(\mathcal{E})$ , it follows that  $\mathcal{K}$  is closed in  $F(\mathcal{E})$  (and therefore in  $V$ ), and there exists a  $K \in K(\mathcal{E})$  and therefore a  $V \in V_c$  (namely  $\Upsilon_{\mathcal{E}}(K)$ ) such that  $U \subset V$  for all  $U \in \mathcal{K}$ . For  $\impliedby$  suppose that  $\mathcal{K}$  is  $\sigma$ -closed in  $V$  (and therefore closed in  $F(\mathcal{E})$ ), and there exists a  $V \in V_c$  such that  $U \subset V$  for all  $U \in \mathcal{K}$ . Then it follows that  $\mathcal{K}$  is a  $\rho$ -compact subset of  $V_c$ .

This result leads to the following myopic convergence criteria.

**Proposition 2.7** A sequence  $\{U_i\}$  in  $V_c$  converges myopically to  $U$  in  $V_c$  if and only if  $U_i \rightarrow U$  in the hit-miss topology of  $V$  and there exists a  $V \in V_c$  such that  $U_i \subset V$  for all  $i$ .

The myopic topology of  $\mathbf{K}(\mathcal{E})$  is generated by

$$\{\mathbf{K}^F : F \in \mathbf{F}(\mathcal{E})\} \cup \{\mathbf{K}_G : G \in \mathbf{G}(\mathcal{E})\}$$

where  $\mathbf{K}^F = \{K \in \mathbf{K}(\mathcal{E}) : K \cap F = \emptyset\}$  and

$$\mathbf{K}_G = \{K \in \mathbf{K}(\mathcal{E}) : K \cap G \neq \emptyset\}.$$

Introducing the obvious definitions  $[\mathbf{V}_c]^F = \{U \in \mathbf{V}_c : U \cap F = \emptyset\}$  and  $[\mathbf{V}_c]_G = \{U \in \mathbf{V}_c : U \cap G \neq \emptyset\}$ , we accordingly obtain the following:

**Remark 2.16** *The topology generated on  $\mathbf{V}_c$  by the collection*

$$\{[\mathbf{V}_c]^F : F \in \mathbf{F}(\mathcal{E})\} \cup \{[\mathbf{V}_c]_G : G \in \mathbf{G}(\mathcal{E})\}$$

*is precisely the (relative) myopic topology of  $\mathbf{V}_c$  (in  $\mathbf{K}(\mathcal{E})$ ).*

Since the relative hit-miss topology of  $\mathbf{V}_c$  in  $\mathbf{V}$  is generated by

$$\{[\mathbf{V}_c]^K : K \in \mathbf{K}(\mathcal{E})\} \cup \{[\mathbf{V}_c]_G : G \in \mathbf{G}(\mathcal{E})\}$$

and since this collection is strictly smaller than

$$\{[\mathbf{V}_c]^F : F \in \mathbf{F}(\mathcal{E})\} \cup \{[\mathbf{V}_c]_G : G \in \mathbf{G}(\mathcal{E})\}$$

we see that the myopic topology of  $\mathbf{V}_c$  is strictly stronger than its relative hit-miss topology. We may thus conclude the following:

**Remark 2.17** *The one-to-one identity mapping  $U \mapsto U$  of  $(\mathbf{V}_c, \rho)$  into  $(\mathbf{V}, \sigma)$  is continuous.*

Since a continuous image of a compact set is necessarily compact, we see that every  $\rho$ -compact subset  $\mathcal{K}$  of  $\mathbf{V}_c$  is a  $\sigma$ -compact and therefore a  $\sigma$ -closed subset of  $\mathbf{V}$ . Since every  $\rho$ -closed subset of  $\mathcal{K}$  is  $\rho$ -compact, it follows that every  $\rho$ -closed subset of  $\mathbf{V}_c$  is a  $\sigma$ -closed subset of  $\mathbf{V}$ . Hence the myopic and hit-miss topologies agree on the  $\rho$ -compact subsets of  $\mathbf{V}_c$ . Conversely, if  $\mathcal{F} \subset \mathbf{V}_c$  is  $\sigma$ -closed, and the myopic and hit-miss topologies agree on  $\mathcal{F}$ , then  $\mathcal{F}$  is  $\rho$ -compact. In other words,

**Proposition 2.8**  *$\mathcal{K}$  is a  $\rho$ -compact subset of  $\mathbf{V}_c$  if and only if  $\mathcal{K}$  is  $\sigma$ -closed and the relative hit-miss and myopic topologies agree on  $\mathcal{K}$ .*

**Corollary 2.11**  *$V \in \mathbf{V}_c \implies \{U \in \mathbf{V}_c : U \subset V\}$  is  $\rho$ -compact.*

It is clear from all this that  $\rho$  has essentially the same properties as the myopic topology of  $\mathbf{K}(\mathcal{E})$ . Letting  $\widehat{\mathbf{V}}_c = \mathbf{V}_c \cap \widehat{\mathbf{V}}$ , I now show how to put a myopic topology on  $\widehat{\mathbf{V}}_c$ .

**Definition 2.18** Let  $\dot{\mathbf{V}}_c = \{U \in \mathbf{V}_c : U = \dot{U}\}$ .

Note that  $\widehat{\mathbf{V}}_c$  is not a subset of  $\mathbf{V}_c$ ; indeed, there is no umbra in  $\widehat{\mathbf{V}}_c$  that lies in  $\mathbf{V}_c$ . The cores of the umbrae in  $\widehat{\mathbf{V}}_c$  are compact, however, and we see that  $\dot{\mathbf{V}}_c$  is a subset of  $\mathbf{V}_c$  and has the natural one-to-one correspondence  $\dot{U} \leftrightarrow \ddot{U}$  with the umbrae in  $\widehat{\mathbf{V}}_c$ . The relative myopic topology of  $\dot{\mathbf{V}}_c$  in  $\mathbf{V}_c$  may therefore be transferred by identification to  $\widehat{\mathbf{V}}_c$ . This quite clearly also gives a myopic topology to the set of  $f \in \mathcal{U}$  with compact support. In view of Proposition 2.7, we have the following characterization of myopic convergence in  $\dot{\mathbf{V}}_c$ .

**Proposition 2.9** A sequence  $\{U_i\}$  in  $\dot{\mathbf{V}}_c$  converges myopically to  $U$  in  $\dot{\mathbf{V}}_c$  if and only if (a) for each  $(x, t) \in U$  there exist  $(x_i, t_i) \in U_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$  in  $\mathcal{E}$ , (b) if  $\{U_{i_k}\}$  is a subsequence of  $\{U_i\}$ , then every  $\mathcal{E}$ -convergent sequence  $(x_{i_k}, t_{i_k}) \in U_{i_k}$  has its limit in  $U$ , and (c) there exists a  $V \in \dot{\mathbf{V}}_c$  such that  $U_i \subset V \forall i$ .

**Definition 2.19** We denote the set of  $U \in \mathcal{U}$  ( $f \in \mathcal{U}$ ) with compact support by  $\mathbf{U}_c$  ( $\mathcal{U}_c$ ).

**Remark 2.18**  $\Gamma$  maps  $\mathbf{U}_c$  one-to-one onto  $\mathcal{U}_c$ .

Note that no  $U \in \mathbf{U}_c$  lies in  $\mathbf{K}(\mathfrak{R}^{n+1})$ . We nonetheless have the following *lateral compactness* condition.

**Remark 2.19** If  $\{(x_i, t_i)\}$  is a sequence in  $U \in \mathbf{U}_c$ , then  $\{x_i\}$  has a convergent subsequence.

To obtain a myopic topology for  $\mathbf{U}_c$ , we may simply note that the mapping of  $\mathbf{U}_c$  given by  $U \mapsto \overline{U}$ , where the overbar denotes closure in  $\mathcal{E}$ , is one-to-one onto  $\dot{\mathbf{V}}_c$ . We may thus endow  $\mathbf{U}_c$  with a myopic topology by identifying it with  $\dot{\mathbf{V}}_c$ . This again gives a myopic topology to  $\mathcal{U}_c$ , indeed the same one it acquired above from  $\widehat{\mathbf{V}}_c$ . Translating Proposition 2.9 to  $\mathbf{U}_c$ , we obtain the following:

**Proposition 2.10** *A sequence  $\{U_i\}$  in  $\mathcal{U}_c$  converges myopically to  $U$  in  $\mathcal{U}_c$  if and only if the following hold:*

- (a) *For each  $(x, t) \in \bar{U}$  there exist  $(x_i, t_i) \in \bar{U}_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$ .*
- (b) *If  $\{\bar{U}_{i_k}\}$  is a subsequence of  $\{\bar{U}_i\}$ , then every convergent sequence  $(x_{i_k}, t_{i_k}) \in \bar{U}_{i_k}$  has its limit in  $\bar{U}$ .*
- (c) *There exists a  $V \in \dot{\mathcal{V}}_c$  such that  $\bar{U}_i \subset V$  for all  $i$ .*

Myopic convergence in  $\mathcal{U}_c$  therefore has the following characterization.

**Proposition 2.11** *A sequence  $\{f_i\}$  in  $\mathcal{U}_c$  converges myopically to  $f$  in  $\mathcal{U}_c$  if and only if the following hold:*

- (a) *For each  $x \in \Delta f$  and  $t \leq f(x)$ , there exist  $(x_i, t_i)$  for all but at most finitely many  $i$  such that*

$$x_i \in \Delta f_i, t_i \leq f_i(x_i), \text{ and } (x_i, t_i) \rightarrow (x, t) \text{ in } \mathcal{E}.$$

- (b) *If  $x_{i_k} \in \Delta f_{i_k} \forall k, t_{i_k} \leq f_{i_k}(x_{i_k}) \forall k$ , and  $(x_{i_k}, t_{i_k}) \rightarrow (x, t)$  in  $\mathcal{E}$ , then  $x \in \Delta f$  and  $t \leq f(x)$ .*
- (c) *There exists a  $g \in \mathcal{U}_c$  such that  $f_i \leq g$  for all  $i$ .*

## 2.7 Lattice/Poset Structures of $\widehat{\mathcal{V}}$ , $\mathcal{U}$ , and $\mathcal{U}$

It is readily seen that  $(\mathcal{U}, \cap, \cup)$  is a sublattice of  $(\mathbf{F}(\mathfrak{R}^{n+1}), \cap, \cup)$ ; likewise, it follows that  $(\widehat{\mathcal{V}}, \cap, \cup)$  is a sublattice of  $(\mathbf{F}(\mathcal{E}), \cap, \cup)$ . Both lattices are therefore distributive and induce the ordering  $\subset$ . Since  $(\mathcal{U}, \cap, \cup)$  has the universal bounds  $\emptyset$  and  $\mathfrak{R}^{n+1}$ , since  $(\widehat{\mathcal{V}}, \cap, \cup)$  has the universal bounds  $\Pi(-\infty)$  and  $\mathcal{E}$ , and since  $\mathcal{U}$  and  $\widehat{\mathcal{V}}$  are closed under arbitrary intersections, it follows that both lattices are complete. If we let  $\vee$  and  $\wedge$  respectively denote the pointwise supremum and infimum in  $\mathcal{U}$  (i.e., if  $f, g \in \mathcal{U}$ , then  $(f \vee g)(x) \equiv \sup\{f(x), g(x)\} = \max\{f(x), g(x)\} \forall x \in \mathfrak{R}^n$ , and  $(f \wedge g)(x) \equiv \inf\{f(x), g(x)\} = \min\{f(x), g(x)\} \forall x \in \mathfrak{R}^n$ ), it easily follows that  $(\mathcal{U}, \wedge, \vee)$  is a lattice with induced ordering  $\leq$ . Since  $f_{U \cup U'} = f_U \vee f_{U'}$  and  $f_{U \cap U'} = f_U \wedge f_{U'}$  for all  $U, U' \in \mathcal{U}$  [ $U, U' \in \widehat{\mathcal{V}}$ ], it follows that  $\Gamma[\widehat{\Gamma}]$  is a lattice isomorphism of  $(\mathcal{U}, \cap, \cup)$  [ $(\widehat{\mathcal{V}}, \cap, \cup)$ ] onto  $(\mathcal{U}, \wedge, \vee)$  and, of course, a poset isomorphism of  $(\mathcal{U}, \subset)$  [ $(\widehat{\mathcal{V}}, \subset)$ ] onto  $(\mathcal{U}, \leq)$ . Thus  $(\mathcal{U}, \wedge, \vee)$  is distributive and complete. Indeed,  $\mathcal{U}$ ,  $\mathcal{U}$ , and  $\widehat{\mathcal{V}}$  are topologically, lattice algebraically, and poset equivalent. The universal bounds of  $(\mathcal{U}, \wedge, \vee)$  are  $\mathcal{N}_\emptyset$  and  $\mathcal{N}_{\mathfrak{R}^n}$ , where  $(\forall x \in \mathfrak{R}^n)$   $\mathcal{N}_\emptyset(x) = -\infty$  and  $\mathcal{N}_{\mathfrak{R}^n}(x) = \infty$ . Thus  $\Gamma(\emptyset) = \widehat{\Gamma}(\Pi(-\infty)) = \mathcal{N}_\emptyset$  and  $\Gamma(\mathfrak{R}^{n+1}) = \widehat{\Gamma}(\mathcal{E}) = \mathcal{N}_{\mathfrak{R}^n}$ .

