Minimal Fixturing of Frictionless Assemblies: Complexity and Algorithms

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Abstract

In many assembly tasks, it is necessary to insure the stability of a subcollection of contacting objects. To achieve stability, it is often necessary to introduce fixture elements (also called "fingers" in some work) to help hold objects in place. In this paper, the complexity of stabilizing multiple contacting bodies with the fewest number of fixture elements possible is considered. Standard fixture elements of the type explored in previous single-object grasping work are considered, along with two generalized fixture element variants. The types of stability considered are: form-closure (complete immobility of the assembly); stability with respect to a specific external force and torque on each body; and stability in the neighborhood of a specific external force and torque on each body. The major result is that for most of the combinations of fixture element varieties, and types of stability considered, achieving an optimal solution (that is, finding a smallest set of fixture elements yielding stability) is *NP*-hard. However, for many fixturing problems it seems likely that suboptimal, yet acceptably small solutions can be found in polynomial time, and some candidate algorithms are presented.

1 Introduction

Stability of bodies is an important consideration in planning an assembly sequence. While the problem of synthesizing stable grasps for a *single* object has received considerable attention[9, 10, 13, 18, 16, 12], stabilization of multiple bodies has only recently been a topic of interest[19, 14]. In the single-object case, the emphasis has been on synthesizing a robust, stable grasp. Several different types of stability can be considered. A *form-closure* of an object is a grasp that completely immobilizes the object; equivalently, the object will remain motionless under any applied external forces and torque. A less restrictive form of stability requires the grasp to balance only a particular external force and torque. Other forms of stability involving friction have been considered, but we will defer frictional considerations to future work.

In this paper, we take on the task of stabilizing a collection of contacting rigid bodies without friction. We will assume that we are given a collection of *fixtures elements* which are sufficient to stabilize the entire assembly.[1] For brevity, we shorten the phrase "fixture element" to merely "fixture." The actual definition of a fixture is deferred to section 2 for the moment, because we consider several variations of fixturing in this paper. Figure 1 shows one of the fixture variations considered. In general, the set of fixtures available is assumed to be a potentially very large collection; for example, the fixtures might be generated by discretizing all exterior surfaces of the objects with fixture locations. Further, we imagine that every fixture has some nonnegative cost associated with it, and that we would like to select an inexpensive subcollection of fixtures that stabilizes the assembly. For example, the cost of a fixture might depend on:

- *Reachability.* A fixture's cost might indicate the relative difficulty in actually establishing the fixture at that specific point in space. Certain fixtures would then be given a high cost because they occurred on regions of objects which were difficult to reach.

- *Fragility.* There may be reasons to prefer not to fixture particular bodies in the assembly unless absolutely necessary. This could be indicated by making fixtures that occur on those objects expensive.

- *Constant cost.* Every fixture might be equally expensive. If we simply scale the cost so that each fixture costs a unit amount, then a subcollection's cost is simply its size.

In this paper, we are concerned with the last, and simplest, cost function—selecting the smallest possible fixture set that achieves stability (figure 1). A lower bound for the complexity of selecting a minimal size stabilizing fixture set immediately establishes lower bounds for more general cost functions.

A very different line of attack would be to generalize previous works on optimal grasps. For example, Kirkpatrick *et al.*[8] descibes the measure of stability of a grasp based on a geometric interpretation of Steinitz' theorem[16, 5]. Trinkle[18] similarly proposes a stability measure that can be computed in terms of a linear program, and then extends

[1] This means of course that *exceptional* objects, as defined by Mishra *et al.*[16] are restricted from our consideration, since they can never be stabilized by a fixture set in a frictionless environment.

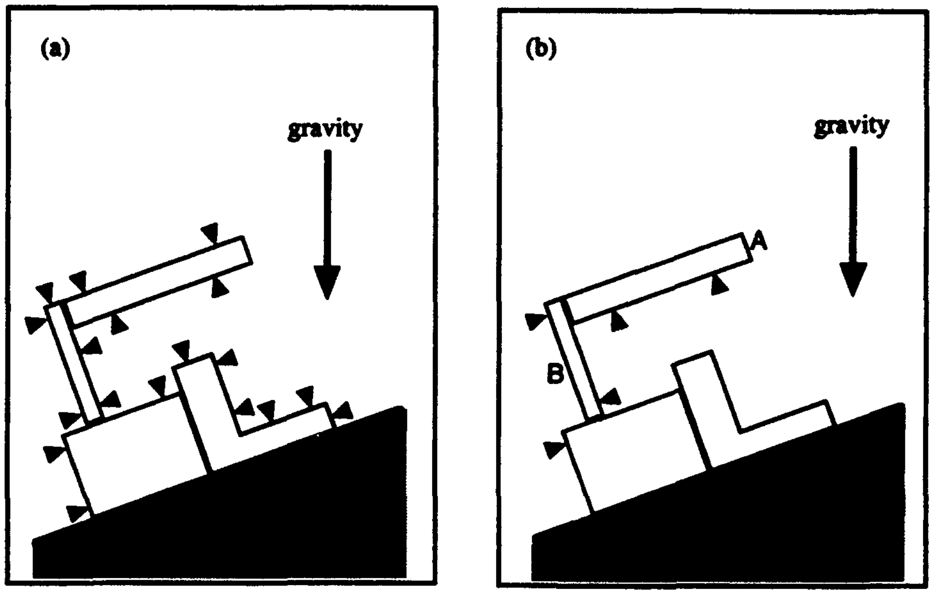


Figure 1: (a) An assembly and its set of potential fixtures. (b) The smallest set of fixtures possible that stabilizes the assembly due to gravity, assuming object A is very massive. The seemingly odd fixturing on object B is because object A slides down and leftward, attempting to force object B to rotate counterclockwise.

this work to measure and synthesize grasps (using, hopefully, a small number of fixtures) for multi-body assemblies. Similarly, Markenscoff and Papadimitriou[13] and Li and Sastry[10] also discuss what constitutes a good grasp for single-object assemblies, given a specific task in mind. We believe that these are all good criterion, and should be extended to the case of multi-body assemblies. However, before addressing these considerations, we believe that the issue of the *number* of fixtures used to stabilize an assembly must be addressed. For single-object assemblies, this is often not a large concern, but for multi-body assemblies, the size of the fixture set is clearly a factor to be considered. Although it is not clear that small fixture sets are in general good fixture sets, the existence, and complexity of finding fixture sets of a specific size seems a reasonable starting point in considering multi-body assemblies. Clearly, the size of the fixture set could be an important consideration when there are multiple fixturing sets yielding equally optimal grasps; in this case, one would probably be interested in the smallest such set. Accordingly, our goal in this paper is to establish lower-bounds on the complexity of finding minimal fixture sets.

In this paper, we define three varieties of fixturing, and consider three different types of stability. Sections 5, 6 and 7 show that it is *NP*-hard to find a minimal stabilizing fixture set for eight of the resulting nine combinations of fixturing and stability. In the ninth case, section 8 shows that a minimal stabilizing set can be found in polynomial time if the contact graph of the assembly is acyclic, but we have not found a lower bound if the graph is cyclic. Polynomial-time algorithms yielding suboptimal, but hopefully small fixture sets, are consider in section 8.

2 Contacts and Fixtures

In this paper, we restrict our consideration to polyhedral objects. Thus, the contact regions between objects are always polygons (or degenerate polygons).[2] Because we do not consider friction, in discussing stability it is sufficient to assume that contact forces arise only at the vertices of the contact region[17]. Thus, we assume that contact between objects can be characterized in terms of only finitely many contact points. At each contact, a surface normal is assumed to be well-defined. Vertex-to-vertex contacts and other degenerate contact geometries are not considered; Palmer[17] discusses the complexity issues that arise when normals are ill-defined.

For illustrative purposes, we will mostly consider planar systems of n objects. All of the results and algorithms in this paper are trivially extended to three-dimensional systems. We will denote matrices and vectors using boldface type; the ith component of a vector $\mathbf{b}$ is the scalar b_i, written in regular type. The symbol $\mathbf{0}$ denotes on appropriately sized vector or matrix of zeros. The notation $\mathbf{b} \geq \mathbf{0}$ indicates that $b_i \geq 0$ for all i. Given vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s \in \mathbf{R}^r$, the $r \times s$ matrix whose ith column is $\mathbf{b}_i$ is written as $[\,\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_r\,]$. Given matrices $\mathbf{A} \in \mathbf{R}^{r\times s}$ and $\mathbf{B} \in \mathbf{R}^{r\times t}$, the notation $[\,\mathbf{A} \mid \mathbf{B}\,]$ denotes the $r \times (s+t)$ matrix whose first s columns are the columns of $\mathbf{A}$, and whose last t columns are the columns of $\mathbf{B}$.

Let $\mathbf{n} \in \mathbf{R}^2$ be a force acting on a body at some point $\mathbf{p} \in \mathbf{R}^2$ in a global coordinate system. The generalized force $\mathbf{q} \in \mathbf{R}^3$ on the body due to $\mathbf{n}$ is

$$\mathbf{q} = (\mathbf{n}, (\mathbf{p} - \mathbf{c}) \times \mathbf{n})$$

where $\mathbf{c} \in \mathbf{R}^2$ denotes the body's center of mass. If the net generalized force acting on body i is $\mathbf{q}_i \in \mathbf{R}^3$, the generalized force for the entire collection of the n bodies is denoted by the vector $\mathbf{Q} \in \mathbf{R}^{3n}$, where $\mathbf{Q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$. We shall refer to both $\mathbf{q} \in \mathbf{R}^3$ and $\mathbf{Q} \in \mathbf{R}^{3n}$ as "forces"; which sort of quantity we are referring to should be clear from the context.

2.1 Contacts

Let the total number of contact points between bodies in an assembly be denoted N_c, and suppose that the ith contact occurs between bodies r and s at point $\mathbf{p}$, with the surface normal $\mathbf{n}$ at $\mathbf{p}$ oriented as show in figure 2a. The generalized force in $\mathbf{R}^3$ acting on body r is in the direction

$$\mathbf{d}_r = (\mathbf{n}, (\mathbf{p} - \mathbf{c}_r) \times \mathbf{n})$$

while the force on body s is in the direction

$$\mathbf{d}_s = -(\mathbf{n}, (\mathbf{p} - \mathbf{c}_s) \times \mathbf{n}).$$

[2]Actually, objects with curved surfaces can also be considered, as long as the resulting contact regions are polygonal. An example of a disallowed situation would be a cylinder resting upright on a plane, so that the contact region is a disk in the plane. Baraff[2] discusses the difficulties that arise in considering nonpolygonal contact regions.

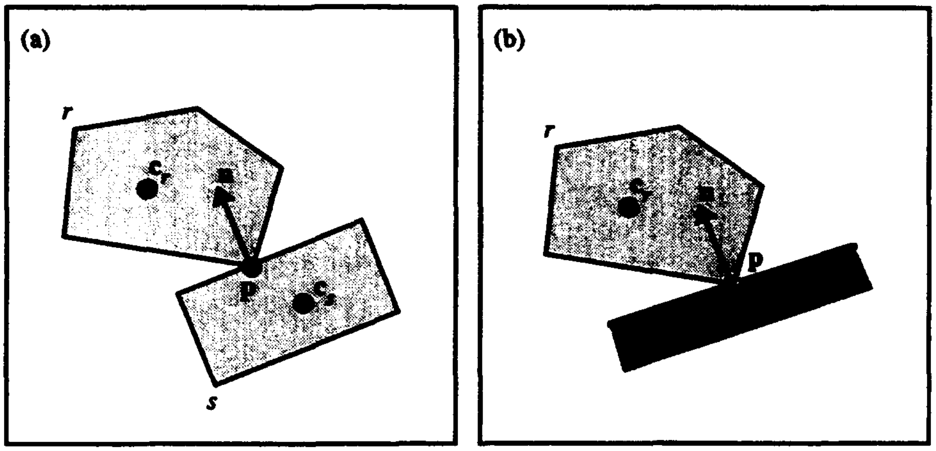


Figure 2: (a) A contact between two objects in the assembly. (b) Contact between an object and an immovable obstacle.

The direction of the generalized force on the entire collection of objects from the ith contact is therefore

$$\mathbf{u}_i = (\overbrace{\mathbf{0},\ldots,\mathbf{0}}^{r-1},\mathbf{d}_r,\overbrace{\mathbf{0},\ldots,\mathbf{0}}^{(s-1)-r},\mathbf{d}_s,\overbrace{\mathbf{0},\ldots,\mathbf{0}}^{n-s})$$

where $\mathbf{0} \in \mathbf{R}^3$.

The actual force $\mathbf{Q}_i$ due to the ith contact has the form

$$\mathbf{Q}_i = \lambda_i \mathbf{u}_i$$

where λ_i is a nonnegative scalar. The nonnegativity of λ_i arises from the restriction that contact forces be compressive.

It is also necessary to represent predetermined mobility constraints; for example, parts of an assembly might be resting on an immovable table. We will handle this by assuming that contact points can occur between some body r and some immovable obstacle (figure 2b). The immovable obstacle is not considered one of the n bodies in our collection. In this case, the force direction $\mathbf{u}_i$ has the form

$$\mathbf{u}_i = (\overbrace{\mathbf{0},\ldots,\mathbf{0}}^{r-1},\mathbf{d}_r,\overbrace{\mathbf{0},\ldots,\mathbf{0}}^{n-r}) \tag{1}$$

and, as before, $\mathbf{Q}_i = \lambda_i \mathbf{u}_i$.

The net generalized force due to contact is simply the sum

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_{N_c} \mathbf{u}_{N_c}$$

where all the λ_i's are nonnegative. For notational ease, we define the matrix $\mathbf{U} \in \mathbf{R}^{3n \times N_c}$ as

$$\mathbf{U} = [\, \mathbf{u}_1 \; \mathbf{u}_2 \; \ldots \; \mathbf{u}_{N_c} \,]. \tag{2}$$

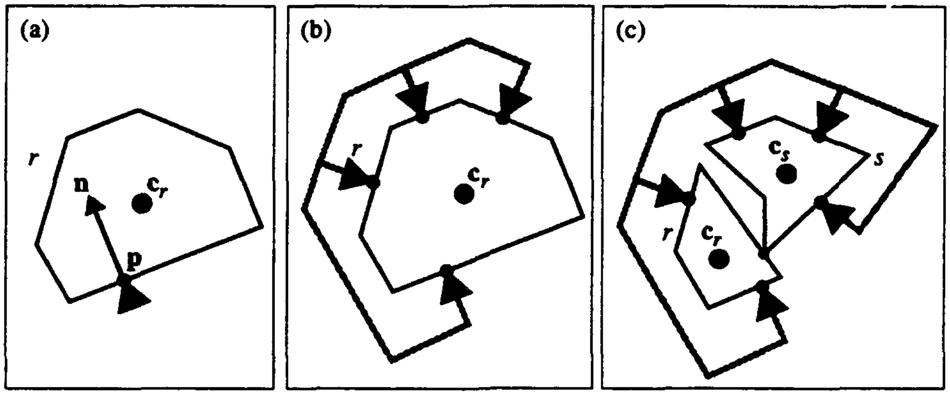


Figure 3: (a) A regular fixture. (b) A clamp, acting on a single object. If the clamp is selected, the object is completely immobilized. (c) A clamp acting on two different objects r and s, which are in contact. Selecting this clamp removes many, but not all degrees of freedom from the two objects.

