RE **AD-A281 909**

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE March 1994	3. REPORT TYPE AND DATES COVERED Technical

4. TITLE AND SUBTITLE How Pooling Failure Data May Reverse Increasing Failure Rates	5. FUNDING NUMBERS DAAH04-93-G-0201
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Florida State University Department of Statistics Tallahassee, FL 32306-3033	8. PERFORMING ORGANIZATION REPORT NUMBER

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9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) U.S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709-2211	10. SPONSORING/MONITORING AGENCY REPORT NUMBER ARO 31029.1-MA

11. SUPPLEMENTARY NOTES
The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.	12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

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In this paper, various types of discrete and continuous mixtures of IFR distributions are considered, and conditions developed for such mixtures to be DFR. These conditions show an unexpected result, that certain mixtures of IFR distributions, even those with very rapidly increasing failure rates (*e.g.* Weibull, Truncated Extreme , etc.), become DFR distributions. It is common practice to pool data from several different IFR distributions to enlarge sample size, for instance. The results of this paper serve as a warning note that such pooling may actually reverse the IFR property of the individual samples to a DFR property for the mixture.

DTIC QUALITY INSPECTED 5

94-21523

14. SUBJECT TERMS	15. NUMBER OF PAGES
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

HOW POOLING FAILURE DATA MAY REVERSE INCREASING FAILURE RATES

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Abstract

Although mixtures of DFR (Decreasing Failure Rate) distributions are always DFR, some mixtures of IFR (Increasing Failure Rate) distributions can also be DFR.

In this paper, various types of discrete and continuous mixtures of IFR distributions are considered, and conditions developed for such mixtures to be DFR. These conditions show an unexpected result, that certain mixtures of IFR distributions, even those with very rapidly increasing failure rates (*e.g.* Weibull, Truncated Extreme , etc.), become DFR distributions. It is common practice to pool data from several different IFR distributions to enlarge sample size, for instance. The results of this paper serve as a warning note that such pooling may actually reverse the IFR property of the individual samples to a DFR property for the mixture.

1   Introduction

Although mixtures of distributions with decreasing failure rate (DFR) are always DFR, some mixtures of distributions with increasing failure rate (IFR) may also be DFR. A well known "border line" example by Proschan (1963) exhibits the DFR property of a mixture of Exponential distributions, which have constant failure rate. Gleser (1989) described arbitrary DFR Gamma distributions, i.e. Gamma distributions with shape parameter less than 1, as explicit mixtures of Exponential distributions. In the light of such examples, and from the standpoint of continuity, it is intuitively clear that a mixture of distributions with "gently increasing" failure rate could be DFR. In this paper we show the unexpected result that mixtures of some IFR distributions, even those with very rapidly increasing failure rates (*e.g.* Weibull, Truncated Extreme, etc.), become DFR distributions. In practice, data from different IFR distributions are sometimes pooled, in order to enlarge sample size, for instance. These results serve as a warning note that such pooling may actually reverse the IFR property of the individual samples to a DFR property for the mixture. This phenomenon is somewhat reminiscent of Simpson's Paradox (Simpson (1951), Blyth (1972)), wherein a positive partial association between two variables may exist at each level of a third variable, yet a negative overall unconditional association holds between the two original variables.

Throughout this paper we denote the survival function, hazard function and failure rate function of a life distribution function $F(t)$ with $t \geq 0$, by $S(t), G(t)$ and $g(t)$, respectively, i.e. $S(t) = 1 - F(t), G(t) = -\log S(t)$ and $g(t) = G'(t)$. When the life distribution function contains subscripts or parameters, we adopt the convention of adding the same to its its survival function, hazard function and failure rate function; thus, S_1, G_1, g_1 are the survival function, hazard function and failure rate function, respectively, of the life distribution function F_1

In Section 2, we consider mixtures of two increasing failure rate (IFR) distributions and give a necessary and sufficient condition, as well as sufficient conditions, for the mixture to be a decreasing failure rate (DFR) distribution.

Many standard families of IFR distributions exhibit the property that mixtures of two distributions from the same family become DFR distributions. We call such families PWMR, *in words, pairwise mixture reversible* families, and give some examples in Section 2.

The above term PWMR refers to a pair of distributions from the same family. It is also possible for a mixture of two distributions from different families of IFR distributions to be DFR. In this paper, we also focus attention on IFR distributions whose mixture with an exponential distribution is a DFR distribution. We call such an IFR distribution an MRE *in words, mixture reversible by the exponential* distribution.

In Section 3, we show that all the well known IFR distributions are MRE distributions. We also show an unexpected result, that even though the Truncated Extreme has an exponentially increasing failure rate function, it is MRE. We also show, by means of an example of an unconventional IFR distribution which is not MRE, that not all IFR distributions are MRE.

A family of survival functions $\mathcal{S} = \{S_\lambda, \lambda > 0\}$ is said to be an IFR *Lehmann family*, or an IFR proportional hazards family, if $S_\lambda(t) = e^{-\lambda G(t)}$, for all $\lambda > 0, t > 0$ and where

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$G(t) \geq 0, G(t) \nearrow \infty$ as $t \nearrow \infty$ and $G(t)$ is a convex function. These conditions imply that $S_\lambda(t)$ has an increasing failure rate function for $\lambda > 0$. In Section 4 we study mixtures of two distributions from IFR Lehmann families and show that they can be DFR distributions, for a wide class of families. In Section 5, the same mixture reversible property is shown to hold for mixtures of infinite sequences of distributions from IFR Lehmann families, when the mixing distribution is the scaled Truncated Poisson.

In Section 6, we consider continuous mixtures of distributions from IFR Lehmann families when the mixing distribution is Gamma, and show that the mixture is DFR for a large class of families. One conspicuous exception leads to the surprising result that a Gamma mixture of distributions from the Lehmann family based on the Truncated Extreme whose parameter is within a certain range and *with an exponentially increasing failure rate*, is a DFR distribution, while a Gamma mixture of distributions from the Lehmann family based on the Gamma, *which has only a linearly increasing failure rate function*, is not DFR.

There are other possibilities for mixtures of IFR distributions which result in the mixture reversing property described above. This paper presents a systematic approach to such a study. As graphic illustrations, we present some figures wherein the graph of the decreasing failure rate function of the mixture distribution is superimposed over the graphs of the increasing failure rate functions of the individual distributions.

2 Mixture of two distributions

We begin by considering mixtures of two arbitrary IFR distribution functions $F_i, i = 1, 2$. Let $\mathbf{p} = (p_1, p_2)$ with $0 \leq p_1, p_2 \leq 1$, $p_1 + p_2 = 1$ denote a mixing vector. Let $F_{\mathbf{p}}(t) \stackrel{def}{=} p_1 F_1(t) + p_2 F_2(t)$ be the mixture of F_1 and F_2 under $\mathbf{p}$. Theorem 1 below gives a necessary and sufficient condition for the mixture $F_{\mathbf{p}}$ to be DFR.

