Effective reflection coefficients for the mean acoustic field between two rough interfaces

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Abstract

A formalism is presented which demonstrates that the mean Green's function for the acoustic field between two rough interfaces can be expressed as a Green's function associated with two flat interfaces with effective reflection coefficients. This result incorporates all orders of the fluctuations in the half-space scattering amplitudes associated with each interface considered separately. From the mean Green's function modal attenuations can be found. To lowest order in the surface height fluctuations it is shown that it is not sufficient to use mean half-space scattering amplitudes as effective reflection coefficients. The formalism is designed to provide approximations for the Green's function in layered media which are based on previously developed approximations for half-space scattering amplitudes.

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I Introduction

A number of good approximations have recently been developed for plane-wave, half-space rough surface scattering amplitudes. The aim of this paper is to show how these scattering amplitudes can be used to describe sound propagation between two rough interfaces. This work is motivated by a certain laziness: it shouldn't be necessary to repeat the discussions used to describe the scattering from one interface when treating two interfaces. One should only be required to treat the interactions between the interfaces.

In an earlier paper [1] the method of smoothing was used to treat this problem to lowest order in scattering amplitude fluctuations. It was indeed shown there that there are effective reflection coefficients for the mean Green's functions which are distinct from the mean half-space scattering amplitudes. Here the work of Ref.[1] is extended to all orders in scattering amplitude fluctuations. It is important here to distinguish scattering amplitude fluctuations from fluctuations in surface heights. In general, scattering amplitudes are non-linear functionals of the surface height. The idea here and in Ref.[1] is to develop a formalism which uses non-perturbative approximations of the scattering amplitudes.

It will be shown here that the existence of effective reflection coefficients for propagation between two rough interfaces is not an artifact of the smoothing approximation. In fact, the effective reflection coefficients can be expressed in terms of self-energies associated with a certain random operator. In constructing the effective reflection coefficients, a systematic way of including higher order moments of the fluctuations of the half-space scattering amplitudes will be developed. In another extension of the work in Ref.[1], it will be shown how correlations between the scattering on the upper and lower interfaces affect the mean field. If there is correlation, the mean Green's function is the sum of two effective Green's functions. Finally, modal attentuations for the mean field will be discussed here. It will be shown that in the case of Dirichlet boundary conditions, in the small roughness limit, the present method reproduces the perturbation results of Bass and Fuks [2]. Bass and Fuks also show that when a waveguide is not wide compared to an acoustic wavelength, then modal attenuations arising from surface roughness using half- space scattering amplitudes, as in the work of Kuperman and Ingenito [3], are considerably different that those calculated using renormalized reflection coefficents. It should be noted that Voronovich has treated the problem of waveguides with a single rough boundary [4] in a somewhat different manner. The results presented here for effective reflection

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coefficients are implicit in his work.

The discussion here concerns only constant sound speed profiles for which up- and down-going plane waves can be used. In section II, scattering states will be introduced and will be used to construct the field of a point source for each realization of an ensemble of rough interfaces. In section III stochastic equations for the Green's function will be discussed. In particular, a self-energy operator will be defined. In section IV the mean Green's function for the case of statistically homogeneous roughness will be developed. Section V discusses modal attenuations. A summary of this work, and an outline of future work will be found in section VI.

II Scattering states and the construction of the Green's function

Consider a homogeneous fluid bounded by two rough interfaces as in Fig. 1. These interfaces are parallel in the mean and may or may not be penetrable. The formalism to be presented here is not tied to any particular boundary conditions. However, assume that half-space scattering amplitudes for each of these surfaces considered separately are known: $T^+(\mathbf{K}|\mathbf{Q})$ for an upgoing plane wave whose wave vector has a horizontal projection $\mathbf{Q}$ scattering at the upper interface into a downgoing plane wave whose wave vector has horizontal component $\mathbf{K}$, and $T^-(\mathbf{K}|\mathbf{Q})$ for a downgoing plane wave described by the horizontal wave vector $\mathbf{Q}$ scattering into an upgoing wave with horizontal wave vector $\mathbf{K}$. In practice, one will have only approximations for $T^{\pm}$. How can one construct the Green's function of the acoustic field using these amplitudes? The idea here is that a lot of physics has gone into developing approximations of the scattering amplitudes. Can this physics be incorporated directly into the two interface problem?

The problem here is to find the Green's function which satisfies

$$(\nabla_r^2 + \omega^2/c^2)G(\mathbf{r}, \mathbf{r}_o) = -\delta(\mathbf{R} - \mathbf{R}_o)\delta(z - z_o) \tag{1}$$

where

$$\mathbf{r} = (\mathbf{R}, z), \mathbf{r}_o = (\mathbf{R}_o, z_o).$$

In general, lower case vectors will denote vectors in 3 dimensions with their horizontal projections given by the corresponding upper case letter. As usual, ω is the acoustic frequency and c is the speed of sound between the interfaces. It will prove convenient to work with the double sided

Fourier transform of G,

$$\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) \equiv \int \exp(-i\mathbf{K} \cdot \mathbf{R} + i\mathbf{Q} \cdot \mathbf{R}_o) G(\mathbf{R}, z; \mathbf{R}_o, z_o) d\mathbf{R} d\mathbf{R}_o. \qquad (2)$$

(In Ref.[1] only one-sided transforms were used. Two-side transforms make the equations below more symmetric, but they are not absolutely necessary.) The two-sided transform satisfies the separated wave equation

$$\partial_z^2 \hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) + (\omega^2/c^2 - K^2)\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) = -(2\pi)^2 \delta(z - z_o)\delta(\mathbf{K} - \mathbf{Q}). \qquad (3)$$

At the rough bounding surfaces $S^{\pm}$, G must respect whatever boundary conditions are imposed there. As a result, $\hat{G}$ will not be diagonal in horizontal wave vector. In order to respect these boundary conditions, whatever they are, G and $\hat{G}$ will be constructed from two sets of scattering states $\chi_{\mathbf{Q}}^{\pm}(\mathbf{r})$ which obey boundary conditions on $S^{\pm}$ respectively. Away from the boundaries, these states can be written as superpostions of up- and down-going plane waves using the associated half-space scattering amplitudes. By staying away from the boundaries, the limits of the Rayleigh hypothesis will not be encountered. In contrast to Ref.[1], here the Fourier transforms of $\chi_{\mathbf{Q}}^{\pm}(\mathbf{r})$ will be used:

$$\chi^{\pm}(\mathbf{K}, \mathbf{Q}, z) = \exp(\pm ik_z z)\delta(\mathbf{K} - \mathbf{Q}) + \exp(\mp ik_z z)T^{\pm}(\mathbf{K}|\mathbf{Q}). \qquad (4)$$

Here k_z is the vertical component of a wavevector with horizontal projection $\mathbf{K}$

$$k_z = \sqrt{(\omega^2/c^2) - K^2}.$$

It was shown in Ref.[5] that the scattering states are sufficient to compute half-space Green's functions. It will be assumed that they are also sufficient to compute the Green's function between two interfaces. In order to respect the upper boundary conditions, the Green's function $\hat{G}$ will be constructed as a continuous superpostion of the states χ^+ when $z > z_o$ and as a different superpostion of the scattering states χ^- when $z < z_o$. The coefficients of the superpositions must depend on z_o and the "incident" wave vector $\mathbf{Q}$. Hence $\hat{G}$ will be written

$$\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) = \begin{cases} \int \chi^+(\mathbf{K}, \mathbf{P}, z)\alpha^+(\mathbf{P}, \mathbf{Q}, z_o)d\mathbf{P} & \text{if } z > z_o \\ \int \chi^-(\mathbf{K}, \mathbf{P}, z)\alpha^-(\mathbf{P}, \mathbf{Q}, z_o)d\mathbf{P} & \text{if } z < z_o \end{cases}. \qquad (5)$$

