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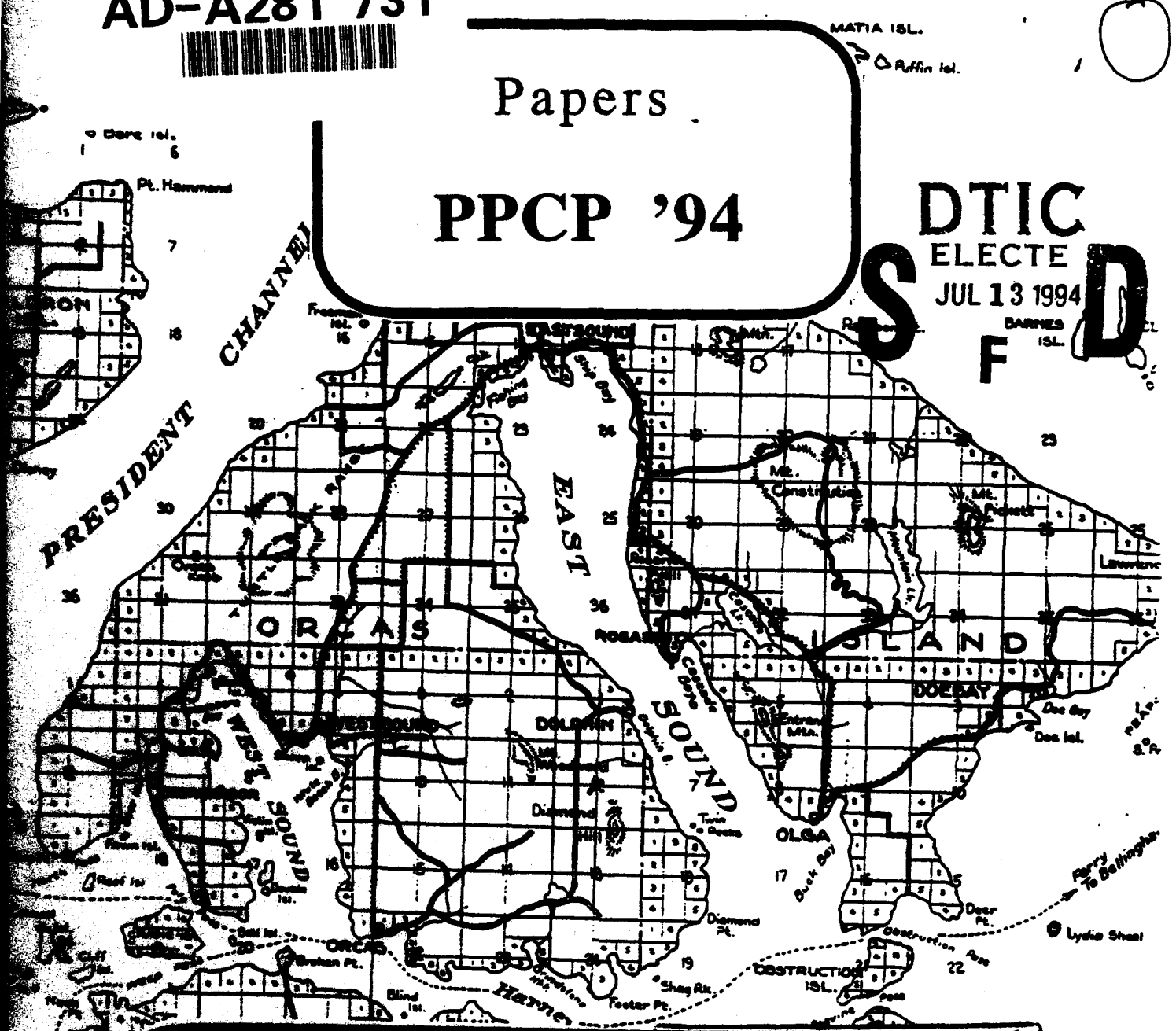

The Second Workshop on Principles and Practice of Constraint Programming

Rosario Resort
Orcas Island, Washington
May 2 - 4, 1994

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The Second Workshop on the Principles and Practice of Constraint Programming is an inter-disciplinary meeting focusing on constraint programming and constraint-based systems. This workshop is held in cooperation with the American Association for Artificial Intelligence and the Association for Logic Programming, and is sponsored in part by the Office of Naval Research.

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Contents

Set Constraints and Set-Based Analysis

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1 Introduction

Set expressions over a signature Σ of function symbols are a natural representation of sets of elements constructed from Σ, and set constraints express basic relationships between these sets. In the literature, set constraints have between used mostly in the context of uninterpreted (or Herbrand) function symbols. Although these applications have used set constraints in quite different ways, a common theme is the use of set constraints to obtain an *approximation* of some aspects of a program.

This paper contains two main parts. The first examines the set constraint calculus, discusses its history, and overviews the current state of known algorithms and related issues. Here we will also survey the uses of set constraints, starting from early work in (imperative) program analysis, to more recent work in logic and functional programming systems.

The second part describes *set-based analysis*. The aim here is a declarative interpretation of what it means to approximate the meaning of a program in just one way: ignore dependencies between variables, and instead, reason about each variable as the set of its possible runtime values. The basic approach starts with some description of the operational semantics, and then systematically replaces descriptions of environments (mappings from program variables to values) by set environments (mappings from program variables to *sets* of values) to obtain an approximate semantics called the *set-based program semantics*. The next step is to transform this semantics into a set constraint problem, and finally, the set constraints are solved.

2 Set Constraints

We present here the general calculus, followed by a brief survey of related work.

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2.1 The Calculus

The set constraint calculus is parameterized by an underlying domain of discourse, and a set of functions Σ. For the purposes of most this paper, we choose Σ to be a given collection of uninterpreted function symbols, and the domain of discourse is then the ground terms constructed from Σ. In addition to Σ, we consider a fixed set of set operators consisting of union, intersection, complementation and projections of Σ functions.

A *set expression* is either a set variable (denoted $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, etc.), or of one of the forms $f(se_1, \ldots, se_n)$ or $op(se_1, \ldots, se_n)$, where $f \in \Sigma$, the se_i are set expressions, and op is a set operator. The set operators include union, intersection, complementation and projection (denoted $f_{(i)}^{-1}$ where f is an n-ary function symbol and $1 \leq i \leq n$). As an example of projection, the operator $cons_{(1)}^{-1}$ denotes the first projection with respect to the constructor *cons*, and is the "set" counterpart of *car*. It is also convenient to include $\top$ and $\bot$ in the definition of set expressions to respectively denote the set of all terms and the empty set (some works use 1 and 0 instead of $\top$ and $\bot$). A *set constraint* is of the form $se \supseteq se'$ where se and se' are set expressions. We write $se = se'$ as an abbreviation for the two constraints $se \supseteq se'$ and $se' \supseteq se$.

A *solution* to a collection $\mathcal{C}$ of set constraints is an assignment of sets to set variables that satisfies each constraint. Specifically, let $\mathcal{I}$ be a mapping from set variables into sets of terms. Such a mapping can be extended to map from set expressions into sets of values:

- $\mathcal{I}(f(se_1, \ldots, se_n)) = \{f(v_1, \ldots, v_n) : v_i \in \mathcal{I}(se_i)\}$;
- $\mathcal{I}(se_1 \cup se_2) = \mathcal{I}(se_1) \cup \mathcal{I}(se_2)$;
- $\mathcal{I}(se_1 \cap se_2) = \mathcal{I}(se_1) \cap \mathcal{I}(se_2)$;
- $\mathcal{I}(f_{(i)}^{-1}(se)) = \{v_i : f(v_1, \ldots, v_n) \in \mathcal{I}(se)\}$;
- $\mathcal{I}(\overline{se})) = \{v : v \notin \mathcal{I}(se)\}$;
- $\mathcal{I}(\top) =$ all values, and $\mathcal{I}(\bot) = \{\}$

$\mathcal{I}$ is a *solution* of a collection of constraints $\mathcal{C}$ if $\mathcal{I}(se) \supseteq \mathcal{I}(se')$ for each constraint $se \supseteq se'$ in $\mathcal{C}$.

For example, let $\mathcal{C}$ denote the single constraint $\mathcal{X} \supseteq c \cup f(f(\mathcal{X}))$, where c is a constant and f is a unary symbol. $\mathcal{C}$ has many models, including the mapping that maps all set variables into the set $\{c, f(c), f(f(c)), \ldots\}$. Another solution of $\mathcal{C}$ is the mapping $\mathcal{I}$ defined by

$$\mathcal{I}(\mathcal{Y}) = \begin{cases} \{c, f^2(c), f^4(c), \ldots\} & \text{if } \mathcal{Y} \text{ is } \mathcal{X} \\ \{\} & \text{if } \mathcal{Y} \text{ is different from } \mathcal{X} \end{cases}$$

where f^n abbreviates n applications of f. This solution is smaller than the first, and is in fact the smallest solution of $\mathcal{C}$. As another example, the smallest solution of the following constraint collection maps $\mathcal{X}$ into $\{a, f^3(a), f^6(a), \ldots\}$, maps $\mathcal{Y}$ into $\{a, f^2(a), f^4(a), \ldots\}$ and maps $\mathcal{Z}$ into $\{f^5(a), f^{11}(a), f^{17}(a), \ldots\}$.

$$\begin{aligned} \mathcal{X} &\supseteq a \cup f^3(\mathcal{X}) \\ \mathcal{Y} &\supseteq a \cup f^2(\mathcal{Y}) \\ \mathcal{Z} &\supseteq f_{(1)}^{-1}(\mathcal{X} \cap \mathcal{Y}) \end{aligned}$$

In general, a collection of set constraints does not always have a unique smallest solution. For example consider the constraint $\mathcal{X} \cup \mathcal{Y} = a$ which has two minimal solutions: one that maps X to $\{a\}$ and $\mathcal{Y}$ to the empty set, and the other that maps X to the emtpy set and $\mathcal{Y}$ to $\{a\}$. For certain kinds of program analysis, it is natural to consider sub-classes of set constraints for which least models always exist. For example, consider constraints of the form $\mathcal{X} \supseteq se$ where $\mathcal{X}$ is a set variable and se is a set expression that does not use complementation. Such constraints always have a least solution. Somewhat more general are the *definite* set constraints, which have the form $a \supseteq se$ where a is a set expression that is "atomic" in the sense that it is constructed solely from set variables and function symbols, and se is a set expression that does not use complementation. A collection of definite set constraints is such that whenever it has a solution, it will in fact have a least solution. Further, it can be shown that this solution is *regular* in the sense that every variable is a regular set, that is, a set accepted by a nondetermistic tree automaton.

2.2 A Brief History

The use of set constraints for analysis of programs dates back to the early works by Reynolds [29] (who presents an analysis for a first-order functional language), and Jones and Muchnick [22] (who present an analysis for a simple imperative language). In both of these works, the set constraints used are quite simple: the only set operations employed are union and projection (there are no intersections or quantified expressions). We say more about these applications in the next subsection.

The general calculus of set constraints, as defined above, was first formalized and studied in a general setting in [17]. This work also presented a decision procedure for the class of definite set constraints (recall that definite constraints do not contain the complement symbol, and are restricted to the form $a \supseteq se$ where the set expression a contain only variables and function symbols). This procedure further provides an explicit representation of the least model of a (satisfiable) collection of definite set constraints. [17] also posed decidability of the satisfiability problem for general set constraints as an open question.

Later, [1] proved the decidability of a different, and incomparable, class: the *positive* set constraints. These are defined simply to be set constraints not involving projection. This procedure reduces the constraints into a simpler form. When reduction terminates without detecting inconsistency, the resulting constraints are evidently satisfiable. Note that satisfiable positive set constraints do not always have a least model. Subsequently, [9] provided an alternative procedure using tree automata techniques. Starting with Rabin's result [28] that the theory of k-successors is decidable, they generalized the Rabin automaton to accomodate positive set constraints. They further showed that satisfiable positive constraints always have a regular solution (all variables are assigned a regular set), and a minimal and maximal regular solution.

While the class of definite constraints and the class of positive classes are not comparable,

the work [5] proved decidability of a class subsuming the two. Briefly, the set constraints considered here are the positive ones, extended to allow projections in a restricted way. The importance of this work probably lies more in the technique used: it is proved that set constraints can be written into equivalent formulas in the *monadic class*, that is, first-order formulas with unrestricted quantification, but no function symbols and only monadic predicate symbols. The transformation is simple and elegant, and gave rise to complexity results on set constraints based on similar results in the monadic class.

The next step was taken by [10], who proved that *negative* set constraints, ie. the extension to positve constraints with negations of subset relationships such as $se_1 \not\subseteq se_2$, remains decidable. Once again, tree automata techniques were used here. An alternative procedure was then given by [4], by reduction to a number-theoretic decision problem. Subsequently, [6] used the abovementioned translation of set constraints to the monadic class to provide a straightforward procedure for deciding negative set constraints. Note that none of these works on negative constraints deal with projections.

In summary, the state of the art for the set constraint decision problem is largely determined by the reduction to the monadic class of formulas. The main question remains how to deal with (unrestricted) projection. At the time of writing, we have verbal communication [26] indicating that the proof in [6] can be extended to solve this problem. Thus the question of whether the general set constraint problem is open, now becomes open!

2.3 Applications

Early works

Two important early works are by Jones and Muchnick [22] and Reynolds [29]. In [22], an analysis is described for an imperative language with LISP-like data structures. The essence here is the construction of set constraints corresponding to a program that capture the flow of values from one variable to another as the program is executed. However, the set constraints here are restricted so that they can be solved by a fairly straightforward algorithm. In particular, the set constraints do not contain a notion of intersection, and their only operation is projection (corresponding to decomposition of data structures). Hence they are not expressive enough to capture a number of important components of programs. For example all information about the conditions in conditional statements is completely omitted. Further, information relating to well definedness of expressions is ignored (for example, after a statement $X = car(Y)$, it must be the case that Y is of the form $cons(\cdots)$ because otherwise the program would have terminated with ar. error).

In contrast, the earlier paper [29] used set constraints to compute data type definitions for program variables in a first order functional language. The constraints used are similar to those used in [22]. Again the only set operation of the constraints is projection, and so the program approximations obtained can be considerably inaccurate.

In summary, the set constraints used in these early works are simple, but the program approximations that they define are not very accurate. These works viewed set constraints as a tool for obtaining information about the program, and the constraints themselves incorpo-

rate a number of *ad hoc* approximations in addition to ignoring inter-variable dependencies. As a result, there is no simple connection between the program and its approximation. This particular shortcoming is one of the motivations for set-based analysis, discussed later in this paper.

Logic Programs

The use of set constraints for the bottom-up analysis of logic programs was first considered in [25]. The set constraints in this relatively early work were rather specialized and used a form of approximation called *tuple-distributive* closure (hereafter just called *closure*). This closure, which was subsequently used in some later works, has the effect of enlarging a set of terms S into S^* as follows:

$$S^* \stackrel{\text{def}}{=} \{c : c \text{ is a constant in } S\} \cup \bigcup_{f \in \Sigma} f\left((f_{(1)}^{-1}(S))^*, \ldots, (f_{(\text{arity}(f))}^{-1}(S))^*\right)$$

where $f(S_1, \ldots, S_n)$ denotes the set $\{f(s_1, \ldots, s_n) : s_i \in S_i\}$ and $f_{(i)}^{-1}(S)$ denotes the set $\{s_i : f(s_1, \ldots, s_n) \in S\}$. Thus for example, closing the set $\{f(a,b), f(c,d)\}$ produces $\{f(a,b), f(a,c), f(b,d). f(c,d)\}$. The set constraints used in [25] are like the general ones defined above, except that the union operation is interpreted to be the closure of the union of sets.

A different approach to approximation starts from the (bottom-up) fixpoint operator T_P of a program P, and the approximate meaning of a program is obtained by imposing closure on each iteration of the operator. For example, [32] defined the operator $Y_P(S) \stackrel{\text{def}}{=} (T_P(S))^*$ and the approximate meaning of the program is the least fixpoint $lfp(Y_P)$ of Y_P (which is always larger than the exact meaning, $lfp(T_P)$). In [16], a more accurate operator τ_P was used. (Roughly, Y_P ignores inter-argument dependencies, while τ_P ignores only inter-variable dependencies.) A more recent work [8] used the closure operators (in conjunction with another approximation technique called widening) to define and compute a program approximation.

The relationship between these closure-based fixpoint operators and set constraints was described in [18]. One result is that the models of the set constraints in [25], essentially correspond to the fixpoints of Y_P. A similar result was that the other fixpoint operator τ_P corresponded to certain formulas obtained from the program. These formulas are similar to but more general than set constraints. The main point here was that the least fixed-point of τ_P provided a more accurate and intuitive notion of approximation, and importantly, the approximation is decidable. It is open as to whether $lfp(Y_P)$ is decidable.

Functional Programs

The general approach of [22, 29] has been extended by [21] to deal with higher-order functions. This approach has been further developed for binding time analysis [24], garbage collection [20] and globalization of function parameters [30]. One presentational difference in these works is the use of various extensions of regular grammars instead of constraints.

Subsequently, a number of set constraint approaches have been developed for the analysis of higher-order functional languages (see, for example, [27, 12, 2, 3, 13]). Perhaps the most developed of these approaches are those by [12, 13] and [2, 3]. The former starts with an operational semantics, and develops a set-based analysis for this semantics. The constraints that arise are briefly sketched in Section 3.3. In the latter, a denotational model of the program inspires the extraction of "type constraints", which are essentially set constraints (involving intersection and complement but not projection) over a domain of downward closed sets of finite elements (essentially the "ideal" model of types). We note that both works include a mechanism for reasoning about non-emptiness of sets (these are called "conditional types" in [3]).

Sorted Unification

Broadly, sorted unification is the problem of unifying two terms in the context of a sort theory, the latter imposing constraints on the values that certain variables can take. The sort theory is typically presented as a sort signature, indicating the hierarchical arrangement of the various sorts, together with a specification on the sorts of the various function symbols. For example,

$$\{even \subseteq int,\ \ odd \subseteq int,\ \ succ : odd \longrightarrow even,\ \ succ : even \longrightarrow odd\}$$

specifies that the sorts *even* and *odd* both belong to *int*, that the function *succ* maps an even integer into an odd one, and vice versa. Such constraints can be naturally specified in set constraints:

$$Int = Odd \cup Even,\ \ Odd = 0 \cup succ(Even),\ \ Even = 0 \cup succ(Odd)$$

In general, sorted unification is decidable only when the sort theory is restricted in some way. In the literature, a typical restriction is that the sorts are regular sets. In [31], a restricted class of set constraints is used to represent the sort theory, and a new sorted unification algorithm is presented. This work shows that further development in set constraints may be useful for sorted unification.

3 Set-Based Analysis

The basic approach of set-based program analysis starts with some description of the operational semantics. Typically, such a description involves environments, which describe the values that each variable may assume at runtime. The next step is a systematic replacement of environments into *set environments*, which map variables into sets of values, as opposed to a single value. This fundamental step gives rise to the notion of a set-based semantics of a program. Next, the set-based semantics is reduced to a set constraint problem, and finally, the set constraints are solved.

In this paper, we will not go through this process in much formal detail. These details can be found in [12]. Instead, we will show by examples how set constraints indeed model the desired approximation from program fragments.

In the following examples, we shall use a simple imperative programming language with basic facilities for data structure creation (e.g. cons and nil for list creation) and data destructuring/projection (e.g. car and cdr for list destructuring). Consider the statement X := cons(Y, X). To model this statement, set variables are introduced to collect the values of the variables X and Y just before and just after the statement (we suppose that these are the only program variables of interest). Let $\mathcal{X}_1$ and $\mathcal{Y}_1$ be the set variables to collect the values of X and Y just before execution of the statement, and let $\mathcal{X}_2$ and $\mathcal{Y}_2$ be the set variables for just after statement execution. Now, the values for X just after execution of the statement include all values $cons(v_Y, v_X)$ such that $v_X \in \mathcal{X}_1$ and $v_Y \in \mathcal{Y}_1$, and so we write $\mathcal{X}_2 \supseteq \{cons(v_Y, v_X) : v_X \in \mathcal{X}_1, v_Y \in \mathcal{Y}_1\}$, which is abbreviated by $\mathcal{X}_2 \supseteq cons(\mathcal{Y}_1, \mathcal{X}_1)$. In contrast, the values for Y just after execution of the statement are exactly those before execution, and so we write the constraint $\mathcal{Y}_2 \supseteq \mathcal{Y}_1$. Hence, from the above program statement, we construct two set constraints: $\mathcal{X}_2 \supseteq cons(\mathcal{Y}_1, \mathcal{X}_1)$ and $\mathcal{Y}_2 \supseteq \mathcal{Y}_1$. Note that for this example, we could have replaced $\supseteq$ by $=$ and written the equations $\mathcal{X}_2 = cons(\mathcal{Y}_1, \mathcal{X}_1)$ and $\mathcal{Y}_2 = \mathcal{Y}_1$. However, for a number of reasons, it is somewhat more convenient to use inequalities rather than equalities[1].

Similarly, for the statement X := cdr(X) we construct the two constraints $\mathcal{X}_2 \supseteq cdr(\mathcal{X}_1)$ and $\mathcal{Y}_2 \supseteq \mathcal{Y}_1$, where $cdr(\mathcal{X}_1)$ abbreviates $\{v_2 : cons(v_1, v_2) \in \mathcal{X}_1\}$, and $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$ are as before. In general, the use of sets to reason about a program leads to an approximation of the program's actual behaviour. This is because the use of sets ignores dependencies between variable values. For example, consider the following program

```
X := car(W);
Y := cdr(W);
W := cons(X, Y);
```

Let $\mathcal{W}_i, \mathcal{X}_i$ and $\mathcal{Y}_i$, $i = 1..4$, be the set variables introduced to collect the values of W, X and Y just before the first statement, just before the second statement, just before the third statement, and just after the third statement respectively. Constructing constraints as before yields:

$$\begin{array}{lll} \mathcal{W}_2 \supseteq \mathcal{W}_1 & \mathcal{W}_3 \supseteq \mathcal{W}_2 & \mathcal{W}_4 \supseteq cons(\mathcal{X}_3, \mathcal{Y}_3) \\ \mathcal{X}_2 \supseteq car(\mathcal{W}_1) & \mathcal{X}_3 \supseteq \mathcal{X}_2 & \mathcal{X}_4 \supseteq \mathcal{X}_3 \\ \mathcal{Y}_2 \supseteq \mathcal{Y}_1 & \mathcal{Y}_3 \supseteq cdr(\mathcal{W}_2) & \mathcal{Y}_4 \supseteq \mathcal{Y}_3 \end{array}$$

Now, suppose that at the start of the program, the variable W is either the list $[1,2]$ or the list $[3,4]$. Then the set for $\mathcal{X}_2$ (and $\mathcal{X}_3$) is $\{1,3\}$, and the set for $\mathcal{Y}_3$ is $\{[2],[4]\}$. Hence, the set for $\mathcal{W}_4$ is $\{[1,2],[3,4],[1,4],[3,2]\}$. In contrast, the only possible values for W after execution of the third statement are $[1,2]$ and $[3,4]$.

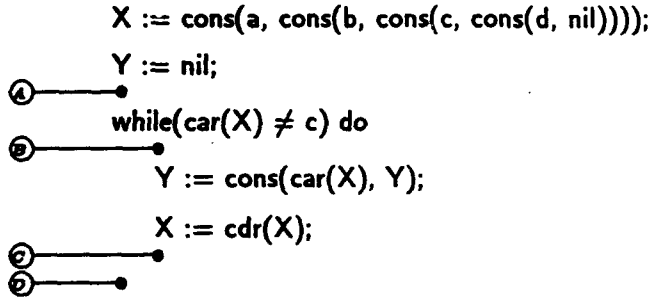
The key property of the constraints constructed from a program is that any solution of the constraints conservatively approximates the operational semantics of the program. This means that to obtain a safe approximation of the program, it is sufficient to construct a solution to the constraints. The constraint solving process will typically compute the minimum solution to the constraints since this is the most accurate approximation (described by the constraints).

[1] In particular, the construction of set constraints is simpler in the presence of statements that change the flow of control.

In summary for this subsection, set constraints can be constructed to approximate the execution of a program by first introducing set variables to capture the values of the program variables at each program point, and then writing constraints between these set variables to approximate the relationships between these variables that are inherent in the program. In effect, the construction of constraints reduces the problem of analyzing the program to the problem of reasoning about set constraints.

3.1 Imperative Programs

The example imperative programs considered above do not illustrate how conditional statements and recursion are handled, and these are probably the most interesting aspects of the analysis. In particular, recursion introduces the possibility of infinite sets of values. Consider the following program



where a, b, c and d are constants. After execution of this program, X is $cons(c, cons(d, nil))$ and Y is $cons(b, cons(a, nil))$; in other words the program reverses the initial segment of X up until the first occurrence of c. The markers Ⓐ, Ⓑ, Ⓒ and Ⓓ indicate points in the program (note that Ⓓ indicates the point at the end of the program). Corresponding to this program, we can construct the following constraints.

$$
\begin{array}{lcl}
\mathcal{X}^A & \supseteq & cons(a, cons(b, cons(c, cons(d, nil)))) \\
\mathcal{Y}^A & \supseteq & nil \\
\mathcal{X}^B & \supseteq & \mathcal{X}^A \cap \overline{cons(c, \mathsf{T})} \\
\mathcal{Y}^B & \supseteq & \mathcal{Y}^A \\
\mathcal{X}^B & \supseteq & \mathcal{X}^C \cap \overline{cons(c, \mathsf{T})} \\
\mathcal{Y}^B & \supseteq & \mathcal{Y}^C
\end{array}
\qquad
\begin{array}{lcl}
\mathcal{X}^C & \supseteq & cdr(\mathcal{X}^B) \\
\mathcal{Y}^C & \supseteq & cons(car(\mathcal{X}^B), \mathcal{Y}^B) \\
\mathcal{X}^D & \supseteq & \mathcal{X}^A \cap cons(c, \mathsf{T}) \\
\mathcal{Y}^D & \supseteq & \mathcal{Y}^A \\
\mathcal{X}^D & \supseteq & \mathcal{X}^C \cap cons(c, \mathsf{T}) \\
\mathcal{Y}^D & \supseteq & \mathcal{Y}^C
\end{array}
$$

The set expression $\overline{cons(c, \mathsf{T})}$ (the complement of the set denoted by $cons(c, \mathsf{T})$) is the set of all values v such that $car(v)$ differs from c. In general, it is useful to introduce a restricted form of complementation in the constraints used to analyze imperative programs. However, these uses are always sufficiently limited that the constraints obtained are still "monotonic". The minimum solution of the above constraints is given by the following mapping:

$$
\begin{array}{lcl}
\mathcal{X}^A & \mapsto & \{cons(a, cons(b, cons(c, cons(d, nil))))\} \\
\mathcal{Y}^A & \mapsto & \{nil\} \\
\mathcal{X}^B & \mapsto & \{cons(a, cons(b, cons(c, cons(d, nil)))),\ \ cons(b, cons(c, cons(d, nil)))\} \\
\mathcal{Y}^B & \mapsto & list_{a,b} \\
\mathcal{X}^C & \mapsto & \{cons(b, cons(c, cons(d, nil))),\ \ cons(c, cons(d, nil))\} \\
\mathcal{Y}^C & \mapsto & non\text{-}nil\text{-}list_{a,b} \\
\mathcal{X}^D & \mapsto & \{cons(c, cons(d, nil))\} \\
\mathcal{Y}^D & \mapsto & list_{a,b}
\end{array}
$$

where $list_{a,b}$ denotes the set of all lists constructed from a and b, and $non\text{-}nil\text{-}list_{a,b}$ denotes the set of all non-empty lists constructed from a and b.

3.2 Logic Programs

The construction of set constraints for logic programs is similar to that for imperative programs. However, for logic programs, there is a choice for the underlying operational semantics used in the analysis. We begin by illustrating the construction of constraints corresponding to a bottom-up execution. Again we introduce a set variable for each program variable. We also introduce set variables Ret_p, for each predicate p, to collect the set of "return" values for that predicate. Consider the following logic program and constraints constructed to model the bottom-up semantics of the program.

```
p(X) :- q(X), r(X).
q(a).
q(b).
r(b).
r(c).
```

$$
\begin{array}{l}
Ret_p \supseteq p(\mathcal{X}) \\
\quad \mathcal{X} \supseteq q_{(1)}^{-1}(Ret_q) \cap r_{(1)}^{-1}(Ret_r) \\
Ret_q \supseteq q(a) \cup q(b) \\
Ret_r \supseteq r(b) \cup r(c)
\end{array}
$$

The minimum solution of these constraints maps Ret_p into $\{p(b)\}$, maps $\mathcal{X}$ into $\{b\}$, maps Ret_q into $\{q(a), q(b)\}$, and maps Ret_r into $\{r(b), r(c)\}$. Now, consider constructing constraints corresponding to a top-down left-to-right execution of the program starting from the goal `?- p(t)` where t is either a, b, c or d. The main change here is the introduction of set variables $Call_p$, for each predicate p, to collect the set of "calls" to that predicate. The program points Ⓐ, Ⓑ and Ⓒ respectively denote the points just before execution of $q(X)$, just before execution of $r(X)$ and just after execution of $r(X)$.

```
p(X) :- Ⓐ, q(X), Ⓑ, r(X), Ⓒ.
q(a).
q(b).
r(b).
r(c).
```

$$
\begin{array}{l}
Call_p \supseteq p(a \cup b \cup c \cup d) \\
Ret_p \supseteq p(\mathcal{X}^C) \\
\mathcal{X}^A \supseteq p_{(1)}^{-1}(Call_p) \\
\mathcal{X}^B \supseteq p_{(1)}^{-1}(Call_p) \cap q_{(1)}^{-1}(Ret_q) \\
\mathcal{X}^C \supseteq p_{(1)}^{-1}(Call_p) \cap q_{(1)}^{-1}(Ret_q) \cap r_{(1)}^{-1}(Ret_r) \\
Call_q \supseteq q(\mathcal{X}^A) \\
Ret_q \supseteq (q(a) \cup q(b)) \cap Call_q \\
Call_r \supseteq r(\mathcal{X}^B) \\
Ret_r \supseteq (r(b) \cup r(c)) \cap Call_r
\end{array}
$$

The minimum solution of these constraints maps $Call_p$ into $\{p(a), p(b), p(c), p(d)\}$, Ret_p into

$\{p(b)\}$, $\mathcal{X}^A$ into $\{a,b,c,d\}$, $\mathcal{X}^B$ into $\{a,b\}$, $\mathcal{X}^C$ into $\{b\}$, $Call_q$ into $\{q(a),q(b),q(c),q(d)\}$, Ret_q into $\{q(a),q(b)\}$, $Call_r$ into $\{r(a),r(b)\}$, and Ret_r into $\{r(b)\}$. As a third alternative, consider constructing constraints corresponding to a top-down parallel execution of the program starting from the same goals. The program points Ⓐ, Ⓑ and Ⓒ respectively denote the points just before execution of $q(X)$, just before execution of $r(X)$ and just after execution of the entire body of the first rule.

```
p(X) :- Ⓐ, q(X), Ⓑ, r(X), Ⓒ.
q(a).
q(b).
r(b).
r(c).
```

$$
\begin{aligned}
Call_p &\supseteq p(a \cup b \cup c \cup d)\\
Ret_p &\supseteq p(\mathcal{X}^C)\\
\mathcal{X}^A &\supseteq p^{-1}_{(1)}(Call_p)\\
\mathcal{X}^B &\supseteq p^{-1}_{(1)}(Call_p)\\
\mathcal{X}^C &\supseteq p^{-1}_{(1)}(Call_p) \cap q^{-1}_{(1)}(Ret_q) \cap r^{-1}_{(1)}(Ret_r)\\
Call_q &\supseteq q(\mathcal{X}^A)\\
Ret_q &\supseteq (q(a) \cup q(b)) \cap Call_q\\
Call_r &\supseteq r(\mathcal{X}^B)\\
Ret_r &\supseteq (r(b) \cup r(c)) \cap Call_r
\end{aligned}
$$

The minimum solution of these constraints maps $Call_p$ into $\{p(a),p(b),p(c),p(d)\}$, Ret_p into $\{p(b)\}$, $\mathcal{X}^A$ and $\mathcal{X}^B$ into $\{a,b,c,d\}$, $\mathcal{X}^C$ into $\{b\}$, $Call_q$ into $\{q(a),q(b),q(c),q(d)\}$, Ret_q into $\{q(a),q(b)\}$, $Call_r$ into $\{r(a),r(b),r(c),r(d)\}$, and Ret_r into $\{r(b),r(c)\}$.

Observe that in all three examples, the use of set constraints has lead to an exact analysis, and that the sets obtained were finite. Neither observation holds in general, as is illustrated by the following bottom-up analysis example:

```
p(f(X), f(Y)) :- p(X, Y).
p(a,b).
```

$$
\begin{aligned}
Ret_p &\supseteq p(f(\mathcal{X}), f(\mathcal{Y})) \cup p(a,b)\\
\mathcal{X} &\supseteq p^{-1}_{(1)}(Ret_p)\\
\mathcal{Y} &\supseteq p^{-1}_{(2)}(Ret_p)
\end{aligned}
$$

In the least model of the constraints, Ret_p is mapped into the set $\{p(a,b)\} \cup \{p(f^i(a), f^j(b)) : i \geq 1, j \geq 1\}$, and this set contains elements such as $p(f(a), f(f(b)))$ which are not part of the program's (exact) meaning.

So far, we have made no mention of variables that appear in the head of a rule and not in the body of a rule. Such variables can take on any value. Hence they are modeled using the $\top$ constant, as illustrated in the following example.

```
p(X, Y) :- q(X).
q(a).
```

$$
\begin{aligned}
Ret_p &\supseteq p(\mathcal{X}, \mathcal{Y})\\
\mathcal{X} &\supseteq q^{-1}_{(1)}(Ret_q)\\
\mathcal{Y} &\supseteq \top\\
Ret_q &\supseteq q(a)
\end{aligned}
$$

We conclude this discussion of the analysis of logic programs by noting that the accuracy of the information obtained using set constraints can be improved by using more complex set operators. For example, consider the following program and its (bottom-up) set constraints:

```
p(X, Y) :- q(X, Y), r(X, Y).
q(a, b).
q(b, a).
r(a, a).
```

$$
\begin{aligned}
Ret_p &\supseteq p(\mathcal{X}, \mathcal{Y}) \\
\mathcal{X} &\supseteq q_{(1)}^{-1}(Ret_q) \cap r_{(1)}^{-1}(Ret_r) \\
\mathcal{Y} &\supseteq q_{(2)}^{-1}(Ret_q) \cap r_{(2)}^{-1}(Ret_r) \\
Ret_q &\supseteq q(a,b) \cup q(b,a) \\
Ret_r &\supseteq r(a,a)
\end{aligned}
$$

The minimum solution of these constraints maps Ret_p into $p(a,a)$, $\mathcal{X}$ and $\mathcal{Y}$ into $\{a\}$, Ret_q into $\{q(a,b), q(b,a)\}$, and Ret_r into $\{r(a,a)\}$. Another way of constructing constraints is to introduce quantified set expressions, which have the form $\{X : \exists X_1 \ldots \exists X_m (t_1 \in se_1 \wedge \cdots \wedge t_n \in se_n)\}$ where $X, X_1, \ldots, X_m$ are program variables, $t_1, \ldots, t_n$ are atom or terms whose variables are from $X, X_1, \ldots X_m$, and $se_1, \ldots, se_n$ are set expressions. T constraints using quantified set expressions that are constructed for the (bottom-up) analysis of the above program are:

$$
\begin{aligned}
Ret_p &\supseteq p(\mathcal{X}, \mathcal{Y}) \\
\mathcal{X} &\supseteq \left\{X : \exists Y \left(q(X,Y) \in Ret_q \ \wedge \ r(X,Y) \in Ret_r\right)\right\} \\
\mathcal{Y} &\supseteq \left\{Y : \exists X \left(q(X,Y) \in Ret_q \ \wedge \ r(X,Y) \in Ret_r\right)\right\} \\
Ret_q &\supseteq q(a,b) \cup q(b,a) \\
Ret_r &\supseteq r(a,a)
\end{aligned}
$$

and the minimum solution of these constraints maps Ret_p, $\mathcal{X}$ and $\mathcal{Y}$ into the empty set, Ret_q into $\{q(a,b), q(b,a)\}$, and Ret_r into $\{r(a,a)\}$. The more complex constraints using quantified expressions not only provide more accurate program approximation, but they are also more faithful to the notion of set-based analysis. In particular, they have closer and much simpler relationship to the underlying operational semantics (see [12, 16] for further details).

3.3 Functional Programs

To analyze functional languages such as Standard ML [23], set constraints must be extended with a mechanism to deal with higher-order functions. In essence, this is achieved by the addition of three new components. First, the set of underlying values is enriched to include a new collection of constants to denote functions. In the following examples, we shall use function identifiers for this purpose; in more formal presentations, it is convenient to use abstractions in an appropriate lambda calculus. Second, for each function constant f, we introduce two set variables $Call_f$ and Ret_f to capture the values on which f is called, and the values that calls to f return, respectively. Third, a new set operator *apply* is introduced to model function application. The meaning of a set expression $apply(se_1, se_2)$ under a mapping $\mathcal{I}$ is defined as follows:

$$
\mathcal{I}(apply(se_1, se_2)) \stackrel{\text{def}}{=} \bigcup_{f \in \mathcal{I}(se_1)} Ret_f, \quad \text{provided } \mathcal{I}(Call_f) \supseteq \mathcal{I}(se_2) \text{ for all } f \in \mathcal{I}(se_1)
$$

If the side condition is not met then $\mathcal{I}(apply(se_1, se_2))$ is not defined. The notion of solution of a collection of set constraints is appropriately modified so that $\mathcal{I}$ is a solution of the

constraints if it is defined on each set expression and satisfies each constraint. Note that the meaning of this expanded class of set expressions involving *apply* is somewhat unusual, because now set expressions themselves may impose restrictions on solutions, independent of the constraints in which they appear. Importantly, unique minimum solutions are still guaranteed to exist.

To illustrate the construction of set constraints to analyze function programs, consider the following program and its constraints. The set variable $\mathcal{E}$ is introduced to capture the set of values resulting from program evaluation. The minimum solution of the constraints maps $\mathcal{X}$, $Call_{id}$, Ret_{id}, and $\mathcal{E}$ into $\{c\}$.

```
let fun id X = X
in
   id c
end
```

$$\begin{aligned} \mathcal{X} &\supseteq Call_{id} \\ Ret_{id} &\supseteq \mathcal{X} \\ \mathcal{E} &\supseteq apply(id, c) \end{aligned}$$

Again, more complex set operators can be introduced to provide more accurate modeling of certain aspects of the language (particularly case statements). See [12, 13] for further details. The complexity of solving the set constraints is $O(n^3)$ [13]. This basic formulation of constraints has been extended to deal with arrays, continuations and exceptions.

3.4 Comparison with Other Analysis Techniques

A key advantage of set-based analysis (and, more generally, the use of set constraints to perform program analysis), in comparison to standard abstract interpretation techniques [7], is that there is no underlying abstract domain. When using an abstract domain, the requirement of "finite ascending chains" is typically required for termination, and this limits the usable abstract domains. A remedy is to use techniques of "narrowing" and "widening". Even so, termination continues to place a fundamental restriction on the accuracy of the treatment of values. Avoiding the use of abstract domains leads to important advantages in terms of accuracy and uniformity. In particular, set-based analysis does not use "depth-limits" or other *a priori* restrictions on the sets of values that can be manipulated. We contend that this reduces the potential for chaotic and unintuitive behaviour.

Another benefit of the simplicity and uniformity of the approximation embedded in set-based analysis is that the analysis is extensible and flexible. In the course of implementing a number of prototype set-based analysis systems, we have observed that modifications to incorporate new features are often straightforward. For example, during the development of a system for the analysis of ML programs, the treatment of continuations, side-effects and exceptions required only minor modifications. There appear to be two reasons for this. First, because set-based analysis has a simple and intuitive definition, it is usually straightforward to determine how to treat new features. Second, because the analysis has a uniform definition, the treatment of one component of a language is largely independent of the treatment of other aspects of the language, and so the analysis can be extended in a modular manner.

Of course, the main limitation of set-based analysis is that all inter-variable dependencies are ignored. Such dependencies can be crucial for some kinds of analysis such as

mode analysis (see [19] for a discussion of this issue). In contrast, abstract interpretation techniques can retain a limited amount of information about dependencies (although there is, of course, additional computational cost associated with maintaining information about dependencies). Motivated by this observation, hybrid approaches that combine aspects of set constraints with abstract interpretation have been developed [19].

3.5 Efficiency Issues

It is difficult to quantify a comparison between set-based analysis and standard analysis techniques. While worst case complexity costs can be obtained, it is not clear what conclusions we can draw from these results about the practicality of the various approaches. Moreover, the technology for implementing set constraints is still in its infancy. With this in mind, we now briefly describe results from implementations of set-based analysis for two different languages.

The first deals with analysis of logic programs [11], and computes type, mode and sharing information. This analysis has a worst case exponential complexity. While substantial progress was made during the development of this implementation, the results indicate that we are still some distance from practical analysis of medium to large programs. Currently, top-down analysis of programs of the order of 50 rules can be achieved in a few seconds. As expected, analysis based on bottom-up semantics is considerably cheaper that for top-down semantics. One of the main lessons of this implementation is the expense of solving set constraints involving intersection. Much of the work of the implementation was directed at reducing this cost.

The second implementation effort provides a contrasting experience. This implementation [13] focussed on the analysis of ML programs. The core algorithm for this analysis is $0(n^3)$ on the size of the input program. Typical execution times are in the range of 200–400 lines per second for programs up to several thousand lines in length. The main reason for the substantial difference between the results from the two implementations seems to hinge on the fact that intersection is not used in the constraints generated from ML programs. Based on this observation, we are currently investigating ways of constructing constraints for logic programs that provide similar levels of accuracy, but either eliminate or substantially reduce the use of intersection.

The results from the second implementation out-perform current implementations of comparable abstract interpretation based approaches. There appear to be a number of reasons for this. In set-based analysis there is only one pass over the program text. In essence, this performs a "pre-compilation" of the program into a convenient computation form (set constraints). In contrast, many abstract interpretation systems repeatedly pass over (some representation of) program text during the iterative fixed-point computation. In set-based analysis, all approximation is carried out in the translation to set constraints, and so no approximation operations need to be done during the main computational component of the analysis (solving set constraints). Furthermore, set constraints are inherently more incremental than the iterative fixed-point computations of abstract interpretation. In essence, constraints provide a compact implicit representation of information. This representation

supports computation over partial information that is particularly well suited to efficient program analysis. We refer to [15] for a deeper discussion of this issue.

3.6 Extensions

So far we have focussed on the use of set constraints to obtain an approximation of the possible run-time values of variables in a program. However, the basic process of constructing set constraints from a program and then solving these set constraints preserves numerous structural properties of a program. It is therefore possible, with only minor modifications to the set constraint algorithm, to compute approximations to a variety of other program properties. We now illustrate this.

Mode Analysis (for Logic Programs)

To adapt set constraints to compute mode information for logic programs, we first change the underlying set of values from the set of all "ground" terms to the set of all terms. Then we replace the constant $\top$ by two new constants *ground* and *any*, which shall denote the set of all *ground* terms and the set of all terms respectively. Finally, we modify the definition of solutions of set constraints to account for these changes. For example, the minimum solution of $\mathcal{X} \supseteq f(ground, any)$ maps $\mathcal{X}$ into the set of all terms of form $f(t_1, t_2)$ such that t_1 is ground. The minimum solution of $\mathcal{X} \supseteq f(ground, any, a) \cap f(any, ground, any)$ maps $\mathcal{X}$ into the set of terms $f(t_1, t_2, a)$ such that t_1 and t_2 are both ground. The constraints generated for mode analysis are essentially unchanged, excepting that *any* and *ground* may be used to describe the initial goals. The modifications for solving these new constraints involve steps such as simplifying $ground \cap any$ into *ground*, and $f(any) \cap ground$ into $f(ground)$. See [12, 15] for further details. Note that the constants *ground* and *any* behave in essentially the same way as $\top$, and may appear in the output of the algorithm (that is, they may appear in the explicit representations that are computed by the algorithm). For example, when the program

```
app(nil, Y, Y).
app(cons(X', X), Y, cons(X', Z)) :- app(X, Y, Z).
```

is analyzed in the context of the goal ?- app(ground, ground, any), the output of the algorithm relevant to $Call_{app}$ and Ret_{app} is

$$\begin{aligned} Call_{app} &= app(ground, ground, any) \\ Ret_{app} &= app(nil, ground, ground) \cup app(cons(ground, \mathcal{X}), ground, cons(ground, \mathcal{Z})) \\ \mathcal{X} &= nil \cup cons(ground, \mathcal{X}) \\ \mathcal{Z} &= ground \cup cons(ground, \mathcal{Z}) \end{aligned}$$

Structure Sharing Analysis

Structure sharing analysis seeks information of the following form: given two variables, determine whether the bindings of these variables can "share" sub-structures (in the sense that the sub-structures have the same heap location). Such information can be used to

determine when data structures can be updated in place or when they can be garbage collected. This kind of analysis may be performed by first giving each occurrence of a function symbol a unique label. Then set constraints are constructed as before, with care to preserve the labels on function symbols – call the resulting constraints *labeled* set constraints. The meaning of these constraints is defined by mapping set expressions into sets of *labeled* terms. We refer to [12, 15] for further details.

Interpreted Function Symbols

The set constraints considered so far deal with uninterpreted symbols so as to correspond to the data constructors of the language at hand. For analysis of programs involving operations such as arithmetic, this approach must be generalized. One possibility is to compute descriptions of *how* arithmetic values are obtained. These descriptions are essentially terms built from arithmetic operations and integers. For example, the description of computations for a program variable x might be given by

$$\mathcal{X} = 0 \cup (\mathcal{X} + 1)$$

that is, the set of computations $\{0, 0+1, (0+1)+1, \ldots\}$. Clearly, the actual values of x are included in the set $\{0, 1, 2, \ldots\}$. [14] describes how this approach can be applied to the problem of removing array bounds checks, and this requires that the analysis also reason about arithmetic tests. An example of the kinds of descriptions that arise in this context is:

$$\mathcal{X} = 0 \cup [LE\ 10](X + 2)$$

where $[LE\ 10]$ is a "restriction" operator that essentially picks those elements from a set that are less than 10 (in general a restriction operator is of the form $[op\ se]$ where op is some arithmetic comparison operation and se is some set expression). The least model of the above equation maps $\mathcal{X}$ into $\{0, 2, 4, 6, 8\}$.

4 Conclusion

The calculus of set constraints was presented, and its history of basic results and applications briefly described. The approach of set-based analysis was then presented in an informal style, with a focus on the breadth of applicability of the technique. The relationship between set constraints and set-based analysis is roughly that the approximation of a program by ignoring inter-variable dependencies can be captured by set constraints. It was then argued that set-based analysis can provide accurate and efficient program analysis.

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A Substitution Operation for Constraints

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Abstract

In order to reduce the search space in finite constraint satisfaction problems, a number of different preprocessing schemes have been proposed. This paper introduces a 'substitution' operation for constraints. This new operation generalizes both the idea of enforcing consistency and the notion of label substitution introduced by Freuder. We show that the constraints in a problem may be replaced by substitutable subsets in order to simplify the problem without affecting the existence of a solution. Furthermore, we show how substitutability may be established locally, by considering only a subproblem of the complete problem.

1 Introduction

The finite constraint satisfaction problem (or consistent labeling problem) is known to be NP-complete [7]. Such problems may always be solved by an exhaustive search strategy, but this is generally very inefficient.

The search space may be reduced by enforcing some level of 'consistency' [5] in the problem. This involves strengthening the given constraints by disallowing labels or combinations of labels which can be eliminated using other constraints. A number of efficient algorithms have been proposed for achieving various levels of consistency in a given problem [2, 8, 9].

For some applications of constraints, notably problems arising in machine vision [3, 10, 11], it is not necessary to calculate all possible solutions to a given problem, only to determine whether a solution exists, and if so to output a single possible solution. When only a single solution is required it is possible to generalize the notion of enforcing consistency to obtain a more powerful constraint simplification strategy, which will be called 'substitution'. The substitution operation simplifies the given constraints by removing labels or combinations of labels which can be shown to be unnecessary when seeking a single solution.

The idea that one label may be substituted for another in some problems, without affecting the existence of solutions was first proposed by Freuder in [6]. In this paper we generalize this idea to apply to arbitrary sets of labels for arbitrary sets of variables. This opens up a wider range of possible substitutions and allows us to apply substitution operations directly to the constraints in a problem.

The motivation for the work described here is to extend the range of simplification operations which may be applied to constraints, in order to identify more precisely the features of a constraint satisfaction problem which give rise to intractability [4].

2 Definitions

A *finite constraint satisfaction problem* (CSP) [7, 10] consists of a number of *variables* which must be assigned *labels* from associated *domains*, subject to a number of *constraints*. Each constraint specifies

allowed combinations of labels for some subset of the variables, referred to as the *scope* of the constraint.

We now give a formal definition:

Definition 2.1 *A* finite constraint satisfaction problem, $\mathcal{P}$, *consists of a pair* (X, C), *where:*

- X *is a finite set of variables.*
- *Each* $x \in X$ *is associated with a finite set of labels,* $\delta(x)$, *called the* domain *of* x.
- C *is a finite set of constraints.*
- *Each* $c \in C$ *is associated with a subset,* $\Sigma(c)$, *of* X, *called the* scope *of* c.

A mapping t *from* $Y \subseteq X$ *such that* $t(x) \in \delta(x)$, *for all* $x \in Y$ *is called a* labeling *of* Y.

Each constraint $c \in C$ *is a set of labelings of* $\Sigma(c)$.

Definition 2.2 *Let* $\mathcal{P} = (X, C)$ *be a constraint satisfaction problem.*

- *Given any constraint,* $c \in C$, *a labeling* t *of* $\Sigma(c)$ *is said to* "satisfy" c *if and only if* $t \in c$.
- *A labeling* t *of* X *is said to be a* "solution" *to* $\mathcal{P}$ *if and only if for every* $c \in C$, *the restriction of* t *to* $\Sigma(c)$ *satisfies* c.

The set of all solutions to $\mathcal{P}$ *is denoted Sol*$(\mathcal{P})$.

To illustrate these definitions, we now give an example of a specific constraint satisfaction problem which will be used as a running example.

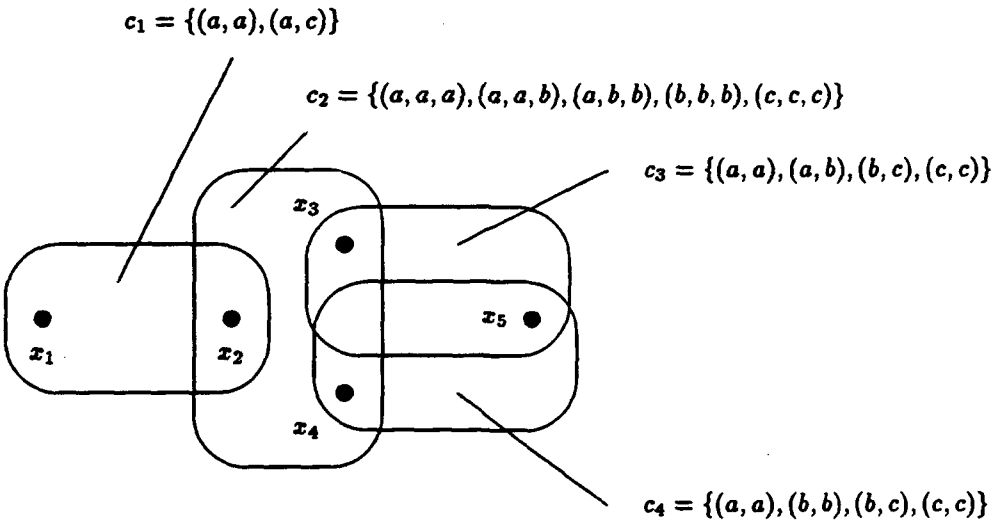

Figure 1: An example of a constraint satisfaction problem

Example 2.3 Let $\mathcal{P} = (X, C)$ be the constraint satisfaction problem illustrated in Figure 1, in which:

- $X = \{x_1, \ldots, x_5\}$
- $\delta(x_i) = \{a, b, c\}$, $i = 1, 2, \ldots, 5$
- $C = \{c_1, \ldots, c_4\}$
- The constraint scopes are as follows:

$$\begin{aligned} \Sigma(c_1) &= \{x_1, x_2\} \\ \Sigma(c_2) &= \{x_2, x_3, x_4\} \\ \Sigma(c_3) &= \{x_3, x_5\} \\ \Sigma(c_4) &= \{x_4, x_5\} \end{aligned}$$

For this problem, a labeling is a mapping from a subset Y of X into the set $\{a, b, c\}$. For instance if $Y = \{x_1, x_4\}$, then the mapping $t : Y \rightarrow \{a, b, c\}$ with $t(x_1) = a$ and $t(x_4) = c$ is a labeling of Y. If we fix a nominal order for the variables in Y, then we can denote a labeling of Y by an n-tuple where n is the size of Y. Using the natural subscript ordering of the variables the labeling t can be written as (a, c).

¿From now on, for convenience, we shall assume that the variables of $\mathcal{P}$ have this natural subscript order. Using the notation just described, we define the constraints of $\mathcal{P}$ to be as follows:

$$\begin{aligned} c_1 &= \{(a,a),(a,c)\} \\ c_2 &= \{(a,a,a),(a,a,b),(a,b,b),(b,b,b),(c,c,c)\} \\ c_3 &= \{(a,a),(a,b),(b,c),(c,c)\} \\ c_4 &= \{(a,a),(b,b),(b,c),(c,c)\} \end{aligned}$$

To complete this example we will compute $\mathrm{Sol}(\mathcal{P})$, the set of all solutions to $\mathcal{P}$. By a simple search we find that it is composed of four elements. As solutions are simply labelings of the complete set of variables X, we can write them as follows:

$$\begin{gathered} (a,a,a,a,a) \\ (a,a,a,b,b) \\ (a,a,b,b,c) \\ (a,c,c,c,c) \end{gathered}$$

□

We will sometimes want to deal with *subproblems* of a given constraint satisfaction problem which arise from considering subsets of the set of constraints. We therefore make the following definition:

Definition 2.4 *Let $\mathcal{P} = (X, C)$ be a constraint satisfaction problem and let D be any subset of C. The* reduced subproblem of $\mathcal{P}$ generated by D *is the constraint satisfaction problem $\mathcal{P}|_D = (X|_D, D)$, where:*

$$X|_D = \bigcup_{c \in D} \Sigma(c)$$

We will make use of the following operations from relational algebra [1]:

Definition 2.5 *Let Y, Z be sets of variables with $Z \subseteq Y$. For any labeling t of Y, the* projection *onto Z of t, denoted $t[Z]$, is the restriction of t to Z. Similarly, for any set S of labelings of Y, the* projection *onto Z of S, denoted $\pi_Z(S)$, is the set $\{t[Z] \mid t \in S\}$.*

Definition 2.6 *Let Y, Z be sets of variables with $Z \subseteq Y$. For any set T of labelings of Z, and any set S of labelings of Y, the* selection *by T from S, denoted $\sigma_T(S)$, is the set $\{t \in S \mid t[Z] \in T\}$.*

3 Substitutability

Freuder [6] defined the concept of substitutability for labels in a CSP as follows: given two possible labels a and b for a variable x, a is *substitutable* for b iff substituting the value a for b at variable x in any solution yields another solution.

We now generalise Freuder's definition to apply to sets of labelings of arbitrary subsets, rather than just individual labels for single variables:

Definition 3.1 *Let $\mathcal{P}$ be a constraint satisfaction problem with variables X, and let R be a subset of X. Given any two sets T_1, T_2 of labelings of R, we say that T_2 is* substitutable *for T_1 in $\mathcal{P}$ if*

$$\pi_{X-R}(\sigma_{T_1}(Sol(\mathcal{P}))) \subseteq \pi_{X-R}(\sigma_{T_2}(Sol(\mathcal{P})))$$

If T_2 is substitutable for T_1 in $\mathcal{P}$, then we will write $T_1 \overset{\mathcal{P}}{\preceq} T_2$.

In other words, given two sets of labelings, T_1 and T_2, for the same variables, we say that T_2 is substitutable for T_1 if the following condition holds: the elements of T_2 may be extended to complete solutions in all the same ways as the elements of T_1.

Note that for any problem $\mathcal{P}$ and any sets of labelings, T_1, T_2, we have

$$T_1 \subseteq T_2 \Rightarrow T_1 \overset{\mathcal{P}}{\preceq} T_2.$$

The following example illustrates the definition:

Example 3.2 Consider the constraint satisfaction problem $\mathcal{P}$ in Example 2.3. None of the possible labels for any of the individual variables is substitutable for any other in this example, according to Freuder's original notion of substitutability.

However, using Definition 3.1 and the list of solutions given in Example 2.3, we can show that the set of labelings $\{(a,a,a),(c,c,c)\}$ for the variables x_3, x_4 and x_5 is substitutable in $\mathcal{P}$ for $\{(a,b,b)\}$, i.e $\{(a,b,b)\} \overset{\mathcal{P}}{\preceq} \{(a,a,a),(c,c,c)\}$. □

The next lemma indicates that a constraint in a constraint satisfaction problem may always be replaced by a substitutable set of labelings without eliminating all of the solutions:

Lemma 3.3 *Let $\mathcal{P} = (X, C)$ be a constraint satisfaction problem . If we replace any constraint $c \in C$ by a new constraint c' with the same scope, such that $c \overset{\mathcal{P}}{\preceq} c'$, then we obtain a new constraint satisfaction problem $\mathcal{P}'$ such that*

$$Sol(\mathcal{P}') = \emptyset \Longrightarrow Sol(\mathcal{P}) = \emptyset$$

Proof: Note that $\text{Sol}(\mathcal{P}) = \sigma_c(\text{Sol}(P))$ and $\text{Sol}(\mathcal{P}') = \sigma_{c'}(\text{Sol}(P))$. Hence, if $\text{Sol}(\mathcal{P}) \neq \emptyset$ then $\sigma_c(\text{Sol}(\mathcal{P})) \neq \emptyset$, so if $c \overset{\mathcal{P}}{\preceq} c'$, then by Definition 3.1 we have $\sigma_{c'}(\text{Sol}(\mathcal{P})) \neq \emptyset$, hence $\text{Sol}(\mathcal{P}') \neq \emptyset$, and the result follows. ∎

For the special case of substitutable *subsets* of a given constraint, Lemma 3.3 has the following important corollary:

Corollary 3.4 *Any constraint in a constraint satisfaction problem may be replaced by a substitutable subset without affecting the existence of solutions.*

Furthermore, in this case, the solutions to the new problem will simply be a subset of the solutions to the original problem.

Replacing a constraint with a substitutable subset will be called a 'substitution' operation. The following example illustrates how this substitution operation may be used to tighten the constraints in a constraint satisfaction problem.

Example 3.5 Reconsider the constraint satisfaction problem $\mathcal{P}$ defined in Example 2.3.
No proper subset of c_1 is substitutable for c_1 in $\mathcal{P}$,
The following proper subsets are substitutable for c_2 in $\mathcal{P}$:

$$\{(a,a,a),(a,a,b),(b,b,b),(c,c,c)\}$$
$$\{(a,a,a),(a,a,b),(b,b,b),(a,b,b)\}$$
$$\{(a,a,a),(a,a,b),(a,b,b),(c,c,c)\}$$
$$\{(a,a,a),(a,a,b),(a,b,b),\}$$
$$\{(a,a,a),(a,a,b),(c,c,c)\}$$

The following proper subsets are substitutable for c_3 in $\mathcal{P}$:

$$\{(a,a),(b,c),(c,c)\}$$
$$\{(a,a),(a,b),(c,c)\}$$

The following proper subsets are substitutable for c_4 in $\mathcal{P}$:

$$\{(b,b),(b,c),(c,c)\}$$
$$\{(a,a),(b,c),(c,c)\}$$

□

Definition 3.1 implies that if a set of labelings T_1 contains any labeling t which cannot be extended to a solution of $\mathcal{P}$, then $T_1 \overset{\mathcal{P}}{\preceq} (T_1 - t)$. This gives us the following result:

Proposition 3.6 *Any tuple which may be eliminated from a constraint in a constraint satisfaction problem by enforcing consistency may be removed by a substitution operation.*

This means that the substitution operation is a true generalization of the notion of enforcing consistency.

Calculating the smallest substitutable subset of a constraint is as difficult as solving the original problem. However, the next result shows that it is sufficient to establish substitutability within certain subproblems.

Definition 3.7 *Let $\mathcal{P} = (X, C)$ be a constraint satisfaction problem.*
For any $c \in C$ define the closure *of c, $\bar{c}$, as follows:*

$$\bar{c} = \{c' \in C \mid \Sigma(c') \cap \Sigma(c) \neq \emptyset\}$$

Lemma 3.8 *Let $\mathcal{P} = (X, C)$ be a constraint satisfaction problem.*
For any $c \in C$ and any set c' of labelings of $\Sigma(c)$, we have

$$c \overset{\mathcal{P}|_{\bar{c}}}{\preceq} c' \Rightarrow c \overset{\mathcal{P}}{\preceq} c'$$

Proof: Assume that $c \overset{\mathcal{P}}{\not\preceq} c'$. By Definition 3.1, this means that

$$\pi_{X-\Sigma(c)}(\sigma_c(\mathrm{Sol}(\mathcal{P}))) \not\subseteq \pi_{X-\Sigma(c)}(\sigma_{c'}(\mathrm{Sol}(\mathcal{P})))$$

Hence, there is some $s \in \mathrm{Sol}(\mathcal{P})$ such that the restriction of s to $X - \Sigma(c)$ is not compatible with any element of c'. In other words, any labeling s' of X which satisfies c' and agrees with s on $X - \Sigma(c)$ must fail to satisfy some constraint in C.