We use the notation

$$g_i'(t) \stackrel{def}{=} G_i''(t) \quad \text{and} \quad Q_i(t) \stackrel{def}{=} p_i S_i(t), \; i = 1, 2.$$

Theorem 1 *Let F_1, F_2 be two IFR distribution functions. The mixture $F_{\mathbf{p}}$ is DFR if and only if*

$$(Q_1 + Q_2)(Q_1 g_1' + Q_2 g_2') \leq Q_1 Q_2 (g_1 - g_2)^2. \tag{2.1}$$

Proof. Note that

$$S_{\mathbf{p}}(t) = p_1 S_1(t) + p_2 S_2(t),$$

$$g_{\mathbf{p}}(t) = \frac{p_1 S_1(t) g_1(t) + p_2 S_2(t) g_2(t)}{p_1 S_1(t) + p_2 S_2(t)} = \frac{Q_1 g_1 + Q_2 g_2}{Q_1 + Q_2}$$

and

$$
\begin{aligned}
(Q_1 + Q_2)^2 g_{\mathbf{p}}'(t) &= (Q_1 + Q_2)[Q_1 g_1' + Q_2 g_2' - Q_1 g_1^2 - Q_2 g_2^2] \\
&\quad + (Q_1 g_1 + Q_2 g_2)^2 \\
&= (Q_1 + Q_2)(Q_1 g_1' + Q_2 g_2') - Q_1 Q_2 (g_1 - g_2)^2,
\end{aligned}
$$

by using the fact that $Q_i'(t) = -Q_i(t) g_i(t)$ in the last step. This shows that condition (2.1) is necessary and sufficient for $g_{\mathbf{p}}'(t) \leq 0$ and thus, for the mixture $F_{\mathbf{p}}$ to be DFR. $\quad\square$

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Remark 1 In some instances, condition (2.1) in Theorem 1 may hold only for sufficiently large t, say $t \geq t_0$. To cover such instances, we could coin a new term and say that the mixture is an *ultimately* DFR distribution. However, we can truncate all such distributions to the region $t \geq t_0$, and the resulting mixture will be DFR. Since, in most applications, it is the aging aspect of failure which is of primary importance, we will simply refer to such mixtures as DFR, with this understanding.

We now give a precise definition of the term PWMR introduced in Section 1.

Definition 1 Let $\mathcal{F}$ be a family of IFR distribution functions. We say that $\mathcal{F}$ is a PWMR (*in words, pair-wise mixture reversible*) family, if mixtures $F_\mathbf{p}$ of two distributions from this family are DFR distributions for some values of the mixing vector $\mathbf{p}$.

To give some examples of PWMR families, we rework condition (2.1) to give a sufficient condition for a mixture of two IFR distributions to be DFR.

Corollary 1 *Let $F_\mathbf{p}$ be the mixture of two IFR distributions F_1, F_2 as in Theorem 1. Then*

$$Q_1(t) \leq Q_2(t) \text{ and } 2(Q_1(t)g_1'(t) + Q_2(t)g_2'(t)) \leq Q_1(t)(g_1(t) - g_2(t))^2 \text{ for all } t \quad (2.2)$$

is a sufficient condition for $F_\mathbf{p}$ to be DFR.

Proof.: When $Q_1(t) \leq Q_2(t)$ for all t, the harmonic mean $2Q_1Q_2/(Q_1 + Q_2)$ of Q_1 and Q_2 is always greater than or equal to Q_1. Corollary 1 follows immediately from condition (2.1) of Theorem 1. $\qquad\square$

Remark 2 As indicated in Remark 1, the condition (2.1) may hold for only sufficiently large t. The same comment can be made about condition (2.2), in particular $Q_1(t) \leq Q_2(t)$ may hold for only sufficiently large t.

Examples of PWMR families

Example 1. Truncated Logistic distributions with survival function $\frac{2e^{-ct}}{1+e^{-t}}$: Here $G_i(t) = c_i t + \log(1 + e^{-t}) - \log 2, i = 1, 2$. Let $c_1 > c_2 > 0$. It is easy to see that both conditions in (2.2) are satisfied for sufficiently large t. Thus the mixture is DFR and the family is PWMR.

Example 2. Distributions with survival function $\frac{e^{-ct}}{\cosh t}$: Here $G_i(t) = c_i t + \log \cosh t$, $i = 1, 2$. Let $c_1 > c_2 > 0$. As in the previous example, it is easy to see that both conditions in (2.2) are satisfied for sufficiently large t. Thus the mixture is DFR and the family is PWMR.

3 Mixtures of an arbitrary distribution with an Exponential distribution

We now specialize Theorem 1 by putting $G_2(t) = \lambda t$, that is by restricting the second distribution to be Exponential. Then $g_2'(t) \equiv 0$ and condition (2.1) simplifies considerably, yielding the following corollaries, which we state without proof.

Corollary 2 *The following is a necessary and sufficient condition for the mixture $F_{\mathbf{p}}$ to be DFR when $G_2(t) = \lambda t$:*

$$g_1'(t) \leq \frac{Q_2(t)}{Q_1(t) + Q_2(t)}(g_1(t) - \lambda)^2 \text{ for all } t. \tag{3.1}$$

Corollary 3 *Suppose, further, that*

$$Q_1(t) \leq Q_2(t) \text{ for all } t. \tag{3.2}$$

The following are progressively stronger sufficient conditions for the mixture $F_{\mathbf{p}}$ to be DFR when $G_2(t) = \lambda t$:

$$g_1'(t) < \frac{1}{2}(g_1(t) - \lambda)^2 \text{ for all } t, \tag{3.3}$$

$$\frac{g_1'(t)}{g_1(t)} < \frac{1}{2}g_1(t) - \lambda \text{ for all } t. \tag{3.4}$$

Remark 2 which appears after Corollary 1 also applies to all the conditions in Corollary 3.

We now give a precise definition of the term MRE introduced in Section 1.

Definition 2 An IFR distribution F is said to be MRE (*in words, mixture reversible by the Exponential*) if the mixture $F_{\mathbf{p}}$ with some Exponential distribution is DFR for some mixing vector **p**.

An MRE distribution can also be described in terms of "contamination", a term that will be used in the following sense. Suppose that a distribution function F_1 has a certain characteristic, but when it is mixed with another distribution function F_2, the resulting mixture reverses (or negates) the characteristic possessed by F_1. We describe this by saying that F_1 *is contaminable* by F_2. In particular we can refer to an MRE distribution as one which is contaminable by an Exponential.

We now demonstrate that many well known IFR distributions are contaminable by the Exponential. In fact, all the usual classes of IFR distributions found in a typical textbook such as Barlow and Proschan (1965), and even some with very rapidly increasing failure rates, are seen to be MRE distributions. This should not lead us to conclude, however, that all IFR distributions are MRE. At the end of this section we give an example of a rather unconventional IFR distribution which is not MRE.

To verify whether a distribution F_1 is MRE, we consider its mixture with $F_2(t) = 1 - e^{-\lambda t}$ for appropriate values of λ and the mixing vector $\mathbf{p}$, and use, as convenience dictates, the necessary and sufficient condition (3.1) or the sufficient conditions (3.2) and (3.3) or (3.2) and (3.4) from Corollary 3.

Examples of MRE distributions

Example 3. Exponential distribution: It is clear that the Exponential distribution is MRE since condition (3.1) is obviously satisfied from the fact that $g_1'(t) \equiv 0$.