**Theorem 2.18** *If  $\{U_\alpha\}$  is any subset of  $\mathbf{U}$  or  $\widehat{\mathbf{V}}$ , then*

1.  $\inf\{U_\alpha\} = \bigcap_\alpha U_\alpha$  and  $\sup\{U_\alpha\} = \overline{\bigcup_\alpha U_\alpha}$ .
2.  $\inf\{f_{U_\alpha}\} \equiv \bigwedge_\alpha f_{U_\alpha}$  and  $(\bigwedge_\alpha f_{U_\alpha})(x) = \inf\{f_{U_\alpha}(x)\}$  for all  $x \in \mathbb{R}^n$ .
3.  $\sup\{f_{U_\alpha}\} \equiv \bigvee_\alpha f_{U_\alpha}$  and  $\bigvee_\alpha f_{U_\alpha}$  is the least function in  $\mathcal{U}$  such that

$$(\bigvee_\alpha f_{U_\alpha})(x) \geq \sup\{f_{U_\alpha}(x)\} \text{ for all } x \in \mathbb{R}^n.$$

4. *If  $\{U_\alpha\}$  is a subset of  $\mathbf{U}$ , then*
  - (a)  $\Gamma(\inf\{U_\alpha\}) = \inf\{f_{U_\alpha}\}$ .
  - (b)  $\Gamma(\sup\{U_\alpha\}) = \sup\{f_{U_\alpha}\}$ .
5. *If  $\{U_\alpha\}$  is a subset of  $\widehat{\mathbf{V}}$ , then*
  - (a)  $\widehat{\Gamma}(\inf\{U_\alpha\}) = \inf\{f_{U_\alpha}\}$ .
  - (b)  $\widehat{\Gamma}(\sup\{U_\alpha\}) = \sup\{f_{U_\alpha}\}$ .

**Corollary 2.12**  $\Gamma$  [ $\widehat{\Gamma}$ ] *is a complete lattice isomorphism of  $(\mathbf{U}, \cap, \cup)$  [ $(\widehat{\mathbf{V}}, \cap, \cup)$ ] onto  $(\mathcal{U}, \wedge, \vee)$ ; i.e.,  $\Gamma$  [ $\widehat{\Gamma}$ ] preserves the infima and suprema of arbitrary subsets. (Actually, all lattice isomorphisms are complete.)*

**Theorem 2.19** *The topological properties in  $\mathcal{U}$  of the ordering relation  $\leq$  and the lattice operations  $\wedge$  and  $\vee$  are as follows:*

1.  $\leq$  *is a closed order in  $\mathcal{U}$ ; i.e.,  $\{(f, g) \in \mathcal{U} \times \mathcal{U} : f \leq g\}$  is closed.*
2.  $\vee$  *is a continuous mapping of  $\mathcal{U} \times \mathcal{U}$  onto  $\mathcal{U}$ .*
3.  $\wedge$ , *however, is only a USC mapping of  $\mathcal{U} \times \mathcal{U}$  onto  $\mathcal{U}$ .*

*Moreover, a similar result holds for  $\mathbf{U}$  [ $\widehat{\mathbf{V}}$ ],  $\subset$ ,  $\cap$ , and  $\cup$ .*

**Remark 2.20**  $(\mathcal{U}, \gamma, \leq)$  *is a compact ordered space and  $(\mathcal{U}, \gamma, \wedge, \vee)$  is a compact closed-order lattice; hence,*

1.  $\widehat{\Gamma}$  and  $\Gamma$  *are compact-ordered-space isomorphisms, respectively, of  $(\widehat{\mathbf{V}}, \widehat{\sigma}, \subset)$  and  $(\mathbf{U}, \omega, \subset)$  onto  $(\mathcal{U}, \gamma, \leq)$ .*
2.  $\widehat{\Gamma}$  and  $\Gamma$  *are closed-order-lattice isomorphisms of  $(\widehat{\mathbf{V}}, \widehat{\sigma}, \cap, \cup)$  and  $(\mathbf{U}, \omega, \cap, \cup)$ , respectively, onto  $(\mathcal{U}, \gamma, \wedge, \vee)$ .*

## 2.8 Minkowski Sum and Difference in $\mathbf{U}$ and $\mathcal{U}$

We can also define a Minkowski sum ( $\oplus$ ) and difference ( $\ominus$ ) of umbrae—indeed, more straightforwardly in  $\mathbf{U}$  than in  $\widehat{\mathbf{V}}$ . The Minkowski difference turns out to be such that  $\mathbf{U}$  is closed under  $\ominus$ ; however, for the Minkowski sum to lie in  $\mathbf{U}$ , one of the summands must have compact support. This parallels the case with  $\mathbf{F}(\mathfrak{R}^n)$ ; i.e., we can Minkowski sum two sets in  $\mathbf{F}(\mathfrak{R}^n)$  and obtain a set in  $\mathbf{F}(\mathfrak{R}^n)$ , provided that one of the summands is compact. We can, moreover, isomorphically define  $\oplus$  and  $\ominus$  in  $\mathcal{U}$  using the identification  $\Gamma$ . As with  $\mathbf{F}(\mathfrak{R}^n)$ , it turns out that  $\ominus$  is USC from  $\mathbf{U} \times \mathbf{U}$  to  $\mathbf{U}$ . Unlike with  $\mathbf{F}(\mathfrak{R}^n)$ , however, it is not true that  $\oplus$  is continuous from  $\mathbf{U} \times \mathbf{U}_c$  or  $\mathbf{U}_c \times \mathbf{U}$  to  $\mathbf{U}$ , even when we use the myopic topology of  $\mathbf{U}_c$ . These matters are considered in detail in what follows.

If  $U, V \in \mathbf{U}$ , then  $U \oplus V \equiv \{(x + y, t + \tau) : (x, t) \in U, (y, \tau) \in V\}$  and  $U \ominus V \equiv \{x : x - y \in U \forall y \in V\}$  are well-defined subsets of  $\mathfrak{R}^{n+1}$ . In this light, consider the following.

**Proposition 2.12** *If  $U$  and  $V$  are in  $\mathbf{U}$ , then*

1.  $U \oplus V$  and  $U \ominus V$  are umbrae in  $\mathfrak{R}^{n+1}$ .
2.  $U \ominus V \in \mathbf{U}$ .
3. If either  $U$  or  $V$  is in  $\mathbf{U}_c$ , then  $U \oplus V \in \mathbf{U}$ .

**Proof** Assume that  $(x, t') \in U \oplus V$  and that  $t' > t \in \mathfrak{R}$ . We show that  $(x, t) \in U \oplus V$ . There are  $(\xi, \mu) \in U$  and  $(\eta, \lambda) \in V$  such that  $\xi + \eta = x$  and  $\mu + \lambda = t'$ . Let  $2\delta = t' - t$ . Then  $2\delta$  is real and positive, and it follows that  $(\xi, \mu - \delta) \in U$  and  $(\eta, \lambda - \delta) \in V$ . Thus  $(x, t' - 2\delta) = (x, t) \in U \oplus V$ , and it follows that  $U \oplus V$  is an umbra in  $\mathfrak{R}^{n+1}$ . Now suppose that  $(x, t') \in U \ominus V$  and that  $t' > t \in \mathfrak{R}$ . We show that  $(x, t) \in U \ominus V$ . By definition,

$$(\eta, \lambda) \in U \ominus V \iff (\eta - \xi, \lambda - \tau) \in U \forall (\xi, \tau) \in V.$$

Thus  $(x - \xi, t' - \tau) \in U$  for all  $(\xi, \tau) \in V$ . Let  $t' - t = \delta > 0$ . Then  $(x - \xi, t' - \delta - \tau) = (x - \xi, t - \tau) \in U$  for all  $(\xi, \tau) \in V$ , and it follows that  $(x, t) \in U \ominus V$ . Thus  $U \ominus V$  is an umbra in  $\mathfrak{R}^{n+1}$ .

To prove (2) it is sufficient to show that  $U \ominus V$  is closed in  $\mathfrak{R}^{n+1}$ . Suppose accordingly that  $(x_i, t_i)$  is in  $U \ominus V$  for all  $i$  and that  $(x_i, t_i) \rightarrow (x, t)$  in  $\mathfrak{R}^{n+1}$ . For each  $i$  it follows that for all  $(\xi, \tau) \in V$  we have  $(x_i - \xi, t_i - \tau) \in U$ . Since  $U$  is closed in  $\mathfrak{R}^{n+1}$  and

$t_i \rightarrow t \in \mathfrak{R} \implies (x_i - \xi, t_i - \tau) \rightarrow (x - \xi, t - \tau)$  in  $\mathfrak{R}^{n+1}$  for all  $(\xi, \tau) \in V$ , we see that  $(x, t) \in U \oplus V$ . This proves (2).

To prove (3), suppose that  $(x_i, t_i) \in U \oplus V$  for all  $i$  and that  $(x_i, t_i) \rightarrow (x, t)$  in  $\mathfrak{R}^{n+1}$ . There are  $(\xi_i, \mu_i) \in U$  and  $(\eta_i, \lambda_i) \in V$  such that  $\xi_i + \eta_i = x_i$  and  $\mu_i + \lambda_i = t_i$ . Since either  $U$  or  $V$  is in  $U_c$  it follows that there is a subsequence  $\{i_k\}$  such that  $\xi_{i_k} \rightarrow \xi$  in  $\mathfrak{R}^n$ ,  $\eta_{i_k} \rightarrow \eta$  in  $\mathfrak{R}^n$ ,  $\xi + \eta = x$ ,  $\mu_{i_k} \rightarrow \mu$  in  $\mathfrak{R}$ ,  $\lambda_{i_k} \rightarrow \lambda$  in  $\mathfrak{R}$ , and  $\mu + \lambda = t$ . Since  $U$  and  $V$  are closed in  $\mathfrak{R}^{n+1}$ , it follows that  $(x, t) \in U \oplus V$ . This completes the proof.

If  $U \in \mathbf{U}$  and  $V \in \mathbf{U}_c$ , we define their commutative *Minkowski sum* as  $U \oplus V = V \oplus U$ . Likewise, if  $U$  and  $V$  are in  $\mathbf{U}$ , we let  $U \ominus V$  and  $V \ominus U$  define their *Minkowski differences*. The Minkowski sum is thus a mapping  $(U, V) \mapsto U \oplus V$  of either  $\mathbf{U} \times \mathbf{U}_c$  or  $\mathbf{U}_c \times \mathbf{U}$  to  $\mathbf{U}$ , and the Minkowski difference is a mapping  $(U, V) \mapsto U \ominus V$  of  $\mathbf{U} \times \mathbf{U}$  to  $\mathbf{U}$ .