By construction, s' satisfies c' and all elements of $C - \bar{c}$, so s' must fail to satisfy some element of $\bar{c} - c'$. Hence $c \overset{\mathcal{P}|_{\bar{c}}}{\not\preceq} c'$. ∎

Any labelling which is substitutable for a constraint c in $\mathcal{P}|_{\bar{c}}$ will be said to be 'locally' substitutable for c. Combining Lemma 3.8 with Corollary 3.4 shows that we may replace any constraint c in a constraint

satisfaction problem $\mathcal{P}$ by a locally substitutable subset without affecting the existence of a solution. For many problems $\mathcal{P}$, local substitutability may be calculated much more efficiently than substitutability in $\mathcal{P}$, since it requires solutions to be calculated only for the subproblems generated by the constraint closures.

However, local substitutability is not implied by (global) substitutability, so using local substitutability is not guaranteed to find all possible constraint substitutions, as the following example shows:

Example 3.9 Reconsider the constraint satisfaction problem $\mathcal{P}$ defined in Example 2.3. The set $c_3' = \{(a,a),(b,c),(c,c)\}$ is substitutable in $\mathcal{P}$ for c_3 (Example 3.5).

However, if we consider the subproblem $\mathcal{P}|_{c_3}$, we find that $\mathrm{Sol}(\mathcal{P}|_{c_3})$ contains the element (b,b,b,c) so $c_3 \overset{\mathcal{P}|_{c_3}}{\preceq} c_3'$. □

4 Propagation of Substitution

Substitution operations may be propagated to obtain further reductions in the constraints, as the following example indicates. Note that in this example the use of substitution operations and propagation is sufficient to obtain a complete solution to the problem.

Example 4.1 Reconsider the constraint satisfaction problem $\mathcal{P}$ defined in Example 2.3. It was shown in Example 3.2 that the set of labelings

$$c_2' = \{(a,a,a),(a,a,b),(a,b,b)\}$$

is substitutable in $\mathcal{P}$ for c_2 (it is also locally substitutable).

If we replace c_2 with c_2' then we obtain a new constraint satisfaction problem $\mathcal{P}'$, and now we find that $c_1' = \{(a,a)\}$ is substitutable for c_1 in $\mathcal{P}'$.

If we replace c_1 with c_1' then we obtain a new constraint satisfaction problem $\mathcal{P}''$, and we find that $c_3' = \{(a,a),(a,b)\}$ is substitutable for c_3 in $\mathcal{P}''$.

If we replace c_3 with c_3' then we obtain a new constraint satisfaction problem $\mathcal{P}'''$, and we find that $c_4' = \{(a,a)\}$ is substitutable for c_4 in $\mathcal{P}'''$.

Finally, if we replace c_4 with c_4' then we obtain a new constraint satisfaction problem with only a single solution, (a,a,a,a,a). Further substitution operations may therefore be carried out on all of the constraints to reduce them to a single element, which is the projection of this solution. □

As with the various methods for enforcing different levels of consistency, it is possible to organise the propagation of substitution operations according to a number of different schemes. One naive algorithm for repeatedly applying local substitutability and propagating the results is as follows:

Algorithm 4.2

```
Repeat
        For each constraint c
            For each t ∈ c
                    If c ⪯^{P|_c} c − {t} then set c = c − {t}
Until no further changes to constraints.
```

The complexity of this algorithm depends on the maximum size of a constraint closure, say k, and the maximum number of labelings permitted by a constraint, say m. The main repeat loop may be executed at most $m|C|$ times, since at least one element is removed from a constraint on each iteration. The complexity of checking for substitutability for each constraint element is $O(m^k)$, since each possible extension must be checked against each other element of c. Hence the overall complexity is $O(|C|^2 m^{k+2})$.

However, unlike operations which simply enforce consistency, the repeated application of substitution operations until no more substitution is possible does not always give an invariant result. More surprisingly, the *number of solutions* to the resulting problem is not always invariant either, as the following example shows:

Example 4.3 Reconsider the constraint satisfaction problem $\mathcal{P}$ defined in Example 2.3. It was shown in Example 3.2 that the set of labelings

$$c_2' = \{(a,a,a),(a,a,b),(c,c,c)\}$$

is substitutable in $\mathcal{P}$ for c_2 (note that this is a different substitutable set to the one considered in Example 4.1).

If we replace c_2 with c_2' then we obtain a new constraint satisfaction problem $\mathcal{P}'$, and now we find that $c_4' = \{(a,a),(c,c)\}$ is substitutable for c_4 in $\mathcal{P}'$.

If we replace c_4 with c_4' then we obtain a new constraint satisfaction problem $\mathcal{P}''$, and we find that $c_3' = \{(a,a),(c,c)\}$ is substitutable for c_3 in $\mathcal{P}''$.

If we replace c_3 with c_3' then we obtain a new constraint satisfaction problem $\mathcal{P}'''$, and we find that $c_2'' = \{(a,a,a),(c,c,c)\}$ is substitutable for c_2' in $\mathcal{P}'''$.

Finally, if we replace c_2' with c_2'' then we obtain a new constraint satisfaction problem with two solutions, (a,a,a,a,a) and (a,c,c,c,c). This constraint satisfaction problem cannot be further reduced using substitution operations. □

The implication of this lack of invariance is that some sequences of substitution operations may be much more effective than others in reducing the search space. It is an open question whether an efficient algorithm exists for choosing the most effective sequence of substitution operations, although we strongly suspect that this problem is as difficult as solving the original problem.

5 Conclusion

We have presented a substitution operation which is a true generalization of Freuder's notion of label substitution, and also generalizes all forms of consistency enforcement.

Although this substitution operation is most useful when searching for a single solution, it may also be useful in the case where we want to find all solutions. In such cases, it can reduce search time by showing more quickly that a branch of the search tree leads to no solutions. Substitution operations may be worth applying to any constraint satisfaction problem that has a high probability of having no solutions.

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Contradicting Conventional Wisdom in Constraint Satisfaction

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Abstract. Constraint satisfaction problems have wide application in artificial intelligence. They involve finding values for problem variables where the values must be consistent in that they satisfy restrictions on which combinations of values are allowed. Two standard techniques used in solving such problems are backtrack search and consistency inference. Conventional wisdom in the constraint satisfaction community suggests: 1) using consistency inference as preprocessing before search to prune values from consideration reduces subsequent search effort and 2) using consistency inference during search to prune values from consideration is best done at the limited level embodied in the forward checking algorithm. We present evidence contradicting both pieces of conventional wisdom, and suggesting renewed consideration of an approach which fully maintains arc consistency during backtrack search.

1 INTRODUCTION

Constraint satisfaction problems (*CSPs*) involve finding values for problem variables subject to constraints that are restrictions on which combinations of values are allowed [15]. They have many applications in artificial intelligence. (We restrict our attention here to binary CSPs, where the constraints involve two variables.)

The basic solution method is *backtrack search*. Often *consistency inference* (constraint propagation) techniques are used to prune values before or during search. The basic pruning technique involves establishing or restoring some form of *arc consistency*. If a value v for a variable V is not consistent with any value for some other variable U, then v is *arc inconsistent* and can be removed. *Full arc consistency* is achieved when all arc inconsistent values are removed.

One of the most successful forms of backtrack search has proven to be *forward checking* [8]. Forward checking combines backtrack search with a limited form of arc consistency maintenance. Some values are removed that become inconsistent when the problem is modified by the choices made during the search process.

This paper provides strong experimental evidence contradicting two well-established pieces of conventional wisdom in the CSP community:

- Conventional CSP wisdom says that using consistency inference in a preprocessing step, to prune values before search, will reduce the subsequent search effort. There has been some question as to the degree of consistency preprocessing that is desirable - additional preprocessing effort may outweigh subsequent search savings [2]. However, it seems an obvious article of faith that removing values from consideration during a preprocessing step will lead to savings during the subsequent search step - or at the very least do no harm. We demonstrate that there are circumstances in which pruning values by consistency preprocessing can in fact greatly *increase* subsequent search effort.
- Conventional CSP wisdom says that using consistency inference during search, to prune values that become inconsistent after making search choices, is best limited to the minimal inference embodied in the forward checking algorithm. The feeling is that additional search savings produced by pruning more values will be offset by the additional inference cost. We show that maintaining *full* arc consistency during search is often in fact very cost effective.

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To contradict the first piece of conventional wisdom we tested the effects of arc consistency preprocessing on one of the most popular and successful CSP algorithms: forward checking combined with dynamic domain size variable ordering. (Dynamic domain size variable ordering prefers to consider variables that have fewer values left to choose from. It is a popular ordering heuristic. In a probabilistic analysis, it was shown optimal under certain assumptions by Haralick and Elliott [8]. It has proven particularly useful in conjunction with forward checking search, and we believe it to be effective on our test problems.)

Another counterintuitive demonstration that pruning values can increase search effort, was obtained recently by Prosser. He showed that pruning values can degrade performance for algorithms that employ "intelligent backtracking" (though the actual exhibited effects were small) [14]. However, even Prosser concluded that "We should now assume that increased consistency, or the removal of redundancies, can only guarantee a reduction in search effort if that search is unintelligent (such as a chronological backtracker)."

Forward checking is a chronological backtracker. However, we found that removing values by arc consistency preprocessing made some problems an order of magnitude more difficult to solve by our ordered forward checking search. (In fairness to Prosser though one might argue that "unintelligent" should rule out dynamic search ordering.) Note we are not merely saying that the effort to do the preprocessing plus the effort to do the subsequent search was an order of magnitude greater than the effort to do the search without preprocessing. We are saying that even if you ignore preprocessing effort, searching the preprocessed problems, which had fewer values, was still an order of magnitude harder than searching the original problems.

The explanation for this counterintuitive phenomenon is that arc consistency preprocessing is counterproductive when it interferes with the functioning of the search ordering heuristic. We interpret our results as implying that eliminating values can move a problem far enough away from the assumptions needed to demonstrate the "optimality" of dynamic domain size search ordering that the advantage of having fewer values is more than offset by the deterioration of the ordering heuristic's performance.

We believe this experience is a useful object lesson in the need to exercise some care in combining CSP methods: two rights may make a wrong. This lesson is particularly relevant now as new constraint programming environments are making it easier to combine techniques for customized algorithms.

To contradict the second piece of conventional wisdom we compared ordered forward checking with an algorithm that established and maintained full arc consistency. These two algorithms represent extreme points on a spectrum of algorithms that maintain various amounts of arc consistency during search.

The conventional wisdom expressed to us by some members of the constraint programming community already runs counter to the second piece of CSP conventional wisdom. Our experiments suggest that the constraint programming community has been conventionally wiser in this regard than the CSP community.

The combination of consistency pruning with backtrack search has a long history [5], [7], [10]. Various degrees of consistency processing interleaved with backtrack search were studied experimentally in [8], [11], [13]. A variety of algorithms were considered that alternate choosing a value for a variable with "looking ahead", via a constraint propagation process, to infer the consequences of that choice for pruning the values available for the as yet uninstantiated variables. The algorithms differed in how much constraint propagation they performed, and thus in the degree of arc consistency they achieved.

Forward checking is an algorithm which does a minimal amount of constraint propagation, in the sense that it performs the minimal amount of lookahead needed to avoid having to "look back", i.e. to avoid the need to check new choices against previous ones. In experimental studies forward checking repeatedly proved superior to algorithms interleaving more constraint propagation.

Of course, the limitations of these experiments were recognized. However, the repeated success of forward checking began to bias the conventional wisdom in the CSP community in the direction of "less is more". For example, in a recent survey of CSP algorithms [9], the section on "How Much Constraint Propagation Is Useful?" concludes: "Experiments by other researchers [in addition to Nadel] with a variety of problems also indicate that it is better to apply constraint propagation

only in a limited form".

Earlier studies were limited, however, in several key ways:

- Many of the experiments were limited to special case problems, especially the Queens problem, a problem in which constraints exist between all possible pairs of values.
- Random problem experiments were conducted before the recent understanding that most random problems appear in fact to be easy problems.
- Small sample sets decreased the likelihood of encountering difficult problems.
- The AC-4 approach to arc consistency [12], which is particularly well suited to consistency maintenance, was not employed.
- The implementation of consistency maintenance may have been less than optimal. In order to maintain arc consistency one does not need to restart an arc consistency algorithm from scratch each time backtrack search chooses a value; one only needs to propagate the effects of the removal of the unchosen values.

In our laboratory several studies began to suggest that "more could be more". Gevecker studied full arc consistency maintenance [6] and Freuder and Wallace studied a range of hybrid algorithms based on a notion of "selective" or "bounded" constraint propagation [4]. However, these results were still limited in their understanding of the random problem space. Also, they did not employ the powerful search ordering scheme we alluded to above.

We conduct here experiments on random problems, focusing on the "hard problem ridge" identified in recent studies of "really hard" random problems [1], [17]. Problems that contradict the conventional CSP wisdom appear to be pervasive, and orders of magnitude effects are found.

There are, of course, significant caveats to these experimental results. In particular, problems of different structure or size may behave differently. We assume individual constraint checks can be efficiently computed. (Each time we ask if a value v for a variable X and a value u for a variable Y satisfy the constraint between X and Y we are performing a *constraint check*.) If this were not the case, maintaining full arc consistency could conceivably require some very expensive constraint check computation that backtracking or forward checking avoided. While we test two extremes of arc consistency processing, optimality may lie between these extremes.

Nevertheless, the performance exhibited here for a variety of difficult random problems suggests that establishing full arc consistency and maintaining it during search may often be more efficient than limiting inconsistency removal to the partial arc consistency maintenance provided by forward checking. Significantly, the full arc consistency approach was particularly effective for "really hard" problems. As a result the full arc consistency algorithm was rather stable in comparison to forward checking: it was a bit more costly for some very easy problems, but remained relatively efficient on problems where the difficulty encountered by forward checking shot way up.

Section 2 describes the algorithms we compared. Section 3 describes our experimental objectives and how we generated test problems. Sections 4 and 5 present the experimental results and our summary observations. Section 6 is a brief conclusion.

2 ALGORITHMS

Forward checking, which we implement here in an algorithm FC, combines backtrack search with a very limited form of arc consistency maintenance. The main idea is to project forward the consequences of variable assignments during search. When a variable X is assigned a value, v, from *domain(X)*, the set of available values for X, v is checked against the domains of each variable Y that is as yet unassigned and for which there is a constraint between X and Y. All values inconsistent with v are removed. This way a limited form of arc consistency is maintained. (If, during this process, the domain of some variable becomes empty, then no complete extension of the current assignment set to a solution is possible, and the current assignment for X must be discarded.) For details on forward checking consult [8].

We describe next an algorithm that combines backtrack search with full arc consistency maintenance. We call the algorithm *MAC* for *Maintaining Arc Consistency*. The algorithm is a combination of old ideas, which we give a new name because the combination is unique and the name is evocative. However, it is essentially a modern version of Gaschnig's CS2 [5].

We describe MAC in some detail, in order to make this paper more self-contained, and to clarify precisely how we combined search with constraint propagation, for anyone wishing to replicate or extend our experiments.

MAC uses the same basic framework as forward checking, alternating search and consistency inference steps, but differs conceptually in two aspects:

- The constraint network is made arc consistent initially.
- When during the search a new variable X is instantiated to a value v, all the other values in the domain are eliminated and the effects of removing them are propagated through the constraint network as necessary to restore full arc consistency.

As underlined in [12], arc consistency is based on the notion of *support*. Let v be a value in domain of X. Value v has support as long as for each of the variables Y for which there is a constraint between X and Y, there is at least one value u such that the pair *(v, u)* satisfies that constraint. Once there exists a variable for which no remaining value is consistent with v, then v must be eliminated from the domain of X.

The algorithm proposed in [12], known as AC-4, keeps track of this support explicitly, by maintaining a counter for each arc-value pair, $Counter[(X_i, X_j), a]$, representing the number of values in the domain of X_j supporting (X_i, a), the value a for X_i. Whenever the counter for some assignment becomes 0, that domain value has to be eliminated.

To make this work efficiently, AC-4 keeps track of which values support which other values. For each value b in the domain of X_j a set $S_{X_j,b} = \{(X_i, a) |\ (X_j, b) supports (X_i, a)\}$ is constructed. Then, if value b is eliminated from domain of X_j, $Counter[(X_i, X_j), a]$ must be decremented for each (i, a) in $S_{X_j,b}$. Two additional data structures are used by AC-4 besides those already mentioned. The table $Marked[X_i, b] = 1$ if b has been eliminated from the domain of X_i. The list *Agenda* maintains all pairs *(X_i, b)*, where value b has been deleted from the domain of X_i but the effects of the deletion have not yet been propagated. The process of propagating the effects of deletions is guided by the list, which specifies which deletion to process next.

Basically, MAC uses the same data structures as AC-4 and consists of three main components:

- initialization: construct and initialize the support counters
- propagation: prune the inconsistent domain elements and propagate arc-consistency through the constraint graph
- search:

The algorithm is presented in Figure 1.

We also combined arc consistency and search in a simpler manner than that embodied by FC and MAC. A single preprocessing pass to achieve some form of consistency has often been used before some form of subsequent search. Waltz's well-known scene labeling experiments [16] are an early example of the success of this basic approach. We will refer to the combination of arc consistency preprocessing followed by forward checking search as AC-FC. (The AC algorithm employed in AC-FC is also AC-4-based, though not identical in implementation to the arc consistency processing employed by MAC.)

The order in which variables are considered for instantiation during search has been found to be extremely important. The simple heuristic that chooses a variable with minimal domain size to process next can be very effective when used with forward checking. We employ this heuristic here for FC, MAC and AC-FC. As we always use this heuristic (except when we explicitly test the effect of eliminating it) we will not bother to repeatedly refer to "ordered FC" etc., but the ordering should be kept in mind.

3 EXPERIMENTAL DESIGN

Our objective was to address the following questions:

- Is the conventional wisdom sometimes wrong?
- Can it be "very wrong"? By orders of magnitude?
- How are the results that run counter to conventional wisdom distributed in random problem space? Where do they occur, how often and at what magnitude?

```
procedure INITIALIZE( variables ) is:
  for each variable X_i ∈ variables do
    for a ∈ domain(X_i) do
      Marked[X_i, a] ← 0
      Supported[X_i, a] ← φ
  Agenda ← φ
  for each constraint (X_i, X_j) ∈ the constraint graph do
    for a ∈ domain(X_i) do
      total ← 0
      for b ∈ domain(X_j) do
        if (a, b) satisfies constraint (X_i, X_j) then
          total ← total + 1
          Supported[X_j, b] ← Supported[X_j, b] ⋃ (X_i, a)
      if total = 0 then
        Marked[X_i, a] ← 1
        Agenda ← Agenda ⋃ (X_i, a)
        if Marked[X_i, b] = 1, ∀ b ∈ domain(X_i) then
          return FAILURE
      else
        Counter[(X_i, X_j), a] ← total
  return SUCCESS

procedure PROPAGATE( Agenda ) is:
  while Agenda ≠ φ do
    select and remove (X_j, b) from Agenda
    for (X_i, a) ∈ Supported[X_j, b] do
      Counter[(X_i, X_j), a] ← Counter[(X_i, X_j), a] - 1
      if Counter[(X_i, X_j), a] = 0 and Marked[X_i, a] = 0
then
        Agenda ← Agenda ⋃ (X_i, a)
        Marked[X_i, a] ← 1
        if Marked[X_i, c] = 1, ∀ c ∈ domain(X_i) then
          return FAILURE
  return SUCCESS

procedure SEARCH( variables, solution ) is:
  if variables = φ then
    report solution
    return SUCCESS
  Xi ← select one variable ∈ variables
  if Marked[X_i, a] = 1, ∀ a ∈ domain(X_i) then
    return FAILURE
  a ← next unmarked value ∈ domain(X_i)
  save values of Marked and Counter data structures
  for b ∈ domain(X_i) \{a} and Marked[X_i, b] = 0
    Agenda ← Agenda ⋃ (X_i, b)
    Marked[X_i, b] ← 1
  if PROPAGATE( Agenda ) = SUCCESS and
    SEARCH( variables \{X_i}, solution ⋃ (X_i, a) ) =
      SUCCESS then
    return SUCCESS
  restore values of Marked and Counter data structures
  Agenda ← (X_i, a)
  Marked[X_i, a] ← 1
  return PROPAGATE( Agenda ) and
        SEARCH( variables, solution )

algorithm MAC( variables ) is:
  if INITIALIZE( variables ) = FAILURE then
    return FAILURE
  return SEARCH( variables, φ)
```

Figure 1. MAC Algorithm

- How do these results relate to problem difficulty?
- Are these results significant for "really hard" problems?

We performed tests with FC, MAC and AC-FC to address these questions. We addressed the problem of finding a single solution to a CSP (or determining that no solution exists).

The test problems are random binary CSPs: each constraint is a relation involving two variables. They are generated according to a (constant) probability of inclusion model, which we will describe briefly.

A problem is generated given several parameters, whose meaning we will explain:

- the number of variables
- the number of values for a variable (initially the same number for each variable)
- the expected constraint density
- the expected constraint tightness

One way to represent binary constraints is with *constraint graphs*, vertices corresponding to variables and edges to constraints. Since we want to deal only with connected constraint graphs (connected components of unconnected graphs can be solved independently), the number of edges for a graph with N vertices is at least N-1 (for a tree) and at most $\frac{N(N-1)}{2}$ (for a complete graph). As a consequence, we define constraint *density* as the fraction of the possible constraints, beyond the minimum N-1, that the problem has. For example, a CSP with a tree structured constraint graph has a constraint density of 0 and a CSP with a complete constraint graph, containing all possible edges, has a constraint density of 1. In the general case, the number of edges for a CSP with a constraint graph with N vertices and a constraint density of D (a number between 0 and 1) is $N - 1 + D(\frac{N(N-1)}{2} - (N - 1))$.

Constraint *tightness* is defined as the fraction of all possible pairs of values from the domains of two variables, that are not allowed by the constraint. For example, if the constraint between

two variables with domains $\{a, b\}$ and $\{c, d\}$ does not allow the pairs (a, c) and (a, d) and (b, c), then the constraint tightness is .75.

Basically, in our problems a specific constraint is present, or a specific pair of values is permitted by a constraint, with a probability based on the expected density and tightness specified for the problem. This problem generation method permits some variation in actual values for the density and tightness compared with the expected ones. Averaged over many constraints we expect the actual values to be close to the expected values, but it should be noted, in particular, that the tightness of an individual constraint within a problem can vary.

We do not allow problems to contain any null constraints (that do not allow any pair of values, and make the problem trivially unsolvable) or any trivial "constraints" (that allow all pairs of values, and are not usually represented by an edge in the constraint graph). We insure that constraint graphs are connected by initially randomly generating a tree of constraints.

The main experiments reported below used problems with 50 variables, each having a domain of 8 values. There is nothing magic about these numbers; we simply wanted problems of a size large enough to permit us to exhibit significant savings and small enough so that they would not require great amounts of processing time. We experimented some with different size problems, but a more systematic study is left for future work.

Based on recent research on really hard problems we expected to find that many random problems are easy, but that if we hold one of either tightness or density fixed, and vary the other sufficiently, that we will encounter a complexity "peak". Together these peaks form a complexity "ridge" in "tightness/density" space. We were particularly interested in performance on this ridge.

When we compared FC with AC-FC we used constraint checks as our measure of effort. Since constraint checks are not an appropriate measure for MAC (the only constraint checks are done during the initializing phase) we used CPU time to measure its performance and to compare it with FC and AC-FC.

4 ON PRUNING CONSIDERED HARMFUL

The data reported here are for problem parameter values chosen to exhibit the phenomenon dramatically. Our intuition, which requires further exploration, is that the phenomenon is more likely to occur at low densities and near, but not at, peak difficulty areas.

We used problems with 50 variables and an initial domain size of 8 values for each variable. For each of four density values .06, .07, .08 and .09 five random problems were generated with a tightness of .50. We measured constraint check effort for AC preprocessing, for FC search after AC preprocessing and for FC search without AC preprocessing. We also measured CPU time for AC-FC, FC and MAC. Table 1 present the results.

Our main observations:

- AC-FC performed worse than FC on average for some problem sets and an order of magnitude worse on some problems. Pruning the search tree by eliminating some domain values can sometimes greatly increase subsequent search effort for the popular combination of forward checking and dynamic variable ordering based on minimal domain size.
- FC was sometimes superior to AC-FC because the preprocessing effort for the AC phase was larger than any possible savings in the FC phase, indeed larger than the entire FC effort with or without AC preprocessing.
- More significantly the FC search effort itself, after preprocessing, was sometimes much greater than the FC search effort without preprocessing.
- MAC, which employs the more extensive full arc consistency maintenance, was superior to both FC and AC-FC except on some very simple problems. (We will have further data on the comparison of MAC and FC in the next section.) Since MAC incorporates an AC preprocessing, we have a situation where adding some additional consistency processing, in the form of AC preprocessing alone, can decrease performance, but adding even more consistency processing, in the form of AC preprocessing plus full AC maintenance, can help.

In order to verify that it is indeed the ordering that is at issue, we took some other easy problems where AC-FC was inferior to FC, and ran AC-FC and FC on them without any variable

Table 1. Performance of AC-FC, FC, MAC

			density			
			.06	.07	.08	.09
#1	AC	P	15,370	16,352	18,465	16,640
	+	S	40,506	160,695	15,006	331,319
	FC	T	55,876	177,047	33,471	347,959
	FC		1,078,271	1,421,487	79,245	7,998
#2	AC	P	13,968	14,840	19,080	19,220
	+	S	847	119,886	150,656	18,287
	FC	T	14,815	134,726	169,736	37,507
	FC		1,083	282,143	56,551	8,921
#3	AC	P	15,526	15,664	16,971	21,899
	+	S	14,470	1,815,053	173,303	14,471
	FC	T	29,996	1,830,717	190,274	36,370
	FC		10,006	1,574,023	14,659	8,396
#4	AC	P	15,816	15,712	19,722	18,193
	+	S	1,461,280	1,118,572	13,519	5,477
	FC	T	1,477,096	1,134,284	33,241	23,670
	FC		1,302,927	474,484	4,993	444,553
#5	AC	P	14,752	14,104	18,146	17,857
	+	S	6,412	765	87,451	67,056
	FC	T	21,164	14,869	105,597	84,913
	FC		2,349	910	50,137	100,312

P–preprocessing(AC) S–search(FC) T–total(AC–FC)

a) Performance of AC-FC, FC, expressed in terms of constraint checks.

		density			
		.06	.07	.08	.09
#1	AC-FC	10	32	6	65
	FC	202	264	14	2
	MAC	1	2	1	3
#2	AC-FC	2	25	30	6
	FC	0	51	10	2
	MAC	1	2	3	1
#3	AC-FC	5	336	33	6
	FC	2	289	3	2
	MAC	1	4	3	1
#4	AC-FC	273	216	5	3
	FC	241	89	1	8
	MAC	5	7	1	1
#5	AC-FC	4	2	19	14
	FC	1	0	9	18
	MAC	1	1	2	2

b) Performance of AC-FC, FC, MAC, expressed in terms of total CPU time.

ordering heuristic (in fact lexical ordering). Without the ordering heuristic the phenomenon of preprocessing making matters worse did indeed disappear. (Without the ordering, however, performance was much worse than either FC or AC-FC with the ordering.)

5 MORE IS MORE

Again we used problems with 50 variables, each having a domain of 8 values. For this experiment, however, we used more combinations of density and tightness values to provide broad coverage of the "density/tightness space". For each combination of density and tightness values, we generated ten random problems, for a total of 1,200 problems.

Figures 2a - 2e present the performance of FC and MAC, as average values over the ten problems generated for each pair (density, tightness). We used five values for the tightness parameter: .150, .325, .500, .675 and .850. For tightnesses .150, .325, .850, 20 equally distanced density values were taken throughout the entire range [0.05, 1]. For tightness .500, 20 equally distanced density values were taken throughout the entire range [0.015, 0.965]. For tightness .675, 40 equally distanced density values were taken throughout the entire range [0, 0.975]. The performance is expressed as seconds of CPU time (on a SUN 4) necessary either to find a solution or to discover that there is none.

Just viewing averages can be misleading. For example, one problem in a set of ten can be so much harder than the others that it dominates the result. For each tightness value (except .850 for which all the problems were very easy to solve), Table 2 presents data on the ten individual problems in the problem set with the highest average difficulty.

Our main observations:

- Overall, establishing and maintaining full arc consistency during search was more efficient than limiting inconsistency removal to the partial arc consistency maintenance embodied in forward checking. MAC performed better than FC throughout the density/tightness space, except on some very easy problems.
- MAC was often at least an order of magnitude better than FC on the complexity peaks.
- The advantage of MAC along the complexity ridge exhibited in Table 2 increased as we moved toward the less dense, more tightly constrained end.

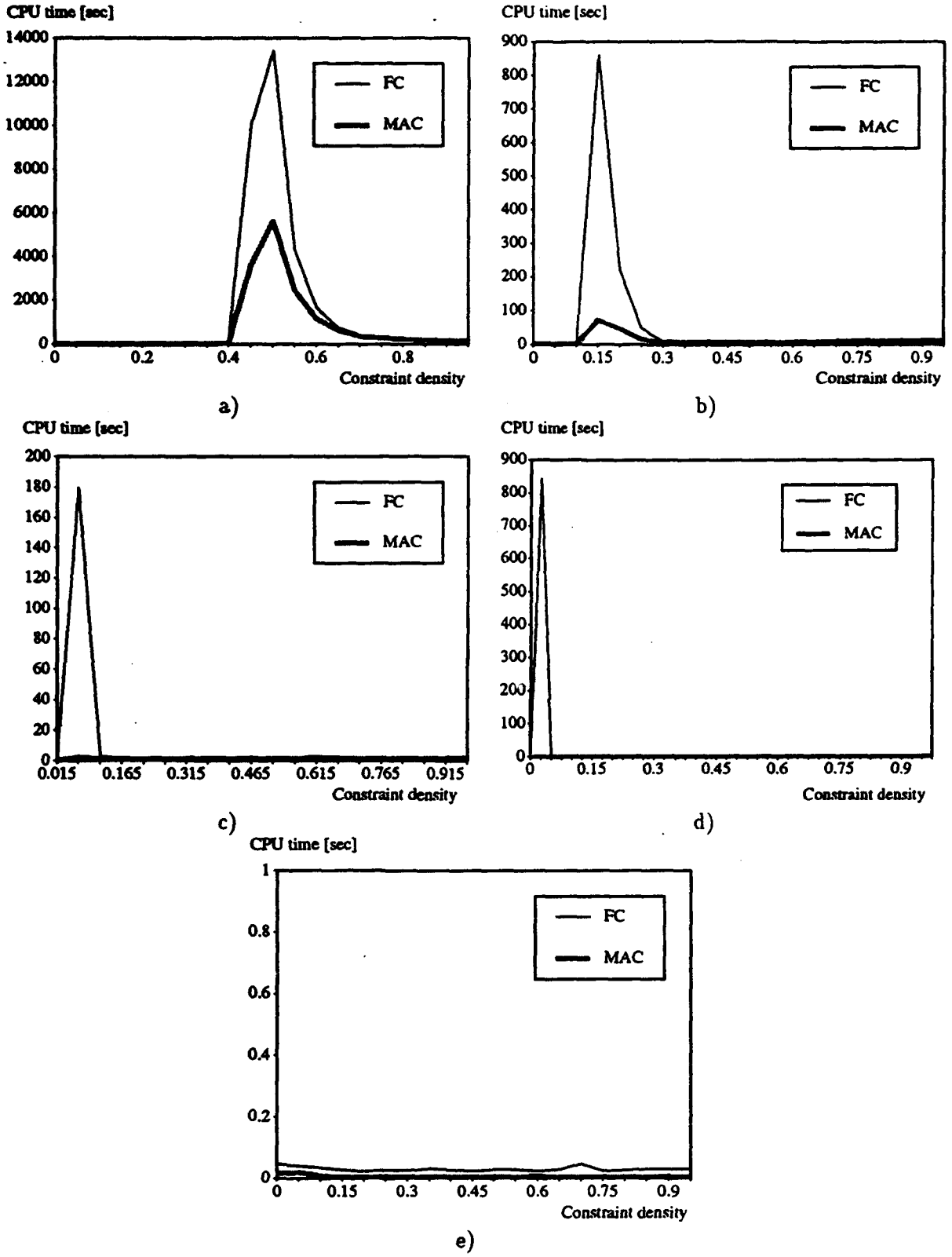

Figure 2. Comparison FC-MAC on random problems with tightness .150 (a), .325 (b), .500 (c), .675 (d) and .850 (e), using CPU time to measure the performance

Table 2. Performance of FC, MAC, expressed in terms of total CPU time

	tightness	.150	.325	.500	.675
	density	.500	.150	.065	.018
#1	FC	19,545	115.5	30.0	45.2
	MAC	8,018	5.3	1.4	0.1
#2	FC	19,135	5.3	5.9	4.6
	MAC	8,217	1.0	3.3	0.1
#3	FC	2,881	7,346.2	95.7	1.4
	MAC	1,365	611.5	1.3	0.1
#4	FC	15,288	2.0	86.0	0.2
	MAC	6,213	2.8	0.5	0.1
#5	FC	16,281	750.2	6.5	28.6
	MAC	6,854	42.1	1.0	0.1
#6	FC	1,795	59.0	11.6	8356.0
	MAC	773	1.8	1.2	0.1
#7	FC	13,496	257.1	1,425.3	0.6
	MAC	5,871	32.4	0.6	0.1
#8	FC	25,541	40.6	0.3	0.8
	MAC	10,264	1.9	0.3	0.1
#9	FC	14,490	41.0	110.0	6.0
	MAC	5,849	6.7	7.8	0.1
#10	FC	5,915	3.4	29.4	0.8
	MAC	2,902	0.7	5.5	0.1

6 CONCLUSION

We have demonstrated that preprocessing to prune values can counterintuitively increase search effort under the right circumstances. We have demonstrated that more arc consistency processing than embodied in forward checking can reduce search effort unexpectedly often.

"Two rights can make a wrong" and "a little knowledge can be a dangerous thing". Arc consistency preprocessing before ordered forward checking search can degrade performance significantly; however, when, in addition, arc consistency is fully maintained during search, performance can be enhanced significantly.

The performance, in our experiments, of an algorithm that operates by maintaining full arc consistency throughout backtrack search suggests that it be reconsidered by the CSP community as an alternative to algorithms that only obtain partial or temporary arc consistency. This algorithm seems especially worth considering for situations where difficult, as opposed to merely large, problems may be encountered with some frequency. Indeed, for these problems we speculate that it may prove profitable to reexamine next the utility of maintaining even higher levels of consistency [3].

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No-good backmarking with min-conflict repair in constraint satisfaction and optimization

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Abstract

There are generally three approaches to constraint satisfaction and optimization: domain-filtering, tree-search labelling and solution repair. The main attractions of repair-based algorithms over domain-filtering and/or tree-search algorithms seem to be their *scalability, reactivity* and *applicability* to optimization problems. The main detraction of the repair-based algorithms appear to be their failure to *guarantee* optimality. In this paper, a repair-based algorithm, that guarantees to find an optimal solution if one exists, is presented. The search space of the algorithm is controlled by *no-good backmarking*, a learning process that records generic patterns of no-good *partial* labels[3] in order to stop the repeated traversing of those failed paths of a search tree. Unlike some similar repair-based methods which usually work on *complete* (but possibly inconsistent) labels, the proposed algorithm works on partial (possibly inconsistent) labels by repairing those variables that contribute to the violation of constraints in the spirit of *dependency-directed backjumping*. In addition, the algorithm *will* accept a repair if it can minimise the conflicts of a label even if it does *not* eliminate them. To control the space of no-good patterns, we propose to generate the most generic no-good pattern as early as possible. To support dynamic constraint satisfaction, we introduce several strategies to maintain no-good patterns on the tradeoffs between space, efficiency and overheads. In particular, through the comparisons with other works, we suggest possible strategies to improve the proposed method.

Keywords : Constraint Satisfaction and Optimization, Backmarking, Learning, Backjumping, Repair-based Methods, Simulated Annealing, Tabu Search, No-good recording and No-good Justification.

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1 Introduction

The importance of constraint satisfaction and optimization is well-recognized [Fox & Sadeh 93]. Scheduling is perhaps the most characteristic real-world applications of this field of research [Atabakhsh 91]. A *constraint problem* can be specified as consisting of an (possibly empty) *objective function* and a set of constraints on n variables (X_1, .., X_n) each of which can be assigned a value from its associated domain (D_1, .., D_n). A *complete* (cf. *partial*) label for a constraint problem is simply an assignment of a value for every (cf. some) variable from its associated domain. A *consistent* label is a label that satisfies all the constraints. *Labelling* is the process of finding a consistent label for

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[3]That is a partial assignment to variables

a constraint problem. A *solution* label is a complete and consistent label. An *optimal* label is a solution label that optimizes the objective function. *Constraint satisfaction* problems aim to find a solution label; while *constraint optimization problems* try to look for an optimal label. The *objective constraint* of a constraint problem is the constraint associated with the objective function. It is a *soft* constraint in the sense that the value of the function is *not* necessarily fixed and is intended to be optimized.

Many different techniques have been developed over the years [Nadel 89]. They have all proved empiracally and theoretically successful for many applications although it is a commonly acknowledged fact that *no* single technique is universally good for all the constraint problems. Despite the diversity of these techniques and their hybrid nature, it may be possible to classify them among three classes: *domain-filtering*, *tree-search* and *repair-based techniques*.

Domain-filtering techniques seek to filter out elements of the domains of variables that do not participate in any solutions of a constraint problem. They are generally *incomplete* in the sense that not all such elements are filtered out. For efficiency reasons, local consistency domain-filterings [Mackworth 77] such as arc-consistency and path-consistency are usually adopted. Domain-filtering techniques however do not produce a solution for a constraint problem. Even if it is complete, not every combination of the filtered domains of the variables is necessarily a solution. Labelling is thus usually performed at a separate stage that involves some tree-search techniques [Nadel 89].

Tree-search techniques follow the paths of a search tree in some *regular* fashion by constructing and extending *partially consistent* labels [Freuder & Wallace 92]. They usually work in a backtracking fashion together with *backmarking* and/or *backjumping*. Backmarking marks the combination of values that have been proven to be satisfiable or unsatisfibale in order to reduce the redundant (*thrashing*) and normally expensive constraint checks. Backjumping performs *dependency-directed backtracking* to the highest point of a search tree that contributes to the current failure in order to prune the search paths.

Repair-based techniques usually work on *complete* but possibly inconsistent labels by repairing them gradually towards a correct or optimal solution. It is naturally extensible to *reactive scheduling* [Minton et al 92] since it always repairs on a complete label or schedule. It is also easily extensible to *optimization* because the repair process is usually based on some estimated cost. Notable representatives of repair-based techniques are *hill-climbing*, *simulated annealling* and *genetic algorithms*. A key issue here is to avoid the trapping of a *local minimum* in a repair process. Hill-climing generally involves repairing variables in conflict in such a fahsion as to minimize the conflicts or to reduce the cost [Minton et al 92]. Simulated annealling [Kirkpatrick et al 83] on the other hand provides a temperature control to enable the repair method to jump out of a local minimum by allowing the possibility of a locally repaired label with a higher cost. To avoid looping in the repair process, *tabu search* [Hertz & de Werra 87] keeps track of a buffer of forbidden moves between *complete* labels. To be on the safe side, genetic algorithms [Schraudolph 91] maintain a pool of potentially "healthy" and *complete* labels which can be jointly (via a *cross-over* operation) or individually (via a *mutation* operation) repaired.

Unlike tree-search techniques which usually have a complete search space, repair-based techniques do not normally enjoy this luxury. This is of course *double-edged*. On one hand, the repair techniques are often *easily* jumping around the search space and greedily expect to find an *approximately* optimal solution in a fairly quick time for some large constraint problems[4]. On the other hand, the repair-based techniques regretably do not guarantee to find the optimal solution of a constraint problem.

The purpose of this paper is to present a repair-based technique (*NG-Backmarking*) that com-

[4]This is one reason why repair-based techniques are generally regarded as scalable.

bines the advantages of the three classes of techniques mentioned above. This technique performs an *indirect* domain filtering by no-good backmarking - a process that records the most generic partial labels that are known to have violated some constraints of a constraint problem. The technique can also accommodate a domain-filtering technique both in a preprocessing phase and in the labelling phase of a repair process.

Like tree-search techniques, the proposed technique also incorporates a backjumping strategy so that only culprit variables that contribute to the violation of some constraints are being repaired. However, unlike tree-search techniques, the new technique neither backtracks nor backjumps nor backmarks along a fixed *regular search structure* (eg. chronological backtracking or dependency-directed backtracking). Rather it jumps about the search space and prunes where it can using no-good backmarking that records learnt information about "bad" partial labels. In particular, the technique attempts to produce the most generic no-good patterns as early as possible in order to reduce both the space and time overheads of the no-good patterns. Unlike a simple learning strategy (e.g. Dechter 90]), the techqnique is also equiped with dynamic support of no-good patterns to deal with constraint maintenance. We propose several strategies to control such dynamic support.

Unlike the *no-good* justification method [Maruyma et al 91, 92] that inspired the proposed technique here, the NG-Backmarking technique *can* accept a local repair if it minimises the conflicts (or the cost) of a label even if the resultant label does *not* eliminate all the conflicts. However unlike similar repair-based methods (eg. *min-conflict* Hill-climbing repair) which usually work on *complete* (but possibly inconsistent) labels, the NG-Backmarking technique can repair *partial* and *inconsistent* labels. In particular, the no-good backmarking process can be seen as a generalization of the *tabu* search in *simulated annealing* as it can forbid moves between *partial* (not just complete) labels. Finally it will be noted that the new technique can be incorporated with some *genetic algorithms* as the no-good backmarking process maintains a pool of generic partial labels that require repairs. These partial labels can be jointly repaired or individually repaired.

This paper is organized as follows. In Section 2, two constraint problems are defined. They will be used to illustrate the ideas of the proposed method later on. In Section 3, the NG-Backmarking technique is presented. In Section 4, the control of the backmarking process in the NG-Backmarking technique is discussed. In Section 5, we propose several strategies to deal with the maintanance of no-good patterns in dynamic constraint satisfaction. In Section 6, we provide our experimental results and analysis of the technique. In Section 7, comparisons and contrasts with other well-known techniques are made and some hybrid modifications that might enhance the NG-Backmarking technique are suggested.

2 Travelling Salesman Problem and Capital Budget Problem

In this section, we define two constraint problems; the Travelling Salesman Problem (TSP) and the Capital Budget Problem [Taha 92]. These problems are chosen purely for illustrative purpose. They are not intended for real-world applications to be repreresentative of the kind that is best solved by the proposed NG-Backmarking technique.

Problem 1 (n-city TSP) ***The n-city TSP problem is to construct a least costly tour visiting each city exactly once in a n-city map.***

For reasons of clarity, we index the cities in a n-city TSP by numbers ranging from 0 to n-1 and we specify each tour/label of a n-city TSP by a sequence of n numbers ranging from 0 to n-1. For example, 01123 is a label (in this case, an inconsistent one) of a 5-city TSP. From a constraint satisfaction point of view, the variables are then the first city to be travelled to, the second city to be travelled to,.., and the nth city to be travelled to. If a variable is undefined in a partial label, we will use U to indicate its status, eg. 00U12.

Problem 2 (Capital Budget) *Five projects are being considered for execution over the next 3 years. The expected returns for each project and the annual expenditure (in £K) are tabulated below. The problem seeks to decide which of the five projects should be executed over the 3-year planning period. In this regard, the problem reduces to a "yes-no" decision for each project. We formalize the decision problem by treating each project as a variable whose domain is {0,1} where the value 0 represents "no" and the value 1 represents "yes".*

	Expenditures			
Project	Year 1	Year 2	Year 3	Returns
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Available funds	25	25	25	

The constraint satisfaction and optimization specification then become

$$\text{maximize } z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

subject to the following resource constraints where $x_i \in \{0,1\}$ for $i = 1,2,..,5$.

$$\begin{array}{rcrcrcrcrcl} 5x_1 & + & 4x_2 & + & 3x_3 & + & 7x_4 & + & 8x_5 & \leq & 25 \\ x_1 & + & 7x_2 & + & 9x_3 & + & 4x_4 & + & 6x_5 & \leq & 25 \\ 8x_1 & + & 10x_2 & + & 2x_3 & + & x_4 & + & 10x_5 & \leq & 25 \end{array}$$

For simplicity, we represent a label for the capital budget problem as a sequence of integers in the set {0,1}, eg. 01011.

3 No-good backmarking with min-conflict repair

NG-Backmarking is a complete repair-based method that works on partial and possibly inconsistent labels. Its architecture is based on the no-good justification algorithm which involves assigning and designing variables of a partial label until a complete and consistent label is generated. However instead of using a dynamically evolving and rather costly set of justifications (or constraints) as in the no-good justification approach, the new algorithm works on a *fixed* set of initial constraints together with a *dynamically generated but simple* set of no-good patterns. *These patterns are partial labels (generated by no-good backmarking) to indicate that these partial labels violate some constraints and should be repaired.* They are used to prune the search space of a constraint problem.

In the No-good justification approach, the set of no-good justifications and the size of each no-good justification can grow combinatorially large. This point is supported by our implementation of the algorithm applied to the TSP problem and the Capital Budget problem. Since checking a no-good justification can be an expensive operation, the size of the no-good justification set has a significant impact on the performance of the algorithm. Although it is possible to remove some of them because they are subsumed by others, subsumption check can be very inefficient and the set can still be rather large at some intermediate stages of the algorithm.

In contrast, while the set of no-good patterns can still grow combinatorily large, the size of each no-good pattern stays the same (if not smaller). In addition, the no-good patterns are simple to check and the subsumption check is also a straightforward operation. For example, given {01UU2, 01UU4} in the database of no-good patterns , if a new no-good pattern 01UUU is generated, the subsumption check will remove {01UU2, 01UU4} before inserting 01UUU into the database. It is

important to note that no-good patterns are *not* permutations of all the labels that violate some constraints. They are generic patterns that correspond to the most general partial labels that so far violate some constraints. The objective is to reduce both the spatial and temporal overheads of the no-good patterns in the spirit of the no-good justifications in ATMS [de Kleer 90] where the minimum partial labels that violate some constraints are created.

Although it is claimed in [Maruyama et al 91] that no-good justifications provide more generic constraints than no-good patterns in some cases, the extra efforts in checking the satisfiability of these justifications appear to far outweigh their advantage of generality. For example, when a partial label L is firstly known to be no good, L will be generated as a no-good pattern. So if the label L pops up again in later repairs, it will be immediately eliminated by the no-good pattern for L. On the other hand, the no-good justification approach will generate a no-good justification J for L when it is firstly detected to be no-good. So if L pops up again in later repairs, the no-good justification approach still has to *re-evaluate* J which can be a rather lengthy conjunction of several previously generated no-good justifications. Even if one such lengthy no-good justification may also prune some other labels, we still have to search through this possibly large set of no-good justifications and evaluate every one of them. In contrast, the NG-Backmarking approach maintains a static set of constraints. If several other labels are also meant to be pruned by one lengthy no-good justification, it will be picked out in the NG-Backmarking approach by checking this fixed set of constraints.

Unlike the no-good justification approach which randomly chooses any defined variable to repair, the NG-Backmarking algorithm only randomly repairs a defined variable that contributes to the violation of some constraints in the spirit of dependency-directed backjumping. For example, to repair the tour 01123 in a 5-city TSP problem, the proposed algorithm will choose either the 2nd or 3rd variable to repair. In the case of choosing the second variable, it can assign the value 4 to the variable if it does not violate any constraints and does not match any no-good patterns.

In addition, the NG-Backmarking algorithm can be regarded as a genuine repair algorithm. Instead of looking for a value that makes the locally repaired label to satisfy all the constraints, the proposed algorithm simply chooses an alternative value that minimizes the number of constraint violations and the resultant label does not match any no-good patterns. For example, to repair the tour 0112233 in a 7-city TSP problem, if the second variable is chosen to repair, we can assign a new value, say 4. Although this value still does not eliminate all the constraint violations, it is the best possible repair. This approach of repairing (in the spirit of min-conflict repair [Minton et al 92]) is very useful for reactive scheduling applications where the change of a schedule is required to be as minimum as possible to the original schedule when some new circumstances arise.

Although randomness is often a virtue in constraint solving as evidenced by some simulated annealing applications [Kirkpatrick et al 83], it is still commonly recognized that constrained heuristics [Fox & Sadeh 89] can greatly improve the performance of many real-world applications. This point is also noted in Zweben et al's anytime scheduling algorithm [90] where a heuristically controlled simulated annealing is shown to be more effective. For these reasons, we have not indicated in the following specification of the proposed algorithm the particular selection strategies of defined variables, undefined variables and domain values. Since we are trying to provide the principles in this paper, we have therefore not elaborated any particular heuristics here which are application dependendent anyway. To name a few, we can choose the defined variable to be the most conjested in an application or the one that most likely to reduce the cost of the objective function. The point to note however is that the NG-Backmarking algorithm is *complete* whatever selection strategies are adopted.

Under a heuristically controlled search strategy, it is not strictly true any more that the proposed

method randomly jumps about the search space. Still the method does not follow any *regular* tree structure in a search process. The selections of variables and domain values are normally dynamically changing as well. So effectively, we still move around the search space fairly opportunistically and prune search space where we can.

Definition 1 (No-good backmarking with min-conflict repair) Given a partial label (which can be complete or inconsistent) and an initial bound on the objective function.

1. check if there is any no-good pattern that matches the label. If there is, go to the Repair Process in 4;
2. else check if there is any constraint violated by the label. If there is, generate a no-good pattern for the label and go to the Repair Process in 4; else go to the Labelling Process in 3.
3. Labelling Process
   - If there is no undefined variable, the current label is a solution and go to the Optimization Process in 5;
   - Else select an undefined variable and check if it is possible to assign a value to it that satisfies all the constraints and the resultant label does not match any no-good patterns. If it is, choose such a value and go back to the Labelling Process in 3; Else go to the Repair Process in 4.
4. Repair Process
   - If there is no defined variable left, the algorithm terminates with no solution.
   - Else select a defined variable that contributes to the violation of some constraints[5] and check if it is possible to assign an alternative value that reduces the number (or cost) of constraint violations and the resultant label does not match any of the no-good patterns.
     - If it is possible, choose such a value that minimizes the number (or cost) of constraint violations.
       If the value can eliminate all the conflicts, go to the Labelling Process in 3; else generate a no-good pattern for the label and go back to the Repair Process in 4;
     - Else make the variable undefined and generate a no-good pattern for the resultant label; go back to the Repair Process in 4.
5. Optimization Process
   - If optimization is not required, then terminate with the current label as a solution.
   - Else calculate the new cost of the objective function against the current label and reset the bound of the constraint on the objective function to the new cost; generate a no-good pattern for the current label and go to the Repair Process in 4.

The major advantages of the no-good backmarking algorithm are

1. it can repair a partial (including complete) label that is inconsistent.
2. it randomly jumps around the search space in assigning and deassigning values of variables. The choice of variable for repair is essentially random and so is not confined to chronological backtracking or dependency-directed backtracking along some regular search structure.
3. it is *complete* in satisfaction and optimization while the search space is controlled by no-good patterns. It is guaranteed to find an optimal solution (if one exists) for a constraint problem.

[5] A variable can contribute the violation of some constraints in two aspects. One is that the variable is assigned in a no-good pattern that matches the current label. The other is that the variable participates in the violation of some constraints.

4. No-good patterns are simple to generate and to check. Subsumption check of no-good patterns are also easy to perform.

Theorem 1 *The algorithm is sound and complete for finite domain constraint satisfaction problems (CSP). That is, for any finite domain CSP, every complete label that is generated on the termination of the algorithm is a solution label of the CSP and the algorithm will find a solution label if one exists for the CSP.*

The algorithm however does not find all solutions of a constraint problem. It can be easily amended by iteratively backmarking every current solution label to be a no-good pattern. This will trigger the algorithm to find alternative solutions until no more solution is found.

Theorem 2 *The algorithm is sound and complete for finite domain constraint optimization problems (COP). That is, for any COP, the last complete label that is generated on the termination of the algorithm is an optimal solution of the COP and the algorithm will find an optimal solution if one exists for the COP.*

4 Controlling the generation of no-good patterns

As noted in the last section, *no-good backmarking* simply records partial labels that violate some constraints. Despite their simplicity to check, it is still essential to maintain only the most generic no-good patterns. This raises the question of *subsumption check*. Even if we allow subsumption check, it is still important to generate more generic no-good patterns as early as possible in order to avoid the accumulation of the database of no-good patterns during the intermediate repair process of the proposed method. As we have experienced from our constraint logic programming implementation, a large set of no-good patterns can significantly hinder the performance of the NG-Backmarking method.

First lets address the subsumption check. Every time a partial label is found to be no-good, we first perform a retract operation from the no-good database of those patterns that unify with the partial label whose undefined values are viewed as "don't care" variables. We then simply insert the partial label where an undefined value is replaced by a "don't care" variable. To check if a partial label matches a no-good pattern, we simply treat the partial label as a goal against the no-good database. Note here that the undefined value U in the partial label is *not* treated as a variable in the goal; otherwise a partial label, say, 1UU1U will match a no-good pattern such as 11111.

To generate more generic no-good patterns as early on as possible, the generation process need be related to specific applications. Here we use the TSP and Capital Budget problems to illustrate the point.

Consider for example a solution tour 02341 in a 5-city TSP problem where the cost of the tour is C. Suppose we try to find the optimal solution, then the bound associated with the constraint on the objective function will be reset to C. The solution label 02341 is no longer a good tour. Normally we will simply add 02341 as a no-good pattern and repair the label. However for the TSP problem, we can generate n no-good generic patterns with one value of the last solution label to be undefined. For the above example, we would immediately generate five generic no-good patterns {U2341, 0U341, 02U41, 023U1, 0234U } since there is no alternative value for repair for any variable given that the other variables' values remain the same. In particular, we can choose any one of the partial labels to proceed repairing.

Consider another example in the Capital Budget Problem where a partial label L = 10UUU is to be repaired. Suppose previously, we have already found a solution label 01111 with the benefit of the objective function as 95. Normally, we would simiply continue to label L. However for this particular problem, we can immediately treat L as a no-good pattern since whatever values we assign to the undefined variables, the objective function of the resultant label cannot be more than 95, ie. the objective constraint is always violated.

The generation of generic no-good patterns is also dependent on the kind of constraint being violated in a specific application. Consider the Capital Budget problem again. Given an initial label 11001, although the objective constraint is not violated, a resource constraint (the third one) is violated. Instead of simply generating 11001 as a no-good pattern, we produce the more generic label 11UU1 as a no-good pattern since the only alternative values (ie. 1) to repair the variables with 0 values would only increase the violation of the label. Consider another label 00111. This is a solution label, but not an optimal one. During the optimizatin process, although this label satisfies all the resource constraints, it no longer satisfies the objective constraint with the current cost. Instead of making the label to be no-good, we generate the more generic no-good pattern 00UUU since whatever values chosen for U will not improve the benefit of the objective function. To summerize, in the Capital Budget problem, if a label violates a resource constraint, we make all the 0 values in the label to be undefined and then generate the resultant label as a no-good pattern; if a label violates an objective constraint, we make the 1 values in the label to be undefined and then generate the resultant label as a no-good pattern. This shows that the generation of no-good patterns can be controlled by exploring the characteristics of the problem.

5 Dynamic Support of no-good patterns

No-good patterns are knowledge learnt during a search process. They can be used in subsequent search and new constraint problems. In a dynamic environment, such knowledge are still valid when new constraints (eg. new jobs, new machine restrictions) are added or old constraints (eg. dealine is put forward) are tightened. This is because we do not create good patterns. However, they may no longer be valid when some old constraints (eg. cancellations of jobs, deadline is delayed) are removed (eg. cancellations of jobs) or relaxed (eg. deadline is delayed).

To incorporate constraint relaxation and removal, we propose to support every no-good pattern by the set of minimum set of constraints that violates the pattern. Subsumption check in this case will also involve checking if the supporting set of set of constraints is also subsumed. In this way, if a constraint is removed, then any set in the supporting set of a no-good pattern that contains the constraint is removed from the support set. If a constraint is relaxed, then any set in the supporting set of a no-good pattern that contains the constraint relaxed will be rechecked by the no-good pattern. If the no-good pattern is no longer supported by the relaxed set, relaxed set is removed from the support set. If the support set is empty, then the no-good pattern is removed.

Like the generation of no-good patterns, uncontrolled generation of the supporting set of a no-good pattern can also be very costly. To control this problem, we suggest several strategies to approximate the supporting set. One is to build the supporting set by the total number of constraints that the no-good pattern is involved with. If any constraint is removed or relaxed from the set, the no-good pattern will be withdrawn. The other strategy is to simply remove a pattern if a constraint that involves the variables of the pattern is removed/relaxed. The advantage of this strategy is that it does not require any space to store the supporting set of a no-good pattern. Subsumption check is just like that in a static environment. Both strategies do not affect the completeness of the no-good backmarking method, but may remove some learnt knowledge (or

no-good patterns) even though the constraint relaxation or removal does not affect the knowledge. However this trade-off between the overheads of maintaining no-good patterns and the loss of no-good patterns is often necessary in practice.

6 Implementation and Experimentation

The proposed technique has been implemented in Eclipse - a constraint logic programming language that extends the CHIP developed at ECRC. We have tested the technique against the TSP and the Capital Budget problem. For these problems, the performance of our no-good backmarking methods compares about 100 times faster than the no-good justification approach. When compared with a pure simulated annealling implementation of TSP, the method also fares much better for 10 city TSP problem. For larger TSPs, it is observed that the method performs still better than simulated annealling if we compare the times of the two methods in obtaining the best cost of the simulated annealling method. This is partly expected as the no-good backmarking method also incorporates an element of randomness. Since the NG-Backmarking method adopts a min-max strategy in optimization, it particularly performs better if the initial cost of a problem is set low.

The current implementation did not explore any constraint handling primitives in Eclipse at the moment and it essentially just runs the Prolog part of the Eclipse language. This is satisfactory for the problems we have tested. However we are now looking at the well-known 10-job and 10-machine scheduling problem [Jiang et al 94] and the British Airways flight allocation problem [Lever & Richards 94]. Our previous experience in these problems indicate that dynamic domain-filtering can play an important role in improving the performance of these problems. It is our intention to explore the full-power of Eclipse, using constraint handling mechanisms within the overall architecture proposed in this paper.

Our experiments have also confirmed the overheads of no-good patterns when the database of such patterns grows very big. Even with subsumption check, the random search strategy can still take quite a long time (and hence lead to possible a huge space) to eventually perform a subsumption reduction. To solve this problem, we integrate random search with regular search. This has proved to be extremely effective with impressive speed improvement of the order of one magnititude. By analysing the results, we discover that when using the random search strategy first, since the repair method jumps about randomly, it will quite quickly settle for a reasonably good solution before it is overwhelmed by the overheads of the size of the database of no-good patterns. By then, if we switch to a regular search strategy, we immediately see substantial amount of subsumption reductions which results a more directed search space towards the optimal solution.

We are currently implementing the dynamic support of no-good patterns. Eclipse is particularly suitable for this task as it can easily find out what constraints are violated by a no-good pattern and what variables are involved in this constraints. We hope to report some of these results at the presentation.

7 Comparisons and Hybrid algorithms

The generation of no-good patterns corresponds to Dechter's learning idea [90] in the search process. Whilst the support set of a no-good pattern is similar to Schiex & Verfaillie's [93] no-good recording algorithm for dynamic constraint satisfaction. These algorithms however did not address the problem of repairing a label which is additionally pursued in the paper. The no-good backmarking method can be seen as an integration of the no-good justification search architecture with

backjumping, no-good pattern learning, dynamic maintenance and min-conflict repair. Recently, it has come to our attention that the no-good justification search architecture with backjumping is similar to Ginsberg's dynamic backtracking search architecture [Ginsberg 93] although he did not address dynamic support for no-good patterns.

Traditional backmarking usually makes good and no-good patterns following a tree structure. NG-Backmarking on the other hand randomly moves about the search space while generating only no-good patterns. There is no need to generate good-patterns because they may not be good any more when tigher bounds are set for the objective function in the optimization process. However it is possible to generate good patterns for each current objective function and remove them when the current objective function is reset to a new bound. Of course, we can apply the concept of a supporting set to good patterns as well, but the overheads are not worthwhile.

Domain filtering methods remove inconsistent domain values during propagation. They are usually incomplete in the sense that not every value that does not contribute to a solution of the constraint problem is eliminated. Even if they are complete under some restricted domain, labelling still has to be done in a separate stage. No-good backmarking on the other hand perform labelling as well as filtering. Here filtering however does not eliminate values from the domain rather they are indirectly eliminated by no-good patterns. Admittedly, such an indirect filtering can be very ineffective especially with large domains and densely connected variables in a constraint system. However the great strength of the algorithm is that it is repair-based and hence can be easily applied to reactive applications such as scheduling.

Nevertheless the proposed NG-Backmarking algorithm is not orthognal to a domain-filtering method. In fact, such a filtering method can be applied as a preprocessing phase to reduce the domains of the variables in the constraint problem. This will certainly improve the efficiency of the algorithm as its performance is significantly dependent on the size of the domains of the variables. In particular, in the optimization process of the algorithm, since the soft constraint on the objective funtion is effectively turned into a hard constraint each time a new solution is found or a new bound is set, using a domain-filtering technique can continously or iteratively reduce the domains of variables in the constraint problem.

We can even perform domain-filtering for each partial labels in the labelling process provided we can maintain the previous domains when we are repairing the labels. This idea is particularly good for the air flight allocation problem [Lever & Richards 94] we have looked at. There a constraint logic programming implmentation [Van Hentenryck et al 92] is applied which provides the natural backtracking or maintenance of previous domains for you.