Example 4. Gamma distribution: Let the density function of the IFR Gamma distribution be

$$f_1(t) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha t} t^{\beta-1}, \quad \text{where} \quad \alpha > 0, \ \beta > 1, \ t > 0.$$

The survival function S_1 can be written as

$$S_1(t) = \frac{\alpha^\beta}{\Gamma(\beta)} \int_t^\infty e^{-\alpha x} x^{\beta-1} dx = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha t} I(t, \beta - 1)$$

where

$$I(t, \theta) = \int_0^\infty e^{-\alpha u} (u + t)^\theta du.$$

The following facts concerning the function $I(t, \theta)$ are easily verified:

$$I'(t, \theta) = \alpha I(t, \theta) - t^\theta, \tag{3.5}$$

$$\alpha I(t, \theta) \geq t^\theta \quad \text{and} \quad \alpha I(t, \theta) \sim t^\theta \quad \text{as } t \to \infty. \tag{3.6}$$

The failure rate of the Gamma distribution is given by

$$g_1(t) = \frac{t^{\beta-1}}{I(t, \beta - 1)}$$

and it satisfies,

$$\frac{g_1'(t)}{g_1(t)} = \frac{\beta - 1}{t} - \alpha + \frac{t^{\beta-1}}{I(t, \beta - 1)}$$

which approaches 0 as $t \to \infty$, as can be shown by using (3.5) and (3.6). Hence, condition (3.4) is satisfied. Further, since

$$\frac{Q_1(t)}{Q_2(t)} = \frac{p_1}{p_2} e^{-[G(t) - \lambda t]} \leq 1$$

for t sufficiently large and $\alpha > \lambda$, it follows that condition (3.2) is satisfied for t sufficiently large. (Note Remark 2.) This shows that the Gamma distribution is MRE.

Example 5. Weibull distribution: The hazard function of the Weibull distribution is given by

$$G_1(t) = \theta t^\gamma$$

where $\gamma > 0$, $\theta > 0$ and $g_1 = \theta \gamma t^{(\gamma-1)}$. For $\gamma > 1$ this distribution is IFR and Condition (3.2) is satisfied for sufficiently large t. Condition (3.4) reduces in this case to

$$\frac{\gamma - 1}{t} \leq \frac{1}{2} \theta \gamma t^{(\gamma-1)} - \lambda$$

which holds for sufficiently large t. This shows that the Weibull is MRE.

Let $g(t, p_2)$ be the failure rate of the mixture, $(1 - p_2)F_1(t) + p_2 F_2(t)$, of a Weibull distribution with an Exponential distribution where $0 < p_2 < 1$. We have shown in the above that there is a turning point $t_0 = t_0(p_2)$ such that $g(t, p_2)$ is decreasing for $t \geq t_0$. We examine this phenomenon visually in Figures 1a and 1b.

Both figures contain plots of the increasing failure rate $g_1(t)$ of the Weibull, the constant failure rate $g_2(t)$ of the Exponential, and the failure rates $g(t, p_2)$ of the mixture, for $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$. In Figure 1a, the Weibull and Exponetial parameter values are $\gamma = 3, \theta = 2.5, \lambda = .25$ and in Figure 1b, they are $\gamma = 1.5, \theta = 21.5, \lambda = .1$.

On examining the curves in Figure 1a, it is evident that for $p_2 = 0.05$, the curve for $g(t, .05)$ decreases for $t > t_0 = .975$. Since $S_1(.975) = .10$ it is not surprising that the failure rate $g(t, .05)$ of the mixture is DFR in the region of the top 10% long-lived units from the Weibull distribution, under contamination by an Exponential distribution. It is also evident (as intuitively expected) from the curves for $g(t, p_2)$ that the turning point $t_0(p_2)$ decreases as p_2 increases, since the Exponential plays an increasingly important role in the mixture, and accordingly fewer items have failed.

As we examine the curves for $g(t, p_2)$ in Figure 1b, which corresponds to $\gamma = 1.5, \theta = 21.5, \lambda = .1$, it is also evident that the turning point t_0 decreases as p_2 increases. For an illustration, we focus our attention on $g(t, .3)$, for which the turning point t_0 is equal to .1 and $S_1(.1) = .51$. This is remarkable in that half of the units from the Weibull distribution have survived to time $t_0 = .1$, yet the failure rate of the mixture has started to decrease from $t_0 = .1$, although the mixture is predominantly (70%) Weibull. To a person dealing with pooled failure data for which the underlying survival function is predominantly Weibull (with $\gamma > 1$), this apparently deceasing failure rate of the mixture could be disconcerting.
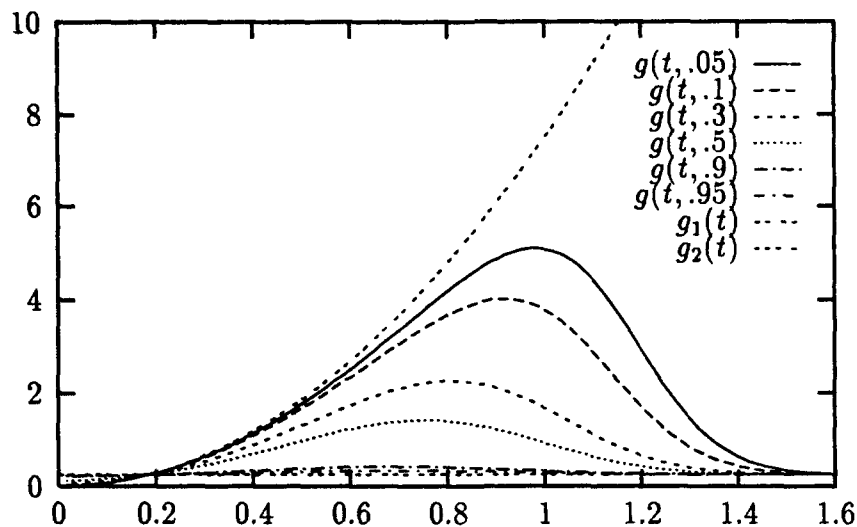
Figure 1a
Mixture of a Weibull distribution ($\gamma = 3, \theta = 2.5$) with an Exponential($\lambda = .25$).
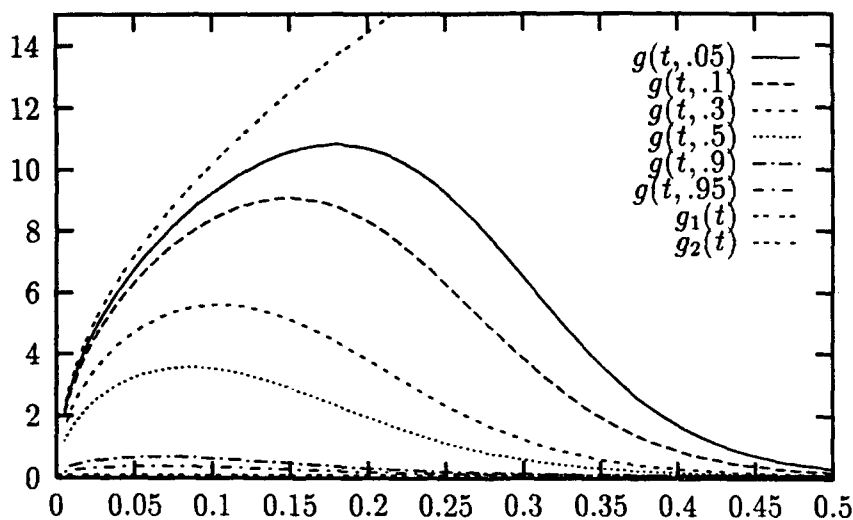Graph of $g(t, p_2), g_1(t)$ and $g_2(t)$ where $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$.