**Definition 2.20** We define  $\oplus$  and  $\ominus$  in  $\mathcal{U}$  with  $\Gamma$  as follows.

1. If  $f \in \mathcal{U}$  and  $g \in \mathcal{U}_c$ , then

$$f \oplus g = g \oplus f \equiv \Gamma[\Gamma^{-1}(f) \oplus \Gamma^{-1}(g)].$$

2. If  $f, g \in \mathcal{U}$ , then

$$f \ominus g \equiv \Gamma[\Gamma^{-1}(f) \ominus \Gamma^{-1}(g)].$$

**Remark 2.21**  $\Gamma$  is an isomorphism relative to  $\oplus$  and  $\ominus$ . In addition and in particular, we have the following.

1. If either  $U$  or  $V$  is empty, then  $U \oplus V = \emptyset$ ; equivalently,

$$f \oplus N_\emptyset = N_\emptyset \oplus f = N_\emptyset \quad \forall f \in \mathcal{U}.$$

2.  $U \ominus \emptyset = \mathfrak{R}^{n+1}$ ; equivalently,  $f \ominus N_\emptyset = N_{\mathfrak{R}^n} \quad \forall f \in \mathcal{U}$ .

3. If  $V \neq \emptyset$ , then  $\emptyset \ominus V = \emptyset$ ; equivalently,

$$f \in \mathcal{U} \text{ and } f \neq N_\emptyset \implies N_\emptyset \ominus f = N_\emptyset.$$

**Theorem 2.20**  $\ominus$  is a USC mapping of  $\mathbf{U} \times \mathbf{U}$  to  $\mathbf{U}$ .

**Proof** Let  $U_i \rightarrow U$  and  $V_i \rightarrow V$  in  $\mathcal{U}$ . We show that

$$\begin{aligned} (x_{i_k}, t_{i_k}) \in U_{i_k} \oplus V_{i_k} \quad \forall k \text{ and } (x_{i_k}, t_{i_k}) \rightarrow (x, t) \\ \implies (x, t) \in U \oplus V. \end{aligned}$$

Note that  $(x_{i_k}, t_{i_k}) \in U_{i_k} \oplus V_{i_k}$  is equivalent to

$$(x_{i_k} - y, t_{i_k} - \tau) \in U_{i_k} \text{ for all } (y, \tau) \in V_{i_k}.$$

Let  $(\xi, \mu)$  be any point in  $V$  and let  $(\xi_i, \mu_i) \in V_i$  (for all but at most finitely many  $i$ ) converge to  $(\xi, \mu)$ . Since

$$(x_{i_k} - \xi_{i_k}, t_{i_k} - \mu_{i_k}) \rightarrow (x - \xi, t - \mu)$$

shows that  $(x - \xi, t - \mu) \in U \quad \forall (\xi, \mu) \in V$ , it finally follows that  $(x, t) \in U \oplus V$ .

**Corollary 2.13**  $\oplus$  is a USC mapping of  $\mathcal{U} \times \mathcal{U}$  to  $\mathcal{U}$ .

**Theorem 2.21**  $\oplus$  is an LSC mapping of either  $\mathcal{U} \times \mathcal{U}_c$  or  $\mathcal{U}_c \times \mathcal{U}$  to  $\mathcal{U}$  with respect to the relative hit-miss and myopic topologies of  $\mathcal{U}_c$ .

**Proof** Let  $U_i \rightarrow U$  in  $\mathcal{U}$  and let  $V_i \rightarrow V$  in the relative hit-miss or myopic topology of  $\mathcal{U}_c$ . We show that if  $(x, t) \in U \oplus V$ , then there is a sequence  $(x_i, t_i) \in U_i \oplus V_i$  for all but at most finitely many  $i$  such that  $(x_i, t_i) \rightarrow (x, t)$ . There are  $(\xi, \mu)$  and  $(\eta, \lambda)$  in  $U$  and  $V$ , respectively, such that  $(\xi + \eta, \mu + \lambda) = (x, t)$ . For all but at most finitely many  $i$ , there are thus  $(\xi_i, \mu_i) \in U_i$  and  $(\eta_i, \lambda_i) \in V_i$  such that  $(\xi_i, \mu_i) \rightarrow (\xi, \mu)$  and  $(\eta_i, \lambda_i) \rightarrow (\eta, \lambda)$ . In the case of myopic convergence, since  $\lambda \in \mathfrak{R}$ , it follows that all but at most finitely many of the  $\lambda_i \in \mathfrak{R}$ , so that  $(\eta_i, \lambda_i) \in V_i$  for all but at most finitely many  $i$ . Hence in either case and for all but at most finitely many  $i$ , we have  $(\xi_i + \eta_i, \mu_i + \lambda_i) \in U_i \oplus V_i$  and  $(\xi_i + \eta_i, \mu_i + \lambda_i) \rightarrow (x, t)$ .

**Corollary 2.14**  $\oplus$  is an LSC mapping of either  $\mathcal{U} \times \mathcal{U}_c$  or  $\mathcal{U}_c \times \mathcal{U}$  to  $\mathcal{U}$  with respect to the relative hit-miss and myopic topologies of  $\mathcal{U}_c$ .

**Proposition 2.13**  $\oplus$  is not USC on  $\mathcal{U} \times \mathcal{U}_c$  relative to either the myopic or relative hit-miss topology of  $\mathcal{U}_c$ .

**Proof** Define the functions  $f_i$ ,  $i = 1, 2, 3, \dots$ , and  $f$  as follows:

$$f_i(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -i & \text{if } 1 < x \leq 2 \\ -\infty & \text{otherwise} \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -\infty & \text{otherwise.} \end{cases}$$

Then  $f_i \rightarrow f$  in the myopic topology of  $\mathcal{U}_c$ . Let

$$g(x) = \begin{cases} -\infty & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and for all  $i$  let  $g_i = g$ . Then

$$g_i \rightarrow g \text{ in } \mathcal{U}$$

$$(g_i \oplus f_i)(x) = \begin{cases} \infty & \text{if } 0 \leq x \leq 2 \\ -\infty & \text{otherwise} \end{cases}$$

and

$$(g \oplus f)(x) = \begin{cases} \infty & \text{if } 0 \leq x \leq 1 \\ -\infty & \text{otherwise.} \end{cases}$$

This completes the proof.

If we let  $\mathcal{L}$  denote the set of ERV LSC functions defined on  $\mathfrak{R}^n$ , then the mapping  $f \mapsto -f$  of  $\mathcal{U}$  is one-to-one and onto  $\mathcal{L}$ . One defines the *dual topology*  $\gamma^*$  on  $\mathcal{L}$  by declaring this mapping to be a homeomorphism. The duality theory associated with  $\mathbf{F}(\mathfrak{R}^n)$  and  $\mathbf{G}(\mathfrak{R}^n)^{5,1}$  manifests itself in the function setting in the pair of spaces  $(\mathcal{U}, \gamma)$  and  $(\mathcal{L}, \gamma^*)$  and in the lattice- and Minkowski-algebraic duality between them.

## 2.9 Translations of Umbrae and Functions

If  $A$  is a nonempty subset of  $\mathfrak{R}^{n+1}$  and  $(x, t) \in \mathfrak{R}^{n+1}$ , then we define  $A + (x, t) = \{(y, \tau) + (x, t) : (y, \tau) \in A\}$ . Note that if  $U \in \mathbf{U}$  and  $(x, t) \in \mathfrak{R}^{n+1}$ , then the *translate*  $U + (x, t)$  of  $U$  by  $(x, t)$  is an umbra in  $\mathbf{U}$ . Indeed, if  $U$  is any umbral subset of  $\mathfrak{R}^{n+1}$ , then  $U + (x, t)$  is an umbra in  $\mathfrak{R}^{n+1}$ . The translates of functions are defined so that

$$f_{U+(x,t)} = f_U + (x, t).$$

**Definition 2.21** If  $f \in \mathcal{U}$  and  $(y, t) \in \mathfrak{R}^{n+1}$ , then  $f + (y, t)$  denotes the function  $g$  in  $\mathcal{U}$  whose graph is  $\{(x + y, f(x) + t) : x \in \mathfrak{R}^n\}$ , that is,

$$g(x) = f(x - y) + t \quad \forall x \in \mathfrak{R}^n.$$

We of course have a similar definition of  $f + (y, t)$  when  $f$  is merely an ERV function defined on  $\mathfrak{R}^n$ .

**Proposition 2.14** If  $f_i \rightarrow f$  in  $\mathcal{U}$  and  $(x_i, t_i) \rightarrow (x, t)$  in  $\mathfrak{R}^{n+1}$ , then  $f_i + (x_i, t_i) \rightarrow f + (x, t)$  in  $\mathcal{U}$ .

**Proof** Let  $g_i = f_i + (x_i, t_i)$  and  $g = f + (x, t)$ . For each  $\xi \in \mathfrak{R}^n$  we then have  $g_i(\xi) = f_i(\xi - x_i) + t_i$  and  $g(\xi) = f(\xi - x) + t$ . If  $\xi_{i_k} \rightarrow \xi$ , then  $\limsup g_{i_k}(\xi_{i_k}) = \limsup [f_{i_k}(\xi_{i_k} - x_{i_k}) + t_{i_k}] \leq$

$$\limsup f_{i_k}(\xi_{i_k} - x_{i_k}) + t \leq f(\xi - x) + t = g(\xi).$$

Thus condition (b) of Theorem 2.15 is satisfied. For condition (a) of that theorem, we can show that there exists a sequence  $\xi_i \rightarrow \xi$  such that  $g_i(\xi_i) = f_i(\xi_i - x_i) + t_i \rightarrow f(\xi - x) + t = g(\xi)$ . For this, it is enough that there exist  $\xi_i \rightarrow \xi$  such that

$$f_i(\xi_i - x_i) \rightarrow f(\xi - x).$$

Since  $\eta_i \rightarrow \xi - x$  exist such that  $f_i(\eta_i) \rightarrow f(\xi - x)$ , let  $\xi_i = \eta_i + x_i$ .

**Corollary 2.15** If  $f \in \mathcal{U}$  and  $(x_i, t_i) \rightarrow (x, t) \in \mathfrak{R}^{n+1}$ , then

$$f - (x_i, t_i) \rightarrow f - (x, t) \text{ in } \mathcal{U}.$$

**Remark 2.22** Let  $f_{x,t}$  denote the function in  $\mathcal{U}_c$  defined by

$$f_{x,t}(y) = -\infty \text{ for all } y \neq x \text{ and } f_{x,t}(x) = t.$$

If  $t$  is finite, then  $f \oplus f_{x,t} = f + (x, t)$  for all  $f \in \mathcal{U}$ .

### 3 The Transform Space $\mathcal{M}(\mathcal{U})$

I now take up the central issue of defining and analyzing the  $\mathcal{M}$ -transforms of  $\mathcal{U}$ , the set of which I denote by  $\mathcal{M}(\mathcal{U})$ . The definition of the  $\mathcal{M}$ -transforms will allow us to use the kernel theory of Maragos<sup>7</sup> to obtain a version of Matheron's closed kernel theorem for  $\mathcal{M}(\mathcal{U})$ . This will in turn lead to a hit-miss topology that converts  $\mathcal{M}(\mathcal{U})$  into a transform space with a complete lattice structure.

**Definition 3.1** Let  $\mathcal{P}_U$  ( $\mathcal{P}_{\mathcal{U}}$ ) denote the class of umbrae in  $\mathfrak{R}^{n+1}$  (ERV functions on  $\mathfrak{R}^n$ ). A mapping  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_U$  is said to be TI if

$$\Phi(U + (x, t)) = \Phi(U) + (x, t) \text{ for all } U \in \mathcal{U} \text{ and } (x, t) \in \mathfrak{R}^{n+1}.$$

A mapping  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_{\mathcal{U}}$  is said to be TI if

$$\Phi(f + (x, t)) = \Phi(f) + (x, t) \text{ for all } f \in \mathcal{U} \text{ and } (x, t) \in \mathfrak{R}^{n+1}.$$

**Definition 3.2** An  $\mathcal{M}$ -transformation of (or on)  $\mathcal{U}$  is a TI USC mapping of  $\mathcal{U}$  to itself. We denote the set of  $\mathcal{M}$ -transformations of  $\mathcal{U}$  by  $\mathcal{M}(\mathcal{U})$ . An  $\mathcal{M}$ -transform of (or on)  $\mathcal{U}$  is a TI USC mapping of  $\mathcal{U}$  to itself. The set of all such transforms will be denoted  $\mathcal{M}(\mathcal{U})$ .