The α's can be determined from jump conditions across the plane containing the source, $z = z_o$. $\hat{G}$ must be continuous across this plane and the vertical derivative of $\hat{G}$ must have

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a jump discontinuity to produce the right hand side of Eq. 3. More explicitly these jump conditions are

$$\hat{G}(\mathbf{K}, z_o^+; \mathbf{Q}, z_o) - \hat{G}(\mathbf{K}, z_o^-; \mathbf{Q}, z_o) = 0 \tag{6}$$

$$\frac{1}{ik_z}\partial_z\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o)|_{z=z_o^+} - \frac{1}{ik_z}\partial_z\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o)|_{z=z_o^-} = -\frac{(2\pi)^2}{ik_z}\delta(\mathbf{K} - \mathbf{Q}) \tag{7}$$

These conditions can be written as a matrix-integral equation for the unknown coefficients α. Define elements of a matrix of integral kernels, M by

$$M_{1,1}(\mathbf{K}, \mathbf{P}, z_o) = \chi^+(\mathbf{K}, \mathbf{P}, z_o) \tag{8}$$

$$M_{1,2}(\mathbf{K}, \mathbf{P}, z_o) = -\chi^-(\mathbf{K}, \mathbf{P}, z_o) \tag{9}$$

$$M_{2,1}(\mathbf{K}, \mathbf{P}, z_o) = (1/ik_z)\partial_z\chi^+(\mathbf{K}, \mathbf{P}, z_o) \tag{10}$$

$$M_{2,2}(\mathbf{K}, \mathbf{P}, z_o) = (-1/ik_z)\partial_z\chi^-(\mathbf{K}, \mathbf{P}, z_o). \tag{11}$$

The jump conditions across z_o can now be written as

$$\int d\mathbf{P} \begin{pmatrix} M_{1,1}(\mathbf{K}, \mathbf{P}, z_o) & M_{1,2}(\mathbf{K}, \mathbf{P}, z_o) \\ M_{2,1}(\mathbf{K}, \mathbf{P}, z_o) & M_{2,2}(\mathbf{K}, \mathbf{P}, z_o) \end{pmatrix} \begin{pmatrix} \alpha^+(\mathbf{P}, \mathbf{Q}, z_o) \\ \alpha^-(\mathbf{P}, \mathbf{Q}, z_o) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{(2\pi)^2}{ik_z}\delta(\mathbf{K} - \mathbf{Q}) \end{pmatrix}. \tag{12}$$

This equation is essentially the Fourier transform with respect to the horizontal position of the source, $\mathbf{R}_o$ of Eq. 14 in Ref.[1]. To condense the notation write this equation as

$$M(z_o)\alpha(z_o) = s, \tag{13}$$

with formal solution

$$\alpha(z_o) = M^{-1}(z_o)s. \tag{14}$$

which can be written out as

$$\begin{pmatrix} \alpha^+(\mathbf{P}, \mathbf{Q}, z_o) \\ \alpha^-(\mathbf{P}, \mathbf{Q}, z_o) \end{pmatrix} = \int d\mathbf{K} \begin{pmatrix} M_{1,1}^{-1}(\mathbf{P}, \mathbf{K}, z_o) & M_{1,2}^{-1}(\mathbf{P}, \mathbf{K}, z_o) \\ M_{2,1}^{-1}(\mathbf{P}, \mathbf{K}, z_o) & M_{2,2}^{-1}(\mathbf{P}, \mathbf{K}, z_o) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{(2\pi)^2}{ik_z}\delta(\mathbf{K} - \mathbf{Q}) \end{pmatrix}. \tag{15}$$

Note that $M_{1,1}^{-1}(\mathbf{P}, \mathbf{K}, z_o)$ does not denote the inverse of $M_{1,1}$. Rather $M_{1,1}^{-1}$ is the $(1, 1)$ (operator) element of M^{-1}.

The coefficients α^+ are now readily expressed in terms of $M_{1,2}^{-1}$ and α^- in terms of $M_{2,2}^{-1}$. The resulting expression for the Green's function $\hat{G}$ is

$$\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) = \begin{cases} \int M_{1,1}(\mathbf{K}, \mathbf{P}, z)M_{1,2}^{-1}(\mathbf{P}, \mathbf{Q}, z_o)d\mathbf{P}[-(2\pi)^2/(iq_z)] & \text{if } z > z_o \\ \int -M_{1,2}(\mathbf{K}, \mathbf{P}, z)M_{2,2}^{-1}(\mathbf{P}, \mathbf{Q}, z_o)d\mathbf{P}[-(2\pi)^2/(iq_z)] & \text{if } z < z_o \end{cases} \tag{16}$$

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To this point everything presented has been equivalent, but perhaps more explicit, to the development in Ref.[1]. For flat surfaces, the operator M is diagonal in wavenumber so that inversion is simply the inversion of a 2 × 2 ordinary matrix. As shown in Ref.[1], the resulting Green's function is the same as that given by Brekhovskikh for flat interfaces.

III   Finding the Mean Green's Function

The Green's function is random because the scattering amplitudes $T^\pm$ are random. The primary difficulty in determining, for example, the moments of $\hat{G}$ is in finding the moments of M^{-1}. In Ref.[1] this was done by the method of smoothing which gives results which only involve second moments of the fluctuation. Here the problem of treating M^{-1} is addressed with methods of quantum field theory. The idea is that if one performs a perturbation expansion of an operator about a free-space operator, long-range interactions can cause difficulties in the covergence of the expansion, and even place the existence of some of the terms in the expansion in doubt. However, intuitively, one supposes that propagation between scattering events is damped at long ranges because a third (higher order) event may come between the two events in question. One might suppose that propagation between scattering events ought to be described using a mean propagator, which is damped because of the randomness of the medium. Of course the mean propagator is what one is trying to find, so the resulting formalism will present a problem in self-consistency. The formal aspects of such a proceedure will now be presented. The operator algebra is straight forward, even if the motivation may be somewhat obscure. A more thorough presentation of this algebra is presented in the appendix of Ref.[6]. In contrast to the smoothing formalism, the algebra presented below results in formal expressions which include all orders of the fluctuations.

Denote the inverse of M by

$$\mathcal{G}(z_o) \equiv M^{-1}(z_o), \quad \mathcal{G}_{i,j}(z_o) = M_{i,j}^{-1}(z_o). \tag{17}$$

The explanation of this notation is that M^{-1} is the inverse of a linear operator, as is a Green's function. In order not to confuse this inverse with the Green's function G, the inverse of the Helmholtz operator, a calligraphic $\mathcal{G}$ is used.