Figure 1b
Mixture of a Weibull distribution ($\gamma = 1.5, \theta = 21.5$) and an Exponential ($\lambda = .1$).
Graph of $g(t, p_2), g_1(t)$ and $g_2(t)$, where $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$.

Example 6. Truncated Extreme distribution: Consider the Truncated Extreme distribution with hazard function $G_1(t) = \theta(e^t - 1)$ and failure rate function $g_1(t) = \theta e^t$. For this distribution, with an exponentially increasing failure rate, clearly, condition (3.2) is satisfied for sufficiently large t. Condition (3.4) reduces in this case to

$$1 < \frac{1}{2}\theta e^t - \lambda$$

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which is satisfied for sufficiently large t. This shows that the Truncated Extreme distribution is MRE.

Let $g(t, p_2)$ be the failure rate of the mixture, $(1 - p_2)F_1(t) + p_2F_2(t)$, of a Truncated Extreme distribution with an Exponential distribution where $0 < p_2 < 1$. We have shown in the above that there is a turning point $t_0 = t_0(p_2)$ such that $g(t, p_2)$ is decreasing for $t \geq t_0$. We examine this phenomenon visually in Figures 2a and 2b.

Both figures contain plots of the increasing failure rate $g_1(t)$ of the Truncated Extreme, the constant failure rate $g_2(t)$ of the Exponential, and the failure rates $g(t, p_2)$ of the mixture, for $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$. In Figure 2a, the Truncated Extreme and Exponential parameter values are $\theta = 20, \lambda = 1$ and in Figure 2b, they are $\theta = 4, \lambda = .1$.

For the first example, consider the graph of $g(t, .05)$ in Figure 2a. In the neighborhood of $t = 0$, it is almost constant, and actually begins to decrease at $t_0 = .0054$ (as can be found by computation; a visual examination of the graph would lead one to believe that $g(t, .05)$ is decreasing on the whole range $t > 0$). What is, indeed, remarkable is that the failure rate $g_1(t)$ increases exponentially, yet $g(t, .05)$, corresponding to a mixture only slightly contaminated by an Exponential, begins to decrease just beyond $t = 0$. This decreasing failure rate of such a mixture could be quite confusing to a person dealing with slightly contaminated failure data.

For the second example, consider the graph of $g(t, .90)$ in Figure 2a. One might think of this as contamination in reverse, that is, the mixture is predominantly (90%) Exponential and is contaminated (10%) by a Truncated Extreme. It is, indeed, strange, that this failure rate decreases at $t = 0$ and continues to do so.

For the third example, consider the graph of $g(t, .05)$ in Figure 2b. This example is quite similar to that corresponding to $g(t, .05)$ in Figure 1a. Here, $t_0 = .4375$ and $S_1(t_0) = .11$, and the decreasing nature of $g(t, .05)$ in the region $t > .4375$ is not unexpected. It is also evident, as in Example 5, that the turning point $t_0(p_2)$ decreases as p_2 increases.
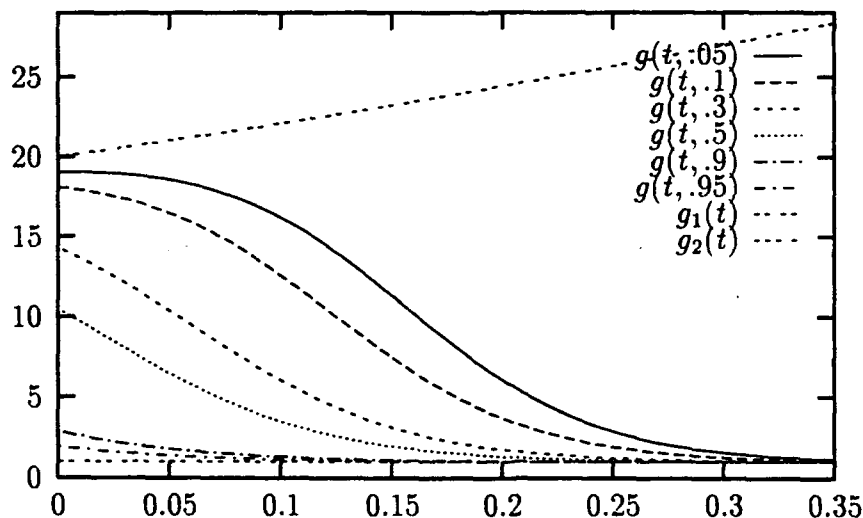

Figure 2a

Mixture of a Truncated Extreme distribution ($\theta = 20$) with an Exponential ($\lambda = 1$).
Graph of $g(t, p_2), g_1(t)$ and $g_2(t)$, where $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$.

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Figure 2b

Mixture of a Truncated Extreme distribution ($\theta = 4$) with an Exponential ($\lambda = 1$).
Graph of $g(t, p_2), g_1(t)$ and $g_2(t)$, where $p_2 = 0.05, 0.1, 0.3, 0.5, 0.9, 0.95$.

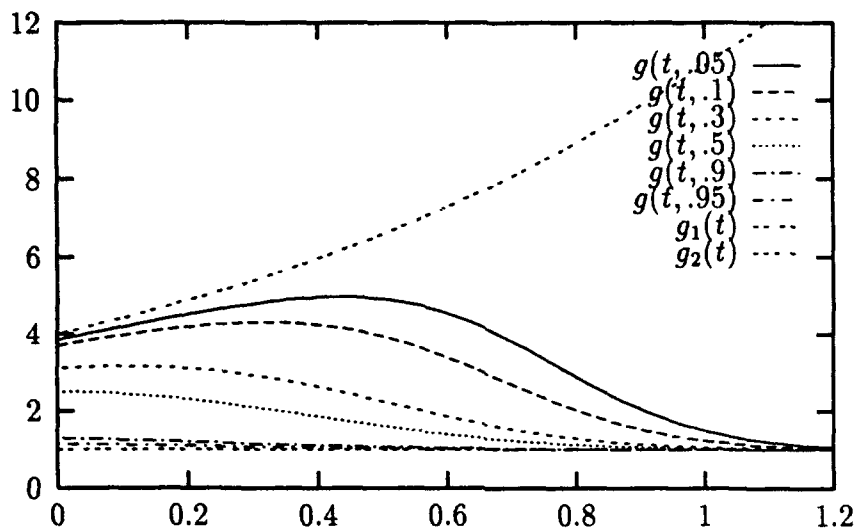**Example 7.** Truncated Normal distribution: The survival function of the Truncated Normal distribution is given by

$$
\begin{aligned}
S_1(t) &= \sqrt{\frac{2}{\pi}} \int_t^\infty e^{-x^2/2} dx \\
&= \sqrt{\frac{2}{\pi}} e^{-t^2/2} J_1(t)
\end{aligned}
\tag{3.7}
$$