#### 3.1 Maragos' Kernel Theory

**Definition 3.3** If  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_U$  is TI, then

$$\ker(\Phi) \equiv \{U \in \mathcal{U} : (0, 0) \in \Phi(U)\}.$$

If  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_{\mathcal{U}}$  is TI, then

$$\ker(\Phi) \equiv \{f \in \mathcal{U} : \Phi(f)(0) \geq 0\}.$$

The first part of this definition is the obvious analog of the definition of the kernel of a TI mapping of  $\mathbf{F}(\mathfrak{R}^n)$ . Note that to each  $\Phi \in \mathcal{M}(\mathcal{U})$  there corresponds a unique  $\Gamma(\Phi) \in \mathcal{M}(\mathcal{U})$  such that  $\Gamma(\Phi)(f_U) = \Gamma(\Phi(U))$  for all  $U \in \mathcal{U}$ . Every transform in  $\mathcal{M}(\mathcal{U})$ , moreover, is the correspondent in this way of a unique transformation in  $\mathcal{M}(\mathcal{U})$ . Thus  $\Gamma : \mathcal{M}(\mathcal{U}) \rightarrow \mathcal{M}(\mathcal{U})$  is one-to-one and onto. The second part of the above definition is framed so that  $\Gamma[\ker(\Phi)] = \ker[\Gamma(\Phi)] \forall$  TI  $\Phi$  on  $\mathcal{U}$ ; i.e., so that  $\Gamma$  becomes a kernel isomorphism of  $\mathcal{M}(\mathcal{U})$  onto  $\mathcal{M}(\mathcal{U})$ . From now on the discussion is restricted to  $\mathcal{M}(\mathcal{U})$ . Note, however, that there are  $\mathcal{M}(\mathcal{U})$ -versions of all the results to follow.

**Proposition 3.1** *If  $\mathcal{K}$  is the kernel of a TI mapping of  $\mathcal{U}$  to  $\mathcal{P}_{\mathcal{U}}$ , then*

$$f \in \mathcal{K} \iff f + (0, t) \in \mathcal{K} \quad \forall t > 0.$$

**Definition 3.4** *A subset  $\mathcal{K}$  of  $\mathcal{U}$  that satisfies*

$$f \in \mathcal{K} \iff f + (0, t) \in \mathcal{K} \quad \forall t > 0$$

*will be called up-closed.*

**Proposition 3.2** *If  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_{\mathcal{U}}$  is TI, then ( $f \in \mathcal{U}$  and  $x \in \mathbb{R}^n$ )*

$$\Phi(f)(x) = \sup\{t : f - (x, t) \in \ker(\Phi)\}.$$

*If  $\mathcal{K}$  is any up-closed subset of  $\mathcal{U}$ , then ( $f \in \mathcal{U}$  and  $x \in \mathbb{R}^n$ )*

$$\Phi(f)(x) = \sup\{t : f - (x, t) \in \mathcal{K}\}$$

*defines a TI mapping  $\Phi$  of  $\mathcal{U}$  to  $\mathcal{P}_{\mathcal{U}}$  whose kernel is  $\mathcal{K}$ .*

### 3.2 Closed Kernel Theorem

**Remark 3.1** *If  $A \subset \mathbb{R}^{n+1}$ , let  $\mathcal{U}^A = \{f \in \mathcal{U} : \Gamma^{-1}(f) \cap A = \emptyset\}$  and let  $\mathcal{U}_A = \{f \in \mathcal{U} : \Gamma^{-1}(f) \cap A \neq \emptyset\}$ .*

1.  $\gamma$  is generated subbasically on  $\mathcal{U}$  by

$$\{\mathcal{U}^K : K \in \mathbf{K}(\mathbb{R}^{n+1})\} \cup \{\mathcal{U}_G : G \in \mathbf{G}(\mathbb{R}^{n+1})\}.$$

2.  $\{\mathcal{U}^K : K \in \mathbf{K}(\mathbb{R}^{n+1})\} \cup \emptyset$  is a base for an upper  $\gamma$ -topology of  $\mathcal{U}$ .

3.  $\{\mathcal{U}_G : G \in \mathbf{G}(\mathbb{R}^{n+1})\} \cup \mathcal{U}$  is a subbase for a companion lower  $\gamma$ -topology of  $\mathcal{U}$ .

To prove a function analog of Matheron's closed kernel theorem, it will be convenient to approach the result sought (Thm. 3.1) in several steps.

**Lemma 3.1** *If  $\Phi$  is a USC TI mapping of  $\mathcal{U}$  to  $\mathcal{U}$ , then  $\ker(\Phi)$  is a closed subset of  $\mathcal{U}$ .*

**Proof** Let  $K \in \mathbf{K}(\mathbb{R}^{n+1})$  and note that  $\mathcal{U}_K$  is the complement of  $\mathcal{U}^K$  in  $\mathcal{U}$  and is therefore closed in the upper  $\gamma$ -topology of  $\mathcal{U}$  noted in the above remark. Thus if  $\Phi$  is into  $\mathcal{U}$  and USC, it follows that  $\Phi^{-1}(\mathcal{U}_K)$  is closed in  $\mathcal{U}$ . Since  $\{\vec{0}\} \equiv \{(0, 0)\} \in \mathbf{K}(\mathbb{R}^{n+1})$  and

$$\begin{aligned} \Phi^{-1}(\mathcal{U}_{\vec{0}}) &= \Phi^{-1}(\{f \in \mathcal{U} : \Upsilon(G_f) \cap \{\vec{0}\} \neq \emptyset\}) = \\ &= \{f \in \mathcal{U} : \Phi(f)(0) \geq 0\} = \ker(\Phi), \end{aligned}$$

it finally follows that  $\ker(\Phi)$  is closed in  $\mathcal{U}$ .

**Lemma 3.2** *If  $\Phi$  is a TI mapping of  $\mathcal{U}$  to  $\mathcal{P}_{\mathcal{U}}$  and  $\ker(\Phi)$  is closed in  $\mathcal{U}$ , then  $\Phi$  is into  $\mathcal{U}$ .*

**Proof** We show that  $f \in \mathcal{U}$ ,  $x \in \mathfrak{R}^n$ , and  $x_i \rightarrow x$  in  $\mathfrak{R}^n$  imply that

$$\Phi(f)(x) \geq \limsup \Phi(f)(x_i)$$

or equivalently that

$$\sup\{t : f - (x, t) \in \ker(\Phi)\} \geq$$

$$\limsup \sup\{\tau_i : f - (x_i, \tau_i) \in \ker(\Phi)\}.$$

Put  $s_i = \Phi(f)(x_i) = \sup\{\tau_i : f - (x_i, \tau_i) \in \ker(\Phi)\}$ . If  $\limsup s_i = -\infty$ , there is nothing to prove. We can therefore assume without loss of generality that  $s_i > -\infty$  for all  $i$ . We then have two cases to consider: (1) all but finitely many (without loss of generality all) of the  $s_i < \infty$  and (2) infinitely many of the  $s_i = \infty$ .

In case (1), it follows from Corollary 2.15 that  $f - (x_i, s_i) \in \ker(\Phi)$  for each  $i$  because  $\ker(\Phi)$  is closed. Case (1) has the two subcases: 1(a)  $\limsup s_i < \infty$  and 1(b)  $\limsup s_i = \infty$ . Letting

$$s_{i_k} \rightarrow \limsup s_i \equiv s$$

we have in case 1(a) (again because  $\ker(\Phi)$  is closed) that

$$f - (x, s) \in \ker(\Phi) \iff \Phi(f)(x) \geq s.$$

This disposes of 1(a). For 1(b), there is a subsequence  $s_{i_k} \rightarrow \infty$ , and here we must show that  $\Phi(f)(x) = \infty$ . We now observe that

$$\Phi(f)(x) = \infty \iff f - (x, t) \in \ker(\Phi) \text{ for all } t \in \mathfrak{R}$$

because  $\ker(\Phi)$  is up-closed. For the same reason,

$$f - (x_{i_k}, t) \in \ker(\Phi) \text{ for all } t \in (-\infty, s_{i_k}].$$

Since  $s_{i_k} \rightarrow \infty$ , it follows for any given  $t \in \mathfrak{R}$  that there exist  $t_k < s_{i_k}$  such that  $t_k \rightarrow t$ . Thus  $f - (x_{i_k}, t_k) \in \ker(\Phi)$  for all  $k$  and  $f - (x_{i_k}, t_k) \rightarrow f - (x, t)$  in  $\mathcal{U}$  by Corollary 2.15. Since  $\ker(\Phi)$  is closed, it follows that  $f - (x, t) \in \ker(\Phi)$  for all real  $t$ . This disposes of case 1(b). For case (2), there is a subsequence  $\{s_{i_k} = \infty\}$ . We now observe that  $f - (x_{i_k}, t) \in \ker(\Phi)$  for all  $t \in (-\infty, \infty)$ . Since  $s_{i_k} \rightarrow \infty$ , it follows for any given  $t \in \mathfrak{R}$  that there exist  $t_k < s_{i_k}$  such that  $t_k \rightarrow t$ . Thus  $f - (x_{i_k}, t_k) \in \ker(\Phi)$  for all  $k$  and  $f - (x_{i_k}, t_k) \rightarrow f - (x, t)$  in  $\mathcal{U}$  by Corollary 2.15. Since  $\ker(\Phi)$  is closed, it follows that  $f - (x, t) \in \ker(\Phi)$  for all real  $t$ . This disposes of case (2) and shows that  $\Phi$  is into  $\mathcal{U}$ .

**Theorem 3.1** (Closed kernel theorem) *A TI mapping  $\Phi$  of  $\mathcal{U}$  to  $\mathcal{P}_{\mathcal{U}}$  is into  $\mathcal{U}$  and USC if and only if  $\ker(\Phi)$  is closed in  $\mathcal{U}$ .*

**Proof** It is sufficient to show that a TI  $\Phi$  with a closed kernel is USC.

To prove that  $\Phi$  is USC, we show that  $g_i \rightarrow g$  in  $\mathcal{U}$ ,  $\Phi(g_{i_k})(x_k) \geq t_k$  for all  $k$ , and  $(x_k, t_k) \rightarrow (x, t)$  together imply that  $\Phi(g)(x) \geq t$ . We consider three cases: (1)  $t = -\infty$ , (2)  $t \in \mathfrak{R}$ , and (3)  $t = \infty$ . In case (1), there is nothing to prove. In case (2), all but at most finitely many of the  $t_k$  are real. For these  $k$ , let  $f_k = g_{i_k} - (x_k, t_k)$ . Then  $\Phi(f_k)(0) = [\Phi(g_{i_k}) - (x_k, t_k)](0) = \Phi(g_{i_k})(x_k) - t_k \geq 0$  and it follows that  $f_k \in \ker(\Phi)$ . Since  $\ker(\Phi)$  is closed and  $f_k \rightarrow g - (x, t)$  in  $\mathcal{U}$ , we see that  $g - (x, t) \in \ker(\Phi)$ , i.e.,  $\Phi(g)(x) - t \geq 0$ . This disposes of case (2). Case (3) resolves into two subcases: (a) infinitely many of the  $t_k$  are real and (b) all but finitely many of the  $t_k = \infty$ . For subcase (a), let  $\{t_{k_j} \in \mathfrak{R}\}$  be a subsequence of  $\{t_k\}$  and put  $f_j = g_{i_{k_j}} - (x_{k_j}, t_{k_j})$ . As before, it follows that each  $f_j \in \ker(\Phi)$  and hence that  $g_{i_{k_j}} - (x_{k_j}, t_{k_j} - \tau) \in \ker(\Phi)$  for all  $\tau > 0$ , because  $\ker(\Phi)$  is up-closed. Given any real number  $\lambda$ , there is a positive sequence  $\tau_j \rightarrow \infty$  such that  $t_{k_j} - \tau_j \rightarrow \lambda$ . Thus  $g - (x, \lambda) \in \ker(\Phi)$  for all  $\lambda \in \mathfrak{R}$  or  $\Phi(g)(x) \geq \lambda$  for all real  $\lambda$ , and it follows that  $\Phi(g)(x) = \infty$ . This proves subcase (a).