Any attainable net contact force can then be written in the form

$$\mathbf{U}\boldsymbol{\lambda}$$

where each component λ_i of $\boldsymbol{\lambda} \in \mathbf{R}^{N_c}$ is nonnegative.

2.2 Fixtures

In stabilizing assemblies, we will consider three varieties of fixtures. In previous single-object fixturing work, a fixture has been defined in terms of a supporting, immobile finger that touches an object at a specific point. We will call this sort of fixture a *regular fixture*. A regular fixture on body r at location $\mathbf{p}$ produces a generalized force on the assembly in the direction

$$\mathbf{v} = (\overbrace{\mathbf{0}, \ldots, \mathbf{0}}^{r-1}, \mathbf{d}, \overbrace{\mathbf{0}, \ldots, \mathbf{0}}^{n-r}) \tag{3}$$

where $\mathbf{d} = (\mathbf{n}, (\mathbf{p} - \mathbf{c}_r) \times \mathbf{n})$ and $\mathbf{n}$ is the inwards pointing surface-normal of body r at $\mathbf{p}$ (figure 3a). We assume that fixtures generate only compressive forces, so that the fixture force generated has the form $\alpha\mathbf{v}$ for some nonnegative scalar α. Regular fixtures, and the single-object contact occurring in figure 2b appear to produce identical sorts of forces. Note however that the force due to contact in figure 2b is always available to us, whereas the fixture force $\alpha\mathbf{v}$ is available only if we have decided to select that particular fixture in stabilizing our assembly.

The force vector $\mathbf{Q} = \alpha\mathbf{v}$ generated by a regular fixture is very sparse, since a regular fixture produces a force on only a single body. For multi-body systems, we might consider problems in which a single fixture could act on more than one body at a time. Accordingly, we offer the notion of a *generalized fixture*. We will define a generalized fixture as a force that acts compressively on multiple bodies, but but in a uniform manner; that is, the force due to a single fixture of this type has the form

$$\mathbf{Q} = \alpha \mathbf{v}$$

where $\alpha \geq 0$ and $\mathbf{v}$ is an arbitrary vector in $\mathbf{R}^{3n}$. Note that the forces generated by a single generalized fixture occupy a one-dimensional subspace of $\mathbf{R}^{3n}$. Generalized fixtures might arise from a gripping mechanism designed to always impart an equal force to a number of different bodies, at different points and different directions. It is not clear to us that this notion of a generalized fixture has any practical correspondence to real-world problems. However, the complexity bounds obtained in section 5 for generalized fixturing furnish valuable insights into regular fixturing of single-object assemblies.

A completely different variety of fixture arises by grouping a number of regular fixture together. Consider a set of k regular fixtures, with the ith such fixture generating a generalized force in the direction $\mathbf{v}_i \in \mathbf{R}^{3n}$, with each $\mathbf{v}_i$ being sparse, as in equation (3). We can group the k fixtures together into a generalized fixture, which can then generate forces of the form

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

where the λ's are all nonnegative. Thus, the set of forces produced by this single fixture spans an k-dimensional space (assuming the vectors $\mathbf{v}_1$ through $\mathbf{v}_k$ are linearly independent). Unlike regular fixtures, the cost of using all k fixtures is unit (even if some of those fixtures turn out to be superfluous). We will call such a fixture a *clamp*. A clamp may act on a single object (figure 3b), or on two or more objects (figure 3c).

The reason for this terminology is because this generalization does in fact capture the concept of stabilizing an assembly by inserting mechanical clamps. Imagine that we have available to us gripping mechanisms which can be attached to a single object and achieve form-closure (complete immobilization of that object). If we were interested in the minimum number of such mechanisms (that is, the minimum number of clamps) necessary to attain stability, we would count each clamp as having unit cost, no matter how many degrees of freedom the clamp actually eliminated. Note that the definition we have given for a clamp allows a clamp to apply to two or more objects, as in figure 3c; However, our complexity proofs concerning clamps are based on fixture groups that do in fact act on only a single body, as in figure 3b. Thus, we will show that the complexity of fixturing with clamps, as we have defined them, holds even if each clamp affects only a single body.

2.3 Fixture Forces

Given a set of N_f potential fixtures (either regular or generalized), we will denote the direction of the force due to the ith fixture as $\mathbf{v}_i \in \mathbf{R}^{3n}$. If all the fixtures are used in stabilizing the assembly, attainable fixture forces have the form

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{N_f} \mathbf{v}_{N_f}$$

where each α_i is nonnegative. Accordingly, we define the matrix $\mathbf{V} \in \mathbf{R}^{3n \times N_f}$ by writing

$$\mathbf{V} = [\, \mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \; \mathbf{v}_{N_f} \,]. \tag{4}$$

Attainable fixture forces, using the entire set of fixtures, have the form

$$\mathbf{V}\boldsymbol{\alpha}$$

where $\boldsymbol{\alpha} \in \mathbf{R}^{N_f}$ is a vector satisfying $\boldsymbol{\alpha} \geq \mathbf{0}$.

We will be concerned with studying the forces attainable using only a subset of all the fixtures. Let us denote a subset of the entire fixture collection as an index set $F \subseteq \{1, 2, \ldots, N_f\}$. Given an index set F, if $i \in F$ we say that the fixture set F contains the ith fixture. For a given fixture set F, only those fixtures in the set can contribute to the total fixture force acting on the assembly. Let f_i be the ith element of F, let $k = |F|$, and define the matrix $\mathbf{V}_F \in \mathbf{R}^{3n \times k}$ by

$$\mathbf{V}_F = [\, \mathbf{v}_{f_1} \; \mathbf{v}_{f_2} \; \ldots \; \mathbf{v}_{f_k} \,]. \tag{5}$$

(Without loss of generality, we may assume that $f_1 < f_2 < \cdots < f_k$.) The fixture forces attainable using only the fixture set F can then be written as

$$\mathbf{V}_F \boldsymbol{\alpha}$$

where now $\boldsymbol{\alpha} \in \mathbf{R}^k$ and $\boldsymbol{\alpha} \geq \mathbf{0}$.

For clamps, almost the same notation can be used—the difference is essentially bookkeeping. In the case of fixturing with clamps, the matrix $\mathbf{V}$ is the matrix one would obtain if each clamp was treated as a set of unrelated regular fixtures. Thus, if the number of regular fixtures grouped to form the ith clamp is c_i and there are r clamps, $\mathbf{V}$ contains $\sum_{i=1}^{r} c_i$ columns. Given a selection F of clamps, the matrix $\mathbf{V}_F$ is obtained in a similar manner; the regular fixtures comprising all the selected clamps are treated as completely unrelated, and used to form the columns of $\mathbf{V}_F$. Note that the number of columns of $\mathbf{V}_F$ may depend on the specific set F, since $\mathbf{V}_F$ will have

$$\sum_{i=1}^{|F|} c_{f_i}$$

columns.

Having defined the ways in which the fixture and contact forces can combine, we can proceed to introduce the various types of stability considered in this paper. Following that, we will consider the complexity of finding fixture sets F of minimum size that achieve stability for a given assembly.

3 Types of Stability

In this paper, we consider three different types of stability. We present them in the order of least restrictive to most restrictive; that is, each successive notion of stability subsumes its predecessors. We describe the stability of the assembly given that a particular subset F of the available fixtures has been selected to help stabilize the assembly. For completeness, we give a formal definition of each type of stability, although these ideas are certainly not new[12, 16].

3.1 Directional Stability

The first type of stability we consider is stability of the assembly with respect to a given generalized external force $\mathbf{Q}_{ext}$. We will call this type of stability *directional stability*:

DEFINITION. *An assembly with a fixture set F has directional stability with respect to an applied force $\mathbf{Q}_{ext}$ if the contact and fixture forces that arise in response to $\mathbf{Q}_{ext}$ sum to exactly $-\mathbf{Q}_{ext}$, resulting in a net force of zero on the assembly.*

Since our assemblies are frictionless and have well-defined contact normals at each contact point, the acceleration of an assembly in response to a given applied force is unique[7, 4]. In particular, for any choice of $\mathbf{Q}_{ext}$, directional stability is easily determined.[3] A simple result is that for a frictionless assemblies, if it is possible to attain contact and fixture forces which sum to $-\mathbf{Q}_{ext}$, then in fact the contact and fixture forces will sum to $-\mathbf{Q}_{ext}$[4, 1].[4] If this is the case, then the applied force $-\mathbf{Q}_{ext}$ is completely canceled, the net force on each body is zero, and no part of the assembly moves. Directional stability with respect to a given $\mathbf{Q}_{ext}$ is therefore easily determined by seeing if the linear program

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} + \mathbf{Q}_{ext} = \mathbf{0}, \qquad \boldsymbol{\lambda} \geq \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\alpha} \geq \mathbf{0} \tag{6}$$

possesses a solution $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$. If this linear program has no solution, then the assembly is unstable and will begin moving under the applied force $\mathbf{Q}_{ext}$; in this case, the actual motion can be determined by solving a quadratic program[11, 1].

Unfortunately, directional stability admits situations which we would consider to be inherently unstable. Consider the assembly in figure 4a consisting of a single object A in contact with an immovable horizontal surface. Potential fixtures are indicated in the figure. If the external force applied to the system acts straight down upon A, then the assembly is directionally stable if we choose $F = \emptyset$; that is, if no fixtures are chosen. However, the slightest perturbation of the external force to one side or the other renders the assembly directionally unstable, given the choice $F = \emptyset$. A more extreme example is shown in figure 4b. In this case, given the fixture set $F = \{2\}$, the assembly is directionally stable

[3] Palmer[17]'s result that determining stability of frictionless system is *NP*-hard does not apply because of our assumption of well-defined contact normals.

[4] This is not to say that the forces at individual contacts and fixtures are well-determined: typically, these forces will be indeterminate. The net effect of the forces on bodies however, will always be well-defined.

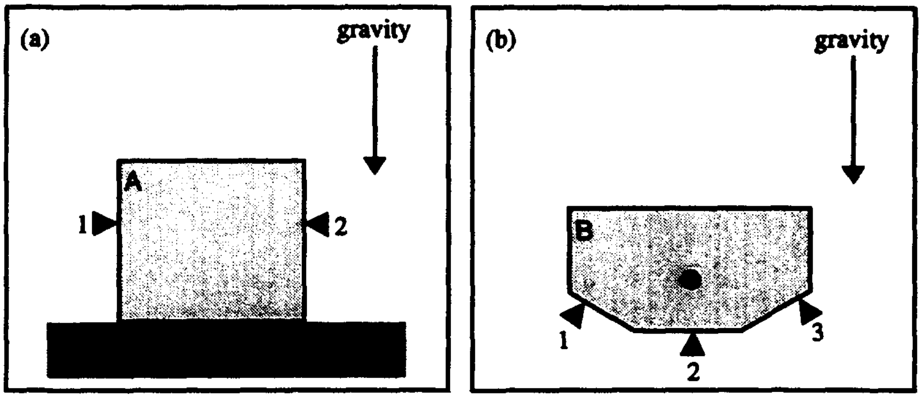


Figure 4: (a) The assembly is stable without any fixtures, but only if the external force is directed straight down. (b) The center of mass of the object lies directly above the central fixture. The assembly is stable using only fixture 2, but if gravity, or the object is perturbed slightly, fixtures 1 and 3 are required.

with respect to a downward applied force through the center of mass. Now however, either a perturbation in the applied force *or* a perturbation in object B's configuration (either a rotation or translation) renders the assembly directionally unstable.

3.2 Robust Directional Stability

In the next section, we will prove lower bound results with respect to directional stability. It can be reasonably argued that if these proofs depend on configurations such as figure 4b (that is, configurations where even the smallest perturbations alter the stability of the system), then the complexity bound obtained is too contrived to be useful. To counter this, we offer a second, less restrictive measure of stability called *robust directional stability*:

DEFINITION. *Given an applied force $\mathbf{Q}_{ext}$, we will say a system with a fixture set F has robust directional stability with respect to $\mathbf{Q}_{ext}$ if the assembly is directionally stable for all external forces in some neighborhood of $\mathbf{Q}_{ext}$. More precisely, for each direction $\mathbf{d} \in \mathbf{R}^{3n}$, there must exist a positive scalar ϵ such that the assembly is directionally stable with respect to the force $\mathbf{Q}_{ext} + \epsilon\mathbf{d}$.*

An assembly which has robust directional stability can endure perturbations of $\mathbf{Q}_{ext}$ without becoming unstable. Thus, in figure 4b, the fixture set $F = \{2\}$ induces directional stability, but the set $F = \{1,2,3\}$ is required for robust directional stability. Note that the definition of robust directional stability says nothing about how large a perturbation in $\mathbf{Q}_{ext}$ can be before stability is lost (figure 5). Our proofs concerning robust directional stability will show that minimal fixture sets yielding robust directional stability are *NP*-hard to compute, even if the assembly can undergo large perturbations in $\mathbf{Q}_{ext}$ without becoming unstable.

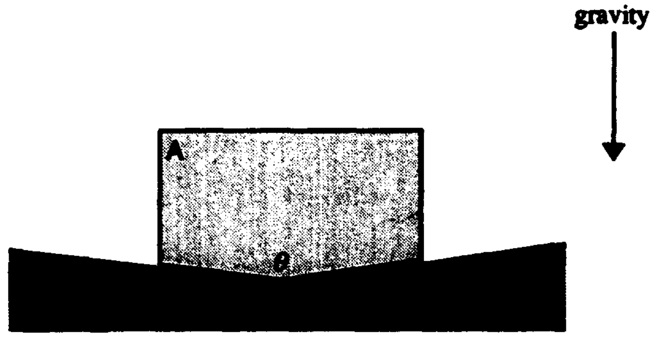


Figure 5: The assembly has robust directional stability without any fixtures. The amount of perturbation in the external force required to make the assembly unstable can be made arbitrarily small by letting the angle θ be arbitrarily close to 180°.