Let

$$M = < M > + \Delta M. \tag{18}$$

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The angular brackets denote an ensemble average over the realizations of scattering amplitudes $T^{\pm}$. Fluctuations about the mean are denoted by ΔM. Just as the inverse of M is written

$$\mathcal{G} = \frac{1}{<M> + \Delta M}, \tag{19}$$

so the mean of $\mathcal{G}$ is written

$$<\mathcal{G}> = \frac{1}{<M> + <\Sigma>}. \tag{20}$$

$<\Sigma>$ is called the average self-energy (or mass) operator or simply the self-energy. The terminology comes from quantum field theory. Use $\mathcal{G}_o$ to denote the inverse of the average of M:

$$\mathcal{G}_o = \frac{1}{<M>}. \tag{21}$$

Following Ref.[6] it is simple to show from Eqs. 20 and 21 that the mean of M^{-1} and the inverse of $<M>$ are related by

$$<\mathcal{G}> = \mathcal{G}_o - \mathcal{G}_o <\Sigma> <\mathcal{G}>. \tag{22}$$

This is Dyson's equation, eq. A14 of Ref. 6. Fluctuations of $\mathcal{G}$ are found from

$$\mathcal{G} - <\mathcal{G}> = \frac{1}{\mathcal{G}_o^{-1} + \Delta M} - \frac{1}{\mathcal{G}_o^{-1} + <\Sigma>}, \tag{23}$$

which using simple operator algebra can be written as

$$\mathcal{G} - <\mathcal{G}> = -\mathcal{G}(\Delta M - <\Sigma>)<\mathcal{G}> = -<\mathcal{G}>(\Delta M - <\Sigma>)\mathcal{G}. \tag{24}$$

Finally, $\mathcal{G}$ can be expressed entirely in terms of $<\mathcal{G}>$ and a scattering operator T:

$$\mathcal{G} = <\mathcal{G}> + <\mathcal{G}> T <\mathcal{G}>, \tag{25}$$

where the scattering operator is given by

$$T \equiv -(\Delta M - <\Sigma>)\frac{1}{1 + <\mathcal{G}>(\Delta M - <\Sigma>)}. \tag{26}$$

Expansion of the operator T shows that the "propagation" between "scattering events," $(\Delta M - <\Sigma>)$, occurs via the mean of the inverse of M, $<\mathcal{G}>$, not via the inverse of the mean of M, $\mathcal{G}_o$.

Since $<\mathcal{G} - <\mathcal{G}>> = 0$, it follows that the mean of the scattering operator T, vanishes.

$$<T> = 0. \tag{27}$$

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The vanishing of the mean of the scattering operator T can be used to develop an expansion of the self energy in powers of the fluctuations ΔM. Write

$$< \Sigma > = \sigma_o + \sigma_1 + \sigma_2 + \dots \tag{28}$$

where the σ_i is the contribution to $< \Sigma >$ from terms ith order in powers of ΔM. Equation 27 and the definition of the scattering operator T in Eq. 26 imply that

$$\sigma_o = 0 \tag{29}$$

$$\sigma_1 = 0 \tag{30}$$

$$\sigma_2 = - < \Delta M < \mathcal{G} > \Delta M > \tag{31}$$

$$\sigma_3 = < \Delta M < \mathcal{G} > \Delta M < \mathcal{G} > \Delta M > \tag{32}$$

$$\sigma_4 = - < \Delta M < \mathcal{G} > \Delta M < \mathcal{G} > \Delta M < \mathcal{G} > \Delta M >$$
$$+ \sigma_2 < \mathcal{G} > \sigma_2 - < \Delta M < \mathcal{G} > \sigma_2 < \mathcal{G} > \Delta M > \tag{33}$$

For Gaussian fluctuations, σ_3 will vanish. The subtractions in σ_4 show that it is represented by a skeleton diagram, or an irreducible diagram [6].

Substituting the expression for $\mathcal{G}$ in Eq.25 in the identities $< M M^{-1} > = < M^{-1} M > = 1$ shows that $< \mathcal{G} >$ can be written as

$$< \mathcal{G} > = \frac{1}{< M > + < \Delta M < \mathcal{G} > T >} = \frac{1}{< M > + < T < \mathcal{G} > \Delta M >}. \tag{34}$$

Comparing with the definition of the self energy shows

$$< \Sigma > = < \Delta M < \mathcal{G} > T > = < T < \mathcal{G} > \Delta M > . \tag{35}$$

In this equation $< \mathcal{G} >$ on the right hand side depends on $< \Sigma >$, and T can't be found until $< \mathcal{G} >$ is known. Hence, although this result is exact, it is purely formal. To make progress, the quantities on the right must be approximated. It is to be hoped that approximation of the operators on the right will be a more effective proceedure than approximating $< \mathcal{G} >$ directly.

It is to be understood here that all operators depend on z_o. In fact the fluctuation ΔM has a simple dependence on z_o.

$$\Delta M(\mathbf{K}, \mathbf{Q}, z_o) = \exp(-ik_z z_o) \Delta T^+(\mathbf{K}|\mathbf{Q}) \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} - \exp(+ik_z z_o) \Delta T^-(\mathbf{K}|\mathbf{Q}) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$
$$\tag{36}$$

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Thus, with summation over repeated indices and integration over repeated wavevectors understood, the self-energy can be written as

$$< \Sigma(\mathbf{P}, \mathbf{Q}, z_o) >=$$

$$\exp(-ip_z z_o) \begin{pmatrix} \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,1} & \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,2} \\ -\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,1} & -\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,2} \end{pmatrix}$$

$$- \exp(+ip_z z_o) \begin{pmatrix} \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,1} & \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,2} \\ \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,1} & \Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,2} \end{pmatrix}. \tag{37}$$

where

$$\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,1} = < \Delta T^+(\mathbf{P}|\mathbf{K}) < \mathcal{G} >_{1,j}(\mathbf{K}, \mathbf{K}', z_o) T_{j,1}(\mathbf{K}', \mathbf{Q}, z_o) > \tag{38}$$

$$\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{1,2} = < \Delta T^+(\mathbf{P}|\mathbf{K}) < \mathcal{G} >_{1,j}(\mathbf{K}, \mathbf{K}', z_o) T_{j,2}(\mathbf{K}', \mathbf{Q}, z_o) > \tag{39}$$

$$\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,1} = < \Delta T^-(\mathbf{P}|\mathbf{K}) < \mathcal{G} >_{2,j}(\mathbf{K}, \mathbf{K}', z_o) T_{j,1}(\mathbf{K}', \mathbf{Q}, z_o) > \tag{40}$$

$$\Lambda(\mathbf{P}, \mathbf{Q}, z_o)_{2,2} = < \Delta T^-(\mathbf{P}|\mathbf{K}) < \mathcal{G} >_{2,j}(\mathbf{K}, \mathbf{K}', z_o) T_{j,2}(\mathbf{K}', \mathbf{Q}, z_o) > . \tag{41}$$

Even though the averages here appear to depend on the source level, z_o, they, in fact, are independent of z_o, as is shown in the Appendix. The functions Λ will turn out to be those parts of the effective reflection coefficients not contained in the mean half-space scattering amplitudes.