Suppose, then, that all but finitely many of the  $t_k = \infty$ . For all such  $k$ , we have that  $\Phi(g_{i_k})(x_k) = \infty$  and hence that there are real  $\tau_k \rightarrow \infty$  such that  $\Phi(g_{i_k})(x_k) \geq \tau_k$ . This puts us in the situation of subcase (a) and completes the proof.

### 3.3 Identification of $\mathcal{M}(\mathcal{U})$ with $\mathbf{F}^\uparrow(\mathcal{U})$

According to the closed kernel theorem, there is a one-to-one correspondence between the elements of  $\mathcal{M}(\mathcal{U})$  and the  $\mathcal{U}$ -closed, up-closed subsets of  $\mathcal{U}$ . Let us denote this class by  $\mathbf{F}^\uparrow(\mathcal{U})$ . Because  $\mathcal{U}$  is a compact LCS space, we may give  $\mathbf{F}(\mathcal{U})$  the usual hit-miss topology and topologize  $\mathcal{M}(\mathcal{U}) \leftrightarrow \mathbf{F}^\uparrow(\mathcal{U})$  with the hit-miss topology that  $\mathbf{F}^\uparrow(\mathcal{U})$  inherits from  $\mathbf{F}(\mathcal{U})$ ; this we call the *hit-miss topology of  $\mathcal{M}(\mathcal{U})$* .

**Theorem 3.2**  $\mathbf{F}^\uparrow(\mathcal{U})$  is a closed (and therefore a compact) subspace of  $\mathbf{F}(\mathcal{U})$  and is closed under intersections and finite unions.

**Proof** Let  $\{\mathcal{K}_i\}$  be a sequence in  $\mathbf{F}^\uparrow(\mathcal{U})$  such that  $\mathcal{K}_i \rightarrow \mathcal{K}$  in  $\mathbf{F}(\mathcal{U})$ .

We show that  $\mathcal{K}$  is up-closed, i.e.,

$$f \in \mathcal{K} \iff f + (0, t) \in \mathcal{K} \quad \forall t > 0.$$

If  $f \in \mathcal{K}$ , there exist  $f_i \in \mathcal{K}_i$  such that  $f_i \rightarrow f$  in  $\mathcal{U}$ . If  $t > 0$ , then  $f_i + (0, t) \in \mathcal{K}_i$  and  $f_i + (0, t) \rightarrow f + (0, t)$  in  $\mathcal{U}$ , so that  $f + (0, t) \in \mathcal{K}$ . On the other hand, if  $f \in \mathcal{U}$  and  $f + (0, t) \in \mathcal{K}$  for all  $t > 0$ , then  $f + (0, t_i) \in \mathcal{K}$  and  $f + (0, t_i) \rightarrow f$  in  $\mathcal{U}$  whenever  $\{t_i\}$  is a positive sequence tending to zero. Hence  $f \in \mathcal{K}$ , and the first part of the theorem is proved.

For the second part, let  $\{\mathcal{K}_\alpha\}$  be a family of elements of  $\mathbf{F}^\dagger(\mathcal{U})$ . It is clear that  $\mathcal{K} \equiv \bigcap_\alpha \mathcal{K}_\alpha \in \mathbf{F}^\dagger(\mathcal{U})$ . To see that  $\mathcal{K}$  is up-closed, first note that the validity of

$$f \in \mathcal{K} \implies f + (0, t) \in \mathcal{K} \quad \forall t > 0$$

is obvious. Suppose, on the other hand, that  $f \in \mathcal{U}$  and that  $f + (0, t) \in \mathcal{K} \quad \forall t > 0$ . If  $\{t_i\}$  is a positive sequence with limit zero, then  $f + (0, t_i) \in \mathcal{K}$  for all  $i$  and  $f + (0, t_i) \rightarrow f$  in  $\mathcal{U}$ . Since  $\mathcal{K}$  is closed in  $\mathcal{U}$ , it therefore follows that  $f \in \mathcal{K}$ . The proof for finite unions is similar.

**Corollary 3.1**  $(\mathbf{F}^\dagger(\mathcal{U}), \cap, \cup)$  is a complete distributive lattice with induced ordering  $\subset$ .

**Proof** It follows from Theorem 3.2 that  $(\mathbf{F}^\dagger(\mathcal{U}), \cap, \cup)$  is a lattice. That this lattice has induced ordering  $\subset$  and is distributive is obvious. Note again that a lattice with a universal upper bound in which every subset has an infimum is complete. Since  $\mathcal{U}$  is up-closed, it follows that  $(\mathbf{F}^\dagger(\mathcal{U}), \cap, \cup)$  has the universal upper bound  $\mathcal{U}$ . Since  $\mathbf{F}^\dagger(\mathcal{U})$  is closed under arbitrary intersections, it follows that all its subsets have infima. This completes the proof.

### 3.4 Lattice/Poset Structure of $\mathcal{M}(\mathcal{U})$

As with  $\mathcal{M}(\mathbf{F})$ , the transform space  $\mathcal{M}(\mathcal{U})$  has a natural lattice and poset structure that it acquires from  $\mathbf{F}^\dagger(\mathcal{U})$  through the correspondence  $\Phi \leftrightarrow \ker(\Phi)$ . If  $\Phi$  and  $\Phi'$  are members of  $\mathcal{M}(\mathcal{U})$ , then the transforms  $\Phi \wedge \Phi'$  and  $\Phi \vee \Phi'$  are defined in terms of their kernels by

$$\ker(\Phi \wedge \Phi') = \ker(\Phi) \cap \ker(\Phi') \quad \text{and} \quad \ker(\Phi \vee \Phi') = \ker(\Phi) \cup \ker(\Phi').$$

Thus it is clear that  $\Phi \wedge \Phi', \Phi \vee \Phi' \in \mathcal{M}(\mathcal{U})$ . Indeed it follows that  $(\mathcal{M}(\mathcal{U}), \wedge, \vee)$  is a lattice and that  $\Phi \mapsto \ker(\Phi)$  is a lattice isomorphism of  $(\mathcal{M}(\mathcal{U}), \wedge, \vee)$  onto  $(\mathbf{F}^\dagger(\mathcal{U}), \cap, \cup)$ . Consequently,  $(\mathcal{M}(\mathcal{U}), \wedge, \vee)$  is complete and distributive.

**Theorem 3.3** *The ordering  $\preceq$  induced in  $\mathcal{M}(\mathcal{U})$  by its lattice operations can be characterized as follows: If  $\Phi$  and  $\Phi'$  are transforms in  $\mathcal{M}(\mathcal{U})$ , then*

$$\Phi \preceq \Phi' \iff \Phi(f) \leq \Phi'(f) \text{ for all } f \in \mathcal{U} \iff \ker(\Phi) \subset \ker(\Phi').$$

*In addition, if  $\{\Phi_\alpha\}$  is any subset of  $\mathcal{M}(\mathcal{U})$ , then*

1.  $\inf\{\Phi_\alpha\} \equiv \bigwedge_\alpha \Phi_\alpha$  has the kernel  $\bigcap_\alpha \ker(\Phi_\alpha)$  and

$$(\bigwedge_\alpha \Phi_\alpha)(f) = \bigwedge_\alpha \Phi_\alpha(f) \quad \forall f \in \mathcal{U}.$$

2.  $\sup\{\Phi_\alpha\} \equiv \bigvee_\alpha \Phi_\alpha$  has the kernel  $\overline{\bigcup_\alpha \ker(\Phi_\alpha)}$  and is the least  $\mathcal{M}$ -transform such that

$$(\bigvee_\alpha \Phi_\alpha)(f) \geq \bigvee_\alpha \Phi_\alpha(f) \text{ for all } f \in \mathcal{U}.$$

3. If  $\{\Phi_\alpha\} = \{\Phi_k\}$  is a finite set of  $\mathcal{M}$ -transforms, then

$$(\bigvee_k \Phi_k)(f) = \bigvee_k \Phi_k(f) \text{ for all } f \in \mathcal{U}.$$

Let  $\tau^\dagger$  denote both the hit-miss topology of  $\mathcal{M}(\mathcal{U})$  and the relative topology of  $\mathbf{F}^\dagger(\mathcal{U})$  in  $\mathbf{F}(\mathcal{U})$ .

**Corollary 3.2** *The poset  $(\mathcal{M}(\mathcal{U}), \preceq)$  is isomorphic as such to the poset  $(\mathbf{F}^\dagger(\mathcal{U}), \subset)$  under the correspondence  $\leftrightarrow$ . In fact,  $(\mathcal{M}(\mathcal{U}), \tau^\dagger, \preceq)$  is a compact ordered space and is isomorphic as such to the compact ordered space  $(\mathbf{F}^\dagger(\mathcal{U}), \tau^\dagger, \subset)$ ; moreover,  $(\mathcal{M}(\mathcal{U}), \tau^\dagger, \wedge, \vee)$  is a compact closed order lattice that is isomorphic as such to the compact closed order lattice  $(\mathbf{F}^\dagger(\mathcal{U}), \tau^\dagger, \cap, \cup)$ .*

**Theorem 3.4** *The topological properties in  $\mathcal{M}(\mathcal{U})$  of the ordering relation  $\preceq$  and the lattice operations  $\wedge$  and  $\vee$  are as follows:*

1.  $\preceq$  is a closed order in  $\mathcal{M}(\mathcal{U})$ .
2.  $\vee$  is a continuous mapping of  $\mathcal{M}(\mathcal{U}) \times \mathcal{M}(\mathcal{U})$  onto  $\mathcal{M}(\mathcal{U})$ .
3.  $\wedge$ , however, is only a USC mapping of  $\mathcal{M}(\mathcal{U}) \times \mathcal{M}(\mathcal{U})$  onto  $\mathcal{M}(\mathcal{U})$ .

### 3.5 Increasing Transforms

Unlike the case with TI set transformations, it is not profitable to entertain the notion of a purely decreasing TI function transform, as this turns out to be a nearly vacuous concept.

**Definition 3.5** If  $\Phi : \mathcal{U} \rightarrow \mathcal{P}_{\mathcal{U}}$ , we say that  $\Phi$  is increasing if

$$\Phi(f)(x) \geq \Phi(g)(x) \quad \forall x \in \mathbb{R}^n$$

whenever  $f$  and  $g$  are in  $\mathcal{U}$  and  $f(x) \geq g(x) \quad \forall x \in \mathbb{R}^n$ . Order preserving is a synonym for increasing.

The corresponding definition of a decreasing transform would require that  $\Phi(f) \geq \Phi(g)$  whenever  $f \leq g$ . But since  $f \leq f + (\mathbf{0}, t) \quad \forall t \geq 0$  is true for all ERV functions, and we are interested only in TI transforms  $\Phi$ , it follows that  $\Phi(f + (\mathbf{0}, t)) = \Phi(f) + (\mathbf{0}, t) \geq \Phi(f)$  for all  $t \geq 0$  and all  $f \in \mathcal{U}$ . A decreasing transform  $\Phi$  would therefore have to satisfy  $\Phi(f) + (\mathbf{0}, t) = \Phi(f)$  for all  $f \in \mathcal{U}$  and all  $t > 0$ , and this shows that our only two candidates are  $\Phi(f) \equiv \mathcal{N}_{\emptyset}$  and  $\Phi(f) \equiv \mathcal{N}_{\mathbb{R}^n}$ , which are indeed (trivially) decreasing (and increasing too). It is clear, then, that the meager concept of a decreasing TI transform is virtually useless.

**Remark 3.2** The mappings  $f \mapsto \mathcal{N}_{\emptyset}$  and  $f \mapsto \mathcal{N}_{\mathbb{R}^n}$ , both  $\forall f \in \mathcal{U}$ , are increasing  $\mathcal{M}$ -transforms. We denote them  $\Phi_{\emptyset}$  and  $\Phi_{\mathbb{R}^n}$  and call them the trivial transforms.