Note that the definition of robust directional stability clearly subsumes the notion of mere directional stability. As a practical matter, determining if an assembly with fixture set F has robust stability is no harder than seeing if the assembly is merely directionally stable. Borrowing from Mishra *et al.*[16]'s work on form-closure, we prove the following:

THEOREM 3.1 *Given an assembly with fixture set F, the assembly has robust directional stability with respect to* $\mathbf{Q}_{ext}$ *if and only if*

1. *The set of vectors* $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_c}, \mathbf{v}_{f_1}, \ldots, \mathbf{v}_{f_{|F|}}\}$ *spans* $\mathbf{R}^{3n}$ *(equivalently, the composite matrix* $[\,\mathbf{U} \mid \mathbf{V}_F\,]$ *has rank* $3n$*) and*

2. *There exist strictly positive vectors* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\lambda}$ *such that* $\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} + \mathbf{Q}_{ext} = \mathbf{0}$.

PROOF. Consider an assembly with fixture set F satisfying the conditions of the theorem for a given external force $\mathbf{Q}_{ext}$. Let $\mathbf{d} \in \mathbf{R}^{3n}$ be an arbitrary vector. We must show that there exists $\epsilon > 0$ such that the assembly is directionally stable with respect to $\mathbf{Q} + \epsilon\mathbf{d}$. From the first condition, there exists a solution $\boldsymbol{\alpha}_s$ and $\boldsymbol{\lambda}_s$ such that

$$[\,\mathbf{U} \mid \mathbf{V}_F\,]\begin{pmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\alpha}_s \end{pmatrix} = -\mathbf{d},$$

or equivalently, such that

$$\mathbf{U}\boldsymbol{\lambda}_s + \mathbf{V}_F\boldsymbol{\alpha}_s + \mathbf{d} = \mathbf{0}$$

although $\boldsymbol{\alpha}_s$ and $\boldsymbol{\lambda}_s$ will not necessarily be positive. From the second condition, there exist vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ satisfying

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} + \mathbf{Q}_{ext} = \mathbf{0}, \quad \boldsymbol{\lambda} > \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\alpha} > \mathbf{0}.$$

Then for some sufficiently small $\epsilon > 0$,

$$\boldsymbol{\lambda} + \epsilon\boldsymbol{\lambda}_s > \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\alpha} + \epsilon\boldsymbol{\alpha}_s > \mathbf{0}$$

and

$$\mathbf{U}(\boldsymbol{\lambda}+\epsilon\boldsymbol{\lambda}_s)+\mathbf{V}_F(\boldsymbol{\alpha}+\epsilon\boldsymbol{\alpha}_s)+(\mathbf{Q}_{ext}+\epsilon\mathbf{d}) = (\mathbf{U}\boldsymbol{\lambda}+\mathbf{V}_F\boldsymbol{\alpha}+\mathbf{Q}_{ext})+\epsilon(\mathbf{U}\boldsymbol{\lambda}_s+\mathbf{V}_F\boldsymbol{\alpha}_s+\mathbf{d}) = \mathbf{0}.$$

Thus, the assembly is directionally stable with respect to $\mathbf{Q} + \epsilon\mathbf{d}$.

Conversely, consider an assembly with robust directional stability, with respect to a force $\mathbf{Q}_{ext}$, and let $\boldsymbol{\lambda}_0$ and $\boldsymbol{\alpha}_0$ be nonnegative vectors such that $\mathbf{U}\boldsymbol{\lambda}_0 + \mathbf{V}_F\boldsymbol{\alpha}_0 = -\mathbf{Q}_{ext}$. To show that the first condition of the theorem holds, let $\mathbf{b} \in \mathbf{R}^{3n}$ be an arbitrary vector. Since the assembly is robustly stable, there exists $\epsilon > 0$ such that

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} = -(\mathbf{Q}_{ext} + \epsilon\mathbf{b}).$$

But then

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} = -(\mathbf{U}\boldsymbol{\lambda}_0 + \mathbf{V}_F\boldsymbol{\alpha}_0 + \epsilon\mathbf{b}).$$

so $\mathbf{U}(\boldsymbol{\lambda} + \boldsymbol{\lambda}_0) + \mathbf{V}_F(\boldsymbol{\alpha} + \boldsymbol{\alpha}_0) = -\epsilon\mathbf{b}$, or equivalently,

$$[\,\mathbf{U} \mid \mathbf{V}_F\,]\begin{pmatrix} \dfrac{-1}{\epsilon}(\boldsymbol{\lambda} + \boldsymbol{\lambda}_0) \\ \dfrac{-1}{\epsilon}(\boldsymbol{\alpha} + \boldsymbol{\alpha}_0) \end{pmatrix} = \mathbf{b}.$$

Since $\mathbf{b}$ was arbitrary, $\text{rank}([\,\mathbf{U} \mid \mathbf{V}_F\,]) = 3n$. To show that the second condition holds, let $\mathbf{e}$ denote the vector $e_i = 1$ for all i; the dimension of $\mathbf{e}$ will vary according to its use. Let $\mathbf{y} \in \mathbf{R}^{3n}$ be defined by $\mathbf{y} = \mathbf{U}\mathbf{e} + \mathbf{V}_F\mathbf{e}$. Then for some $\epsilon > 0$, there exist nonnegative $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ satisfying

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} = -(\mathbf{Q}_{ext} + \epsilon\mathbf{y}) = -\left(\mathbf{Q}_{ext} + \epsilon(\mathbf{U}\mathbf{e} + \mathbf{V}_F\mathbf{e})\right)$$

which yields

$$\mathbf{U}(\boldsymbol{\lambda} + \epsilon\mathbf{e}) + \mathbf{V}_F(\boldsymbol{\alpha} + \epsilon\mathbf{e}) + \mathbf{Q}_{ext} = \mathbf{0}.$$

Since $\epsilon > 0$ and $\mathbf{e}$ is strictly positive, both $\boldsymbol{\lambda} + \epsilon\mathbf{e}$ and $\boldsymbol{\alpha} + \epsilon\mathbf{e}$ are strictly positive as well, and condition 2 is seen to hold. □

Based on this theorem, robust directional stability is also easily determined. First, the rank of the matrix $[\,\mathbf{U} \mid \mathbf{V}_F\,]$ must be determined (say, by employing an SVD decomposition, or Gaussian elimination). Assuming that the matrix has rank $3n$, one can employ linear programming to determine if the second condition of theorem 3.1 holds. Although the strict positivity constraints on $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ cannot be directly enforced in a linear program, we can work around this restriction; clearly, the magnitude of $\mathbf{Q}_{ext}$ has no bearing on the problem. Let us arbitrarily constrain all components of $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ to be one or larger, by requiring $\boldsymbol{\lambda} \geq \mathbf{e}$ and $\boldsymbol{\alpha} \geq \mathbf{e}$. Letting s denote an unknown scalar, strictly positive vectors $\boldsymbol{\lambda}$

and α satisfying $\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}\boldsymbol{\alpha} + \mathbf{Q}_{ext} = \mathbf{0}$ exist if and only if the linear program

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} + s\mathbf{Q}_{ext} = \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{e}, \qquad \boldsymbol{\alpha} \geq \mathbf{e} \qquad \text{and} \qquad s \geq 1 \tag{7}$$

possesses a solution for $\boldsymbol{\alpha}$, $\boldsymbol{\lambda}$ and s.

One can also show that if an assembly has robust directional stability, then there must exist a lower bound on the ϵ's needed in the above definition, assuming a bound on the $\mathbf{d}$'s. That is, there exists a positive scalar ϵ such that the assembly is directionally stable with respect to $\mathbf{Q}_{ext} + \epsilon\mathbf{d}$ for all $\mathbf{d} \in \mathbf{R}^{3n}$ such that $\|\mathbf{d}\| = 1$ (under any vector-norm). This follows from convexity: if an assembly is directionally stable with respect to $\mathbf{Q}_{ext} + \mathbf{d}_1$ and also $\mathbf{Q}_{ext} + \mathbf{d}_2$ then the assembly is directionally stable with respect to

$$\mathbf{Q}_{ext} + t(\mathbf{d}_2 - \mathbf{d}_1)$$

for any $0 \leq t \leq 1$. This result furnishes a practical way to determine if an assembly will be stable over an entire range of applied forces. Given a set S of external force vectors, if the assembly is directionally stable with respect to each force vector in the convex hull of S, then the assembly is directionally stable with respect to every force vector in S (and robustly stable for any force vector in the interior of the convex hull of S).

3.3 Form-closure

The last form of stability we consider is *form-closure*, where the contact and fixture forces are sufficient to balance any and all external forces.

DEFINITION. *An assembly with a fixture set F has form-closure (or is form-closed) if the contact and fixture forces that arise in response to an external force $\mathbf{Q}_{ext}$ sum to exactly $-\mathbf{Q}_{ext}$, for all $\mathbf{Q}_{ext} \in \mathbf{R}^{3n}$.*

Form-closure of an assembly means that the assembly is completely immobile, and will not move in response to any force. Clearly, form-closure implies both direction stability and robust directional stability with respect to all external forces.[5] Following the form of theorem 3.1, and from Mishra *et al.* [16]'s work, the following theorem is easily proved:

THEOREM 3.2 *Given an assembly with fixture set F, the assembly is form-closed if and only if*

1. *The set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_c}, \mathbf{v}_{f_1}, \ldots, \mathbf{v}_{f_{|F|}}\}$ spans $\mathbf{R}^{3n}$ and*
2. *There exist strictly positive vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ such that $\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} = \mathbf{0}$.*

[5]Note that finding a fixture set inducing form-closure is at least as hard as finding a fixture set inducing directional or robust directional stability. However, one cannot immediately say anything about the relative difficulty of finding *minimal* fixture sets for the three different types of stability. In fact, we suspect that finding a minimal set of regular fixtures inducing form-closure is *easier* than finding a minimal set that produces directional or robust directional stability.

Once again, form-closure is easily detected by checking the rank of the matrix [$\mathbf{U}$ | $\mathbf{V}_F$], and then seeing if the linear program

$$\mathbf{U}\boldsymbol{\lambda} + \mathbf{V}_F\boldsymbol{\alpha} = \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{e} \qquad \text{and} \qquad \boldsymbol{\alpha} \geq \mathbf{e} \tag{8}$$

has a solution.

Having formally defined these notions of stability, we can now consider the complexity of finding minimal stabilizing fixture sets. Following this, we will consider algorithms for finding minimal stabilizing sets.

4 Size Bounds on F

For form-closure, work by Mishra *et al.*[16] and Markenscoff *et al.*[12] establishes that a single planar object always requires at least four fixtures. However, never more than six fixtures are required for form-closure. For three-dimensional objects, those bounds are respectively seven and twelve. These bounds are obtained from the theorems of Carathéodory, and Steinitz[16, 5].

In particular, Steinitz' theorem establishes that if $Z \subseteq \mathbf{R}^k$ and a point $\mathbf{b} \in \mathbf{R}^k$ is in the interior of the convex hull of Z, then there exists a subset $X \subseteq Z$ with at most $2k$ elements such that $\mathbf{b}$ is interior to the convex hull of X. In matrix-parlance, this means that if

$$[\,\mathbf{U} \mid \mathbf{V}\,]\begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \mathbf{b}$$

for some strictly positive vector $(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \in \mathbf{R}^{N_c+N_f}$ then there exists a matrix $\mathbf{M}$ containing $6n$ or fewer of the $N_c + N_f$ columns of $[\,\mathbf{U} \mid \mathbf{V}\,]$ such that

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$

for some strictly positive vector $\mathbf{x}$.

We can apply this result to planar multi-body assemblies. If we take $\mathbf{b}$ to be the origin, $\mathbf{0}$, this gives us the result that never more than $2 \cdot 3n = 6n$ contacts and fixtures are necessary for form-closure. Similarly, if we take $\mathbf{b}$ to be a specific external force $\mathbf{Q} \in \mathbf{R}^{3n}$, then again, never more than $6n$ contacts and fixtures are required for robust directional stability with respect to any external force. For directional stability, the upper bound is much lower; a basic theorem of linear programming is that if there exists $\mathbf{x}$ satisfying

$$\mathbf{M}\mathbf{x} = \mathbf{b} \qquad \text{and} \qquad \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{M} \in \mathbf{R}^{r \times s}$ then there exists a solution $\mathbf{x}$ with at most r nonzero elements. The maximum number of contacts and fixtures needed for directional stability is therefore $3n$.

In the other direction, both form-closure and robust directional stability require a fixture set F such that the matrix $[\,\mathbf{U} \mid \mathbf{V}_F\,]$ has rank $3n$, and thus at least $3n$ columns. Because of this, both types of stability require a total of at least $3n$ contacts and fixtures. For form-closure, this lower bound is improved to $3n + 1$: if the matrix $[\,\mathbf{U} \mid \mathbf{V}_F\,]$ has rank $3n$ and exactly $3n$ columns, it is nonsingular and the equation

$$[\,\mathbf{U} \mid \mathbf{V}_F\,]\mathbf{y} = \mathbf{0}$$

has only the single nonpositive solution $\mathbf{y} = \mathbf{0}$.