These formal expressions for $\mathcal{G}$ allow computation of the averaged Green's function for the Helmholtz equation given in Eq.16. For example, for the case $z > z_o$

$$< \hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) >= [-(2\pi)^2/(iq_z)] \int < M_{1,1}(\mathbf{K}, \mathbf{P}, z) >< \mathcal{G}_{1,2}(\mathbf{P}, \mathbf{Q}, z_o) >$$

$$+ < \Delta M_{1,1}(\mathbf{K}, \mathbf{P}, z) \Delta \mathcal{G}_{1,2}(\mathbf{P}, \mathbf{Q}, z_o) > d\mathbf{P}. \tag{42}$$

The first product of averages is easily expressed in terms of $< \Sigma >$, and therefore Λ's, and the averaged half-space scattering amplitudes, $< T^\pm >$. The average of the product of fluctuations in Eq. 42 appears more problematic, but it too can be expressed in terms of the $\Lambda's$, as follows. The fluctuation of $M_{1,1}$ is

$$\Delta M_{1,1}(\mathbf{K}, \mathbf{P}, z) = \exp(-ik_z z) \Delta T^+(\mathbf{K}|\mathbf{P}). \tag{43}$$

The fluctuation of $\mathcal{G}$ from eq.25 is just $< \mathcal{G} > T < \mathcal{G} >$. The average of the product of these fluctuations is (again with summations and integrations understood for repeated indices and wavevectors)

$$< \Delta M_{1,1}(\mathbf{K}, \mathbf{P}, z)\Delta \mathcal{G}_{1,2}(\mathbf{P}, \mathbf{Q}, z_o) >=$$

$$\exp(-ik_z z) < \Delta T^+(\mathbf{K}|\mathbf{P}) < \mathcal{G}(\mathbf{P}, \mathbf{P}', z_o)_{1,j} > T(\mathbf{P}', \mathbf{P}'', z_o)_{j,l} >< \mathcal{G}(\mathbf{P}'', \mathbf{Q}, z_o)_{l,2} >$$

$$= \exp(-ik_z z)\Lambda(\mathbf{K}, \mathbf{P})_{1,j} < \mathcal{G}(\mathbf{P}, \mathbf{Q}, z_o)_{j,2} > . \tag{44}$$

This general expression for the average of the products of the fluctuations was missed in ref.[1]. The point here is that the mean Green's function for the Helmholtz equation, $\hat{G}$, is now entirely expressible in terms of the averaged half-space scattering amplitudes $< T^\pm >$ and in terms of the functions $\Lambda_{i,j}$. When there is statistical homogeneity it will be shown that $\hat{G}$ can be expressed as the Green's function for a duct with flat interfaces and effective reflection coefficients which are combinations of $< T^\pm >$ and the Λ's. In terms of the Λ's and the mean of $\mathcal{G}$ the Green's function for the Helmholtz equation is now given by

$$\langle \hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o)\rangle =$$

$$-\frac{(2\pi)^2}{iq_z} \int \left[\exp(+ik_z z)\delta(\mathbf{K}-\mathbf{P}) + \exp(-ik_z z)\{\langle T^+(\mathbf{K}|\mathbf{P})\rangle + \Lambda_{1,1}(\mathbf{K}, \mathbf{P})\}\right] \langle \mathcal{G}(\mathbf{P}, \mathbf{Q}, z_o)_{1,2}\rangle$$

$$+ \exp(-ik_z z)\Lambda_{1,2}(\mathbf{K}, \mathbf{P})\langle \mathcal{G}(\mathbf{P}, \mathbf{Q}, z_o)_{2,2}\rangle d\mathbf{P}, \tag{45}$$

for the case $z > z_o$. For the case $z < z_o$, $\langle \hat{G} \rangle$ is given by

$$\langle \hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o)\rangle =$$

$$-\frac{(2\pi)^2}{iq_z} \int \left[\exp(-ik_z z)\delta(\mathbf{K}-\mathbf{P}) + \exp(+ik_z z)\{\langle T^-(\mathbf{K}|\mathbf{P})\rangle + \Lambda_{2,2}(\mathbf{K}, \mathbf{P})\}\right] \langle \mathcal{G}(\mathbf{P}, \mathbf{Q}, z_o)_{2,2}\rangle$$

$$+ \exp(+ik_z z)\Lambda_{2,1}(\mathbf{K}, \mathbf{P})\langle \mathcal{G}(\mathbf{P}, \mathbf{Q}, z_o)_{1,2}\rangle d\mathbf{P}. \tag{46}$$

IV   Statistically Homogeneous Roughness

In this this section the results of the previous section are specialized to the case of statistically homogeneous surface roughness. This case is tractable because when the statistics are homogeneous, the inverse of the mean Green's function $< \mathcal{G} >^{-1} = < M > + < \Sigma >$ becomes diagonal in wavenumber. Finding $< \mathcal{G} >$ itself then requires only finding the inverse of a 2×2 matrix.

As discussed in the appendix to Ref.[1], statistically homogeneous roughness means that if $z = h(\mathbf{R})$ is a member of a ensemble of surfaces, then $z = D_{\mathbf{a}}h(\mathbf{R}) \equiv h(\mathbf{R} - \mathbf{a})$ is an equally probable member of that ensemble, where $\mathbf{a}$ is any constant translation vector in the x-y plane. In Ref.[1] it is also argued that scattering amplitudes are functionals of the surface function $h(\mathbf{R})$. (Here there are two surface functions, one for the upper surface and one for the lower surface. When necessary h will be understood to include both surfaces, and translations will apply to both simultaneously.) If this functional dependence is made explicit, as in $T(\mathbf{K}|\mathbf{Q}, [h])$, then simple physical arguments show that the scattering amplitudes for translated and untranslated surfaces, $D_{\mathbf{a}}h$ and h are related by

$$T^{\pm}(\mathbf{K}|\mathbf{Q}, [h]) = \exp(i(\mathbf{K} - \mathbf{Q}) \cdot \mathbf{a}) T^{\pm}(\mathbf{K}|\mathbf{Q}, [D_{\mathbf{a}}h]). \tag{47}$$

Applying the same translation to both the upper and lower surfaces shows that the matrix operator M behaves similarly:

$$M(\mathbf{K}, \mathbf{Q}, z_o, [h]) = \exp(i(\mathbf{K} - \mathbf{Q}) \cdot \mathbf{a}) M(\mathbf{K}, \mathbf{Q}, z_o, [D_{\mathbf{a}}h]). \tag{48}$$

Since $\mathcal{G}$ is the inverse of M, it too behaves this way under horizontal translations of the waveguide:

$$\mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [h]) = \exp(i(\mathbf{K} - \mathbf{Q}) \cdot \mathbf{a}) \mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [D_{\mathbf{a}}h]). \tag{49}$$

Because of the assumed statistical homogeneity, it follows that

$$< \mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [h]) > = < \mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [D_{\mathbf{a}}h]) >, \tag{50}$$

for all translations $\mathbf{a}$ and all wavevectors $\mathbf{K}$ and $\mathbf{Q}$. This means that $< \mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [h]) >$ must be proportional to a delta function in $\mathbf{K} - \mathbf{Q}$, i.e. it must be diagonal in wavevectors, and that it can be written as

$$< \mathcal{G}(\mathbf{K}, \mathbf{Q}, z_o, [h]) > = \delta(\mathbf{K} - \mathbf{Q}) g(\mathbf{K}, z_o). \tag{51}$$

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When eq. 25 is used to compare $\mathcal{G}$'s for translated and untranslated surfaces, it follows from eqs. 49 and 51 that $\mathcal{T}$ has the same properties under translation:

$$\mathcal{T}(\mathbf{K}, \mathbf{Q}, z_o, [h]) = \exp(i(\mathbf{K} - \mathbf{Q}) \cdot \mathbf{a}) \mathcal{T}(\mathbf{K}, \mathbf{Q}, z_o, [D_\mathbf{a}h]). \tag{52}$$

The matrix elements of the average self energy Σ, the Λ's (treated generically) satisfy

$$\Lambda = < \Delta T(\mathbf{K}|\mathbf{P}, [h]) g(\mathbf{P}, z_o) T(\mathbf{P}, \mathbf{Q}, z_o, [h] > =$$

$$\exp[i(\mathbf{K} - \mathbf{Q}) \cdot \mathbf{a}] < \Delta T(\mathbf{K}|\mathbf{P}, D_\mathbf{a}[h]) g(\mathbf{P}, z_o) T(\mathbf{P}, \mathbf{Q}, z_o, D_\mathbf{a}[h] > . \tag{53}$$