**Theorem 3.5** If  $\Phi$  is a TI mapping of  $\mathcal{U}$  to  $\mathcal{P}_{\mathcal{U}}$ , then  $\Phi(\mathcal{N}_{\emptyset})$  is either  $\mathcal{N}_{\emptyset}$  or  $\mathcal{N}_{\mathbb{R}^n}$ , and likewise for  $\Phi(\mathcal{N}_{\mathbb{R}^n})$ . Furthermore, if  $\Phi$  is nontrivial and increasing, then  $\Phi(\mathcal{N}_{\emptyset}) = \mathcal{N}_{\emptyset}$  and  $\Phi(\mathcal{N}_{\mathbb{R}^n}) = \mathcal{N}_{\mathbb{R}^n}$ .

**Theorem 3.6** A transform  $\Phi \in \mathcal{M}(\mathcal{U})$  is increasing if and only if  $\ker(\Phi)$  is an increasing set.

Maragos<sup>7</sup> has established the analogs of Theorems 1.7 and 1.8 for increasing  $\mathcal{M}$ -transforms (Thm. 3.7 and Cor. 3.3 below) by means of the following natural definition of function erosion.

**Definition 3.6** If  $g$  is ERV on  $\mathbb{R}^n$ , let  $S(g) = \{x : g(x) > -\infty\}$ . If  $f, g \in \mathcal{U}$ , then define  $(f, g) \mapsto \mathcal{E}(f, g)$  by

$$\mathcal{E}(f, g)(x) = \sup\{t : t + g(y - x) \leq f(y) \text{ for all } y - x \in S(g)\}.$$

$\mathcal{E}(f, g)$  is ERV on  $\mathbb{R}^n$  and is called the erosion of  $f$  by  $g$ .

**Theorem 3.7** (Maragos) *Let  $\mathcal{M}_\uparrow(\mathcal{U})$  denote the subspace of increasing transforms in  $\mathcal{M}(\mathcal{U})$ . If  $\Phi \in \mathcal{M}_\uparrow(\mathcal{U})$ , then for all  $f \in \mathcal{U}$*

$$\Phi(f) = \bigvee \{ \mathcal{E}(f, h) : h \in \ker(\Phi) \}.$$

The *minimal basis kernel*<sup>12</sup>  $\mathcal{K}_{\min}(\Phi)$  of a transform  $\Phi \in \mathcal{M}_\uparrow(\mathcal{U})$  is the collection of minimal elements of  $\ker(\Phi)$  under the ordering  $\leq$ , i.e., the set of  $h \in \ker(\Phi)$  such that no  $f \in \ker(\Phi)$  is strictly less than  $h$ . The nonempty existence of this collection for  $\ker(\Phi) \neq \emptyset$  is guaranteed by *Zorn's lemma* and the fact that  $\ker(\Phi)$  is closed in  $\mathcal{U}$ . In fact,

**Lemma 3.3** *If  $\mathcal{L}$  is a totally ordered subset of  $\mathcal{U}$ , then  $\bigwedge \mathcal{L}$  and  $\bigvee \mathcal{L}$  lie in the closure of  $\mathcal{L}$ .*

**Proof** Banon and Barrera<sup>8</sup> have shown that if  $\mathcal{L}'$  is a totally ordered subset of  $\mathbf{F}(S)$ , then  $\inf \mathcal{L}'$  and  $\sup \mathcal{L}'$  lie in the closure of  $\mathcal{L}'$ .  $\mathcal{L}' \equiv \Gamma^{-1}(\mathcal{L})$  is a totally ordered subset of  $\mathbf{U}$  and hence of  $\mathbf{F}(\mathbb{R}^{n+1})$ . Thus the infimum and supremum of  $\Gamma^{-1}(\mathcal{L})$  (which lie in  $\mathbf{U}$ ) lie in the  $\mathbf{U}$ -closure of  $\Gamma^{-1}(\mathcal{L})$ . Since  $\Gamma^{-1}$  is both a poset isomorphism and a homeomorphism, the lemma follows.

**Theorem 3.8** *If  $\mathcal{K}$  is a nonempty increasing closed subset of  $\mathcal{U}$ , then the set  $\mathbf{M}_\wedge$  of minimal elements of  $\mathcal{K}$  in  $\mathcal{K}$  is nonempty and such that  $\mathcal{K} = \{f \in \mathcal{U} : \lambda \leq f, \lambda \in \mathbf{M}_\wedge\}$ . Similarly, if  $\mathcal{K}$  is a nonempty decreasing closed subset of  $\mathcal{U}$ , then the set  $\mathbf{M}_\vee$  of maximal elements of  $\mathcal{K}$  in  $\mathcal{K}$  is nonempty and such that  $\mathcal{K} = \{f \in \mathcal{U} : f \leq \mu, \mu \in \mathbf{M}_\vee\}$ .*

**Proof** Let  $h$  be an element of  $\mathcal{K}$  and let  $\mathcal{A}_h = \{f \in \mathcal{K} : f \leq h\}$ . Every totally ordered subset of  $\mathcal{A}_h$  has an infimum in  $\mathcal{U}$ . Because  $\mathcal{K}$  is closed in  $\mathcal{U}$ , it follows from the above lemma that said infimum lies in  $\mathcal{K}$  and hence in  $\mathcal{A}_h$ . We accordingly see that every totally ordered subset of  $\mathcal{A}_h$  has a lower bound in  $\mathcal{A}_h$ ; hence, by Zorn's lemma,  $\mathcal{A}_h$  has a minimal element that is also a minimal element of  $\mathcal{K}$ . Let  $\mathbf{M}_\wedge$  be the set of all such minimal elements as  $h$  ranges over  $\mathcal{K}$ . Since  $\mathcal{K}$  is increasing and every  $h \in \mathcal{K}$  is bounded from below by a  $\lambda \in \mathbf{M}_\wedge$ , we see that  $\mathcal{K} = \{f \in \mathcal{U} : \lambda \leq f, \lambda \in \mathbf{M}_\wedge\}$ . The proof for decreasing  $\mathcal{K}$  is similar.

**Corollary 3.3** (Maragos) *If  $\Phi \in \mathcal{M}_\uparrow(\mathcal{U})$ , then for all  $f \in \mathcal{U}$*

$$\Phi(f) = \bigvee \{ \mathcal{E}(f, h) : h \in \mathcal{K}_{\min}(\Phi) \}.$$

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<sup>12</sup>P. Maragos, *A unified theory of translation-invariant systems with applications to morphological analysis and coding of images*, Ph. D. thesis, Georgia Inst. of Tech., Atlanta, GA (1985).

## 4 Morphological Mappings of $\mathcal{U}$ to $\mathbf{F}(\mathfrak{R}^{n+1})$

This final section introduces and develops the space  $\mathcal{H}(\mathcal{U})$  of morphological mappings of  $\mathcal{U}$  to  $\mathbf{F}(\mathfrak{R}^{n+1})$ . It is shown that  $\mathcal{H}(\mathcal{U})$  isomorphically includes and considerably generalizes  $\mathcal{M}(\mathcal{U})$ , and is a proper setting in which to generalize the Banon-Barrera representation theorems to apply to morphological mappings of the functions in  $\mathcal{U}$ .

### 4.1 The Morphological Mapping Space $\mathcal{H}(\mathcal{U})$

**Definition 4.1** A mapping  $\sigma : \mathcal{U} \rightarrow \mathcal{P}(\mathfrak{R}^{n+1})$  is TI if

$$\sigma(f + (x, t)) = \sigma(f) + (x, t) \text{ for all } f \in \mathcal{U} \text{ and } (x, t) \in \mathfrak{R}^{n+1}.$$

If  $\sigma : \mathcal{U} \rightarrow \mathcal{P}(\mathfrak{R}^{n+1})$  is TI, then  $\ker(\sigma) \equiv \{f \in \mathcal{U} : (\mathbf{0}, 0) \in \sigma(f)\}$ .

With this we obtain a result analogous to §1.23 of section 1.3 and Proposition 3.2.

**Proposition 4.1** If  $\sigma : \mathcal{U} \rightarrow \mathcal{P}(\mathfrak{R}^{n+1})$  is TI and  $f \in \mathcal{U}$ , then

$$\sigma(f) = \{(x, t) \in \mathfrak{R}^{n+1} : f - (x, t) \in \ker(\sigma)\}.$$

If  $\mathcal{K}$  is any subset of  $\mathcal{U}$ , then  $f \mapsto \{(x, t) \in \mathfrak{R}^{n+1} : f - (x, t) \in \mathcal{K}\}$  defines a TI mapping of  $\mathcal{U}$  to  $\mathcal{P}(\mathfrak{R}^{n+1})$  whose kernel is  $\mathcal{K}$ .

As in the analogous instances, then, there is a one-to-one correspondence  $\sigma \leftrightarrow \mathcal{K} = \ker(\sigma)$  between the TI mappings of  $\mathcal{U}$  to  $\mathcal{P}(\mathfrak{R}^{n+1})$  and the subsets of  $\mathcal{U}$ . In addition, the following version of Matheron's closed kernel theorem is easily obtained.

**Theorem 4.1** A TI mapping  $\sigma$  of  $\mathcal{U}$  to  $\mathcal{P}(\mathfrak{R}^{n+1})$  is into  $\mathbf{F}(\mathfrak{R}^{n+1})$  and USC if and only if  $\ker(\sigma)$  is closed in  $\mathcal{U}$ .

The set of USC TI mappings of  $\mathcal{U}$  to  $\mathbf{F}(\mathfrak{R}^{n+1})$  will be denoted  $\mathcal{H}(\mathcal{U})$ . According to the above theorem, then, there is a one-to-one correspondence  $\sigma \leftrightarrow \mathcal{K} = \ker(\sigma)$  between the  $\sigma \in \mathcal{H}(\mathcal{U})$  and the closed subsets of  $\mathcal{U}$ .  $\mathcal{H}(\mathcal{U})$  thus obtains its hit-miss topology  $\theta$  by identification with  $\mathbf{F}(\mathcal{U})$ . Let us now observe that the closed subsets of  $\mathcal{U}$ , together with the intersection ( $\cap$ ) and union ( $\cup$ ) operations, form a complete distributive lattice. An isomorphic lattice structure is thus imposed on  $\mathcal{H}(\mathcal{U})$  by the one-to-one correspondence  $\sigma \leftrightarrow \ker(\sigma)$ . This is essentially the same situation as those respectively outlined in sections 1.3 and 3.3 for  $\mathcal{M}(\mathbf{F}) \leftrightarrow \mathbf{F}(\mathbf{F})$  and  $\mathcal{M}(\mathcal{U}) \leftrightarrow \mathbf{F}^1(\mathcal{U})$ . With (it is hoped) no confusion, the imposed lattice operations in  $\mathcal{H}(\mathcal{U})$  will be denoted by  $\cap$  and  $\cup$ .

**Theorem 4.2** *The ordering  $\subseteq$  induced in  $\mathcal{H}(\mathcal{U})$  by its lattice operations can be characterized as follows: if  $\sigma$  and  $\sigma'$  are maps in  $\mathcal{H}(\mathcal{U})$ , then*

$$\sigma \subseteq \sigma' \iff \sigma(f) \subset \sigma'(f) \text{ for all } f \in \mathcal{U} \iff \ker(\sigma) \subset \ker(\sigma').$$

*In addition, if  $\{\sigma_\alpha\}$  is any subset of  $\mathcal{H}(\mathcal{U})$ , then*

1.  $\inf\{\sigma_\alpha\} \equiv \bigcap_\alpha \sigma_\alpha$  has the kernel  $\bigcap_\alpha \ker(\sigma_\alpha)$  and

$$(\bigcap_\alpha \sigma_\alpha)(f) = \bigcap_\alpha \sigma_\alpha(f) \quad \forall f \in \mathcal{U}.$$

2.  $\sup\{\sigma_\alpha\} \equiv \bigcup_\alpha \sigma_\alpha$  has the kernel  $\overline{\bigcup_\alpha \ker(\sigma_\alpha)}$  and is the least map in  $\mathcal{H}(\mathcal{U})$  such that  $(\bigcup_\alpha \sigma_\alpha)(f) \supset \overline{\bigcup_\alpha \sigma_\alpha(f)}$  for all  $f \in \mathcal{U}$ .