where

$$
J_1(t) = \int_0^\infty \frac{u e^{-u^2/2}}{\sqrt{u^2 + t^2}} du
\tag{3.8}
$$

as can be seen by the substitution $x^2 = y^2 + t^2$. Differentiating (3.7) with respect to t and equating it to $-\sqrt{\frac{2}{\pi}} e^{-t^2/2}$, which is another expression for the same, coming directly from the density of the Normal distribution, we obtain the useful identity

$$
J_1'(t) = t J_1(t) - 1.
\tag{3.9}
$$

By examining (3.8), we can obtain the following useful inequality for $J_1(t)$, known in the literature as Mill's ratio:

$$
\frac{1}{t} - \frac{1}{t^3} \le J_1(t) \le \frac{1}{t}.
\tag{3.10}
$$

The upper bound in (3.10) is obtained by using the bound $\frac{u e^{-u^2/2}}{\sqrt{u^2+t^2}} \le \frac{u e^{-u^2/2}}{t}$ for $u \ge 0$, in (3.8). The lower bound in (3.10) is obtained by integrating (3.8) by parts and using the same bound. Again, from (3.7), we see that

$$
G_1(t) = -\log\left(\sqrt{\frac{2}{\pi}}\right) + \frac{t^2}{2} - \log J_1(t).
\tag{3.11}
$$

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By virtue of (3.9), the failure rate function $g_1(t) = \frac{1}{J_1(t)}$ satisfies

$$\frac{g_1'(t)}{g_1(t)} = \frac{1}{J_1(t)} - t \qquad (3.12)$$

and condition (3.4) reduces to

$$\frac{1}{J_1(t)} - t \le \frac{1}{2J_1(t)} - \lambda$$

which is equivalent to

$$\frac{1}{2tJ_1(t)} \le 1 - \frac{\lambda}{t}.$$

This condition is satisfied for large t, since $tJ_1(t) \to 1$ as $t \to \infty$ as can be easily seen from (3.8). Since $0 \le J_1(t) \le \frac{1}{t}$, it is easy to see from (3.11) that $G_1(t) \ge \lambda t$ for sufficiently large t, which in turn implies that condition (3.2) is satisfied for such t. This shows that the Truncated Normal is MRE.

Example 8. Truncated Logistic: Here $S_1(t) = 2/(e^t + 1)$, $t \ge 0$ and it can be seen that $g_1'(t)/g_1(t) = 1/(e^t + 1)$ which approaches 0 as $t \to \infty$. This shows that condition (3.4) is satisfied for large t. It can also be seen that condition (3.2) is satisfied for $\lambda < 1$ and large t. This shows that the Truncated Logistic distribution is MRE.

We can carry on in a similar manner and show that many more distributions — some well known, and others not so well known — are all contaminable by the Exponential distribution. We list a few more, without proof, just to illustrate this point.

Example 9. Distributions with $G_1(t) = [t - \log(1 + t)]$, $t \ge 0$.

Example 10. Distributions with $G_1(t) = t + \log \cosh t$, $t \ge 0$.

Example 11. Distributions with $G_1(t) = t \log(a + t)$, $t \ge 0$, $a \ge 1$.

It is remarkable that all of the distributions considered above are MRE. In fact, we have examined several other distributions, only to find they have the same property. It is tempting to conjecture that all "well-behaved" distributions are MRE; but one would need to specify in precise terms what is meant by "well-behaved". We are considering this question, and expect to report on it at a later time.

We give below, in Example 12, a not so "well-behaved" distribution F_1 which is not MRE, *i.e.* not contaminable by an Exponential distribution; the mixture of this F_1 with an Exponential is neither IFR nor DFR. The failure rate of this distribution is increasing, but its derivative oscillates between 0 and ∞, a feature not found among the common IFR distributions.

Example 12. Let $1 - F_1(t) = S_1(t) = e^{-G_1(t)}$ where $g_1(t) = G_1'(t), g_1'(t) = G_1''(t)$. Set

$$g_1'(t) = \begin{cases} n, & t \in (n - \frac{1}{n^3}, n + \frac{1}{n^3}], n = 2, 3, \ldots \\ 0, & \text{for all other } t \in [0, \infty). \end{cases}$$

Note that $g_1(t) \nearrow 2(\frac{\pi^2}{6} - 1) > 0$ as $t \to \infty$. Let $1 - F_2(t) = S_2(t) = e^{-\lambda t}$ be the survival function of an Exponential distribution. From condition (3.1) the mixture $F_{\mathbf{p}}(t)$ is DFR if and only if

$$g_1'(t) \le \frac{Q_2(t)}{Q_1(t) + Q_2(t)} [g_1(t) - \lambda]^2 \tag{3.13}$$

holds for sufficiently large t. Note that the right-hand side of the above is bounded by a finite constant, but $g_1'(t)$ oscillates wildly. Thus there is no region of the form $[t_0, \infty)$ where condition (3.13) holds and hence F_1 is not an MRE distribution.

4 Mixtures of a pair of distributions from Lehmann families

Let $\mathcal{S} = \{S_\lambda = e^{-\lambda G(t)}, \lambda > 0\}$ where $G(t)$ is an increasing convex function such that $G(0) = 0$ and $G(t) \nearrow \infty$ as $t \nearrow \infty$. Then $\mathcal{S}$ is said to be an IFR Lehmann family based on the hazard function $G(t)$. In this section we consider the mixture $S_{\mathbf{p},\boldsymbol{\lambda}}(t) = p_1 S_{\lambda_1}(t) + p_2 S_{\lambda_2}(t)$ of two members from an IFR Lehmann family with mixing vector $\mathbf{p}$, and give conditions for the mixture to be DFR.

Theorem 2 *Let $S_{\mathbf{p},\boldsymbol{\lambda}}(t) = p_1 S_{\lambda_1}(t) + p_2 S_{\lambda_2}(t)$ be a mixture based on S_{λ_1} and S_{λ_2} from an IFR Lehmann family with mixing vector $\mathbf{p}$. Let $\rho = \frac{\lambda_2}{\lambda_1} > 1$ and $k = \frac{p_2}{p_1}$. The mixture $S_{\mathbf{p},\boldsymbol{\lambda}}$ is DFR if and only if*

$$\frac{g'(t)}{g^2(t)} \le \frac{k\lambda_1(1-\rho)^2 S_{\lambda_2 - \lambda_1}(t)}{(1 + k S_{\lambda_2 - \lambda_1}(t))(1 + k\rho S_{\lambda_2 - \lambda_1}(t))} \tag{4.1}$$

Proof.: Although this condition is obtainable by simplifying condition (2.1) of Theorem 1, the following approach provides insight into the structure of the failure rate of the mixture.