These lower bounds give us a lower bound on the size of the fixture set alone. Since both form-closure and robust stability require $\text{rank}([\,\mathbf{U} \mid \mathbf{V}_F\,]) = 3n$, we obtain

$$3n = \text{rank}([\,\mathbf{U} \mid \mathbf{V}_F\,]) \leq \text{rank}(\mathbf{U}) + \text{rank}(\mathbf{V}_F)$$

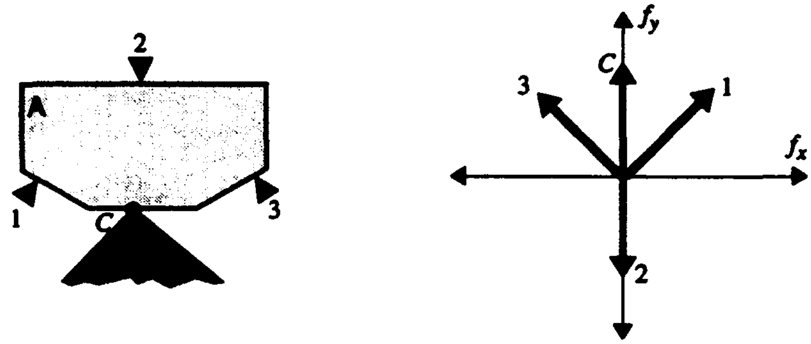


Figure 6: An object with only translational freedom in contact with an immovable obstacle. The directions of the forces in $\mathbf{R}^2$ generated by the fixtures are shown, as is the direction of the force generated by the contact. To positively sum these vectors to zero, vectors 1, 2 and 3 must all be used.

or equivalently $\text{rank}(\mathbf{V}_F) \geq 3n - \text{rank}(\mathbf{U})$. Thus, at least

$$3n - \text{rank}(\mathbf{U})$$

fixtures are required to achieve form-closure. It is tempting to think that contacts always help to decrease the number of fixtures needed for form-closure (or for robust directional stability, if the contacts are "underneath" an object) but this is not always so. For simplicity, consider a single object with only translational freedom, as show in figure 6. Without any contact, three fixtures would be required for form-closure. But even though contact occurs, all three fixtures are still required for form-closure. The direction of the contact and fixture forces in $\mathbf{R}^2$ are shown next to the object. Clearly, the only strictly positive sum of the vectors that is zero requires using vectors 1, 2, and 3. The use of vector C is not required.

Although we will see that finding a minimal fixture set is in general hard, it is not difficult at all to find a fixture of size $6n$ or less that imparts form-closure. We present such an algorithm below. Let us assume a predicate function *positive-span*($\mathbf{M}, \mathbf{b}$) that takes a matrix $\mathbf{M}$ and a vector $\mathbf{b}$ and returns a value of *TRUE* if and only if there exists a vector $\mathbf{x}$ such that

$$\mathbf{Mx} = \mathbf{b} \qquad \text{and} \qquad \mathbf{x} > \mathbf{0}.$$

Note that $\mathbf{x}$ is required to be strictly positive. As discussed in section 3, *positive-span* can be implemented as a linear program. The following algorithm returns a fixture subset F that achieves form-closure (assuming such a subset exists):

Algorithm *prune-fixtures*

$F = \{ 1, 2, \ldots, N_f \}$
for $i = 1$ **to** N_f
do
  $F' = F - \{ i \}$
  $\mathbf{S} = [\, \mathbf{U} \mid \mathbf{V}_{F'} \,]$
  if $\mathrm{rank}(\mathbf{S}) = 3n$ **and** *positive-span*$(\mathbf{S}, \mathbf{0}) =$ *TRUE*
    $F = F'$
done
return F

Assuming that a fixture set F imparting form-closure exists, the fixture set found by *prune-fixture-set* is guaranteed to be of size $6n$ or less. The correctness of the algorithm is a consequence of Steinitz' theorem. If the set of N_c contacts and all N_f fixtures can achieve form-closure, then no more than $6n$ of the fixtures are required for form-closure. Initially, the fixture set F contains all the fixtures, so F imparts form-closure. As the algorithm progresses, fixture i is removed from F only if it is demonstrated that the ith fixture is not required for form-closure. Thus, as long as F contains more than $6n$ elements, Steinitz' theorem applies and there must exist some element of F that can be removed. As a result, the final set F returned must of size $6n$ or less.

Clearly, the same algorithm can be used to find a set of $6n$ or less fixtures that yields robust directional stability with respect to an external force $\mathbf{Q}_{ext}$. The exact same algorithm is used, except that

$$\textit{positive-span}(\mathbf{S}, \mathbf{0})$$

is replaced by

$$\textit{positive-span}(\mathbf{S}, -\mathbf{Q}_{ext}).$$

An algorithm that finds a fixture set of size $3n$ or less for directional stability is also easily obtained. Assuming directional stability with respect to a force $\mathbf{Q}_{ext}$ is possible, the simplex algorithm, applied to the linear program

$$[\, \mathbf{U} \mid \mathbf{V} \,] \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = -\mathbf{Q}_{ext} = \mathbf{0} \qquad \text{and} \qquad \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} \geq \mathbf{0}, \tag{9}$$

yields a solution $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ with at most $3n$ positive elements. The fixture set F defined by

$$F = \{ i \mid \alpha_i \neq 0 \} \tag{10}$$

has at most $3n$ elements, and yields directional stability with respect to $\mathbf{Q}_{ext}$.

5 Complexity of Minimal Generalized Fixturing

We now turn our attention to the complexity of minimal fixturing. For generalized fixtures (as we have chosen to define them), there is no restriction on the fixture vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{|F|}\}$, and hence no restriction on the matrix $\mathbf{V}$. Because of this, complexity results concerning generalized fixturing are simply obtained. Note that for a single rigid body, regular and generalized fixtures are essentially the same, since any arbitrary force direction $\mathbf{v}$ in either $\mathbf{R}^3$ (for a planar solid) or $\mathbf{R}^6$ (for a three-dimensional solid) can be obtained from a single fixture.[6] Thus, the problem of selecting a minimal set of regular fixtures for a single rigid body (either for form-closure, robust directional stability or directional stability) can be considered a generalized fixturing problem (albeit of very low dimension).

Since the largest minimal fixture set for a planar object will always be no larger than six, a minimal fixturing set can found by an exhaustive search in time $O(N_f{}^6)$. For a three-dimensional object, an exhaustive search could require up to $O(N_f{}^{12})$ time. Although these complexity bounds are polynomial, a smaller upper bound would clearly be better (especially in fixturing three-dimensional objects). The result of this section that generalized fixturing is *NP*-hard suggests that minimal fixturing of a single object requires, in the worst case, exhaustive search to obtain a minimal set (assuming of course that $P \neq NP$). Although generalized fixturing may not be physically motivated enough to be a useful measure for multi-body fixturing, it is useful at least in pointing out that minimal single-object fixturing algorithms are likely to have worst-case performances equivalent to exhaustive search.

For brevity, we will make the following assumptions in all our complexity proofs concerning regular and generalized fixturing:

- The contact set for an assembly is indicated by the matrix $\mathbf{U}$, as described in section 2.
- The set of fixtures we can select from in trying to impart stability is indicated by the matrix $\mathbf{V}$, as described in section 2.
- The assembly in question can be stabilized by selecting *all* the fixtures indicated by $\mathbf{V}$, so that there is in fact at least one minimal fixture set F which imparts stability.

Note that given a choice F of a fixture set, any of the three types of stability previously described can be tested for by solving a linear program and computing the rank of matrix. Since both linear programming and computing the rank of a matrix are polynomial time problems, it is clear that all the varieties of minimal fixturing we consider are in *NP*.

5.1 Generalized Fixtures and Directional Stability

The simplest complexity result to obtain is that finding a minimal size fixture set F imparting directional stability is *NP*-hard.

THEOREM 5.1 *Given a vector $\mathbf{Q}_{ext}$ and a set of generalized fixtures, finding a minimal subset of those fixtures that yields directional stability with respect to $\mathbf{Q}_{ext}$ (assuming such a set exists) is NP-hard.*

[6] The exception being that a pure torque cannot be obtained by a single regular fixture.

PROOF. The proof involves a restriction to assemblies with no contact (thus our assertion that optimal fixturing of a single object is likely to require exhaustive search). Garey and Johnson[6, section A6, problem MP5] establishes that the following variation of linear programming is *NP*-complete:

> Given a matrix $\mathbf{M}$, an n-vector $\mathbf{b}$, and an integer $k < n$, does there exist a vector $\mathbf{x}$ with no more than k nonzero components satisfying
>
> $$\mathbf{Mx} = \mathbf{b} \qquad \text{and} \qquad \mathbf{x} \geq \mathbf{0}?$$

This linear programming variation is trivially reducible to our stability problem. Given a matrix $\mathbf{M} \in \mathbf{R}^{3n \times m}$ and a vector $\mathbf{b} \in \mathbf{R}^{3n}$, we can produce an assembly of n bodies, none of which contact, and set of m generalized fixtures such that $\mathbf{V} = \mathbf{M}$.[7] We choose an external force $\mathbf{Q}_{ext} = -\mathbf{b}$ to act on the assembly. Since there is no contact, a fixture set F yields directional stability if and only if

$$\mathbf{V}_F \boldsymbol{\alpha} = -\mathbf{Q}_{ext} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\alpha} \geq \mathbf{0}$$

has a solution $\boldsymbol{\alpha}$. Clearly then, finding a minimal F is equivalent to finding a vector $\boldsymbol{\alpha}$ with the least number of nonzero elements such that

$$\mathbf{V}\boldsymbol{\alpha} = -\mathbf{Q}_{ext} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\alpha} \geq \mathbf{0}.$$

Since this problem is *NP*-hard, we conclude that finding a minimal set of generalized fixtures yielding directional stability is *NP*-hard as well. □

A proof of the complexity of obtaining robust directional stability with a minimal number of general fixtures will follow from a study of regular fixturing, in section 6.

5.2 Generalized Fixtures and Form-Closure

For form-closure of an assembly without contact, we are interested in the following problem: what is the smallest set F such that $\text{rank}(\mathbf{V}_F) = 3n$ and

$$\mathbf{V}_F \boldsymbol{\alpha} = \mathbf{0}$$

possesses a strictly positive solution $\boldsymbol{\alpha}$? As in the above proof, given any matrix $\mathbf{M} \in \mathbf{R}^{3n \times m}$, we can produce an assembly of n bodies without contact, and a generalized set of m fixtures such that $\mathbf{V} = \mathbf{M}$. Thus, finding a minimal set of generalized fixtures yielding form-closure is equivalent in complexity to the following problem, which we call *positive minimal span*:

[7] Note that the assumption of generalized fixtures is key here. If we are limited to regular fixtures, for most matrices $\mathbf{M}$, we cannot design a fixture set such that $\mathbf{V} = \mathbf{M}$.

THEOREM 5.2 *Let* $\mathbf{M}$ *be an* $r \times s$ *matrix with* $r < s$ *such that* $\operatorname{rank}(\mathbf{M}) = r$ *and* $\mathbf{Mx} = \mathbf{0}$ *for some strictly positive vector* $\mathbf{x}$*. Given an integer* $k \geq r$*, determining if there exists an index set* $C \subseteq \{1, 2, \ldots s\}$ *of size* k *such that*

1. $\operatorname{rank}(\mathbf{M}_C) = r$ *and*
2. $\mathbf{M}_C\mathbf{y} = \mathbf{0}$ *for some vector* $\mathbf{y} > \mathbf{0}$

is NP-complete.[8,9]

PROOF. To begin, it is clear that the problem is in *NP* since given C, the matrix $\mathbf{M}_C$ can be tested to see if it satisfies the conditions of the theorem in polynomial time by linear programming and a computation of the rank of $\mathbf{M}_C$. To show that the problem is *NP*-hard, we use a reduction from the following *NP*-complete problem, known as the *minimum set cover* problem:

> Let A be a set $A = \{a_1, a_2, \ldots a_r\}$ of r elements. Let A_1 through A_s be subsets of A. Does there exist a k-element subset $Z \subseteq \{A_1, A_2, \ldots A_s\}$ such that $\bigcup_{z \in Z} z = A$?

Given A and the subsets A_1 through A_s, the reduction is simple. We will construct a matrix $\mathbf{M}$ of size $r \times (r+s)$ such that A can be covered by k of the A_i subsets if and only if there exists a $(k+s)$-element index set C satisfying the conditions of theorem 5.2. We construct $\mathbf{M}$ as follows: let $\mathbf{M}$ be written in the form

$$\mathbf{M} = [\,\mathbf{m}_1\ \mathbf{m}_2\ \ldots\ \mathbf{m}_{r+s}\,]$$

where each $\mathbf{m}_i$ is a column vector of length r. The first s columns of $\mathbf{M}$, that is, m_1 through m_s, consist entirely of zeroes and ones. For $1 \leq j \leq s$, let the ith component of $\mathbf{m}_j$ equal one if $a_i \in A_j$ and zero otherwise.[10] The last r columns form a negated identity matrix; that is,

$$\mathbf{m}_{s+1} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{m}_{s+2} = \begin{pmatrix} 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \mathbf{m}_{s+r} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix},$$

[8] Note that the inequality on $\mathbf{y}$ in this theorem is strict.

[9] We would not be surprised to find that a proof of this theorem already exists in the literature. (Actually, we were surprised *not* to find a proof already in the literature!) If any of the referees know of a reference for this proof, we would be happy to eliminate the proof of this theorem and simply cite the reference instead.

[10] As an example, for $s = 6$, $A_1 = \{a_2, a_3, a_5\}$ and $A_2 = \{a_1, a_3, a_4\}$, we would have

$$\mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We must show that A can be covered by a union of k of the A_i's if and only if there exists a $(k+s)$-element index set C satisfying the second condition of theorem 5.2.

Suppose that A can be covered by the k-element collection $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$. Then if we let $C = \{n_1, n_2, \ldots, n_k, r+1, r+2, \ldots, r+s\}$, we claim that $\mathbf{M}_C$ satisfies the conditions of theorem 5.2. First, since C contains the indices $r+1$ through $r+s$, $\mathbf{M}_C$'s rightmost r columns form the negated identity matrix, establishing that $\text{rank}(\mathbf{M}) = r$. Second, since A is covered by the collection $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$, the vector

$$\mathbf{y} = \mathbf{m}_{n_1} + \mathbf{m}_{n_2} + \cdots + \mathbf{m}_{n_k}$$

satisfies $y_i \geq 1$ for all i, since y_i indicates the number of times a_i appears in the collection $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$, and this collection covers A. Letting $\mathbf{I}$ denote the $s \times s$ identity matrix and $\mathbf{e} \in \Re^k$ be the vector $e_i = 1$ for all i, we obtain

$$\begin{aligned} \mathbf{M}_C \begin{pmatrix} \mathbf{e} \\ \mathbf{y} \end{pmatrix} &= [\,\mathbf{m}_{n_1}\ \mathbf{m}_{n_2}\ \ldots\ \mathbf{m}_{n_k} \mid -\mathbf{I}\,] \begin{pmatrix} \mathbf{e} \\ \mathbf{y} \end{pmatrix} \\ &= \mathbf{m}_{n_1} + \mathbf{m}_{n_2} + \cdots + \mathbf{m}_{n_k} - \mathbf{I}\mathbf{y} \\ &= \mathbf{y} - \mathbf{y} = \mathbf{0} \end{aligned}$$

where the vector $(\mathbf{e}, \mathbf{y})$ is strictly positive.