3. If  $\{\sigma_\alpha\} = \{\sigma_k\}$  is a finite subset of  $\mathcal{H}(\mathcal{U})$ , then

$$(\bigcup_k \sigma_k)(f) = \bigcup_k \sigma_k(f) \quad \forall f \in \mathcal{U}.$$

**Corollary 4.1** *Hence we have the following:*

1.  $(\mathcal{H}(\mathcal{U}), \theta, \subseteq)$  is a compact ordered space.
2.  $(\mathcal{H}(\mathcal{U}), \theta, \cap, \cup)$  is a compact closed-order lattice.

**Corollary 4.2** *The mapping  $\sigma \mapsto \ker(\sigma)$  is a complete lattice isomorphism of  $(\mathcal{H}(\mathcal{U}), \cap, \cup)$  onto  $(\mathbf{F}(\mathcal{U}), \cap, \cup)$ ; i.e.,  $\sigma \mapsto \ker(\sigma)$  preserves the infima and suprema of arbitrary subsets. (See Cor. 2.12.)*

**Theorem 4.3** *The topological properties in  $\mathcal{H}(\mathcal{U})$  of the ordering relation  $\subseteq$  and the lattice operations  $\cap$  and  $\cup$  are as follows:*

1.  $\subseteq$  is a closed order in  $\mathcal{H}(\mathcal{U})$ .
2.  $\cup$  is a continuous mapping of  $\mathcal{H}(\mathcal{U}) \times \mathcal{H}(\mathcal{U})$  onto  $\mathcal{H}(\mathcal{U})$ .
3.  $\cap$ , however, is only a USC mapping of  $\mathcal{H}(\mathcal{U}) \times \mathcal{H}(\mathcal{U})$  onto  $\mathcal{H}(\mathcal{U})$ .

**Proposition 4.2** *Let  $\{\sigma_\alpha\}$  be a family of mappings in  $\mathcal{H}(\mathcal{U})$  such that  $\bigcup_\alpha \ker(\sigma_\alpha)$  is  $\mathcal{U}$ -closed. Then  $\bigcup_\alpha \sigma_\alpha(f)$  is closed for all  $f \in \mathcal{U}$  and  $(\bigcup_\alpha \sigma_\alpha)(f) = \bigcup_\alpha \sigma_\alpha(f)$  for all  $f \in \mathcal{U}$ .*

**Proof** Note that if  $f \in \mathcal{U}$ , then

$$\bigcup_\alpha \sigma_\alpha(f) = \bigcup_\alpha \{(x, t) \in \mathfrak{R}^{n+1} : f - (x, t) \in \ker(\sigma_\alpha)\}.$$

If  $\{(x_i, t_i)\}$  is an  $\mathfrak{R}^{n+1}$ -convergent sequence in the above union, then  $\{f - (x_i, t_i)\}$  is a  $\mathcal{U}$ -convergent sequence lying in  $\bigcup_\alpha \ker(\sigma_\alpha)$ .

Since  $\cup_{\alpha} \ker(\sigma_{\alpha})$  is closed, it follows that  $f - (\xi, \tau) \in \cup_{\alpha} \ker(\sigma_{\alpha})$  where  $(\xi, \tau) = \lim(x_i, t_i)$ . Thus  $(\xi, \tau) \in \cup_{\alpha} \sigma_{\alpha}(f)$ , and it follows that  $\cup_{\alpha} \sigma_{\alpha}(f)$  is closed for all  $f \in \mathcal{U}$ . The mapping  $\lambda$  of  $\mathcal{U}$  given by  $f \mapsto \cup_{\alpha} \sigma_{\alpha}(f)$  is TI and  $\ker(\lambda) \supset \cup_{\alpha} \ker(\sigma_{\alpha}) = \ker((\cup_{\alpha} \sigma_{\alpha}))$ . But since  $\cup_{\alpha} \sigma_{\alpha}$  satisfies  $(\cup_{\alpha} \sigma_{\alpha})(f) \supset \cup_{\alpha} \sigma_{\alpha}(f)$  for all  $f \in \mathcal{U}$ , it also follows that  $\ker(\lambda) \subset \ker((\cup_{\alpha} \sigma_{\alpha}))$ . Hence  $\ker(\lambda) = \ker((\cup_{\alpha} \sigma_{\alpha}))$  and the proposition follows.

**Proposition 4.3** *If  $\sigma : \mathcal{U} \rightarrow \mathcal{P}(\mathfrak{R}^{n+1})$  is a TI mapping with an up-closed kernel, then  $\sigma$  is into  $\mathcal{P}_U$ . A TI mapping  $\sigma : \mathcal{U} \rightarrow \mathbf{F}(\mathfrak{R}^{n+1})$  is into  $\mathbf{U}$  if and only if  $\ker(\sigma)$  is up-closed.*

**Proof** First suppose that

$$f \in \ker(\sigma) \iff f + (\mathbf{0}, t) \in \ker(\sigma) \quad \forall t > 0.$$

We show that  $\sigma(f) \in \mathcal{P}_U$  for all  $f \in \mathcal{U}$ . Let  $(x, t) \in \sigma(f)$  and suppose that  $t' < t$ . Since

$$\sigma(f) = \{(x, t) \in \mathfrak{R}^{n+1} : f - (x, t) \in \ker(\sigma)\}$$

it follows that  $f - (x, t) \in \ker(\sigma)$  and therefore that

$$f - (x, t) + (\mathbf{0}, \tau) \in \ker(\sigma) \quad \text{for all } \tau > 0.$$

Since  $f - (x, t) + (\mathbf{0}, \tau) = f - (x, t - \tau)$ , we see that  $f - (x, t')$  is in the kernel of  $\sigma$ . Hence  $(x, t') \in \sigma(f)$  and  $\sigma(f) \in \mathcal{P}_U$ . Now assume that  $\sigma(f) \in \mathcal{P}_U$  for all  $f \in \mathcal{U}$ . If  $f \in \ker(\sigma)$ , then  $(\mathbf{0}, 0) \in \sigma(f)$ . Since  $\sigma(f + (\mathbf{0}, t)) = \sigma(f) + (\mathbf{0}, t)$  and  $(\mathbf{0}, \tau) \in \sigma(f)$  for all  $\tau \leq 0$ , we see that  $(\mathbf{0}, 0) \in \sigma(f + (\mathbf{0}, t))$  for all positive  $t$ . Hence

$$f \in \ker(\sigma) \implies f + (\mathbf{0}, t) \in \ker(\sigma) \quad \forall t > 0.$$

If, on the other hand,  $f + (\mathbf{0}, t) \in \ker(\sigma) \quad \forall t > 0$ , then

$$(\mathbf{0}, 0) \in \sigma(f) + (\mathbf{0}, t) \quad \text{for all positive } t;$$

i.e.,  $(\mathbf{0}, -t) \in \sigma(f)$  for all positive  $t$ . Thus if  $\sigma(f) \in \mathbf{F}(\mathfrak{R}^{n+1})$ , then  $(\mathbf{0}, 0) \in \sigma(f)$  and  $f \in \ker(\sigma)$ . This completes the proof.

**Corollary 4.3** *The  $\sigma \in \mathcal{H}(\mathcal{U})$  with range in  $\mathbf{U}$  are precisely those whose kernels lie in  $\mathbf{F}^{\uparrow}(\mathcal{U})$ .*

Let  $\widetilde{\mathcal{M}}(\mathcal{U})$  denote the set of TI USC mappings of  $\mathcal{U}$  to  $\mathbf{U}$ . Since the bijection  $\Phi \mapsto \Gamma^{-1} \circ \Phi$  of  $\mathcal{M}(\mathcal{U})$  onto  $\widetilde{\mathcal{M}}(\mathcal{U})$  serves to define the latter as a compact closed-order lattice, isomorphic as such to  $\mathcal{M}(\mathcal{U})$ , we see that there is essentially no difference between the spaces  $\mathcal{M}(\mathcal{U})$  and  $\widetilde{\mathcal{M}}(\mathcal{U})$ . According to the corollary, then, the closed subspace of  $\mathcal{H}(\mathcal{U})$  identified with  $\mathbf{F}^1(\mathcal{U})$  is precisely the space  $\widetilde{\mathcal{M}}(\mathcal{U})$ . It is therefore clear that  $\mathcal{H}(\mathcal{U})$  is essentially an extension of  $\mathcal{M}(\mathcal{U})$  to a larger space of morphological mappings.

## 4.2 Banon-Barrera Representations

**Definition 4.2** Let  $g, h \in \mathcal{U}$  and let  $g \leq h$ . Then we define the bracket mapping  $\sigma_{[g,h]}(\cdot) = \cdot \Delta (g, h)$  for all  $f \in \mathcal{U}$  by

$$\sigma_{[g,h]}(f) = f \Delta (g, h) = \{(x, t) \in \mathfrak{R}^{n+1} : g \leq f - (x, t) \leq h\}.$$

**Lemma 4.1** If  $g, h \in \mathcal{U}$  and  $g \leq h$ , then  $\{f \in \mathcal{U} : g \leq f \leq h\}$  is a closed subset of  $\mathcal{U}$ .

**Proof** Let  $f_i \rightarrow f$  in  $\mathcal{U}$  and suppose that  $g \leq f_i \leq h$  for all  $i$ . We employ the convergence criteria of Theorem 2.15 for  $\{f_i\}$  to show that  $g \leq f \leq h$ . For each  $x \in \mathfrak{R}^n$ , there is a sequence  $x_i \in \mathfrak{R}^n$  with limit  $x$  such that  $f_i(x_i) \rightarrow f(x)$ . Since  $f_i(x_i) \leq h(x_i)$  for all  $i$ , we see that  $f(x) \leq \limsup h(x_i) \leq h(x)$  because  $h$  is USC. On the other hand, since  $g(x) \leq f_i(x)$  for all  $i$ , it follows that  $g(x) \leq \limsup f_i(x) \leq f(x)$ .

**Definition 4.3** If  $g, h \in \mathcal{U}$  and  $g \leq h$ , then

$$[g, h] \equiv \{f \in \mathcal{U} : g \leq f \leq h\}$$

is called a closed interval of or in  $\mathcal{U}$ .

**Remark 4.1**  $\sigma_{[g,h]} \in \mathcal{H}(\mathcal{U})$  and  $\ker(\sigma_{[g,h]}) = [g, h]$ .

As with the set-mapping case, the Banon-Barrera representation theory for  $\mathcal{H}(\mathcal{U})$  stems from the following simple lemma.

**Lemma 4.2** If  $\mathcal{K} \subset \mathcal{U}$ , then  $\mathcal{K} = \bigcup\{[g, h] : [g, h] \subset \mathcal{K}\}$ .

**Proof**  $\mathcal{K} \supset \bigcup\{[g, h] : [g, h] \subset \mathcal{K}\}$  and  $f \in \mathcal{K} \implies [f, f] \subset \mathcal{K}$ .

The Banon-Barrera representation of the  $\sigma \in \mathcal{H}(\mathcal{U})$  is now easily obtained. We simply reproduce their proof in the present context.<sup>8</sup>

**Theorem 4.4** *If  $\sigma \in \mathcal{H}(\mathcal{U})$ , then  $\sigma = \bigcup \{\sigma_{[g,h]} : [g,h] \subset \ker(\sigma)\}$ . Moreover, if  $f \in \mathcal{U}$ , then  $\sigma(f) = \bigcup \{\sigma_{[g,h]}(f) : [g,h] \subset \ker(\sigma)\}$ .*

**Proof** By the last lemma,  $\ker(\sigma) = \bigcup \{[g,h] : [g,h] \subset \ker(\sigma)\}$ . Since  $\ker(\sigma_{[g,h]}) = [g,h]$ , it therefore follows that

$$\ker(\sigma) = \bigcup \{ \ker(\sigma_{[g,h]}) : [g,h] \subset \ker(\sigma) \}.$$

We may now obtain  $\sigma = \bigcup \{\sigma_{[g,h]} : [g,h] \subset \ker(\sigma)\}$  from the fact that  $\sigma \mapsto \ker(\sigma)$  is a complete lattice isomorphism (Cor. 4.2). Moreover, since

$$\bigcup \{ \ker(\sigma_{[g,h]}) : [g,h] \subset \ker(\sigma) \} = \ker(\sigma)$$

is closed, the rest of the theorem follows by Proposition 4.2.