Traditional repair methods usually work on complete (but possibly inconsistent) labels, the propose NG-Backmarking method however can work on partial (but still possibly inconsistent) labels. It is also worth noting that no-good backmarking subsumes *tabu* search. While tabu search only forbid moves from one complete label to another, no-good backmarking can *additionally* forbid moves from one partial label to another.

Traditional repair methods often do not guarantee to find the optimal solution although some of them can avoid local minimum. The proposed NG-Backmarking method guarantees to find the optimal solution. The search space in this case is controlled by "jumping" about the search tree where any part of the tree that leads to no-good patterns are pruned.

In real-world applications, we often have an idea about the rough bound of the objective function. For example, we may well know that driving from London to Camridge cannot be more than 3 hours and all we want to find is the quickest route to take. Unfortunately, traditional repair methods are not particularly benefitable by a good initial bound for the objective function. This is because it may make the methods to be trapped in a local minimum. The proposed NG-

Backmarking method on the other hand can be greatly benefitted by a good initial bound. This is because a good bound can prune the search space through the generation of many generic no-good patterns earlier on in the backmarking process.

The NB-backmarking algorithm only allows repair of a partial solution to another with lower cost using a hill-climbing strategy. Although the method guarantees to find a global optimum, it can take a rather long time to move away from local minimum. Especially, the number of no-good patterns generated at every step of Hill-climbing is exponential in space complexity. Hill-climbing itself can also be very expensive since it involves the selection of the best measurement of the relative cost of a repair. Furthermore, each no-good pattern can be regarded as an extra constraint despite its simplicity. So learning such a constraint may not be always effective if the size of the no-good pattern is not controlled.

To deal with this problem, in [Li & Jiang 94], we have presented a hybrid method called $NG-Backmarking^{ST}$ that integrates simulated annealing and tabu search with NG-Backmarking. The basic idea is to allow the repair of a partial label or solution with higher cost depending on some probability measure. Although this could lead to repairs of higher cost earlier in the search process, it may well improve the search later on. In particular, compared with Hill-climbing, the simulated annealing strategy does not suffer from the computationally expensive overheads of choosing a least costly repairs.

The proposed repair method can even be adapted to accommodate genetic algorithms (GA). The no-good database essentially contains all possible partial labels that need to be repaired. We can combine two *partial* labels (via a cross-over operation between two partial labels) to form a new partial label to be repaired. Or we can choose any individual label in parallel or alternatingly to repair via a mutation operation. This approach generalizes genetic algorithm which only perform combination or mutatation on *complete* labels to form new labels. In the hybrid approach, GA helps the NG-Backmarking to perform repair on more "healthy" labels, while the NG-Backmarking helps GA to ensure the optimal label to be found.

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Locally Simultaneous Constraint Satisfaction

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Abstract

Local propagation is often used in graphical user interfaces to solve constraint systems that describe structures and layouts of figures. However, algorithms based on local propagation cannot solve simultaneous constraint systems because local propagation must solve constraints individually. We propose an efficient algorithm that satisfies systems of constraints with strengths, even if they must be solved simultaneously, by 'dividing' them as much as possible. In addition to multi-way constraints, it handles various other types of constraints, for example, constraints solved with the least squares method. Furthermore, it unifies the treatment of different types of *constraints* in a single system. We implemented a prototype constraint solver based on this algorithm, and evaluated its performance.

1 Introduction

Local Propagation is an efficient constraint satisfaction algorithm that takes advantage of potential locality of constraint systems. It is often used in graphical user interfaces (GUIs) to solve constraint systems that describe structures and layouts of figures.

Early constraint solvers based on local propagation handle *one-way constraints* because the algorithm is simple [6]. A one-way constraint always outputs a value to a certain variable. For example, consider a constraint system with the constraints $v = w \times x$, $w = y$, and $x = y + z$. Figure 1 shows a *constraint graph* representing this system, where circles and squares represent variables and constraints respectively. If these constraints are one-way, they are always solved for certain variables, e.g. $v \leftarrow w \times x$, $w \leftarrow y$, and $x \leftarrow y + z$. This case is illustrated by the *correct solution graph* in Figure 2, where arrows from constraints point to variables to which the constraints output values. A solution graph is a constraint graph extended so that it dictates how constraints will be solved. A correct solution graph is a solution graph that can produce correct solutions, and satisfies the following two properties: the value of each variable must be determined by at most one constraint, that is, the graph should have no *conflicts*, and all the constraints must be partially ordered, that is, the graph must have no *cycles*. Applying local propagation to a correct solution graph is, in short, equivalent to solving necessary constraints in the order consistent with the partial order dictated by the graph. For example, if the value of variable y is changed in Figure 2, local propagation solves this solution graph by first computing $w \leftarrow y$, *next* $x \leftarrow y + z$, and finally $v \leftarrow w \times x$. Since this order is easily obtained with topological sort, and also since constraints are individually solved at most once, local propagation is an extremely efficient algorithm.

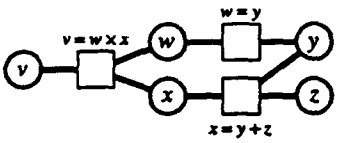

Figure 1: A Constraint Graph

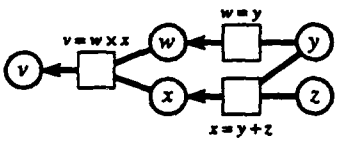

Figure 2: A Correct Solution Graph

However, one-way constraints are often insufficient because they cannot change dependencies among variables. To cope with this problem, *multi-way constraints* are proposed [1]. A multi-way constraint has

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multiple candidates for its output variable. For example, the constraints in the above example can be multi-way because they have multiple variables whose values can be uniquely determined. By contrast, logical formula such as $a = b \wedge c$ are not multi-way constraints since they lack such a property. A system of multi-way constraints is solved as follows: First, output variables are selected for each constraint, that is, a solution graph is generated out of the system so that the graph has no conflicts and no cycles[1]. Then, local propagation is applied to the solution graph.

Multi-way constraints also embody a problem that output variables are not determined uniquely. Figure 3 illustrates a solution graph for the constraint graph in Figure 1, but is different in the output variables from the graph in Figure 2. Both solution graphs are correct because they have no conflicts and no cycles, but such ambiguity is not preferable in GUIs since it may cause unexpected behavior to the user. The ad-hoc solution is to provide additional constraints: when the value of y is edited in the above example, a constraint that fixes the values of variable v or z will determine a unique solution graph. However, such a solution is obviously not desirable since it would easily result in over-constrained systems.

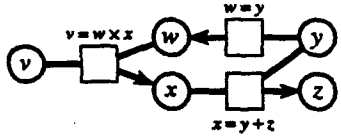

Figure 3: Another Correct Solution Graph

Borning et al. proposed *constraint hierarchies* to cope with this problem [2, 10]. A constraint hierarchy is a system of constraints with hierarchical *strengths*. If the system is over-constrained, it is solved so that there are as many satisfied strong constraints as possible. In Figure 4a, for example, the constraints $x = 1$ and $x = 3$ conflict. However, if $x = 1$ and $x = 3$ are associated with strong and weak respectively, the constraint system is solved by satisfying only $x = 1$ as shown in Figure 4b. Blue and DeltaBlue were first proposed as algorithms that solve constraint hierarchies with multi-way constraints [4, 8]. The DeltaBlue algorithm determines output variables of constraints incrementally when a constraint is added or removed, and realizes constraint satisfaction without losing the efficiency of local propagation.

Although constraints have become powerful as described above, local propagation has a serious problem: constraint systems employed in real applications often result in solution graphs with cycles or conflicts. For example, consider a constraint system with the constraints $a - b = l$, $(a + b)/2 = m$, stay(l), and edit(m). This system represents a typical situation where the midpoint of two points is moved with a mouse, but its solution graphs contain cycles by necessity, e.g. as illustrated in Figure 5. As another example, suppose a constraint hierarchy with the constraints strong $x = 1$ and strong $x = 3$. Even if one wants to apply the least squares method to these constraints and to obtain the solution $x = 2$, local propagation will fail. The resulting solution graph contains a conflict as shown in Figure 6. Generally, in constraint systems that result in solution graphs with cycles or conflicts, constraints need to be solved simultaneously.

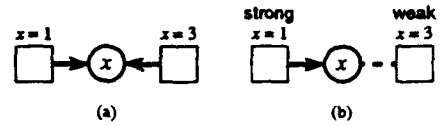

Figure 4: Solution Graphs (a) for an Over-Constrained System and (b) for a Constraint Hierarchy

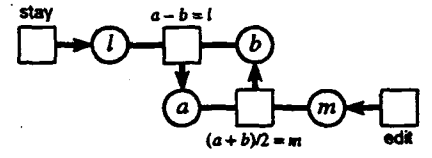

Figure 5: A Solution Graph with a Cycle

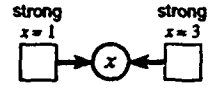

Figure 6: A Solution Graph with a Conflict

We propose an efficient algorithm that satisfies constraint hierarchies, even if constraints must be solved simultaneously, by 'dividing' them as much as possible. This algorithm is efficient enough to be applied to constraint-based GUIs since it incrementally finds parts of constraint systems that must be solved simultaneously. In addition to multi-way constraints, it handles various other kinds of constraints, for example, constraints solved with the least squares method. Furthermore, it unifies the treatment of different types of constraints in a single hierarchy. We implemented a prototype constraint solver based on

[1] Few practical algorithms try to eliminate cycles.

this algorithm, and evaluated its performance. Taking advantage of this solver, we developed the IMAGE system, which generates GUIs by generalizing multiple visual examples.

2 Locally Simultaneous Constraint Satisfaction

In this section, we present an extended theory of constraint hierarchies and an efficient algorithm that incrementally finds parts of constraint hierarchies that must be solved simultaneously.

2.1 Overview

In our extended constraint hierarchy theory, constraints are categorized into *solution types*, which are determined by how the constraints are solved. For example, there is a solution type of constraints that will be ignored if they cannot be solved exactly, as is with the Blue and DeltaBlue algorithms. Also, there is another solution type of constraints that must be solved even in such a case by minimizing their errors with the least squares method. Alternatively, we can consider a solution type of constraints to layout graphs, etc.

All constraints with an equal strength must belong to a single solution type. Intuitively, this requirement is necessary because it is difficult to treat constraints equally if they have different solution types.

Based on this theory, our algorithm solves constraint hierarchies with the following restrictions:

- If solved individually, constraints are single-output and multi-way and can select any of their constrained variables as outputs.
- All constraints in a constraint hierarchy are independent[2]. For example, a hierarchy must not contain the constraints strong $x + y = 1$ and weak $x + y = 1$.

2.2 Theory

By extending the theory described in [10], we formulated constraint hierarchies that contain multiple solution types of constraints. A constraint hierarchy H is a pair (V, C), where V is a set of variables that range over some domain $\mathcal{D}$, and C is a set of constraints on variables in V. Each constraint is associated with a strength i where $0 \leq i \leq n$. Strength 0 represents the strength of required constraints, and the larger the number of a strength, the weaker it is. All constraints with an equal strength i are categorized into a solution type τ_i. C is divided into lists $C_0, C_1, ..., C_n$, where C_i contains constraints with strength i in some arbitrary order.

Solutions to a constraint hierarchy are defined as a set of valuations. A valuation θ is a function that maps variables in V to their values in $\mathcal{D}$. An error function e_τ returns a non-negative real by evaluating the error for θ of a constraint c of a solution type τ. The error $e_\tau(c\theta) = 0$ if and only if c is exactly satisfied by θ. The function E_{τ_i} returns the list of errors of a list of constraints $C_i = [c_1, c_2, ..., c_k]$, i.e.,

$$E_{\tau_i}(C_i\theta) = [e_{\tau_i}(c_1\theta), e_{\tau_i}(c_2\theta), ..., e_{\tau_i}(c_k\theta)].$$

Each element $e_{\tau_i}(c_i\theta)$ can be weighted by a positive real w_i. An error sequence $R(C\theta)$ is the error of C except C_0:

$$R(C\theta) = [E_{\tau_1}(C_1\theta), E_{\tau_2}(C_2\theta), ..., E_{\tau_n}(C_n\theta)].$$

A combining function g_{τ_i} combines $E_{\tau_i}(C_i\theta)$. Two combined errors $g_{\tau_i}(E_{\tau_i}(C_i\theta))$ and $g_{\tau_i}(E_{\tau_i}(C_i\varphi))$ are compared by a reflexive and symmetric relation $<>_{g_{\tau_i}}$, and an irreflexive, antisymmetric, and transitive relation $<_{g_{\tau_i}}$. The function G combines an error sequence $R(C\theta)$:

$$G(R(C\theta)) = [g_{\tau_1}(E_{\tau_1}(C_1\theta)), ..., g_{\tau_n}(E_{\tau_n}(C_n\theta))].$$

Two combined error sequences $G(R(C\theta))$ and $G(R(C\varphi))$ are compared by a lexicographic ordering relation $<_G$:

$$\begin{aligned} & G(R(C\theta)) <_G G(R(C\varphi)) \\ \equiv\ & \exists k \in 1...n.\ \forall i \in 1...k-1. \\ & \quad g_{\tau_i}(E_{\tau_i}(C_i\theta)) <>_{g_{\tau_i}} g_{\tau_i}(E_{\tau_i}(C_i\varphi)) \ \wedge \\ & \quad g_{\tau_k}(E_{\tau_k}(C_k\theta)) <_{g_{\tau_k}} g_{\tau_k}(E_{\tau_k}(C_k\varphi)). \end{aligned}$$

We say that θ is *better* than φ if and only if $G(R(C\theta)) <_G G(R(C\varphi))$.

The set S of solutions to H is defined as follows:

$$\begin{aligned} S_0 &= \{\theta \mid \forall c \in C_0.\ e_{\tau_0}(c\theta) = 0\} \\ S &= \{\varphi \in S_0 \mid \forall \theta \in S_0. \\ &\qquad \neg(G(R(C\theta)) <_G G(R(C\varphi)))\}. \end{aligned}$$

The main difference from the original formulation in [10] is existence of solution types. In [10], all constraints in a constraint hierarchy are categorized into some single solution type, and therefore, for each strength i, e_{τ_i} and g_{τ_i} are some e and some g respectively. Since errors of constraints with different strengths are never compared directly, we can safely assign various solution types to each strength.

[2]The reason is that our algorithm mainly uses information on graphical structures of constraint hierarchies.

Two error functions are presented in [10]: Given a constraint c and a valuation θ, the *predicate* error function returns 1 if c is exactly satisfied for θ and 0 otherwise. Also, the *metric* error function returns c's metric, e.g. for the constraint $x = y$, the distance between x and y.

Also in [10], several combining functions and associated relations are provided. Since it does not introduce multiple solution types in a constraint hierarchy, an instance of $<_G$ is determined by single instances of e and g. For an instance of $<_G$ called *least-squares-better*, given lists of errors $\mathbf{v} = [v_1, v_2, ..., v_k]$ obtained with the metric error function, $g(\mathbf{v}) = \sum_{i=1}^{k} w_i v_i^2$, $<_g$ is $<$ and $<>_g$ is $=$ for reals. For instances of $<_G$ called *locally-better*, given $\mathbf{v} = [v_1, v_2, ..., v_k]$ and $\mathbf{u} = [u_1, u_2, ..., u_k]$, $g(\mathbf{v}) = \mathbf{v}$ and $<_g$ and $<>_g$ are defined as follows:

$$\mathbf{v} <_g \mathbf{u} \equiv \forall i.\ v_i \leq u_i \ \wedge\ \exists j.\ v_j < u_j$$
$$\mathbf{v} <>_g \mathbf{u} \equiv \forall i.\ v_i = u_i.$$

Locally-predicate-better is the locally-better using the predicate error function, and *locally-metric-better* is the one employing the metric error function.

In the rest of this paper, we refer to the solution type associated with least-squares-better as τ_{LSB} and constraints of τ_{LSB} as least-squares-better constraints, and correspondingly locally-predicate-better as τ_{LPB} and locally-predicate-better constraints[3].

2.3 Solution Graphs

Local propagation cannot solve conventional solution graphs that have cycles or conflicts. To cope with this problem, we propose a new definition of solution graphs. Before presenting the definition, we define constraint graphs of constraint hierarchies:

Definition 1 (Constraint Graph) Given a constraint hierarchy $H = (V, C)$, a bipartite graph $B = (V, C, E)$, where V and C are sets of nodes and E is a set of edges, is a constraint graph of H if and only if

$$E = \{(v, c) \in V \times C \mid v \text{ is constrained by } c\}. \quad \square$$

By regarding constraint graphs as bipartite graphs, we can use theorems and algorithms presented in graph theory.

We introduce *constraint cells* to overcome the defects of conventional solution graphs. A constraint cell is defined as follows:

[3]These names may sound strange because 'better' is associated with $<_G$ (not $<_g$), but we use them instead of introducing new terminologies.

Definition 2 (Constraint Cell) Let $H = (V, C)$ be a constraint hierarchy, and $B = (V, C, E)$ a constraint graph of H. For $X \subseteq V$, define Γ as follows:

$$\Gamma(X) = \{c \mid (v, c) \in E \ \wedge\ v \in X\}.$$

A pair $p = (V_p, C_p)$ is a constraint cell in B if and only if $V_p \subseteq V$, $C_p = \emptyset$, and $|V_p| = 1$, or $V_p \subseteq V$, $C_p \subseteq C$, the subgraph of B induced by V_p and C_p is connected, and

$$\forall X \subseteq V_p.\ |X| \leq |\Gamma(X) \cap C_p|.$$

We say that p is *over-constrained* if and only if $|V_p| < |C_p|$. $\square$

For example, the box with round corners in Figure 7a illustrates a constraint cell with a single variable. Also, Figure 7b illustrates a constraint cell with a single constraint, and Figure 7c shows a constraint cell with a variable and a constraint.

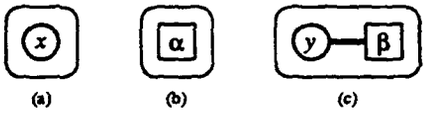

Figure 7: Constraint Cells

Values of variables in a constraint cell are obtained by evaluating constraints in the cell. In Figure 7c, for example, the value of the variable y is determined by the constraint β. Because of Definition 2, this is always possible for constraints that we handle. Definition 2 is based on Hall's theorem, which describes the condition on existence of perfect matchings of bipartite graphs in graph theory. Intuitively, Definition 2 means that given a constraint cell $p = (V_p, C_p)$, the value of each variable in V_p can be determined by at least one constraint in C_p.

We can regard constraint graphs 'divided' by constraint cells as solution graphs:

Definition 3 (Solution Graph) Given a constraint graph $B = (V, C, E)$ and a set P of constraint cells in B, a quadruple $B_S = (V, C, E, P)$ is a solution graph for B if and only if:

1. each variable in V belongs to only one constraint cell in P,
2. each constraint in C belongs to only one constraint cell in P, and
3. there are no cyclic dependencies among constraint cells in P. $\square$

For example, Figure 8 shows a solution graph equivalent to the one in Figure 2[4]. We can apply local

[4]For readability, we often draw arrowheads in constraint cells although they are not essential.

propagation to such solution graphs in the same way as conventional solution graphs.

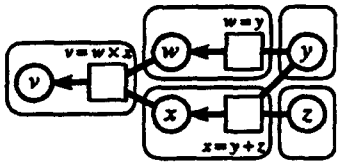

Figure 8: A Solution Graph with Constraint Cells

Solution graphs with constraint cells support constraint hierarchies that conventional solution graphs do not because of cycles and conflicts. For example, consider a constraint hierarchy with the constraints α, β, γ, δ, ϵ, ζ, η, and θ. Let α be required $t = 0$, β weak $t = u$, γ weak $v = 1$, δ strong $t + v = w$, ϵ weak $w = x$, ζ strong $x + y = z$, η required $x + 1 = y$, and θ medium $z = 7$, where strong and medium constraints are locally-predicate-better constraints, and weak constraints are least-squares-better constraints. Figure 9a shows the constraint graph of this hierarchy, and Figure 9b illustrates a conventional solution graph for this constraint graph. This solution graph is not correct since it has a cycle with ζ, x, η, and y, and a conflict of δ and ϵ at w. To begin with, we create a solution graph so that constraint cells contain the cycle and the conflict as shown in Figure 9c. Satisfying constraints locally in these cells, the corresponding valuation Θ is obtained as $\{t \mapsto 0,\ u \mapsto 0,\ v \mapsto 1,\ w \mapsto 1,\ x \mapsto 3,\ y \mapsto 4,\ z \mapsto 7\}$. The combined error sequence is:

$$\begin{array}{ll} \textsf{strong} & g_{\tau_{LPB}}(E_{\tau_{LPB}}([\delta,\zeta]\Theta)) \\ & \quad = g_{\tau_{LPB}}([0,0]) = [0,0] \\ \textsf{medium} & g_{\tau_{LPB}}(E_{\tau_{LPB}}([\theta]\Theta)) = g_{\tau_{LPB}}([0]) = [0] \\ \textsf{weak} & g_{\tau_{LSB}}(E_{\tau_{LSB}}([\beta,\gamma,\epsilon]\Theta)) \\ & \quad = g_{\tau_{LSB}}([0,0,2]) = 0^2 + 0^2 + 2^2 = 4. \end{array}$$

Merging over-constrained cells with other cells sometimes creates a 'better' solution graph, i.e. the corresponding valuation is better, because the new cell may acquire more freedom to determine the values of its variables. For example, by merging the over-constrained cell W and the cell V into the new cell W', we obtain the solution graph in Figure 9d[5]. Then, the corresponding valuation Φ is $\{t \mapsto 0,\ u \mapsto 0,\ v \mapsto 2,\ w \mapsto 2,\ x \mapsto 3,\ y \mapsto 4,\ z \mapsto 7\}$, and the

[5]We do not merge constraint cells simply because they contain constraints of similar solution types or constraints with equal strengths. For example, W' in Figure 9d contains multiple solution types of constraints with multiple strengths

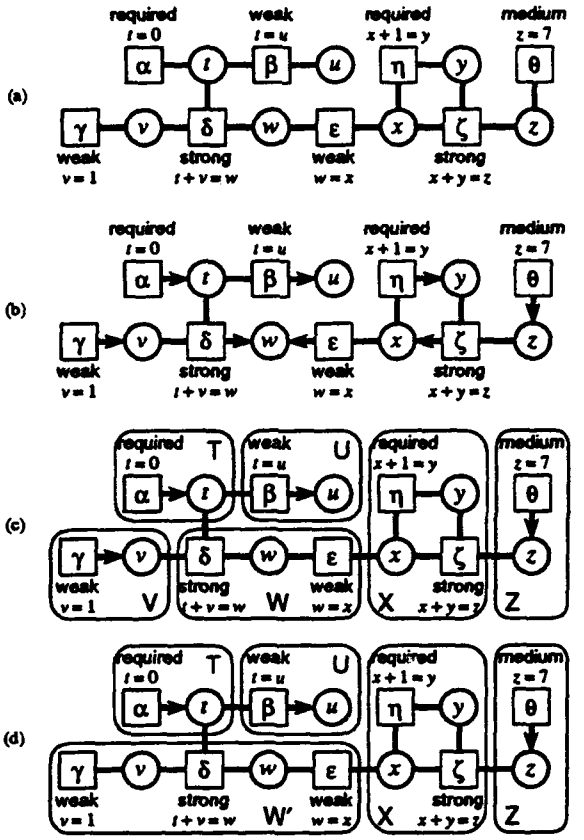

Figure 9: A Constraint Graph and its Solution Graphs

combined error sequence is:

$$\begin{array}{ll} \textsf{strong} & g_{\tau_{LPB}}(E_{\tau_{LPB}}([\delta,\zeta]\Phi)) \\ & \quad = g_{\tau_{LPB}}([0,0]) = [0,0] \\ \textsf{medium} & g_{\tau_{LPB}}(E_{\tau_{LPB}}([\theta]\Phi)) = g_{\tau_{LPB}}([0]) = [0] \\ \textsf{weak} & g_{\tau_{LSB}}(E_{\tau_{LSB}}([\beta,\gamma,\epsilon]\Phi)) \\ & \quad = g_{\tau_{LSB}}([1,0,1]) = 1^2 + 0^2 + 1^2 = 2. \end{array}$$

This indicates that Φ is better than Θ.

We define correct solution graphs so that they can produce solutions to constraint hierarchies. Before presenting the definition, we define *internal strengths* and *walkabout strengths* of constraint cells. Walkabout strengths were first introduced as walkabout strengths of variables in the DeltaBlue algorithm, but for our purpose, we modify the definition.

Definition 4 (Internal Strength) Let p be a constraint cell (V_p, C_p). The internal strength of p is weakest if $C_p = \emptyset$, and the weakest among strengths of constraints in C_p otherwise. □

Definition 5 (Walkabout Strength) Given a constraint cell p, the walkabout strength of p is the

weakest among p's internal strength and walkabout strengths of cells with variables adjacent to p. □

For example, the internal strength of the constraint cell W' in Figure 10 is weak, the weakest among γ's strength weak, δ's strength strong, and ϵ's strength weak. The walkabout strength of W' is also weak, the weakest among W''s internal strength weak, T's walkabout strength required, and X's walkabout strength medium.

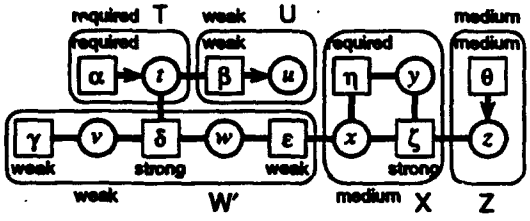

Figure 10: Walkabout Strengths of a Solution Graph

A correct solution graph is defined as follows:

Definition 6 (Correct Solution Graph) A solution graph is correct if and only if:

1. for each constraint cell with multiple constraints, the pair of the set of its variables and the set of its non-weakest constraints is not a constraint cell, and
2. for each over-constrained cell, its internal strength is weaker than the walkabout strengths of any other cells with the variables adjacent to the constraints in the over-constrained cell.

Intuitively, condition 1 means that constraint cells must use the weakest constraints to determine the values of their variables, and condition 2 means that it is impossible to create better solution graphs by merging such over-constrained cells with others. The reason for the latter is as follows: Suppose an over-constrained cell p that satisfies condition 1 but not condition 2. Because of condition 1, p uses the weakest constraints to determine the values of its variables, and by definition, its internal strength i_p is the strength of the weakest constraints. Since p does not satisfy condition 2, i_p is equal to or stronger than the walkabout strengths w_q of an adjacent cell q. Also by definition, the values of the variables in q are determined by one or more constraints with strength w_q. Therefore, merging p with q, it may be possible to decrease the errors of the constraints with strength i_p in p by increasing the errors of the constraints with strength w_q and then to create a better valuation. However, if p satisfies condition 2, it is useless to merge p with q: since i_p is weaker than w_q, it is impossible to reduce the errors of the constraints with strength i_p by increasing the errors of the constraints with strength w_q.

For example, the solution graph in Figure 11 is not correct because the cell W does not use the weakest constraint ϵ to determine the value of the variable w, and also since the internal strength weak of the over-constrained cell W is equal to the walkabout strength of the cell V. By contrast, in Figure 10, the over-constrained cell W' needs the weakest constraints γ and ϵ to compute the values of the variables v and w, and W''s internal strength weak is weaker than the walkabout strength required of the cell T and medium of X. This solution graph is correct by definition.

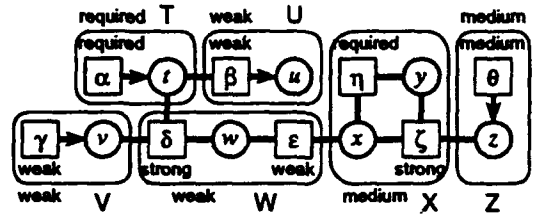

Figure 11: Walkabout Strengths of a Incorrect Solution Graph

2.4 Algorithm

It is desirable that sizes of constraint cells in correct solution graphs are minimized since local propagation can be efficiently applied to such solution graphs. Our algorithm creates such solution graphs incrementally when invoked with the following five operations: adding a variable, removing a variable, adding a constraint, removing a constraint, and updating a variable value. The former four operations cause the corresponding solution graph to be modified, and the last operation applies local propagation to the solution graph as described earlier. We call the former *planning* and the latter *execution*.

The algorithm for adding or removing a variable is quite simple: to add a variable, we only create a new constraint cell with the variable, and to remove a variable, we delete the constraint cell with the variable after verifying that the variable is not adjacent to constraints. In the rest of this section, we describe the algorithm for adding or removing a constraint to a hierarchy.

2.4.1 Adding a Constraint

Initially, there is a correct solution graph whose constraint cells are minimized. When a new constraint

is added to this hierarchy, one or more constraints with an equal or weaker strength may be 'victimized,' that is, their associated errors will be increased. In such a case, the algorithm re-constructs the solution graph incrementally to keep it correct and its constraint cells minimal by modifying the necessary set of cells.

This algorithm handles locally-predicate-better constraints differently from other solution types of constraints since they may be ignored if they cannot be exactly satisfied. For example, suppose that weak constraints were locally-predicate-better constraints in the constraint hierarchy presented in the previous subsection. In the constraint cell W' in Figure 9d, only one of the constraints γ and ϵ would have to be exactly satisfied because they would be locally-predicate-better constraints. Therefore, in this case, the solution graph in Figure 9c could also produce a correct solution although it is not a correct solution graph by definition. In addition, this solution graph could be solved more efficiently than the graph in Figure 9d. Accordingly, we treat locally-predicate-better constraints specially by permitting 'equal to' as well as 'weaker than' in condition 2 of Definition 6.

Figure 12 shows the algorithm that adds a constraint *con* to a constraint hierarchy, and Figure 13 describes the algorithm to decompose a constraint cell at lines 13 and 19 in Figure 12. Let us explain the former algorithm briefly: First, we create a constraint cell with c at line 1. Second, we find the strength of the 'victim' constraint at line 2. Next, we follow the path in the graph of the constraint cells from c to the victim at lines 5–21, reversing the dependency between these cells. After this process, c becomes active. Then, we eliminate cycles of constraint cells generated in the previous process at line 22, and update walkabout strengths correctly at line 23. Finally, we merge over-constrained cells with others at line 25 so that they can minimize the errors of their constraints.

Figure 14 shows an example of the execution of this algorithm. Initially, there is a correct solution graph illustrated in Figure 14a. When a constraint θ is added to the constraint hierarchy, this algorithm works as follows:

1. A constraint cell H with θ is created (Figure 14b). The strength of the victim is found to be weak.
2. After the variable z is removed from the cell G, it is added to H (Figure 14c).
3. The variable x is deleted from the cell E, and is added to G (Figure 14d). The constraint ϵ in E is found to be the victim.
4. The constraint cells G and F are merged because

```
1   cl ← a new cell with con;
2   wastr ← the weakest of walkabout strengths of cells
        with variables adjacent to cl;
3   if wastr is weaker than con's strength then
4     str ← con's strength;
5     while str is stronger than wastr do
6       nextcl ← a cell that contains a variable adjacent
            to cl and that has walkabout strength wastr;
7       var ← a variable in nextcl that connects to cl;
8       remove var from nextcl;
9       add var to cl;
10      if nextcl is empty then
11        str ← weakest;
12      else if nextcl's internal strength is wastr then
13        cls ← cells generated by decomposing nextcl;
14        cl ← an over-constrained cell in cls;
15        str ← cl's internal strength;
16      else
17        bordercon ← a constraint in nextcl adjacent to
              a cell with walkabout strength wastr;
18        remove bordercon from nextcl;
19        decompose nextcl;
20        cl ← a new cell with bordercon;
21        str ← cl's internal strength;
22    merge cyclic cells dependent on con;
23  update walkabout strengths of cells dependent on con;
24  if wastr is weakest or constraints with strength wastr
        are not locally-predicate-better constraints then
25    merge cells that cl depends on and
          that have the same walkabout strength as cl;
```

Figure 12: Adding a Constraint *con* to a Constraint Hierarchy.

```
1   for each variable var in cl do
2     remove var from cl;
3     create a cell with var;
4   for each constraint con stronger than wastr in cl do
5     remove con from cl;
6     var ← a variable initially in cl that forms
          a cell alone and that con depends on;
7     reverse the dependency between con and var;
8   for each constraint con with strength wastr in cl do
9     remove con from cl;
10    if there is a variable initially in cl that forms
          a cell alone and that con depends on then
11      var ← the variable found above;
12      reverse the dependency between con and var;
13    else
14      create a cell with con;
```

Figure 13: Decomposing a Constraint Cell *cl* with Walkabout Strength *wastr*

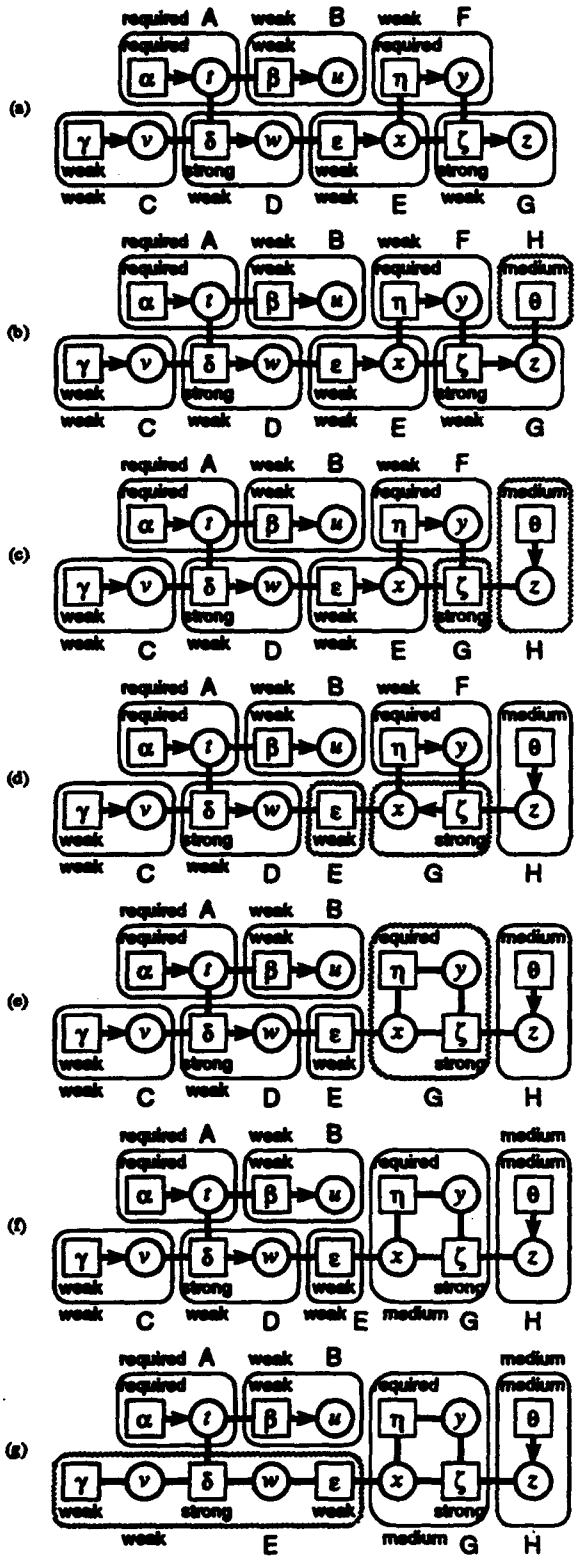

Figure 14: Adding a Constraint

they form a cycle (Figure 14e).

5. Walkabout strengths are updated (Figure 14f).
6. Since E is over-constrained, it is joined with the constraint cells D and C, which have the same walkabout strength as E (Figure 14g).

It is sometimes necessary to decompose 'large' constraint cells that contain multiple constraints. For example, suppose that a constraint ν is added to a constraint hierarchy as shown in Figure 15a. It is not sufficient to remove the variable w from the constraint cell A and then to add it to the new cell N as illustrated in Figure 15b, because this solution graph is not correct by definition. The correct solution graph is created by decomposing A as shown in Figure 15c. Figure 13 describes the algorithm that decomposes such 'large' constraint cells into 'small' ones. The basic idea is to match variables with constraints in solution graphs, employing a perfect matching algorithm for bipartite graphs. For example, in Figure 15c, the constraint cells A_0, A_2, and A_3 are matched pairs. To leave the weakest constraints such as β and ϵ unsatisfied, our algorithm later tries to match the weakest constraints in over-constrained cells. The definition of constraint cells guarantees that there are no undetermined variables after decomposing cells with one or more constraints. Even if constraint cells that do not satisfy condition 3 in Definition 3 or condition 2 in Definition 6 are generated, they will be merged by the caller algorithm in Figure 12.

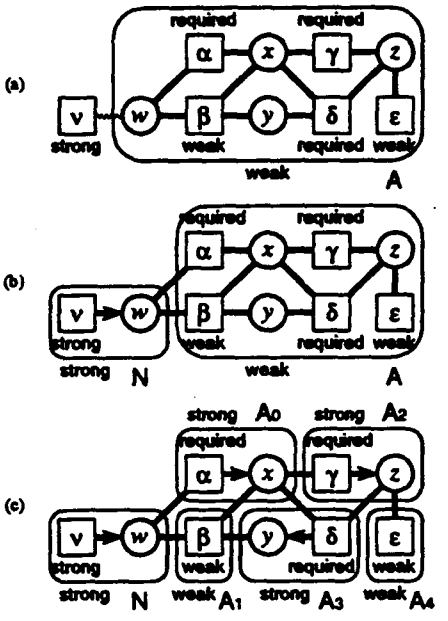

Figure 15: Decomposing a Constraint Cell

2.4.2 Removing a Constraint

Removing a constraint from a constraint hierarchy may cause one or more constraints with an equal or weaker strength to decrease their errors. This is performed in the similar way to adding a constraint by reversing the dependency between the cell with such constraints and the cell that has been contained the removed constraint.

3 Implementation

Based on the algorithm presented in the previous section, we implemented a constraint solver in Objective-C. This constraint solver consists of two layers called a *solver* and *subsolvers*. A *solver* produces correct solution graphs, and applies local propagation to them. *Subsolvers* obtain values of variables by solving constraint systems locally in individual constraint cells. During local propagation, the *solver* invokes appropriate *subsolvers* based on solution types of constraints in cells. For example, if a cell contains only locally-predicate-better constraints, the *solver* calls the *subsolver* for τ_{LPB}. This architecture enables us to introduce a new solution type of constraints only by implementing a new *subsolver*.

We implemented three *subsolvers*: one that handles locally-predicate-better constraints represented as linear equations or multi-way constraints, one that treats least-squares-better linear equation constraints, and one that generates graph layouts based on the spring model [3].

4 Comparison with SkyBlue

SkyBlue is a successor of the DeltaBlue algorithm [7]. Like DeltaBlue, SkyBlue solves hierarchies of multi-way constraints using *locally-graph-better*, a variation of locally-predicate-better. Moreover, it supports constraints with multi-output methods and cycles of constraints. A method of a constraint is a procedure used to satisfy the constraint, and a multi-output method is a method that outputs values to multiple variables. For example, the constraint $p = (x, y)$, which equates a point variable p with two real variables x and y, has a single-output method $p \leftarrow (x, y)$ and a multi-output method $(x, y) \leftarrow p$. SkyBlue treats cycles of constraints by invoking cycle solvers, which solve constraints in cycles simultaneously.

It is interesting to compare cycle solvers of the SkyBlue constraint solver with *subsolvers* of our constraint solver: Cycle solvers solve two or more constraints at once because methods solve associated constraints individually. By contrast, *subsolvers* can solve individual constraints, cycles of constraints, and even over-constrained sets of constraints. In addition, *subsolvers* allow constraints to have methods as done in SkyBlue[6]. Therefore, *subsolvers* are more functional than cycle solvers.

The critical differences between our algorithm and the SkyBlue algorithm are summarized as follows:

- Our algorithm handles various solution types of constraints in single hierarchies while SkyBlue treats only multi-way constraints solved using locally-graph-better.
- SkyBlue supports multi-output methods while our algorithm does not.

It will depend on applications which algorithm is over the other.

5 Performance Measurements

Using the chain benchmark [8], we compared the performance of our constraint solver implemented in Objective-C with that of DeltaBlue implemented in C[7]. Initially, the constraint hierarchy contains the required constraints $x_0 = x_1, x_1 = x_2, ..., x_{n-2} = x_{n-1}$ and the constraint weak stay(x_0) (Figure 16a). The chain benchmark measures the planning time to add the constraint strong edit(x_{n-1}) to the hierarchy (Figure 16b), and also measures the execution time to compute values of variables when the value of x_{n-1} is changed through edit(x_{n-1}). Both of the planning and the execution are the worst cases where the overall solution graph must be processed.

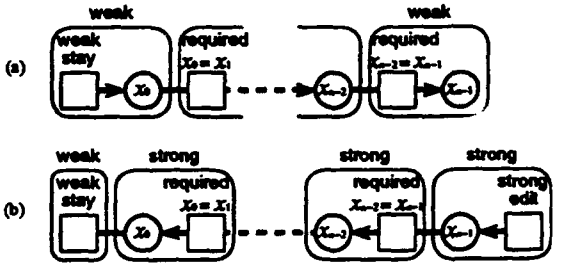

Figure 16: The Chain Benchmark

Figure 17 shows the result[8]: while the planning time of our solver is almost four times as long as

[6] In fact, the *subsolver* implemented for locally-predicate-better constraints handles multi-way constraints with single-output methods.

[7] Since SkyBlue is implemented only in Lisp, we did not compared our constraint solver with SkyBlue.

[8] Precisely speaking, the separation of planning and execution is slightly different from the description presented in Section 2. In both our constraint solver and DeltaBlue, the

n		1000	2000	3000	4000	5000
Planning	DeltaBlue	67	16[illegible]	250	350	434
(msecs)	Our Solver	283	617	933	1183	1817
Execution	DeltaBlue	2.5	4.3	6.7	8.7	10.8
(msecs)	Our Solver	36.7	68.3	105.0	140.0	176.7

Figure 17: Results of the Chain Benchmark

that of DeltaBlue, the execution time is nearly twenty times as long. The planning time is probably acceptable for GUI applications, but the execution time is extremely too long for such applications. The main handicaps of our solver are the complex data structure of constraint cells, and dynamic bindings of methods in Objective-C[9]. We believe that dynamic bindings caused such slow execution because the source program involves numerous message sendings with dynamic bindings[10]. If we re-implement our solver in C++, its performance is expected to approach that of DeltaBlue.

6 Conclusions and Status

We proposed an efficient algorithm that incrementally solves multiple solution types of constraints in single constraint hierarchies by grouping together cyclic or conflicting constraints into constraint *cells*. We implemented a constraint solver based on this algorithm, and provided a promising result on its performance.

Using this solver, we developed the IMAGE system, which generates GUIs by generalizing multiple visual examples [5]. This system takes advantage of the ability of our solver to handle hierarchies of simultaneous constraints. Also, we are planning on applying our constraint solver to our algorithm animation system based on declarative specification [9].

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planning time includes the time of topological sort for local propagation.

[9] Objective-C does not support static bindings like C++.

[10] The execution time became almost ten-percent shorter when we removed only one message sending that would be executed n times from the source code by ignoring encapsulation of objects.

Analyzing and Debugging Hierarchies of Multi-Way Local Propagation Constraints

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Abstract

Multi-way local propagation constraints are a powerful and flexible tool for implementing applications such as graphical user interfaces. We have built constraint solvers that maintain sets of preferential multi-way constraints, and integrated them into user interface development environments. These solvers are based on the formal theory of constraint hierarchies, leaving weaker constraints unsatisfied in order to solve stronger constraints if all of the constraints cannot be satisfied.

Our experience has indicated that large constraint networks can be difficult to construct and understand. To investigate this problem, we have developed a system for interactively constructing constraint-based user interfaces, integrated with tools for displaying and analyzing constraint networks. This paper describes the debugging facilities of this system, and presents a new algorithm for enumerating all of the ways that the solver could maintain a set of constraints.

1 Introduction

A multi-way local propagation constraint is represented by a set of *method* procedures that read the values of some of the constrained variables, and calculate values for the remaining constrained variables that satisfy the constraint. A set of such constraints can be maintained by a constraint solver that chooses one method for each constraint so that no variable is set by more than one selected method (i.e., there no *method conflicts*). If there are no cycles in the selected methods, the solver can order them and execute them to satisfy all of the constraints. For example, given the constraint $A + B = C$ (represented by three methods $C \leftarrow A + B$, $A \leftarrow C - B$, and $B \leftarrow C - A$) and the constraint $C + D = E$ (represented by three similar methods), the two constraints could be satisfied by executing the methods $C \leftarrow A + B$ and $E \leftarrow C + D$ in order.

For a given set of constraints, it may not be possible to choose methods for all constraints so there are no method conflicts, or there may be multiple ways to select methods. The theory of constraint hierarchies [1] offers a way to control the behavior of a constraint solver in these situations. Given a constraint hierarchy, a set of constraints where each constraint has an associated *strength*, a constraint solver can leave weaker constraints unsatisfied in order to solve stronger constraints. Research on constraint hierarchies has produced several formal definitions for the "best" solution to a constraint hierarchy that are useful in different applications.

The DeltaBlue and SkyBlue incremental constraint solvers can be used to maintain hierarchies of multi-way local propagation constraints in applications such as user interfaces. Both of these solvers select constraint methods to construct a *method graph* (or *mgraph*) with no method conflicts where stronger constraints are enforced (have a selected method) in favor of weaker constraints. More formally, they construct a locally-graph-better (or LGB) method graph, where a method graph MG is an LGB method graph if it has no method conflicts and for each unenforced constraint C in MG there exists no conflict-free method graph for the same constraints where C is enforced and all of the enforced constraints of MG with the same or stronger strength as C are enforced.

DeltaBlue was the basis of the ThingLab II interactive user interface development environment [5, 10]. SkyBlue is more general successor to DeltaBlue that satisfies cycles of methods by calling external solvers

and supports multi-output methods (methods that set multiple output variables) [7, 8]. The Multi-Garnet package [9] uses the SkyBlue solver to add support for multi-way constraints and constraint hierarchies to the Garnet user interface toolkit [6].

As constraint solvers have been applied to larger problems it has become clear that there is a need for constraint network debugging tools. In order to debug a constraint network, the programmer needs tools to examine the constraint network, determine why a given solution is produced, and change the network to produce the desired solution. We have created a system for interactively constructing graphical user interfaces based on constraints (maintained by SkyBlue), and debugging the constraint networks created. The remainder of this paper describes the debugging facilities of this system and presents a new algorithm for generating all LGB method graphs for a set of constraints that promises to be more efficient and useful for debugging common constraint networks.

2 Debugging Constraint Networks

Figure 1 shows two views of a simple user interface constructed using our system. In Figure 1a, the two horizontal lines are lined up. In Figure 1b, the mouse has moved the left endpoint of the bottom line. Constraints keep the width of the line constant, so the right endpoint is moved by the same amount.

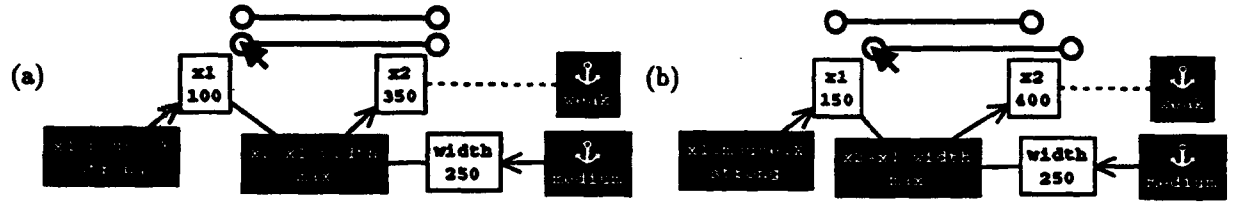

Figure 1: *Moving the left endpoint of a constant-width horizontal line.*

This figure also displays the constraints relating the variables $x1$ and $x2$, the X-coordinates of the two ends of the bottom horizontal line, to the variable *width*, the difference between the two X-coordinates. The constraint $x1 = mouse.X$ sets $x1$ to the X-coordinate of the mouse position. The *medium* stay constraint (displayed with an anchor symbol) on *width* prevents the solver from changing this variable. The *weak* stay constraint on $x2$ is not enforced, since the solver cannot satisfy this constraint without revoking a stronger constraint. As the mouse is moved, the width of the line is kept constant.

The constraint diagrams present information about the constraints (black boxes) and variables (white boxes) including names, constraint strengths[1] and variable values. The connection between the graphic objects and the variables specifying their positions (such as $x1$) is shown by positioning the variables next to their graphics (though these variable boxes can be moved by the user, if the display gets too complicated). These diagrams also show how each variable value is calculated: the arrows indicate the variables currently determined by the selected method of each constraint. If the constraint is not enforced by any method, the lines to its variables are dashed (i.e., the stay constraint on $x2$). It is possible to explain the derivation of a variable value by examining all of constraints and variables *upstream* of the given variable.

This system can be used to construct complex constraint graphs, and experiment with the behavior of the user interface as constraints are added and removed. We have developed a set of debugging tools that present additional information such as disjoint subgraphs, directed cycles, or directed paths between two variables. A more sophisticated tool analyzes the constraint network to determine why a particular constraint cannot be satisfied, identifying those stronger constraints that prevent the given constraint from being enforced. Similar tools have been developed for the QOCA toolkit [2] and the Geometric Constraint Engine [4]. The following section describes a tool for examining the different possible LGB mgraphs that the solver may

[1] The examples in this paper will use the strengths *max*, *strong*, *medium*, and *weak*, in order from strongest to weakest.

produce, and presents a new algorithm for generating these mgraphs. Research continues on developing new debugging tools, improving the facilities for invoking them, and presenting the results of their analyses.

3 Examining Multiple LGB Method Graphs

When debugging a constraint network, the programmer may want to know whether the constraints specify a unique solution, or whether the solver might produce different solutions at different times. Some constraint solvers can produce different possible solutions for a set of constraints, such as the CLP($\mathcal{R}$) system that generates symbolic expressions representing sets of multiple solutions, and produces alternate solutions upon backtracking [3]. Examining the different solutions can help the programmer understand the constraint network, and determine what constraints should be added to control the solver.

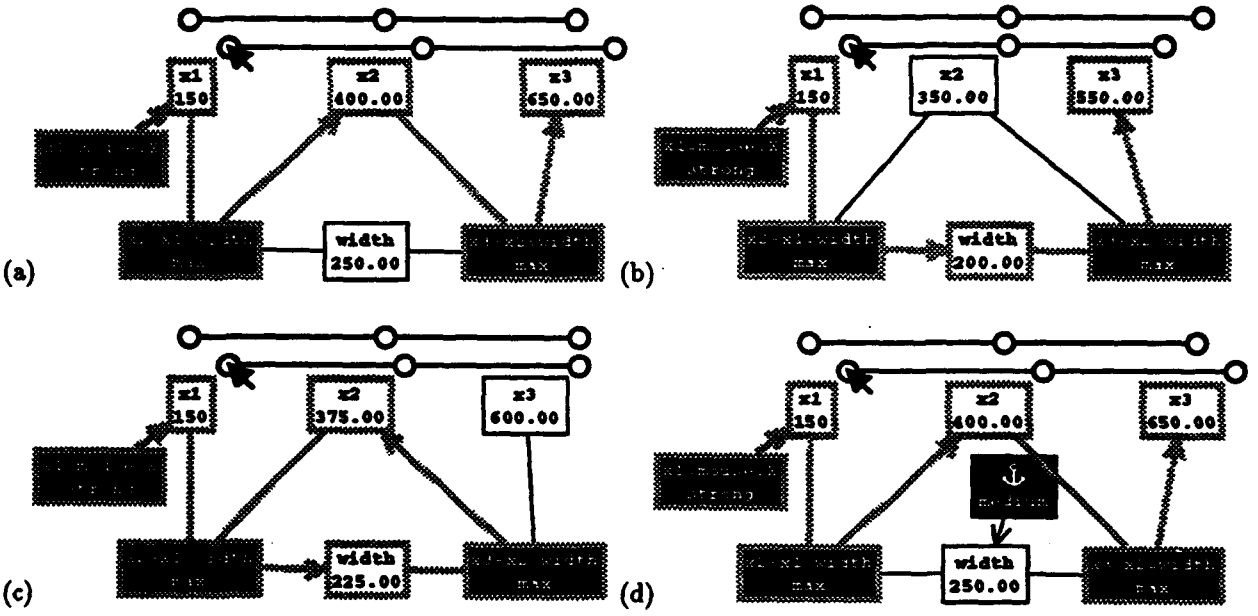

Figure 2: Moving the left endpoint of a horizontal line with a midpoint. The top line shows the initial positions of the three points. Constraints and variables downstream of the mouse constraint are highlighted.

Given a hierarchy of multi-way local propagation constraints, there may be more than one possible LGB method graph that the solver could use to maintain the constraints. For example, consider the constraint network shown in Figure 2. In this situation, there are three ways for the solver to maintain the constraints: by keeping the width variable constant and moving the line (2a), keeping $x2$ constant and moving the two endpoints inward (2b), or keeping $x3$ constant and solving the cycle of linear constraints to position $x2$ between $x1$ and $x3$ (2c). The solver can be forced to choose one of these behaviors by adding *stay* constraints to variables that the user would prefer stay constant (2d). Different strength stay constraints can be used to specify relative preferences for which variables should be constant.

In this example, it is easy to manually generate the possible LGB method graphs. This is much more difficult for large constraint networks: it may not be clear whether there are *any* alternate LGB method graphs. We have developed an algorithm that enumerates all possible LGB method graphs that SkyBlue could produce for a set of constraints. The different method graphs in Figure 2 were generated in this way. Work continues on developing better ways of examining a set of LGB method graphs, such as automatically partitioning them into subclasses depending on which variables are constant. Given two method graphs, it may not be obvious how they differ. Tools have been created for comparing two or more method graphs, highlighting the similarities and differences.

4 An Algorithm for Generating All LGB Method Graphs

We have developed an algorithm for enumerating the possible LGB method graphs for a set of constraints. This algorithm systematically calls the SkyBlue solver to increase the strength of unenforced constraints, searching for alternate method graphs where these constraints are enforced. SkyBlue incrementally updates the current LGB method graph as a constraint is added, removed, or has its strength changed, so it is practical to change constraint strengths repeatedly. The following subsections present this algorithm in stages. First, we present an algorithm for generating all sets of constraints that can be simultaneously enforced in an LGB mgraph. Then, this algorithm is extended to generate all LGB mgraphs.

4.1 LGB Enforced Sets

The *enforced set* (or *E-set*) of an mgraph is the set of constraints that are enforced in the mgraph. The E-set of an LGB mgraph are known as an LGB E-set. For example, Figure 3 shows the two possible LGB mgraphs for the three constraints. These mgraphs have E-sets of $\{C1, C2\}$ and $\{C2, C3\}$ respectively. Sometimes it is useful to speak of the E-set for the constraints with a particular strength. For example, Figure 3a has a *strong* E-set of $\{C2\}$, and a *weak* E-set of $\{C1\}$.

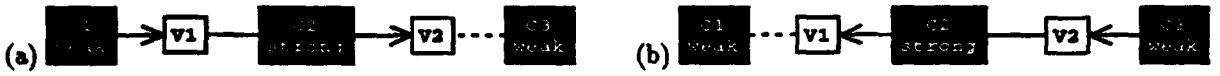

Figure 3: Two LGB mgraphs with different LGB E-sets.

Note that no LGB E-set for a set of constraints can be a proper subset of another LGB E-set. Suppose that E_1 and E_2 are the E-sets for two LGB mgraphs M_1 and M_2 for the same set of constraints. If E_1 were a proper subset of E_2, this would imply that every constraint enforced in M_1 is enforced in M_2, and M_2 contains at least one enforced constraint that is unenforced in M_1, hence M_1 would not be an LGB mgraph.

4.2 Pinning Constraints

Consider an LGB mgraph for a set of constraints. If all of the constraints are enforced, then there is only one possible LGB E-set for these constraints. If some of the constraints are unenforced, then there may be other LGB mgraphs for the constraints where some of the currently-unenforced constraints are enforced and other currently-enforced constraints are unenforced. The question is how to generate them.

Suppose that we start with the LGB mgraph of Figure 3a. All of the *strong* constraints are enforced, so all LGB mgraphs for these constraints will have a strong E-set of $\{C2\}$. Consider the unenforced *weak* constraint $C3$. Suppose that we changed the strength of $C3$ to be slightly stronger than *weak*. In this case, $C3$ would be enforced and $C1$ would be revoked, leading to the mgraph in Figure 4 (the only LGB mgraph for the modified constraints). This mgraph has an E-set of $\{C2, C3\}$, the same as Figure 3 .

Figure 4: Increasing the strength of $C3$ to produce a different E-set.

For each constraint strength *str*, define the *pin-strength* of *str* as another strength that is slightly stronger than *str*, and weaker than the next stronger constraint strength. The act of increasing the strength of a constraint to its pin-strength (i.e., the pin-strength between its normal strength and the next constraint strength) is called "pinning" the constraint. Pinning different constraints can produce LGB mgraphs with different E-sets, as in Figure 4.

An important fact about pinning is that, no matter what combination of constraints are pinned, the selected methods in the resulting LGB mgraph will specify an LGB mgraph for the original (unpinned) constraints. The algorithm described in the next section systematically pins unenforced constraints to generate different LGB mgraphs for the original constraints, and collects their E-sets.

4.3 Generating LGB E-Sets

Figure 5 presents pseudocode that generates all of the LGB E-sets for a set of constraints. `get_esets` simply initializes global variables containing a list of the constraints we are interested in (`*cns*`), a list of the collected E-sets (`*esets*`), and a procedure to be called to save each E-set (`*save_proc*`).[2] In this case `*save_proc*` is set to the procedure `save_eset`, which adds the E-set for the current mgraph to `*esets*` if it isn't there already. After setting the global variables, `get_esets` calls `pin_cns`, which pins different combinations of constraints, calling `*save_proc*` to process each of the resulting LGB mgraphs.

```
get_esets(cns)
   *cns* := cns
   *esets* := {}
   *save_proc* := save_eset
   pin_cns({}, {}, {}, cns)
   return *esets*

save_eset()
   eset := collect list of all enforced constraints in *cns*
   add eset to *esets* if it is not already there

pin_cns(pinned, unpinned, cns, weaker_cns)
   If cns contains any unenforced constraints then
      cn := choose any unenforced cn in cns
       ;; generate esets with cn unpinned
      pin_cns(pinned, unpinned ∪ {cn}, cns - {cn}, weaker_cns)
       ;; generate esets with cn pinned
      pin(cn)
      If pinned ∪ {cn} are all enforced then
         pin_cns(pinned ∪ {cn}, unpinned, cns - {cn}, weaker_cns)
      unpin(cn)
   Else If weaker_cns is not empty then
       ;; pin all unpinned enforced constraints at current strength
      enforced_unpinned := all enforced constraints in unpinned ∪ cns
      For cn in enforced_unpinned do pin(cn)
       ;; process next weaker cns
      next_strength := strongest strength of constraints in weaker_cns
      next_cns := all constraints in weaker_cns with strength next_strength
      pin_cns({}, {}, next_cns, weaker_cns - next_cns)
       ;; unpin constraints
      For cn in enforced_unpinned do unpin(cn)
   Else
       ;; all cns processed: save current state
      call the procedure *save_proc*

pin(cn)
   cn.original_strength := cn.strength
   change_strength(cn, get_pin_strength(cn.strength))

unpin(cn)
   change_strength(cn, cn.original_strength)
```

Figure 5: Pseudocode to generate all LGB E-sets for cns.

[2] All global variables in the pseudocode begin and end with an asterisk. All other variables are local to their procedures.

Most of the work happens in the recursive procedure `pin_cns`. During any call to `pin_cns`, it is processing the set of constraints at a single strength level. The arguments `pinned`, `unpinned` and `cns` are the sets of constraints at the current strength level that have been pinned, left unpinned, and have not been processed. `weaker_cns` contains weaker constraints to be processed later. If `cns` contains any unenforced constraints, one is chosen (`cn`) and `pin_cns` recurses to investigate mgraphs where `cn` is not pinned. When that recursive call returns, `cn` is pinned. If `cn` can be enforced along with all of the other pinned constraints, then `pin_cns` recurses to investigate mgraphs where `cn` is pinned. Finally we unpin `cn`, restoring its original strength.

If there are no unenforced constraints in `cns`, then we have finished processing the constraints at this strength level. Now we are ready to processes the weaker constraints. To ensure that the current strength E-set doesn't change, all of the enforced constraints that haven't been pinned (`enforced_unpinned`) are pinned. Then `pin_cns` recurses, extracting the constraints with the next-weaker strength from `weaker_cns`. Note that when `pin_cns` is initially called from `get_esets` the first three arguments are all empty sets, so `pin_cns` just extracts the strongest constraints from `weaker_cns` and recurses.

When all of the constraints have been processed `*save_proc*` is called to save information about the current LGB mgraph (`get_esets` sets this to `save_eset`, which saves the current LGB E-set).

4.4 Why get_esets Generates All E-Sets

Since `get_esets` only modifies the mgraph by pinning constraints, every E-set collected is a correct LGB E-set for the original constraints. To show that `get_esets` is correct, we need to show that every possible E-set is generated. Suppose that this were not true, and there was a set of constraints *cns* with an LGB E-set E that was not generated by `get_esets(`*cns*`)`. Consider the tree of recursive calls to `pin_cns` caused by `get_esets(`*cns*`)`. Figure 6 shows part of such a tree, where first $C1$, and then $C2$, are found to be unenforced, and then either left unpinned or pinned during different recursive calls.