Conversely, suppose C is an index set of size $k+s$ such that $\mathbf{M}_C$ satisfies the conditions of the theorem. Since $\text{rank}(\mathbf{M}_C) = r$ and $\mathbf{M}_C$ has only r rows, no row of $\mathbf{M}_C$ can contain all zeroes. Thus, for each $1 \leq i \leq r$, $\mathbf{M}_C$ must include at least one column of $\mathbf{M}$ whose ith component is nonzero. But in order to satisfy $\mathbf{M}_C\mathbf{y} = \mathbf{0}$ with $\mathbf{y}$ strictly positive, for each $1 \leq i \leq r$, the matrix $\mathbf{M}_C$ must contain at least *two* columns of $\mathbf{M}$, both of whose ith components are nonzero. Furthermore, for each i, these two components must be of opposite sign. Since for each $1 \leq i \leq r$ the matrix $\mathbf{M}_C$ must contain a column whose ith element is negative and since the only columns of $\mathbf{M}$ with negative entries are columns $s+1$ through $s+r$, clearly $\mathbf{M}_C$ must contain the last r columns of $\mathbf{M}$, It must be then that $\{s+1, s+2, \ldots, s+r\} \subset C$. Since C contains $k+r$ elements, C must therefore have the form

$$C = \{n_1, n_2, \ldots, n_k, s+1, s+2, \ldots s+r\}$$

where the n_i's are all less than $s+1$. Then $\mathbf{M}_C$ can be written in the form

$$\mathbf{M}_C = [\,\mathbf{m}_{n_1}\ \mathbf{m}_{n_2}\ \ldots\ \mathbf{m}_{n_k} \mid -\mathbf{I}\,].$$

Since C satisfies the second condition of the theorem, there exist strictly positive vectors $\mathbf{x}$ and $\mathbf{y}$ such that

$$\mathbf{M}_C \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = [\,\mathbf{m}_{n_1}\ \mathbf{m}_{n_2}\ \ldots\ \mathbf{m}_{n_k} \mid -\mathbf{I}\,] \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = [\,\mathbf{m}_{n_1}\ \mathbf{m}_{n_2}\ \ldots\ \mathbf{m}_{n_k}\,]\mathbf{x} - \mathbf{y} = \mathbf{0},$$

or equivalently,

$$[\,\mathbf{m}_{n_1}\ \mathbf{m}_{n_2}\ \ldots\ \mathbf{m}_{n_k}\,]\mathbf{x} = \mathbf{y}.$$

Since every component of **y** is nonzero, every component of the vector

$$x_1\mathbf{m}_{n_1} + \cdots + x_k\mathbf{m}_{n_k}$$

is nonzero as well. Hence, for each $1 \leq i \leq r$, there must exist $1 \leq j \leq k$ such that the ith component of $\mathbf{m}_{n_j}$ is nonzero as well. By the construction of the vectors $\mathbf{m}_1$ through $\mathbf{m}_s$, the set

$$\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$$

must therefore cover A. We conclude that positive minimal spanning is *NP*-complete. □

This leads us directly to

THEOREM 5.3 *Given a set of generalized fixtures, finding a minimal subset of those fixtures that yields form-closure (assuming such a set exists) is NP-hard.*

PROOF. The proof follows immediately from theorem 5.2. □

6 Complexity of Minimal Regular Fixturing

Complexity bounds for generalized fixturing could be obtained by considering linear programs augmented with combinatorial constraints. For regular fixtures, this approach cannot be used because regular fixtures yield matrices of **V** with a very specific structure. Instead, we will produce actual assemblies, whose minimal fixturing sets will correspond with the solution of a minimum set-covering problem. We will make use of the *NP*-complete problem of minimum set cover used in section 5 to establish complexity bounds. Given a set of elements $A = \{a_1, a_2, \ldots, a_r\}$ and subsets $A_1, A_2, \ldots, A_s$ of A, we will construct an assembly that can be stabilized with k fixtures if and only if there exists a set $Z \subseteq \{A_1, A_2, \ldots A_s\}$ such that $|Z| = k$ and $\bigcup_{z \in Z} z = A$. Without loss of generality, we assume that each subset A_i contains three elements; this restricted form of minimum set cover is still *NP*-hard.

THEOREM 6.1 *Given a vector $\mathbf{Q}_{ext}$, and a set of regular fixtures, finding a minimal subset of those fixtures that yields directional stability with respect to $\mathbf{Q}_{ext}$ (assuming such a set exists) is NP-hard.*

PROOF. The proof is by reduction of the minimum set covering problem. Given the set $A = \{a_1, a_2, \ldots, a_r\}$, and subsets $A_1, A_2, \ldots, A_s$ of A, we construct an assembly of $r + s$ objects. Objects 1 through r correspond to the r elements of A. Contact constraints with immovable obstacles prevent the first r objects from any motion other than a translation straight down (figure 7). The external force $\mathbf{Q}_{ext}$ acting on the system tries to accelerate each of these r objects downwards. For each $1 \leq i \leq r$, we will say a_i is *covered* when we have prevented object i from moving downwards.

Object $r + j$, for $1 \leq j \leq s$, corresponds to the subset A_j. These objects are also constrained so that they can only move vertically. The set of fixtures for the assembly consists of s fixtures. If the jth fixture is selected then object $r+j$ is prevented from moving downwards (figure 7). However, no external force (or rather, an external force of zero) acts on objects $r+1$ through $r+s$. Thus, not all of these s objects necessarily need to be fixtures; in particular, if object $r+j$ is not used to support any of the first r objects, object $r+j$ can remain unfixtured.

Each object $r + j$ touches three other of the first r objects: if $A_j = \{a_s, a_t, a_u\}$, then object $r + j$ contacts objects s, t and u. For example, the contacts on objects $r + 1$ and $r + 2$ in figure 8 indicate that $A_1 = \{a_1, a_2, a_4\}$ and $A_2 = \{a_2, a_5, a_r\}$. Selecting fixture j prevents object $r + j$ from moving downwards, thus covering object i for all i such that $a_i \in A_j$. Conversely, if for some $i \leq r$, none of the fixtures j satisfying $a_i \in A_j$ are selected, then the external force will cause object i to move downwards, consequently causing all such objects j to move downwards as well.

Clearly then, a fixture set F that yields directional stability with respect to the chosen $\mathbf{Q}_{ext}$ induces a covering set Z, by defining

$$Z = \{A_j \mid j \in F\}.$$

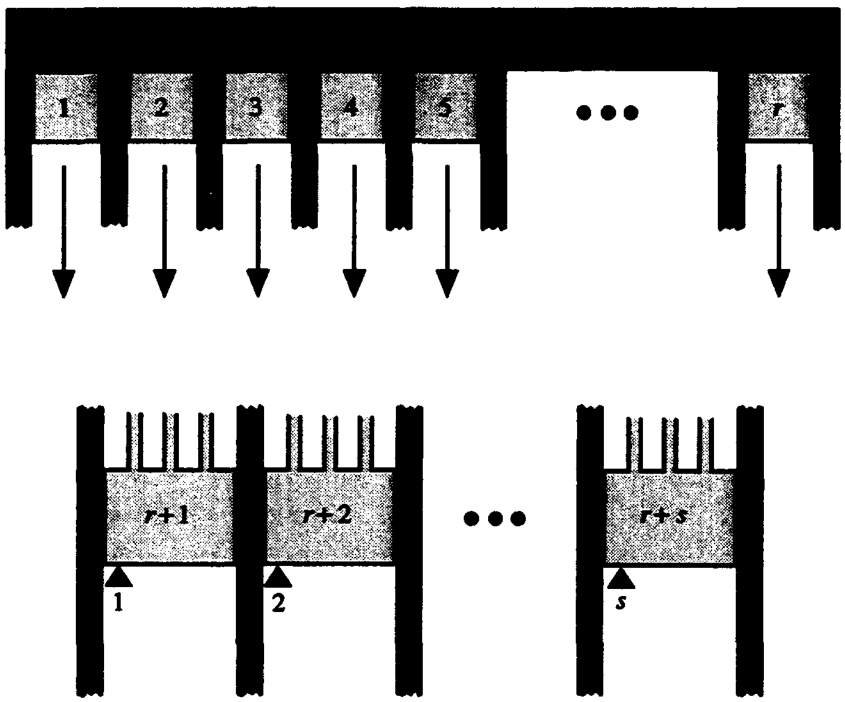


Figure 7: Objects 1 through r can only move downwards; a downwards external force acts on each of these objects. Objects $r+1$ through $r+s$ can move downwards but are not subjected to an external force. An object $r+j$ can be prevented from moving by selecting fixture j.

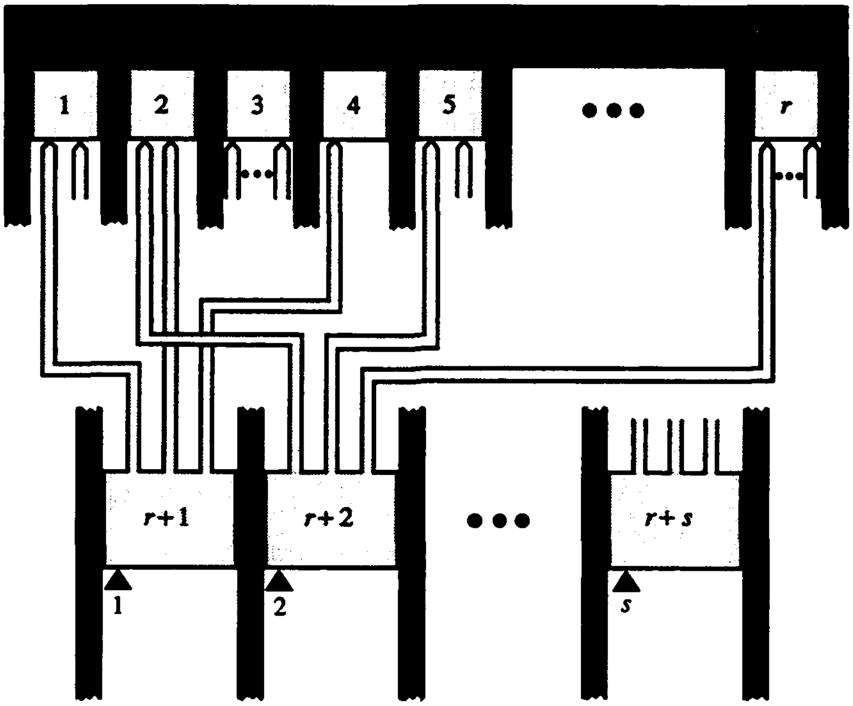


Figure 8: Contact between object $r+j$ and object i occurs if and only if $a_i \in A_j$.

Conversely, a covering set Z induces a fixture set yielding directional stability[11] by defining

$$F = \{j \mid A_j \in Z\}. \tag{11}$$

Since minimum covering is *NP*-hard, we conclude that finding a minimal fixture set yielding directional stability is *NP*-hard as well. □

As it stands, this result is somewhat unsatisfactory, in the sense that the external force $\mathbf{Q}_{ext}$ is somewhat unnatural. Essentially, the first r objects are acted upon by gravity, but objects $r+1$ through $r+s$ are weightless. Because of this, the stability achieved is unstable with respect to perturbations of $\mathbf{Q}_{ext}$. Suppose for example, that we choose a fixturing set yielding stability that does not include fixture 1. Then object $r+1$ remains motionless only because the external force on it is zero; if $\mathbf{Q}_{ext}$ is perturbed so that even an infinitesimal downwards force acts on object $r+1$, the object will move downwards.

However, we can modify the construction of figure 8 so that minimum covering sets Z correspond to minimal fixture sets F that yield robust directional stability. In particular, the external force acting on the modified system will be an ordinary gravity field. A set of fixtures yielding stability will do so for relatively large external force variations (for

[11] A subtle point in this construction concerns the behavior of unfixtured objects. Suppose Z covers A, and F is defined by equation (11). Let $A_j \notin Z$, so $j \notin F$ and object $r+j$ is unfixtured. Could some object $i \in A_j$ exert a force on object $r+j$, causing a downwards motion? The answer to this is "no." Since Z covers A, object j is covered and cannot move. If object i exerted a force on object $r+j$, object $r+j$ would move away (downwards), breaking contact with object i. The contact force at a point where contact is broken however must be zero[7].

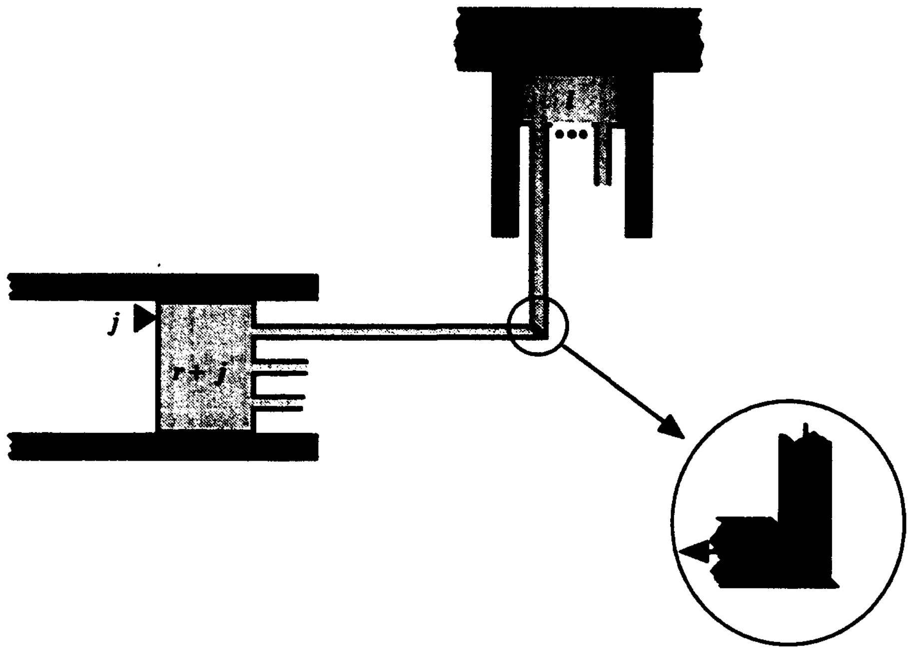


Figure 9: A downwards force on object i results in a horizontal force on object $r+j$. The external gravity force acting on object $r+j$ has no effect, as long as gravity points exactly downwards.

example, if gravity acts straight down on one object, but is inclined 15° with respect to another object).