We now derive the minimal-representation form of this theorem.

**Definition 4.4** *If  $\sigma \in \mathcal{H}(\mathcal{U})$ , then a collection  $\mathcal{B}$  of closed intervals contained in  $\ker(\sigma)$  is said to satisfy the representation condition for  $\sigma$  if every closed interval contained in  $\ker(\sigma)$  is contained in an interval of  $\mathcal{B}$ . The class of maximal closed intervals contained in  $\ker(\sigma)$  is denoted  $\mathcal{B}(\sigma)$  and is called the basis of  $\sigma$ .*

**Proposition 4.4** *If  $\sigma \in \mathcal{H}(\mathcal{U})$  and if  $\mathcal{B}$  satisfies the representation condition for  $\sigma$ , then for all  $f \in \mathcal{U}$*

$$\sigma(f) = \bigcup \{ \sigma_{[g,h]}(f) : [g,h] \in \mathcal{B} \}.$$

**Proof** Since  $[g,h] \in \mathcal{B} \implies [g,h] \subset \ker(\sigma)$ , Theorem 4.4 shows that

$$\sigma(f) \supset \bigcup \{ \sigma_{[g,h]}(f) : [g,h] \in \mathcal{B} \} \quad \forall f \in \mathcal{U}.$$

On the other hand, since every closed interval contained in  $\ker(\sigma)$  is contained in an interval of  $\mathcal{B}$ , we see that

$$\sigma(f) \subset \bigcup \{ \sigma_{[g,h]}(f) : [g,h] \in \mathcal{B} \} \quad \forall f \in \mathcal{U}.$$

**Remark 4.2** *If  $\sigma \in \mathcal{H}(\mathcal{U})$ , then  $\mathcal{B}(\sigma)$  is contained in every  $\mathcal{B}$  that satisfies the representation condition for  $\sigma$ .*

Banon and Barrera<sup>8</sup> have shown that the basis of each  $\Psi \in \mathcal{M}(\mathbf{F})$  satisfies the representation condition for  $\Psi$  by establishing three lemmas which we now proceed to show are valid in the present setting. The first is Lemma 3.3. The second and third are as follows.

**Lemma 4.3** *Let  $\{g_i\}$  and  $\{h_i\}$  be sequences in  $\mathcal{U}$  such that  $g_i \downarrow g$  in  $\mathcal{U}$ ,  $h_i \uparrow h$  in  $\mathcal{U}$ , and  $g_i \leq h_i$  for all  $i$ . Further, suppose that  $f \in \mathcal{U}$  is such that  $f \in [g, h]$ . Then there is a sequence  $\{f_i\}$  in  $\mathcal{U}$  such that  $f_i \in [g_i, h_i]$  for all  $i$  and  $f_i \rightarrow f$  in  $\mathcal{U}$ .*

**Proof** Following Banon and Barrera, we let  $f_i = (g_i \vee f) \wedge h_i$ . Then for all  $i$  we see that  $f_i \in \mathcal{U}$ ,  $g_i \leq f_i \leq h_i$ , and  $f_i = g_i \vee (f \wedge h_i)$ . Thus it follows that  $f_i \rightarrow f$ .

**Lemma 4.4** *Let  $\mathcal{A} \subset \mathcal{U}$  and let  $\mathcal{C}$  be a totally ordered set of closed intervals of  $\mathcal{U}$  contained in  $\mathcal{A}$ . Then  $\bigvee \mathcal{C}$  lies in the  $\mathcal{U}$ -closure of  $\mathcal{A}$ .*

**Proof** Following Banon and Barrera, we let

$$\mathcal{L} = \{f \in \mathcal{U} : \exists (g \in \mathcal{U}) [(f, g) \in \mathcal{C}] \vee (g, f) \in \mathcal{C}\}.$$

First we prove that  $\mathcal{L}$  is totally ordered. Let  $f, f' \in \mathcal{L}$ . Then there are  $g, g' \in \mathcal{U}$  such that one of the following holds:

1.  $[f, g] \in \mathcal{C}$  and  $[f', g'] \in \mathcal{C}$ .
2.  $[f, g] \in \mathcal{C}$  and  $[g', f'] \in \mathcal{C}$ .
3.  $[g, f] \in \mathcal{C}$  and  $[f', g'] \in \mathcal{C}$ .
4.  $[g, f] \in \mathcal{C}$  and  $[g', f'] \in \mathcal{C}$ .

Since  $\mathcal{C}$  is totally ordered, we see in any case that either  $f \leq f'$  or  $f' \leq f$ . By Lemma 3.3,  $\bigwedge \mathcal{L}$  and  $\bigvee \mathcal{L}$  lie in  $\overline{\mathcal{L}}$ . There are thus sequences  $\{g_i\}$  and  $\{h_i\}$  in  $\mathcal{U}$  such that  $g_i \downarrow \bigwedge \mathcal{L}$  in  $\mathcal{U}$  and  $h_i \uparrow \bigvee \mathcal{L}$  in  $\mathcal{U}$ ; moreover, the  $g_i$  and  $h_i$  can be chosen so that  $g_i \leq h_i$  for all  $i$ . Let  $f \in \bigvee \mathcal{C}$ . Then  $\bigwedge \mathcal{L} \leq f \leq \bigvee \mathcal{L}$ , and it follows that there is a sequence  $\{f_i\}$  converging to  $f$  in  $\mathcal{U}$  such that  $g_i \leq f_i \leq h_i$ . It furthermore follows that there is a closed interval in  $\mathcal{C}$  that contains  $[g_i, h_i]$ ; hence,  $f_i \in \mathcal{A}$  for all  $i$  and  $f \in \overline{\mathcal{A}}$ .

The minimal-representation form of Theorem 4.4 can now be stated and proved as follows. (Again, we simply reproduce the proof of Banon and Barrera<sup>8</sup> in the present setting.)

**Theorem 4.5** *If  $\sigma \in \mathcal{H}(\mathcal{U})$ , then  $\mathcal{B}(\sigma)$  satisfies the representation condition for  $\sigma$ ; hence,  $\sigma(\cdot) = \bigcup \{\cdot \Delta (g, h) : [g, h] \in \mathcal{B}(\sigma)\}$  is a minimal representation of  $\sigma$  as a supremum of bracket mappings.*

**Proof** Let  $[g, h] \in \ker(\sigma)$ . Since there is a totally ordered set  $\mathcal{L}$  of closed intervals contained in  $\ker(\sigma)$  such that  $[g, h] \in \mathcal{L}$ , it follows by Lemma 2.1 of Maragos<sup>12</sup> that there exists a maximal totally ordered set  $\mathcal{N}$  of closed intervals contained in  $\ker(\sigma)$  such that  $\mathcal{L} \subset \mathcal{N}$ . Therefore,  $\bigvee \mathcal{N}$  is a closed interval  $[g', h']$  of  $\mathcal{U}$  such that

$$[g, h] \subset \bigvee \mathcal{L} \subset \bigvee \mathcal{N} = [g', h'].$$

By the last lemma and the closedness of  $\ker(\sigma)$  in  $\mathcal{U}$ , it finally follows that  $[g', h'] \in \mathcal{B}(\sigma)$ . This completes the proof.

## 5 Conclusion

This report has given the results of my research aimed at discovering and clarifying the fundamentals of greyscale-image morphology for its application to the image-processing tasks of ATR. Specifically, the report develops the elements of a topologized greyscale-image morphology on the basis of closed-set morphology by rigorously pursuing the umbra method. I conclude with a summary retracing of the path followed.

First it was shown that the set  $\mathcal{U}$  of ERV USC functions of  $n$  real variables (whose bounded nonnegative members were chosen to represent greyscale images) can be given a morphologically characteristic hit-miss topology and complete lattice structure by being identified with either of two spaces of umbrae of the functions in  $\mathcal{U}$ . One is the subspace  $\mathbf{U}$  of umbrae in  $\mathbf{F}(\mathfrak{R}^{n+1})$  (the closed subsets of  $\mathfrak{R}^{n+1}$  equipped with Matheron's hit-miss topology), and the other is a certain subspace  $\widehat{\mathbf{V}}$  of the subspace  $\mathbf{V}$  of umbrae in  $\mathbf{F}(\mathfrak{R}^n \times \mathfrak{R}^{(e)})$  (the closed subsets of  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$  equipped, likewise, with Matheron's hit-miss topology). Recall that  $\widehat{\mathbf{V}}$  consists of the umbrae in  $\mathbf{V}$  that include the entire horizontal plane in  $\mathfrak{R}^n \times \mathfrak{R}^{(e)}$  at  $-\infty$ . I demonstrated that  $\mathcal{U}$  admits a unique morphological structure consisting of a natural hit-miss topology and a similarly natural (complete) lattice algebra by showing (1) that  $\mathbf{U}$  and  $\widehat{\mathbf{V}}$  are topologically equivalent and lattice isomorphic under the natural correspondence between the elements of these spaces, and (2) that the mappings of  $\mathbf{U}$  and  $\widehat{\mathbf{V}}$  to  $\mathcal{U}$  provided by the supremum formula  $f_U(x) = \sup\{t : (x, t) \in U\}$  are each not only one-to-one onto  $\mathcal{U}$ , but in fact the same bijections relative to the above natural correspondence. Apart from the myopic topology considerations of section 2.6, this was the accomplishment in place at the end of section 2.7. The remainder of section 2 was devoted to the establishment (in  $\mathbf{U}$  and  $\mathcal{U}$ ) of an algebra of Minkowski sums and differences, and, equally importantly, to the determination of the semicontinuity properties of this algebra relative to both the hit-miss and myopic topologies of  $\mathbf{U}$  and  $\mathcal{U}$ .

Next, and on the basis of the foregoing, the morphological transform space  $\mathcal{M}(\mathcal{U})$  was defined and studied in section 3. As with the morphological transformation space  $\mathcal{M}(\mathbf{F})$  of euclidean morphology,  $\mathcal{M}(\mathcal{U})$  consists of the TI USC transformations of  $\mathcal{U}$ . Indeed, the parallelism between  $\mathcal{M}(\mathbf{F})$  and  $\mathcal{M}(\mathcal{U})$  is quite far reaching. By taking advantage of the kernel theory of Maragos, I showed that the transforms constituting  $\mathcal{M}(\mathcal{U})$  are characterized by two features of their kernels, namely, up-closedness and topological closedness in  $\mathcal{U}$ . In other words, there is a one-to-one correspondence between  $\mathcal{M}(\mathcal{U})$  and the subspace  $\mathbf{F}^\uparrow(\mathcal{U})$  of up-closed elements of  $\mathbf{F}(\mathcal{U})$  (the space of closed subsets of  $\mathcal{U}$ ). Because  $\mathbf{F}^\uparrow(\mathcal{U})$  is both topologically closed in  $\mathbf{F}(\mathcal{U})$  and a complete sublattice of  $(\mathbf{F}(\mathcal{U}), \cap, \cup)$  (results proved in sect. 3), the kernel-based correspondence between  $\mathbf{F}^\uparrow(\mathcal{U})$  and  $\mathcal{M}(\mathcal{U})$  invests the latter with a hit-miss topology and a complete lattice structure. Again, the parallelism with  $\mathcal{M}(\mathbf{F})$  is evident. Section 3.5 continued to pursue this parallelism by going on to establish Maragos' extension to the increasing transforms in  $\mathcal{M}(\mathcal{U})$  of the Matheron-Maragos representations of the increasing transformations in  $\mathcal{M}(\mathbf{F})$ .

Finally, in section 4, the space  $\mathcal{H}(\mathcal{U})$  of morphological mappings of  $\mathcal{U}$  to  $\mathbf{F}(\mathfrak{R}^{n+1})$  was introduced, developed, and shown (a) to isomorphically include and considerably generalize  $\mathcal{M}(\mathcal{U})$  and (b) to be a proper setting in which to generalize the Banon-Barrera representation theorems to the realm of greyscale morphology. This generalization, detailed in section 4.2, is perhaps the most important contribution of this report. It now remains to use  $\mathcal{H}(\mathcal{U})$  and the generalized Banon-Barrera representations to increase and clarify the capabilities of greyscale morphology for the image-processing tasks of ATR.

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