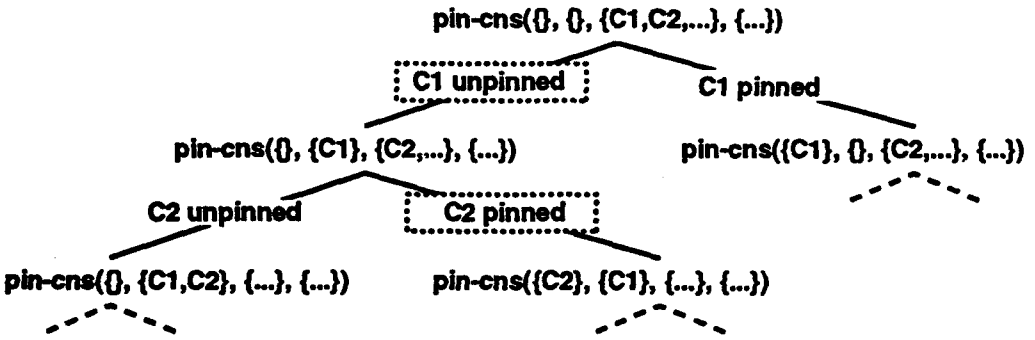

Figure 6: Partial tree of recursive calls to `pin_cns`.

Consider tracing down this tree, following each branch that pins a constraint in E, and each branch that leaves unpinned a constraint that is not in E. For example, if E contained $C2$ and did not contain $C1$, one would follow the branches with boxed labels in Figure 6. Note that every time a constraint in E is pinned it *must* be possible to enforce it along with the other pinned constraints, since E is the E-set for an LGB mgraph, so all of the constraints in E must be simultaneously enforcible. Eventually, you will reach a leaf of the tree, and `*save_proc*` will be called to process the current mgraph.

We claim that the E-set of this leaf mgraph is exactly E. First, all of the constraints in E that were found unenforced in `cns` and pinned are enforced. Consider some other constraint *cn* that is in E but was not found unenforced in `cns`. It must not have been removed from `cns`, since the only way constraints are removed from `cns` is when they are considered for pinning, and if this had happened then *cn* would have been explicitly pinned. Since it wasn't removed from `cns`, then it must have been enforced when that strength level was processed, and hence it was pinned when going to the next strength level. Therefore, it must be enforced in the final mgraph, and all of the constraints in E must be enforced. Finally, consider some constraint *cn'*

that is not in E. If it was enforced, then there would be an LGB mgraph where all of the constraints in E plus another constraint cn' are enforced, in which case E would not be the E-set of an LGB mgraph. Thus, we have shown that exactly those constraints in E are enforced in the leaf mgraph.

4.5 Generating Results Multiple Times

The procedure **save_eset** is written to add the current E-set to the list ***esets*** only if it is not there already. This is necessary because **get_esets** may generate the same E-set multiple times if constraints have multi-output methods. For example, suppose we call **get_esets** on the three *strong* constraints $C1$, $C2$, and $C3$, whose current mgraph is shown in Figure 7a. The clock diagram indicates that $C1$ has a single method which outputs to both $V1$ and $V2$ (this diagram is not shown for constraints with a single-output method outputting to each of their variables). If we pin $C2$ and not $C3$, we produce the mgraph of Figure 7b, and collect its E-set. On backtracking, if we leave $C2$ unpinned, and pin $C3$, we will produce Figure 7c, which has the same E-set.

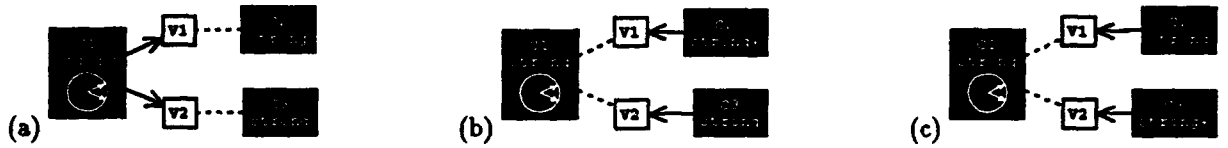

Figure 7: Starting with mgraph (a), **get_esets** may collect the same E-set multiple times (b,c).

4.6 Collecting Some LGB Method Graphs by Adding Stay Constraints

It would be possible to modify **save_eset** to collect the enforced constraints along with their current selected methods when it is called within **get_esets**. If the given set of constraints had exactly one LGB mgraph for each LGB E-set, this would collect all of the LGB mgraphs. However, if there are multiple LGB mgraphs that have the same E-set (Figure 8), there is no guarantee that they would all be generated by **get_esets**.

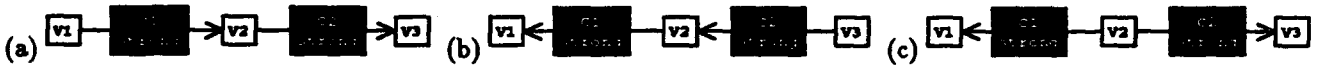

Figure 8: Three possible LGB mgraphs for $\{C1, C2\}$.

One thing that distinguishes different LGB mgraphs with the same E-set is the sets of variables that are determined and undetermined. This observation can be used to generate these different LGB mgraphs: Given a set of constraints *cns*, let *v-weak* be a strength weaker than any of these constraints. For each of the variables that can be determined by any of the constraints' methods (the *potential outputs* of the constraints), add a new stay constraint with strength *v-weak*. Consider an LGB mgraph for this extended set of constraints, *cns'*. The selected methods for *cns* in the extended mgraph define an LGB mgraph for *cns* alone, since none of the *v-weak* stay constraints can effect which stronger constraints are enforced, but they can effect the selected methods used to enforce stronger constraints. Calling **pin_cns**(*cns'*) will pin all of the constraints including the *v-weak* stay constraints, generating different LGB mgraphs for *cns*. For example, Figure 9 shows how extra *v-weak* stay constraints added to the constraints from Figure 8 can be pinned to generate the mgraphs in Figure 8a and 8c.

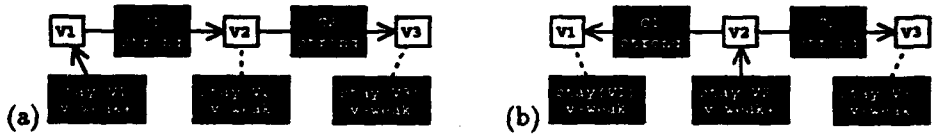

Figure 9: Pinning extra stay constraints to generate different LGB mgraphs for $\{C1, C2\}$.

Figure 10 presents pseudocode that creates the extra variable stay constraints and passes all of these constraints to **pin_cns**, which will generate different LGB mgraphs for **cns**. Note that ***save_proc*** is set to the procedure **save_mgraph**, so it will be called to save the current mgraph (including selected methods)

within pin_cns. *cns* does not include the extra variable stay constraints, since we are only concerned with collecting the method graphs for the original constraints.

```
get_some_lgb_mgraphs(cns)
   *cns* := cns
   *mgraphs* := {}
   *save_proc* := save_mgraph
    ;; add stays to output variables
   var_stay_strength := any strength weaker than all of the constraints in cns
   potential_outputs := a list of all potential output variables for cns
   output_var_stays := a stay constraint with strength var_stay_strength
                       for each var in potential_outputs
   For cn in output_var_stays do add_constraint(cn)
    ;; generate mgraphs for constraints, including extra stays
   pin_cns({}, {}, {}, cns ∪ output_var_stays)
    ;; remove added stays
   For cn in output_var_stays do remove_constraint(cn)
   return *mgraphs*

save_mgraph()
   mgraph := For cn in *cns* collect cn and its current selected mt
   add mgraph to *mgraphs* if it is not already there
```

Figure 10: Pseudocode to generate some LGB mgraphs for cns.

4.7 Collecting All LGB Method Graphs by Adding Method Variables

There are two situations where get_some_lgb_mgraphs may not generate all possible LGB mgraphs for a set of constraints: (1) There are directed cycles of methods. The constraints $\{C1, C2\}$ in Figure 11a have two LGB mgraphs, one with a directed cycle in each direction. Pinning extra stay constraints on the variables will not choose one mgraph over the other. (2) There are constraints with "subset methods," where the outputs of one constraint method are a subset of the outputs of another method for the same constraint. This is rare, but it is not prohibited by the definition of multi-way local propagation constraints. For example, constraint $C3$ in Figure 11b has one method that outputs to $V7$ and $V8$, and another method that outputs to $V8$. If the constraint solver always chooses the second method, get_some_lgb_mgraphs will never generate an mgraph containing the first method.

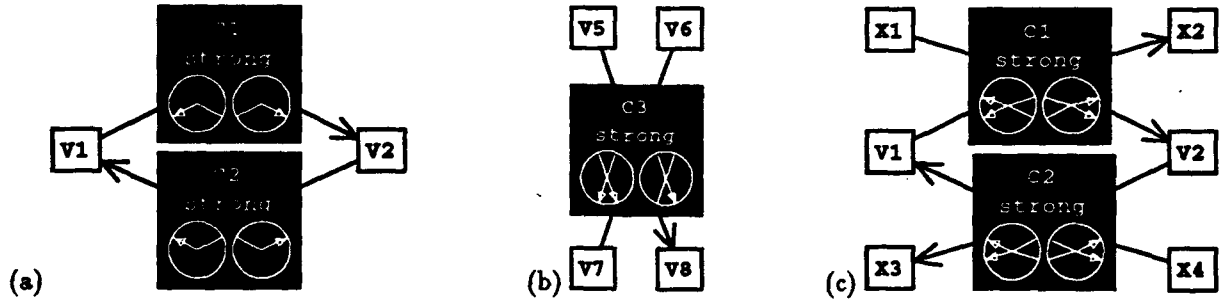

Figure 11: (a) A method graph with a directed method cycle. (b) A constraint with subset methods. (c) Adding extra variables to (a). Method diagrams are shown in (a) for comparison with (c).

Given cycles or subset methods, it is possible to generate all possible mgraphs using the pseudocode of Figure 12. This code modifies every constraint that has more than one method, creating an extra variable for each constraint method, and adding it as an output to all of the *other* methods of the constraint. Applying this to Figure 11a produces Figure 11c, where the new variable $X1$ is only set when $C1$ is enforced with a method other than its second method (setting $V2$ and $X2$). When the modified constraints are passed to

get_some_lgb_mgraphs, and *v-weak* stays are added to these extra variables, pinning these extra stays will try all of the methods of each constraint, if they are allowed in an LGB mgraph. Note that constraints with only a single method do not need to have any extra variables added, since such a constraint's single method must be used whenever the constraint is enforced.

```
get_all_lgb_mgraphs(cns)
    ;; add extra variables to methods
   For all constraints cn in cns with more than one method do
      remove_constraint(cn)
      For mt in cn.methods do
         v := create a new variable
         add v to cn.variables
         add v to the outputs of all of cn's methods except mt
      add_constraint(cn)
    ;; add extra stays to variables, and generate mgraphs
   mgraphs := get_some_lgb_mgraphs(cns)
    ;; remove extra variables from constraints and methods
   For all constraints cn in cns with more than one method do
      remove_constraint(cn)
      restore cn.variables
      restore outputs for all of cn's methods
      add_constraint(cn)
   return mgraphs
```

Figure 12: Pseudocode to generate all LGB mgraphs for cns.

The pseudocode removes all of the constraints before adding the additional variables to the methods, and then re-adds the constraints. Likewise, the constraints are removed before removing these additional variables. If the constraint solver had an entry for modifying methods, this would not be necessary.

4.8 Evaluating the Algorithms

We are currently comparing the performance of get_all_lgb_mgraphs to alternate algorithms for generating LGB mgraphs. An earlier algorithm enumerated all possible combinations of selected methods without method conflicts, collecting all mgraphs that were LGB. This worked well for small networks, but was much too slow for large networks (taking time exponential in the number of constraints). Testing shows that get_all_lgb_mgraphs is much faster than the earlier algorithm for many sets of constraints encountered in actual practice (2 seconds versus 22 minutes for one set of 17 constraints), but we have been able to construct constraint networks where it is significantly slower than the earlier algorithm. get_all_lgb_mgraphs appears to be most efficient when there are only a few possible LGB mgraphs. In the absence of a simple way to predict which algorithm is faster, it might be reasonable to run both algorithms in parallel.

The different algorithms described in this paper may be useful at different points during debugging. get_osets can be used to determine whether a given constraint is always enforced, or never enforced. If there are no subset methods and the programmer doesn't care about the directions of cycles, get_some_lgb_mgraphs can be called instead of get_all_lgb_mgraphs.

These algorithms call the SkyBlue constraint solver to manipulate the constraints. Therefore, any future performance improvements to SkyBlue (or other algorithms that maintain LGB mgraphs) will directly improve the performance of these algorithms.

5 Conclusions and Future Work

We have described some of the debugging tools included within our system for interactively constructing constraint-based user interfaces, and presented a new algorithm for generating all of the LGB method graphs for a set of constraints. This algorithm is the basis for a powerful debugging tool that allows the programmer to explore the different behaviors that can be produced by a set of constraints.

In the future we want to continue developing new debugging tools, improving the facilities for invoking them and presenting the results of their analyses. We also want to conduct user testing, to determine which debugging tools are particularly helpful to programmers when constructing large constraint networks.

Acknowledgements

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Inferring 3-dimensional constraints with DEVI

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Abstract

Constraints can be used to specify and maintain spatial relationships among objects in a geometric design. In the 3-D geometric design domain, the diversity of possible relationships among objects makes it difficult for the designer to specify useful or intended relationships in a productive and intuitive manner. We have built a constraint-based 3D geometric editor called DEVI that infers possible or intended relationships among objects of the design. DEVI's database of relationships between design primitives can be extended using a descriptive language which enables the developer to specify a set of rules made up of conditions, to be satisfied, and inferences to be made. Each rule has two parts; the first is a boolean condition wherein a certain situation is described; the second part is an instruction to the system to infer the specified constraint (or set of constraints) if the boolean condition is true.

1 Introduction

Constraints have proven useful in automatically keeping spatial relationships satisfied among geometric objects in a geometric design. They alleviate

much of the tedium involved in making small changes and then propagating their effect.

Constraint-based geometric-design is not a new area. Rossignac's CSG system [Rossignac86] allows the user to specify models in terms of unevaluated constraints that are evaluated sequentially, during the construction process, in a user-specified order. Constraints are evaluated by performing rigid-body motions. In their paper, Kapur *et.al.* [Kapur91] describe a generic model for representing polyhedra as a network of nodes and constraints. Van Emmerik's solid modeling system [vanEmmerik90] is an example of a constraint-based modeling system with a graphical front-end that allows the specification of constraints via popup and cascading menus.

Constraints thus represent a significant advantage in geometric-design systems. To gain this advantage, designers have to invest extra effort to specify constraints. Geometric-design systems usually allow a fixed number of pre-defined relationships among the geometric objects. In a typical interactive geometric design environment, the designer would create the geometric objects using menus, positioning and alignment tools, and tell the system how he wants them constrained to each other. The designer normally knows, in advance, where the new geometric object that he is creating should be located and how it should be related to its' neighbours. He therefore tends to create a situation close to the end-result that he has in mind.

To make the process of specifying relationships simpler, earlier systems have used some of the following approaches. VanWyk's automatic drawing beautifier [vanWyk85] looks at a design to check if any predefined relationships exist and makes them persistent. This approach is not interactive. Converge [Sistare90] provides a "locus" input mode for constraints wherein newly created geometric objects are automatically constrained in a desired way to a specified existing geometric object. It is an improvement over the previous approach but is still rather limiting because it forces the user to switch modes constantly and is not a general solution. A more general approach is to use geometric context, users' actions and knowledge of geometric objects and their possible relationships to infer what the user is trying to do. Variations of this approach have been used successfully by a few systems, primarily in the two-dimensional domain. Peridot [Myers86] uses the 'demonstration' metaphor to help specify constraints. It infers the relationships of the users' actions to user-interface elements during a demonstration sequence, and generates code to handle this action in a real situation. Briar [Gleicher92] augments snap-dragging [Bier86] by making the relationships persistent. Rockit [Karsenty92] also uses augmented snap-dragging

and maintains a database of relationships and a static inference-rule base — it allows the user to dynamically change the conditions that determine which rules to execute.

The last approach has a limitation: it is difficult to extend and customize. We propose to extend this method by allowing the designer to write inference rules in a descriptive language. Each rule has two parts; the first is a boolean condition wherein a certain situation is described in terms of geometric objects and the geometric constraints relating them; the second part of the rule is an instruction to the system to infer a specified constraint (or set of constraints) if the boolean condition is satisfied. These rules are applied in response to interactive events like creation or perturbation of geometric objects.

In this paper we describe in detail how DEVI infers constraints using it's knowledge database and how we have augmented it with our inference-rule approach. For a general introduction to DEVI, please see [Thennarangam93].

2 DEVI's Approach

DEVI is a constraint-based, interactive 3D geometric editing environment that uses flexible user-interface techniques to simplify the task of editing 3D geometry (see Figure 1). DEVI provides an interpretive language to specify geometric objects and the constraints between them. It infers constraints among geometric objects as they are created and manipulated. These inferred constraints subsequently become persistent and are maintained by the system. In order to help understand and debug the design in a graphical fashion, DEVI presents the network of constraints and geometry in a constraint-network browser that is useful in determining relationships and debugging the constraint network (see Figure 1).

DEVI organizes its' constraint network as a partitioned directed graph. When an event occurs in the interactive geometry editor, DEVI quickly isolates the portion of the design that will be affected by the event. By default, DEVI only tries to infer constraints between selected objects and newly created or newly perturbed objects — a newly created object is added to the current selection. This limits the number of inferences the system has to consider. Experience with the Druid UIMS [Singh90] shows that designers tend to create designs incrementally — they usually create related parts of the design one after the other, rather than randomly. DEVI exploits this fact. In case it makes more than one inference, it prompts the user to make

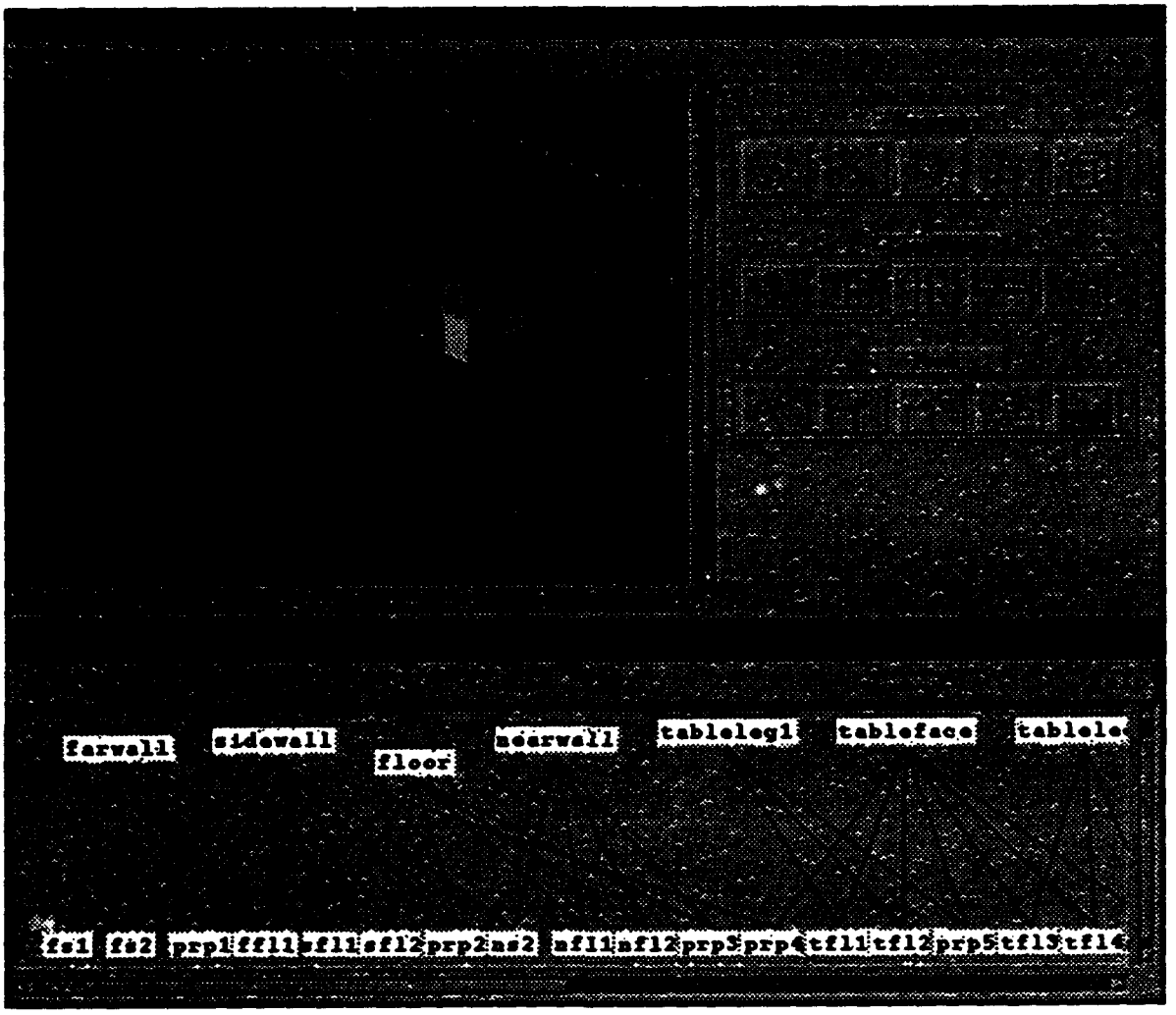

Figure 1: DEVI's User-Interface - The top-left window is the design work-area. To the right are the geometry, constraint and tools palettes.The window at the bottom depicts the constraint-network browser. The user can edit and manipulate constraints using the browser.

a choice. This scheme also serves as an aid in case the designer forgets to specify some relationships that he had intended to.

Consider the process of solving a network of constraints. DEVI's hybrid solver propagates known values about the constraint graph until it satisfies each constraint node. Failing to do so, it resorts to an iterative approach, using Newton-Raphson's iteration to solve the set of algebraic functions that describe the constraint network.

A minimization function is defined for each constraint. When this function has a value close to zero, that instance of the constraint is considered satisfied. In fact, DEVI calls the algebraic solver with a single constraint as an argument, demanding only a single iteration of the solver, when it is propagating values in the constraint network; the solver tells DEVI whether the constraint is satisfied or not. DEVI uses this property to infer constraints. Consider a scenario where we want to infer a constraint between geometric objects A and B. Assume that constraints of type C_1 and C_2 can exist between the geometric classes of A and B. We compute the minimization function values for those two classes of constraints i.e. $f_{C_1}(A,B)$ and $f_{C_2}(A,B)$. We choose those values that are smaller than some threshold value. If there is more than one such value then the user is asked to make a choice.

For the sake of generality and robustness of the solving process, one can have the minimization function return an euclidean value that is representative of the constraint. See [Sutherland80] and [Sistare90] for further discussion on this point. For example, to constrain a point to lie on a plane, one measures the distance of the point to the plane. Algebraically, this can be expressed by substituting the point in the equation of the plane. With more complicated constraints, it is not easy to see this relationship.

Consider another constraint that fixes the angle between two planes A and B to be θ. Let $k_A = |n_A|, k_B = |n_B|$, where n_A and n_B are unit normal vectors to planes A and B respectively. We could express the function we want to minimize as follows:

$$f_\theta = \sqrt{k_A^2 + k_B^2 - 2k_Ak_B\cos\theta} - |n_A - n_B|$$

The geometrical interpretation of this equation is as follows:

$\sqrt{k_A^2 + k_B^2 - 2k_Ak_B\cos\theta}$ is the desired length between the tips of the vectors n_A and n_B, when they are constrained at an angle θ, and $|n_A - n_B|$ is the current length. Obviously, when this value reaches zero, the constraint is satisfied.

The problem with this approach is how to choose an appropriate thresh-

old value, which is essentially a real number. This number might not have much intuitive value. Since the designer may not be able to relate too well to these seemingly obscure numbers, he might find it difficult to customize the inferencing process. All he has are a set of real numbers that he can manipulate back and forth. The significance of this number is determined by how the system translates geometric constraints into algebraic equations that can be minimized.

Our method uses familiar terminology like distance, angle etc. Although the solver eventually works with vectors and real numbers, the designer thinks in terms of concepts like angles, adjacencies, equalities and distances. He might want, for example, to specify that angles close to 45° should become 45° angles because, presumably, he has a lot of 45° angle situations in his design and he does not want to go through all the trouble of explicitly constraining them each time.

Here is an example of a rule written in this language:

```
RULE Rp( POLYGON p1, POLYGON p2)
   IF  ( (ABS ( ANGLE ( p1, p2 ) - 45 )) <= 5 )
   THEN
      INFER CONSTRAINT FIXANGLE ( p1, p2, 45 );
```

Based on this rule, the system automatically constrains the planes of two polygons to lie at 45° to each other if it detects that the angle between them is anywhere between 40° and 50°. The magic number 5 here represents the desired angular tolerance and it is easy to understand exactly what it represents and the consequences of changing this number to suit preferences. A significant advantage of this approach is that it provides a way to limit the problem of "over-generalization" [Bos92] with inferencing systems — that they insist on making inferences that the user did not intend.

The syntax of DEVI's inference rules is simple and the effort involved in creating new rules is well worth the effort in terms of the productivity gained from it.

DEVI stores inference rules in memory; for each inference rule, a record is maintained that consists of it's name, it's parameter types and a parse tree that is evaluated each time the rule is triggered. A rule is triggered if it's parameters match with that of the current event. Inference rules are thus treated like geometric or constraint objects. They can be created, deleted or modified.

3 Implementation

DEVI has been implemented in the C++3.0 programming language. It runs on a Silicon Graphics Indigo Elan workstation running IRIX 4.0.5F and the X11R4 Windows system. It uses the Motif 1.1 toolkit and IrisGL graphics libraries.

4 Discussion

We have presented a system that enhances the ease and functionality of using constraints in a geometric-design environment. DEVI uses constraint-inferencing to ease the effort involved in specifying constraints in the 3-D geometry domain. It achieves this by using it's knowledge of geometric objects and their possible relationships. It also provides a powerful means to extend the inferencing process by allowing the user to write inference rules. To our knowledge, DEVI is the first system that infers spatial constraints in the 3-D geometric domain and provides the user with the means to extend and customize the system's constraint - inferencing capability.

Future work on DEVI will concentrate on making it more conversational, especially offering advice on the degree of constraint to the designer and detecting and warning of redundant and circular constraints. We would also like to provide a graphical way to compose and edit inference rules. Metamouse is a 2-D drawing program [Maulsby89] that induces picture-editing procedures from execution traces of the users actions at work — it performs a localized analysis of changes in spatial relationships to isolate constraints and matches action sequences to build a state graph that describes what it has learned. On detecting a repitition, it uses the state graph to predict further actions. We are not sure if this approach can easily scale to three-dimensions, given the inherently much greater complexity and the limitations of even the state-of-art 3-D interface systems. Virtual reality systems will probably allow this area to develop much more.

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Beyond Finite Domains

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1 Introduction

A finite domain constraint system can be viewed as an linear integer constraint system in which each variable has an upper and lower bound. Finite domains have been used successfully in Constraint Logic Programming (CLP) languages, for example CHIP [3], to attack combinatorial problems such as resource allocation, digital circuit verification, etc. In these problems, finite domains allow a natural expression of the problem constraints because bounds on the problem variables are explicit in the problem. In other problems however, for example in temporal reasoning and some scheduling problems, there may not be natural bounds.

For these problems, a standard approach has been to use ad hoc bounds, giving rise to a two-fold problem. If a bound is too tight, then important solutions could be lost. If a bound is too loose, then significant inefficiency may result. This is because the algorithms used in finite domains work by propagating bounds on variables[1] until certain local consistency conditions (for example, arc-consistency [4, 11]) are achieved. These algorithms have the disadvantage that they reason about transitivity of inequalities in an iterative manner; for example, detecting that $x + 1 \leq y \leq x$, $0 \leq x, y \leq k$ is unsatisfiable will require a cost proportional to k.

We thus suggest that it is worthwhile to go beyond finite domains to more general integer constraints. The issue then becomes the trade-off between greater expressiveness and potentially exponential-cost constraint solving. In this abstract, we propose a restricted class of integer constraints which can be solved more efficiently than in the general case, but which remains reasonably expressive. Furthermore, our algorithm can be extended easily to accommodate more general integer constraints (though not in all cases), and it also combines well with traditional propagation-based methods. In this abstract, our presentation level is restricted to high-level algorithmic issues, and we do not address specific implementation considerations.

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[1] And other domain information, in general.

2 Integer Constraints

What are the features we desire of an integer constraint domain and solver for a CLP system? Clearly soundness is essential. Completeness is obviously attractive, but there is no known sufficiently efficient solver. In fact, the satisfiability problem for nonlinear integer constraints is undecidable, so completeness is impossible to achieve. The problem is decidable for linear integer constraints, but it is NP-complete. Thus it appears that, in the context of a CLP system, a constraint solver that handles linear integer constraints will necessarily be either incomplete or inefficient. In practice, the choice taken by implementations is incompleteness and efficiency.

Given an incomplete solver, it is highly desirable to be able to characterize classes of constraints for which the solver is complete. On the practical front, the algorithm should be efficient, incremental and should support backtracking. Other operations that may be required are: the ability to detect groundness, implicit equalities and constraint entailment, the ability to extract constraints from disjunctive information, and the ability to eliminate variables (projection).

Propagation-based solvers (e.g. [4, 5]) are a prime example of the choice of an efficient algorithm which is relatively incomplete. These solvers are complete when each of the constraints they handle involves only a single variable. Call this the class of (linear) single-variable-per-inequality (SVPI) integer constraints [2]. In general, these propagation-based solvers handle constraints by extracting SVPI information from the interaction of non-SVPI constraints with SVPI constraints. These solvers satisfy most of the efficiency criteria mentioned above, but are incomplete and/or inefficient when handling problems with variables which are unbounded or have very large domains.

An obvious generalization of SVPI is the class of (linear) two-variable-per-inequality (TVPI) integer constraints. This class appears to be strictly simpler than the general problem (unlike the three-variable-per-inequality problem). There is a strong analogy here with the corresponding problem over the real numbers. There, current algorithms for deciding real TVPI constraints (e.g. [2]) are more efficient than current algorithms for arbitrary real linear constraint solving. Certainly integer TVPI constraints are far more expressive than SVPI constraints: for example we can encode constraints such as $x \bmod 11 \in \{1,5\}$ by $x \geq 11y+1, x \leq 11y+5, x \geq 11z+5, x \leq 11z+12$. Surprisingly, solving integer TVPI constraints is also NP-complete [9]. However TVPI constraints seem more directly amenable to transitivity-based methods similar to those used for real constraints [1, 2, 14].

A class of constraints intermediate between SVPI and TVPI is the class of TVPI constraints $ax + by \leq d$ with unit coefficients, that is, $a, b \in \{-1, 0, 1\}$. We call these *unit TVPI constraints*. This class is considerably less expressive than the general class of TVPI constraints (for example, it cannot express modulo constraints). However, the constraints are sufficient for many problems in temporal reasoning and scheduling.

Much earlier, Pratt [12] had considered a restricted class of unit TVPI constraints, those of the form $ax + d \leq by$, $a, b \in \{0, 1\}$, and presented an efficient algorithm

[2] In this abstract we ignore disequality ($\neq$) constraints. However we note that the addition of disequalities such as $x \neq y$ to the class of SVPI constraints results in an NP-complete satisfiability problem (see, for example, [13]).

for their solution. (Essentially, the integer and real satisfiability problems for these constraints are equivalent, and hence real-based methods are applicable.) However, unlike unit TVPI constraints, this class is not expressive enough for many problems. For example it cannot express the mutual exclusion $\neg(x \wedge y)$ which has a unit TVPI representation $x + y \leq 1, x \geq 0, y \geq 0$. The generalization of Pratt's class to permit any positive a, b can be solved using arc-consistency techniques, provided all variables have finite domains [4].

3 Unit TVPI Constraints

At the heart of our algorithm is a general framework for implementing transitive-closure in real TVPI inequalities. This is described in the next subsection, together with an adaptation of the framework to deal with integers. In the next section we present our algorithm as an instance of the (modified) transitive-closure algorithm.

3.1 A Transitive-Closure Algorithm

Shostak [14] gave an algorithm for satisfiability of real TVPI problems, not restricted to unit coefficients. In this algorithm every single variable inequality (i.e. bound) is converted to a two variable inequality by adding a dummy variable v_0 as follows: $x \leq 10$ becomes $x + 0v_0 \leq 10$. We give an incremental formulation of this algorithm, which maintains integer coefficients for integer problems, as follows.

A singleton set of TVPI constraints is *transitively closed*. Given a transitively closed set of TVPI constraints C and new TVPI constraint $c \equiv ax + by \leq d$. Let $C_x^a = \{c' : c' \equiv a'x + b'z \leq d', a \times a' < 0, c' \in C\}$. Define C_y^b similarly. The *transitive closure* of $C \cup \{c\}$, is

$$
\begin{aligned}
C \cup \{c\} \;\cup\; & \{|ab''|ez + |a'b|ft \leq |a'b|d'' + |a'b''|d + |ab''|d' \;:\\
& \quad a'x + ez \leq d' \in C_x^a, b''y + ft \leq d'' \in C_y^b\}\\
\cup\; & \{|a'|by + |a|ez \leq |a'|d + |a|d' \;:\; a'x + ez \leq d' \in C_x^a\}\\
\cup\; & \{|b''|ax + |b|ft \leq |b|d'' + |b''|d \;:\; b''y + ft \leq d'' \in C_y^b\}
\end{aligned}
$$

The system $C \cup \{c\}$ is satisfiable (in the reals) iff the transitive closure does not contain a constraint of the form $0 \leq d$ where $d < 0$ [14].

The algorithm is immediately applicable to integer TVPI problems, but it is not complete. For example, consider $2x + 2y \leq 1, -2x + -2y \leq -1$. This is equivalent to $2x + 2y = 1$ which clearly has no integer solutions. The difficulties arises because these inequalities are equivalent (in the integers) to the tighter constraints $x + y \leq 0$ and $-x + -y \leq -1$.

We extend Shostak's algorithm by adding tightening constraints. The *tightening constraints* of C, denoted *tightening*(C), are

$$\{a/kx + b/ky \leq d' \mid ax + by \leq d \in C, gcd(\{|a|, |b|\}) = k, k \neq 1, d' = \lfloor d/k \rfloor, d' < d/k\}$$

For example $tightening(\{2x+2y \leq 1, -2x+-2y \leq -1\})$ is $\{x+y \leq 0, -x-y \leq -1\}$. The tightening constraints give more information so that we are more likely to find unsatisfiability.

Once we have determined tightening constraints they need to be added and their transitive consequences found as above. By interleaving tightening and transitive closure we eventually obtain a transitively closed, tightened set of constraints, given the procedure terminates. We conjecture that, for integer TVPI, this procedure always either detects unsatisfiability or terminates. In the case of unit TVPI constraints, it is easy to show this is true.

Even if the procedure always terminates it is still incomplete. The following system provides a counterexample. It describes a unit cube with several edges cut off, so that no corner remains. It has no integer solution, but each real projection onto two variables has an integer solution.

$$\begin{aligned} 0 \leq x, y, z &\leq 1 \\ 4x + 3y &\leq 6 \\ -3x - 4y &\leq -1 \\ 4x - 3z &\leq 3 \\ -3x + 4z &\leq 3 \\ 4y - 3z &\leq 3 \\ -3y + 4z &\leq 3 \end{aligned}$$

However, the algorithm is clearly "more complete" than bounds propagation which, in the TVPI case, is simply the application of transitivity to one TVPI constraint and one SVPI constraint (possibly with tightening).

Clearly the above procedure is naive in a number of ways. First, corresponding to tightening we also wish to divide the coefficients and constants of each constraint $ax + by \leq d$ so that $gcd(\{a, b, d\})$ is 1. Second, we can eliminate redundant constraints that are generated by the method. Detecting all redundant constraints is just as hard as the satisfaction problem, in general, but some kinds of redundancy are easy to detect. A constraint $exp \leq d$ is *quasi-syntactic redundant* [6] with respect to constraints C if a constraint of the form $exp \leq d'$ appears in C where $d' \leq d$. Quasi-syntactic redundancy is particularly easy to detect. More generally, we can remove any TVPI constraints involving x and y that are redundant (in the reals) with respect to the other x, y constraints.

3.2 The Unit TVPI Solver

When dealing with unit TVPI constraints, the transitive closure algorithm above produces new unit TVPI constraints, except in one case. Consider $C = \{x+y \leq 1, x+z \leq 2\}$ and the addition of $-y+-z \leq 0$. One of the consequences is $2x \leq 3$ which is not of the correct form. But we can always simplify such constraints to have unit coefficients, in this case $x \leq 1$. Moreover this is the only way in which tightening is possible.

For each pair of variables x, y there are at most four possible non quasi-syntactic redundant constraints: $\{x + y \leq d_1, x - y \leq d_2, -x + y \leq d_3, -x - y \leq d_4\}$. Hence the

maximum number of (non-redundant) constraints that can be produced by closure under transitivity and tightening for a system including n variables is $2n^2$. Quasi-syntactic redundancy elimination is very simple, it just requires maintaining the minimal d for each of the above constraint forms. This, together with the fact that no tightened constraints can create further tightened constraints, gives a polynomial time bound on the algorithm.

Given a new constraint $ax + by \leq d$ and a tightened, transitively closed set of constraints C, there are at most $2n$ constraints in C involving $-ax$ and $2n$ constraints involving $-by$. Thus the cost of transitive closure is $O(n^2)$. Tightening will introduce at most $2n$ constraints, all of which are bounds. Further transitive closure will produce only more bounds, and at most $2n$ of them for each initial bound. Thus tightening and further closure also has cost $O(n^2)$. Hence the cost of producing a new tightened and transitively closed set of constraints is $O(n^2)$. It follows that the cost of testing the satisfiability of N unit TVPI constraints with our algorithm is $O(N^3)$ in time and $O(N^2)$ in space.

The key result relating unit TVPI constraints to transitive closure and tightening is as follows:

Theorem 1 *Let C be a set of unit TVPI constraints that is closed under transitivity and tightening. Let $C|_{-x}$ denote the conjunction of constraints in C that do not contain x. Then $\exists x\ C \leftrightarrow C|_{-x}$ $\Box$.*

The proof follows the proof of the corresponding result for (arbitrary) inequalities over the reals, with only minor modifications. It extends to general TVPI constraints only to the extent that all occurrences of the eliminated variable have only unit coefficients.

Given the above result it is easy to show that:

Theorem 2 *Let C be a set of unit TVPI constraints that is closed under transitivity and tightening. Then C is satisfiable iff it does not contain a constraint of the form $0 \leq d$ where $d < 0$. $\Box$.*

This demonstrates the completeness of our algorithm. Note that propagation-based methods are not complete for unit TVPI constraints, even for finite domain problems. Consider the following example:

$$\begin{aligned} x - y &\leq 2 \\ x + y &\leq 1 \\ -x + z &\leq -2 \\ -x - z &\leq -1 \\ -20 \leq x, y, z &\leq 20 \end{aligned}$$

Bounds propagation simply determines that the variables lie in the following ranges $-17 \leq x \leq 20$, $-19 \leq y, z \leq 18$. In fact there is no solution: the first two constraints imply $2x \leq 3$ and hence $x \leq 1$; the second two constraints imply $2x \geq 3$ and hence $x \geq 2$. Our approach discovers the unsatisfiability essentially by following the above argument.

Recently we have learned of a related approach [8] to testing the satisfiability of general integer constraints which is based on extending Fourier's algorithm for the

reals (see, for example, [10]) to integers. The relationship between this approach and our algorithm is quite close, since transitive closure can be thought of as a cumulative form of Fourier's algorithm with redundancy elimination. The algorithm of [8] is not suitable for a CLP solver since it is not incremental. However the work may be a useful basis for extending our TVPI algorithm.

We can expect to improve the efficiency of the approach by treating equations, for example $x + y = 3$, directly rather than as two inequalities $x + y \leq 3, -x - y \leq -3$. Any unit TVPI equation can be used as a substitution to eliminate one of its variable, for example $x + y = 3$ can be used to replace each occurrence of x by $-y + 3$. Note that applying such a substitution to a unit TVPI constraint either maintains the unit TVPI form or creates a constraint of the form $2y \leq k$ which can be simplified to a unit TVPI constraint (possibly with tightening).

To maintain the transitive closure the approach above is modified to treat an equation $x = t$ as follows: add both the inequalities, $x \leq t, x \geq t$, and close under transitivity and tightening, then remove inequalities involving x. The equations are maintained separately in Gauss-Jordan normal form and they are applied as substitutions to constraints that are added later. Note that we need to fail if we detect equations (after substitution) of the form $2x = k$, where k is odd, and to simplify if we detect equations of the form $2x = k$, where k is even.

Given we are keeping the equations in a separate tableau it seems worthwhile to extract implicit equations from the inequalities. When we detect a transitive consequence of the form $0 \leq 0$ this signals that the inequalities which produced it are implicit equations. By marking these and waiting till the closure process terminates we can extract the (marked) implicit equations, place them in the equation tableau and simply remove the inequalities that involve a substituted variable.

It is easy to extend this approach to perform other operations of interest. Let the *active store* denote the current set of TVPI constraints in the computation, closed under transitivity and tightening. Any groundness information that is a consequence of the active store will appear in the equation tableau (perhaps through implicit equations). Constraint entailment can be simply determined because of the following result:

Theorem 3 *Let c be a unit TVPI constraint and let C be a satisfiable set of unit TVPI constraints that is closed under transitivity and tightening. Then $C \rightarrow c$ iff either c is a tautology, c is implied by the SVPI constraints of C, or c is quasi-syntactic redundant with respect to a constraint in C* $\square$.

Hence to determine whether a unit TVPI constraint is entailed by the active store we simply check if it is quasi-syntactically redundant or implied by bounds (after substitution). It is straightforward to make this check incremental. Projecting out a variable is straightforward: if the variable appears in an equation this can be rewritten to eliminate the variable, otherwise all inequalities involving the variable can simply be removed (c.f. Theorem 1).

Using the result of Theorem 3 we can determine unit TVPI consequences of the disjunction of two tightened transitively closed unit TVPI constraint sets C_1 and C_2, as follows. For each inequality form $ax + by \leq ...$, let $ax \leq d_i^x \in C_i$, $by \leq d_i^y \in C_i$, and

$ax + by \leq d_i \in C_i$, for $i = 1, 2$[3]. Then $ax + by \leq d$ is a consequence of $C_1 \vee C_2$, where $d = max(min(d_1, d_1^x + d_1^y), min(d_2, d_2^x + d_2^y))$ if $a, b \neq 0$ and $d = max(d_1, d_2)$ otherwise. The set of such constraints describes the smallest unit TVPI polyhedra that contains both C_1 and C_2. Extending this procedure to handle separate equations is reasonably straightforward. This procedure can be the basis of constructive disjunction in this constraint domain.

3.3 The Solver in a General Setting

In the context of a CLP system we want to handle a larger class of constraints than just unit TVPI constraints. For non-unit TVPI constraints we can use propagation-based methods to extract SVPI information in exactly the same way as finite domain solvers. Note that the bounds on variables are available in the unit TVPI inequalities. Applying propagation to unit TVPI constraints is unnecessary as they are completely handled already. We can use the equation tableau to simplify non unit TVPI constraints by substitution. The resulting constraints (possibly after tightening) may be unit TVPI constraints. For example, applying the substitution $y = z + 1$ to $5x + 3y + 2z \leq 7$ results in $5x + 5z \leq 4$ and thus $x + z \leq 0$.

An alternative is to apply the (incomplete) method of section 3.1 to all TVPI constraints. Non-TVPI constraints would be treated by propagation methods, as above. This would provide a more powerful, but more expensive, integer solver. The choice between these alternatives can only be made after experimental evaluation.

4 Conclusion

Unit TVPI constraints are sufficiently expressive for many problems: for example in scheduling and temporal reasoning. We give an algorithm for incremental satisfiability of unit TVPI constraints. Not only is this algorithm efficiently implementable, it also supports efficient implementation of entailment detection, including entailment of disjunctive constraints, and projection. Finally, for use in a CLP system, constraints more general than unit TVPI must be handled, though not necessarily in a complete way. Our algorithm naturally extends to (non-unit) TVPI constraints, and it can be augmented with a bounds-propagation technique for constraints more general than TVPI. An implementation of the solver is underway as part of the development of CLP($\mathcal{R}$) [7].

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QUAD-CLP(R) : ADDING THE POWER OF QUADRATIC CONSTRAINTS

GILLES PESANT * AND MICHEL BOYER *

Abstract. We report on a new way of handling non-linear arithmetic constraints and its implementation into the QUAD-CLP(R) language. Important properties of the problem at hand are a discretization through geometric equivalence classes and decomposition into convex pieces. A case analysis of those equivalence classes leads to a relaxation (and sometimes recasting) of the original constraints into linear constraints, much easier to handle. Complementing earlier expositions in [18] and [19], the present focus is on applications upholding its worth.

1. Motivation. This paper presents the constraint programming language QUAD-CLP(R) which offers a powerful novel solving strategy for non-linear arithmetic constraints under the computing paradigm of logic programming. Emphasis will be given here to the techniques involved in the constraint solver for quadratic constraints over **R** and to applications making use of this added power.

Despite the enormous potential of non-linear arithmetic constraints in several spheres of scientific activity, typical efforts to provide for them amidst constraint languages have brought mostly disappointments as the resulting solvers either lacked effectiveness or scalability.

The delay strategy implemented in languages such as CLP(R) [10] and PROLOG III [1] yields an incomplete solver which will be effective only if the problem under attack is such that reasoning about linear constraints ultimately becomes sufficient. Unfortunately, this is seldom the case for interesting problems, even very simple ones. One classic example is the multiplication of complex numbers, which can be expressed as `cmult((R1,I1),(R2,I2),(R1*R2-I1*I2,R1*I2+R2*I1))` in predicate calculus. Among interesting queries, "`?- cmult((R,I),(R,I),(-1,0)).`" requires reasoning about non-linear system `-R*I = R*I, I*I -1 = R*R`.
QUAD-CLP(R) can easily handle this, giving the answer:

```
I = 1
R = 0
*** Retry? y

I = -1
R = 0
*** Retry? y

*** No
```

On the other hand, languages like CAL [20] and RISC-CLP(Real) [7] bear witness that the price to pay to achieve a complete solver seems to be the use of costly computational algebra techniques which confine their usefulness to very small (albeit interesting) problems.

Our approach, introduced in [18], takes advantage of the ease with which quadratic constraints can be replaced or approximated by linear constraints. It is therefore especially well-suited to problems involving quadratic and linear constraints. There

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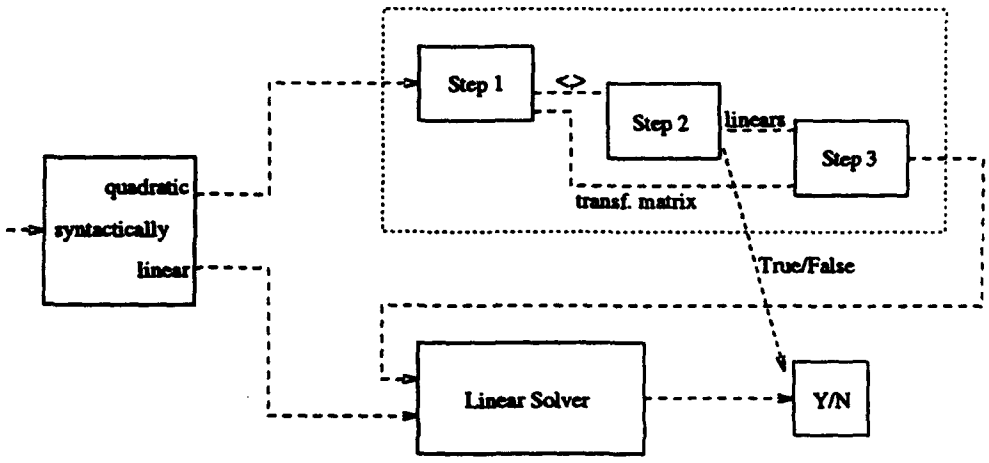
FIG. 1. *The constraint solver*



is nevertheless the possibility of handling general arithmetic constraints by breaking them down into quadratic components through the introduction of auxiliary variables (we address this further in §5).

Even a restriction to quadratic constraints still provides a rich and expressive extension to the domain brought about by linear constraints. Many problems and solutions in CAD/CAM, spatial databases, motion planning and graphics are naturally expressed through them [2][11][4][3]. They have also been used in seemingly unrelated domains such as molecular biology [14], automobile transmission design [15] and electrical engineering [5].

The rest of the paper is organized as follows. The next section outlines the steps involved in the quadratic solver of QUAD-CLP(R) . Some features of the language and system are described in §3. A large part of the paper is devoted to applications in Solid Modeling and Combinatorial Search problems, described and analyzed in §4. Some relations to other work are established in §5.

2. The Quadratic Solver. The aim of this section is to acquaint the reader with the steps taken by the quadratic solver of QUAD-CLP(R) . Details and proofs of the algorithms involved can be found in [17].

Figure 1 illustrates some of the interactions between the quadratic and linear solvers. The latter should be considered here a black box relying on incremental versions of Gaussian elimination and of phase I of the Two-Phase-Simplex method. Upon encountering a constraint in the course of the computation, we first classify it as either linear or quadratic according to its syntax, by considering the number of bound variables in each monomial [1]. In the former case, it is directly fed to a solver for linear constraints. In the latter, it goes through the process summarized below:

Step 1: Discretize. Quadratic arithmetic constraints offer a natural geometric interpretation which leads to a small number of equivalence classes. For example, the constraint $\frac{9}{25}y^2 + \frac{16}{25}x^2 + \frac{24}{25}xy - \frac{4}{5}y + \frac{3}{5}x < -19$ belongs to the class parabola whose

[1] For simplicity, we do not discuss here the case of monomials whose degree is ≥ 3. The corresponding constraints could be broken into quadratic pieces, as mentioned previously, or just delayed.

canonical representative is the algebraic equation of the corresponding locus in standard position, $\frac{x^2}{a^2} - by$ (in this case, parameters a and b would both have a value of 1). Those *geometric equivalence classes* allow us to achieve a discretization of the problem. This first step identifies the geometric equivalence class to which the quadratic constraint belongs, producing the canonical representative and a transformation matrix (whose geometric interpretation is the translation and rotations needed to bring the locus to standard position). The computation amounts to the diagonalization of a real symmetric matrix.

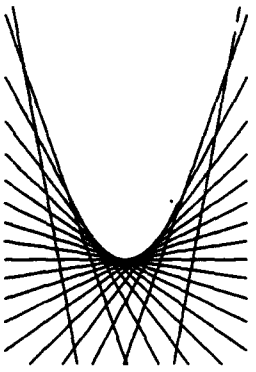
Step 2: Simplify. For several of the possible pairs ⟨*canonical representative*, *relational symbol*⟩, the constraint can be immediately decided or replaced by an equivalent Boolean combination of linear constraints.

Examples

- The geometric equivalence class of $-481x^2+216xy-544y^2+3946x+3272y \geq 37685$ is imaginary ellipse. The pair $\langle -\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \ , \geq (0) \rangle$ reveals that the constraint is trivially false.
- Constraint $x^2-4xy+6xz-8xw+4y^2-12yz+16yw+9z^2-24zw+16w^2-25=0$ is classified as two points, leading to the simplification $(x-2y+3z-4w=5) \vee (x-2y+3z-4w=-5)$. The unexpected feature of this example is that a constraint on four variables falls into a geometric equivalence class seemingly reserved to constraints on one variable. This degenerate case is best viewed through substitution $v = x-2y+3z-4w$, yielding $v^2-25=0$. In this form, it becomes less surprising that its class should be two points ($v = \pm 5$). Note that step 1 described above does not look for such simplifying substitutions but nevertheless produces equivalent results.

Step 2(bis): Approximate. For each remaining pair ⟨*canonical representative*, *relational symbol*⟩, a sound approximation made up of linear constraints is computed. The strategy leading to the efficient and accurate production of linear approximations considers a Boolean combination of convex constraints in place of the original constraint (note that it may already be convex). The convex pieces are approximated and the results recombined. That Boolean combination may be equivalent to the initial constraint, in which case it will be termed a *convex expression*, or constitute a relaxation of it and will then be called *convex approximation*. In both cases the resulting combination of linear constraints constitutes an approximation.

Bringing back the example of step 1, the pair $\langle x^2-y \ , < \rangle$ indicates a constraint which is already convex and for which we can compute a linear approximation such as:



$$
\begin{aligned}
0 &\leq y \quad \wedge \\
2.36522x &\leq y + 1.39857 \quad \wedge \\
-1.39857 &\leq y + 2.36522x \quad \wedge \\
0.662911x &\leq y + 0.109863 \quad \wedge \\
-0.109863 &\leq y + 0.662911x \quad \wedge \\
&\ldots
\end{aligned}
$$

Step 3: ***Realize.*** Map the simplification or approximation for the canonical representative to a simplification or approximation for the original constraint. This is achieved by "multiplying" each linear constraint in the Boolean combination by the transformation matrix. The result of step 3 is in turn sent to the linear solver to decide upon the new collection of constraints. If it was an approximation, the original quadratic constraint is also kept (delayed).

The resulting solver is not a complete solver since it partly relies on approximations. It nevertheless exhibits much less incompleteness than one which unilaterally sets aside non-linear constraints. Some of this incompleteness can actually be driven back by choosing an appropriate size for the approximations, as will be seen in the next section. Note that from a logic programming perspective, the nature of the approximations generated ensures the soundness of the inference.

3. Features of the System. QUAD-CLP(**R**) is built on top of the CLP(**R**) system, which allowed us to concentrate on the non-linear component of the solver. It was written in *C* to facilitate its integration with the host system whose source code is available and also written in *C*. We discuss some of the additional features provided.

3.1. $\wedge$,$\vee$-bounds. Recall that in most cases, from a quadratic constraint is extracted a Boolean combination of linear constraints which is sent to the linear solver. It proves convenient to write that Boolean combination in disjunctive normal form:

$$\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} C_{ij}$$

Each disjunct will give rise to a solver choice point. There are cases where n can be quite large. One may therefore wish to specify an upper bound on n in an effort to control the non-deterministic behavior. QUAD-CLP(**R**) provides the user with a parameter, the $\vee$-*bound*, which has the desired effect. A Boolean combination whose disjunctive normal form exceeds that bound will not be sent to the linear solver. Note that setting it to 1 yields a deterministic solver.

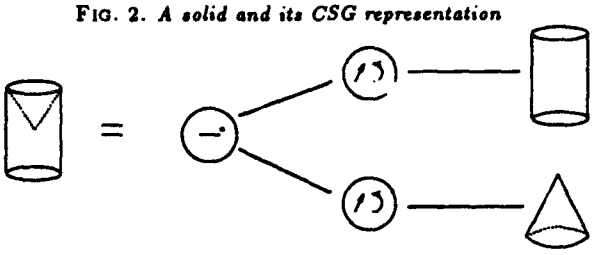
The $\wedge$-*bound* specifies the desired size of an approximation to a convex co nt. Such a name was chosen because it often corresponds to an upper bound on m. A default value of 4 has been found adequate experimentally for a first exploration: refined approximations can always be tried subsequently. For example, the following (unsolvable) system of inequalities,

$$\begin{aligned} x^2 + y^2 &\leq 1 \\ u^2 + v^2 &\leq 1 \\ x + y + u + v &\geq 3 \end{aligned}$$

required approximations of size 7 to decide that there was no solution.

3.2. Output. The simplification of the constraint set may be desirable for efficiency reasons, since it reduces its size, but also to ease the understanding of the result by the user, when the answer takes the form of a collection of constraints. Much research has been devoted for example to quantifier elimination in the special case of an existentially quantified conjunction of linear constraints, in an effort to express the output in terms of query variables only [13][8][12][9][7]. We discuss here another aspect of simplification brought about by non-linear constraints.

FIG. 2. *A solid and its CSG representation*



Seemingly very different answers such as $481x^2 - 216xy + 544y^2 - 3946x - 3272y + 6409 = 0$ and $16x^2 + 25y^2 - 400 = 0$ express a quite similar relationship between the variables (valid pairs (x, y) lie on an ellipse), which is captured by the concept of geometric equivalence classes. An answer like the first one can be complemented by:

```
real ellipse:  foci at (7.4,5.8),(2.6,2.2); principal axis of length 10.
```

Such information thus allows to deepen the understanding of the relationship between the variables of a solution or may help to determine its solvability if no conclusion was reached.

Redundancy in the solution is also an issue of simplification. Let us mention that the detection of a redundant linear inequality with respect to a quadratic constraint can in some cases be reduced to the efficient computation of a supporting hyperplane. Some heuristics can also be applied for redundancy detection between pairs of quadratic constraints [18].

4. Examples. In this section we describe two applications which demonstrate the expressiveness and efficiency of QUAD-CLP(R) .

4.1. Solid Modeling. We consider the Point/Solid Classification and Solid Intersection problems in constructive solid geometry (CSG). In such a representation scheme, a solid is built by combining *primitive solids*, using *regularized Boolean operations* and *rigid motions*(translation and rotations) [6]. These primitive solids are usually chosen among the parallelepiped, triangular prism, sphere, cylinder, cone and torus. The regularized Boolean operations are $\bigcup^*$, $\bigcap^*$ and $-^*$, differing from the set-theoretic operations in that the result is the closure of the operation on the interior of the solids. A solid can be represented as a tree whose leaves are primitive solids and whose internal nodes are the operations on them (an example is given in figure 2).

With the exception of the torus, every primitive solid has an implicit form expressed in terms of quadratic and linear arithmetic inequalities. This makes it particularly attractive to our language. For simplicity, we shall drop the regularization of the Boolean operations: in some applications, this is even desirable. Constraint logic programming allows for an elegant and concise solution to the Point/Solid Classification problem, which consists of deciding if a point lies inside a solid. The first half of this solution follows:

```
%% inside(Point, Solid): Point lies inside Solid.
inside(Point, solid(and(S1,S2))) :-
    inside(Point, solid(S1)),
    inside(Point, solid(S2)).
```

```
inside(Point, solid(or(S1,S2))) :-
    inside(Point, solid(S1));
    inside(Point, solid(S2)).
inside(Point, solid(minus(S1,S2))) :-
    inside(Point, solid(S1)),
    outside(Point, solid(S2)).
```

Our solution also has the advantage of replacing the need to specify rotations and translations to "move" the solid into place by directly giving its position in terms of natural parameters. For example, `solid(cylinder((1,1,1),(3,4,2),5))` defines a cylinder of radius 5 whose axis extends from $(1,1,1)$ to $(3,4,2)$. Additional rules must be written for each of the primitives and we give one of them below:

```
%% point (X,Y,Z) lies inside primitive solid "cylinder".
inside((X,Y,Z), solid(cylinder((X0,Y0,Z0),(X1,Y1,Z1),R))) :-
    % orientation of symmetry axis
    Vx = X1-X0, Vy = Y1-Y0, Vz = Z1-Z0,
    % point (Xp,Yp,Zp) is on the axis of symmetry, ...
    Xp = Vx*T + X0,
    Yp = Vy*T + Y0,
    Zp = Vz*T + Z0,
    %... inside the cylinder ...
    T >= 0, T <= 1,
    % ... and on the plane which contains (X,Y,Z) ...
    % ... and is orthogonal to the axis.
    Vx*(X-Xp) + Vy*(Y-Yp) + Vz*(Z-Zp) = 0,
    % constrain the cylinder
    (X-Xp)*(X-Xp) + (Y-Yp)*(Y-Yp) + (Z-Zp)*(Z-Zp) <= R*R.
```

Solid Intersection problems arise not only when we want to avoid overlapping objects but also when we wish to eliminate redundancies in the representation of a solid. A common approach is to verify a criterion for non-intersection obtained by approximating the shape of the solid, usually through "box approximations" (see figure 3). If the approximations do not intersect then certainly neither do the solids. A simple extension to the above provides a solution:

```
%% solids S1 and S2 intersect.
intersect(S1,S2) :-
    inside(Point, S1),
    inside(Point, S2).
```

The above solution, applied to the Point/Solid Classification problem, worked in a satisfactory manner given a suitable linear solver (non-linearities vanished as enough variables were fixed). The present problem on the other hand retains non-linear constraints. Here the strength of QUAD-CLP(R) is to provide for free a behavior conceptually similar to "box approximations" but with potentially much closer approximations.

FIG. 3. *Box approximation of a cone.*

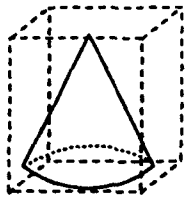
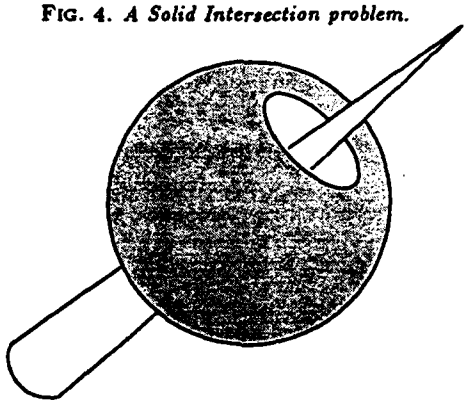


FIG. 4. *A Solid Intersection problem.*



In fact, it generates approximations with "holes" if need be. For example in the following instance, illustrated in figure 4, the conventional approach would have failed to detect the non-intersection, whereas with QUAD-CLP(R) :

```
?- Bead = solid(minus(sphere(0,0,0,4),
                      cylinder((-4,-4,-4),(4,4,4),2))),
   Needle = solid(cone((7,6,6),(-6,-5,-5),1)),
   intersect(Needle, Bead).

*** No
```

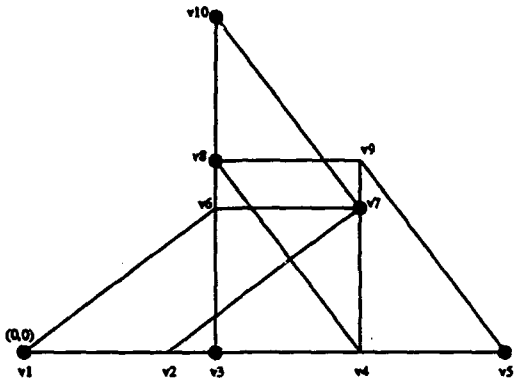
4.2. Combinatorial Search. We examine next a combinatorial search problem involving Euclidean distances and thus quadratic constraints. Instances of respectable size can be solved in a reasonable amount of time through a simple QUAD-CLP(R) program.