THEOREM 6.2 *Given a vector* $\mathbf{Q}_{ext}$ *and a set of regular fixtures, finding a minimal subset of those fixtures that yields robust directional stability with respect to* $\mathbf{Q}_{ext}$ *(assuming such a set exists) is NP-hard.*

PROOF. The proof consists of modifying the construction in theorem 6.1 so that a minimal covering set Z yields a minimal fixture set F that is robustly stable with respect to a uniform gravity force. The modification is made in two steps.

Let $\mathbf{Q}_{ext}$ be an external force acting downwards on each object with strength proportional to an object's mass. In the assemblies of theorem 6.1, contact occurred between object i and object $r+j$ if $a_i \in A_j$. We modify the constraints on objects $r+1$ through $r+s$ so that each object can now move only horizontally. The fixtures are positioned so that selecting fixture j prevents object $r+j$ from moving to the left. Contact between objects i and $r+j$ is modified as shown in figure 9. In order for object i to move downwards, object $r+j$ must move to the left. Clearly, selecting fixture j prevents object i from moving downwards. Note however that if fixture j is unselected, the external force acting on object $r+j$ induces no motion of object $r+j$.

This modification does not change the fact that it is *NP*-hard to pick a minimal fixture set yielding directional stability. This first modification overcomes the objection that the external force chosen for the previous proof was somewhat unnatural. Still, given a covering

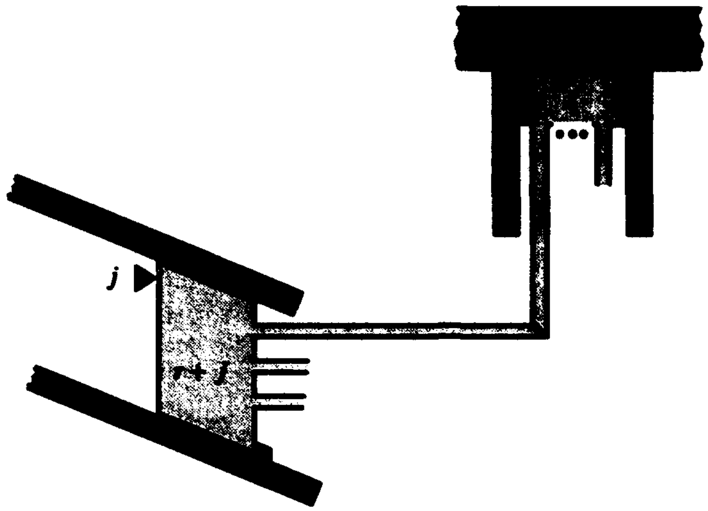


Figure 10: The constraints on object $r+j$ are altered so that perturbations in the gravity direction have no immediate effect on object $r+j$.

set Z, the corresponding F only induces directional stability; any unfixtured object $r+j$ is stable only because the external force points exactly straight down.

A second modification fixes this. In figure 10, the motion constraint on object $r+j$ is altered so that the object can only move to the left and up (if unfixtured). If we assume that objects 1 through r are massive compares with objects $r+1$ through $r+s$, then a fixture set which fails to cover some object $i \leq r$ does not yield stability. However, a fixture set F that yields directional stability also yields robust directional stability. To see this, suppose object $r+j$ is unfixtured. Perturbations on the forces acting on fixtured objects are of no account. In order for object $r+j$ to move, the external force acting on that object must be inclined a sizeable amount with respect to the negative vertical axis in order to cause a motion. Thus, a minimal covering set Z yields a minimal fixture set F yielding robust directional stability. □

THEOREM 6.3 *Given a vector $\mathbf{Q}_{ext}$ and a set of generalized fixtures, finding a minimal subset of those fixtures that yields robust directional stability with respect to $\mathbf{Q}_{ext}$ (assuming such a set exists) is NP-hard.*

PROOF. The proof follows from theorem 6.2 by restriction. □

Unfortunately, we cannot follow theorems 6.1 and 6.2 with a proof that finding a set of minimal fixtures yielding form-closure is *NP*-hard; nor can we show that such a fixture set can be found in polynomial time. We will show however that if an assembly's contact graph (a graph where nodes are object, and edges indicate contact between pairs of objects) is acyclic, a minimal fixture set is easily obtained in polynomial time. The complexity of the more general case of assemblies with cyclic contact graphs eludes us. Our strong suspicion however is that finding minimal sets of regular fixtures yielding form-closure is *not NP*-hard. Before considering contact graphs, we consider the complexity of fixturing with clamps.

7 Complexity of Minimal Fixturing with Clamps

Recall that clamps, which are groupings of regular fixtures, give us the ability to form-close a single object with unit cost. If we consider assemblies where selecting a clamp causes a particular body to be immobilized, we see that determining a set of bodies that when immobilized yields stability of the entire assembly is a clamp-fixturing problem. Accordingly, we can show that fixturing with clamps is *NP*-hard by constructing assemblies where it is *NP*-hard to determine the smallest number of objects whose immobilization yields stability for the entire structure.

The first two theorems concerning fixturing with clamps are trivial modifications to the assemblies constructed in figures 8 and 10.

THEOREM 7.1 *Given a vector* $\mathbf{Q}_{ext}$ *and a set of clamps, finding a minimal subset of those clamps that yields robust directional stability with respect to* $\mathbf{Q}_{ext}$ *(assuming such a set exists) is NP-hard.*

PROOF. Consider the assembly of figure 8. Let us replace each fixture j with a clamp that form-closes body $r+j$ when selected. Each such clamp has no more or less effect when selected than the original regular fixture. Thus, finding a minimal set of clamps required for directional stability is just as hard as finding a minimal set of regular fixtures. □

THEOREM 7.2 *Given a vector* $\mathbf{Q}_{ext}$, *and a set of clamps, finding a minimal subset of those clamps that yields rboust directional stability with respect to* $\mathbf{Q}_{ext}$ *(assuming such a set exists) is NP-hard.*

PROOF. The argument of the previous proof applies to the assembly of figure 10. □

Although we have not obtained a result concerning regular fixturing and form-closure, we can say something about the complexity of minimal fixturing with clamps for form-closure. Our proof will involve a reduction from the following *NP*-complete problem[6, section A1, problem GT 7]:

> Let $G = (N, E)$ be a directed graph with node set N and edge set E, and let $k < |N|$. Let the in- and out-degree of every node be two or less. Does there exist a subset of nodes $N' \subseteq N$ with $|N'| = k$ such that N' contains at least one vertex from every directed cycle in G?

To perform the reduction, we will take a directed graph all of whose nodes have both in- and out-degree of two or less, and build a corresponding assembly. The assembly will be form-closed by immobilizing k objects if and only if a subset N' of size k exists that contains at least one vertex of every directed cycle of the graph.

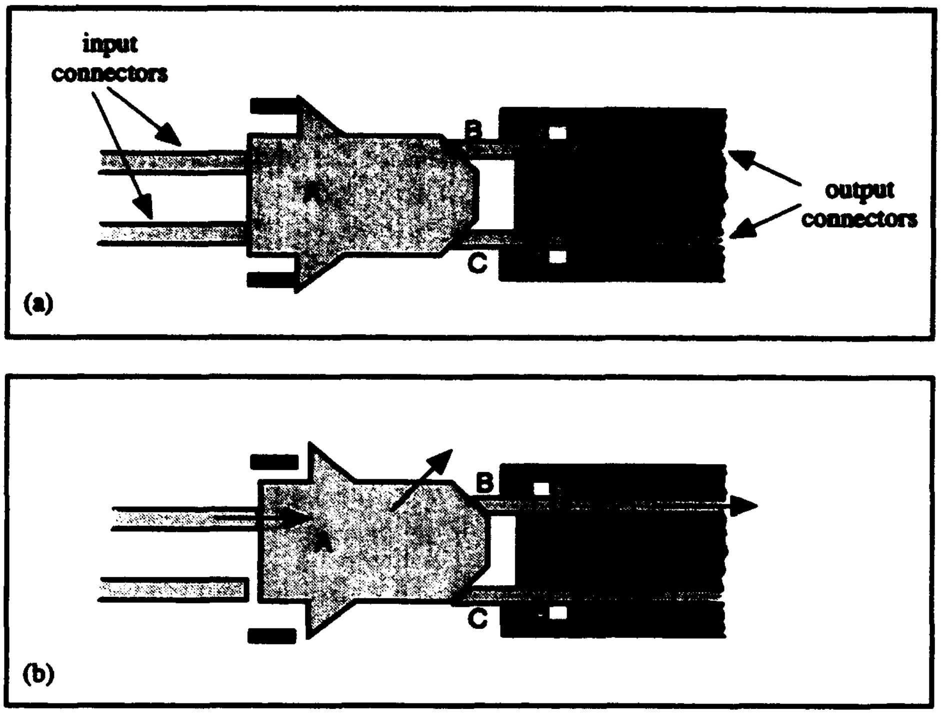
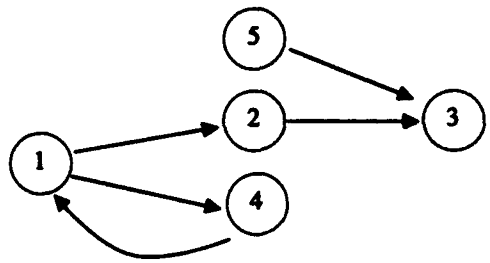


Figure 11: (a) Starting from its initial position, node object A can never slide to the left. Similarly, the the connecting rods B and C can never slide to the left. Object A's input connecting rods can slide forward (to the right) only if object A can. (b) If object A is given a push from an input connecting rod, and object C cannot move, object A can move by sliding to the right and up. Conversely, if B could not move but C could, object A would slide to the right and down. If neither B nor C can move, then A cannot move either, preventing A's input connecting rods from moving as well.

The basic building block of the assembly is shown in figure 11. We will call object A a *node object* and objects B and C *connecting rods.* A node object corresponds to a node of the graph. Connecting rods represent edges, and run from the "output" side of a node (the right side), to the "input" side (the left side) of some other node. In figure 11, the only freedom of motion for the connecting rods is to slide horizontally to the right. We will say this is a "forward" motion of the connecting rod. Connecting rods are always prevented from moving "backward" (to the left) from their initial position because of their spurs.

When can object A slide forward? Suppose that rod B is prevented from moving forward. Then object A could slide rightward and down *only* if the rod C is free to slide forward. Similarly, object A can slight to the right and up only if rod B is free to slide forward. If neither of the rods B and C can move, object A is prevented from moving as well. (Clearly, under no circumstances can object A slide to the left.)

The construction of figure 11 is replicated for each node of a graph. The assembly corresponding to the graph



is shown in figure 12. Consider object 1 in this assembly: as in figure 11, object 1 can slide forward if and only if one or more of its connecting "output" rods can slide forward. (Connecting rods can be made to turn corners corners as shown in the upper-left detail of figure 12. Similarly, nonplanar graphs can be formed by having a connecting rod cross over another rod by turning a corner out of the plane of the paper, and later turning back into the plane of the paper. For clarity, the spurs on the connecting rods preventing any "backwards" movement by the rods are not shown in figure 12.) Since object 1's connecting rods contact objects 2 and 4, object 1 can move if and only if one or both of objects 2 and 4 are free to move.

In order to perform the reduction, we want a node object i to be able to move if and only if one or more of the node objects it is connected to can move. For example, object 1 can move if and only if object 2 or object 4 can move. Object 5 is connected to only one other node object, so an extra constraint is placed object 5, preventing it from moving downwards at all. Clearly, object 5 can move if and only if object 3 can move. Since object 3 is connected to *no* other node objects, it is constrained to have no motion at all. Nodes with zero or one *incoming edge* (such as nodes 5 and 2) need no special treatment.

Thus, given a directed graph $G = (N, E)$ let us construct an assembly. The assembly will have $|N|$ node objects of the type shown in figure 11, and $|E|$ connecting rod mechanisms. (We will describe the set of clamps shortly.) A connecting rod mechanism is placed between node objects i and j for each directed edge $(i,j) \in E$, with the connector running "out" from object i and "into" object j. For each node in the graph with no outputs, additional constraints are added on the node object that immediately prevent it from moving (as illustrated by node 3 in figure 12). For each node with only one output, an additional constraint is placed on the node object i (as illustrated by node 5 in figure 12). The final assembly thus has the property that node object i can move if and only if for some $(i,j) \in E$, node object j can move as well. Note that this implies that if node i has no outgoing edges, node object i can never move.

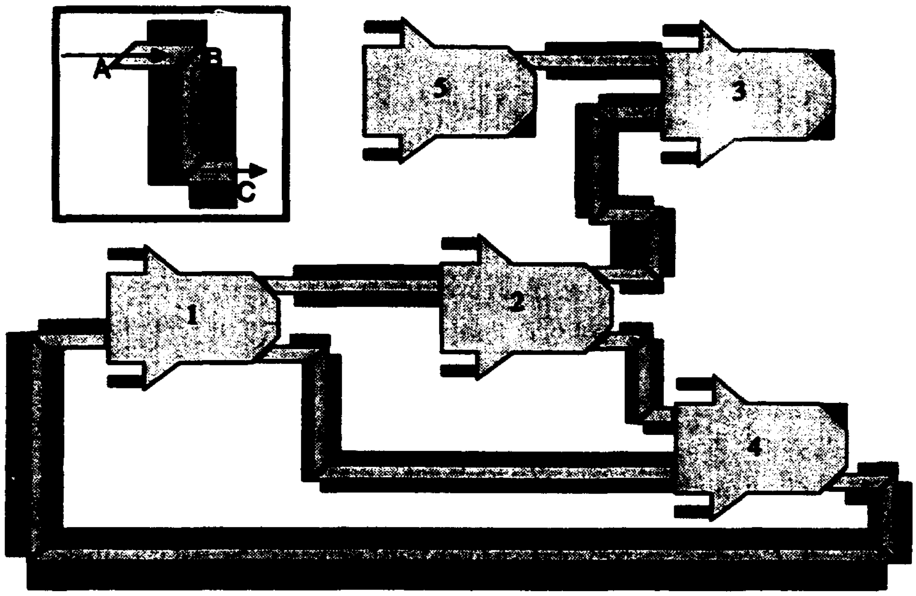


Figure 12: The assembly corresponding to the five node graph on page 29. Connecting rods can bend around corners as shown in the detail in the upper left: assuming rod *C* can move to the right, moving rod *A* to the right causes rod *B* to move downwards, causing rod *C* to move to the right. Nonplanar graphs (requiring edges to "cross over" each other) are modeled by having connecting rods turn corners taking them out of the plane of the assembly.