The survival function of the mixture is given by

$$S_{\mathbf{p},\boldsymbol{\lambda}}(t) = p_1 S_{\lambda_1}(t)[1 + k S_{\lambda_1}^{\rho - 1}(t)]$$

and its failure rate function is given by

$$-\frac{S_{\lambda_1}'(t)}{S_{\lambda_1}(t)}[\rho - \frac{\rho - 1}{1 + k S_{\lambda_1}^{\rho - 1}(t)}]. \tag{4.2}$$

This failure rate function is the product of two factors, the first of which is the increasing failure rate function of $S_{\lambda_1}(t)$ and the second is an adjustment factor, which is decreasing, since $\rho > 1$. Condition (4.1) is obtained by requiring the derivative of (4.2) to be less than 0. $\qquad \square$

Remark 3 Since $S_{\lambda_2 - \lambda_1}(t) \le 1$, the following is a sufficient condition for the mixture $S_{\mathbf{p},\boldsymbol{\lambda}}$ to be DFR:

$$\frac{g'(t)}{g^2(t)} \le \frac{k\lambda_1(1-\rho)^2 S_{\lambda_2 - \lambda_1}(t)}{(1 + k)(1 + k\rho)}. \tag{4.3}$$

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Definition 3 A Lehmann family $\{S_\lambda, \lambda > 0\}$ of IFR distributions is said to be a PWMRL (*in words, pair-wise mixture reversible Lehmann*) family, if the mixture $S_{\mathbf{p}, \boldsymbol{\lambda}}(t)$ is DFR for some values of $\mathbf{p}$ and $\boldsymbol{\lambda}$.

Examples of PWMRL families

We now give examples of PWMRL families.

Example 13. Consider the IFR Lehmann family based on $G(t) = \log(e^t + 1) - \log 2, t \geq 0$. Condition (4.3) becomes

$$e^{-t} \leq A_1 e^{-(\lambda_2 - \lambda_1)[\log(e^t + 1) - \log 2]}$$

for an appropriate constant A_1. This condition is clearly satisfied for t large and $\lambda_1 < \lambda_2 < \lambda_1 + 1$. Thus the Lehmann family based on the Truncated Logistic is a PWMRL family for this range of values.

It may be remarked that the mixture of two distributions from a Lehmann family based on the Truncated Logistic considered here differs from the mixture of two Truncated Logistic distributions considered in Example 1.

Example 14. Consider the IFR Lehmann family based on $G(t) = \log \cosh t$, $t \geq 0$. Condition (4.3) becomes

$$\frac{\text{sech}^2(t)}{\tanh(t)} \leq \frac{A_2}{(\cosh(t))^{\lambda_2 - \lambda_1}}$$

for an appropriate constant A_2. This condition is clearly satisfied for t large and $\lambda_2 - \lambda_1 < 2$; thus this family is PWMRL for this range of values.

We can also see that the Lehmann family based on $G(t) = t + \log \cosh t$ is a PWMRL family following almost the same calculations as in Example 14.

5   Poisson mixtures of distributions from Lehmann families

We begin this section by first defining Poisson mixtures from a Lehmann family. Consider an IFR Lehmann family of distributions with survival functions $\{S_\lambda(t) = e^{-\lambda G(t)}, \lambda > 0\}$ where $G(t)$ is an increasing convex function with $G(0) = 0$ and $G(t) \nearrow \infty$ as $t \nearrow \infty$. We consider the Truncated Poisson distribution $P_{\delta, c}$, with parameters δ and c, which is defined to be the distribution of Y/c where Y has a Truncated Poisson distribution with parameter δ. In other words, $P_{\delta, c}$ places its probability mass at the points $c, 2c, 3c, \ldots$, and its frequency function is given by

$$p_{\delta, c}(rc) = \frac{e^{-\delta} \delta^r}{r!(1 - e^{-\delta})}, r = 1, 2, \ldots.$$

The probability generating function of $P = P_{\delta, c}$ is given by

$$\psi_P(z) = \frac{e^{\delta(z^c - 1)} - e^{-\delta}}{1 - e^{-\delta}}.$$

Let

$$S_{(P)}(t) = \sum_{r=1}^{\infty} S_{rc}(t) p_{\delta,c}(rc). \tag{5.1}$$

We will refer to $S_{(P)}(t)$ as a Poisson mixture of the Lehmann family $\{S_\lambda\}$. We now give a precise definition of TPMRL families of distributions.

Definition 4 An IFR Lehmann family $\{S_\lambda, \lambda > 0\}$ is said to be a TPMRL (*in words, Truncated Poisson mixture reversible Lehmann*) family, if the Poisson mixture $S_{(P)}$ defined in (5.1) is a DFR distribution for some choice of δ and c.

We shall see later that the use of the scale factor c, in the Poisson distribution, gives us extra latitude in constructing TPMRL families. The other scale factor λ in the Lehmann family is being averaged out and does not appear in the Poisson Mixture $S_{(P)}$.

The theorem below gives necessary and sufficient conditions for an IFR Lehmann family to be a TPMRL family.

Theorem 3 *The Poisson mixture $S_{(P)}$ of an IFR Lehmann family $\{S_\lambda(t) = e^{-\lambda G(t)}, \lambda > 0\}$ is DFR if and only if*

$$\frac{1}{c}\frac{g'(t)}{g^2(t)} \leq \frac{e^{\delta w(t)} - 1 - \delta w(t)}{e^{\delta w(t)} - 1} \quad \text{for all} \quad t \tag{5.2}$$

where $w = w(t) = e^{-cG(t)}$.

Proof.: A compact expression for the Poisson mixture survival function is given by

$$S_{(P)}(t) = \psi_P(e^{-G(t)}) = \frac{e^{-\delta}}{1 - e^{-\delta}}(e^{\delta w} - 1).$$

Using the identity

$$w'(t) = -cg(t)w(t)$$

which follows from the definition of $w(t)$, we can verify that the failure rate function $g_{(P)}$ of $S_{(P)}(t)$ satisfies

$$g_{(P)}(t) = \frac{c\delta g(t)w(t)e^{\delta w(t)}}{e^{\delta w(t)} - 1}.$$

Taking the logarithmic derivative, we obtain

$$\frac{g'_{(P)}(t)}{g_{(P)}(t)} = cg(t)\left\{ \frac{g'(t)}{cg^2(t)} - \frac{e^{\delta w(t)} - 1 - \delta w(t)}{e^{\delta w(t)} - 1} \right\}.$$

This shows that condition (5.2) is necessary and sufficient for the family to be a TPMRL family. □

Remark 4 Since $0 \leq w \leq 1$, it follows that

$$\frac{e^{\delta w} - 1 - \delta w}{e^{\delta w} - 1} > \frac{\frac{\delta^2 w}{2}}{e^\delta - 1}.$$

Thus the following is a sufficient condition for the mixture $S_{(P)}$ to be DFR:

$$\frac{1}{c}\frac{g'(t)}{g^2(t)} \leq \frac{\delta^2 e^{-cG(t)}}{2(e^\delta - 1)}. \tag{5.3}$$

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There is a marked similarity between the sufficient conditions (4.3) and (5.3). Suppose that an IFR Lehmann family S_λ satisfies condition condition (4.3), i.e.

$$\frac{g'}{g^2} \leq A_1 e^{-(\lambda_2 - \lambda_1)G}$$

for appropriate constants A_1, λ_1 and λ_2. Then it also satisfies condition (5.3), which can be rewritten as

$$\frac{g'}{g^2} \leq A_2 e^{-cG}$$

for appropriate values of c and A_2. The converse, of course, is also true. Clearly, the examples of PWMRL given in the Section 4 are also examples of TPMRL families.