Graph Geometric Embedding : Given a graph $G(\mathcal{V}, \mathcal{E})$, a label $\ell(e) \in \mathbf{Z}^+$ for each $e \in \mathcal{E}$ and a set P of points in $\mathbf{E^2}$, is there a mapping $f : \mathcal{V} \mapsto \mathbf{E^2}$ such that $P \subseteq \text{codom}(f)$ and $\forall (v, v') \in \mathcal{E}, d(f(v), f(v')) \leq \ell((v, v'))$ (where $d : \mathbf{E^2} \times \mathbf{E^2} \mapsto \mathbf{R}$ is the Euclidean metric)?

Intuitively, we are asked to cover certain points in $\mathbf{E^2}$ with vertices of a labeled graph without "breaking" an edge. When $|\mathcal{V}| = |P|$, a simple generate-and-test approach will solve the problem, although through considering all $|\mathcal{V}|!$ possible pairings. The test-and-generate paradigm associated with constraint programming may accelerate our inspection by pruning the search tree. If $|\mathcal{V}| > |P|$, generating candidate solutions by associating a different vertex with each point in P will leave some vertices

FIG. 5. *The* 10;6 *instance.*

$P = \{(0,0),(10,0),(4,7),(4,0),(4,4),(7,3)\}$,
$\mathcal{V} = \{v1, v2, v3, v4, v5, v6, v7, v8, v9, v10\}$,
$\ell((v1,v2)) = 3, \ell((v1,v6)) = 5, \ell((v2,v3)) = 1, \ell((v2,v7)) = 5, \ell((v3,v4)) = 3,$
$\ell((v3,v6)) = 3, \ell((v4,v5)) = 3, \ell((v4,v7)) = 3, \ell((v4,v8)) = 5, \ell((v5,v9)) = 5,$
$\ell((v6,v7)) = 3, \ell((v6,v8)) = 1, \ell((v7,v9)) = 1, \ell((v7,v10)) = 5, \ell((v8,v9)) = 3,$
$\ell((v8,v10)) = 3.$



"free". Testing those candidates thus requires reasoning about quadratic constraints.

The statistics in table 1 were obtained from a straightforward program implementing the test-and-generate algorithm: state all the distance constraints implicit in the graph, assign vertices to points in P, output candidate solutions. The tests were run on a SUN SPARCstation 10/42. The problem on 10 vertices was generated by hand (the 10;6 instance and its unique solution appear in figure 5). As for the rest, the graphs were randomly generated with an edge-occurrence probability of about 0.4. Points in P were distributed on a square grid and the labels ranged from 1 to the length of the diagonal of the grid.

The first three instances were run on both the QUAD-CLP(R) and CLP(R) systems, in order to compare the performance of the quadratic solver with that of the delay strategy implemented by the latter. Important speed-ups were always observed mainly because of the difference in the number of nodes which were expanded in the search tree, reflecting the amount of pruning that took place. A notable difference between the results for the 10;10 and 10;6 instances is the number of candidate solutions found by CLP(R). These instances share the graph and six points of P: the first one includes four more points so that $|\mathcal{V}| = |P|$. Consequently in the 10;10 instance, a basic pairing procedure guarantees that we will find all and only solutions to the problem, regardless of the strategy used to handle non-linear constraints, since the distance constraints will eventually become ground. However the 10;6 instance brings forth the incompleteness of a solver as some of the constraints may never become ground (or even linear) during the search. Thus we obtain a set of 72 possible solutions.

Larger instances (30 and 50 vertices, about 180 and 500 quadratic constraints respectively) were solved on QUAD-CLP(R) only as the 20;8 instance was already overwhelming for the delay strategy. Despite the surprisingly slight increase in the number of nodes expanded as the problems gain in size, the time taken grows by several orders of magnitude. This should be attributed to the growing system of

TABLE 1
Performance statistics for the Graph Geometric Embedding problem.

size ($\|\mathcal{V}\|;\|P\|$)	*language*	*time (sec)*	*nodes expanded*	*# solutions*
10;10	QUAD-CLP(R)	0.52	38	1
	CLP(R)	4.94	1674	1
10;6	QUAD-CLP(R)	0.54	36	1
	CLP(R)	3.95	1314	72
20;8	QUAD-CLP(R)	13.75	35	1
	CLP(R)	>14 063.00	>2 411 229	>987 546
30;10	QUAD-CLP(R)	276.54	49	0
50;5	QUAD-CLP(R)	2 727.41	51	0

inequalities that the linear solver has to deal with. For example the largest instance, given a conservative $\wedge$-*bound* of 4 (which is what was used in every instance), spawns a dynamically changing system of around 2000 linear inequalities in 100 variables. As the linear solver relies on a Simplex algorithm, basic feasible solutions must constantly be found at the cost of pivoting operations.

The most accurate rendition of the improvements brought by the approach described in the paper must be found in the pruning of the search tree and the number of candidate solutions offered.

5. Related Work. As was noted in §1, computational algebra techniques, currently still very expensive, nevertheless yield a complete solver through a uniform treatment of polynomial constraints. It is not clear how well the approach described in this paper, whose motivation was to solve quadratic constraints, can perform on arbitrary polynomial constraints. The introduction of auxiliary variables fragments the original constraints; separately considering (and most likely approximating) each piece may yield weaker results. Since in general a constraint will admit several possible fragmentations, choosing the best one is an interesting problem in its own right.

As an illustration, consider the following system of non-linear inequalities, borrowed from [7]:

$$\begin{aligned} s &> 0 \\ s^4 - 0.029901s^3 - 247.971s^2 + 396.01s - 245.03 &\geq 0 \\ s^4 - 2.01005s^3 - 247.246s^2 + 400s - 248.246 &\leq 0 \end{aligned}$$

Use of a computer algebra package reveals that s lies somewhere in $[14.93, 15.98]$. One possible fragmentation,

$$\begin{aligned} t &= s^2 \\ s &> 0 \\ t^2 - 0.029901st - 247.971s^2 + 396.01s - 245.03 &\geq 0 \\ t^2 - 2.01005st - 247.246s^2 + 400s - 248.246 &\leq 0, \end{aligned}$$

run on QUAD-CLP(R) , constrains s to the slightly larger interval $[4.61, 16.62]$ whereas

$$\begin{aligned} t_i &= s^2,\ i = 1 \ldots 8 \\ s &> 0 \\ t_1 t_2 - 0.029901st_3 - 247.971t_4 + 396.01s - 245.03 &\geq 0 \\ t_5 t_6 - 2.01005st_7 - 247.246t_8 + 400s - 248.246 &\leq 0, \end{aligned}$$

offers a vastly different result, namely $]0.00, \infty[$.

An even more uniform treatment of constraints is that provided by the language CLP(BNR) [16]. Here relational interval arithmetic is applied to reals, integers and booleans alike. A parallel can be drawn with our approach since interval arithmetic is a form of approximation. Their approach to constraint solving is nevertheless quite different as it is based on the local propagation of bounds on the value of the variables through a constraint network.

We say a few more words on those approximations. In the context of quadratic constraints, they represent a special case of ours. Each bound of an interval can be viewed as a linear inequality. The size of such an approximation for a constraint is consequently determined by the number of variables appearing in it (4 with 2 variables; 6 with 3 variables; ...) and the approximation itself is isothetic (aligned with the coordinate axes). The result is comparable to the "box approximations" of §4.1. It may be sufficient in some cases but is certainly less powerful in general (recall for example figure 4).

6. Conclusion. This report presented a new way of handling non-linear arithmetic constraints and its implementation into the QUAD-CLP(R) language. Important properties of the problem at hand where discretization through geometric equivalence classes and decomposition into convex pieces. A case analysis of those equivalence classes led to a relaxation (and sometimes recasting) of the original constraints into linear constraints, much easier to handle. Applications in Solid Modeling and Combinatorial Search showed both the expressiveness and the efficiency of such a tool within a constraint language.

The latter application revealed a need for more efficient linear solvers when confronted to large systems of inequalities. It proved to be a bottleneck for the speed of the quadratic solver. One must therefore be careful when considering constraints as language primitives: the apparently simple addition or deletion of a constraint may hide a considerable cost in problems involving a large number of constraints.

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Applications in Constraint Logic Programming with Strings

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Abstract

In this paper, we discuss CLP($\mathcal{S}$) which combines logic programming with constraint solving over *strings* and show how CLP($\mathcal{S}$) can be used naturally in several applications ranging from natural language processing, to encoding of genetic operators and DNA grammar rules, to scene analysis in iconic image processing.

Several applications in artificial intelligence require that one deal with information which is not as precisely encodable as required by logic-based systems. In recent years there has been a number of innovative applications in new fields which makes additional demands on the representational efficiency of logic-based automated reasoning. Some of the most challenging applications have come in diverse fields such as processing textual data [24], processing genome sequences (The Genome Project) [3, 20, 21], representing and reasoning with visual data [8, 4], storing and processing musical compositions, natural language processing [6], etc. These applications have some characteristic commonality - they process strings (or streams) of information, the data may be incomplete and may require approximate reasoning. String-based logic provides a tool for developing such automated reasoning systems.

Strings can be loosely defined as concatenations of variables and constants. String unification is difficult and may not lead to a unique most general unifier and in fact there may be an infinite number of maximally general unifiers. The decidability of the string unification problem (also called as the word problem) was established by Makanin [15] and procedures based on his technique have been developed by other researchers: Abdulrab and Pecuchet [1], Koscielski [14] and Jaffar [10]. But such procedures are not suitable for use in an automated reasoning environment or in a logic programming language because of their generation of multiple (maximally general) unifiers and non-termination when there are infinite number of such unifiers. In [18, 16] (see also [17]) we offer a solution to this dilemma through constraint logic programming [11, 12, 23]. is to apply constraint solving techniques, instead In our approach, we solve the problems of string unification by deferring full unification and performing partial unification at resolution step. By adapting this technique, we generate a set of string equations at each step, which subsumes the sets of (possibly infinite) maximally general unifiers. We define a notion of "partially-solved" form of string equations and develop an algorithm for obtaining such partially-solved forms from any given set of string equations. We discuss, in detail, the theoretical and procedural aspects of CLP($\mathcal{S}$) in [18] and define a constraint-solver which can be used to provide a sound and complete query answering system for allowed string logic programs.

In this paper we only provide a brief overview of CLP($\mathcal{S}$). We mainly concentrate on describing applications in CLP($\mathcal{S}$).

1 CLP($\mathcal{S}$) - Constraint Logic Programming with Strings

In CLP($\mathcal{S}$), apart from terms (built from constants, function symbols and variables) which can be used as arguments for building predicates, there is a new set of constructs called strings. Strings are built with *string constants* and *string-variable names*. A special symbol ϵ is used to denote an empty string. Each string-variable name has a parameter associated with it called its *size*, which limits the strings that can be bound to the variable to be of the same size. The size can be defined by a positive, integral arithmetic expression, called the *bounds expression* formed using *bounds-constants* and *bounds-variable names*. In essence, one can think of the set of string-variable names to be typed (or sorted) by their size and are limited to acquiring values of the same type (sort). We denote a string variable in the following way: $\overset{t}{W}$, where W is a string-variable name and t is a bounds expression denoting its size.

A *string* is defined recursively as follows: an empty string ϵ is a string; its size is 0. A string-constant is a string; its size is 1. A string-variable $\overset{t}{W}$ is a string; its size is t. If S_1 and S_2 are strings then so is their concatenation, S_1S_2; the size of S_1S_2 is the sum of the sizes of S_1 and S_2. The notions of *ground strings*, *string-atoms* and *string-literals* are defined as in logic programming.

A string equation (or constraint) is of the form $S_1 = S_2$, where S_1 and S_2 are strings and $=$ is a predicate which does not occur in the vocabulary of the logic programming language. An arithmetic equation is of the form $e_1 = e_2$ where e_1 and e_2 are arithmetic expressions. In [17, 18] we provide a string equational theory for $=$.

A CLP($\mathcal{S}$) program is defined as a finite set of rules of the form:

$A \leftarrow C, B_1, \ldots, B_n$, where $n \geq 0$, $A, B_1, \ldots, B_n$ are string atoms and C is a set of string equations and arithmetic equations. Whenever a string variable occurs in more than once in a rule, we consider it to have the same size at each occurrence. An allowed program rule is a CLP($\mathcal{S}$) program rule in which every variable in A also occurs in $B_1, \ldots, B_n$. A *goal* is of the form:

$\leftarrow C, B_1, \ldots, B_n$ i.e., a rule without a head. Some examples of CLP($\mathcal{S}$) program rules are:

$add(\overset{r}{X}\overset{1}{A}, \overset{m}{Y}0, \overset{p}{Z}\overset{1}{A}) \leftarrow add(\overset{r}{X}, \overset{m}{Y}, \overset{p}{Z})$ % addition as a shift operation

$surftolex(n, \overset{n-3}{W}\overset{1}{N}icat\overset{m}{X}, \overset{n-3}{W}\overset{1}{N}y + at\overset{m}{X}, 1) \leftarrow nc\_val(\overset{1}{N})$. % a C-insertion morphological rule (eg. apply + ation = application)

$same\_obj(\overset{r}{X}, \overset{r}{X})$ % used to check when two objects are identical (unifiable)

$sim\_obj(\overset{r}{X}, \overset{m}{Y}) \leftarrow \{\overset{r}{X}=\overset{m}{Y}\}$ % can be used to check whether objects are approximately identical. This requires using approximate string equality checking.

Jaffar and Lassez [11] show that CLP paradigms can generalize the Horn logic programming semantics based on term structures (operational, algebraic, logical) over to Horn logic programs based on an arbitrary structure which is *solution-compact* and *satisfaction-complete*. The structure $(SU, =)$ (= is equality with associativity) is *solution-compact* and *satisfaction-complete* [11]. The only remaining piece in the puzzle is the definition of constraint-solvers in the string domain for reducing string equations. In [18], we describe such an algorithm, called the reduce algorithm, which reduces a set of string equation into an equivalent set of strings in partially-solved form. The reduce algorithm, used in conjunction with Gaussian elimination (for solving arithmetic equations on string sizes) provides a sound and complete proof procedure for allowed programs, using the constraint logic programming paradigm. These results are shown in [18]. The reduce algorithm is similar to the term rule-based unification algorithm (see eg. [13]). This allows one to easily incorporate approximate string matching techniques during the reduction process. We discuss one such technique later in the paper. We do not provide the definition of the reduce algorithm and the theoretical results due to space constraints!

Prolog III is another example of a string processing constraint language [5]. CLP($\mathcal{S}$) differs from Prolog III in several ways; mainly in the association of an explicit size factor for string variables and in allowing unrestricted concatenation. The integrated structure of string-value and size provides several advantages. First, it provides a notion of 'types' on the string variables and allows one to restrict the domain of values that can be bound for the variable. One can also write equations (both equality and inequalities) on sizes which can ease and speed-up the unification process by allowing one to solve string equations using algebraic equation solvers on size-equations. The size information also allows one to effectively detect inequalities in string equations and fail a derivation earlier than otherwise. Moreover, one can use known string inequalities, such as $aS \neq Sb$ (where S can be taken as a string of arbitrary length), to fail CLP($\mathcal{S}$) derivations. The usage of the reduce algorithm, which is similar to a rule-based unification algorithm and based on the concept of partially-solved forms of string equations, is also unique in our approach and allows unrestricted concatenation and sub-string insertions and deletions. The advantage of this can be seen in the ease with which it can be adapted for approximate reasoning on strings (see [16].) To make the paper more readable, we briefly describe the unify algorithm in the appendix.

2 Applications of CLP($\mathcal{S}$)

We discuss three applications of CLP($\mathcal{S}$): in natural language processing for encoding logic grammar rules and for performing computational morphology; in visual scene processing for picture correspondence and as a picture description language; and, in genetic sequence analysis and for implementing genetic algorithms.

2.1 Natural Language Processing

Natural language sentences are inherently not well-structured. Their processing requires sentences of different types and phraseology to be parsed and analyzed. Even though there are some concrete rules which govern their analyses, for most parts ad hoc analysis needs to be performed. The analysis further deteriorates when one has to deal with spoken and/or colloquial sentences. In such cases, words or even parts of sentences may be missing caused probably by the speaker having a casual locution. Further, in morphological analysis one needs to divide a word into several parts to identify the underlying morphemes; sometimes no division is necessary. For example take the case of the following three transitive verbs, *incite, instigate* and *invent*. The first word has to be divided into a prefix *in* and a transitive verb *cite*, whereas the other two need no such division. In the last case, one can actually divide it into a prefix *in* and a transitive verb *vent*, but the recombined meaning of the morphemes *in+vent* is entirely different from the one given by the full word *invent*. In the above analysis, one is neither dealing with a (indivisible) constant nor building a term from constants in the manner of term-based logic. The use of strings as representation would be useful in naturally encoding the different types of grammatical formations and rules used in the lexical and morphological analysis of natural language. The associative property of string concatenation permits one to cut a string into two or more substrings at arbitrary locations.

Another important advantage of using strings as representation of sentences comes from the 'global' view offered by strings as compared to the 'local' view of term-based structures. When one needs to analyze (or process) several parts of the list as the same time, as may be required for extraposition or discontinuity analysis, one has to move (skip) through the list to perform the analyses. Such analysis can be easily done using strings. The use of strings also eases the operations of 'movement', insertions and deletions which are not easily performed with functions or lists. Such insertions and deletions occur quite often in morphological analysis.

To show how logic grammar rules can be encoded as CLP($\mathcal{S}$) rules, we give an example in Discontinuous grammar [2]. The rules in these grammars are of the form:

$$S, \alpha_0, skip(X_1), \alpha_1, \ldots, skip(X_n), \alpha_n \rightarrow \beta_0, skip(X'_1), \beta_1, \ldots, skip(X'_m), \beta_m$$

where S is a non-terminal and, αs and βs are strings of terminals and non-terminals. (β can also be procedure calls). The Xs denote arbitrary strings which need to be skipped. For example the following rule:

(DG) $Rel\_marker, skip(G), trace \rightarrow Rel\_pronoun, skip(G)$

taken from [2] parses sentences such as "the man that John saw laughed", where it considers the noun phrase "the man that John saw" to be a surface expression of a more explicit statement: "the man [John saw the man]", where the second occurrence of "the man" has been moved to the left and subsumed by the relative pronoun "that". The rule can be translated into a CLP($\mathcal{S}$) program rule as:

$$sentence(\overset{h}{H}\overset{p}{Y}\overset{r}{Z}\overset{t}{T}) \leftarrow rel\_marker(\overset{n}{X}), trace(\overset{m}{W}), sentence(\overset{h}{H}\overset{n}{X}\overset{r}{Z}\overset{m}{W}\overset{t}{T})$$

The atoms $rel\_marker(\overset{n}{X})$ and $trace(\overset{m}{W})$ are conditions which need to be enforced to make the transformation valid. In [16] we provide transformations for several different types of logic grammars and prove their correctness.

The analysis of word structures using computers is called *computational morphology*. In this section we show, through an example, how one can represent morphological rules using string-based logic. In our discussion on computational morphology we follow the book [19] by Ritchie, Russell, Black and Pulman. A sample rule is given below:

+:e <=> { < {c:c | s:s} (h:h) > | z:z | x:x | y:i } __ s:s

The rule[1] states that the surface character e gets deleted and is replaced by a lexical character + if and only if it is is preceded by either ch, sh, z, x or i realized as a lexical y and is followed by a s. The notation c:c denotes that c remains unchanged while transforming from surface level to lexical level. The rule can be used to transform the surface form flies to lexical form fly + s. The equivalent CLP($\mathcal{S}$) program is given by[2]:

$$surftolex(n, \overset{n-3}{X}shes, \overset{n-3}{X}sh{+}s, 1)$$
$$surftolex(n, \overset{n-3}{X}ches, \overset{n-3}{X}ch{+}s, 1)$$
$$surftolex(n, \overset{n-2}{X}xes, \overset{n-2}{X}x{+}s, 1)$$

[1] < *items* > denotes sequential items and {*items*} denotes choice of items

[2] n and 1 are used as markers and flags used in other clauses. See [16] for details.

$surftolex(n, \overset{n-2}{X}zes, \overset{n-2}{X}z+s, 1)$

In [16], we provide translations for some of the other morphographemic rules given in [19]. From the rules shown here and in [16] it can be seen that morphological transformation of a surface character into a lexical character requires character strings of variable lengths on either side and also requires insertion and deletion of characters. These operations are well-suited for a string-based representation.

In [16], we also show how one can use CLP($\mathcal{S}$) to perform word segmentation to identify categories and for encoding feature passing conventions. Rules such as generation and analysis of plural nouns, compound nouns, prefixing, etc, are given in [16].

Next, we point out how the CLP($\mathcal{S}$) system can be used for dealing with sentences with simple errors and sentences that are incomplete. In [26] algorithms for several kinds of approximate string matching are provided. They permit mismatches caused by extra characters, missing characters, altered (substituted) characters and interchanged characters. For example, the sentence "The man tat John saw lauhged" has two errors, one caused by a missing character and another by a pair of interchanged characters. Parsing this sentence normally would lead to failure. If we augment string-matching to reason with such errors and build in the mechanism as part of string-constraint solving one can parse the above statement. In [16] we show how one can augment the reduce algorithm to take care of such errors. The case of incomplete sentences, sentences with gaps in them, is easily treated with CLP($\mathcal{S}$), even though computationally it may not be attractive. The following can be given as a goal, when one knows that there are two gaps in the sentences with one of them of bounded size.

$\leftarrow \{n > 3, n < 10\}, sentence(the\ man\ \overset{n}{X}\ john\ \overset{3}{Y}\ laughed).$

One of the bindings returned may be values "that " and "saw" to X and Y respectively when used with proper rules. The need for parsing such incomplete sentences can be seen in several cases: when parsing old manuscripts with torn or missing segments or when parsing a sentence heard over radio or telephone where one may miss some segments due to noise. Parsing colloquial sentences which may have many missing segments. CLP($\mathcal{S}$) provides a method for parsing such sentences which may not be easily possible with other methods of natural language analysis.

2.2 Image Processing

One of the main areas in image processing deals with picture identification. For example, given a set of pictures one may want to find a picture in which there are two cars or, one may want to check whether another picture is a sub-picture of a picture in the set. A second question may involve approximate reasoning, since the smaller picture may not be a precise sub-picture. One of the ways of representing pictures is to convert them into symbolic forms based on an alphabet of 'icons'. For example, if one wants to represent a map of a region, then one can iconify the objects in the map (such as large lakes (a), mountains (b), forests (c), hills (d),etc.,) and place the icons on a corresponding scaled grid-map. Figure 1 shows such maps. Chang et. al. [4] define a scheme where iconic images are stored as 2-D strings. For example, the 2D-representation of the iconic picture of Figure 1, p is given by $(ad < b < c)(a < bc < d)$.

Note that the symbol $<$ captures the spatial relationship of below and to-the-right-of in the two 2-D string representation. Chang et. al. [4] provide algorithms to translate iconic pictures into 2D-representations and vice versa. In [18] we provide CLP($\mathcal{S}$) programs for performing these translations.

The two-dimensional string representation provides a simple approach to perform subpicture matching. Chang et. al. [4] describe three types of matching with decreasing levels of approximation; the last type (type-2) provides an exact sub-picture of another. In the following $r(a)$ denotes the rank of a non-$<$ symbol in a string and is defined as one plus the number of $<$'s preceding the symbol in s. A string u is a *type-i 1-D subsequence* of a string v if for all a_1s and b_1s, if $a_1S_1b_1$ is a substring of u and $a_2S_2b_2$ is a substring of v (where S_1 and S_2 are strings) and a_1, b_1 match a_2, b_2 resp., then

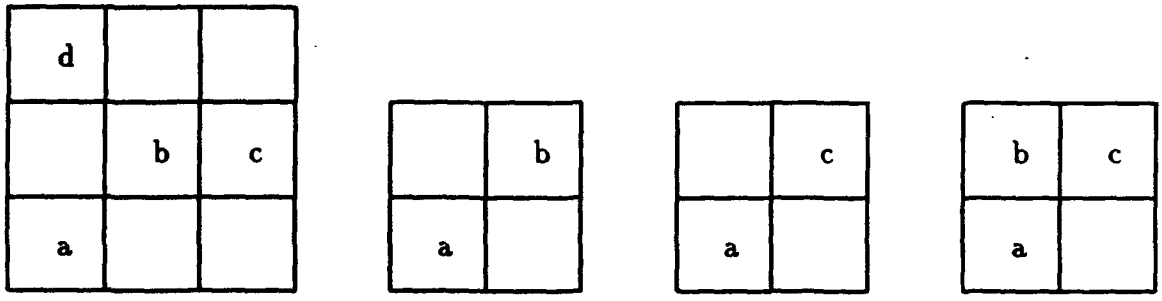
(for type-0) $r(b_2) - r(a_2) \geq r(b_1) - r(a_1)$ or $r(b_1) - r(a_1) = 0$

(for type-1) $r(b_2) - r(a_2) \geq r(b_1) - r(a_1) > 0$ or $r(b_2) - r(a_2) = r(b_1) - r(a_1) = 0$

(for type-2) $r(b_2) - r(a_2) = r(b_1) - r(a_1)$.

Let (u, v) and (u', v') be 2-D representations of pictures p and p' respectively. Then p' is a *type-i 2-D subpicture* of p if u' is a type-i 1-D subsequence of u, and v' is a type-i 1-D subsequence of v. In Figure 2, p_1, p_2 and p_3 are type-0 subpictures of p; p_1 and p_2 are type-1 subpictures of p; and p_1 is a type-2 subpicture

of p.



p p_1 p_2 p_3

figure 1

Chang et. al. [4] provide a complicated pseudo code procedure for performing the three matchings. In [18] we use CLP($\mathcal{S}$) programs to encode their definitions of type-i subsequences and to provide matching procedures for picture identification. We do not discuss them due to space constraints.

CLP($\mathcal{S}$) programs can also be used to encode picture description languages (PDLs) [22, 8]. For example, with proper labeling, the string *ababab* denotes a staircase structure with three stairs. The above description can be captured using string-based logic as follows:

$staircase(ab)$,

$staircase(ab\overset{n}{X}) \leftarrow staircase(\overset{n}{X})$

Similarly, description of compound objects with occlusions can be given succinctly using CLP($\mathcal{S}$):

$occluded\_scene(\overset{n\,m\,p}{WZY}) \leftarrow object(\overset{n\,m\,p}{WXY}), object(\overset{m}{Z})$.

$occluded\_scene(\overset{n\,m}{WZ}) \leftarrow object(\overset{n\,r}{WX}), object(\overset{m}{Z}), m \geq r$.

$occluded\_scene(\overset{m\,n}{ZW}) \leftarrow object(\overset{r\,n}{XW}), object(\overset{m}{Z}), m \geq r$.

The representation can be used to reason about a scene, answering more complicated questions such as 'whether an object is in a scene (probably partially occluded)?' 'whether an object is to the left (or right) of another object?'

2.3 Genetic Operators and DNA Grammar

String-based logic can be used in genetic code processing in two ways. One of the method is to define operators which can be used to *build genetic sequences*. Another method is to use string-based logic to define genetic sequences and use these definitions to *search for sequences* in a given genetic code.

Genetic synthesis can be defined by several operators such as reproduction, crossover, jumping and mutation. These operations can be implemented in CLP($\mathcal{S}$). A somewhat complex cross-overs can easily be defined as follows.

$crossed\_genes(\overset{n\,m\,n}{X_1Y_2Z_1}, \overset{r\,m\,m}{X_2Y_1Z_2}) \leftarrow gene(\overset{n\,m\,n}{X_1Y_1Z_1}), gene(\overset{r\,m\,m}{X_2Y_2Z_2})$.

In [18] we define other operators using CLP($\mathcal{S}$). The new field of Genetic Algorithms is based on these and other operators and [7, 25] shows how Genetic Algorithms can be used to encode and solve several problems including the traveling salesman problem. The advantage gained by encoding the operators using this approach, is that one obtains declarative-procedural duality and a reasoning system based on the logic programming paradigm.

The recent explosion in genetics research, e.g., the *Genome Project*, has lead to the accumulation of a very large database of human (and other species) genetic sequences. Analyzing this massive amount of data would require a vast amount of computation and sophisticated algorithms. As Searls points out in [20, 21] the primary tool currently used to analyze the data is based on linear pattern matching and on viewing the data as a long string [3, 9]. Search for genetic sequences are carried out using regular expressions based on a regular language. In [20, 21] Searls describes a computational linguistic approach where the DNA sequences can be represented using formal grammar, which is better than the linear search techniques.. We translate his

DNA grammar rules into CLP($\mathcal{S}$) rules. He defines complicated genetic structures using simpler structures.

gene ⇒ upstream, xscript, downstream.

upstream ⇒ catBox, 40...50, **tataBox** , 19...27.

xscript ⇒ capSite , ..., **xlate** , ..., **termination.**

These definitions given above are not regular expressions and require encoding of gaps (both unbound (...) and variably-bounded (40...50)) in the sequence. These gaps may be unimportant or untranslatable in that particular gene expression. The above rules given above can be directly represented as the following CLP($\mathcal{S}$) rules:

$$gene(\overset{m\ n\ p}{XYZ}) \leftarrow upstream(\overset{m}{X}), xscript(\overset{n}{Y}), downstream(\overset{p}{Z})$$

$$upstream(\overset{m\ n\ p\ q}{WXYZ}) \leftarrow$$
$$catbox(\overset{m}{W}), n < 50, n > 40, tatabox(\overset{p}{Y}), q < 27, q > 19$$

$$xscript(\overset{m\ n\ p\ q\ r}{VWXYZ}) \leftarrow$$
$$capsite(\overset{m}{V}), xlate(\overset{p}{X}), termination(\overset{r}{Z}).$$

Our representation of the DNA grammar rules have the advantage of being straightforward and declarative translations, whereas Searls' transformation into Prolog are interpreter dependent.

There are other genetic features such as a **repeat, inverted repeat, palindromes, tandem repeats, clover-leaf repeat, copia** and so on, which cause the representation of genetic sequences to be beyond the power of context-free languages. In [18] we show how these features can be defined using CLP($\mathcal{S}$) rules.

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Appendix

A Unify Algorithm

Definition 1 A *partially-solved form of a set of string equations* is a set of string equations $\{R_1 = S_1, \ldots R_n = S_n\}$ such that

$\forall i, 1 \leq i \leq n$, either R_i or S_i is of the form $\overset{t}{X}_i Q_i$, where $\overset{t}{X}_i$ is a string-variable with size t and Q_i is a (possibly empty) string, and

$\forall i, 1 \leq i \leq n$, either R_i or S_i is of the form $P_i \overset{t}{X}_i$, where $\overset{t}{X}_i$ is a string-variable with size t and P_i is a (possibly empty) string. □

Definition 2 A *solved form of a set of string equations* is a set of string equations $\{\overset{t_1}{X}_1 = S_1, \ldots \overset{t_n}{X}_n = S_n\}$ such that $\forall i, 1 \leq i \leq n, \overset{t_i}{X}_i$ is a string-variable, $\forall i, 1 \leq i < j \leq n, X_i \neq X_j$, and $\forall i, 1 \leq i, j \leq n$, X_i is not in any variable in S_j □

Every set of equations in solved-form are also in partially-solved form. We next define an algorithm which transforms a set of string equations into a partially-solved form. Proofs of theorems are given in [18].

Algorithm 1 (Reduce Algorithm)
Input *S, a set of string equations*
Output *On SUCCESS returns S', a set of string equations, else returns FAILURE.*
Non-deterministically choose from the set of equation S an equation of a form below and perform the associated action: (in the following i denotes a natural number.)

(A:1) $a = \epsilon$ *halt with FAILURE.*
(A:2) $S_1 S_2 = \epsilon$ *replace by the equations* $S_1 = \epsilon, S_2 = \epsilon$.
(A:3) $aS_1 = aS_2$ *or* $S_1 a = S_2 a$, *replace by the equation* $S_1 = S_2$.
(A:4) $aS_1 = bS_2$ *or* $S_1 a = S_2 b$, *where* $a \neq b$, *halt with FAILURE.*
(A:5) $S_1 = S_1$ *for any string* S_1, *delete the equation*
(A:6) $\overset{t}{X} = S_1$ *or* $S_1 = \overset{t}{X}$ *where* S_1 *is not identical to* $\overset{t}{X}$ *and* $\overset{t}{X}$ *has another occurrence in the set of equations. If* $\overset{t}{X}$ *appears in* S_1 *then halt with FAILURE. Otherwise substitute* S_1 *wherever* $\overset{t}{X}$ *appears in every other equation.*
(A:7) $S_1 a_1 \dots a_j S_2 = b_1 \dots b_k$ *or* $b_1 \dots b_k = S_1 a_1 \dots a_j S_2$,
if $a_1 \dots a_j$ *is not a sub-string of* $b_1 \dots b_k$ *then halt with FAILURE.,*
else, if $a_1 \dots a_j$ *is a unique substring of* $b_1 \dots b_k$ *such that* $b_1 \dots b_k = b_1 \dots b_r a_1 \dots a_j b_s \dots b_k$, *replace by the equations* $S_1 = b_1 \dots b_r$ *and* $S_2 = b_s \dots b_k$
(A:8) $\overset{i}{X}\, S_1 = a_1 \dots a_j S_2$ *or* $a_1 \dots a_j S_2 = \overset{i}{X}\, S_1$ *where* S_2 *contains at least one variable and* $S_1 \neq \epsilon$
If $j > i$, *replace by the equations* $\overset{i}{X} = a_1 \dots a_i$ *and* $S_1 = a_{i+1} \dots a_j S_2$
If $i \geq j$, *replace by the equations* $\overset{i}{X} = a_1 \dots a_j\, \overset{i-j}{W}$ *and* $S_2 = \overset{i-j}{W}\, S_1$
(A:8') $S_1\, \overset{i}{X} = S_2 a_j \dots a_1$ *or* $S_2 a_j \dots a_1 = S_1\, \overset{i}{X}$ *where* S_2 *contains at least one variable and* $S_1 \neq \epsilon$
If $j > i$, *replace by the equations* $\overset{i}{X} = a_i \dots a_1$ *and* $S_1 = S_2 a_j \dots a_{i+1}$
If $i \geq j$, *replace by the equations* $\overset{i}{X} = \overset{i-j}{W}\, a_j \dots a_1$ *and* $S_2 = S_1\, \overset{i-j}{W}$
(A:9) $\overset{i}{X}\, S_1 = \overset{j}{Y}\, S_2$, *where* S_1 *and* S_2 *are not empty strings.*
If $j > i$, *replace by the equations* $\overset{j}{Y} = \overset{i}{X}\, \overset{j-i}{W}$ *and* $S_1 = \overset{j-i}{W}\, S_2$
If $i \geq j$, *replace by the equations* $\overset{i}{X} = \overset{j}{Y}\, \overset{i-j}{W}$ *and* $S_2 = \overset{i-j}{W}\, S_1$
(A:9') $S_1\, \overset{i}{X} = S_2\, \overset{j}{Y}$, *where* S_1 *and* S_2 *are not empty strings.*
If $j > i$, *replace by the equations* $\overset{j}{Y} = \overset{j-i}{W}\, \overset{i}{X}$ *and* $S_1 = S_2\, \overset{j-i}{W}$
If $i \geq j$, *replace by the equations* $\overset{i}{X} = \overset{i-j}{W}\, \overset{j}{Y}$ *and* $S_2 = S_1\, \overset{i-j}{W}$
(A:10) $S_1 S_2 = S_3 S_4$ *where* $|S_1| = |S_3|$ *or* $|S_2| = |S_4|$, *replace by* $S_1 = S_3$ *and* $S_2 = S_4$
(A:11) If none of the steps (A:1) through (A:10) can be applied, halt with SUCCESS returning the set of equations.

Theorem 1 Let S be a set of string equations which is reduced using the reduce algorithm. Then,

1. The algorithm halts in finite steps;
2. If S' is the output set of string equations then S is equivalent to S';
3. If S' is the output set of string equations then S is in partially-solved form;
4. If the algorithm terminates with FAILURE then S is not unifiable.

Definition 3 A string equation is in *size-constant* form if every string-variable occurring in the equation has an integer as its size parameter. A set of string equations is in size-constant form if every equation in the set is in size-constant form. A string is in size-constant form if every string-variable occurring in the string has an integer as its size parameter.

Lemma 1 Let S be a set of string equations in size-constant form. Then the reduce algorithm results in FAILURE if and only if, S has no unifiers, or else results in a solved-form. Moreover, the set of equations in solved form provides the unique (up to renaming) most general unifier for S. □

Example 1 Let $S = \{\overset{2}{X}\overset{3}{Y}=\overset{3}{Y}\overset{2}{X}\}$
The reduce algorithm proceeds as follows:
$S_1 = \{\overset{3}{Y}=\overset{2}{X}\overset{1}{W}, \overset{3}{Y}=\overset{1}{W}\overset{2}{X}\}$ from $(A:8)$
$S_2 = \{\overset{3}{Y}=\overset{2}{X}\overset{1}{W}, \overset{2}{X}\overset{1}{W}=\overset{1}{W}\overset{2}{X}\}$ from $(A:6)$
$S_3 = \{\overset{3}{Y}=\overset{2}{X}\overset{1}{W}, \overset{2}{X}=\overset{1}{W}\overset{1}{Z}, \overset{2}{X}=\overset{1}{Z}\overset{1}{W}\}$ from $(A:8)$
$S_4 = \{\overset{3}{Y}=\overset{1}{W}\overset{1}{Z}\overset{1}{W}, \overset{2}{X}=\overset{1}{W}\overset{1}{Z}, \overset{1}{W}\overset{1}{Z}=\overset{1}{Z}\overset{1}{W}\}$ from $(A:6)$
$S_5 = \{\overset{3}{Y}=\overset{1}{W}\overset{1}{Z}\overset{1}{W}, \overset{2}{X}=\overset{1}{W}\overset{1}{Z}, \overset{1}{W}=\overset{1}{Z}, \overset{1}{Z}=\overset{1}{W}\}$ from $(A:8)$
$S_6 = \{\overset{3}{Y}=\overset{1}{Z}\overset{1}{Z}\overset{1}{Z}, \overset{2}{X}=\overset{1}{Z}\overset{1}{Z}, \overset{1}{W}=\overset{1}{Z}, \overset{1}{Z}=\overset{1}{Z}\}$ from $(A:6)$
$S_7 = \{\overset{3}{Y}=\overset{1}{Z}\overset{1}{Z}\overset{1}{Z}, \overset{2}{X}=\overset{1}{Z}\overset{1}{Z}, \overset{1}{W}=\overset{1}{Z}\}$ from $(A:5)$
Halts with success. □

An advantage of the rule-based reduce algorithm is that one can add failure-rules to it without compromising the soundness or the completeness of the algorithm. Such additions would enable one to fail a reduction process faster than without the rules. For example, it is well known that the following string equations have no unifier: $a\ \overset{n}{X}=\overset{n}{X}\ b$ and $\overset{m}{X}\ a\ \overset{n}{Y}=\overset{n}{Y}\ b\ \overset{m}{X}$. Such rules can be used as additional conditions for failing the unification process of the reduce algorithm.

B CLP($\mathcal{S}$)-Resolution

A CLP($\mathcal{S}$)-derivation is similar to a CLP-derivation [11] but uses algebraic and string constraint solvers.

Definition 4 (SSLD-Derivation)
Let Let $\mathcal{A}$ be an algebraic constraint solver and let $\mathcal{U}$ be a string-constraint solver. Let P be a CLP($\mathcal{S}$) program and let G be a goal. A String SLD-derivation *is a sequence of CLP($\mathcal{S}$)-goals $G_0 = G, G_1, \ldots$, such that $\forall i \geq 0$, G_{i+1} is obtained from $G_i = \leftarrow C_i \cup E_i, A_1, \ldots, A_n$. as follows:*

1. *A_m is an atom in G_i and is called the selected atom.*
2. *$A \leftarrow B_1, \ldots B_r$ is a program clause in P standardized apart with respect to G_i*
3. *$C_{i+1} = C_i \cup \{S_1 = S_1', \ldots, S_n = S_n'\}$ and $E_{i+1} = E_i \cup \{|S_1| = |S_1'|, \ldots, |S_n| = |S_n'|\}$ when A_m and A are of the form $A_m = p(S_1, \ldots, S_n)$ and $A = p(S_1', \ldots, S_n')$*
4. *G_{i+1}' is the goal $\leftarrow C_{i+1} \cup E_{i+1}, A_1, \ldots, A_{m-1}, B_1, \ldots B_r, A_{m+1}, , \ldots, A_n$*
5. *Let θ_i be the set of string substitutions found by applying $\mathcal{A}$ and $\mathcal{U}$ to $C_{i+1} \cup E_{i+1}$.*
6. *G_{i+1} is the goal $\leftarrow (C_{i+1} \cup E_{i+1}, A_1, \ldots, A_{m-1}, B_1, \ldots B_r, A_{m+1}, , \ldots, A_n)\theta_i$* □

In line 5, θ_i consists only of equations in $C_{i+1} \cup E_{i+1}$ which are in solved form; the rest of the equations in $C_{i+1} \cup E_{i+1}$ may be in partially-solved form.

Definition 5 Let P be a CLP($\mathcal{S}$) program and let G be a goal. A successful SSLD-derivation is an SSLD-derivation which ends in a goal with only (possibly empty) constraints and no predicate goals. □

Definition 6 Let A be an algebraic constraint solver and let U be a string-constraint solver. Let P be a CLP($\mathcal{S}$) program and let G be a goal. Let $\leftarrow C_n \cup E_n$ be the final goal in a successful SSLD-derivation. Then $E_A \cup E_U$ is a *SSLD-computed constraint using A and U* for $P \cup \{G\}$ if
E_A is a set of integer equations obtained from reducing E_n using A,
and let θ_A be the subset of E_A that are variable substitutions
E_U is a set of string equations obtained from reducing $C_n\theta_A$ using U.
If $E_U \cup E_U$ is a set of string substitutions then the substitutions in $E_A \cup E_U$ restricted to the variables occurring in G is an *SSLD-computed answer substitution* for $P \cup \{G\}$. □

Note that it is possible that the algebraic constraint solver or the string constraint solver may halt in failure. In such a case the SSLD-resolution is considered to have ended in failure. The soundness and completeness of SSLD-resolution for allowed CLP($\mathcal{S}$) programs is shown in [18].

The following is an SSLD-derivation for the goal $\leftarrow add(0100, 0101, \overset{s}{P})$ using the program given below:

$\{succ(\overset{r}{X}0, \overset{r}{X}1),$

$succ(\overset{r}{X}1, \overset{m}{Y}0) \leftarrow succ(\overset{r}{X}, \overset{m}{Y}),$

$add(\overset{r}{X}, 0, \overset{r}{X}),$

$add(\overset{r}{X}\overset{1}{A}, \overset{m}{Y}0, \overset{p}{Z}\overset{1}{A}) \leftarrow add(\overset{r}{X}, \overset{m}{Y}, \overset{p}{Z}),$

$add(\overset{r}{X}, \overset{m}{Y}1, \overset{p}{Z}\overset{1}{A}) \leftarrow (succ(\overset{r}{X}, \overset{n}{W}\overset{1}{A}) \wedge add(\overset{n}{W}, \overset{m}{Y}, \overset{p}{Z}))\}$

(Instantiations of the variable symbols in the SSLD-derivation is differentiated through subscripts.)

$\leftarrow \{\}, add(0100, 0101, \overset{s}{P})$

| *using* $add(\overset{r}{X}, \overset{m}{Y}1, \overset{p}{Z}\overset{1}{A}) \leftarrow succ(\overset{r}{X}, \overset{n}{W}\overset{1}{A}), add(\overset{n}{W}, \overset{m}{Y}, \overset{p}{Z})$

| $\theta_1 = \{\overset{4}{X} = 0100, \overset{3}{Y} = 010, r = 4, m = 3\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1\} \cup \theta_1, succ(0100, \overset{n}{W}\overset{1}{A}), add(\overset{n}{W}, 010, \overset{p}{Z})$

| *using* $succ(\overset{r_1}{X_1}0, \overset{r_1}{X_1}1)$

| $\theta_2 = \{\overset{3}{X_1} = 010, \overset{3}{W} = 010, \overset{1}{A} = 1, r_1 = 3, n = 3\}$

$\leftarrow \{\overset{p}{Z}\overset{1}{A} = \overset{s}{P}, s = p + 1\} \cup \theta_1 \cup \theta_2, add(010, 010, \overset{p}{Z})$

| *using* $add(\overset{r_2}{X_2}\overset{1}{A_2}, \overset{m_2}{Y_2}0, \overset{p_1}{Z_1}\overset{1}{A_2}) \leftarrow add(\overset{r_2}{X_2}, \overset{m_2}{Y_2}, \overset{p_1}{Z_1})$

| $\theta_3 = \{\overset{2}{X_2} = 01, \overset{1}{A_2} = 0, \overset{2}{Y_2} = 01, r_2 = 2, m_2 = 2\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3, add(01, 01, \overset{p_1}{Z_1})$

| *using* $add(\overset{r_3}{X_3}, \overset{m_3}{Y_3}1, \overset{p_2}{Z_2}\overset{1}{A_3}) \leftarrow succ(\overset{r_3}{X_3}, \overset{n_1}{W_1}\overset{1}{A_3}), add(\overset{n_1}{W_1}, \overset{m_3}{Y_3}, \overset{p_2}{Z_2})$

| $\theta_4 = \{\overset{2}{X_3} = 01, \overset{1}{Y_3} = 0, r_3 = 2, m_3 = 1\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1, \overset{p_1}{Z_1} = \overset{p_2}{Z_2}\overset{1}{A_3}, p_1 = p_2 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4,$

$succ(01, \overset{n_1}{W_1}\overset{1}{A_3}), add(\overset{n_1}{W_1}, 0, \overset{p_2}{Z_2})$

| *using* $succ(\overset{r_4}{X_4}1, \overset{m_4}{Y_4}0) \leftarrow succ(\overset{r_4}{X_4}, \overset{m_4}{Y_4})$

| $\theta_5 = \{\overset{1}{X_4} = 0, \overset{1}{Y_4} = \overset{n_1}{W_1}, \overset{1}{A_3} = 0, r_4 = 1, m_4 = 1\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1, \overset{p_1}{Z_1} = \overset{p_2}{Z_2}\overset{1}{A_3}, p_1 = p_2 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5,$

$succ(0, \overset{n_1}{W_1})add(\overset{n_1}{W_1}, 0, \overset{p_2}{Z_2})$

| *using* $succ(\overset{r_5}{X_5}0, \overset{r_5}{X_5}1)$

| $\theta_6 = \{\overset{0}{X_5} = \varepsilon, \overset{1}{W_1} = 1, n_1 = 1, r_5 = 0\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1, \overset{p_1}{Z_1} = \overset{p_2}{Z_2}\overset{1}{A_3}, p_1 = p_2 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6,$

$add(1, 0, \overset{p_2}{Z_2})$

| *using* $add(\overset{r_6}{X_6}, 0, \overset{r_6}{X_6})$

| $\theta_7 = \{\overset{1}{X_6} = 1, \overset{1}{Z_2} = 1, p_2 = 1, r_6 = 1\}$

$\leftarrow \{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1, \overset{p_1}{Z_1} = \overset{p_2}{Z_2}\overset{1}{A_3}, p_1 = p_2 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6 \cup \theta_7.$

Applying the Gaussian elimination and the reduce algorithm to the constraint set given by:

$\{\overset{s}{P} = \overset{p}{Z}\overset{1}{A}, s = p + 1, \overset{p_1}{Z_1}\overset{1}{A_2} = \overset{p}{Z}, p = p_1 + 1, \overset{p_1}{Z_1} = \overset{p_2}{Z_2}\overset{1}{A_3}, p_1 = p_2 + 1\} \cup \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6 \cup \theta_7,$

we obtain the answer substitution: $\overset{s}{P} = 1001$. □

Towards CIAO-Prolog - A Parallel Concurrent Constraint System

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1 Introduction

We present an informal discussion on some methodological aspects regarding the efficient parallel implementation of (concurrent) (constraint) logic programming systems, as well as an overview of some of the current work performed by our group in the context of such systems. These efforts represent our first steps towards the development of what we call the CIAO (Concurrent, Independence-based And/Or parallel) system – a platform which we expect will provide efficient implementations of a series of *non-deterministic, concurrent, constraint logic programming languages*, on sequential and multiprocessor machines.

CIAO can be in some ways seen as an evolution of the &-Prolog [17] system concepts: it builds on &-Prolog ideas such as parallelization and optimization heavily based on compile-time global analysis and efficient abstract machine design. On the other hand, CIAO is aimed at adding several important extensions, such as or-parallelism, constraints, more direct support for explicit concurrency in the source language, as well as other ideas inspired by proposals such as Muse [1] and Aurora [27], GHC [39], PNU-Prolog [30], IDIOM [16], DDAS [32], Andorra-I [31], AKL [20], and the extended Andorra model [40]. One of the objectives of CIAO is to offer at the same time all the user-level models provided by these systems.

More than a precisely defined design, at this point the CIAO system should be seen as a target which serves to motivate and direct our current research efforts. This impreciseness is purposely based on our belief that, in order to develop an efficient system with the characteristics that we desire, a number of technologies have to mature and others still have to be developed from scratch. Thus, our main focus at the moment is in the development of some of these technologies, which include, among others, improved memory management and scheduling techniques, development of parallelization technology for non-strict forms of independence, efficient combination of and- and or-parallelism, support of several programming paradigms via program transformation, and the extension of current parallelization theory and global analysis tools to deal with constraint-based languages.

We will start our discussion by dealing with some methodological issues. We will then introduce some of our recent work in the direction mentioned above. Given the space limitations the description will be aimed at providing an overall view of our recent progress and a set of pointers to some relevant recent publications and technical reports which describe our results more fully. We hope that in light of the objective of providing pointers, the reader will be kind enough to excuse the summarized descriptions and the predominance in the references of (at least recent) work of our group.

2 Separation of issues / Fundamental Principles

We begin our discussion with some very general observations regarding computation rules, concurrency, parallelism, and independence. We believe these observations to be instrumental in understanding our approach and its relationship to others. A motivation for the discussions that follow is the fact that many current proposals for parallel or concurrent logic programming languages and models are actually "bundled packages", in the sense that they offer a combined solution affecting a number of issues such as choice of computation rule, concurrency, exploitation of parallelism, etc. This is understandable since certainly a practical model has to offer solutions for all the problems involved. However, the bundled nature of (the description of) many models often makes it difficult to compare them with each other. It is our view that, in order to be able to perform such comparisons,

a "separation analysis" of models isolating their fundamental principles in (at least) the coordinates proposed above must be performed. In fact, we also believe that such un-bundling brings the additional benefit of allowing the identification and study of the fundamental principles involved in a system independent manner and the transference of the valuable features of a system to another. In the following we present some ideas on how we believe the separation analysis mentioned above might be approached.

2.1 Separating Control Rules and Parallelism

We start by discussing the separation of parallelism and computation rules in logic programming systems. Of the concepts mentioned above, probably the best understood from the formal point of view is that of computation rules. Assuming for example an SLD resolution-based system the "computation rules" amount to a "selection rule" and a "search rule." The objective of computation rules in general is to minimize work, i.e. to reduce the total amount of resolutions needed to obtain an answer. We believe it is useful, at least from the point of view of analyzing systems, to make a strict distinction between parallelism issues and computation rule related issues. To this end, we define parallelism as the simultaneous execution of a number of independent sequences of resolutions, *taken from those which would have to be performed in any case as determined by the computation rules*. We call each such sequence a *thread* of execution. Note that as soon as there is an *actual* (i.e., run-time) dependency between two sequences, one has to wait for the other and therefore parallelism does not occur for some time. Thus, such sequences contain several threads. Exploiting parallelism means taking a fixed-size computation (determined by the computation rules), splitting it into independent threads related by dependencies (building a dependency graph), and assigning these segments to different agents. Both the partitioning and the agent assignment can be performed statically or dynamically. The objective of parallelism in this definition is simply to *perform the same amount of work in less time*.

We consider as an example a typical or-parallel system. Let us assume a finite tree, with no cuts or side-effects, and that all solutions are required. In a first approximation we could consider that the computation rules in such a system are the same as in Prolog and thus the same tree is explored and the number of resolution steps is the same. Exploiting (or-)parallelism then means taking branches of the resolution tree (which have no dependencies, given the assumptions) and giving them to different agents. The result is a performance gain that is independent of any performance implications of the computation rule. As is well known, however, if only (any) one solution is needed, then such a system can behave quite differently from Prolog: if the leftmost solution (the one Prolog would find) is deep in the tree, and there is another, shallower solution to its right, the or-parallel system may find this other solution first. Furthermore, it may do this after having explored a different portion of the tree which is potentially smaller (although also potentially bigger). The interesting thing to realize from our point of view is that part of the possible performance gain (which sometimes produces "super-linear" speedups) comes in a fundamental way from a change in the computation rule, rather than from parallel execution itself. It is not due to the fact that several agents are operating but to the different way in which the tree is being explored ("more breath-first").[1]

A similar phenomenon appears for example in independent and-parallel systems if they incorporate a certain amount of "intelligent failure": computation may be saved. We would like this to be seen as associated to a smarter computation rule that is taking advantage of the knowledge of the independence of some goals rather than having really anything to do with the parallelism. In contrast, also the possibility of performing additional work arises: unless non-failure can be proved ahead of time, and-parallel systems necessarily need to be speculative to a certain degree in order to obtain speedups. However such speculation can in fact be controlled so that no slow down occurs [18].

Another interesting example to consider is the Andorra-I system. The basic Andorra principle underlying this system states (informally) that deterministic reductions are performed ahead of time and possibly in parallel. This principle would be seen from our point of view as actually two principles, one related to the computation rules and another to parallelism. From the computation rule point of view the bottom line is that deterministic reductions are executed first. This is potentially very useful in practice since it can result in a change (generally a reduction, although the converse may also be true) of the number of resolutions needed to find a solution. Once the computation rule is isolated the remaining part of the rule is related to parallelism and can be seen

[1] This can be observed for example by starting a Muse or an Aurora system with several "workers" on a uniprocessor machine. In this experiment it is possible sometimes to obtain a performance gain w.r.t. a sequential Prolog system even though there is no parallelism involved – just a *coroutining* computation rule, in this case implemented by the multitasking operating system.

simply as stating that deterministic reductions can be executed in parallel. Thus, the "parallelism part" of the basic Andorra principle, once isolated from the computation rule part, brings a basic principle to parallelism: that of the general convenience of parallel execution of deterministic threads.

We believe that the separation of computation rule and parallelism issues mentioned above allows enlarging the applicability of the interesting principles brought in by many current models.

2.2 Abstracting Away the Granularity Level: The Fundamental Principles

Having argued for the separation of parallelism issues from those that are related to computation rules, we now concentrate on the fundamental principles governing parallelism in the different models proposed. We argue that moving a principle from one system to another can often be done quite easily if another such "separation" is performed: isolating the principle itself from the *level of granularity* at which it is applied. This means viewing the parallelizing principle involved as associated to a generic concept of thread, to be particularized for each system, according to the fundamental unit of parallelism used in such system.

As an example, and following these ideas, the fundamental principle of determinism used in the basic Andorra model can be applied to the &-Prolog system. The basic unit of parallelism considered when parallelizing programs in the classical &-Prolog tools is the subtree corresponding to the complete resolution of a given goal in the resolvent. If the basic Andorra principle is applied at this level of granularity its implications are that deterministic subtrees can and should be executed in parallel (even if they are "dependent" in the classical sense). Moving the notions of determinism in the other direction, i.e. towards a finer level of granularity, one can think of applying the principle at the level of bindings, rather than clauses, which yields the concept of "binding determinism" of PNU-Prolog [30].

In fact, the converse can also be done: the underlying principles of &-Prolog w.r.t. parallelism –basically its independence rules– can in fact be applied at the granularity level of the Andorra model. The concept of independence in the context of &-Prolog is defined informally as requiring that a part of the execution "will not be affected" by another. Sufficient conditions –strict and non-strict independence [18]– are then defined which are shown to ensure this property. We argue that applying these concepts at the granularity level of the Andorra model gives some new ways of understanding the model and some new solutions for its parallelization. In order to do this it is quite convenient to look at the basic operations in the light of David Warren's *extended* Andorra model.[2] The extended Andorra model brings in the first place the idea of presenting the execution of logic programs as a series of simple, low level operations on and-or trees. In addition to defining a lower level of granularity, the extended Andorra model incorporates some principles which are related in part to parallelism and in part to computation rule related issues such as the above mentioned basic Andorra principle and the avoidance of re-computation of goals.

On the other hand the extended Andorra model also leaves several other issues relatively more open. One example is that of when nondeterministic reductions may take place in parallel. One answer for this important and relatively open issue was given in the instantiation of the model in the AKL language. In AKL the concept of "stability" is defined as follows: a configuration (partial resolvent) is said to be stable if it cannot be affected by other sibling configurations. In that case the operational semantics of AKL allow the non-determinate promotion to proceed. Note that the definition is, not surprisingly, equivalent to that of independence, although applied at a different granularity level. Unfortunately stability/independence is in general an undecidable property. However, applying the work developed in the context of independent and-parallelism at this level of granularity provides sufficient conditions for it. The usefulness of this is underlined by the fact that the current version of AKL incorporates the relatively simple notion of strict independence (i.e. the absence of variable sharing) as its stability rule. However, the presentation above clearly marks the way for incorporating more advanced concepts, such as non-strict independence, as a sufficient condition for the independence/stability rule. As will be mentioned, we are actively working on compile-time detection of non-strict independence, which we believe will be instrumental in this context. Furthermore, and as we will show, when adding constraint support to a system the traditional notions of independence are no longer valid and both new definitions of independence and sufficient conditions for it need to be developed. We believe that the view proposed herein allows the direct application of general results concerning independence in constraint systems to several realms, such as the extended Andorra model and AKL.

[2] This is understandable, given that adding independent and-parallelism to the basic Andorra model was one of the objectives in the development of its extended version.

Another way of moving the concept of independence to a finer level of granularity is to apply it at the binding level. This yields a rule which states that dependent bindings of variables should wait for their leftmost occurrences to complete (in the same way as subtrees wait for dependent subtrees to their left to complete in the standard independent and-parallelism model), which is essentially the underlying rule of the DDAS model [32]. In fact, one can imagine applying the principle of non-strict independence at the level of bindings, which would yield a "non-strict" version of DDAS which would not require dependent bindings to wait for bindings to their left which are guaranteed to never occur, or for bindings which are guaranteed to be compatible with them.

With this view in mind we argue that there are essentially four fundamental principles which govern exploitation of parallelism:

- ***independence***, which allows parallelism among non-deterministic threads,
- ***determinacy***, which allows parallelism among dependent threads,
- ***non-failure***, which allows guaranteeing non-speculativeness, and
- ***granularity***, which allows guaranteeing speedup in the presence of overheads.

2.3 User-level Concurrency

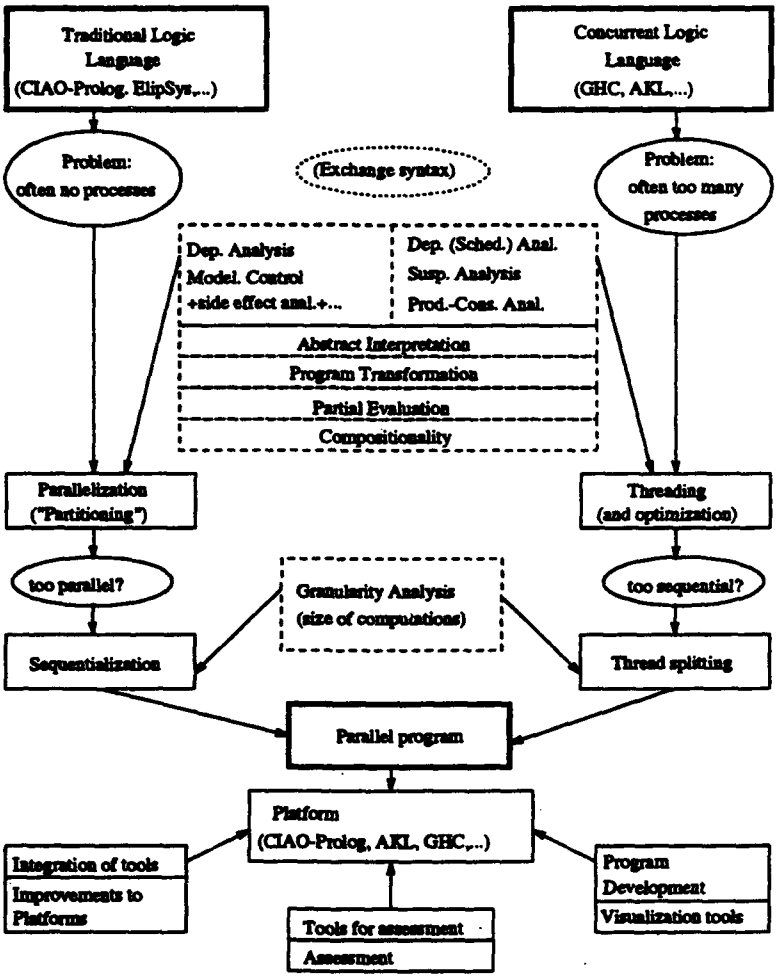
Similarly to the separations mentioned above (parallelism vs. computation rule and principles vs. granularity level of their application) we also believe in a separation of "concurrency" from both parallelism and computation rules. We believe concurrency is most useful when it is explicitly controlled by the user and should be separate from the implicit computation rules. This is in contrast with parallelism, which ideally should be transparent to the user, and with smart computation rules of which the user should only be aware in the sense of being able to derive an upper bound on the amount of computation involved in running a program for a given query using that rule. Space limitations prevent us from elaborating more on this topic or that of the separation between concurrency and parallelism. However, an example of an application of the latter can be seen in ***schedule analysis***, where the maximal essential components of concurrency are isolated and sequenced to allow the most efficient possible execution of the concurrent program by one agent [21]. Schedule analysis is, after all, an application of the concept of dependence (or, conversely, independence) at a certain level of granularity in order to "unparallelize" a program, and is thus based on the same principles as automatic parallelization.