THEOREM 7.3 *Given a graph $G = (V, E)$, there is a movable node object in the constructed assembly if and only if there exists movable node objects $n_1, n_2, \ldots n_k$ such that $(n_i, n_{i+1}) \in E$ for $i < k$ and $(n_k, n_1) \in E$.*

PROOF. The forward direction of the proof is obvious from the construction of the assembly: assuming the hypothesized list $n_1, n_2, \ldots n_k$ of movable objects, each of the k node objects in the list can be moved forward some small amount, if all the movements occur simultaneously.

Conversely, suppose that node object n_1 in the assembly is movable. By construction, a node object is movable if and only if it is connected to another movable node object n_2. Then there must exist a movable node object n_2 such that $(n_1, n_2) \in E$. But in order for n_2 to be movable, it too must be connected to some movable node n_3 such that $(n_2, n_e) \in E$. Clearly, this argument can be continued, generating a list of movable nodes $n_1, n_2, n_3, \ldots$; but since there are only finitely many nodes, at some point we will encounter a movable object n_j such that n_i is already on the list. Then the subsequence $n_i, n_{i+1}, \ldots n_j$ is a list of movable node object satisfying the conditions of the theorem. □

Using theorem 7.3, we can establish the complexity of establishing form-closure with a minimal set of clamps.

THEOREM 7.4 *Given a set of clamps, finding a minimal subset of those clamps that yields form-closure (assuming such a set exists), is NP-hard.*

PROOF. Given a directed graph $G = (N, E)$, we construct the corresponding assembly of theorem 7.3. In addition, $|N|$ clamps are added. The ith clamp, when selected, form-closes node object i.

Given a set N' that contains at least one vertex from every directed cycle in G, the corresponding set of clamps

$$F = \{\, i \mid n_i \in N' \,\}$$

induces form-closure. This follows from theorem 7.3: the connecting rods cannot move unless some node object can move, and a movable node object implies a movable cycle of node objects. However, since at least one node object in every possible cycle of node objects is clamped, no object can move. By the same argument, a set of clamps F inducing form-closure yields a set

$$N' = \{\, n_i \mid i \in F \,\}$$

containing at least one vertex of every cycle in the graph. As a result, minimal sets N' containing a vertex from each cycle correspond to minimal sets F yielding form-closure. □

8 Algorithms

So far, we have shown that for all but one of the fixturing/stability combinations considered, finding minimal fixture set is *NP*-hard. The only exception has been minimal regular fixturing for form-closure. Although *NP*-hardness results can often be cause for disappointment, from the standpoint of implementing algorithms, we do not believe this is the case here.

8.1 Degeneracy

From the theorems of Carathéodory, and Steinitz, we know that form-closure requires between $3n+1$ and $6n$ fixtures and contacts. The pruning algorithm presented in section 4 is a simple polynomial-time algorithm for finding a stabilizing set of not more than $6n$ fixtures. However, $6n$ is the worst-case behavior of the algorithm—the algorithm may manage to prune the set of fixtures to well below $6n$. Characterizing the performance of the algorithm would require a characterization of the set of assemblies one wants to fixture; clearly, this is an application dependent question. However, there is a simple insight into the lower and upper ranges of $3n + 1$ and $6n$ contacts and fixtures required to form-close an assembly. A similar insight applies to robust directional stability.

DEFINITION. *A set of vectors $Z = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\} \subset \mathbf{R}^n$ is said to be degenerate if there exists a subset of size n or less vectors of Z that are linearly dependent. Otherwise, Z is said to be nondegenerate.*

THEOREM 8.1 *Let $Z = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\} \subset \mathbf{R}^n$ be a nondegenerate set of vectors. Then if the origin, $\mathbf{0}$, lies in the interior of the convex hull of Z, there exists a subset $X \subseteq Z$ such that $|X| = n + 1$ and the origin lies in the interior of the convex hull of X.*

PROOF. The proof is geometric in nature. Given Z, let us consider the boundary of the convex hull of Z. Since Z is nondegenerate, its convex hull occupies some n-dimensional volume of $\mathbf{R}^n$. Thus, the convex hull of Z is a convex n-dimensional polyhedron in $\mathbf{R}^n$. Examining the boundary of this polyhedron, it is clear that each facet of this polyhedron is embedded in some $(n-1)$-dimensional hyperplane of $\mathbf{R}^n$. Given a point $p \in \mathbf{R}^n$, if $\mathbf{p}$ lies on the boundary of the convex hull of Z, then either $\mathbf{p}$ is interior to a facet, or $\mathbf{p}$ lies on the boundary of a facet. Since each vertex of the polyhedron is a point in Z, if $\mathbf{p}$ lies on the boundary of a facet, then $\mathbf{p}$ is a linear combination of $n-1$ or fewer points of Z. Otherwise, $\mathbf{p}$ is interior to a facet and $\mathbf{p}$ may be written as a strictly positive linear combination of n linearly independent points in Z.

With this in mind, the proof is simple: given $\mathbf{m}_1$, consider the directed ray emanating from $\mathbf{m}_1$ and passing through the origin (figure 13). Not counting $\mathbf{m}_1$ itself (which might lie on the boundary of the convex hull of Z), no point on the ray between $\mathbf{m}_1$ and $\mathbf{0}$ lies on the boundary of the convex hull, since $\mathbf{0}$ is interior to the convex hull. Let $\mathbf{p}$ be the intersection point of the ray and the convex hull boundary. By construction, $\mathbf{p} = -t\mathbf{m}_1$, where t is a

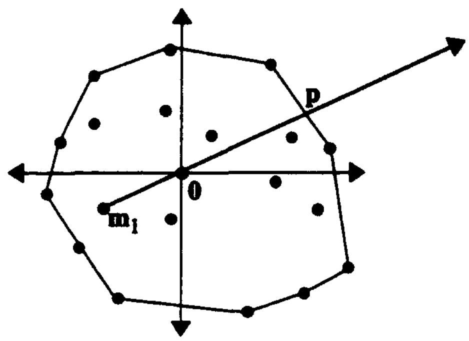


Figure 13: The geometric intuition for the proof. Since the set is nondegenerate, starting at any point and traversing through the origin, a point $\mathbf{p}$ interior to a face of the convex hull is reached.

positive scalar. If $\mathbf{p}$ is not interior to some facet of the convex hull boundary, then $\mathbf{p}$ is a linear combination of $n-1$ or fewer points of Z. But since $\mathbf{p}+t\mathbf{m}_1 = \mathbf{0}$, we would have n or fewer points of Z that are linearly dependent. This would contradict the nondegeneracy of Z. Therefore, it must be that $\mathbf{p}$ is a strictly linear combination of n points of Z. Then since $\mathbf{p}+t\mathbf{m}_1 = \mathbf{0}$ with $t > 0$, there exists a strictly positive set of $n+1$ points of Z that sum to zero. We conclude that some subset of $n+1$ points of Z exists such that the origin lies in the interior of its convex hull. □

The implication of this theorem is that if the contact and fixture set is nondegenerate, the pruning algorithm in section 4 will always find a minimal fixture set. Testing a set of fixtures and contacts for nondegeneracy involves seeing if there exists a nonzero vector $\mathbf{x}$ with n or fewer components such that

$$\mathbf{Mx} = \mathbf{0}$$

where $\mathbf{M} = [\,\mathbf{U} \mid \mathbf{V}\,]$. Unfortunately, this problem is *NP*-complete[15, 3]. Note however that the converse of the theorem is false: even if the fixture and contact force direction vectors form a degenerate set, it still might be possible to form-close the assembly with only $3n+1$ fixtures and contacts. Our intuition then is that if the fixture and contact force direction vectors are not highly degenerate (that is, if most sets of $3n$ or few vectors are linearly independent), then the algorithm *prune-fixtures* from section 4 is likely to come close to finding an optimal solution. The degree to which fixture and contact sets for an assembly are degenerate is of course an application-dependent issue.

8.2 A Bottom-up Approach to Fixturing

The pruning algorithm *prune-fixtures* operates from the top-down: the algorithm starts by initially selecting all the fixtures, then sees which fixtures can be eliminated, subject to the condition that stability is still possible. Theorem 8.1 suggests that a bottom-up strategy could be employed as well: in practice, it might be much faster to start with no fixtures selected, and then continually select fixtures until stability is achieved (or until $6n$ fixtures have been selected, in which case the algorithm halts and the top-down pruning approach is used instead). A bottom-up algorithm would have the basic form:

Algorithm *select-fixtures*
- $F = \emptyset$
- **while** $|F| \leq 6n$
- **do**
  - **let** $i \notin F$ **minimize** $score(F, i)$
  - $F = F \cup \{ i \}$
  - $\mathbf{S} = [\, \mathbf{U} \mid \mathbf{V}_F \,]$
  - **if** $\mathrm{rank}(\mathbf{S}) = 3n$ **and** $\textit{positive-span}(\mathbf{S}, \mathbf{0}) = \textit{TRUE}$
    - **return** F
- **done**
- **return** *ERROR*

The algorithm *select-fixtures* attempts to find a set of $6n$ or less fixtures that establish form-closure. If F grows to have more than $6n$ elements, the algorithm returns an error; in this case, there is no point in continuing, since the pruning algorithm can be used instead to guarantee a fixture set of size $6n$ or less, assuming form-closure can be established at all. (As a further refinement, the set returned by *select-fixtures* could be pruned by the *prune-fixtures* algorithm, since fixtures that are added into F might not be needed later for the form-closure.) The question in a bottom-up approach is is how to go about selecting the next fixture—what function should we choose for $score(F, i)$?

Since theorem 8.1 shows us that it is degeneracy which prevents us from achieving optimal fixture sets, we need to avoid choosing linearly dependent fixture sets. Accordingly, we want to add fixtures into our set that yield forces that are as orthogonal as possible to the current set of fixture/contact-force directions.[12] Thus, a simple choice for *score* might be to measure the deviation of orthogonality between a potential fixture and the current set of fixtures and contacts. One possibility is

$$score(F, i) = \sum_{j \in F} \left(\frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \|\mathbf{v}_j\|} \right)^2 + \sum_{j=1}^{N_c} \left(\frac{\mathbf{v}_i \cdot \mathbf{u}_j}{\|\mathbf{v}_i\| \|\mathbf{u}_j\|} \right)^2 . \tag{12}$$

A fixture with a force direction exactly orthogonal to all the other fixtures and contact force directions is given a score of zero.

Note that both the bottom-up and top-down algorithms can be used to find fixture sets that stabilize an assembly over some range of external forces. As we noted in section 3.2, an assembly with robust directional stability might be stable only over some very small

[12] Actually, this is not exactly true. We would really like to distribute the directions of the vectors chosen as evenly as possible. In a two-dimensional space, we require at least three contact/fixture-force directions for form-closure. The optimal spread of three vectors in two dimensions is to have the vectors exactly 120° apart. In an n-dimensional space, $n + 1$ vectors are evenly spread with angular separation $\cos^{-1} 1/n$. As n grows large, the desired angle between vectors approaches 90°, so we can simply say that in general, we want to pick new fixtures so that they are as orthogonal as possible to all the previously determined fixture/contact-force directions.

range of external forces. However, if an assembly with a fixture set is stabile with respect to some external $\mathbf{Q}_1$ and also $\mathbf{Q}_2$, then the assembly is stable with respect to any convex combination of $\mathbf{Q}_1$ and $\mathbf{Q}_2$. For example, suppose we want an assembly to be stable with respect to gravity, even if gravity is perturbed by up to 15° from vertical. For a planar system, we would need the assembly to be stable with respect to two external forces: gravity perturbed by positive 15° and gravity perturbed by $-15°$. (For a three-dimensional system, the gravity range could be described by four external forces.) In the top-down approach, fixtures are pruned only if their removal allows directional stability with respect to these two external forces. In the bottom-up approach, fixtures are added until the assembly is stable with respect to both external forces.

8.3 Form-closure of Acyclic Assemblies by Regular Fixturing

The one result we have not obtained a complexity measure for is minimal regular fixturing for form-closure. We conclude this paper by showing that the problem has a polynomial-time algorithm, given a restriction in the contact graph of the assembly:

DEFINITION. *Given an assembly with n objects, the contact graph for the assembly is an undirected graph $G = (N, E)$ with $|N| = n$. An edge (i,j) occurs in the graph if objects i and j contact each other.*

We would like to show that if an assembly has an acyclic contact graph, that we can minimally fixture the assembly for form-closure by treating each object of the assembly separately. In essence, while fixturing object i, we will temporarily pretend that all the other objects are already form-closed, and select a set of minimal fixture set F_i that form-closes object i under this pretense. Remarkably, the set $F = \bigcup_i F_i$ will turn out to form-close the entire assembly. Since the number of fixtures required to form-close object i is not increased if we temporarily pretend that all other objects are frozen, $|F_i|$ is a lower bound on the number of fixtures required to form-close object i when all the objects have the potential to move. Thus, if F does indeed induce form-closure, it will be minimal.

The key step in our proof will involve "splitting" a contact between two objects; that is, the contact is ignored, and in its place, a pair of constraints is added (figure 14). That is, in figure 14b, we imagine that objects 1 and 2 no longer touch, but that object 1 touches some immovable obstacle, as does object 2. In making this (temporary) alteration, and splitting the contact at $\mathbf{p}$ into two separate constraints, we are allowing the contact force between objects 1 and 2 to be unbalanced. That is, in figure 14a, a force of the form $\lambda\mathbf{n}$ is applied to object 1, and a force of $-\lambda\mathbf{n}$ is applied to object 2, due to contact. After splitting the contact, a force $\lambda_1\mathbf{n}$ might be applied to object 1, and a force $-\lambda_2\mathbf{n}$ might be applied to object 2. Thus, we will need to show that even though we have treated each contact between objects as two seperate constraints (as in figure 14b), the forces that actually arise at the contact point will be the same as would occur if the contact really was split.