The averages are equal for all translations and it follows again that the Λ's are diagonal in wavevector:

$$\Lambda_{i,j}(\mathbf{K}, \mathbf{Q}) = \delta(\mathbf{K} - \mathbf{Q})\lambda_{i,j}(\mathbf{K}). \tag{54}$$

Obviously the averaged self-energy $< \Sigma >$ is also diagonal in wavevector and can be expressed in terms of the λ's. The same arguments show that the mean scattering amplitudes can be written

$$< T^\pm(\mathbf{K}|\mathbf{Q}) > = \delta(\mathbf{K} - \mathbf{Q}) t^\pm(\mathbf{K}). \tag{55}$$

It is now possible to express the mean Green's function g in terms of the λ's. The function g is found from the inverse of

$$< M(\mathbf{K}, \mathbf{Q}, z_o) > + < \Sigma(\mathbf{K}, \mathbf{Q}, z_o) > = \delta(\mathbf{K} - \mathbf{Q}) \times$$

$$\begin{pmatrix} + \exp(-ik_z z_o)(t^+ + \lambda_{1,1}) + \exp(ik_z z_o)(1 - \lambda_{2,1}), & - \exp(-ik_z z_o)(1 - \lambda_{1,2}) - \exp(+ik_z z_o)(t^- + \lambda_{2,2}) \\ - \exp(-ik_z z_o)(t^+ + \lambda_{1,1}) + \exp(ik_z z_o)(1 - \lambda_{2,1}), & + \exp(-ik_z z_o)(1 - \lambda_{1,2}) - \exp(+ik_z z_o)(t^- + \lambda_{2,2}) \end{pmatrix} . \tag{56}$$

and is given by

$$g(\mathbf{K}, z_o) = \frac{1}{2D(\mathbf{K})}$$

$$\begin{pmatrix} + \exp(-ik_z z_o)(1 - \lambda_{1,2}) - \exp(ik_z z_o)(t^- + \lambda_{2,2}), & + \exp(-ik_z z_o)(1 - \lambda_{1,2}) + \exp(ik_z z_o)(t^- + \lambda_{2,2}) \\ - \exp(ik_z z_o)(1 - \lambda_{2,1}) + \exp(-ik_z z_o)(t^+ + \lambda_{1,1}), & \exp(ik_z z_o)(1 - \lambda_{2,1}) + \exp(-ik_z z_o)(t^+ + \lambda_{1,1}) + \end{pmatrix} . \tag{57}$$

It is understood here that $t^\pm = t^\pm(\mathbf{K})$ and $\lambda_{i,j} = \lambda_{i,j}(\mathbf{K})$. The determinant $2D$ is given by

$$2D(\mathbf{K}) = 2[(1 - \lambda_{1,2}(\mathbf{K}))(1 - \lambda_{2,1}(\mathbf{K})) - (t^+(\mathbf{K}) + \lambda_{1,1}(\mathbf{K}))(t^-(\mathbf{K}) + \lambda_{2,2}(\mathbf{K}))] \tag{58}$$

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These results can now be used to write the mean Green's function of the Helmholtz equation, $\hat{G}$, in terms of $\lambda_{i,j}$ and $t^{\pm}$. Combining Eqs. 42, 44, 51 and 54 gives

$$< \hat{G}(\mathbf{K}, z, \mathbf{Q}, z_o) >= \frac{-(2\pi)^2}{ik_z}\delta(\mathbf{K} - \mathbf{Q})G(\mathbf{K}, z, z_o). \tag{59}$$

For $z > z_o$ the reduced Green's function is

$$G(\mathbf{K}, z, z_o) = [\exp(ik_z z) + \exp(-ik_z z)(t^+(\mathbf{K}) + \lambda_{1,1}(\mathbf{K}))]g_{1,2}(\mathbf{K}, z_o)+$$

$$\exp(-ik_z z)\lambda_{1,2}(\mathbf{K})g_{2,2}(\mathbf{K}, z_o), \tag{60}$$

while for $z < z_o$ it is given by

$$G(\mathbf{K}, z, z_o) = [\exp(-ik_z z) + \exp(ik_z z)(t^-(\mathbf{K}) + \lambda_{2,2}(\mathbf{K}))]g_{2,2}(\mathbf{K}, z_o)+$$

$$\exp(+ik_z z)\lambda_{2,1}(\mathbf{K})g_{1,2}(\mathbf{K}, z_o). \tag{61}$$

Equation 57 gives expressions for the $g_{i,j}$ that appear here in terms of t's and λ's.

Reciprocity in the form

$$G(\mathbf{R}, z; \mathbf{R}_o, z_o) = G(\mathbf{R}_o, z_o; \mathbf{R}, z), \tag{62}$$

implies

$$\hat{G}(\mathbf{K}, z; \mathbf{Q}, z_o) = \hat{G}(-\mathbf{Q}, z_o; -\mathbf{K}, z), \tag{63}$$

and

$$G(\mathbf{K}, z, z_o) = G(-\mathbf{K}, z_o, z). \tag{64}$$

Comparing coefficients of $\exp(ik_z(z - z_o))$ in these expressions shows that

$$(1 - \lambda_{2,1}(\mathbf{K}))/D(\mathbf{K}) = (1 - \lambda_{1,2}(-\mathbf{K}))/D(-\mathbf{K}). \tag{65}$$

If isotropy is assumed in the form

$$\lambda_{i,j}(\mathbf{K}) = \lambda_{i,j}(-\mathbf{K}),$$

and

$$t^{\pm}(\mathbf{K}) = t^{\pm}(-\mathbf{K})$$

then Eq. 65 implies that

$$\lambda_{1,2}(\mathbf{K}) = \lambda_{2,1}(\mathbf{K}).$$

Then, assuming isotropy, it is possible to combine the parts of $G(\mathbf{K}, z, z_o)$ to show what is the principal result of this work,

$$G(\mathbf{K}, z, z_o) = \frac{[e^{ik_z z_>} + e^{-ik_z z_>} V_{eff}^+(\mathbf{K})][e^{-ik_z z_<} + e^{+ik_z z_<} V_{eff}^-(\mathbf{K})]}{2(1 - \lambda_{1,2}(\mathbf{K}))(1 - V_{eff}^+(\mathbf{K}) V_{eff}^-(\mathbf{K}))} +$$

$$\frac{\lambda_{1,2}(\mathbf{K}) \exp(-ik_z|z - z_o|)}{2(1 - \lambda_{1,2}(\mathbf{K}))}. \tag{66}$$

In this equation, $z_>$ is the greater of z and z_o and $z_<$ is the lesser of z and z_o. The effective reflection coefficients, $V_{eff}^\pm$ are given by

$$V_{eff}^+ = \frac{t^+ + \lambda_{1,1}}{(1 - \lambda_{1,2})} \tag{67}$$

$$V_{eff}^- = \frac{t^- + \lambda_{2,2}}{(1 - \lambda_{1,2})}. \tag{68}$$

When there is no correlation between the fluctuations in the scattering amplitudes at the upper and lower interfaces, $\lambda_{1,2}$ will vanish, at least to second order in the fluctuations. Then G is exactly of the form described by Brekhovskikh [7] for flat surfaces with reflection coefficients $V_{eff}^\pm$. This is also the form found in ref.[1] using the method of smoothing. Here the existence of effective reflection coefficients is demonstrated to all orders in the fluctuations of the scattering amplitudes.