2.4 Towards a General-Purpose Implementation

We believe that the points regarding the separation of issues and fundamental principles sketched in the previous sections at the same time explain and are supported by the recent trend towards convergence in the implemenation techniques of systems that are in principle very different, such as the various parallel implementations of Prolog on one hand (see, for example, [17, 27, 2]) and the implementations of the various committed choice languages on the other (see, for example, [7, 8, 14, 19, 24, 35, 38, 39]). The former are based on schemes for parallelizing a sequential language; they tend to be stack-based, in the sense that (virtual) processors allocate environments on a stack and execute computations "locally" as far as possible until there is no more work to do, at which point they "steal" work from a busy processor. The latter, by contrast, are based on concurrent languages with dataflow synchronization; they tend to be heap-based, in the sense that environments are generally allocated on a heap, and there is (at least conceptually) a shared queue of active tasks.

The aforementioned convergence can be observed in that, on one hand, driven by the demonstrated utility of delay primitives in sequential Prolog systems (e.g., the `freeze` and `block` declarations of Sicstus Prolog [6], `when` declarations of NU-Prolog [36], etc.), parallel Prolog systems have been incorporating capabilities to deal with user-defined suspension and coroutining behaviors—for example, &-Prolog allows programmer-supplied *wait*-declarations, which can be used to express arbitrary control dependencies. In sequential Prolog systems with delay primitives, delayed goals are typically represented via heap-allocated "suspension records," and such goals are awakened when the variables they are suspended on get bindings [5]. Parallel Prolog systems inherit this architecture, leading to implementations where individual tasks are stack-oriented, together with support for heap-allocated suspensions and dataflow synchronization. On the other hand, driven by a growing consensus that some form of "sequentialization" is necessary to reduce the overhead of managing fine-grained parallel tasks

on stock hardware (see, for example, [13, 37, 22]), implementors of committed choice languages are investigating the use of compile-time analyses to coalesce fine-grained tasks into coarser-grained sequential threads that can be implemented more efficiently. This, again, leads to implementations where individual sequential threads execute in a stack-oriented manner, but where sets of such threads are represented via heap-allocated activation records that employ dataflow synchronization. Interestingly, and conversely, in the context of parallel Prolog systems, there is also a growing body of work trying to address the problem of automatic parallelizing compilers often "parallelising too much" which appears if the target architecture is not capable of supporting fine grain parallelism. Figure 2.4 illustrates this (and in fact reflects the interactions among the partners of the ParForCE Esprit project, where some of these interactions are being investigated).



This convergence of trends is exciting: it suggests that we are beginning to understand the essential implementation issues for these languages, and that from an implementor's perspective these languages are not as fundamentally different as was originally believed. It also opens up the possibility of having a general purpose abstract machine to serve as a compilation target for a variety of languages. As mentioned before this is precisely one of the objectives of the CIAO system. Encouraging initial results in this direction have been demonstrated in the sequential context by the QD-Janus system [12] of S. Debray and his group. QD-Janus, which compiles down to Sicstus Prolog and uses the delay primitives of the Prolog system to implement dataflow synchronization, turns out to be more than three times faster, on the average, than Kliger's customized implementation of FCP(:) [23] and requires two orders of magnitude less heap memory [11]. We believe that this point will also extend to parallel systems: as noted above, the &-Prolog system already supports stack-oriented parallel execution together with arbitrary control dependencies, suspension, and dataflow synchronization via user-supplied *wait*-declarations, all characteristics that CIAO inherits. This suggests that the dependence graphs and *wait*-declarations of &-Prolog/CIAO can serve as a common intermediate language, and its runtime system can act as

an appropriate common low-level implementation, for a variety of parallel logic programming implementations. We do not mean to suggest that the performance of such a system will be *optimal* for all possible logic programming languages: our claim is rather that it will provide a way to researchers in the community implement their languages with considerably less effort than has been possible to date, and yet attain reasonably good performance. We are currently exploring these points in collaboration with S. Debray.

3 Some of our recent work in this context

We now provide an overview of our recent work in filling some of the gaps that, in our understanding, are missing in order to fulfill the objectives outlined in the previous section.

3.1 Parallelism based on Non-Strict Independence

One of our starting steps is to improve the independence-based detection of parallelism based on information that can be obtained from global analysis using the current state of the art in abstract interpretation. We have had a quite successful experience using this technique for detecting the classical notion of "strict" independence. These results are summarized in [3], which compares the performance of several abstract interpretation domains and parallelization algorithms using the &-Prolog compiler and system.

While these results are quite encouraging there is another notion of independence – "non-strict" independence [18] – which ensures the same important "no slow down" properties than the traditional notion of strict independence and allows considerable more parallelism than strict independence [33]. The support of non-strict independence requires, however, a review of our compile-time parallelization technology which to date has been exclusively based on strict independence. In [4] we describe some of our recent work filling this gap. Rules and algorithms are provided for detecting and annotating non-strict independence at compile-time. We also propose algorithms for combined compile-time/run-time detection, including run-time checks for this type of parallelism, which in some cases turn out to be different from the traditional groundness and independence checks used for strict independence. The approach is based on the knowledge of certain properties about run-time instantiations of program variables —sharing, groundness, freeness, etc.— for which compile-time technology is available, with new approaches being currently proposed. Rather than dealing with the analysis itself, we present how the analysis results can be used to parallelize programs.

3.2 Parallelization in the Presence of Constraints: Independence / Stability

In the CIAO-Prolog system, from the language point of view, we assume a constraint-based, non-deterministic logic programming language. As such, and apart from the concurrency/coroutining primitives, the user language can be viewed as similar to Prolog when working on the Herbrand domain, and to systems such as CLP(R) or CHIP when working over other domains. This implies that the traditional notions of independence / stability need to be evaluated in this context and, if necessary, extended to deal with the fact that constraint solving is occurring in the actual execution in lieu of unification.

Previous work in the context of traditional Logic Programming languages has concentrated on defining independence in terms of preservation of search space, and such preservation has then been achieved by ensuring that either the goals do not share variables (*strict independence*) or if they share variables, that they do not "compete" for their bindings (*non-strict independence*).

In [10] we have shown (in collaboration with Monash University) that a naive extrapolation of the traditional notions of independence to Constraint Logic Programming is unsatisfactory (in fact, wrong) for two reasons. First, because interaction between variables through constraints is more complex than in the case of logic programming. Second, in order to ensure the efficiency of several optimizations not only must independence of the search space be considered, but also an orthogonal issue – "independence of constraint solving." We clarify these issues by proposing various types of search independence and constraint solver independence, and show how they can be combined to allow different independence-related optimizations, in particular parallelism. Sufficient conditions for independence which can be evaluated "a-priori" at run-time and are easier to identify at compile-time than the original definitions, are also proposed. Also, it has been shown how the concepts proposed, when applied to traditional Logic Programming, render the traditional notions and are thus a strict generalization of such notions.

3.3 Extending Global Analysis Technology to CLP

As mentioned before, since many optimizations, including independence / stability detection, are greatly aided by (and sometimes even require) global analysis, traditional global analysis techniques have to be extended to deal with the fact that constraint solving is occurring in the actual execution in lieu of unification. In [9] we present and illustrate with an implementation a practical approach to the dataflow analysis of programs written in constraint logic programming (CLP) languages using abstract interpretation. We argue that, from the framework point of view, it suffices to propose quite simple extensions to traditional analysis methods which have already been proved useful and practical and for which efficient fixpoint algorithms have been developed. This is shown by proposing a simple but quite general extension to the analysis of CLP programs of Bruynooghe's traditional framework, and describing its implementation – the "PLAI" system. As the original, the framework is parametric and we provide correctness conditions to be met by the abstract domain related functions to be provided. In this extension constraints are viewed not as "suspended goals" but rather as new information in the store, following the traditional view of CLP. Using this approach, and as an example of its use, a complete, constraint system independent, abstract analysis is presented for approximating definiteness information. The analysis is in fact of quite general applicability. It has been implemented and used in the analysis of CLP(R) and Prolog-III applications. Results from this implementation are also presented which show good efficiency and accuracy for the analysis.

This framework, combined with the ideas of [10] (and [29]) presented in the previous section, is the basis for our current development of automatic parallelization tools for CLP programs, and, in particular, of the parallelizer for the CIAO-Prolog system.

3.4 Extending Global Analysis Technology for Explicit Concurrency

Another step that has to be taken in adapting current compile-time technology to CIAO systems is to develop global analysis technology which can deal with the fact that the new computation rules allow the specification of concurrent executions. While there have been many approaches proposed in the literature to address this problem, in a first approach we focus on a class of languages (which includes modern Prologs with delay declarations) which provide both sequential and concurrent operators for composing goals. In this approach we concentrate on extending traditional abstract interpretation based global analysis techniques to incorporate these new computation rules. This gives a practical method for analyzing (constraint) logic programming languages with (explicit) dynamic scheduling policies, which is at the same time equally powerful as the older methods for traditional programs.

We have developed, in collaboration with the University of Melbourne, a framework for global dataflow analysis of this class of languages [28]. First, we give a denotational semantics for languages with dynamic scheduling which provides the semantic basis for our generic analysis. The main difference with denotational definitions for traditional Prolog is that sequences of delayed atoms must also be abstracted and are included in "calls" and "answers." Second, we give a generic global dataflow analysis algorithm which is based on the denotational semantics. Correctness is formalized in terms of abstract interpretation. The analysis gives information about call arguments and the delayed calls, as well as implicit information about possible call schedulings at runtime. The analysis is generic in the sense that it has a parametric domain and various parametric functions. Finally, we demonstrate the utility and practical importance of the dataflow analysis algorithm by presenting and implementing an example instantiation of the generic analysis which gives information about groundness and freeness of variables in the delayed and actual calls. Some preliminary test results are included in which the information provided the implemented analyzer is used to reduce the overhead of dynamic scheduling by removing unnecessary tests for delaying and awakening, to reorder goals so that atoms are not delayed, and to recognize calls which are "independent" and so allow the program to be run in parallel.

3.5 Granularity Analysis

While logic programming languages offer a great deal of scope for parallelism, there is usually some overhead associated with the execution of goals in parallel because of the work involved in task creation and scheduling. In practice, therefore, the "granularity" of a goal, i.e. an estimate of the work available under it, should be taken into account when deciding whether or not to execute a goal in parallel as a separate task. Building on the ideas first proposed in [13] we describe in [25] a proposal for an automatic granularity control system, which is based

on an accurate granularity analysis and program transformation techniques. The proposal covers granualrity control of both and-parallelism and or-parallelism. The system estimates the granularities of goals at compile time, but they are actually evaluated at runtime. The runtime overhead associated with our approach is usually quite small, and the performance improvements resulting from the incorporation of grain size control can be quite good. Moreover a static analysis of the overhead associated with granularity control process is performed in order to decide its convenience.

The method proposed requires among other things knowing the size of the terms to which program variables are bound at run-time (something which is useful in a class of optimizations which also include recursion elimination). Such size is difficult to even approximate at compile time and is thus generally computed at run-time by using (possibly predefined) predicates which traverse the terms involved. In [26] we propose a technique based on program transformation which has the potential of performing this computation much more efficiently. The technique is based on finding program procedures which are called before those in which knowledge regarding term sizes is needed and which traverse the terms whose size is to be determined, and transforming such procedures so that they compute term sizes "on the fly". We present a systematic way of determining whether a given program can be transformed in order to compute a given term size at a given program point without additional term traversal. Also, if several such transformations are possible our approach allows finding minimal transformations under certain criteria. We also discuss the advantages and applications of our technique and present some performance results.

3.6 Memory Management and Scheduling in Non-deterministic And-parallel Systems

From our experience with the &-Prolog system implementation [17], the results from the DDAS simulator [32], and from informal conversations with the Andorra-I developers, efficient memory management in systems which exploit and-parallelism is a problem for which current solutions are not completely satisfactory. This appears to be specially the case with and-parallel systems which support don't-know nondeterminism or deep guards. We believe *non-deterministic* and-parallel schemes to be highly interesting in that they present a relatively general set of problems to be solved (including most of those encountered in the memory management of or-parallel only systems) and have chosen to concentrate on their study.

In collaboration with U. of Bristol, we have developed a distributed stack memory management model which allows flexible scheduling of goals. Previously proposed models are lacking in that they impose restrictions on the selection of goals to be executed or they may require a large amount of virtual memory. Our measurements imply that the above mentioned shortcomings can have significant performance impacts, and that the extension that we propose of the "Marker Model" allows flexible scheduling of goals while keeping (virtual) memory consumption down. We also discuss methods for handling forward and backward execution, cut, and roll back. Also, we show that the mechanism proposed for flexible scheduling can be applied to the efficient handling of the very general form of suspension that can occur in systems which combine several types of non-deterministic and-parallelism and advanced computation rules, such as PNU-Prolog [30], IDIOM [16], DDAS [32], AKL [20], and, in general, those that can be seen as an instantiation of the extended Andorra model [40]. Thus, we believe that the results may be applicable to a whole class of and- and or-parallel systems. Our solutions and results are described more fully in [34].

3.7 Incorporating Or-Parallelism: The ACE Approach

Another important issue is the incorporation of Or-parallelism to an and-parallel system. This implies well known problems related to or-parallelism itself, such as the maintenance of several binding environments, as well as new problems such as the interactions of the multiplicity of binding environments and threads of or-parallel computation with the scoping and memory management requirements of and-parallelism. The stack copying approach, as exemplified by the MUSE system, has been shown to be a quite successful alternative for representing multiple environments during or-parallel execution of logic programs. In collaboration with the U. of New Mexico and U. of Bristol we have developed an approach for parallel implementation of logic programs, described more fully in [15], which we believe is capable of exploiting both or-parallelism and independent and-parallelism (as well as other types of and-parallelism) in an efficient way using stack copying ideas. This model combines such ideas with proven techniques in the implementation of independent and-parallelism, such

as those used in &-Prolog. We show how all solutions to non-deterministic and-parallel goals are found without repetitions. This is done through re-computation as in Prolog (and &-Prolog), i.e., solutions of and-parallel goals are not shared. We propose a scheme for the efficient management of the address space in a way that is compatible with the apparently incompatible requirements of both and- and or-parallelism. This scheme allows incorporating and combining the memory management techniques used in (non-deterministic) and-parallel systems, such as those mentioned in the previous section, and memory management techniques of or-parallel systems, such as incremental copying. We also show how the full Prolog language, with all its extra-logical features, can be supported in our and-or parallel system so that its sequential semantics is preserved. The resulting system retains the advantages of both purely or-parallel systems as well as purely and-parallel systems. The stack copying scheme together with our proposed memory management scheme can also be used to implement models that combine dependent and-parallelism and or-parallelism, such as Andorra and Prometheus.

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Encapsulated Search and Constraint Programming in Oz

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Abstract

Oz is an attempt to create a high-level concurrent programming language providing the problem solving capabilities of logic programming (i.e., constraints and search). Its computation model can be seen as a rather radical extension of the concurrent constraint model providing for higher-order programming, deep guards, state, and encapsulated search. This paper focuses on the most recent extension, a higher-order combinator providing for encapsulated search. The search combinator spawns a local computation space and resolves remaining choices by returning the alternatives as first-class citizens. The search combinator allows to program different search strategies, including depth-first, indeterministic one solution, demand-driven multiple solution, all solutions, and best solution (branch and bound) search. The paper also discusses the semantics of integer and finite domain constraints in a deep guard computation model.

1 Introduction

Oz [2, 7, 6, 1] is an attempt to create a high-level concurrent programming language providing the problem solving capabilities of logic programming (i.e., constraints and search). Its computation model can be seen as a rather radical extension of the concurrent constraint model [5] providing for higher-order programming, deep guards, state, and encapsulated search. This paper focuses on the most recent extension, a higher-order combinator providing for encapsulated search. The search combinator spawns a local computation space and resolves remaining choices by returning the alternatives as first-class citizens. The search combinator allows to program different search strategies, including depth-first, indeterministic one solution, demand-driven multiple solution, all solutions, and best solution (branch and bound) search. The paper also discusses the semantics of integer and finite domain constraints in a deep guard computation model, which is an interesting issue since these constraints cannot be realized with their declarative semantics (due to intractability and even undecidability of satisfiability and entailment).

The idea behind our search combinator is simple and new. It exploits the fact that Oz is a higher-order language. The search combinator is given an expression E and a variable x (i.e., a predicate x/E) with the idea that E (which declaratively reads as a logic formula) is to be solved for x. The combinator spawns a local computation space for E, which evolves until it fails or becomes stable (a property known from AKL). If the local computation space evolves to a stable expression $(A \vee B) \wedge C$, the two alternatives are returned as predicates:

$$x/(A \vee B) \wedge C \quad \rightarrow \quad x/A \wedge C,\ x/B \wedge C.$$

If the local computation space evolves to a stable expression C not containing a distributable disjunction, it is considered solved and the predicate x/C is returned.

We now relate Oz to AKL and cc(FD), two first-order concurrent constraint programming languages having important aspects in common with Oz.

AKL [3] is a deep guard language aiming like Oz at the integration of concurrent and logic programming. AKL can encapsulate search. AKL admits distribution of a nondeterminate

choice in a local computation space spawned by the guard of a clause when the space has become stable (a crucial control condition we have also adopted in Oz). In AKL, search alternatives are not available as first-class citizens. All solutions search is provided through an extra primitive. Best solution and demand-driven multiple solution search are not expressible.

cc(FD) [8] is a constraint programming language specialized for finite domain constraints. It employs a Prolog-style search strategy and three concurrent constraint combinators called cardinality, constructive disjunction, and blocking implication. It is a compromise between a flat and a deep guard language in that combinators can be nested into combinators, but procedure calls (and hence nondeterminate choice) cannot. Encapsulated best solution search is provided as a primitive, but its control (e.g., stability) is left unspecified.

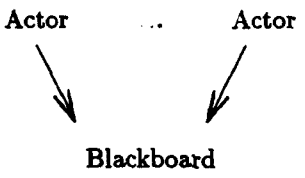
The paper is organized as follows. Section 2 gives an informal presentation of Oz's computation model, and Section 3 relates Oz to logic programming by means of examples. Section 4 shows how encapsulated and demand-driven search can be integrated into a reactive language. Section 5 presents the search combinator, and Section 6 shows how the search strategies mentioned above can be programmed with it. Section 7 discusses how integer and finite domain constraints are accommodated in Oz. Section 8 puts everything together by showing how the N-Queens problem can be solved in Oz.

2 Computation Spaces, Actors, and Blackboards

The computation model underlying Oz generalizes the concurrent constraint model (CC) [5] by providing for higher-order programming, deep guard combinators, and state. Deep guard combinators introduce local computation spaces, as in the concurrent constraint language AKL [3]. Recall that there is only one computation space in CC.

In [6] we give a formal model of computation in Oz, consisting of a calculus rewriting expressions modulo a structural congruence relation, similar to the setup of the π-calculus [4]. For the purposes of this paper, an informal presentation of Oz's computation model, ignoring state, will suffice.

A computation space consists of a number of actors[1] connected to a blackboard.



The actors read the blackboard and reduce once the blackboard contains sufficient information. The information on the blackboard increases monotonically. When an actor reduces, it may put new information on the blackboard and create new actors. As long as an actor does not reduce, it does not have an outside effect. The actors of a computation space are short-lived: once they reduce they disappear. Actors may spawn local computation spaces.

The blackboard stores a constraint (constraints are closed under conjunction, hence one constraint suffices) and a number of named abstractions (to be explained later). Constraints are formulas of first-order predicate logic with equality that are interpreted in a fixed first-order structure called the Oz Universe. For the purposes of this paper it suffices to know that the Oz Universe provides rational trees (as in Prolog II) and integers. The constraint on the blackboard is always satisfiable in the Oz Universe. We say that a blackboard entails a constraint ψ if the implication $\phi \rightarrow \psi$ is valid in the Oz Universe, where ϕ is the constraint stored on the blackboard. We say that a blackboard is consistent with a constraint ψ if the conjunction $\phi \wedge \psi$ is satisfiable in the Oz Universe, where ϕ is the constraint stored on the

[1] Oz's actors are different from Hewitt's actors. We reserve the term agent for longer-lived computational activities enjoying persistent and first-class identity.

blackboard. Since the constraint on the blackboard can only be observed through entailment and consistency testing, it suffices to represent it modulo logical equivalence.

There are several kinds of actors. This section will introduce elaborators, conditionals, and disjunctions.

An elaborator is an actor executing an expression. The expressions we will consider in this section are defined as follows:

$$\begin{array}{rcl} E & ::= & \phi \mid E_1\ E_2 \mid \textbf{local}\ x\ \textbf{in}\ E\ \textbf{end} \\ & \mid & \textbf{proc}\ \{x\ y_1 \ldots y_n\}\ E\ \textbf{end} \mid \{x\ y_1 \ldots y_n\} \\ & \mid & \textbf{if}\ C_1\ []\ \ldots\ []\ C_n\ \textbf{else}\ E\ \textbf{fi} \mid \textbf{or}\ C_1\ []\ \ldots\ []\ C_n\ \textbf{ro} \\ C & ::= & E_1\ \textbf{then}\ E_2 \mid x_1 \ldots x_n\ \textbf{in}\ E_1\ \textbf{then}\ E_2 \end{array}$$

Elaboration of a constraint ϕ checks whether ϕ is consistent with the blackboard. If this is the case, ϕ is conjoined to the constraint on the blackboard; otherwise, the computation space is marked failed and all its actors are cancelled. Elaboration of a constraint corresponds to the eventual tell operation of CC.

Elaboration of a concurrent composition $E_1\ E_2$ creates two separate elaborators for E_1 and E_2.

Elaboration of a variable declaration **local** x **in** E **end** creates a new variable (local to the computation space) and an elaborator for the expression E. Within the expression E the new variable is referred to by x. Every computation space maintains a finite set of local variables.

Elaboration of a procedure definition **proc** $\{x\ y_1 \ldots y_n\}$ E **end** chooses a fresh name a, writes the named abstraction $a{:}\, y_1 \ldots y_n/E$ on the blackboard, and creates an elaborator for the constraint $x = a$. Names are constants denoting pairwise distinct elements of the Oz Universe; there are infinitely many. Since abstractions are associated with fresh names when they are written on the blackboard, a name cannot refer to more than one abstraction.

Elaboration of a procedure application $\{x\ y_1\ \ldots\ y_n\}$ waits until the blackboard entails $x = a$ and contains a named abstraction $a{:}\, x_1 \ldots x_n/E$, for some name a. When this is the case, an elaborator for the expression $E[y_1/x_1 \ldots y_n/x_n]$ is created ($E[y_1/x_1 \ldots y_n/x_n]$ is obtained from E by replacing the formal arguments $x_1, \ldots, x_n$ with the actual arguments $y_1, \ldots, y_n$).

This simple treatment of procedures provides for all higher-order programming techniques. By making variables denote names rather than higher-order values, we obtain a smooth combination of first-order constraints with higher-order programming.

The elaboration of conditional expressions is more involved. We first consider the special case of a one clause conditional with flat guard.

Elaboration of **if** ϕ **then** E_1 **else** E_2 **fi** creates a conditional actor, which waits until the blackboard entails either ϕ or $\neg\phi$. If the blackboard entails ϕ [$\neg\phi$], the conditional actor reduces to an elaborator for E_1 [E_2]. In CC, such a conditional can be expressed as a parallel composition $(\textbf{ask}\,\phi \rightarrow E_1)\ ||\ (\textbf{ask}\,\neg\phi \rightarrow E_2)$ of two ask clauses.

Elaboration of a conditional expression **if** $C_1\ []\ldots[]\ C_n$ **else** E **fi** creates a conditional actor spawning a local computation space for each clause C_i. A clause takes the form

$$x_1 \ldots x_k\ \textbf{in}\ E\ \textbf{then}\ D$$

where the local variables $x_1, \ldots, x_k$ range over both the guard E and the body D of the clause. We speak of a deep guard if E is not a constraint. In Oz, any expression can be used as a guard. This is similar to AKL and in contrast to CC, where guards are restricted to constraints. The local computation space for a clause

$$x\ \textbf{in}\ E\ \textbf{then}\ D$$

(clauses with no or several local variables are dealt with similarly) is created with an empty blackboard and an elaborator for the expression **local** x **in** E **end**.

Constraints from the global blackboard (the blackboard of the computation space the conditional actor belongs to) are automatically propagated to local spaces by elaborating them

in the local spaces (propagation of global constraints can fail local spaces). Moreover, named abstractions from global blackboards are copied to local blackboards (conflicts cannot occur).

We say that a clause of a conditional actor is entailed if its associated computation space S is not failed, S has no actors left, and the global board entails $\exists \overline{y}\, \phi$, where $\overline{y}$ are the local variables of S and ϕ is the constraint of the blackboard of S. Entailment of a local space is a stable property, (i.e., remains to hold when computation proceeds).

A conditional actor must wait until either one of its clauses is entailed or all its clauses (i.e., their associated local spaces) are failed.

If all clauses of a conditional actor **if** C_1 [] ... [] C_n **else** E **fi** are failed, the conditional actor reduces to an elaborator for the expression E (the else constituent of the conditional).

If a clause x_i **in** E_i **then** D_i of a conditional actor is entailed, the other clauses and their associated spaces are discarded, the space associated with the entailed clause is merged with the global space (conflicts cannot occur), and the conditional actor reduces to an elaborator for D_i (the body of the clause).

Elaboration of a disjunctive expression **or** C_1 [] ... [] C_n **ro** creates a disjunctive actor spawning a local computation space for every clause $C_1, \ldots, C_n$. The local spaces are created in the same way as for conditionals. As with conditional clauses, constraints and named abstractions from the global blackboard are automatically propagated to local blackboards.

A disjunctive actor must wait until all but possibly one of its clauses are failed, or until a clause whose body is the trivial constraint **true** is entailed. In the latter case, the disjunctive actor just disappears (justified by the equivalence $A \wedge (A \vee B) \equiv A$). If all clauses of a disjunctive actor are failed, the space of the disjunctive actor is failed (i.e., all its actors are cancelled). If all but one clause of a disjunctive actor are failed, it reduces with the unfailed clause. This is done in two steps. First, the space associated with the unfailed clause is merged with the global space, and then an elaborator for the body of the clause is created. The merge of the local with the global space may fail because the local constraint may be inconsistent with the global constraint. In this case the global space will be failed.

3 Example: Length of Lists

This section clarifies how Oz relates to logic programming and Prolog.

The Horn clauses

```
length(nil,0)
length(X|Xr, s(M)) ← length(Xr,M)
```

define a predicate length(Xs,N) that holds if Xs is a list of length N. Numbers are represented as trees 0, $s(0)$, $s(s(0))$, ..., and lists as trees $t_1|t_2|\ldots|t_n|$nil. The intended semantics of the clauses is captured by the equivalence

$$\begin{aligned} \mathsf{length}(\mathsf{Xs},\mathsf{N}) \quad &\leftrightarrow \quad \mathsf{Xs} = \mathsf{nil} \;\wedge\; \mathsf{N} = 0 \\ &\vee \quad \exists \mathsf{X},\mathsf{Xr},\mathsf{M}\ (\mathsf{Xs} = \mathsf{X}|\mathsf{Xr} \wedge \mathsf{N} = \mathsf{s}(\mathsf{M}) \wedge \mathsf{length}(\mathsf{Xr},\mathsf{M}))\,, \end{aligned}$$

which is obtained from the Horn clauses by Clark's completion. The equivalence exhibits the relevant primitives and combinators of logic programming: constraints (i.e., Xs=nil), conjunction, existential quantification, disjunction, and definition by equivalence. Given the equivalence, it is easy to define the length predicate in Oz:

```
proc {Length Xs N}
   or Xs=nil N=0 then true
   [] X Xr M in Xs=X|Xr N=s(M) then {Length Xr M}
   ro
end
```

There are two things that need explanation. First, the predicate is now referred to by a variable Length, as to be expected in a higher-order language. Second, the two disjunctive clauses have been divided into guards and bodies. The procedure application {Length Xr M} is put into the body to obtain a terminating operational semantics.

To illustrate the operational semantics of Length, assume that the procedure definition has been elaborated. Now we enter the expression

```
declare Xs N in {Length Xs N}
```

whose elaboration declares two new variables Xs and N and reduces the procedure application {Length Xs N} to a disjunctive actor. The **declare** expression is a variant of the **local** expression whose scope extends to expressions the programmer enters later. The disjunctive actor cannot reduce since there is no information about the variables Xs and N on the global blackboard. It now becomes clear why we did not write the recursive procedure application {Length Xr M} into the guard: this would have caused divergence.

Now we enter the constraint ('_' is a variable occurring only once)

```
N = s(s(_))
```

Since N = s(s(_)) is inconsistent with the constraint N=0 on the local blackboard, the first clause of the suspended disjunctive actor can now be failed and the disjunctive actor can reduce with its second clause. This will elaborate the recursive application {Length Xr M} and create a new disjunctive actor whose first clause fails immediately. This will create once more a new disjunctive actor, which this time cannot reduce. The global blackboard now entails

```
Xs = _|_|_   N = s(s(_))
```

Next we enter the constraint

```
Xs = 1|2|nil
```

whose elaboration fails the second clause of the suspended disjunctive actor (since $x = \mathsf{nil}$ is inconsistent with $x = y|z$). Hence the suspended actor reduces with its first clause, no new disjunctive actor is created, and the blackboard finally entails

```
Xs = 1|2|nil   N = s(s(0))
```

The example illustrates important differences between Oz and Prolog: if there are alternatives (specified by the clauses of disjunctions or conditionals), Oz explores the guards of the alternatives concurrently. Only once it is safe to commit to an alternative (e.g., because all other alternatives are failed or because the guard of a conditional clause is entailed), Oz will commit to it. In contrast, Prolog will eagerly commit to the first alternative if a choice is to be made, and backtrack if necessary.

A sublanguage of Oz enjoys a declarative semantics such that computation amounts to equivalence transformation [6]. The declarative semantics of a conditional

$$\textbf{if } x \textbf{ in } E_1 \textbf{ then } E_2 \textbf{ else } E_3 \textbf{ fi}$$

with only one clause is

$$\exists x(E_1 \wedge E_2) \vee (\neg\exists x E_1 \wedge E_3).$$

Hence Oz can express negation $\neg E$ as **if** E **then false else true fi**.

The length predicate can also be defined in a functional manner using a conditional:

```
proc {Length Xs N}
   if Xs=nil then N=0
   [] X Xr M in Xs=X|Xr then N=s(M) {Length Xr M}
   else false fi
end
```

While the functional version has the same declarative reading as the disjunctive formulation, its operational semantics is different in that it will wait until information about its first argument is available. Thus

```
declare Xs N in N=s(s(0)) {Length Xs N}
```

will create a suspending conditional actor and not write anything on the global blackboard. On the other hand,

```
declare Xs N in Xs=_|_|nil {Length Xs N}
```

will write N=s(s(0)) on the global blackboard (although there is only partial information about Xs).

Oz supports functional syntax: the functional version of the length predicate can equivalently be written as:

```
fun {Length Xs}
   case Xs of nil then 0
   [] X|Xr then s({Length Xr})
   end
end
```

4 Encapsulated and Demand-driven Search

Given the length predicate of the previous section, Prolog allows to enumerate all pairs Xs, N such that length(Xs,N) is satisfied. This service can be obtained in Oz in a more flexible form. Oz provides search agents that can be given queries and be prompted for answers. These search agents take the form of objects, the basic concurrency abstraction of Oz.

An object is a procedure O taking a message M as argument. It encapsulates a reference to a data structure acting as the state of the object. A procedure application {O M} (the object is applied to the message) first competes for exclusive access to the object's state (necessary in a concurrent setting) and then applies the method requested by the message:

$$\textit{method}\colon \textit{state} \times \textit{message} \rightarrow \textit{state}.$$

This yields a new state which is released. The message indicates the method to be applied by a name that is mapped to the actual method by the object itself (so-called late binding).

Objects can be expressed in the computation model outlined in Section 2 if one further primitive, called constraint communication, is added. Oz's higher-order programming facilities make it straightforward to obtain multiple inheritance of methods. For more information about objects in Oz we refer the reader to [7, 2, 1].

Now suppose Search is a search object as outlined above (any number of search objects can be created by inheritance from a predefined search object). First, we present it a query using the method query:

```
local Q in
   proc {Q A} local Xs N in A=Xs#N {Length Xs N} end end
   {Search query(Q)}
end
```

The query is specified by a unary predicate, so that solutions can be computed uniformly for one variable. Since we have existential quantification and pairing, this is no loss of generality. Using functional notation, we can write the above expression more conveniently as

```
{Search query(proc {A} local Xs N in A=Xs#N {Length Xs N} end end)}
```

Now we can request computation of the first solution by sending the message

```
{Search next}
```

which will produce the pair nil#0. Sending next (i.e., elaborating {Search next}) once more will produce (_|nil)#s(0), and so on. What happens when an solution is found can be specified by sending Search the message action(P), where P is a unary procedure to be applied to every solution found. The procedure P may, for instance, display solutions in a window or send them to other objects.

We remark that Prolog provides demand-driven search at the user interface, but not at the programming level. Aggregation in Prolog (i.e., bagof) is eager and will diverge if there are infinitely many solutions. In Oz, we can have any number of search objects at the same time and request solutions as required.

5 Solvers

We now introduce solvers, which are higher-order actors providing for encapsulated search. Many different search strategies can be programmed with solvers, ranging from demand-driven depth-first (as exemplified by the search object in the previous section) to best solution (branch and bound) strategies.

The key idea behind search in Oz is to exploit the distributivity law and proceed from $(A \vee B) \wedge C$ to $A \wedge C$ and $B \wedge C$. While Prolog commits to $A \wedge C$ first and considers $B \wedge C$ only upon backtracking, Oz makes both alternatives available as first-class citizens. To do this, the variable being solved for must be made explicit and abstracted from in the alternatives. For instance, if **or** $x = 1$ [] $x = 2$ **ro** is being solved for x, distribution will produce the abstractions **proc** $\{x\}$ $x = 1$ **end** and **proc** $\{x\}$ $x = 2$ **end**.

Solvers are created by elaboration of solve expressions

$$\textbf{solve}[x\text{: } E;\ u]$$

where x (the variable being solved for) is a local variable taking the expression E as scope. The variable u provides for output. The solver created by elaboration of the above expression spawns a local computation space for the expression

$$\textbf{local } x \textbf{ in } E \textbf{ end}$$

As with other local computation spaces, constraints and named abstractions are propagated from global blackboards to the local blackboards of solvers.

A solver can reduce if its local computation space is either failed or stable. A local computation space is called stable if it is blocked and remains blocked for every consistent extension of the global blackboard. A computation space is called blocked if it is not failed and none of its actors can reduce. Stability is known from AKL [3], where it is used to control nondeterministic promotion. Note that a local computation space is entailed if and only if it is stable and has no actor left.

If the local computation space of a solver has failed, the solver reduces to an elaborator for the constraint (u is the output variable)

$$u = \text{failed}.$$

If the local computation space of a solver is stable and does not contain a disjunctive actor, the solver reduces to an elaborator for

$$u = \text{solved}(\textbf{proc } \{x\}\ F \textbf{ end})$$

where F is an expression representing the stable local computation space (the nested procedure definition has been explained in the previous section).[2] Abstracting the solution with respect to x is advantageous in case F does not fully determine x; for instance, if F is **local** z **in** $x = f(z)$ **end**, different applications will enjoy different local variables z. A less general way to return the solution would be to reduce to an elaborator for $u =$ **solved**(x) F.

If the local computation space of a solver is stable and contains a disjunctive actor **or** C_1 [] ... [] C_n **ro**, the solver reduces to an elaborator for

$$u = \textbf{distributed}(\textbf{proc}\ \{x\}\ \textbf{or}\ C_1\ \textbf{ro}\ F\ \textbf{end}\quad \textbf{proc}\ \{x\}\ \textbf{or}\ C_2\ []\ \ldots\ []\ C_n\ \textbf{ro}\ F\ \textbf{end})$$

where F is an expression representing the stable local computation space after deletion of the disjunctive actor. Requiring stability ensures that distribution is postponed until no other reductions are possible. This is important since repeated distribution may result in combinatorial explosion.

For combinatorial search problems it is often important to distribute the right disjunction and try the right clause first. Oz makes the following commitments about order: clauses are distributed according to their static order; solvers distribute the most recently created disjunctive actor; and elaboration proceeds from left to right, where suspended actors that become reducible are given priority (similar to Prologs with freeze). Taking the most recently created disjunctive actor for distribution seems to be more expressive than taking the least recently created one (see the first failure labeling procedure in Section 8).

Solvers cannot express breadth-first search if disjunctions with more than two clauses are used. This can be remedied by also returning the number of remaining clauses when a disjunctive actor is distributed.

Solve expressions are made available through a predefined procedure

```
proc {Solve P U} solve[X: {P X}; U] end
```

6 Search Strategies

We start with a function taking a query (i.e., a unary procedure) as argument and trying to solve it following a depth-first strategy:

```
fun {Depth Q}
   local S = {Solve Q} in
      case S of distributed(L R) then
         case {Depth L} of solved(_)=T then T else {Depth R} end
      else S end
   end
end
```

If no solution is found (but search terminates), failed is returned. If a solution is found, solved(A) is returned, where A is the abstracted solution. A procedure solving a query with Depth and displaying the result can be written as follows:

```
proc {SolveAndBrowse Q}
   case {Depth Q} of failed then {Browse 'no solution found'}
    [] solved(A) then {Browse {A}}
   end
end
```

[2] The reader might be surprised by the fact that local computation spaces can be represented as expressions. This is however an obvious consequence of the fact that Oz's formal model [6] models computation states as expressions.

```
fun {One Q}
   local S = {Solve Q} in
      case S of distributed(L R) then
         if T in {One L}=solved(_)=T then T
         [] T in {One R}=solved(_)=T then T
         else failed fi
      else S end
   end
end
```

Figure 1: Parallel one solution search.

The search performed by Depth is sequential. Figure 1 shows an indeterministic search function One that explores alternatives in parallel guards.[3] The use of deep parallel guards provides a high potential for parallel execution.

Combinatorial optimization problems (e.g., scheduling) often require best solution search. Following a branch and bound strategy, this can be done as follows: once a solution is found, only solutions that are better with respect to a total order are searched for. With every better solution found, the constraints on further solutions can be strengthened, thus pruning the search space.

```
fun {Best Q R}
   local
      fun {BAB Fs Bs S}
         case Fs of nil then
            case Bs of nil then S
            [] B|Br then {BAB (proc {X} {R {S} X} {B X} end)|nil Br S}
            end
         [] F|Fr then
            case {Solve F} of failed then {BAB Fr Bs S}
            [] solved(T) then {BAB nil {Append Fr Bs} T}
            [] distributed(L R) then {BAB L|R|Fr Bs S}
            end
         end
      end
   in {BAB Q|nil nil R failed} end
end
```

Figure 2: Best solution search.

Figure 2 shows a function Best searching the best solution of a query Q with respect to a total order R (a binary procedure). The local function BAB takes two stacks Fs and Bs of alternatives and the best solution found so far as arguments (if no solution has been found so far, failed is taken as last argument) and returns the best solution. Alternatives which are already constrained to produce a better solution than S reside on the foreground stack Fs, and the remaining alternatives reside on the background stack Bs. If the foreground stack is empty, an alternative B from the background stack is taken. The query A obtained from constraining B to solutions better than S (the best solution so far) is expressed as follows:

[3] This search function was suggested to us by Sverker Janson.

```
create Search from UrObject
   meth action(A)  action←A end
   meth query(Q)  stack←Q|nil end
   meth next
      case @stack of nil then {@action failed}
      [] N|Nr then
         case {Solve N} of failed then stack←Nr ⟪next⟫
         [] solved(S) then stack←Nr {@action solved(S)}
         [] distributed(L R) then stack←L|R|Nr ⟪next⟫
         end
      end
   end
end
```

Figure 3: Demand driven depth-first search.

```
A = proc {X} {R {S} X} {B X} end
```

If a new and better solution is obtained, all nodes from the foreground stack are moved to the background stack so that they will be correctly constrained before they are explored.

The program in Figure 3 defines an object Search realizing the functionality described in Section 4. The object must be initialized with messages query(Q) and action(A) fixing the query to be solved and the action to be taken when a solution is found, respectively. The attribute stack stores the unexplored alternatives. If a solution is requested with the method next, the alternatives on the stack are explored following a depth-first strategy. If no alternatives are left on the stack, the specified action is applied to the atom failed.

The search object illustrates object-oriented constraint programming in Oz. More sophisticated search strategies, for instance iterated depth-first search, can be obtained by refining Search using inheritance.

7 Integers and Finite Domains

An implementation of the presented computation model must come with efficient and incremental algorithms for deciding satisfiability and entailment of constraints. This means that a programming language must drastically restrict the constraints a programmer can actually use. For instance, addition and multiplication of integers cannot be made available as purely declarative constraints since satisfiability of conjunctions of such constraints is undecidable (Hilbert's tenth problem).

The usual way to deal with this problem is to base the implementation on incomplete algorithms for satisfiability and entailment (e.g., delay nonlinear arithmetic constraints until they are linear). Consequently, constraints are not anymore fully characterized by their declarative semantics, and the programmer must understand their operational semantics.

In Oz, we make a distinction between basic and virtual constraints. Basic constraints are what has been called constraints so far. Their semantics is given purely declaratively by the Oz Universe. Oz is designed such that the programmer can only write basic constraints whose declarative semantics can be faithfully realized by the implementation (i.e., sound and complete algorithms for satisfiability and entailment). Virtual constraints are procedures whose operational semantics is sound but incomplete with respect to the declarative semantics of the corresponding logic constraint. A typical example of a virtual constraint is the length predicate for lists defined in Section 3.

Most constraints expressible over the Oz Universe are only available through predefined virtual constraints (i.e., with incomplete operational semantics). A typical example is addition

```
proc {`≤'' X Y}
   if {FdIn X Inf Sup} {FdIn Y Inf Sup} then
     local
        proc {LE Xl Xu Yl Yu}
           if X=Y then true
           [] Xu≤Yl then true
           [] {FdIn X Xl+1 Sup} then {FdIn Y Xl+1 Sup} {LE Xl+1 Xu Yl Yu}
           [] {FdIn X Inf Xu−1} then {LE Xl Xu−1 Yl Yu}
           [] {FdIn Y Yl+1 Sup} then {LE Xl Xu Yl+1 Yu}
           [] {FdIn Y Inf Yu−1} then {FdIn X Inf Yu−1} {LE Xl Xu Yl Yu−1}
           fi
        end
     in {LE Inf Sup Inf Sup} end
   else false fi
end
```

Figure 4: The virtual constraint $X \leq' Y$.

of integers, whose definition is as follows:

```
proc {`+` X Y Z}
   if int(X) int(Y) isdet[X] isdet[Y] then plus(X,Y,Z) else false fi
end
```

Here plus(X,Y,Z) is the basic constraint expressing integer addition (partial functions are avoided by using relations), int(X) is the basic constraint expressing that X is an integer, and isdet[X] creates an actor that disappears as soon as there is a constant a in the signature of the Oz Universe such that X=a is entailed by the blackboard. Clearly, there is no difficulty in implementing the virtual constraint {`+` X Y Z}. Moreover, its semantics is fully defined in terms of the computation model outlined in Section 2 (extended with the isdet[X] actor, of course).

The virtual constraint

```
proc {IsInt X}
   if int(X) isdet[X] then true else false fi
end
```

will fail if the blackboard entails that X is no integer, and disappear (important for deep guards) if there is an integer n such that the blackboard entails X=n.

A further example is the predefined virtual constraint

```
proc {`≤` X Y}
   if {IsInt X} {IsInt Y} then le(X,Y) else false fi
end
```

where and le(X,Y) is the basic constraint expressing the canonical order on integers.

The predefined virtual constraint

```
proc {FdIn X L U}
   if {IsInt L} {IsInt U} then le(L,X) le(X,U) le(Inf,L) le(U,Sup) else false fi
end
```

makes it possible to constrain a variable X to a finite domain L..U (i.e., the value of X must be an integer between L and U). There variables Inf and Sup are predefined by the implementation and fix the maximal size of finite domains (i.e., there are only finitely many finite domains).

Another important predefined virtual constraint is

```
proc {FdNec X C}
   if {FdIn X Inf Sup} {IsInt C} then X ≠ C else false fi
end
```

whose declarative reading says that X is a finite domain variable different from C ($X \neq C$ is a basic constraint).

Figure 4 shows the definition of a virtual constraint $X \leq' Y$ enforcing domain consistency for finite domain variables (the infix operators $\leq$, $+$, and $-$ expand to applications of the corresponding virtual constraints). For instance, elaboration of the expression

```
local X Y in
   {FdIn X 3 7} {FdIn Y 7 24}
   if X ≤' Y then {Browse yes} else {Browse no} fi
end
```

will reduce the conditional actor to {Browse yes}, and elaboration of

```
{FdIn X 3 7} {FdIn Y 7 24} Y ≤' X
```

will constrain X and Y to 7.

With the outlined techniques we can formally define all finite domain constraints as virtual constraints such that a faithful and efficient implementation is possible. To our knowledge, this is the first formal semantics for finite domain constraints in a deep guard computation model.

To define heuristics such as first failure labeling (see next section), we need a reflective primitive. The actor

$$\textbf{reflect}[x; y]$$

can reduce as soon as the blackboard constrains the variable x to a finite domain. It will then reduce to an elaborator for the constraint

$$y = n_1|\ldots|n_k|\mathsf{nil},$$

where $n_1|\ldots|n_k|\mathsf{nil}$ is the shortest list in ascending order such that the blackboard entails the constraint $x = n_1 \vee \ldots \vee x = n_k$. Note that the reflection actor is different from all other actors in that its reduction may have different effect if it is postponed.

8 Example: N-queens

Figure 5 shows an Oz program solving the n-queens problem (place n queens on an $n \times n$ chessboard such that no queen is attacked by another queen). The predicate {Queens N Xs} is satisfied iff the list Xs represents a solution to the n-queens problem. The list Xs has length N, where every element is an integer between 1 and N. The ith element of Xs specifies in which row the queen in the ith column is placed. The solutions to the 100-queens problem, say, can be obtained by providing the search object of Section 6 with the query

```
{Search query(proc {Xs} {Queens 100 Xs} end)}
```

The procedure {Consistent Xs Ys} iterates through the columns of the board, where Ys are the columns already constrained and Xs are the columns still to be constrained. Since a queen only imposes its constraints once it is determined (i.e., {IsInt X} can reduce), there are at most N actors spawned before a distribution.

The procedure {Label Xs} labels the elements of Xs. Different labeling strategies are possible. Figure 6 shows a labeling procedure realizing the first-fail heuristic (label variables with fewest remaining values first). The procedure FdSize yields the number of values still possible for a finite domain variable, and FdMin yields the minimal value still possible. Both procedures can be expressed with the reflection actor of Section 7.

After all determined elements of Xs have been dropped with the higher-order procedure Filter, the remaining elements are sorted according to the current size of their domain. If X is the variable with the smallest domain, the disjunction

```
local
   proc {NoAttack Xs Y I}
      case Xs of nil then true
      [] X|Xr then {FdNec X Y} {FdNec X Y+I} {FdNec X Y-I} {NoAttack Xr Y I+1}
      end
   end
   proc {Consistent Xs Ys}
      case Xs of nil then true
      [] X|Xr then
         if {IsInt X} then {NoAttack Xr X 1} {NoAttack Ys X 1} fi  {Consistent Xr X|Ys}
      end
   end
   proc {Board I N Xs}
      if I=0 then Xs=nil
      else local X Xr in Xs=X|Xr {FdIn X 1 N} {Board I-1 N Xr} end fi
   end
in
   proc {Queens N Xs}
      {Board N N Xs}
      {Consistent Xs nil}
      {Label Xs}
   end
end
```

Figure 5: The n-queens problem.

```
or X=M then {Label Xr} [] {FdNec X M} then {Label X|Xr} ro
```

is created, where M is the minimal possible value for X, and Xr are the remaining variables to be labeled.

Because of the use of the reflective procedures FdSize and FdMin, it is important that the labeling procedure is elaborated only after all constraints have been propagated. This is ensured by the fact that suspended actors are given priority once they become reducible, and that the application of Label appears last. Since the most recently created disjunctive actor is distributed, the latter ensures that the disjunctive actor created by the labeling procedure is distributed even if there are further disjunctive actors (which is not the case in our example).

```
proc {Label Xs}
   case {Sort {Filter Xs proc {X} {FdSize X} > 1 end}
              proc {X Y} {FdSize X} < {FdSize Y} end}
   of nil then true
   [] X|Xr then
      local M={FdMin X} in
         or X=M then {Label Xr} [] {FdNec X M} then {Label X|Xr} ro
      end
   end
end
```

Figure 6: First-failure labeling.

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Remark

The Oz System and its documentation are available through anonymous ftp from `duck.dfki.uni-sb.de` or through www at `http://www.dfki.uni-sb.de/`.

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Towards a Concurrent Semantics based Analysis of CC and CLP

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1 Introduction

We present in an informal way some preliminary results on the investigation of efficient compile-time techniques for Constraint Logic and Concurrent Constraint Programming. These techniques are viewed as source-to-source program transformations between the two programming paradigms and are based on a concurrent semantics of CC programs [MR91].

Previous work [BH92] showed that it is possible to perform program transformations from Prolog to AKL[1] [JH91], allowing the latter to fully exploit the Independent And-Parallelism (IAP) [HR93] present in Prolog programs. When extending the transformation techniques to the CLP paradigm [JL87, Col90, VanH89], some issues have to be initially solved. First, the notion of independence has to be extended [GHM93]. Second, compile-time tools based on the extended notions have to be developed in order to capture the independence of goals, allowing such transformation. For this purpose an analysis of the programs turns out to be needed.

Our analysis will be based on a semantics [MR91] which, although originally intended for CC programming, can be also applied to CLP, if suitably extended [BGHMR94]. Such semantics allows us to capture the dependencies present in a CLP program at a finer level of granularity than ever proposed to date in the literature. This provides the knowledge for performing a transformation of the program which will force an execution-time scheduling of processes which preserves those dependencies. When the transformed program is run in a concurrent environment, parallel execution of concurrent processes will be exploited, except for the cases where an explicit ordering has been annotated at compile-time based on the dependencies identified.

The same semantics can also be used to identify dependencies in CC programs. Based on such dependencies, an analysis of parallel and sequential threads in the concurrent computation can be performed, establishing the basis for a transformation into parallel CLP programs (with explicit dynamic scheduling). A similar approach (although not based on program transformation) has recently been proposed in [KS92], in which a static analysis of concurrent languages is proposed based on an algebraic construction of execution trees from which dependencies are identified.

The needed extension of the semantics (for dealing with CLP instead of CC programs) is non-trivial [BGHMR94]. In fact, it consists in capturing the *atomic* (instead of the eventual) interpretation of the tell operation: constraints are added only if they are consistent with the current store. This implies the need of having the possibility of knowing immediately if a set of constraints is consistent or not. Thus it may seem that the semantics construction would have to go back to the usual notion of a constraint system as a black box which can answer yes/no questions in one step (which is what is most generally used in all the semantics other than [MR91]). However, this is not really true. In fact, the semantic structure still shows all the atomic entailment steps of the underlying constraint system, thus allowing to derive the correct dependencies among agents.

[1] AKL is a CC language based on the Extended Andorra Model, which is able to exploit the determinate-goals-first principle as well as various kinds of parallelism.

2 Independence in CLP

The general, intuitive notion of independence between goals is that the goals' executions do not interfere with each other, and do not change in any "observable" way. Observables include the solutions and/or the time that it takes to compute them.

Previous work in the context of traditional Logic Programming languages [Con83, DeG84, HR93] has concentrated on defining independence in terms of preservation of search space, and such preservation has then been achieved by ensuring that either the goals do not share variables (*strict independence*) or if they share variables, that they do not "compete" for their bindings (*non-strict independence*).

Recently, the concept of independence has been extended to CLP [GHM93]. It has been shown that search space preservation is no longer sufficient for ensuring the efficiency of several optimizations when arbitrary CLP languages are taken into account. The reason is that while the number of reduction steps will certainly be constant if the search space is preserved, the cost of each step will not: modifying the order in which a sequence of primitive constraints is added to the store may have a critical influence on the time spent by the constraint solver algorithm in obtaining the answer, even if the resulting constraint is consistent (in fact, this issue is the core of the reordering application described in [MS92]). This implies that optimizations which vary the intended execution order established by the user, such as parallel or concurrent execution, must also consider an orthogonal issue – *independence of constraint solving* – which characterizes the properties of the constraint solver behavior when changing the order in which primitive constraints are considered

3 A Concurrent Semantics for CC and CLP

Usually the semantics of CC programs [Sar89] is given operationally, following the SOS-style operational semantics, and thus suffering from the typical pathologies of an interleaving semantics. On the other hand, the concurrent semantics approach introduced in [MR91] presents a non-monolithic model of the shared store and of its communication with the agents, in which the behavior of the store and that of the agents can be uniformly expressed by context-dependent rewrite rules (i.e. rules which have a left hand side, a right hand side and a context), each of them being applicable if both its left hand side and its context are present in the current state of the computation. An application removes the left hand side and adds the right hand side. In particular, the context is crucial in faithfully representing asked constraints, which are checked for presence but not affected by the computation.

From such rules a semantics structure is then obtained. Such structure is called a contextual net [MR93] and it is constructed by starting from the initial agent and applying all rules in all possible ways. A contextual net is just an acyclic Petri net where the presence of context conditions, besides pre- and post-conditions, is allowed. In a net obtained from a CC program, transitions are labelled by the rule applied for them.

Three relations can be defined on the items (conditions and events) of the obtained net: two items are *concurrent* if they represent objects which may appear together in a computation state, they are *mutually exclusive* if they represent objects which can not appear in the same computation, and they are *dependent* if they represent objects which may appear in the same computation but in different computation steps.

For each computation of the CC program, the net provides a partial order expressing the dependency pattern among the events of the computation. As a result, all such computations are represented in a unique structure, where it is possible to see the maximal degree of both concurrency (via the concurrency relation) and indeterminism (via the mutual exclusion relation) available both at the program level and at the underlying constraint system.

Nevertheless, such semantics is not able to handle failure, in the sense of detecting inconsistencies generated by tell operations, since constraints are added without any consistency check (i.e., the "eventual" interpretation of the tell operation is modelled). We extended such semantics to include the case of failure [BGHMR94]. We showed that the new semantics can be obtained from the old one either by pruning some parts of the semantic structure, or by not generating them at all. On one hand, the semantic structure can be built up by first generating the net as before, and then propagating the failure information through the net by introducing a notion of *mutual inconsistency* between items. The inconsistent items are then

pruned out. On the other hand, the net can be generated from scratch with a new computation rule for the semantics which takes mutual inconsistency into account.

The mutual inconsistency relation extends the mutual exclusion relation, in the sense of capturing more objects which are not allowed to be present in the same computation. In fact, in the original semantics, if two objects were mutually exclusive, they could not be present in the same deterministic computation, even at different computation steps, because they belonged to two different nondeterministic (in the sense of "don't-care" nondeterminism, or indeterministic) branches of the program execution. Now, two items exclude one another also when they are mutually inconsistent, that is, when they represent (or generate) objects which are inconsistent.

When introducing an explicit representation for failure in the original semantics, what is achieved in fact is a faithful model for capturing backtracking. In other words, failing branches in a computation are also captured, allowing us to make a step further and exchange nondeterminism for indeterminism. In the extended semantics, two different branches will be mutually inconsistent if they lead to failure. Otherwise, if they are mutually exclusive, they will represent two different deterministic computations yielding distinct solutions, i.e., a nondeterministic choice.

Thus the new semantics, although originally intended for CC programs, can be used also for describing the behavior of (pure) CLP programs. The only difference is the interpretation of the mutual exclusion relation, which expresses indeterminism when applied to CC programs, and nondeterminism when applied to CLP programs.

4 Local Independence and CLP Parallelization

The semantics obtained above, while being maximally parallel, could be very inefficient if implemented directly as an operational model for CLP. One reason for this is that branches of the search tree may be explored which would have been previously pruned by another goal in the sequential execution. The general problem of finding a rule to avoid the exploration of such branches is directly related to the concept of independence and has been previously addressed in Section 2. In order to avoid such efficiency problems we propose to apply those independence rules, but at the finest possible level of granularity (as proposed in [BGH93]). This is now possible because we have a structure in which all intermediate atomic steps in the execution of a goal and their dependencies are clearly identifiable.

Capturing independence is achieved by identifying dependencies which occur due to subcomputations which affect each other, in the sense of the constraint independence notions above. In our nets, these notions are applied not only at the level of whole computations of different goals, but also at the finer level of subcomputations of those goals, i.e., the actual subcomputations which can affect each other. This new notion of independence (*local independence*) is, to our knowledge, the most general proposed so far (in the sense that it allows the greatest amount of parallelism) which, at the same time, preserves the efficiency of the sequential execution.

A drawback of local independence is that it requires an oracle, since mutual inconsistency of branches is not known a priori, and thus suitable scheduling strategies for AND-OR parallelism must be devised which make sure that the added dependency links are respected (i.e. the strategy is *consistent*), while still taking advantage of the remaining parallelism (i.e. the strategy is, more or less, *efficient*). Such an oracle can be devised at compile-time by means of abstract interpretation based analysis, and a scheduling strategy can be obtained for instance by a suitable program transformation (as that presented in Section 6).

5 A Meta-interpreter of the Concrete Semantics

A meta-interpreter has been implemented which takes as input a CC program and a concrete query, and builds up the associated contextual net as defined by the true concurrency semantics of [MR91], presented in Section 3. The computation of the concrete model is performed in several steps:

1. A program is read in and transformed into a suitable set of context-dependent rules.

2. Starting from the initial (concrete) agent – the query – rules are applied one at a time, until no rule application is possible.

3. Relations of mutual exclusion, causal dependency and concurrency are constructed from the structure given by the previous step.

4. The contextual net giving the program semantics can be visualized in a windows environment, as well as the resulting relations.

Although the construction of the net is completely deterministic, a fixpoint computation based on memoization is performed in order to ensure termination (whenever the semantics model is finite).

Once the computation is finished, the structure giving the model of the program resembles an event structure [Ros93]. An event structure is a set of events (together with *conflict* and *dependency* relations), where each of them represents a single computation step, i.e., a rule application, and contains all the history of the subcomputation leading to the particular step represented. The events represent either program agents, which will be consumed by applying the program rules, or constraint tokens which will be asked for in such rule applications. The former are represented by usual conditions in the net, the latter by *context* conditions.

For simplicity, the current implementation only implements the Herbrand constraint system, leaving to the underlying Prolog machinery much of the entailment relation.

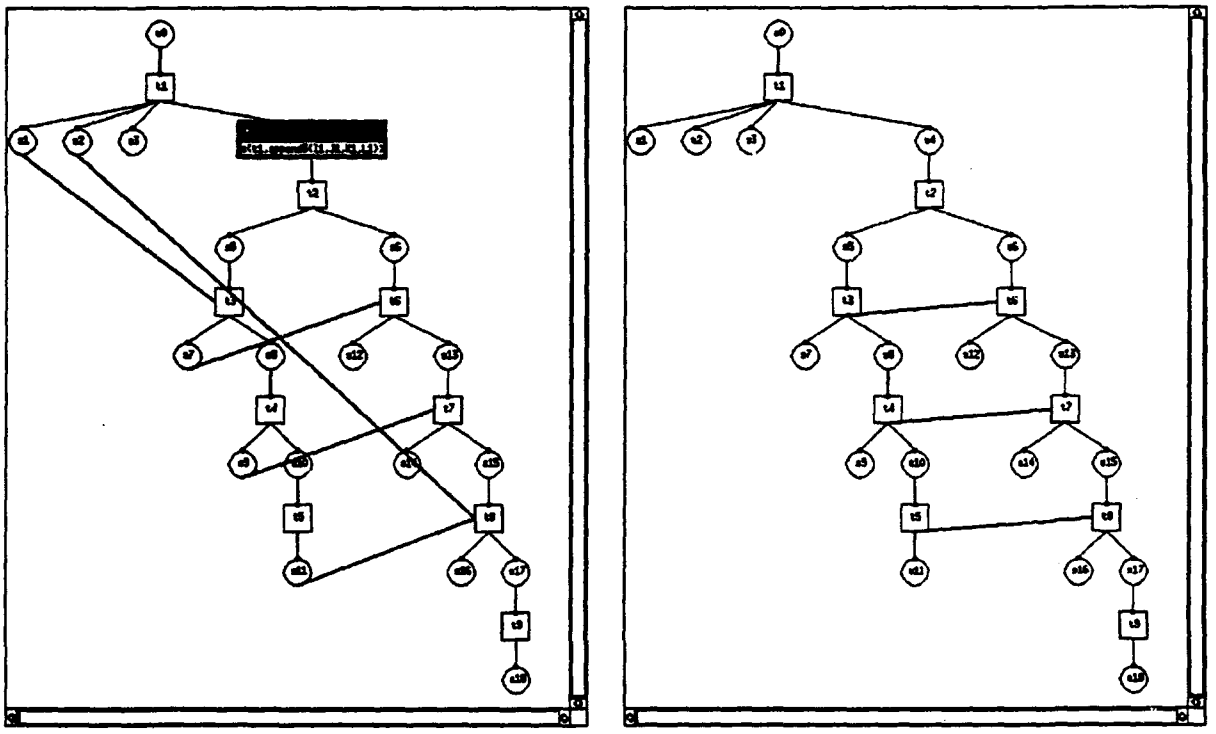

Figure 1: Contextual Net of the append3/4 example.