6 Continuous mixtures of distributions from Lehmann families

Let $\{S_\lambda(t) = e^{-\lambda G(t)}, \lambda > 0\}$ be an IFR Lehmann family where as before $G(t)$ is an increasing convex function with $G(0) = 0$ and $G(t) \nearrow \infty$ as $t \nearrow \infty$. Let $C(\lambda)$ be an arbitrary distribution function on $(0, \infty)$ with moment generating function $\phi(s)$. Consider the mixture survival function

$$S_{(C)}(t) = \int_0^\infty S_\lambda(t) dC(\lambda). \tag{6.1}$$

The theorem below gives necessary and sufficient conditions for the mixture $S_{(C)}$ to be DFR.

Theorem 4 *Let $S_{(C)}(t)$ be the survival function of the mixture of an IFR family $\{S_\lambda(t)\}$ by a distribution $C(\lambda)$, as given in (6.1). Let the moment generating function of $C(\lambda)$ be $\phi(s)$. Then $S_{(C)}$ is DFR if and only if*

$$\frac{g'(t)}{g^2(t)} \leq \frac{\{\phi(-G(t)\phi''(-G(t)) - [\phi'(-G(t))]^2\}}{\phi(-G(t))\phi'(-G(t))}. \tag{6.2}$$

Proof.: Rewriting the definition in (6.1) we get

$$S_{(C)}(t) = \int_0^\infty S_\lambda(t) dC(\lambda) = \int_0^\infty \epsilon^{-\lambda G(t)} dC(\lambda) = \phi(-G(t)). \tag{6.3}$$

Hence the failure rate function of the mixture is given by

$$g_{(C)}(t) = \frac{g(t)\phi'(-G(t))}{\phi(-G(t))}$$

where $g(t)$ is the failure rate function of $S(t)$. Thus

$$g'_{(C)}(t) = \frac{\phi(-G(t))[\phi'(-G(t))g'(t) - \phi''(-G(t))g^2(t)] + g^2(t)[\phi'(-G(t)]^2}{[\phi(G(t))^2]}.$$

Consequently, $S_{(C)}$ is DFR if and only if the numerator of the above is nonpositive, which is the same as condition (6.2). □

Note that Theorem 3 can be derived from Theorem 4 by substituting the moment generating function $\frac{e^\delta}{(1-e^\delta)}e^{\delta(e^\lambda-1)}$ for $\phi(\lambda)$.

We now specialize the mixing distribution $C(\lambda)$ to a Gamma distribution and examine the mixture reversal properties of IFR Lehmann families.

Definition 5 The IFR Lehmann family $\{S_\lambda\}$ is said to be a GMRL family, (*in words, Gamma mixture reversible Lehmann*) family, if the mixture $S_{(C)}$ above is DFR when $C(\lambda)$ is the Gamma distribution with scale parameter α and shape parameter β, for some va¹ of α and β.

We now use the fact that the moment generating function $\phi(s)$ of the Gamma distribution $C(t)$ with scale parameter α and shape parameter β (see Example 4) is given by $\phi(s) = (1 - \frac{s}{\alpha})^{-\beta}$.

Corollary 4 *The IFR Lehmann family $\{S_\lambda(t) = e^{-\lambda G(t)}, \lambda > 0\}$ is a GMRL family if and only if*

$$\frac{g'(t)}{g^2(t)} \leq \frac{1}{\alpha + G(t)}. \tag{6.4}$$

Proof.: Note that the parameter β does not enter condition (6.4). This can be explained by the fact that the mixture $S_{(C)}$ becomes a DFR Lehmann family in the parameter β. In fact, substituting $\phi(s) = (1 - \frac{s}{\alpha})^{-\beta}$ in (6.3), we find that

$$S_{(C)}(t) = e^{-\beta \log(1+\frac{G(t)}{\alpha})}. \tag{6.5}$$

Hence $S_{(C)}$ is a DFR survival function if and only if $\log(1 + \frac{G(t)}{\alpha})$ is concave, which is equivalent to condition (6.4).

One could also give a proof of this result by simplifying the necessary and sufficient condition (6.2) pertaining to this case. □

Let us now verify or refute the GMRL properties of several well known IFR Lehmann families by verifying or negating condition (6.4) of Corollary 4.

Example 15. The Exponential family, which forms an IFR Lehmann family with $G(t) = t$: This family is clearly GMRL since $g' = 0$.

Example 16. The IFR Lehmann family based on a Gamma survival function is *not a GMRL family* if the shape parameter satisfies $\beta > 2$. To verify this, refer to the expression for $S_1(t)$, $g(t)$ and $\frac{g'(t)}{g(t)}$ in Example 4. We see that

$$G(t) = -\log S_1(t) = \log \Gamma(\beta) + \alpha t - \beta \log \alpha - \log I(t, \beta - 1) \text{ and}$$

$$\frac{g'(t)}{g^2(t)} = \frac{I(t, \beta-1)}{t^{\beta-1}}\left[\frac{\beta-1}{t} - \alpha\right] + 1.$$

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Substituting these in condition (6.4) and using the fact that $\alpha I(t, \theta) \sim t^{\theta}$ as $t \to \infty$ (see (3.6)), we find that the Lehmann family based on the Gamma is GMRL if and only if

$$\frac{\beta - 1}{\alpha t} \leq \frac{1}{\alpha + \log \Gamma(\beta) + \alpha t - (\beta - 1) \log \alpha - (\beta - 1) \log t},$$

for large t, that is, if and only if

$$\frac{\beta - 2}{\beta - 1} \alpha t + \alpha + \log \Gamma(\beta) \leq (\beta - 1) \log(\alpha t).$$

For sufficiently large t, this inequality is satisfied for $1 < \beta \leq 2$ and the reverse inequality is satisfied for $\beta > 2$. Thus, the Lehmann family based on the Gamma is GMRL for $1 < \beta \leq 2$ and *is not* GMRL for $\beta > 2$.

Example 17. Weibull family of distributions: When the shape parameter $\gamma > 1$ is fixed, the Weibull distributions form an IFR Lehmann family in terms of the scale parameter. Here $G(t) = t^{\gamma}$ and $\gamma > 1$. In this case condition (6.4) reduces to