The mean field will be described by normal modes which are determined from the poles of $G(\mathbf{K})$. Apparently these poles can arise from the zeros of $1 - \lambda_{1,2}$ or from the zeros of $1 - V_{eff}^+(\mathbf{K}) V_{eff}^-(\mathbf{K})$. However, as $\lambda_{1,2} \to 1$, $V_{eff}^+ \to \infty$. Keeping track of dominant terms shows that in fact, G is not singular as $\lambda_{1,2} \to 1$. Thus the normal modes are determined only by the zeros of $1 - V_{eff}^+(\mathbf{K}) V_{eff}^-(\mathbf{K})$. These zeros occur for complex K and the imaginary parts of K determine modal attenuations.

In the next section the λ's will be discussed in more detail in the lowest order approximation, when only second order terms in the fluctuations of scattering amplitudes are kept. It turns out that the effective reflection coefficients describe all processes in which a wavevector is forward scattered. The mean half-space scattering amplitude includes only processes than involve a single interface. When a second interface is present, there are additional processes by which a wave may be forward scattered. These processes are contained in the $\lambda_{1,1}$ and $\lambda_{2,2}$ additions to the mean half-space scattering amplitudes, to form the effective reflection coefficients for the mean field between two interfaces.

V Second order calculations

In this section the simplest approximation for the effective reflection coefficients will be developed. In effect, this means finding the average self-energy. It will be shown that if this result is further approximated by using pertubative approximations for the half-space scattering amplitudes, then the approximation of Bass and Fuks [2] is obtained.

To find the self-energy, an approximation for the scattering operator T is required. Of course, to find T, $< \Sigma >$ is required, but one has to start somewhere. Equations 28-33 show that the self-energy will be at least second order in fluctuations of the half-space scattering amplitudes. This means that a first order approximation for T (see Eq.26) is simply

$$T \approx -\Delta M, \tag{69}$$

and this is the approximation that will be used throughout this section. This approximation does not require that ΔT be expressed in powers of the surface roughness. One could approximate ΔT, for example, by the first-order small slope approximation.

In any event, if for the purposes of this section we assume Eq. 69, then

$$\Lambda_{i,j}(\mathbf{P}, \mathbf{Q}) = -\int d\mathbf{K} g_{i,l}(\mathbf{K}, z_o) < \Delta T^{\pm}(\mathbf{P}|\mathbf{K})\Delta M_{l,j}(\mathbf{K}, \mathbf{Q}, z_o) >, \tag{70}$$

where ΔT^{+} is to be used when $i = 1$ and ΔT^{-} when $i = 2$. For example, from Eq.36 it follows that

$$\Lambda_{1,1}(\mathbf{P}, \mathbf{Q}) = \int d\mathbf{K}[-g_{1,1}(\mathbf{K}, z_o) + g_{1,2}(\mathbf{K}, z_o)] \exp(-ik_z z_o) < \Delta T^{+}(\mathbf{P}|\mathbf{K})\Delta T^{+}(\mathbf{K}|\mathbf{Q}) > . \tag{71}$$

From eq. 57, the sum of the g's here is

$$- g_{1,1}(\mathbf{K}, z_o) + g_{1,2}(\mathbf{K}, z_o) = \exp(ik_z z_o)\frac{(t^{-}(\mathbf{K}) + \lambda_{2,2}(\mathbf{K}))}{D(\mathbf{k})}, \tag{72}$$

so that $\Lambda_{1,1}(\mathbf{P}, \mathbf{Q})$ is given by

$$\Lambda_{1,1}(\mathbf{P}, \mathbf{Q}) = \int d\mathbf{K}\frac{(t^{-}(\mathbf{K}) + \lambda_{2,2}(\mathbf{K}))}{D(\mathbf{k})} < \Delta T^{+}(\mathbf{P}|\mathbf{K})\Delta T^{+}(\mathbf{K}|\mathbf{Q}) > \tag{73}$$

Note how the exponential dependence on the source level has dissapeared. Statistical homogeneity implies that the average of the fluctuations of the scattering amplitudes can be written as

$$< \Delta T^{\pm}(\mathbf{P}|\mathbf{K})\Delta T^{\pm}(\mathbf{K}|\mathbf{Q}) >= \delta(\mathbf{P} - \mathbf{Q})c^{\pm,\pm}(\mathbf{P}, \mathbf{K}), \tag{74}$$

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so that

$$\Lambda_{1,1}(\mathbf{P},\mathbf{Q}) = \delta(\mathbf{P}-\mathbf{Q}) \int d\mathbf{K} \frac{(t^-(\mathbf{K}) + \lambda_{2,2}(\mathbf{K}))}{D(k)} c^{+,+}(\mathbf{P},\mathbf{K})$$

$$= \delta(\mathbf{P}-\mathbf{Q})\lambda_{1,1}(\mathbf{P}). \tag{75}$$

One can show similarly that

$$\lambda_{2,2}(\mathbf{P}) = \int d\mathbf{K} \frac{(t^+(\mathbf{K}) + \lambda_{1,1}(\mathbf{K}))}{D(k)} c^{-,-}(\mathbf{P},\mathbf{K}) \tag{76}$$

$$\lambda_{1,2}(\mathbf{P}) = \int d\mathbf{K} \frac{(1 - \lambda_{1,2}(\mathbf{K}))}{D(k)} c^{+,-}(\mathbf{P},\mathbf{K}). \tag{77}$$

These three equations form a coupled set of integral equations; one can't find $\lambda_{1,1}$ without knowing $\lambda_{2,2}$, for example. If there is no correlation between the fluctuations in scattering amplitudes at the upper and lower interfaces, then $c^{+,-}$ will vanish and $\lambda_{1,2}$ along with it. Furthermore, $\lambda_{1,1}$ and $\lambda_{2,2}$ are second order in the fluctuations and should be small compared to $t^\pm$. Thus, the λ's are further approximated by

$$\lambda_{1,1}(\mathbf{P}) = \int d\mathbf{K} \frac{t^-(\mathbf{K})}{D(k)} c^{+,+}(\mathbf{P},\mathbf{K}) \tag{78}$$

$$\lambda_{2,2}(\mathbf{P}) = \int d\mathbf{K} \frac{t^+(\mathbf{K})}{D(k)} c^{-,-}(\mathbf{P},\mathbf{K}). \tag{79}$$

To this same order the determinant $D(\mathbf{K})$ should be written as

$$D(\mathbf{K}) = 1 - t^+(\mathbf{K})t^-(\mathbf{K}). \tag{80}$$

There might be zeros in $D(\mathbf{K})$ near the effective modal wave numbers. However, since $t^\pm$ are the *mean* half-space scattering amplitudes, the zeros of $D(\mathbf{K})$ should occur for complex K.

One can expect that Eqs. 78 and 79 provide reasonable approximations if the λ's determined by these equations are small compared to t^+ and t^-. It is also necessary that the zeros of $D(\mathbf{K})$ not be shifted significantly by the use of the approximation in Eq. 80. Without doing detailed calculations it is difficult to determine when the λ's will be small. However, consider the following "toy" problem. Suppose that the fluctuations in scattering amplitudes are sharply peaked in the forward direction, corresponding to an ensemble of nearly flat surfaces. In fact suppose that

$$c^{+,+}(\mathbf{P},\mathbf{K}) = \sigma^+\delta(\mathbf{P}-\mathbf{K}) \tag{81}$$

$$c^{-,-}(\mathbf{P},\mathbf{K}) = \sigma^-\delta(\mathbf{P}-\mathbf{K}). \tag{82}$$

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Then Eqs. 78 and 79 give

$$\lambda_{1,1}(\mathbf{P}) = \frac{t^-(\mathbf{P})\sigma^+}{D(\mathbf{P})} \tag{83}$$

$$\lambda_{2,2}(\mathbf{P}) = \frac{t^+(\mathbf{P})\sigma^-}{D(\mathbf{P})} \tag{84}$$

The approximations in Eqs. 78 and 79 will be self-consistent provided

$$\frac{t^-(\mathbf{P})\sigma^+}{D(\mathbf{P})} \ll t^+(\mathbf{P}) \tag{85}$$

$$\frac{t^+(\mathbf{P})\sigma^-}{D(\mathbf{P})} \ll t^-(\mathbf{P}). \tag{86}$$

These two inequalities in turn require that

$$\frac{\sigma^+\sigma^-}{D(\mathbf{P})^2} \ll 1. \tag{87}$$

If the zeros of D occur for real values of $\mathbf{P}$ this inequality will be violated. Otherwise, it would seem possible to satisfy this constraint if the strengths of the fluctuations are sufficiently small.