As an example, consider the following definition of append/3, which appends two lists into another one, and then splits it into another two. It can be run either first appending and then splitting or "backwards" (first splitting and then appending).

```
:- tell(X=[1,2]), tell(Y=[3]), tell(Z=[4]), append3(X,Y,Z,W).

append3(A, B, D, E) :- app(A, B, C), app(C, D, E).
```

```
app(X, Y, Z) :-  ask(X = []),    tell(Y = Z).
app(X, Y, Z) :-  ask(X = [A|B]), tell(Z = [A|D]), app(B, Y, D).
app(X, Y, Z) :-  ask(Z = []),    tell(X = []),    tell(Y = Z).
app(X, Y, Z) :-  ask(Z = [_|_]), tell(X = []),    tell(Y = Z).
app(X, Y, Z) :-  ask(Z = [A|D]), tell(X = [A|B]), app(B, Y, D).
```

A query has been included which performs the "forward" computation, where the second app/3 goal in the body of the **append3/4** clause has to wait on the first goal to proceed at each step while the resulting list C is being constructed to consume it. The semantic structure resulting for the computation with this query can be seen in Figure 1.

Circles in the figure correspond to agents (either program agents or tokens) and squares correspond to steps. Context conditions corresponding to the constraint tokens told to the store in the computation can be seen, and the use of such contexts by subsequent transitions are denoted by links between the corresponding tokens and transitions (Figure 1.a). The partial order subsumed in the net corresponds to the causal dependency relation, plus additional dependencies due to the "use" of contexts, which appear in Figure 1.b.

In this way, the causal dependency relation captures an optimal scheduling of processes based on producer/consumer relations on the tokens added to the store. This can be augmented with the local independence relation (as explained in Section 4) to capture and-parallel scheduling based on mutually inconsistent computations.

6 Parallelization of CLP via Program Transformation to CC

One possible application of our semantics can be achieved by program transformation from CLP to CC. The purpose of the transformation will be to allow CLP programs to run under CC machinery with an optimal scheduling of processes which ensures no-slowdown and allows for maximal parallelism. In doing this, the target language should allow for the features of CC, including synchronization and indeterminism (although this latter is not needed for our purposes), and also for additional nondeterminism (in the sense of backtracking - which is indeed needed to embed CLP). Examples of such languages are AKL[2] and concurrent (constraint) Prologs (i.e. Prologs with explicit delay).

The transformation will proceed as follows. First, the CLP program is rewritten into a CC program. This first step will embed a CLP program into CC syntax, by (possibly) normalizing goals and head unifications, and make all constraint operations explicit as tell agents. Second, inconsistency dependencies are identified within the (abstract) semantics via program analysis, and then the program is augmented with sequentialization arguments where required, and suitable ask and tell operations for this are incorporated to the program clauses.

Let t_1 *dep* t_2 denote an existing inconsistency dependency link between transitions t_1 and t_2. The corresponding rules applied in those transitions are identified, and also the program declarations related to such rules. Let these be p_1 and p_2, respectively, where As represent ask agents, At tell agents, and Ag other agents. The transformation required for sequentialization maps these declarations into the corresponding p_1' and p_2'.

$$p_1 ::= \mathrm{p1}(\overline{X}) : -As_1, At_1, Ag_1 \qquad p_1' ::= \mathrm{p1}'(\overline{X}, Y) : -As_1, At_1, tell(c(Y)), Ag_1$$
$$p_2 ::= \mathrm{p2}(\overline{X}) : -As_2, At_2, Ag_2 \qquad p_2' ::= \mathrm{p2}'(\overline{X}, Y) : -ask(c(Y)), As_2, At_2, Ag_2$$

where Y is a completely new variable and $c(Y)$ is some arbitrary constraint token over Y. Instances of agents p1 and p2 are also mapped into the corresponding p1' and p2' by augmenting their number of arguments accordingly and matching this additional argument to the same variable wherever both agents appear together in the same declaration.

The transformed program will allow for or-parallelism (which is captured in the semantics by the mutual exclusion relation) and *locally independent* and-parallelism (which is captured by means of relations derived from the mutual inconsistency relation). An efficient strategy for parallel execution is thus achieved.

[2]However, in AKL computations are encapsulated in the so called *deep* guards, an issue that our semantics does not capture yet.

7 Static Scheduling in CC via Program Transformation to CLP

Another complementary application of the independence detection based on our semantics is schedule analysis. We propose to perform the linearization associated to schedule analysis by means of program transformation from CC to CLP, achieving in addition an efficient parallelization of concurrent goals. In order to do this the intended target language should allow "delay" features able to support concurrency.

The basic idea is related to the approach of [BGH93] and QD-Janus [Deb93]. However, we propose to perform a more "intelligent" transformation (see also [BGH93]), which is based on the results of the analysis performed over the CC program.

Let us illustrate our approach with the append3/4 example of Section 5. Assume the following query:

```
:- tell(W=[1]), append3(X,Y,Z,W).
```

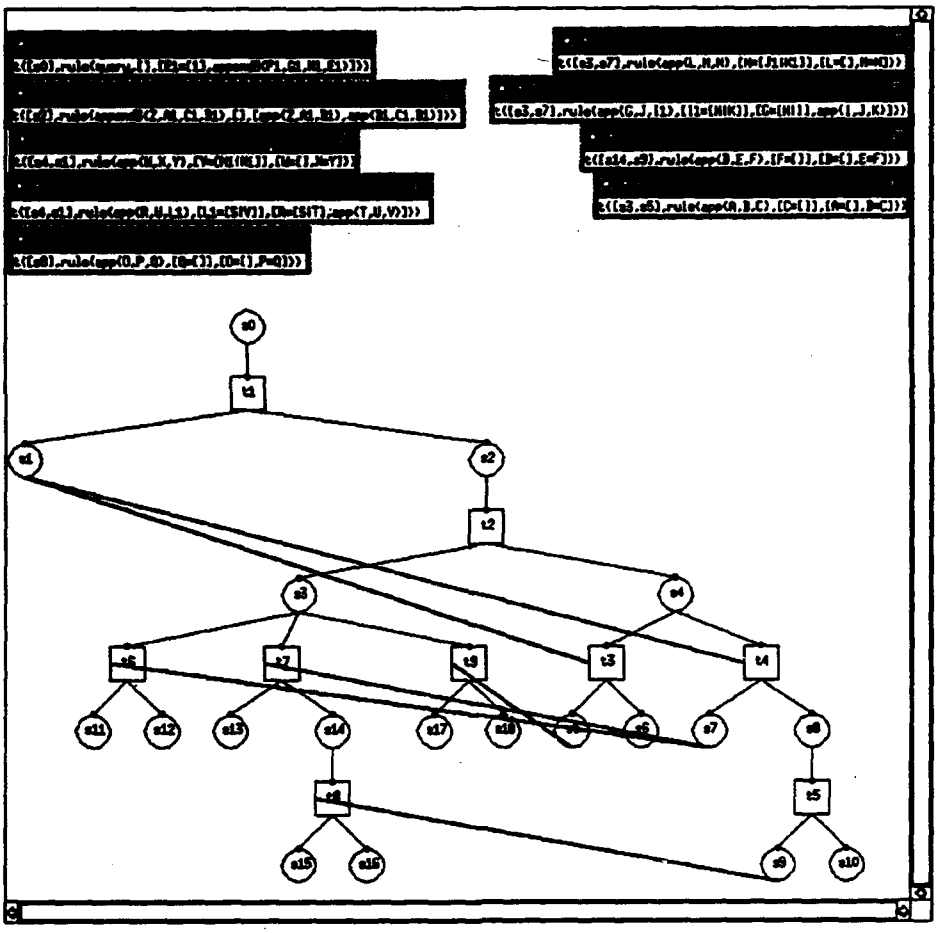

Figure 2: Contextual net for append3/4 running backwards.

The resulting contextual net given by our meta-interpreter is that of Figure 2, where the context dependencies links are shown, and the information corresponding to each rule application ($t_1, t_2, \ldots$) appears explicitly at the top. From the net, it can be seen that only the "backwards" version of the predicate app/3 is used: while the second app/3 goal in the body of the append3/4 clause (corresponding to agent s_4) can proceed without suspending, as no context other than the told constraints in the query is needed, the first goal and the goals occurring in its subcomputation always suspend until the third argument becomes instantiated. An identical behavior will occur in all queries in which the three first arguments of

append3/4 are free and the forth is instantiated to a non-incomplete list. With this knowledge the following transformed CLP program can be obtained:

```
append3(A, B, D, E) :-
        when(nonvar(C),app(A, B, C)),
        app(C, D, E).

app(X, Y, Z) :- X = [],    Y = Z.
app(X, Y, Z) :- Z = [A|D], X = [A|B], app(B, Y, D).
```

Our aim is to develop an analysis able to infer such invariants based on the semantics. Such analyzer will guarantee that the transformations applied to a CC program in the spirit above are correct.

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CC Programs with both In- and Non-determinism: A Concurrent Semantics

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Abstract

We present a concurrent semantics for concurrent constraint (cc) programming framework with both ("committed choice") indeterminism and ("backtracking") nondeterminism. The semantics extends the previous semantics for Indeterminate cc by (1) allowing each state to contain different or-parallel components and (2) splitting the concurrency relation into two to distinguish between and- and or-concurrency. Thereby, the construction produces a single representation (an *And-or contextual net*) that captures the important relationships between events in program runs: concurrency, causal dependency, indeterminism and nondeterminism.

We believe this is a first step towards the formal analysis of the concurrent semantics of practical cc languages containing both in- and non-determinism, such as AKL [HJ90].

1 Introduction

The paper proposes a simple concurrent semantics for concurrent constraint (cc) programs [Sar93] which may contain *both* indeterminism ("don't care" or "committed-choice" nondeterminism) and nondeterminism ("don't know" or "search" non-determinism). Prolog-style nondeterminism is obviously of considerable value in allowing simple, perspicuous representations of search-spaces. Indeterminism arises naturally in reactive distributed contexts, where the relative speeds of processors and relative communication delays across the network are unpredictable. Thus, the combination of indeterminism and nondeterminism we discuss in this paper arises naturally when one seeks to implement simple representations for search problems that are to be solved in a distributed, reactive context. Moreover, it also appears whenever, for any reason, one decides to make some of the choices backtrackable (that is, nondeterministic, or collective), and others committed (that is, indeterministic). Examples of cc programs containing both indeterminism and nondeterminism can be found in [Sar93].

To define the operational behaviour of cc programs, we represent each computation state as a collection of sets of agent and constraint occurrences, where different sets in a collection represent situations which are achieved by making different nondeterministic choices. Then, each state is rewritten via rewriting rules, which specify (1) conditions under which they can be executed, and (2) the new configuration (collection of sets) that results on execution of the rule. The operational semantics then associates with each agent the sequence of configurations that arise as a result of the applications of the rewrite rules generated from the program and the underlying constraint system.

The *concurrent* semantics we develop is derived from the operational semantics by internalizing the history of the computation in the states. The resulting objects in the semantic domain (called *contextual nets* [MR93a]) contain information about concurrency, causal dependency and mutual exclusion. Contextual nets generalize Petri net [Rei85] by allowing each event to have context conditions, in addition to the usual pre- and post-conditions: for the event to occur the context conditions must be present. Causal dependency describes the necessary sequentialization in the program (as introduced by ask conditions).

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The concurrency relation describes possible parallelism between events. The mutual exclusion relation describes conflict, that is, the impossibility of the related events being in the same computation [MR93b].

To model nondeterminism, we split the concurrency relation into two: *and-concurrency* and *or-concurrency*, obtaining a new kind of net that we call an *and-or contextual net*. Such a refinement of the model is unavoidable if one wants to distinguish between in-determinism, non-determinism, concurrency and dependency.

The and-or contextual net is derived from a cc program by using just one inference rule, which states that whenever the left hand side and the context of the rule are already represented in the net, then we can add new items to represent its application and its right hand side, and link them suitably to the other elements of the net via the four relations. The applications of such inference rule are Church-Rosser, in the sense that the resulting net does not depend on the order in which they occur. Moreover, it is easy to see that such application is very similar to that of the rules in the operational semantics. In fact, as noted above, the only real difference between the operational and the concurrent semantics (that is, the contextual net) is that the latter generates an object which contains both the final state and the history of the computation, with the appropriate dependencies among the computations steps. This allows to reason more profoundly about several properties of cc programs.

From the obtained net, it is possible to recover all and only the computations as defined by the operational semantics. Moreover, much more information, about concurrency of the steps involved in each computation, is contained in the net. In fact, both causal and functional dependencies between the items involved in a computation are explicitely expressed in the net.

The net representing the concurrent semantics of a given cc program contains many events and conditions which are uninteresting, like those related to the expansion of a declaration. Therefore an abstraction phase, which removes all such items, is needed if one wants to use in practice such nets to analyse cc programs.

2 Syntax

In the cc paradigm, we consider the usual description of the chosen constraint system as a *system of partial information* [SRP91] $(D, \vdash)$ where D is a set of *tokens* (or primitive constraints) and $\vdash \subseteq \wp(D) \times D$ is the entailment relation which states which tokens are entailed by which sets of other tokens. The language (we consider a propositional language just for simplicity of the technical developments) is concretely described by the following grammar, where P ranges over programs, F over sequences of procedure declarations, A over agents, and c over constraints:

(Programs)	P	::=	$F.A$	
(Declarations)	F	::=	$p :: A \mid F.F$	
(Agents)	A	::=	*success*	
			$\mid$ *failure*	
			$\mid c$	(Tell)
			$\mid c \rightarrow A$	(Ask)
			$\mid A + A$	(Indeterminism)
			$\mid A \parallel A$	(Parallel composition)
			$\mid A \vee A$	(Non-determinism)
			$\mid p$	(Procedure Call)

3 Operational Semantics

Each state of a cc computation consists of a collection of sets $(M_1, \ldots, M_n)$, where each set M_i is called an *or-state* and contains occurrences of agents and constraints. Intuitively, each set M_i represents the

(intermediate) result of one nondeterministic branch of a computation. Therefore a state represents the (intermediate) result of all nondeterministic branches occurring in a computation.

Then, each computation step models either the evolution of a single agent, or the entailment of a new token through the $\vdash$ relation. Such a change in the state of the computation will be performed via the application of a rewrite rule

$$r : L(r) \overset{c(r)}{\rightsquigarrow} R_1(r); \ldots; R_k(r)$$

where $L(r)$ is an agent, $c(r)$ is a constraint, and each $R_i(r)$ is a set of agent and constraint occurrences. The intuitive meaning of a rule is that $L(r)$, called the left hand side of the rule, is deleted from one of the or-states of the current state, say M_j, and k copies of the so obtained or-state are produced. Then, in each of such copies, say copy i, $R_i(r)$ (called a right hand side of the rule) is added. All this is done only if $c(r)$ is present in M_j.

We have as many rewrite rules as the number of agents and declarations in a program (which is finite), plus the number of pairs of the entailment relation (which can be infinite):

$$\begin{aligned}
(c \rightarrow A) &\overset{c}{\rightsquigarrow} A \\
A_1 \parallel A_2 &\rightsquigarrow A_1, A_2 \\
A_1 + A_2 &\rightsquigarrow A_1 \\
A_1 + A_2 &\rightsquigarrow A_2 \\
A_1 \vee A_2 &\rightsquigarrow A_1; A_2
\end{aligned}$$

In addition, there is a rule

$$p \rightsquigarrow A$$

for every program clause $p :: A$ and a rule

$$\overset{S}{\rightsquigarrow} t$$

for every entailment pair $S \vdash t$ in the underlying constraint system.

Formally, rule application works as follows. A rule $r : L(r) \overset{c(r)}{\rightsquigarrow} R_1(r); \ldots; R_k(r)$ is said to be applicable in a state $S_1 = \langle M_1, \ldots, M_n \rangle$ if there exists M_i such that $(L(r) \cup c(r)) \subseteq M_i$. In such a case, applying r to S_1 yields the state $S_2 = \langle M_1, \ldots, M_{i-1}, (M_i \setminus L(r)) \cup R_1(r), \ldots, (M_i \setminus L(r)) \cup R_k(r), M_{i+1}, \ldots, M_n \rangle$.

The operational semantics of a given cc program P consists of all the computations for P, i.e., the sequences of computations steps which apply rules representing agents and constraints of P. We will also need the concept of *non-redundant* computations, which are those computations where no entailment rule is ever applied more than once on the same constraint occurrence.

Note that this mechanism of making copies of the current world whenever a nondeterministic choice is accomplished is the usual way to give an operational semantics to languages with nondeterminism. In this way, a computation may contain nondeterminism, and different computations are instead originated by different indeterministic choices.

Let us now consider an example of a simple cc program with both nondeterminism and indeterminism in order to see how these two choice mechanisms interact. Suppose to have the parallel composition

$$(c_1 + c_2) \parallel (c_3 \vee c_4).$$

Although it could seem at first sight that the two choices are independent, and that c_1 and c_2 cannot appear in the same computation, in reality they can, and this depends on the order in which the two choices are made. In fact, if the indeterministic choice is made first, then we have two computations, one with final state $\langle \{c_1, c_3\}, \{c_1, c_4\} \rangle$ and one with final state $\langle \{c_2, c_3\}, \{c_2, c_4\} \rangle$. If instead the nondeterministic choice is made first, then we have other four computations: two of them produce the results written

above, and the other two have final state $\langle\{c_1, c_3\}, \{c_2, c_4\}\rangle$ and $\langle\{c_2, c_3\}, \{c_1, c_4\}\rangle$ respectively. Thus in this second case c_1 and c_2 belong to the same computation. In fact, once the state has been divided into two or-states, the computation may proceed in different ways in the two or-states and thus in particular it may choose to evolve to c_1 in one or-state and to c_2 in the other or-state.

We will consider this example again later on in the paper. In fact, although being very simple, it is enough to understand the relationship between nondeterminism and indeterminism from the concurrency point of view and to check whether usual partial order structures may be enough to represent concurrency in cc programs with both kinds of choices.

4 And-Or Contextual Nets

Contextual nets [MR93a] extend standard Petri nets (actually, C/E systems) with the possibility, for each event, of having context-conditions besides pre- and post-conditions. While pre-conditions are deleted by the event occurrence, and post-conditions are created, context conditions are needed for such an occurrence but are left unchanged. Contextual nets are able to specify three relations among their elements: causal dependency, concurrency, and mutual exclusion, which in terms of cc programming can be interpreted as necessary sequentialization, possible concurrency, and indeterminism, respectively.

In order to be able to model cc programs with nondeterminsm as well, we have to extend the semantic structure so that it can express also nondeterminism. To this end we introduce the notion of and-or contextual nets, which add to contextual nets the possibility of stating when some items of the net are "or-concurrent", that is, they belong to different nondeterministic branches.

We will write such nets as $\langle B, E; F_1, F_2, F_3\rangle$, where B is the set of conditions, E the set of events, F_1 gives the direct causal dependencies, F_2 states the context conditions for each event, and F_3 contains pairs of postconditions of the same event (which have to be considered as or-concurrent). In terms of cc programming, conditions are agents and/or tokens, while events are computation steps.

Each and-or contextual net induces four relations on its elements: causal dependency, mutual exclusion, or-concurrency, and and-concurrency. Causal dependency ($\leq$) is derived from F_1 and F_2, mutual exclusion (#) originates from events sharing a precondition and it is propagated via the $\leq$ relation, or-concurrency (or-co) originates from the F_3 relation and it is propagated via $\leq$, and and-concurrency (and-co) is what is left from the other relations: two items are and-concurrent if they are not in any of the other relations. Two elements which are not concurrent may be in more than one of the other relations. For our semantics we will consider only occurrence nets, i.e., nets where the $\leq$ relation does not have cycles.

And-or context-dependent nets will be graphically represented in the same way as classical and contextual nets. Thus, conditions are circles, events are boxes, and the flow relation F_1 is represented by directed arcs from circles to boxes or viceversa. We choose to represent the context relation F_2 by undirected arcs (since the direction of such relation is unambiguous, i.e. from elements of B to elements of E) and the or-concurrency relation F_3 by undirected labelled arcs (whose label is *or*). An and-or contextual net can be seen in Figure 1. In this net, for example, events e_1 and e_2 are mutually exclusive, while e_2 and e_4 are and-concurrent. Also, a and b and or-concurrent, and c is a context for both e_2 and e_4.

5 Concurrent Semantics

In order to give a concurrent semantics to cc programs with both in- and non-determinism, we follow the same idea used for indeterministic cc programs, that is, to associate a net to each program. However, while the nets used for indeterministic cc programs are contextual nets, here we need and-or nets. Nevertheless, the generating mechanism is very similar: we take the rewrite rules associated to a given cc program and by using them we incrementally construct an and-or contextual net plus a mapping which relates the items of the net to the agents, constraints, and rules of the program. Such incremental construction is achieved via the use of one inference rule (plus another one to start). Each time the inference rule

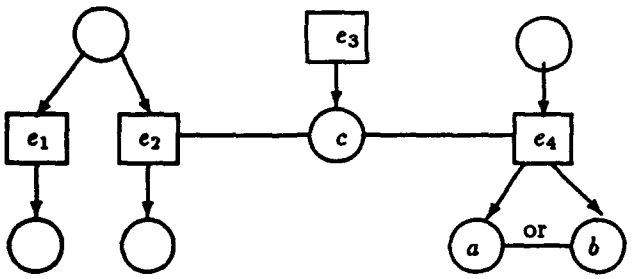

Figure 1: An and-or contextual net.

is applied, a rewrite rule is chosen whose left hand side and context are already present in the partially built net. Such elements have to be and-concurrent (which means that they can appear together in the same or-state). Then, a new element representing the rule application is added (as an event), as well as new elements representing the right hand sides of the rule (as conditions).

The elements of the net are structured in such a way that elements generated by using different sequences of rules are indeed different. That is, each element contains its "history". The way this is achieved consists in defining an element as a pair, of which the first element is the *type* of the term, and represents the rule or agent or constraint that the term corresponds to, and the second element is its *history*.

More precisely, assuming the net to be obtained is $\langle B, E, F_1, F_2, F_3 \rangle$, the starting inference rule is:

$$\frac{P = F.A}{\langle A, \emptyset \rangle \in B}$$

which means that we start with one element, which is a condition corresponding to agent A and with empty history. Instead, the main inference rule is:

$$\frac{\begin{array}{c}\{s_0, \ldots, s_{n-1}\} \subseteq B \\ s_i \text{ and-co } s_j \quad (i,j < n) \\ s_i = \langle a_i, e_i \rangle \quad (i < n) \\ a_i \neq a_j \quad (i,j < n) \\ \exists r \in RR(P) \text{ such that } L(r) = \{a_0, \ldots, a_{m-1}\} \text{ and } c(r) = \{a_m, \ldots, a_{n-1}\}\end{array}}{\begin{array}{c}e = \langle r, \{s_0, \ldots, s_{n-1}\} \rangle \in E \\ s_i F_1 e \quad (i < m) \\ s_i F_2 e \quad (m \leq i < n) \\ \forall i = 1, \ldots, k, \text{ if } a \in R_i(r) \text{ then } \langle a, e \rangle \in B \text{ and } e F_1 \langle a, e \rangle \\ a \in R_i(r) \text{ and } b \in R_j(r) \text{ and } i \neq j \text{ implies } \langle a, b \rangle \in F_3\end{array}}$$

That is, if we find items of the net which correspond to the left hand side and the context of a rule and which are and-concurrent, then we add a new event corresponding to the rule application, and new conditions corresponding to the elements of all right hand sides of the rule. Then we also suitably link such new objects among them via the F_1 (dependency), F_2 (context), and F_3 (or-concurrency) relations. In particular, the F_3 relation is set to hold among any pair of items representing elements belonging to different right hand sides of the rule. For example, the concurrent semantics of the program described at the end of Section 3 is the and-or contextual net in Figure 2.

Note that the above inference rule is a simplified version of what is actually needed to correctly generate the and-or net corresponding to a given cc program. In fact, we assume here that no rule has the same agent more then once in its right hand sides. However, if this should happen, a straightforward extension of the term coding, written as triple instead of pairs (where the added element is a natural number used to distinguish the different occurrences of an agent) would be enough (see [MR93b]).

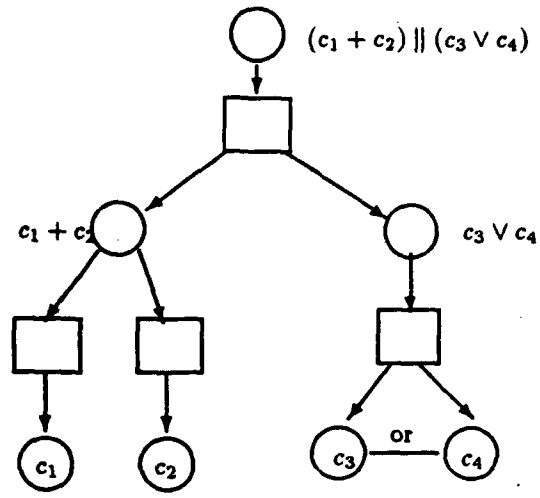

Figure 2: The and-or contextual net giving the concurrent semantics of a cc program.

6 Concurrent vs. Operational Semantics

It is important now to understand the relationship between the operational semantics defined in Section 3 and the concurrent semantics defined in Section 5. In particular, it is important to be able to show that from the concurrent semantics it is possible to recover all and only the computations of the operational semantics.

In previous studies concerning the concurrent semantics of indeterministic cc programs via contextual nets [MR93b] such relationship is very simple: any linearization (that is, a total order of the events which is compatible with the partial order) of each (maximal and left-closed) subnet of the semantics structure which does not contain any pair of mutual exclusive elements represents one (non-redundant) computation; and viceversa, each (non-redundant) computation is represented by one linearization of one of such subnets.

When however nondeterminism and indeterminism coexist, the representation of computations via subnets is not possible any more. Consider again the simple cc program whose computations are described at the end of Section 3 and whose semantics structure is depicted in Figure 2. Then, it is easy to see that there is no collection of subnets which may represent all its computations. In fact, if we consider all its subnets which do not contain any pair of mutually exclusive elements (which can be seen in Figure 3), then we are able to represent only those computations where the same branch of the indeterministic choice has been taken in both or-branches (either because the indeterministic choice was done before the or-choice, or because the choices in the two or-branches coincide). But we are not able to represent those computations where the or-choice has been made first, and where each or-branch evolved via a different indeterministic branch (one chose c_1 and the other one c_2).

A possible solution would be to consider subnets that do not contain pairs of mutually exclusive elements. However, to allow two in-braches to appear in the same computation, we would have to specify that or-choices must occur before in-choices. Unfortunately, recent approaches to the semantics of some extensions of Petri nets have shown that in presence of some specific features it is not possible to represents concurrency via partial orders, instead pairs of partial orders are required. In the present case, the first partial order would give a subnet, and the second would specify the order of choices. However, this would still not distinguish between a computation in which the first in-choice (c_1) is taken in the first or-state (c_3), and the second (c_2) with the second or-state (c_4) and one in which c_1 is taken with c_4 and c_2 with c_3.

Therefore the usual method of relating semantic nets to programs (using subnet selection), does not

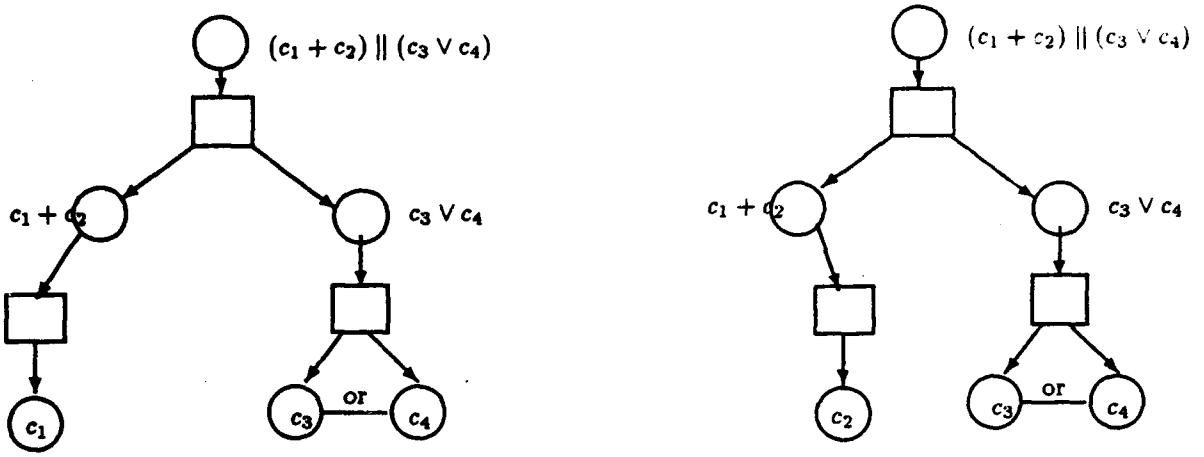

Figure 3: The non-mutually-exclusive subnets of the and-or contextual net in Figure 2.

seem applicable in the current setting. So we look for alternative ways in which the semantic structure may still represent all and only the computations of the operational semantics of cc programs. To this end we introduce the concept of *net execution*.

Informally, a net execution is a sequence of steps which starts with the entire net and at each intermediate stage reaches a collection of subnets of the original net. Each step *executes* one of the events of the net, among those whose pre- and context-conditions are in the current collection of nets (and are minimals), and the result is that the events and its preconditions are cancelled, together with every item which is mutually exclusive with that event. Moreover, if the event is an or-choice, a replication of the net containing the executed event is made, in the same manner as for the operational semantics.

More precisely, consider an and-or contextual net N. Then let us call $min(N)$ the set of all items of N which are minimal w.r.t. the causal dependency relation. Also, for any event e in N, let us call $pre(e)$ (resp., $con(e)$, $post(e)$) the set of preconditions (resp., context conditions, postconditions) of e. Moreover, given any condition s in a net, let us call *or-rest(s)* the set S of all conditions which are siblings of s and or-concurrent with it, plus the set S' which contains all elements (conditions and/or events) which depend on some element of S as well as all events which have some element of S as a context. Finally, given a net N and a set of items S in N, let us call $ex(N, S)$ the net which is obtained from N by deleting all the items which are mutually exclusive with any element in S.

Consider now a cc program P and the corresponding concurrent semantics N, and assume to have a collection of subnets of N, say $\langle N_1, \ldots, N_n \rangle$ (at the beginning we just have $\langle N \rangle$). Then an execution step is accomplished by choosing an event $e \in N_i$ such that $pre(e) \cup con(e) \in min(N_i)$. There are now two cases that can occur :

- there are no $s_1, s_2 \in post(e)$ such that s_1 and s_2 are or-concurrent; then the new collection of nets is

$$\langle N_1, \ldots, N_{i-1}, ex(N_i - (pre(e) \cup e), e), N_{i+1}, \ldots, N_n \rangle$$

- the other case, where there are instead pairs of or-concurrent postconditions of e, is simplified by the fact that this can arise only from the application of the nondeterminism rule, which only generates two postconditions; in this case, assuming $post(e) = \{s_1, s_2\}$, the new collection of nets is
$\langle N_1, \ldots, N_{i-1}, ex(N_i - (pre(e) \cup e), e) - \text{or-rest}(s_1), ex(N_i - (pre(e) \cup e), e) - \text{or-rest}(s_2), N_{i+1}, \ldots, N_n \rangle$

Consider now the set OS all the (non-redundant) computations of a cc program P and the set CS of all the executions of the corresponding and-or contextual net N. Then there is a bijection between OS and CS.

This means that it is possible to recover all computations from the and-or net representing the concurrent semantics of P. Thus the concurrent semantics does not loose any information w.r.t. the operational semantics. Indeed, it adds much information, concerning the possible concurrency of execution steps, as well as the causal and the functional dependencies, are explicitely represented.

7 Abstraction

The contextual net corresponding to a given cc program can be used to analyse cc programs in terms of concurrency, agent's dependency, choice points, parallelism level, and many others. However, the net as defined in the previous section has many events and conditions which are uninteresting for any reasonable analysis. Therefore one could think of abstracting from the information given by such items, and obtain a net, or a similar structure, where only the relevant information is contained.

A choice that has been adopted also in many operational semantic approaches for cc programs is to say that only ask and tell agents are important. Therefore, in our terms, it would mean that we only want to keep those events which represent the evolution of ask agents or tell agents[1].

To do that, consider an and-or net $\langle B, E, F_1, F_2, F_3 \rangle$, plus the mapping to the cc program rules, and the corresponding relations $\leq$, $\#$, or-co, and and-co. Now, consider the set of events $E' \subseteq E$ such that $E' = \{e \in E \mid e = \langle e_1, e_2 \rangle . e_1$ is an ask or a tell rule$\}$. Then the structure $\langle E', \leq_{|E'}$, or-co $_{|E'} \rangle$ relates the interesting events via the same relations as above, but projected over E'. This structure is obviously not an and-or contextual net, because it does not contain any conditions. However, it is nevertheless able to provide causality, indeterminism, and-concurrency, and or-concurrency information among the selected events.

Note also that the abstract structure so obtained is able to represent the computations and the dependencies present in many programs, possibly very different among them. Therefore reasoning on the net instead of on the program allows one to focus on the crucial issues regarding causality and functionality, and to be independent of the particular recursive set of agent definitions which have been chosen to represent such causality information.

Acknowledgements

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[1] A tell agent is any agent which adds a constraint. Similarly, a tell rule is any rule which adds a constraint.

Efficient and Complete Tests for Database Integrity Constraint Checking*

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1 Introduction

An important feature of modern database management systems is the automatic checking of *integrity constraints*. An integrity constraint is a predicate or query such that if the predicate holds on a state of the data, or equivalently if the query produces an empty answer, then the database is considered *valid*. When an integrity constraint is *violated*, i.e., when the predicate does not hold or the query produces a non-empty answer, then the update creating the undesirable database state must be rejected or some other compensating action must be taken.

We are interested in efficient methods for checking integrity constraints (hereafter called constraints) as a database is updated. Here, general efficiency is measured both in the amount of data that needs to be accessed in order to check a constraint, and in whether the check can be performed by submitting a query to the database system (rather than running an algorithm directly on the data). In terms of complexity, we are not interested in methods that are exponential in the size of the data or in the number of constraints, but we are willing to accept methods that are exponential in the size of the constraints themselves since, in databases, constraints tend to be short.

Suppose that we have a constraint C, and a database update occurs. We need to ensure that C still holds after the update. Assume that we have available at least the update itself and the definition of C. In addition to this information, there are three levels in the amount of data we might use to check the constraint: *none*, *some*, or *all*. Using none of the data corresponds to the *query independent of update* problem, which has been studied in its generality in [BT88, Elk90, LS93] and with respect to constraints by us in [GSUW94]. Using all of the data amounts to efficient evaluation of predicates or queries over the database [BC79, GMS93, HD92, Nic82, QW91, UO92]. We study the case where *some* of the data is used to check the constraint. This scenario arises whenever certain data involved in the constraint is very expensive or impossible to access, such as in distributed database systems or collaborative design [TH93, GT93]. Hereafter we refer to the portion of the data used to check a constraint as *accessible* data, and we refer to the portion of the data involved in a constraint but not used to check the constraint as *inaccessible* data.

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Note that, unless all of the relevant data is accessible, our constraint checking methods will be conservative. That is, by looking at only some of the data, we may be able to determine that the constraint still holds, or we may determine that it is necessary to look at all of the data to check the constraint. A check is *correct* if, whenever it determines that a constraint still holds, then indeed the constraint holds. We also want our checks to be *complete*, but completeness is with respect to the accessible data. A check is complete in this sense if, whenever we determine that a constraint may not hold, there is some configuration of the inaccessible data for which the constraint indeed does not hold.

In the remainder of this short paper we outline the languages we have been considering for database constraints and we solidify the notion of using *some* of the data to check a constraint. We then give several examples that illustrate when and how our constraint checking methods apply. (Due to space limitations, complete technical results are not included.) The examples serve to bring out a number of problems we have not yet solved, which are enumerated at the end of the paper.

2 Problem Definition

We consider relational databases where relations are modeled as predicates and queries are expressed as logical rules that derive a *result* predicate, as in, e.g., *Datalog* [Ull88]. Examples are given below. A *constraint* is expressed as a query whose result is a special 0-ary predicate that we call **panic**. If the query produces $\emptyset$ on a given database D, then the constraint holds for D. If the query produces **panic** then the constraint is violated. The difficulty of checking constraints depends on the language that we use to express constraint queries. Examples of interesting languages for expressing constraint queries are:

1. Conjunctive queries [CM77].
2. Nonrecursive Datalog, or unions of conjunctive queries [SY80].
3. Conjunctive queries with arithmetic comparisons [Klu88].
4. Datalog with negation [Ull88].
5. Recursive Datalog, possibly with arithmetic comparisons and/or negation and/or arithmetic operators [Ull88].

For some combinations of a language above and an amount of information used (*none*, *some*, or *all*), the constraint checking problem can be reduced to other problems that have been studied in the literature; see [GSUW94] for a discussion. For instance, conjunctive query containment results can be used to check constraints when only updates and constraint definitions are used [LS93].

In this paper we focus our discussion on constraints expressed as conjunctive queries with arithmetic comparisons, we suppose the only accessible relation is the updated relation, and we consider updates that are insertions of a single tuple. The general form of a conjunctive query constraint is:

$$\textbf{panic} :- l \;\&\; r_1 \;\&\; \ldots \;\&\; r_n \;\&\; c_1 \;\&\; \ldots \;\&\; c_k.$$

Here, l is the predicate for which the corresponding relation L is accessible, the relation R_i for each of the r_i's is inaccessible, and each c_i is an arithmetic comparison involving one of $<, \leq, >, \geq, =$.[1]

[1] The use of L and R refers to the fact that, in distributed databases, the "Local" data is accessible and the "Remote" data is inaccessible.

Let tuple t be inserted into relation L and assume constraint C holds before the insertion. We want to use L, C, and t to infer that C is not violated after the insertion. We derive a condition that relation L needs to satisfy in order for t not to violate C. We refer to this condition as the *test* condition. If the test condition is satisfiable, then relations $R_1, \ldots, R_n$ do not need to be accessed. The test condition is obtained by reducing the problem outlined above to the problem of checking if a conjunctive query is contained in a union of conjunctive queries; details are in [GSUW94].

3 Examples

EXAMPLE 1. Consider an employee-department relational database with two relations:

emp(E, D, S) *% employee number E in department D has salary S*
dept(D, MS) *% some manager in department D has salary MS*

Let the constraint assert that every employee earns less than every manager in the same department. This constraint is expressed as a conjunctive query C such that if C produces **panic** then the constraint is violated:

C: **panic** :- emp(E, D, S) & dept(D, MS) & $S \geq MS$.

Let relation EMP for predicate emp be accessible and relation DEPT be inaccessible. Suppose tuple emp$(e, d1, 50)$ is inserted into relation EMP. Constraint C will be violated if department $d1$ has a manager whose salary is ≤ 50. However, suppose department $d1$ already has an employee whose salary is 100. Since constraint C is not violated before the insertion, we can infer that no manager in $d1$ earns as little as 100, and therefore emp$(e, d1, 50)$ does not violate constraint C.

The above inference procedure can be formalized by specifying a test condition on the relation EMP and the inserted tuple, such that if EMP satisfies the test condition, then the inserted tuple does not violate the constraint. For constraint C, the test condition is the following Datalog query that derives insertion_ok if and only if the inserted tuple does not violate C, independent of the value of relation DEPT.

insertion_ok :- inserted(E, D, S) & emp(X, D, Y) & $Y \geq S$.

Relation INSERTED contains only the inserted tuple and EMP does not contain the inserted tuple. This test is complete with respect to the accessible data, as defined in Section 1. □

Note, the test condition in Example 1 is a single Datalog rule and was derived without considering the actual value of the inserted tuple. We now give two examples that illustrate the complexity that simple arithmetic comparison operators $<, >, \leq, \geq$ introduce. Example 2 shows that the complete test could be a recursive Datalog program. The constraint in Example 3 also has a complete test in the form of a recursive Datalog program, but illustrates the computational complexity of evaluating the test.

EXAMPLE 2. We shall refer to this example as *forbidden intervals.*

C: **panic** :- l(X, Y) & r(Z) & $X \leq Z \leq Y$.

Each pair in the accessible relation L can be thought of as the ends of an interval that no Z in the inaccessible relation R may occupy.

Suppose relation L has the tuples $(3, 6)$ and $(5, 10)$. The tuples of relation R that violate the constraint given tuple $1(3, 6)$ lie in the interval $[3, 6]$ and similarly, the tuples of relation R that violate the constraint given tuple $1(5, 10)$ lie in the interval $[5, 10]$. If the constraint is not violated

then we can infer that the tuples of the inaccessible relation lie outside the *forbidden intervals* $[3, 6]$ and $[5, 10]$ and therefore outside the combined forbidden interval $[3, 10]$.

Let tuple $\mathtt{l}(a, b)$ be inserted into relation L. If $a \geq 3$ and $b \leq 10$, then the forbidden interval for $\mathtt{l}(a, b)$ is contained in the union of the forbidden intervals of one or more existing tuples, and relation R need not be accessed in order to infer that constraint C is not violated. Note, the complete test may need to access multiple existing tuples in order to make the above inference. An incomplete, but sufficient, test would be to check that the forbidden interval for some single existing tuple contains the forbidden interval for the inserted tuple. That corresponds to using a single tuple in the accessible relation as opposed to using an arbitrary number of tuples, and was the approach taken in our initial work [GW93]. The sufficient test is linear in the number of tuples in L whereas the complete test could be exponential, if implemented naively. With some preprocessing, the complete test can also be evaluated in linear time [GSUW94].

The complete test for this example is the following recursive Datalog program that derives `insertion_ok` if and only if the inserted tuple does not violate C, assuming that C was not violated before the insertion.

$\mathtt{insertion\_ok} :- \mathtt{inserted}(A, B)\ \&\ \mathtt{forbidden\_int}(C, D)\ \&\ A \geq C\ \&\ B \leq D.$
$\mathtt{forbidden\_int}(C, D) :- \mathtt{l}(C, D).$
$\mathtt{forbidden\_int}(C, D) :- \mathtt{forbidden\_int}(C, X)\ \&\ \mathtt{forbidden\_int}(Y, D)\ \&\ X \geq Y.$

□

EXAMPLE 3. Consider a constraint C that involves two variables in the inaccessible relation:

C: **panic** $:- \mathtt{l}(U, V, W, Z)\ \&\ \mathtt{r}(X, Y)\ \&\ U \leq X \leq V\ \&\ W \leq Y \leq Z.$

Intuitively, the above constraint is the *forbidden interval* constraint in two dimensions. A tuple in relation R defines a point in a two dimensional space and a tuple in relation L defines a rectangular region in this 2-D space. Constraint C requires that all the points defined by the inaccessible relation R lie outside every rectangular region defined by the accessible tuples. Therefore, an inserted tuple $\mathtt{l}(a, b, c, d)$, does not violate C if the rectangle defined by $\mathtt{l}(a, b, c, d)$ is contained in the union of the rectangles defined by the existing tuples in L. The test for determining when a rectangle is contained in a set of other rectangles can still be represented as a recursive Datalog program. However, building the program is not as straightforward as in Example 2. In addition, the complexity of the test is high even with preprocessing. Without preprocessing the test is exponential in the number of tuples in L. □

4 Discussion

In [GSUW94] we identify some subclasses of conjunctive query constraints with arithmetic comparisons for which the complete test is a (recursive) Datalog program. We also identify some subclasses where the complete test does not need to consider multiple tuples from the accessible relation, but can consider tuples one at a time (i.e., there is no need to consider combinations of tuples, as in Example 2). The test condition for conjunctive query constraints, including the subclasses, is NP-complete. However, in at least some cases, the exponential behavior is only in the size of the constraint specifications, which we believe will be relatively small. In other cases the tests may be exponential in the size of the database or the number of constraints in the system. In such cases sufficient tests, instead of complete tests, may be preferable.

For conjunctive query constraints that use function symbols like $+, -$ (instead of only arithmetic comparisons), the complete test is an implication condition where both sides of the implication use

disjunction and the function symbols. Even though the implication condition can be derived in time exponential in the size of the constraint, evaluating the implication may be undecidable or have very high complexity. However, for some subclasses the ideas outlined in this paper can be extended to derive sufficient decidable tests.

5 Future Research Directions

Many interesting avenues remain unexplored in making constraint checking efficient following the framework we outlined above. We plan to:

- Consider more expressive constraint languages. Aggregate functions like MAX, SUM, AVG, *etc.* make the constraints more general. For instance, we might want a constraint requiring the average gradepoint of a graduating student to be at least 3.
- Use different amounts and type of information. For instance, constraint C_1 might be checked using constraints C_2 and C_3, possibly together with some functional dependency information. In distributed database systems, such algorithms can be used to increase the amount of constraint checking that can be done locally, without accessing remote data.
- Devise algorithms to efficiently perform local tests. As the examples in this paper illustrate, the test conditions often have high complexity. Techniques from constraint logic programming, operations research, and other areas provide ways of evaluating the tests efficiently. For instance, in Examples 2 and 3, algorithms from computational geometry are useful for efficient evaluation.
- For constraints where the complexity of local checking is inherently very high, it is useful to look for sufficient tests that are efficient to implement even though they may not be complete.

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Linear vs. Polynomial Constraints in Database Query Languages

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Abstract

We prove positive and negative results on the expressive power of the relational calculus augmented with linear constraints. We show non-expressibility of some properties expressed by polynomial constraints. We also show expressibility of some queries involving existence of lines, when the query output has a simple geometrical relation to the input. Finally, we compare the expressive power of linear vs. polynomial constraints in the presence of a discrete order.

1 Introduction

An active area of recent research is concerned with integrating constraints into logical formalisms for programming languages [DG,JL87,Ma87,Sa] and database query languages [BJM93,KKR90, Kup90,Kup93,Re90]. Constraints are incorporated in logic programming systems such as CLP, Prolog III and CHIP. The class of *linear* constraints is of particular interest, because of its applicability and the potential for efficient implementation [HJLL90,JL87,La90].

Kanellakis et.al.[KKR90] describe a methodology to combine constraint programming with database query languages. They propose several generalizations of the traditional relational database calculus (first-order logic). One of the more powerful languages described in [KKR90] is the relational calculus augmented with polynomial constraints, *FO+poly*. This language is powerful enough to express many geometric problems, and has NC data complexity; however, the complexity of quantifier elimination (over real closed fields) makes it impractical for most purposes. A natural question therefore is to ask what happens if constraints are restricted to be linear.

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In this paper we study the expressive power of the relational calculus augmented with linear constraints, *FO+linear*. We first give some negative results, showing that there exist properties in *FO+poly* which are not expressible in *FO+linear*. We use the well-known technique of *Ehrenfeucht-Fraissé* games [Eh61,Fr54]. We show that, when constraints are introduced to first-order logic, games can be appropriately adapted to prove non-definability; by contrast, techniques such as the compactness theorem (from first-order logic) or locality and 0/1 laws (from finite model theory) fail with constraints [GS94].

A natural subset of *FO+poly* singled out in [KKR90] is *FO+lines*; it extends *FO+linear* with variables ranging over lines. We show that there exist properties in *FO+lines* which are not expressible in *FO+linear*. We also show that some natural queries in *FO+lines* can be expressed in *FO+linear* when the output of the query has a simple geometrical relation to the input.

Maybe the most basic query expressed by line variables is "*compute the set of lines contained in the database*". We do not know if it is expressible in *FO+linear*. We show, however, that it is *linear*, i.e., if the input is defined by linear constraints, the output is defined by linear constraints as well. Linearity is a desirable property of query languages with linear constraints, because it makes it possible to cascade queries. It can be shown that queries expressed in *FO+linear* are linear. Also, queries expressed in a fragment of *FO+poly* described in [HJLL90, La90] (the *parametric* queries) are linear. It is an interesting open problem to find the most general fragment of *FO+poly* which expresses only linear queries.

We also compare the expressiveness of linear vs. polynomial constraints in a different context, namely in the presence of a discrete order. We show that including addition in first-order logic increases its expressive power. Adding multiplication increases the expressive power further. Neither is the case for Datalog, because of the availability of recursion. Results in a similar perspective are presented in [NS93], where it is shown that no formula of first order logic using linear ordering and the logical relation $y = 2x$ can define the property that the size of a finite model is divisible by 3.

2 Background

Databases are subsets of the k-dimensional Euclidean space $\mathcal{R}^k$ ($\mathcal{R}$ is the real line). *Queries* are functions from databases to databases; *Boolean* queries are functions from databases to $\{true, false\}$.

FO+poly [KKR90] is the set of first-order formulas (with equality) over atomic formulas as follows:

(i) $S(x_1, \ldots, x_k)$, meaning the point $(x_1, \ldots, x_k)$ is in the database S.
(ii) *Polynomial constraints* of the form

$$f(x_1, \ldots, x_k) \quad \theta \quad 0$$

where f is a k-variable polynomial (with real coefficients) and $\theta \in \{>, =\}$.

Note that $\geq, \neq$ are expressed as Boolean combinations of $>, =$. Also, when writing *FO+poly* formulas we will use abbreviations such as $x < e$ (instead of $-x + e > 0$) and $S(x + 1, y)$ (instead of $\exists z.\{z = x + 1 \wedge S(z, y)\}$).

FO+linear is the subset of *FO+poly* obtained be restricting constraints to be linear.

Formulas of *FO+poly* with free variables define queries: the output is the set of tuples satisfying the formula. Sentences define Boolean queries. If the input to a *FO+poly* query is defined by a Boolean combination of polynomial constraints, the output is also defined by such a combination [KKR90].

A *linear* database is a subset of $\mathcal{R}^k$ defined by a Boolean combination of linear constraints. A *linear* query is a function from linear databases to linear databases.

Formulas of *FO+linear* with free variables define linear queries. To see this, consider the formula obtained by substituting the definition of the input (by linear constraints) into the query formula. Now the quantifiers can be eliminated, as follows: if C is a set of linear constraints

$$\begin{aligned} x &> f_i \\ x &< f_j \\ x &= f_k \\ x &= f_l \end{aligned}$$

(where x does not occur in the f's), then the formula $\exists x. \bigwedge C$ is equivalent to the formula $\bigwedge C'$, where C' is the set of linear constraints

$$f_i < f_k = f_l < f_j.$$

Formulas of *FO+poly* do not in general define linear queries, as can be seen by standard geometric arguments (consider, for instance, the set of pairs (x, y) satisfying $x^2 + y^2 = 1$). The *parametric* queries [La90,HJLL90] is a class of formulas of *FO+poly* which define linear queries (by the *Subsumption Theorem* and variable elimination [La90]).

FO+lines is an extension of *FO+linear* with variables ranging over points and lines in $\mathcal{R}^k$. Atomic formulas $S(p), S(l)$ mean the point p (resp. the line l) is contained in the database S; $p \in l$ means the point p lies on the line l. It can be seen that *FO+lines* queries can be expressed in *FO+poly*. More generally, one can consider variables of higher dimension. It can be seen that extending *FO+poly* in this way does not increase its expressive power [KKR90].

3 Linear constraints are less expressive than polynomial

3.1 Games and the expressiveness of *FO+poly*

Definition 1 *The n-round* Ehrenfeucht-Fraissé *game is played between two players on two databases $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{R}^k$. At round r player I picks a point $p_r \in \mathcal{R}$ and associates it to either $\mathcal{D}$ or $\mathcal{D}'$; player II responds by picking $q_r \in \mathcal{R}$ and associating it to the other database.*

For each r let t_r, t'_r be the points associated with $\mathcal{D}, \mathcal{D}'$ respectively; $\{t_r, t'_r\} = \{p_r, q_r\}$. Player II wins *the game iff*

(i) $t_i = t_j$ iff $t'_i = t'_j$ and
(ii) $(t_{i_1}, \ldots, t_{i_k}) \in \mathcal{D}$ iff $(t'_{i_1}, \ldots, t'_{i_k}) \in \mathcal{D}'$.

The above condition is extended, given a set C of constraints over n variables, by the clause

(iii) $c(t_{i_1}, \ldots, t_{i_n})$ iff $c(t'_{i_1}, \ldots, t'_{i_n})$, for every constraint c in C.

The well-known theory of Ehrenfeucht-Fraissé games [Eh61,Fr54] gives the following results:

Theorem 2 *Let Q be a property of databases. For each n and each finite set of linear constraints C (over n variables), the following are equivalent:*

(a) Q is not expressible in FO+linear *with quantifier depth at most n and constraints from C.*
(b) There exist databases $\mathcal{D}_{n,C}, \mathcal{D}'_{n,C}$ which differ wrto Q such that player II wins the n-round Ehrenfeucht-Fraissé game on $\mathcal{D}_{n,C}, \mathcal{D}'_{n,C}$.

Corollary 3 *Let Q be a property of databases. The following are equivalent:*

(a) Q is not expressible in FO+linear.
(b) For each n and each finite set of linear constraints C (over n variables), there exist databases $\mathcal{D}_{n,C}, \mathcal{D}'_{n,C}$ which differ wrto Q such that player II wins the n-round Ehrenfeucht-Fraissé game on $\mathcal{D}_{n,C}, \mathcal{D}'_{n,C}$.

Consider databases consisting of a subset U of the real line. We will use Corollary 3 to show:

Theorem 4 *The set of databases satisfying*

$$\exists x.\exists y. \quad \{U(x) \wedge U(y) \wedge x^2 + y^2 = 1\}$$

is not expressible in FO+linear.

Proof: (Sketch) Given n and C as in Corollary 3, we will find points $\delta, \delta', \epsilon$ such that

$$\begin{aligned} \delta^2 + \epsilon^2 &= 1 \\ (\delta')^2 + \epsilon^2 &\neq 1 \end{aligned}$$

and player II has a winning strategy for the game played on the databases

$$\begin{aligned} \mathcal{D} &= \{\delta, \epsilon\} \\ \mathcal{D}' &= \{\delta', \epsilon\}. \end{aligned}$$

Let t_r, t'_r be the points associated (at round r) with $\mathcal{D}, \mathcal{D}'$ respectively (Definition 1). For each r, $0 \leq r \leq n$, we define sets of linear constraints $\mathcal{C}_r, \mathcal{C}'_r$ on the points $\{t_1, \ldots, t_r, \delta, \epsilon\}$ and $\{t'_1, \ldots, t'_r, \delta', \epsilon\}$ respectively. A constraint $c(t_1, \ldots, t_r, \delta, \epsilon)$ is in $\mathcal{C}_r$ iff the *corresponding* constraint $c(t'_1, \ldots, t'_r, \delta', \epsilon)$ is in $\mathcal{C}'_r$. We proceed by induction on r:

$$\begin{aligned} r = n: \ \mathcal{C}_n \ = \ & \{t_i = t_j, i, j = 1, \ldots, n\} \cup \\ & \{t_i = \delta : i = 1, \ldots, n\} \cup \\ & \{t_i = \epsilon : i = 1, \ldots, n\} \cup \\ & \{c(t_{i_1}, \ldots, t_{i_n}) : \text{ where } c \in \mathcal{C}, 1 \leq i_j \leq n\}. \end{aligned}$$

$$\begin{aligned} 0 \leq r < n: \ \mathcal{C}_r \ = \ & \{t_i = t_j, i, j = 1, \ldots, r\} \cup \\ & \{t_i = \delta : i = 1, \ldots, r\} \cup \\ & \{t_i = \epsilon : i = 1, \ldots, r\} \cup \\ & \{c(t_{i_1}, \ldots, t_{i_n}) : \text{ where } c \in \mathcal{C}, 1 \leq i_j \leq r\} \cup \\ & \Delta, \end{aligned}$$

where Δ is the set of constraints obtained by eliminating t_{r+1} from the set C_{r+1}.

We say that C_r, C'_r are *equisatisfied* iff a constraint in C_r is true just in case the corresponding constraint in C'_r is true.

Claim: If C_r, C'_r are equisatisfied, then for any choice of t_{r+1} (resp. t'_{r+1}) there is a choice of t'_{r+1} (resp. t_{r+1}) such that C_{r+1}, C'_{r+1} are equisatisfied.

It follows that, if C_0, C'_0 are equisatisfied, player II can play so that C_n, C'_n are equisatisfied. I.e., by the definition of C_n, C'_n player II can win the n-round game, since

$$x \in \mathcal{D} \text{ iff } x = \delta \vee x = \epsilon$$

(resp. $x \in \mathcal{D}'$ iff $x = \delta' \vee x = \epsilon$).

We now show how to pick $\delta, \delta', \epsilon$ so that C_0, C'_0 are equisatisfied. Write the constraints in C_0 in the form $\delta\ \theta\ f_m(\epsilon)$, where $\theta \in \{>, <, =\}$. Pick ϵ so that $f_m(\epsilon)^2 + \epsilon^2 \neq 1$ for every m. Pick δ so that $\delta^2 + \epsilon^2 = 1$. Now $\delta \neq f_m(\epsilon)$ for every m, and by choosing δ' close enough to δ we can make sure that C_0, C'_0 are equisatisfied. ■

3.2 The expressiveness of *FO+lines*

We consider databases consisting of a binary relation S. The Boolean query *in-line* asks whether S is contained in a line. It is expressible in *FO+poly*, by the formula

$$\exists u.\exists v.\exists w.\forall x.\forall y.\{S(x,y) \rightarrow ux + vy + w = 0\}.$$

In particular, it is expressible in *FO+lines* by the formula

$$\exists l.\forall p.\{S(p) \rightarrow p \in l\}.$$

Theorem 5 *The* in-line *query is not expressible in* FO+linear.

Proof: We show that, if the *in-line* query is expressible in *FO+linear*, then the set of tuples (x, y, z) satisfying $z = xy$ is definable in *FO+linear*; the latter can be shown to be false.

Given x, y, z, let S be a binary relation containing three tuples:

$$S = \{[1, x], [0, 0], [y, z]\}.$$

It is easy to verify that the three points are on a line if and only if $z = xy$. ■

4 Expressibility of some *FO+lines* queries

We consider databases consisting of a binary relation S. The Boolean query *exists-line* asks whether S contains a line; it is expressible in *FO+lines* by the formula

$$\exists l.S(l).$$

The more general *lines* query returns the set of lines contained in S. The output of *lines* is a set of tuples (u, v, w), each specifying the set of points (x, y) satisfying $ux + vy + w = 0$. Both *lines* and *exists-line* are expressible in *FO+poly*, using the formula

$$\forall x.\forall y.\{ux + vy + w = 0 \rightarrow S(x,y)\}.$$

The *line-intersection* query returns all points p which are intersections of (pairs of distinct) lines contained in S; it is expressible in *FO+lines* by the formula

$$\exists l_1.\exists l_2.\ \{S(l_1) \wedge S(l_2) \wedge l_1 \neq l_2 \wedge p \in l_1 \wedge p \in l_2\}.$$

We will show that *exists-line* and *line-intersection* can be expressed in *FO+linear* for databases of certain geometric shapes. We will also show that *lines* is a linear query.

Definition 6 *A* two-slope *database* $S(x,y)$ *has the form*

$$\begin{array}{llll} & (x \geq \beta_1 & \wedge & y \leq \alpha_1 x + s_1) \\ \vee & (\beta_2 < x < \beta_1 & \wedge & y \leq \gamma) \\ \vee & (x \leq \beta_2 & \wedge & y \leq \alpha_2 x + s_2) \end{array}$$

(see Figure 1).

A two-slope database contains a line iff $\alpha_1 \geq \alpha_2$.

Theorem 7 *The* exists-line *query is expressible in* FO+linear *for two-slope databases.*

Proof: (Sketch) Let φ be the formula

$$\exists z.\exists w.\ \{\neg S(x,z) \wedge \neg S(-x,w) \wedge z + w \leq y \wedge x > b_1 \wedge -x < b_2\}.$$

Suppose S is a two-slope database with parameters $\beta_1, \beta_2, \alpha_1, \alpha_2, s_1, s_2, \gamma$ as in Definition 6. For $b_1 > \beta_1$, $b_2 < \beta_2$, the formula φ is equivalent to

$$\begin{array}{lll} y & > & (\alpha_1 - \alpha_2)x + s_1 + s_2 \\ x & > & b_1, -b_2. \end{array}$$

It follows that, for $b_1 > \beta_1$, $b_2 < \beta_2$, the formula $\forall y.\exists x.\varphi$ is true iff $\alpha_1 - \alpha_2 < 0$ (since $b_2 < b_1$ implies $x > 0$).