$$\frac{(\gamma - 1)}{\gamma t^{\gamma}} \leq \frac{1}{\alpha + t^{\gamma}}$$

which is equivalent to $(\gamma - 1)\alpha \leq t^{\gamma}$. This is satisfied for sufficiently large t; thus, the Weibull Lehmann family is GMRL. The failure rate function of the Weibull Lehmann family can be a rapidly increasing function. It is surprising to see that *this family is mixture reversible under Gamma mixtures*. This can be seen graphically in Figure 3, where we superimpose the plot of the decreasing failure rate function, $g_{(C)}(t)$, of the Gamma mixture with the plots of the increasing failure rate functions, $g_i(t), i = 1 \ldots, 8$, of a sampling of the Weibull distributions that are being mixed, namely $g_i(t) = \lambda \gamma t^{(\gamma - 1)}$, where $\lambda = ih, i = 1, \ldots, 8$ with $h = .06$ and $\gamma = 1.5$. The parameter, α, of the mixing Gamma distribution was chosen to be equal to 2.
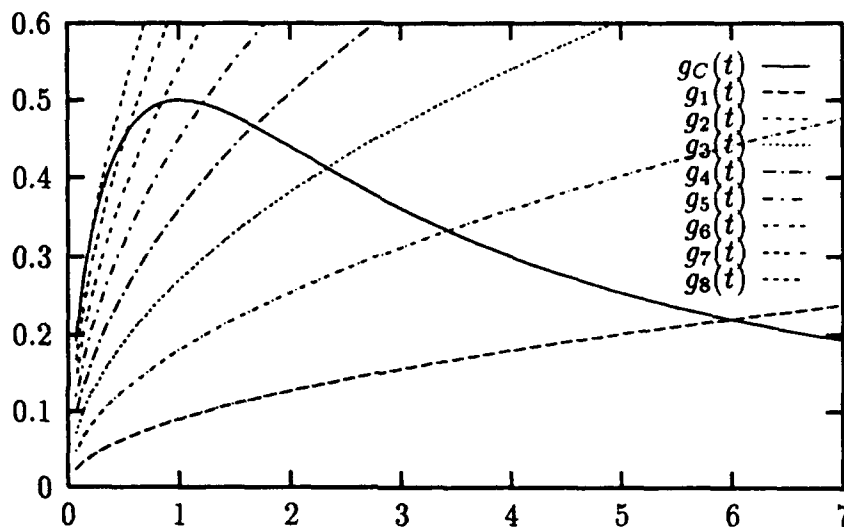
Figure 3
Mixture of a Weibull family of distributions by an Exponential.
Graphs of the failure rate functions of the mixture, $g_{(C)}(t)$,
and those of several of the original Weibull distributions $g_i(t), i = 1, \ldots, 8$.

Example 18. The IFR Lehmann family based on the Truncated Extreme distribution: Here $G(t) = \theta(e^t - 1), t \geq 0$. Condition (6.4) reduces to

$$(\alpha - \theta)e^{-t} \leq 0.$$

This is satisfied for $\alpha \leq \theta$, and for such values, this family is also a GMRL family. This continues our surprise expressed in the previous example, since the failure rate of the Truncated Extreme increases even faster than that of the Weibull. However, for the range of values $\alpha > \theta$, the reverse inequality to the inequality in Condition (6.4) holds and the Lehmann family based on the Truncated Extreme *is not a GMRL family* for such values of α. This phenomenon is similar to the one in Example 16.

Example 19. The IFR Lehmann family based on the Truncated Normal distribution: From Example 7, we see that $G(t) = -\log(\sqrt{\frac{2}{\pi}}) + \frac{t^2}{2} - \log J(t)$ where $J(t)$ is the same as $J_1(t)$ defined in (3.8). From (3.10) we find

$$G(t) \leq -\log(\sqrt{\frac{2}{\pi}}) + \frac{t^2}{2} + \log t - \log(1 - \frac{1}{t^2}). \qquad (6.6)$$

The calculations in Example 7 and (3.10) give the inequality

$$\frac{g'(t)}{g^2(t)} = 1 - tJ(t) \leq \frac{1}{t^2}. \qquad (6.7)$$

The inequalities (6.6) and (6.7) show that

$$G(t) + \alpha \leq \alpha - \log(\sqrt{\frac{2}{\pi}}) + \frac{t^2}{2} + \log t - \log(1 - \frac{1}{t^2}) \leq t^2 \leq \frac{g^2(t)}{g'(t)}$$

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for large t and any $\alpha > 0$. This verifies condition (6.4) in its equivalent form, namely, $G(t) + \alpha \leq \frac{g^2(t)}{g'(t)}$. Thus the Lehmann family based on the Truncated Normal is GMRL.

Example 20. The IFR Lehmann family based on the Truncated Logistic distribution: Here $G(t) = \log(e^t + 1) - \log 2$. It is easy to verify that condition (6.4) holds for large t and hence that this is a GMRL family.

In a similar manner it can be shown that the Lehmann families based on $G(t)$ for $G(t) = t\log(a + t)$ and $G(t) = t - \log(1 + t)$ are GMRL families.

7 Conclusion

In our search for IFR distributions which are mixture reversible, certain types of mixing have been employed and some relevant conditions have been developed. To delineate certain families of such distributions, some notation has been introduced and some examples of distributions belonging to such families have been presented. For the purpose of discussion it is helpful to associate the various families of distributions with the particular sufficient conditions employed to obtain the illustrative examples, and to summarize the results obtained.

Table 1 lists the 10 distributions considered here. The first six are from the list of "typical useful failure laws" in Barlow and Proschan (1965) and include the Truncated Logistic. The remaining four have been chosen for convenience and simplicity.

Table 2 summarizes the results obtained. As evident from Table 2, all of the distributions in Table 1 are GMRL except for the Gamma (with $\beta > 2$) and the Truncated Extreme (with $\alpha > 1$). It is curious that the Truncated Extreme, with failure rate of order $O(e^t)$, the Weibull with failure rate of order $O(t^\gamma), \gamma > 1$, and the Truncated Normal with failure rate of order $O(t)$, are all GMRL; yet the Gamma, with failure rate of order $O(1)$ is not GMRL for $\beta > 2$. As a matter of fact, Gamma mixtures of the Lehmann family based on the Gamma with shape parameter β are IFR for $\beta > 2$.

It is also interesting that the only distributions from Table 1 which are TPMR or PWMRL are the Truncated Logistic and the distributions with $G(t) = \log \cosh t$ and $G(t) = t + \log \cosh t$.

It is also remarkable that all of the distributions considered here are MRE, i.e. are contaminable by the Exponential. One is tempted to conclude that this class is very far ranging. From a practical standpoint, this contaminable aspect could be quite serious when analyzing pooled data. It could conceivably happen that much of the data conforms to an IFR distribution, such as the Weibull or Truncated Extreme, for example, but the remainder of the data conforms to an Exponential, in which case the overall pooled data would conform to a DFR distribution.

In the light of the examples in this paper, it is conceivable that the failure rate, as commonly employed, could be quite misleading when dealing with heterogeneous data. The situation is not unlike the presence of outliers in regression analysis, where a few (or even one) observations can seriously affect conclusions based on the regression function. In the case of regression, the analysis can be performed separately, with and without the

outliers, and the different conclusions assessed. In the case of heterogeneous failure data, however, the different underlying populations in the mixture are usually not capable of being separated out; hence, the effect of contamination may not be assessable. It is an open and challenging question of how to deal with pooled or heterogeneous failure data.

TABLE 1

DISTRIBUTIONS CONSIDERED

Exponential
Gamma
Weibull
Truncated Extreme
Truncated Normal
Truncated Logistic
$G(t) = t\log(a+t), a \geq 1$
$G(t) = t - \log(1+t)$
$G(t) = \log\cosh t$
$G(t) = t + \log\cosh t$

TABLE 2

SUMMARY OF RESULTS

Condition	Family	Examples
Condition (2.2)	PWMR	$G_j = c_j t + \log\cosh t, \ j = 1,2,$ and $G_j = c_j t + \log\frac{(1+e^{-t})}{2}, \ j = 1,2$
Conditions (3.2) and (3.4)	MRE	All distributions from TABLE 1
Condition (4.3)	PWMRL	Lehmann families based on the Truncated Logistic, the $G = \log\cosh t$ and the $G = t + \log\cosh t$
Condition (5.3)	TPMRL	The three examples above
Condition (6.4)	GMRL	Lehmann families based on all distributions in TABLE 1, including the Gamma distribution (with $\beta \leq 2$) and the Truncated Extreme distribution (with $\alpha \leq \theta$)

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