The corresponding approximations for the effective reflection coefficients now become

$$V_{eff}^+(\mathbf{P}) = t^+(\mathbf{P}) + \int d\mathbf{K}\frac{t^-(\mathbf{K})}{1 - t^+(\mathbf{K})t^-(\mathbf{K})}c^{+,+}(\mathbf{P},\mathbf{K}) \tag{88}$$

$$V_{eff}^-(\mathbf{P}) = t^+(\mathbf{P}) + \int d\mathbf{K}\frac{t^+(\mathbf{K})}{1 - t^+(\mathbf{K})t^-(\mathbf{K})}c^{-,-}(\mathbf{P},\mathbf{K}). \tag{89}$$

If there is a small correlation between ΔT^+ and ΔT^+, then

$$\lambda_{1,2}(\mathbf{P}) = \int d\mathbf{K}\frac{1}{1 - t^+(\mathbf{K})t^-(\mathbf{K})}c^{+,-}(\mathbf{P},\mathbf{K}), \tag{90}$$

and the effective reflection coefficients will be reduced by $1 - \lambda_{1,2}$. These expressions for the effective reflection coeficients were derived in ref.[1] by the method of smoothing. Here they have been shown to be low-order approximations for more general expressions. The result for $\lambda_{1,2}$ is new here. It is not known yet how important $\lambda_{1,2}$ is, but it seems likely that it is important in scattering in thin sedimented layers where interfaces tend to follow one another.

The expressions for the effective reflection coefficients can be specialized even further in the case of small surface roughness. This specialization will be shown here to compare the pres ent results with those of Bass and Fuks [2] and to illustrate the contribution of the self-energy to the effective reflection coefficients.

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Consider two Dirichlet surfaces on which the field vanishes, and which are described by

$$z = H^{\pm} + h^{\pm}(\mathbf{R}). \tag{91}$$

The nominal width of the layer between these sufaces is $L = H^{+} - H^{-} > 0$ and the mean values of roughnesses $h^{\pm}$ vanish. To second order in the surface roughness the scattering amplitude for the upper surface with a plane wave incident from below is [8] (with exponentials appearing because the mean surface is not at $z = 0$)

$$T^{+}(\mathbf{K}|\mathbf{P}) \approx -e^{2ik_z H^{+}} \delta(\mathbf{K} - \mathbf{P}) - 2ip_z e^{i(p_z + k_z)H^{+}} \hat{h}^{+}(\mathbf{P} - \mathbf{K})$$

$$+ e^{i(p_z + k_z)H^{+}} \int d\mathbf{Q} 2p_z q_z \hat{h}^{+}(\mathbf{Q} - \mathbf{K}) \hat{h}^{+}(\mathbf{P} - \mathbf{Q}), \tag{92}$$

where the Fourier transform of the surface roughness is defined by

$$\hat{h}^{\pm}(\mathbf{K}) = \frac{1}{(2\pi)^2} \int e^{i\mathbf{K}\cdot\mathbf{R}} h^{\pm}(\mathbf{R})d\mathbf{R}. \tag{93}$$

For a plane wave incident from above on the lower surface whose mean is at H^{-}, the scattering likewise is

$$T^{-}(\mathbf{K}|\mathbf{P}) \approx -e^{-2ik_z H^{+}} \delta(\mathbf{K} - \mathbf{P}) + 2ip_z e^{-i(p_z + k_z)H^{-}} \hat{h}^{-}(\mathbf{P} - \mathbf{K})$$

$$+ e^{-i(p_z + k_z)H^{+}} \int d\mathbf{Q} 2p_z q_z \hat{h}^{-}(\mathbf{Q} - \mathbf{K}) \hat{h}^{-}(\mathbf{P} - \mathbf{Q}). \tag{94}$$

The spectra of the surface roughnesses, $S^{\pm,\pm}$, are defined from

$$< \hat{h}^{\pm}(\mathbf{Q} - \mathbf{K}) \hat{h}^{\pm}(\mathbf{P} - \mathbf{Q}) > = \delta(\mathbf{P} - \mathbf{K}) S^{\pm,\pm}(\mathbf{P}, \mathbf{Q}). \tag{95}$$

Through second order in the surface roughness, the mean half-space scattering amplitudes are seen to be

$$t^{\pm}(\mathbf{K}) = e^{\pm 2ik_z H^{\pm}} \{-1 + 2k_z \int d\mathbf{Q} q_z S^{\pm,\pm}(\mathbf{K}, \mathbf{Q})\}. \tag{96}$$

On the other hand, the correlations of the scattering amplitudes, $c^{\pm,\pm}$ are given by

$$c^{+,+}(\mathbf{K}, \mathbf{P}) = -4k_z p_z S^{+,+}(\mathbf{K}, \mathbf{P}) e^{+2ik_z H^{+} + 2ip_z H^{+}} \tag{97}$$

$$c^{-,-}(\mathbf{K}, \mathbf{P}) = -4k_z p_z S^{-,-}(\mathbf{K}, \mathbf{P}) e^{-2ik_z H^{-} - 2ip_z H^{-}} \tag{98}$$

$$c^{+,-}(\mathbf{K}, \mathbf{P}) = +4k_z p_z S^{+,-}(\mathbf{K}, \mathbf{P}) e^{+ik_z (H^{+} - H^{-}) + ip_z (H^{+} - H^{-})} \tag{99}$$

To compute the self-energies, $\lambda_{i,j}$, approximate the determinant D using the flat surface reflection coefficients, $t_o^{\pm}$

$$D(\mathbf{K}) \approx 1 - t_o^{+} t_o^{-} = 1 - e^{2ik_z (H^{+} - H^{-})} = 1 - e^{2ik_z L}. \tag{100}$$

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In this perturbative approximation, the self-energies are given by

$$\lambda_{1,1}(\mathbf{K}) = 4k_z e^{2ik_z H^+} \int p_z \frac{e^{-2ip_z L}}{1 - e^{2ip_z L}} S^{+,+}(\mathbf{K},\mathbf{P})d\mathbf{P} \tag{101}$$

$$\lambda_{2,2}(\mathbf{K}) = 4k_z e^{-2ik_z H^-} \int p_z \frac{e^{2ip_z L}}{1 - e^{2ip_z L}} S^{-,-}(\mathbf{K},\mathbf{P})d\mathbf{P} \tag{102}$$

$$\lambda_{1,2}(\mathbf{K}) = -4k_z \int p_z \frac{e^{2i(k_z+p_z)L}}{1 - e^{2ip_z L}} S^{+,-}(\mathbf{K},\mathbf{P})d\mathbf{P}. \tag{103}$$

As indicated in the previous section, the pole structure in the integrands of these expressions is capturing the fact that in a layer, forward scattering can occur not only by first scattering to an intermediate wave number $\mathbf{P}$ and then back to $\mathbf{K}$ by a single surface, as expressed in the various contributions to $< t^\pm >$, but also that forward scattering can occur by scattering into $\mathbf{P}$, then having any number of unperturbed forward scatterings when the energy bounces between the interfaces of the layer, and then rescattering into the incident wavevector $\mathbf{K}$. The denominators in these expressions for λ are the phases acquired in bouncing between interfaces. One can see that the self-energy contribution to effective scattering amplitudes will have a much stronger frequency dependence, because of the denominators in the integrands, than the half-space scattering amplitudes. . These contributions are of the same order of magnitude as the perturbations of the flat surface mean scattering amplitudes.