Given the original assembly shown in figure 15a, we would split the contact between objects 1 and 2 as shown in figure 15b. We then separately fixture object 1 by itself, ignoring

(a) (b)

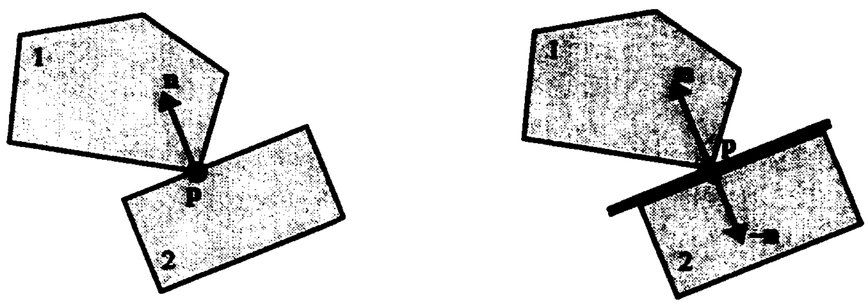


Figure 14: (a) Contact between two movable objects. The contact force exerted at **p** acts equally and oppositely on both objects. (b) The contact is replaced by a pair of contacts with immovable obstacles. Now the force due to contact at **p** on the two objects can be of different strengths.

its contact with object 2 but making use of the constraint on object 1 induced by the splitting. As the example in figure 15b shows, object 1 requires only 2 fixtures for form-closure. Next, we separately fixture the assembly consisting of objects 2 through n, as denoted in figure 15b (the fixtures are not shown in this figure). Again, we ignore the contact between objects 1 and 2 and instead make use of the constraint induced by the splitting. The final fixture set produced is simply the union of the two fixture sets.

THEOREM 8.2 *Given an assembly of n planar objects with an acyclic contact graph, and such that a pair of objects has at most one mutual contact point,*[13] *a minimal set of regular fixtures yielding form-closure can be found in $O(nN_f{}^6)$ time.*

PROOF. The proof is by induction on the number of contact points. Clearly, an assembly with n objects and no contact points can be minimally fixtured for form-closure by applying an exhaustive search algorithm separately to each object. Given a set of N_f regular fixtures, the maximum time to minimally fixture object i is $O(N_f{}^6)$. All n objects can be minimally fixtured for form-closure in at most $O(nN_f{}^6)$ time.

Let us assume that the the theorem applies for assemblies with m contacts, and consider an assembly with $m+1$ contacts. Since the contact graph of the assembly is acyclic, there must be some object that contacts only one other object. (Note that contact with immovable obstacles has no effect on the contact graph; hence in figure 15a, object 1 is said to contact only one other object, namely object 2.) Without loss of generality, let us assume that object 1 contacts object 2, and no other objects. We shall label this contact as contact $m+1$.

Let us split the assembly, as shown in figure 15a. Separately fixturing object 1 requires at most $O(N_f{}^6)$ time. Likewise, separately fixturing the assembly consisting of objects 2

[13]We believe that the theorem applies even if a pair of objects contact each other at multiple points.

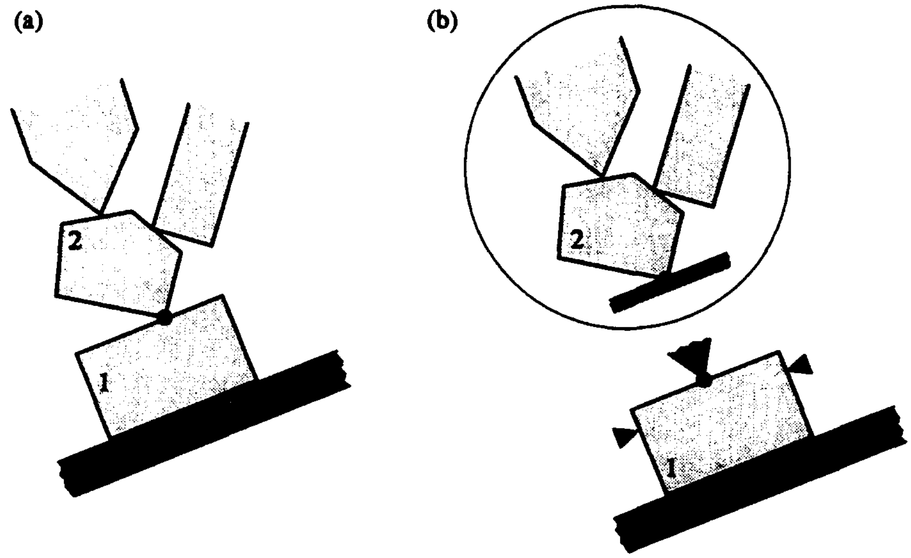


Figure 15: (a) An assembly with $m + 1$ contact points. Object 1 touches object 2, and an immovable obstacle. (b) The assembly is split, and in the new assembly, object 1 is fixtured. Due to the contact constraints on object 1, only two fixtures are required for form-closure. Objects 2 through n are fixtured as a separate group (fixtures for these objects are not drawn).

through n, ignoring contact $m + 1$ but taking the new constraint on object 2 into account requires $O((n - 1)N_f{}^6)$ time, by inductive assumption. We claim that the union of the fixtures chosen for the two separate problems will in fact yield form-closure for the entire assembly. As discussed above, if this is so, then the union of the fixtures is also a minimal fixturing set.

Let the matrix of contact force directions for the original, unsplit problem be written as

$$\mathbf{U} = [\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_{m+1}\,]$$

where $\mathbf{u}_{m+1}$ denotes the direction of force due to m + 1st contact, $\mathbf{U}_1$ denotes all other contact forces that effect object 1, and $\mathbf{U}_2$ denotes all other contact forces that effect objects 2 through n. Note that the first three rows of $\mathbf{U}_2$ are zero, since $\mathbf{U}_2$ denotes only contact forces not involving object 1. Similarly, all *but* the first three rows of $\mathbf{U}_1$ are zero. The vector $\mathbf{u}_{m+1}$ has the form

$$\mathbf{u} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{0}, \mathbf{0}, \ldots)$$

where $\mathbf{d}_1$ and $\mathbf{d}_2$ denote the direction in $\mathbf{R}^3$ of the generalized contact force acting on objects 1 and 2 respectively.

In the first step, we choose a set of fixtures F_A that minimally form-close object 1 by itself. Let $\mathbf{V}_1 = \mathbf{V}_{F_A}$. Similarly, define the fixture matrix $\mathbf{V}_2$ by $\mathbf{V}_2 = \mathbf{V}_{F_B}$. where F_B is the minimal fixture set chosen to minimally form-close objects 2 through n. The matrix $\mathbf{V}_1$, like $\mathbf{U}_1$, only describes forces involving object 1, so all but the first three rows of $\mathbf{V}_1$ are zero. Similarly, the first three rows of $\mathbf{V}_2$ are zero as well.

We claim that $F_A \cup F_B$ induces form-closure on the entire assembly. The contact matrix for the original, unsplit assembly has the form $\mathbf{U} = [\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_{m+1}\,]$, the contact matrix $\mathbf{U}'$ for the split assembly has the form

$$\mathbf{U}' = [\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_A \mid \mathbf{u}_B\,]$$

where

$$\mathbf{u}_A = (\mathbf{d}_1, \mathbf{0}, \mathbf{0}, \ldots) \qquad \text{and} \qquad \mathbf{u}_B = (\mathbf{0}, \mathbf{d}_2, \mathbf{0}, \ldots).$$

Now consider the assembly with contact $m+1$ split, as in figure 15b. Since F_A achieved form closure on a single object (namely, object 1), the composite matrix $[\,\mathbf{U}_1 \mid \mathbf{u}_A \mid \mathbf{V}_1\,]$ must have rank 3. Similarly, F_B form-closed the other $n-1$ objects, so the composite matrix $[\,\mathbf{U}_2 \mid \mathbf{u}_B \mid \mathbf{V}_2\,]$ has rank $3(n-1)$. Since $\mathbf{u}_{m+1} = \mathbf{u}_A + \mathbf{u}_B$, this implies that

$$\text{rank}[\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_{m+1} \mid \mathbf{V}_1 \mid \mathbf{V}_2\,] = \text{rank}\,[\,\mathbf{U} \mid \mathbf{V}_{F_A \cup F_B}\,] = 3n.$$

Thus, $F_A \cup F_B$ satisfies the first condition of theorem 3.2.

The fixture set $F_A \cup F_B$ also satisfies the second condition of theorem 3.2. Since the fixture sets F_A and F_B form-close all the objects in the split assembly, by definition

$$\text{rank}\,[\,\mathbf{U}' \mid \mathbf{V}_{F_A \cup F_B}\,] = 3n$$

and there exist strictly positive vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ such that

$$\mathbf{U}'\boldsymbol{\lambda} + \mathbf{V}_{F_A \cup F_B}\boldsymbol{\alpha} = \mathbf{0}. \tag{13}$$

Let us partition $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ by writing $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ and $\boldsymbol{\lambda} = (\mathbf{x}_1, \mathbf{x}_2, \lambda_1, \lambda_2)$ where λ_1 and λ_2 are scalars. Then from equation (13), we can write

$$\mathbf{U}_1\mathbf{x}_1 + \mathbf{U}_2\mathbf{x}_2 + \mathbf{u}_A\lambda_1 + \mathbf{u}_B\lambda_2 + \mathbf{V}_1\boldsymbol{\alpha}_1 + \mathbf{V}_2\boldsymbol{\alpha}_2 = \mathbf{0}. \tag{14}$$

Since all but the first three rows of $\mathbf{U}_1$, $\mathbf{V}_1$, and $\mathbf{u}_A$ are zero, and the first three rows of $\mathbf{U}_2$, $\mathbf{V}_2$ and $\mathbf{u}_A$ are zero, equation (14) is separable. Thus,

$$\mathbf{U}_1\mathbf{x}_1 + \mathbf{u}_A\lambda_1 + \mathbf{V}_1\boldsymbol{\alpha}_1 = \mathbf{0}$$

and

$$\mathbf{U}_2\mathbf{x}_2 + \mathbf{u}_B\lambda_2 + \mathbf{V}_2\boldsymbol{\alpha}_2 = \mathbf{0}. \tag{15}$$

Since λ_1 is strictly positive, we can write

$$\frac{\lambda_2}{\lambda_1}\mathbf{U}_1\mathbf{x}_1 + \frac{\lambda_2}{\lambda_1}\mathbf{u}_A\lambda_1 + \frac{\lambda_2}{\lambda_1}\mathbf{V}_1\boldsymbol{\alpha}_1 = \frac{\lambda_2}{\lambda_1}\mathbf{U}_1\mathbf{x}_1 + \mathbf{u}_A\lambda_2 + \frac{\lambda_2}{\lambda_1}\mathbf{V}_1\boldsymbol{\alpha}_1 = \mathbf{0}. \tag{16}$$

Adding equations (15) and (16), we obtain

$$\frac{\lambda_2}{\lambda_1}\mathbf{U}_1\mathbf{x}_1 + \mathbf{U}_2\mathbf{x}_2 + \mathbf{u}_A\lambda_2 + \mathbf{u}_B\lambda_2 + \frac{\lambda_2}{\lambda_1}\mathbf{V}_1\boldsymbol{\alpha}_1 + \mathbf{V}_2\boldsymbol{\alpha}_2 = \mathbf{0}.$$

But since $\mathbf{u}_{m+1} = \mathbf{u}_A + \mathbf{u}_B$, we can express $\mathbf{u}_A\lambda_2 + \mathbf{u}_B\lambda_2$ as $\mathbf{u}_{m+1}\lambda_2$, which yields

$$[\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_{m+1}\,] \begin{pmatrix} \mathbf{x}_1 \dfrac{\lambda_2}{\lambda_1} \\ \mathbf{x}_2 \\ \lambda_2 \end{pmatrix} + [\,\mathbf{V}_1 \mid \mathbf{V}_2\,] \begin{pmatrix} \boldsymbol{\alpha}_1 \dfrac{\lambda_2}{\lambda_1} \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \mathbf{0}.$$

Since $\mathbf{U} = [\,\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{u}_{m+1}\,]$ and $\mathbf{V}_{F_A \cup F_B} = [\,\mathbf{V}_1 \mid \mathbf{V}_2\,]$, this yields

$$\mathbf{U} \begin{pmatrix} \mathbf{x}_1 \dfrac{\lambda_2}{\lambda_1} \\ \mathbf{x}_2 \\ \lambda_2 \end{pmatrix} + \mathbf{V}_{F_A \cup F_B} \begin{pmatrix} \boldsymbol{\alpha}_1 \dfrac{\lambda_2}{\lambda_1} \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \mathbf{0}. \tag{17}$$

Since λ_1 and λ_2 are positive scalars, and $\mathbf{x}_1$, $\mathbf{x}_2$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are strictly positive vectors, the vectors $(\mathbf{x}_1, \lambda_2/\lambda_1, \mathbf{x}_2, \lambda_2)$ and $(\boldsymbol{\alpha}_1\lambda_2/\lambda_1, \boldsymbol{\alpha}_2)$ are strictly positive as well. Thus, equation (17) establishes that $F_A \cup F_B$ satisfies the second condition of theorem 3.2. We conclude that $F_A \cup F_B$ does indeed induce form-closure. Since the two fixturing steps require at most $O(N_f{}^6)$ and $O((n-1)N_f{}^6)$ time, the entire algorithm requires at most $O(nN_f{}^6)$ time. □

The theorem applies directly to assemblies without any cycles. However, given an assembly with only a small number of cycles, one could "break" the cycles using the following approach: suppose that an assembly has a single cycle in its contact graph, and the removal of some node i from the graph eliminates that cycle. If fixtures acting solely on object i are introduced to form close object i, treating object i as if it had no contact with any other object in the assembly, then thereafter, object i is an immovable obstacle. Treating object i as such, the new assembly is acyclic and can be quickly fixtured. Clearly, this approach can be used only as long as the number of cycles in the graph is small.

9 Conclusion

We have defined three varieties of fixturing for multi-body frictionless assemblies with contact. We have explored the complexity of finding smallest sets of fixtures inducing various types of stability on the assemblies. For the most part, finding minimal fixture sets is shown to be *NP*-hard. This establishes that under the model where each fixture has some preassigned cost, minimal-cost fixturing is mostly *NP*-hard as well. Based on the theorems of Carathéodory, and Steinitz, simple polynomial-time algorithms have been presented for finding small, but not necessarily optimal fixture sets. Finally, a characterization of when lower bounds on fixturing sets can be realized is given in the guise of geometric degeneracy in the force-space of the contacts and fixtures.

Much work remains however; choosing to minimize the number of fixtures is only one of the many ways to optimize fixture design. For some applications, a minimal set of fixtures may not necessarily be a good set of fixtures in practice. Following the lines of research on single-object grasping, there are many factors to consider: for example, one might want to consider the magnitude of forces exerted by the fixtures and contacts, to avoid situations in which the fixture set theoretically yields stability, but only by exerting an enormous force on an object. (This can arise if a pair of fixtures point in almost exactly opposite directions.) Clearly, for some applications, the assumption of frictionless contact is too restrictive, and attention needs to be given to the tangential friction forces arising at contacts and fixtures.

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