In fact, t^+ and $\lambda_{1,1}$ can be added to show how the result of Bass and Fuks can be obtained from this formalism,

$$t^+(\mathbf{K}) + \lambda_{1,1}(\mathbf{K}) = e^{2ik_z H^+}\{-1 + 2k_z \int d\mathbf{P} p_z [\frac{1 + e^{2ip_z L}}{1 - e^{2ip_z L}}]S^{+,+}(\mathbf{K},\mathbf{P})\}. \tag{104}$$

This expression can be written in terms of the Green's function between two flat Dirichlet surfaces at $H^\pm$. From Brekhovski or from eq.66 above this Green's function is

$$G_o(\mathbf{K},z,z_o) = \frac{(2\pi)^2}{-2ik_z} \frac{[e^{ik_z z_>} - e^{ik_z(2H^+ - z_>)}][e^{-ik_z z_<} - e^{-ik_z(2H^- - z_<)}]}{1 - e^{2ik_z L}}. \tag{105}$$

It is now straight forward to show that the integrand in the effective reflection coefficient can be expressed in terms of a second derivative of G_o:

$$\lim_{z_o \to H^+} \lim_{z \to H^+} \partial_z \partial_{z_o} G_o(\mathbf{K},z,z_o) \equiv G_o''(\mathbf{K})$$

$$= (2\pi)^2 i k_z [\frac{1 + e^{2ik_z L}}{1 - e^{2ik_z L}}]. \tag{106}$$

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The effective reflection coefficient now can be written as

$$V_{eff}^{+}(\mathbf{K}) = e^{2ik_z H^{+}} \{-1 - \frac{2ik_z}{(2\pi)^2} \int d\mathbf{P} p_z G_o''(\mathbf{P}) S^{+,+}(\mathbf{K},\mathbf{P})\}. \qquad (107)$$

Bass and Fuks calculate an effective impedance, which is the ratio of the mean field to the z-derivative of the mean field. When the effective reflection coefficient is V_{eff}^{+} the effective impedance m is

$$m(\mathbf{K}) = \frac{2}{(2\pi)^2} \int d\mathbf{P} p_z G_o''(\mathbf{P}) S^{+,+}(\mathbf{K},\mathbf{P}). \qquad (108)$$

Apart from factors of π and 2 which arise from differences in normalization of the Green's function and the Fourier transforms, this is precisely the impedance of Bass and Fuks.

VI Summary

This work is an extension of the ideas presented in Ref.[1]. The primary result is Eq. 66 which shows that the mean Green's function in a medium between two rough interfaces can be expressed as a weighted sum of two familiar Green's functions. The first is the Green's function in a layer bounded by flat interfaces with effective reflection coefficients, $V_{eff}^{\pm}$. The second is a free-space Green's function whose exponent has a sign opposite to that of the conventional free-space Green's function. The z-derivative of the combined Green's function has the correct jump discontinuity across the plane $z = z_o$. This result extends that of Ref.[1] in that it is not restricted to be second order in fluctuations of half-space scattering amplitudes. It seems to be a quite general result that surface roughness can be incorporated into the mean field by effective reflection coefficients. In addition, here the possibility of correlated surface roughness is considered. To show that the methods used here are not unreasonable, results were specialized to first include only second-order terms and second, to include only terms second order in surface roughness. In this case, the results of Bass and Fuks were obtained.

The motivation for this inquiry is to find a way to incorporate what is already known about half-space scattering into the waveguide problem. One shouldn't have to do the half-space problem again in the waveguide. For example, with the results presented here, one could consider propagation in a fluid layer bounded by an elastic solid without repeating some fairly messy computations.

The effective reflection coefficients are not simply the mean half-space scattering amplitudes. Instead, these effective reflection coefficients account for channels of forward scattering that are

not available when there is only one interface. Modal wave numbers are found from the dispersion relation

$$1 - V_{eff}^{+}(\mathbf{K})V_{eff}^{-}(\mathbf{K}).$$

As discussed by Bass and Fuks, the modal wave numbers can be quite different from those found using only the half-space scattering amplitudes.

Acknowledgements

Appendix

It is argued here that in the self-energy given in Eq. 37, the Λ's must be independent of the source level, z_o. This is important for it means that the effective reflection coefficients for the mean field will be independent of the source level, a desirable property. When $\lambda_{1,2} \neq 0$ it appears that this will not be the case, even though the λ's are independent of z_o.

First note that the derivative of $M(z)$ satisfies

$$\frac{dM(z)}{dz} = ik_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(z). \qquad (A1)$$

Since by definition

$$M(z)M(z)^{-1} = 1, \qquad (A2)$$

the derivative of the inverse of M satisfies

$$\frac{dM^{-1}(z)}{dz} = -M^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ik_z. \qquad (A3)$$

Averaging this equation shows that

$$\frac{d < \mathcal{G}(z) >}{dz} = - < \mathcal{G} > \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ik_z. \qquad (A4)$$

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Repeating these arguments shows first that

$$\frac{d < \mathcal{G}(z) >^{-1}}{dz} = ik_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} < \mathcal{G} >^{-1} \qquad (A5)$$

This equation obviously holds for $< M >$, and since $< \mathcal{G} >^{-1} = < M > + < \Sigma >$,

$$\frac{d < \Sigma >}{dz} = ik_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} < \Sigma >. \qquad (A6)$$

Using Eq. 37 for $< \Sigma >$ shows that Λ must satisfy

$$\exp(-ip_z z)\frac{d\Lambda_1(z)}{dz} - \exp(+ip_z z)\frac{d\Lambda_2(z)}{dz} = 0, \qquad (A7)$$

where Λ_1 is the first matrix in eq.37 and Λ_2 the second.

Equation (A6) can be differentiated again to show that the self-energy satisfies the separated wave equation:

$$\frac{d^2 < \Sigma >}{dz^2} + p_z^2 < \Sigma > = 0. \qquad (A8)$$

This means that the Λ's must be of the form

$$\Lambda_{1,j} = A_j + B_j \exp(2ip_z z) \qquad (A9)$$

$$\Lambda_{2,j} = C_j + D_j \exp(-2ip_z z) \qquad (A10)$$

with A, B, C, D constant. Then eq.A7 implies

$$B_j \exp(ip_z z) - D_j \exp(-ip_z z) = 0. \qquad (A11)$$

The only way this can hold over a continuous range of z is for the B's and D's to vanish, which implies that the Λ's are independent of source level.

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Figure Captions

Figure 1. A sketch of the geometry used in this paper. The z-axis is positive upward. The dashed line indicates an imaginary plane through the source. Between the planar surfaces shown here, the field can be expanded in half-space plane-wave scattering states, $\chi^{\pm}$. These states describe scattering from a wavevector $\mathbf{K}$ into a number of plane-waves with wavevectors $\mathbf{Q}$.