Now the formula

$$\exists B_1.\exists B_2.\forall b_1.\forall b_2.\{(b_1 > B_1 \wedge b_2 < B_2) \rightarrow (\forall y.\exists x.\varphi)\}$$

is true iff $\alpha_1 - \alpha_2 < 0$, i.e., iff S does not contain a line. ■

Theorem 8 *The* line-intersection *query is expressible in* FO+linear *for databases consisting of at most two lines.*

Proof: (Sketch) Suppose S consists of exactly two lines, neither parallel to the x-axis, intersecting at (a,b) (it is easy to remove these assumptions). The database

$$S'(x,y) \stackrel{\text{def}}{=} S(x,y) \wedge S(x+1,y)$$

consists of two points (x_1, y_1), (x_2, y_2) (see Figure 2). By a simple geometrical argument,

$$\begin{aligned} x_1 + x_2 &= 2a - 1 \\ y_1 + y_2 &= 2b. \end{aligned}$$

Therefore, the formula

$$\begin{aligned} \exists x_1.\exists y_1.\exists x_2.\exists y_2. \quad & \{S'(x_1, y_1) \wedge S'(x_2, y_2) \wedge (x_1 \neq x_2 \vee y_1 \neq y_2) \\ & \wedge u = \tfrac{x_1+x_2+1}{2} \wedge v = \tfrac{y_1+y_2}{2}\} \end{aligned}$$

is true iff $(u, v) = (a, b)$. ■

Theorem 9 *The* lines *query is linear.*

Proof: (Sketch) Write $S(x, y)$ in conjunctive normal form: $\wedge_i \vee_j C_{ij}(x, y)$, where C_{ij} is a linear constraint. Write the formula

$$\forall x.\forall y.\{ux + vy + w = 0 \rightarrow S(x, y)\}$$

in the form

$$\forall x.\{\bigwedge_i \bigvee_j C_{ij}(x, -\frac{ux + w}{v})\},$$

equivalently

$$\bigwedge_i \{\neg\exists x.(\bigwedge_j \neg C_{ij}(x, -\frac{ux + w}{v}))\}$$

(it is easy to deal with the case $v = 0$). Now consider eliminating x from the set of linear constraints $\wedge_j \neg C_{ij}(x, -\frac{ux+w}{v})$. Eliminating x from

$$\begin{aligned} d_1 x + d_2(-\frac{ux + w}{v}) + d_3 \quad \theta_1 \quad 0 \\ d'_1 x + d'_2(-\frac{ux + w}{v}) + d'_3 \quad \theta_2 \quad 0 \end{aligned}$$

gives, after simplification and cancellation of a common factor v, a constraint

$$(d_2 d'_3 - d'_2 d_3)u + (d_3 d'_1 - d'_3 d_1)v + (d_1 d'_2 - d'_1 d_2)w \quad \theta \quad 0$$

which is linear in the free variables of the query, u, v, w. ■

5 Addition, multiplication, and discrete order

In this section we consider first-order logic and Datalog with a discrete (linear) order. We denote by FO (FO($\leq$), FO($\leq$,+), FO($\leq$, +, $\times$)) first-order logic with equality (and order, and addition, and multiplication). We use corresponding notation for Datalog and the corresponding extensions. The version of Datalog we are considering allows first-order queries on the input predicates.

It is easy to see that Datalog($\leq$,+) = Datalog($\leq$). We first use $\leq$ and negation to define a successor relation *succ*. Addition can then be defined as a ternary predicate, **PLUS**, as follows:

$$\mathbf{PLUS}(0, x, x) \leftarrow,$$
$$\mathbf{PLUS}(x', y, z') \leftarrow succ(x, x') \wedge succ(z, z') \wedge \mathbf{PLUS}(x, y, z).$$

Further, Datalog($\leq$,+,$\times$) = Datalog($\leq$,+). Multiplication can be easily defined as a ternary predicate, **MULT**, using + as follows:

$$\mathbf{MULT}(0, x, 0) \leftarrow,$$
$$\mathbf{MULT}(x', y, z') \leftarrow succ(x, x') \wedge z' = z + y \wedge \mathbf{MULT}(x, y, z).$$

Therefore, in the presence of discrete order, recursion can be used to show that addition and multiplication do not add expressive power to Datalog. We next see that this is not the case in first-order logic. The following query is (i) not expressible in FO($\leq$), but (ii) expressible in FO($\leq$,+).

Example 10 *Consider the schema $\sigma = (R)$, where R is a binary relation. The universe is the set of natural numbers. The query answers true if and only if (i) the cardinality of the projection of R on the first attribute, R_1, is even, and (ii) the second projection of R, R_2, contains the order of x in R_1 (i.e. $R(x, y)$ iff x is the y^{th} element of R_1).*

It is easy to express the query in FO($\leq$,+).

$$\Big(\forall x_1\ x_2\ y_1\ y_2\ \big(\neg\exists x\ ((x_1 < x < x_2) \wedge R_1(x)) \wedge R_1(x_1) \wedge R_1(x_2) \wedge R(x_1, y_1) \wedge R(x_2, y_2)\big) \rightarrow (y_2 = y_1 + 1)\Big)$$
$$\wedge min_{R_2}(1) \wedge \exists n\ (max_{R_2}(n) \wedge \exists m\ (n = m + m)).$$

Here $min_{R_2}(1)$ expresses the fact that the smallest element in the second column of R is 1 and $max_{R_2}(n)$ the fact that the largest element in the second column of R is n. The proof that it cannot be expressed in FO($\leq$) is based on Ehrenfeucht-Fraïssé games.

The query "is the cardinality of the domain a prime number" is expressible in FO($\leq, +, \times$) but not in FO($\leq, +$). We can therefore conclude with the following result.

Theorem 11
$$\begin{array}{ccccccc} FO & \subset & FO(\leq) & \subset & FO(\leq,+) & \subset & FO(\leq,+,\times) \\ Datalog & \subset & Datalog(\leq) & = & Datalog(\leq,+) & = & Datalog(\leq,+,\times) \end{array}$$

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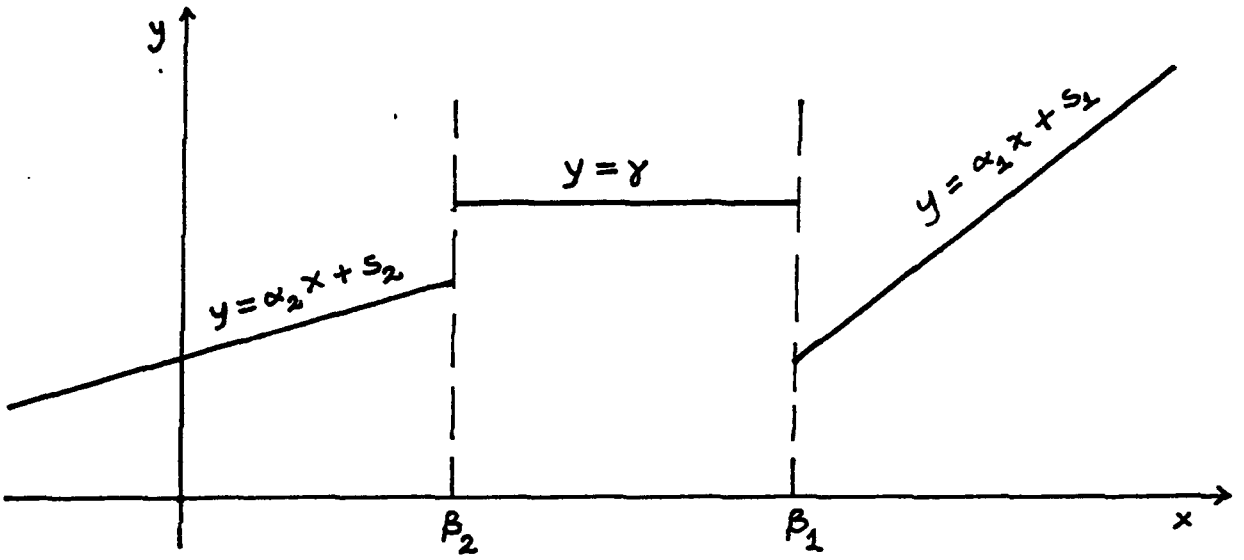

Figure 1 : A two-slope database

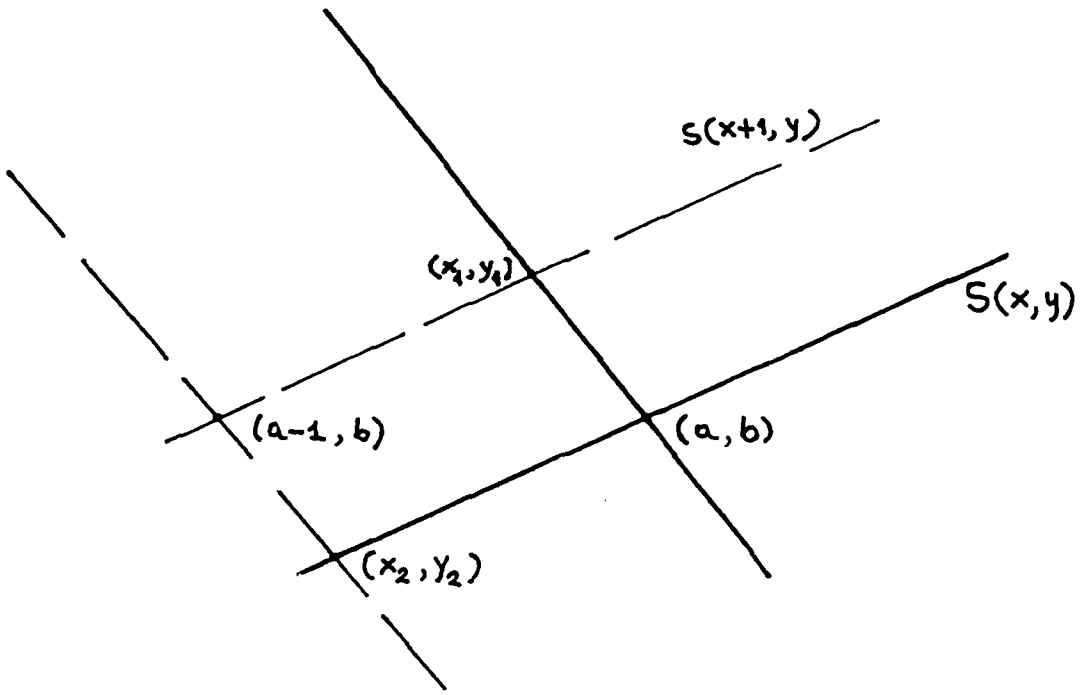

Figure 2 : Intersection of two lines

Foundations of Aggregation Constraints

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1 Introduction

Database query languages, such as SQL and Coral [RSS92], use the grouping construct in conjunction with aggregate operations (such as *min*, *max*, *sum*, *count* and *average*) to obtain summary information from the database. These database query languages also allow constraints on values, such as the results of aggregate operations, to restrict the collection of answers to a query. As an example, consider the following program/query pair (using the notation of [MPR90]):

$$q\_depts(Dept, M1, M2, C, S) :- groupby(employee(Emp, Dept, Sal), [Dept],$$
$$[M1 = min(Sal), M2 = max(Sal), C = count(Sal), S = sum(Sal)]),$$
$$C \leq 10, M1 > 0, M2 \leq 10000.$$

Query: ?-$q\_depts(D, M1, M2, C, S)$.

Intuitively, the program rule scans all tuples in the *employee* relation (the first argument of the *groupby*), and for each department (the variable within [ ] in the second argument of the *groupby*), computes the *min*, *max*, *count* and *sum* of the salaries of the employees in that department. Tuples corresponding to departments where the minimum salary is > 0, where the maximum salary is ≤ 10000 and where the number of employees is ≤ 10 are answers to the query ? $q\_depts(D, M1, M2, C, S)$.

The variables $M1, M2, C$ and S are related by the fact that they are all obtained by performing an aggregate operation on the same multiset. Thus, constraints such as $M1 \leq M2$ are implicitly present on this set of variables, and act in conjunction with the other explicitly specified constraints on these variables.

A fundamental operation on any constraint domain is checking if a conjunction of constraints is solvable. Given a query ? $q\_depts(D, M1, M2, C, S), S > 100000$, it is possible to determine that there are no answers to this query. To do this, we need to determine that the conjunction of (aggregation) constraints:

$$min(S) > 0 \ \wedge \ count(S) \leq 10 \ \wedge \ max(S) \leq 10000 \ \wedge \ sum(S) > 100000$$

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is unsolvable, where S can be any finite multiset of unbounded cardinality.[1] The techniques described in this paper can be used to efficiently check for solvability of such constraints.

Checking solvability of aggregation constraints can be used much like checking solvability of ordinary arithmetic constraints in a constraint logic programming system. Aggregate operations are typically applied only after multisets have been constructed. However, checking solvability of aggregation constraints even before the multisets have been constructed can be used to restrict the search space by not generating subgoals that are guaranteed to fail, as illustrated by the above program.

The contributions of this paper are as follows:

- We introduce a new constraint domain, *aggregation constraints*, which is extremely useful in database query languages and in constraint logic programming languages that incorporate aggregate operations [MS94] (Section 2).
- We discuss the factors that determine the complexity of checking for the *solvability* of conjunctions of aggregation constraints. Further, we show intractability results for checking solvability of conjunctions of certain simple kinds of aggregation constraints (Section 3).
- We present a reduction from the domain of aggregation constraints to the domain of mixed integer/real, non-linear arithmetic constraints (Section 4). This reduction enables us to use existing techniques to check solvability of aggregation constraints. However, these techniques cannot be in polynomial time since checking solvability of aggregation constraints is intractable in general. We point out interesting special cases of aggregation constraints where the reduction-based approach does, in fact, allow for tractable checks for solvability.
- We describe a polynomial-time algorithm that checks for solvability of a useful class of aggregation constraints, where the reduction-based approach generates non-linear, mixed integer/real constraints. Our technique operates directly on the aggregation constraints, rather than on the reduced form; it is not clear how to operate directly on the reduced form to attain the same complexity.

Our work provides the foundations of the area of aggregation constraints. We believe there is a lot of interesting research to be done. To illustrate the possibilities, consider the following example. Given a query ? $q\_depts(D, \_, M2, \_, \_), M2 \geq 5000$, i.e., the user is interested only in departments where the maximum salary is ≥ 5000, this constraint can be used as a *filter* on the tuples of the underlying *employee* relation; *employee* tuples that do not satisfy this criterion need not be considered for the *groupby* operation. This fact has been noted by Sudarshan and Ramakrishnan [SR91] and by Levy et al. [LMS94], who look at some simple cases of query optimization in the presence of aggregate operations. Using more general aggregation constraints in such situations remains to be studied.

2 Syntax and Semantics

In this section, we present an overview of the syntax and semantics of aggregation constraints.

The *primitive terms* of this constraint domain are integer constants, real constants and *aggregation terms*, which are formed using aggregate functions on multiset variables that range over finite multisets. Thus, $7, 3.142$

[1] The cardinality of the multiset S depends on the number of employees in a given department, which can be unbounded.

and $max(S)$ are primitive terms, where S is a multiset variable that ranges over finite multisets. For simplicity, we do not allow integer and real-valued variables as primitive terms in our treatment.[2] *Complex terms* are constructed using primitive terms and arithmetic functions such as $+, -, *$ and $/$. Thus, $min(S_1) + max(S_2) - 3.142 * count(S_2)$ is a complex term.

A ***primitive aggregation constraint*** is constructed using complex terms and arithmetic predicates such as $\leq, <, =, \neq, >$ and $\geq$. Thus, $sum(S_1) \leq min(S_1) + max(S_2) + 3$ is a primitive aggregation constraint. Complex aggregation constraints can be constructed using conjunctions and disjunctions of primitive aggregation constraints, in the usual manner. In the sequel, we often use "aggregation constraints" to loosely refer to primitive aggregation constraints.

The fundamental problem that we are interested in, in this paper, is as follows:

> **Solvability**: Given a conjunction C of primitive aggregation constraints, does there exist an assignment σ of finite multisets to the multiset variables in C, such that $C\sigma$ is satisfied?

Checking for solvability of more complex aggregation constraints can be reduced to this fundamental problem. The other important problems of checking *implication* (or entailment) and *equivalence* of pairs of conjunctions of aggregation constraints can be reduced to checking solvability of (collections of other) conjunctions of aggregation constraints, in polynomial-time.

3 Complexity of Solvability

We present some intractability results for checking solvability of conjunctions of certain simple kinds of aggregation constraints to illustrate the difficulty of the problem in general.

There is a straightforward linear-time, linear-space reduction from integer arithmetic constraints to aggregation constraints, where, (1) the multiset elements can be from any domain, and (2) only the *count* aggregate function needs to be used. For each (integer) variable X_i in the conjunction of integer arithmetic constraints, the reduction algorithm creates two new multiset variables S_{i1} and S_{i2}, and replaces each occurrence of X_i by $count(S_{i1}) - count(S_{i2})$. The difference of counts is needed to simulate negative integers. Similarly, if the multisets range over integers, we can create one new multiset variable S_i for each integer variable X_i, and replace each occurrence of X_i by $min(S_i)$ (or $max(S_i)$ or $sum(S_i)$).

It is easy to see that the resulting conjunction of aggregation constraints is solvable iff the original conjunction of integer arithmetic constraints is solvable. Further, the algorithm preserves the linear/non-linear nature of the original integer arithmetic constraints. Since checking for solvability of integer linear arithmetic constraints is NP-complete [Sch86], we have the following result:

Theorem 3.1 *Checking solvability of a conjunction of linear aggregation constraints involving just the count aggregate function is NP-hard. If the multiset elements are drawn from the integers, then checking solvability of a conjunction of linear aggregation constraints involving just min or max or sum is NP-hard.* □

3.1 Special Cases: A Taxonomy

Although checking for solvability of aggregation constraints is NP-hard in the general case, there are many special cases that are tractable. We present below several factors that affect the complexity, and in later sections present

[2] If desired, these can be simulated using the primitive terms allowed; for example, a real-valued variable X_i can be replaced by $min(S_i)$, where S_i is a new multiset variable that ranges over finite multisets of reals.

tractable special cases defined on the basis of these factors.

Domain of multiset elements : This determines the feasible assignments to the multiset variables in checking for solvability. Possibilities include integers and reals; correspondingly, the multiset variables range over finite multisets of integers or reals. In general, restricting the domain to integers increases the difficulty of the problem.

Aggregate functions : This determines the possible aggregation terms that are allowed. Possibilities include *min*, *max*, *sum*, *count*, *average*, etc. In general, the complexity of checking for solvability increases if more aggregate functions are allowed.

Class of constraints : This determines the form of the primitive aggregation constraints considered, which affects the complexity of the solvability problem. There are at least two factors that are relevant:

1. **Linear vs. Non-linear** constraints: Checking for solvability of linear constraints is, in general, easier than for non-linear constraints. By restricting the form even further, such that each primitive aggregation constraint has at most one or two aggregation terms, the problem can become even simpler.
2. **Constraint predicates** allowed: The complexity of checking for solvability also depends on which types of the constraint predicates are allowed. We can choose to allow only equational constraints ($=$) or add inequalities ($<, \leq$) or possibly even disequalities ($\neq$). In general, the difficulty of the solvability problem increases with each new type.

Separability : This also determines the form of the primitive aggregation constraints considered. The two possible dimensions in this case are:

1. **Multiset variables**: A conjunction of primitive aggregation constraints is *multiset-variable-separable* if each primitive aggregation constraint involves only one multiset variable. For example, the conjunction $min(S_1) + max(S_1) \leq 5 \ \wedge \ sum(S_2) \geq 10$ is multiset-variable-separable, while $min(S_1) + min(S_2) \leq 10$ is not. In general, multiset-variable-separability makes the solvability problem easier since one can check solvability of the aggregation constraints separately for each multiset variable.
2. **Aggregate functions**: A conjunction of primitive aggregation constraints is *aggregate-function-separable* if each primitive aggregation constraint involves only one aggregate function. For example, the conjunction $min(S_1) \leq min(S_2) \ \wedge \ sum(S_1) \geq sum(S_2) + 2$ is aggregate-function-separable, although it is not multiset-variable-separable.

4 A Reduction-based Approach To Solvability

Our first approach to checking for the solvability of a conjunction of aggregation constraints is to try and reduce aggregation constraints to an *existing* constraint domain. The advantage of this approach is that, if successful, solvability checking techniques from previously known constraint domains can be used to check for solvability in our novel constraint domain. In this section, we present some preliminary results in this direction.

The key idea behind our reduction algorithm is to add to the conjunction of aggregation constraints a *complete* set of relationships between the aggregate operations on a *single* multiset. The intuition here is that the constraint domain of "aggregation constraints" only allows primitive aggregate operations on individual multisets. Interactions between different multisets is possible only via arithmetic constraints between the results

of the aggregate operations on individual multisets. Consequently, relationships between the results of aggregate operations on different multisets can be *inferred* using techniques from the domain of ordinary arithmetic constraints (see [Sch86]).

Theorem 4.1 *The following relationships provide a* correct, complete *and* minimal *axiomatization of the relationships between the aggregate operations min, max, sum, count and average on a single multiset S.*

(1) $count(S)$ is an integer ≥ 0.
(2) if $(count(S) = 0)$ then $min(S)$ and $max(S)$ are undefined.[3]
(3) if $(count(S) > 0)$ then $min(S) \leq max(S)$.
(4) if $(count(S) = 0)$ then $sum(S) = 0$.
(5) if $(count(S) > 0)$ then $(count(S) - 1) * min(S) + max(S) \leq sum(S)$.
(6) if $(count(S) > 0)$ then $sum(S) \leq min(S) + (count(S) - 1) * max(S)$.
(7) if $(count(S) = 0)$ then $average(S)$ is undefined.
(8) if $(count(S) > 0)$ then $sum(S) = average(S) * count(S)$.

Proof: We first prove correctness and completeness of the set of relationships (1)–(8).

The multiset S clearly has 0 or more elements. If S has 0 elements, relationships (2), (4) and (7) are obviously correct and complete. If S has 1 element, then S can be represented as $\{X_1\}$. In this case, we have:

$$min(S) = X_1 \ \wedge \ max(S) = X_1 \ \wedge \ sum(S) = X_1 \ \wedge \ average(S) = X_1.$$

Projecting out the variable X_1, we have:

$$min(S) = max(S) \ \wedge \ min(S) = sum(S) \ \wedge \ min(S) = average(S).$$

It is easy to verify that the conjunction of relationships (3), (5), (6) and (8) are equivalent to the above conjunction, when $count(S) = 1$.

If S has $n \geq 2$ elements, then S can be represented as $\{X_1, X_2, \ldots, X_n\}$, where $X_1 \leq X_2 \leq \ldots \leq X_n$. In this case, we have $min(S) = X_1 \ \wedge \ max(S) = X_n \ \wedge \ sum(S) = X_1 + X_2 + \ldots + X_n \ \wedge \ average(S) = sum(S)/n$. Aggregation constraints involving *min*, *max*, *sum*, *count* or *average* do not allow direct reference to any of the values $X_2, \ldots, X_{n-1}$. Without loss of generality, we can assume that $m \leq n-2$ of these values are identical (say $= X_2$) and $n-m-2$ of these values are identical (say $= X_{n-1}$), where $X_1 \leq X_2 \leq X_{n-1} \leq X_n$.[4] Consequently, we can simplify the above relationships as follows:

$$X_1 \leq X_2 \ \wedge \ X_2 \leq X_{n-1} \ \wedge \ X_{n-1} \leq X_n \ \wedge$$
$$min(S) = X_1 \ \wedge \ max(S) = X_n \ \wedge \ average(S) = sum(S)/n \ \wedge$$
$$sum(S) = X_1 + m * X_2 + (n - m - 2) * X_{n-1} + X_n,$$

where $0 \leq m \leq n-2$. We can now replace the variables X_1 and X_n by $min(S)$ and $max(S)$. Since X_2 and X_{n-1} cannot be directly referenced in the aggregation constraints, and the only other constraints known about these variables are their bounds, we can project them out to obtain:

[3] An alternative suggestion, made in [RS92], is to take $min(\emptyset) = \infty$ and $max(\emptyset) = -\infty$. While this is useful in an inductive characterization of the *min* and *max* aggregate operations, it violates our intuition that $min(S) \leq max(S)$.

[4] If the elements of the multiset S are drawn from the reals, we can assume that all the $n-2$ values $X_2, \ldots, X_{n-1}$ are identical. If the elements are drawn from the integers, we may need two distinct values X_2 and X_{n-1}, such that there are $m \leq n-2$ copies of X_2 and $n-m-2$ copies of X_{n-1}. The reason for this has to do with computing the *sum* of all the elements of the multiset.

$min(S) \leq max(S) \ \wedge \ sum(S) = average(S) * n \ \wedge$
$sum(S) \leq min(S) + (n-1) * max(S) \ \wedge \ sum(S) \geq (n-1) * min(S) + max(S).$

It is easy to verify that the conjunction of relationships (3), (5), (6) and (8) are equivalent to the above conjunction, when $count(S) = n$. This completes the proof of correctness and completeness. Minimality follows from the fact that none of the relationships is entailed by the others. □

Other relationships between the results of aggregate operations can be inferred using these basic relationships. For example, we can infer that $count(S) = 1$ implies that $min(S) = max(S)$. Similarly, we can infer that the constraint $max(S) < average(S)$ is unsolvable.

The above reduction results in non-linear, mixed integer/real constraints, even when applied to linear aggregation constraints. Such constraints are harder to solve than linear constraints.

Consider the linear aggregation constraint $min(S) = max(S)$, where S ranges over finite multisets of reals/integers. For this aggregation constraint to be solvable, $min(S)$ and $max(S)$ must be defined. Hence, this implies the additional constraint $(count(S) > 0) \ \wedge \ (sum(S) = count(S) * max(S))$. Since the values of $min(S)$ and $max(S)$ are not constrained any further, this is not equivalent to any finite collection of linear constraints. The following theorem formalizes this idea.

Theorem 4.2 *There is no finite collection of linear arithmetic constraints over the reals and integers that correctly and completely axiomatizes the relationships between the aggregate operations min, max, sum and count.* □

4.1 Efficient Special Cases

In general, checking for solvability of aggregation constraints, even after the reduction, is intractable. In this section, we briefly describe two cases where the reduction-based approach leads to polynomial-time algorithms for checking solvability.

The first case is when the conjunction of constraints involves only min and max. If we want such constraints to be satisfiable, we must make the assumption that $min(S)$ and $max(S)$ are defined, and hence $count(S) > 0$. Hence, in this case, only the relationship $min(S) \leq max(S)$ (which assumes $count(S) > 0$) needs to be added. If the original conjunction of aggregation constraints is linear and the multiset elements are drawn from the reals, the transformed conjunction of arithmetic constraints is also linear over the reals; solvability can now be checked in time polynomial in the size of the aggregation constraints, using any of the standard techniques (see [Sch86]) for solving linear arithmetic constraints over the reals.

The second case is when the conjunction of linear aggregation constraints explicitly specifies the cardinality of each multiset, i.e., for each multiset variable S_i, we know that $count(S_i) = k_i$, where k_i is a constant. In this case, each of the non-linear constraints in our axiomatization can be simplified to linear constraints; checking for solvability again takes time polynomial in the size of the aggregation constraints if the multiset elements are drawn from the reals.

5 Linear Separable Aggregation Constraints

In this section, we examine a very useful class of aggregation constraints, and present a polynomial-time algorithm to check for solvability of constraints in the class. Our technique operates directly on the aggregation constraints, rather than on their reduction to arithmetic constraints. The reduced form of this class includes

mixed integer/real constraints, and is non-linear; it is not clear how to operate directly on the reduced form and attain the same complexity as our algorithm.

We specify the class of constraints in terms of the factors, described in Section 3, that affect the complexity of checking for solvability. We require that: (1) the domain of multiset elements is the *reals*, (2) the only aggregate functions present are *min*, *max*, *sum* and *count*, (3) the constraints are linear and specified using $\leq, <, =, >$ and $\geq$, and (4) the constraints are aggregate-function-separable and multiset-variable-separable. Intuitively, the above four restrictions ensure that we can simplify the given conjunction of aggregation constraints to range constraints on each aggregate function on each multiset variable. In addition, we require that: (5) for each multiset variable S_i, the ranges for $min(S_i)$ and $max(S_i)$ are identical. This semantic condition ensures that the multisets can contain *any* finite collection of elements from the given ranges for $min(S_i)$ and $max(S_i)$. We refer to this class of aggregation constraints as $\mathcal{LS}$-aggregation-constraints.

Most aggregation constraints occurring in practice are multiset-variable-separable since typically a single grouping literal appears in each rule. Only when we consider constraint propagation or fold/unfold transformations are we likely to obtain non-multiset-variable-separable aggregation constraints. The further restrictions for $\mathcal{LS}$-aggregation-constraints are not onerous; the example in the introduction is such a constraint.

5.1 Multiset Ranges

The heart of our algorithm is a function Multiset_Ranges that takes three ranges, two real ranges $\langle m_l, m_h \rangle$ and $\langle v_l, v_h \rangle$, and an integer range $\langle k_l, k_h \rangle$, along with information about whether each side of each range is open or closed, and answers the following question:

> Do there exist $k \geq 0$ numbers, k between k_l and k_h, each number between m_l and m_h, such that the sum of the k numbers is between v_l and v_h?

For simplicity of exposition, we present a special case of the algorithm below, where each range is assumed to be finite (i.e., no value is infinite), closed on both sides, and feasible. The general case does not add to the intuition, but makes the algorithm more verbose.

---

function Multiset_Ranges $(m_l, m_h, v_l, v_h, k_l, k_h)$

{

/* we assume finite numbers: $k_l \geq 0$, $k_l \leq k_h$, $m_l \leq m_h$ and $v_l \leq v_h$, and closed ranges. */

(1) if ($m_l \leq 0$ and $m_h \geq 0$) then /* Case 1: $[m_l, m_h]$ includes 0. */

(a) if ($v_h < m_l * k_h$ or $v_l > m_h * k_h$) then /* sum is too low or too high. */

return 0.

(b) else return 1.

(2) if ($m_h < 0$) then /* Case 2: m_l and m_h are both < 0. switch everything. */

(a) $temp = -m_l$; $m_l = -m_h$; $m_h = temp$. /* both m_l and m_h become positive and $m_l \leq m_h$. */

(b) $temp = -v_l$; $v_l = -v_h$; $v_h = temp$.

/* Case 3: m_l and m_h are both > 0. */

(3) if ($v_h < k_l * m_l$ or $v_l > k_h * m_h$) then /* sum is too low or too high. */

return 0.

(4) define k_1 and k_2 by $v_l = k_1 * m_h - k_2, 0 \leq k_2 < m_h$.

/* k_1 is the smallest number of possible values from $[m_l, m_h]$, whose sum is $\geq v_l$. */

(5) define k_3 and k_4 by $v_h = k_3 * m_l + k_4, 0 \leq k_4 < m_l$.

```
        /* k3 is the largest number of possible values from [m_l, m_h], whose sum is ≤ v_h. */
    /* check if the [k_l, k_h] range overlaps with the [k_1, k_3] range. */
    (6) if (k_1 ≤ k_3 and k_1 ≤ k_h and k_l ≤ k_3) then
        return 1.    /* the intersection gives a possible value for k */
    (7) else return 0.
}
```

Theorem 5.1 *Function Multiset_Ranges returns* 1 *iff there exist* $k \geq 0$ *real numbers,* $k_l \leq k \leq k_h$*, each number is greater than or equal to* m_l *and less than or equal to* m_h*, such that the sum of the* k *numbers is greater than or equal to* v_l *and less than or equal to* v_h*.*

Proof: The algorithm has three cases, based on the location of the $[m_l, m_h]$ range with respect to zero. The first case is when this range includes zero; in this case the sum can take any value in the continuous range $[k_h * m_l, k_h * m_h]$. The second case is when the $[m_l, m_h]$ range includes only negative numbers, and the third case is when this range includes only positive numbers. These two cases are symmetric, and we transform the second case into the third case, and consider only the third case in detail.

In the third case, the sum lies within the continuous range $[k_l * m_l, k_h * m_h]$, but it *cannot* take all values within this range; it can take values only from the union of the ranges $[k_l * m_l, k_l * m_h], [(k_l + 1) * m_l, (k_l + 1) * m_h], \ldots, [k_h * m_l, k_h * m_h]$. This union of ranges need not be convex; there may be gaps. Iff the $[v_l, v_h]$ range lies outside the $[k_l * m_l, k_h * m_h]$ range, or entire within one of the gaps, then the conjunction of constraints is unsolvable. This concludes the proof. □

5.2 Checking for Solvability

Recall the class of $\mathcal{LS}$-aggregation-constraints. Since the conjunction of aggregation constraints is multiset-variable-separable, the primitive aggregation constraints can be partitioned based on the multiset variable, and the conjunction of aggregation constraints in each partition can be solved separately; the overall conjunction is solvable iff the conjunction in each partition is separately solvable.

By definition, we can simplify a conjunction of $\mathcal{LS}$-aggregation-constraints on a single multiset variable S_i to range constraints on each aggregate function. We can then check whether each range is feasible, and whether the ranges for $min(S_i)$ and $max(S_i)$ are identical. If so, the algorithm Check_LS_Solvability that checks for solvability first takes into account the special case of $count(S_i) = 0$. It then calls function Multiset_Ranges with the range for $min(S_i)$ (equivalently $max(S_i)$), the range for $sum(S_i)$ and the range for $count(S_i)$.

If, for each multiset variable S_i, function Multiset_Ranges returns 1, then algorithm Check_LS_Solvability returns SOLVABLE.

Theorem 5.2 *Given a conjunction of* $\mathcal{LS}$*-aggregation-constraints, algorithm* Check_LS_Solvability *returns SOLVABLE iff the conjunction is solvable.*

Further, it takes time polynomial in the size of the conjunction of $\mathcal{LS}$*-aggregation-constraints.* □

Though $\mathcal{LS}$-aggregation-constraints are significantly restricted, they are strong enough to usefully entail new aggregate constraint information. They can be used to infer information about an *arbitrary* aggregation constraint, C, by determining an $\mathcal{LS}$-aggregation-constraint, H, that is implied by C; any aggregation constraints entailed by H are then also entailed by C.

6 Conclusions and Future Work

We presented a new and extremely useful class of constraints, *aggregation constraints*. We studied the complexity of the problem of checking for solvability of conjunctions of aggregation constraints, and described some simple cases that are intractable. We identified interesting classes of aggregation constraints that are tractable, and presented novel algorithms for checking for solvability.

There are many interesting directions to pursue. An important direction of active research is to significantly extend the class of aggregation constraints for which solvability can be efficiently checked. We believe that our algorithm works on a larger class of aggregation constraints than presented here—for instance, we believe that our algorithm will work correctly even if we relax the conditions to not require *min* and *max* to be separated; characterizing this class will be very useful.

Combining aggregation constraints with multiset constraints that give additional information about the multisets (using functions and predicates such as $\cup, \in, \subseteq$, etc.) will be very important practically.

Another important direction is to examine how this research can be used to improve query optimization and integrity constraint checking in database query languages such as SQL. Sudarshan and Ramakrishnan [SR91] and Levy et al. [LMS94] consider how to use simple aggregate conditions for query optimization; it would be interesting to see how their work can be generalized. Stuckey and Sudarshan [SS94] present compilation techniques for query constraints in logic programs, essentially extending Magic sets to handle general query constraints, not just equality constraints on queries. It would be interesting to see how to use aggregation constraints in conjunction with their techniques.

We believe that we have identified an important area of research, namely aggregation constraints, in this paper and have laid the foundations for further research in the area.

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Set Constraints: Results, Applications and Future Directions

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Abstract

Set constraints are a natural formalism for many problems that arise in program analysis. This paper provides a brief introduction to set constraints: what set constraints are, why they are interesting, the current state of the art, open problems, applications and implementations.

1 Introduction

Set constraints are a natural formalism for describing relationships between sets of terms of a free algebra. A set constraint has the form $X \subseteq Y$, where X and Y are *set expressions*. Examples of set expressions are 0 (the empty set), α (a set-valued variable), $c(X, Y)$ (a constructor application), and the union, intersection, or complement of set expressions.

Recently, there has been a great deal of interest in program analysis algorithms based on solving systems of set constraints, including analyses for functional languages [AWL94, Hei94, AW93, AM91, JM79, MR85, Rey69], logic programming languages [AL94, HJ92, HJ90b, Mis84], and imperative languages [HJ91]. In these algorithms, sets of terms describe the possible values computed by a program. Set constraints are generated from the program text; solving the constraints yields some useful information about the program (e.g., for type-checking or optimization).

Set constraints have proven to be a very successful formalism. On the theoretical side, rapid progress has been made in understanding the algorithms for and complexity of solving various classes of set constraints. On the practical side, several program analysis systems based either entirely or partially on set constraint algorithms have been implemented. In addition, the use of set constraints has simplified previously known, but rather complicated, program analyses and set constraints have led directly to the discovery of other, previously unknown, analyses.

Much of the work on set constraints is very recent. Consequently, many of the results are not well known outside of the community of researchers active in the area. The purpose of this paper is to provide a brief, accessible survey of the area: what set constraints are, why they are useful, what is and isn't known about solving set constraints, the important open problems, and likely directions for future work. Section 2 gives definitions of the basic set constraint formalism and some illustrative examples. Section 3 presents a survey of results on the satisfiability, complexity, and solvability of various set

constraint problems; open problems are also discussed. In Section 4 a brief, informal description of algorithms for solving systems of set constraints is given; this discussion also points out basic trade-offs between expressive power and computational complexity for various classes of set constraint problems. Section 5 surveys applications of set constraints to program analysis. Section 6 concludes with a discussion of current implementations and likely directions for future work.

2 Set Constraints

Let C be a set of constructors and let V be a set of variables. Each $c \in C$ has a fixed arity $a(c)$; if $a(c) = 0$ then c is a constant. The *set expressions* are defined by the following grammar:

$$E ::= \alpha \mid 0 \mid c(E_1, \ldots, E_{a(c)}) \mid E_1 \cup E_2 \mid E_1 \cap E_2 \mid \neg E_1$$

In this grammar, α is a variable (i.e., $\alpha \in V$) and c is a constructor (i.e., $c \in C$). Set expressions denote sets of *terms*. A term is $c(t_1, \ldots, t_{a(c)})$ where $c \in C$ and every t_i is a term (the base cases of this definition are the constants). The set H of all terms is the Herbrand universe. An *assignment* is a mapping $V \to 2^H$ that assigns sets of terms to variables. The meaning of set expressions is given by extending assignments from variables to set expressions as follows:

$$\begin{aligned}
\sigma(0) &= \emptyset \\
\sigma(c(E_1, \ldots, E_n)) &= \{c(t_1, \ldots, t_n) | t_i \in \sigma(E_i)\} \\
\sigma(E_1 \cup E_2) &= \sigma(E_1) \cup \sigma(E_2) \\
\sigma(E_1 \cap E_2) &= \sigma(E_1) \cap \sigma(E_2) \\
\sigma(\neg E_1) &= H - \sigma(E_1)
\end{aligned}$$

A *system of set constraints* is a finite conjunction of constraints $\bigwedge_i X_i \subseteq Y_i$ where each of the X_i and Y_i is a set expression. A *solution* of a system of set constraints is an assignment σ such that $\bigwedge_i \sigma(X_i) \subseteq \sigma(Y_i)$ is true. A system of set constraints is *satisfiable* if it has at least one solution. The following result was proven first in [AW92]. Simpler proofs have been discovered since [BGW93, AKVW93].

Theorem 2.1 It is decidable whether a system of set constraints is satisfiable. Furthermore, all solutions can be finitely presented.

From the definition above, it is easy to see that the set expressions consist only of elementary set operations plus constructors—simply put, it is a set theory of terms. The constraint language is rich enough, however, to describe all of the data types commonly used in programming, and this is the property that makes set constraints a natural tool for program analysis. For example, programming language data type facilities provide "sums of products" data types, which means simply unions of (usually distinct) data type constructors. All such data types can be expressed as set constraints.

Let $X = Y$ stand for the pair of constraints $X \subseteq Y$ and $Y \subseteq X$. Consider the constraint

$$\beta = \mathtt{cons}(\alpha, \beta) \cup \mathtt{nil}$$

If **cons** and **nil** are interpreted in the usual way, then the solution of this constraint assigns to β the set of all lists with elements drawn from α. This example also shows that a special operation for recursion is not required in the set expression language—recursion is obtained naturally through recursive constraints.

The set of non-nil lists (with elements drawn from α) can be defined as $\gamma = \beta \cap \neg\mathtt{nil}$, where β is defined as above. The set γ is useful because it describes the proper domain of the function that selects the first element of a list; such a function is undefined for empty lists. This example also illustrates that set constraints can describe proper subsets of standard sums of products data types.

The final example shows a non-trivial set of constraints where some work is required to derive the solutions. Consider the universe of the natural numbers with one unary constructor **succ** and one nullary constructor **zero**. Let the system of constraints be:

$$\mathtt{succ}(\alpha) \subseteq \neg\alpha \bigwedge \mathtt{succ}(\neg\alpha) \subseteq \alpha$$

These constraints say that if $x \in \alpha$ (resp. $x \in \neg\alpha$) then $\mathtt{succ}(x) \in \neg\alpha$ (resp. $\mathtt{succ}(x) \in \alpha$). In other words, these constraints have two solutions, one where α is the set of even integers and one where α is the set of odd integers. The solutions are described by the following equations:

$$\alpha = \mathtt{zero} \cup \mathtt{succ}(\mathtt{succ}(\alpha))$$
$$\alpha = \mathtt{succ}(\mathtt{zero}) \cup \mathtt{succ}(\mathtt{succ}(\alpha))$$

Note that the two solutions are incomparable; in general, there is no least solution of a system of set constraints.

3 Results and Open Problems

The set constraint language defined in Section 2 is henceforth called the *basic language*. There are several interesting extensions to the basic language, each of which substantially alters the set constraint problem. Three extensions are discussed in this paper: projections, function spaces, and negative constraints.

For every constructor c of arity n, a family of *projections* $c^{-1}, \ldots, c^{-n}$ can be defined such that

$$\sigma(c^{-i}(E)) = \{t_i | \exists t_1, \ldots, t_n.\, c(t_1, \ldots, t_n) \in \sigma(E)\}$$

Projections are used primarily in set constraint analyses for logic programming languages [HJ90b].

A separate extension is adding sets of functions $X \rightarrow Y$ to the set expressions. This is a major change, because it not only enriches the language, but also requires a new domain. The construction of a suitable domain with function spaces is beyond the scope of this paper; somewhat surprisingly, however, given such a domain, set constraint techniques still apply. In an appropriate domain, the meaning of $X \rightarrow Y$ is

$$X \rightarrow Y = \{f | x \in X \Rightarrow f(x) \in Y\}$$

Function spaces are used primarily in the analysis of functional programming languages [AW93, AWL94].

Finally, negative constraints are strict containments $X \not\subseteq Y$. Negative constraints can express the set of non-solutions of a system of positive constraints:

$$\neg \bigwedge_i (X_i \subseteq Y_i) = \bigvee_i X_i \not\subseteq Y_i$$

Since conjunctions of positive constraints correspond to an existential property (i.e., is any assignment a solution of the constraints) disjunctions of negative constraints can express universal properties (i.e., is every assignment a solution of the constraints) [AKW93, GTT93].

Four proofs of decidability of the satisfiability problem for the basic language are known [AW92, GTT92, BGW93, AKVW93]. Remarkably, each proof is based on completely different techniques. A particularly elegant proof is due to Bachmair, Ganzinger, and Waldmann [BGW93]; their result shows set constraints are equivalent to the *monadic class*, the class of first order formulas with arbitrary quantification but only unary predicates and no function symbols. In addition to satisfiability, constraint resolution algorithms are known that construct explicit representations of the solutions of systems of set constraints for the basic language.

The situation with the various extensions is less clear. Table 1 summarizes the current state of knowledge. Of the open problems in Table 1, decidability of the satisfiability of set constraints with projections has been open for the longest time [HJ90a]. Constraint resolution algorithms for restricted forms of the general problem are known [HJ90a, Hei92]; the current state of the art permits the full basic language and restricts only projections [BGW93].

Work on set constraints extended with negative constraints has been motivated in part because it appears to be an intermediate step toward handling projections. To see this, consider the expression $c^{-1}(c(X,Y))$. Note that if $Y = 0$, then $c(X,Y) = 0$, since constructors function as cross products. Therefore, the meaning of this expression can be characterized as

$$c^{-1}(c(X,Y)) = \begin{cases} 0 & \text{if } Y = 0 \\ X & \text{if } Y \neq 0 \end{cases}$$

Thus, even a restricted form of projection implicitly involves negative constraints ($Y \neq 0$ in the right-hand side above). Two independent proofs of the decidability of set constraints with negative constraints have been discovered [AKW93, GTT93]. These are decision procedures only, however, and do not characterize the solution sets.

Set constraints extended with function spaces have been used to develop very expressive subtype inference systems for functional languages. Currently, constraint solving algorithms for a fairly general class of set constraints with function types are known [AW93, AWL94]. Damm has proven the surprising result that satisfiability of set constraints with function spaces is decidable [Dam94].

Set constraint resolution algorithms are computationally expensive in general. For the basic problem, deciding satisfiability is NEXPTIME-complete [BGW93] and even if the language is restricted to the set operations over constants satisfiability remains NP-complete [AKVW93]. By restricting the set

Problem	*Satisfiability*	*Constraint Resolution*
basic	yes	yes
basic + projections	?	with restrictions
basic + function spaces	yes	with restrictions
basic + negative constraints	yes	?

Table 1: Status of set constraint problems.

operations (instead of the arity of constructors) it is possible to achieve polynomial time algorithms for interesting classes of constraints [JM79, MR85, Hei92].

4 Algorithms

At the current time, the literature on set constraint algorithms is very diverse in many dimensions, with a wide variety of notation and algorithmic techniques in use. Unfortunately, no reference provides a systematic introduction to more than a small portion of the body of existing work. This section gives a very brief and relatively informal overview of the basic algorithmic issues in solving systems of set constraints. For a more detailed treatment of the various algorithms, the interested reader should consult sources listed in the bibliography.

All set constraint resolution algorithms have the same basic structure. An initial system of constraints is systematically transformed until the constraints reach a particular syntactic *solved form*. In most cases, the solved form is equivalent to one or more regular tree grammars. More precisely, the final result is a set of equations

$$\alpha = c(X_1, \ldots, X_n) \cup \ldots \cup d(Y_1, \ldots, Y_m)$$

which can viewed equivalently as the productions of a grammar

$$\alpha ::= c(X_1, \ldots, X_n) \mid \ldots \mid d(Y_1, \ldots, Y_m)$$

The language generated by the tree grammar then describes the solution of the constraints.

Unfortunately, this simple explanation of the solutions of set constraints is a bit oversimplified. In reality, set constraints are more general than tree grammars. In the solutions of set constraints, this extra generality appears as "free" variables in the solved form equations. A free variable is one that does not appear on the left-hand side of any equation. Thus, a more accurate description of the solutions of set constraints is that they are tree grammars that may include free variables.

At their core, all set constraint algorithms have two characteristic forms of constraints: transitive constraints and structural constraints. Transitive constraints arise from combining upper and lower bounds on variables:

$$X \subseteq \alpha \ \wedge \ \alpha \subseteq Y \Rightarrow X \subseteq Y$$

Because of the need to resolve transitive constraints, most interesting set constraint problems have at least $\mathcal{O}(n^3)$ time complexity.

Structural constraints are constraints between constructor expressions:

$$c(X_1, \ldots, X_n) \subseteq c(Y_1, \ldots, Y_n)$$

In general, there may be many incomparable solutions of such a constraint. For example, because the semantics of a constructor is essentially a cross product, a constructor expression is 0 if any component is 0, and therefore the constraint is satisfied if $X_i = 0$ for any i. Of course, the constraint is also satisfied if $X_i \subseteq Y_i$ for all i. Thus, the complete set of solutions is

$$c(X_1, \ldots, X_n) \subseteq c(Y_1, \ldots, Y_n) \Leftrightarrow X_1 = 0 \vee \ldots \vee X_n = 0 \vee (X_1 \subseteq Y_1 \wedge \ldots \wedge X_n \subseteq Y_n)$$

Searching for a solution of such a constraint requires guessing a disjunct that can be satisfied. This non-deterministic choice increases the complexity of set constraint problems above the complexity of the corresponding tree automata problems. For example, deciding whether the language of one tree automata is a subset of another is complete for EXPTIME [Sei90]; solving a general system of set constraint inclusions is complete for NEXPTIME.

If it is known that the system of constraints under consideration has a least solution and the goal is to compute only the least solution, then it is easy to see that the cases $X_i = 0$ need not be considered and the last case can be chosen deterministically. Thus, more efficient algorithms are possible in the special case that a system of constraints has a least solution.

Finally, the set operators $\cap, \cup$, and $\neg$ play roles very similar to their roles in other logics. There are some distributive laws involving constructors, but these are not surprising:[1]

$$\begin{aligned}
c(X_1, \ldots, X_n) \cap c(Y_1, \ldots, Y_n) &= c(X_1 \cap Y_1, \ldots, X_n \cap Y_n) \\
c(X_1 \cup Y_1, Z_2, \ldots, Z_n) &= c(X_1, Z_2, \ldots, Z_n) \cup c(Y_1, Z_2, \ldots, Z_n) \\
\neg c(X_1, \ldots, X_n) &= c(\neg X_1, 1, \ldots, 1) \cup \ldots \cup c(1, \ldots, 1, \neg X_n) \cup \bigcup_{d \neq c} d(1, \ldots, 1)
\end{aligned}$$

For set constraint problems with restricted set operations and where the constraints have least solutions, it is possible to design polynomial time algorithms to compute the least solution; for examples, see [JM79, MR85, Hei92, Hei94]. If the set operations are not restricted, then it becomes possible to describe some complex sets of terms very succinctly with set expressions, which raises the computational complexity of constraint resolution to exponential time.

[1] As written, the law for negation appears to require that the set of all constructors d such that $d \neq c$ can be enumerated and thus the set of constructors must be finite. In fact, this restriction is not necessary, and it is a simple matter to implement negation for infinite sets of constructors.

5 Applications

Set constraints have a long history and, in fact, their use predates the term "set constraints" by many years. The basic language of set constraints is now known to be equivalent to the monadic class of logical formulas [BGW93]; the first decision procedure for the monadic class was given by Löwenheim in 1915 [Lö15]. Within the realm of computer science, Reynolds was the first to develop a resolution algorithm for a class of set constraints [Rey69]. Reynolds was interested in the analysis and optimization of Lisp programs. In this application, set constraints were used to compute a conservative description of the data structures in use at a program point. Using this information, a Lisp program could be optimized by, for example, eliminating run-time type checks where it was provably safe to do so.

Independently of Reynolds, Jones and Muchnick developed a different analysis system for Lisp programs based on solving systems of set equations [JM79]. This analysis was used not only to eliminate dynamic type checks but also to reduce reference count operations in automatic memory management systems based on reference counting. Recently Wang and Hilfinger have proposed another analysis method for Lisp based on set equations [WH92].

A different set of applications provide type inference algorithms for functional languages that verify the type correctness of a larger class of programs than the standard Hindley/Milner type system. Mishra and Reddy described a type system based on a set constraint resolution algorithm that could handle considerably more complex constraints than previous algorithms [MR85]. Thatte introduced *partial types* [Tha88], the type inference problem for which, while substantially different from earlier systems, is also reducible a set constraint resolution problem. The most recent work in this area is due to Wimmers and the author [AW93, AWL94], who provide a type inference system that generalizes the results in [MR85, Tha88]. An implementation of this last system is publicly available (see Section 6).

A natural application area for set constraints is the analysis of logic programs. The idea was first explored by Mishra [Mis84]; more recently, this line of work has been well developed in a series of papers by Jaffar and Heintze [HJ90b, HJ90a, HJ92], as well as in Heintze's thesis [Hei92]. Many of the techniques developed in [Hei92] have been fruitfully applied to compile time analysis in other areas, especially the compile-time analysis of ML programs [Hei94].

6 Conclusions and Directions

Interest in set constraints originally arose from the needs of researchers working in program analysis. Currently, there is a lively, continuing interplay between the theoretical and practical efforts in the area. Future work is most likely to proceed along three lines. First, the open problems in Table 1 may be resolved; in particular, there is considerable interest in understanding the combination of projections and the basic language. Second, efforts to apply set constraints to new problems will lead to additional variations on the basic language. Third, there will be additional effort devoted to the efficient implementation of set constraint resolution algorithms. This is likely to include not only new engineering techniques, but also exploration of restricted classes of constraints for which good worst-case complexity

results can be obtained.

Besides a number of prototype or special purpose systems, there are currently two substantial, complete set constraint resolution implementations, one by Nevin Heintze at CMU [Hei92] and one by the author and colleagues at IBM. The latter implementation is available by anonymous ftp and comes with a type inference system for a functional language based on solving systems of set constraints [AWL94]. To get this system, retrieve the file `pub/personal/aiken/Illyria.tar.Z` from the machine `s2k-ftp.cs.berkeley.edu`.

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Experiences with Constraint-based Array Dependence Analysis

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1 Introduction

When two memory accesses refer to the same address, and at least one of those accesses is a write, we say there is a *data dependence* between the accesses. In this case, we must be careful not to reorder the execution of the accesses during optimization, if we are to preserve the semantics of the program being optimized. We therefore need accurate array data dependence information to determine the legality of many optimizations for programs that use arrays. Array dependence testing can be viewed as constraint analysis. For example, in Figure 1, determining whether or not any array element is both written by `A[i, j+1]` and read by `A[100, j]`, is equivalent to testing for the existence of solutions to the constraints shown on the right of the figure.

Since integer programming is an NP-complete problem, ([GJ79]), production compilers employ techniques that are guaranteed to be fast but give conservative answers: they might report a possible solution when no solution exists. We have explored the use of exact constraint analysis methods for array data dependence analysis. We have gone beyond simply checking for satisfiability of conjunctions of constraints to being able to manipulate arbitrary Presburger formulas. This has allowed us to address problems beyond traditional dependence analysis.

In our previous papers [Pug92, PW93a], we have presented timing results for our system on a variety of benchmark programs, and argued that our techniques are not prohibitively slow. In fact, using exact techniques to obtain standard kinds of dependence information requires about $1\% - 10\%$ of the total time required by simple workstation compilers that do no array data dependence analysis of any kind.

Our techniques are based on an extension of Fourier variable elimination to integers. Many other researchers in the constraints field [Duf74, LL92, Imb93, JMSY93] have stated that direct application of Fourier's technique is impractical because of the number of redundant constraints generated. We have not experienced any significant problems with Fourier elimination generating redundant constraints, even though we have not implemented methods suggested [Duf74, Imb93, JMSY93] to control this problem. We believe that our extension of Fourier elimination to integers is much more efficient that described by [Wil76].

In this paper, we summarize some of the constraint manipulation techniques we use for dependence analysis, and discuss some of the reasons for our performance results.

```
for i = 1 to n
  for j = i to n
    A[i, j+1] = A[n, j]
```

$1 \le i_w \le j_w \le n$ (write iteration in bounds)
$1 \le i_r \le j_r \le n$ (read iteration in bounds)
$i_w = n$ (first subscripts equal)
$j_w + 1 = j_r$ (second subscripts equal)

Figure 1: Dependence testing and associated constraints

2 The Omega Test

The Omega test [Pug92] was originally developed to check if a set of linear constraints has an integer solution, and was initially used in array data dependence testing. Since then, its capabilities and uses have grown substantially. In this section, we describe the various capabilities of the Omega test.

The Omega test is based on an extension of Fourier variable elimination [DE73] to integer programming. Other researchers have suggested the use of Fourier variable elimination for dependence analysis [WT92, MHL91b] but only as a last resort after exact and fast, but incomplete, methods have failed to give decisive answers. We proved [Pug92] that in cases where the fast but incomplete methods of Lam et al. [MHL91b] apply, the Omega test is guaranteed to have low-order polynomial time complexity.

2.1 Eliminating an existentially quantified variable

The basic operation of the Omega test is the elimination of an existentially quantified variable, also referred to as shadow-casting or projection. For example, given a set of constraints P over x, y and z that define, for example, a dodecahedron, the Omega test can compute the constraints on x and y that define the shadow of the dodecahedron. Mathematically, these constraints are equivalent to $\exists z$ s.t. P. But the Omega test is able to remove the existentially quantified variables, and report the answer just in terms of the free variables (x and y).

Over rational variables, projection of a convex region always gives a convex result. Unfortunately, the same does not apply for integer variables. For example, $\exists y$ s.t. $1 \leq y \leq 4 \wedge x = 2y$ has $x = 2$, $x = 4$, $x = 6$ and $x = 8$ as solutions. Sometimes, the result is even more complicated. For example, the solutions for x in:

$$\exists i, j \text{ s.t. } 1 \leq i \leq 8 \wedge 1 \leq j \leq 5 \wedge x = 6i + 9j - 7$$

are all numbers between 8 and 86 (inclusive) that have remainder 2 when divided by 3, except for 11 and 83.

In general, the Omega test produces an answer in disjunctive normal form: the union of a finite list of clauses. A clause may need to describe a non-convex region. There are two methods for describing these regions:

Stride format The Omega test can produce clauses that consist of affine constraints over the free variables and stride constraints. A stride constraint $c|e$ is interpreted as "c evenly divides e". In this form, the above solution could be represented as:

$$x = 8 \ \vee \ (\ 14 \leq x \leq 80 \wedge 3|(x + 1) \ ) \ \vee \ x = 86$$

Projected format Alternatively, the Omega test can produce clauses that consist of a set of linear constraints over a set of auxiliary variables and an affine 1-1 mapping from those variables to the free variables. Using this format, the above solution could be represented as

$$x = 8 \ \vee \ (\exists \alpha \text{ s.t. } 5 \leq \alpha \leq 27 \wedge x = 3\alpha - 1) \ \vee \ x = 86$$

These two representations are equivalent and there are simple and efficient methods for converting between them.

2.1.1 Our extension of Fourier elimination to integers

If $\beta \leq bz$ and $az \leq \alpha$ (where a and b are positive integers), then $a\beta \leq abz \leq b\alpha$. If z is a real variable, $\exists z$ s.t. $a\beta \leq abz \leq b\alpha$ if and only if $a\beta \leq b\alpha$. Fourier variable elimination eliminates a variable z by combining together all pairs of upper and lower bounds on z and adding the resulting constraints to those constraints that do not involve z. This produces a set of constraints that has a solution if and only if there exists a real value of z that satisfies the original set of constraints.

In [Pug92] and Figure 2 we show how to compute the "dark shadow" of a set of constraints: a set of constraints that, if it has solutions, implies the existence of an integer z such that the original set of constraints is satisfied. Of course, not all solutions are contained in the dark shadow.

Eliminate z from C, the conjunction of a set of inequalities
R = **False**
C' = **all constraints from** C **that do not involve** z
$C'' = C$
for each lower bound on z: $\beta \leq bz$
  for each upper bound on z: $az \leq \alpha$
    $C' = C' \wedge a\beta + (a-1)(b-1) \leq b\alpha$
    % **Misses** $a\beta \leq abz \leq b\alpha < a\beta + (a-1)(b-1)$
    % **Misses** $\beta \leq bz < \beta + \frac{(a-1)(b-1)}{a}$
  let $a_{\max}$ = **max coefficient of** z **in upper bound on** z
  for $i = 0$ **to** $((a_{\max} - 1)(b-1) - 1)/a_{\max}$ **do**
    $R = R \vee C \wedge \beta + i = bz$
% C' **is the dark shadow**
% R **contains the splinters**
% $C' \vee (\exists$ **integer** z **s.t.** $R) \equiv \exists$ **integer** z **s.t.** C

Figure 2: Extension of Fourier variable elimination to integers

For example, consider the constraints:

$$\exists y \text{ s.t. } 0 \leq 3y - x \leq 7 \wedge 1 \leq x - 2y \leq 5$$

Using Fourier variable elimination, we find that $3 \leq x \leq 27$ if we allow y to take on non-integer values. The dark shadow of these constraints is $5 \leq x \leq 25$. In fact, this equation has solutions for $x = 3, 5 \leq x \leq 27$ and $x = 29$.

In [Pug92] and Figure 2 we give a method for generating an additional sets of constraints that would contain any solutions not contained in the dark shadow. These "splinters" still contain references to the eliminated variable, but also contain an equality constraint (i.e., are flat). This equality constraint allows us to eliminate the desired variable exactly. For the example given previously, the splinters are:

$$\exists y \text{ s.t. } x = 3y \wedge 0 \leq 3y - x \leq 7 \wedge 1 \leq x - 2y \leq 5$$

$$\exists y \text{ s.t. } x + 1 = 3y \wedge 0 \leq 3y - x \leq 7 \wedge 1 \leq x - 2y \leq 5$$

$$\exists y \text{ s.t. } x - 5 = 2y \wedge y \text{ s.t. } 0 \leq 3y - x \leq 7 \wedge 1 \leq x - 2y \leq 5$$

Simplifying these produces clauses in projected form:

$$\exists y \text{ s.t. } x = 3y \wedge 1 \leq y \leq 5$$

$$\exists y \text{ s.t. } x = 3y - 1 \wedge 2 \leq y \leq 6$$

$$\exists y \text{ s.t. } x = 2y + 5 \wedge 5 \leq y \leq 12$$

2.2 Verifying the existence of solutions

The Omega test also provides direct support for checking if integer solutions exist to a set of linear constraints. It does this by treating all the variables as existentially quantified and eliminating variables until it produces a problem containing a single variable; such problems are easy to check for integer solutions. The Omega test incorporates several extensions over a naive application of variable elimination.

2.3 Removing redundant constraints

In the normal operation of the Omega test, we eliminate any constraint that is made redundant by any other single constraint (e.g., $x + y \leq 10$ is made redundant by $x + y \leq 5$). Upon request, we can use more aggressive techniques to eliminate redundant constraints. We use fast but incomplete tests that can flag a constraint as definitely redundant or definitely not redundant, and a backup complete test. This capability is used when verifying implications and simplifying formulas involving negation.

We also use these techniques to define a "gist" operator: informally, (gist P given Q) is what is "interesting" about P, given that we already know Q. We guarantee that $((\text{gist } P \text{ given } Q) \wedge Q) \equiv P \wedge Q$ and try to make the result of the gist operator as simple as possible. More formally, gist P given Q returns a subset of the constraint of P such that none of the constraints returned are implied by the constraints of Q and the other constraints in the result.

2.4 Simplifying formulas involving negation

There are two problems involved in simplifying formulas containing negated conjuncts, such as

$$-10 \leq i + j, i - j \leq 10 \ \wedge \ \neg(2 \leq i, j \leq 8 \wedge 2|i + j)$$

Naively converting such formulas to disjunctive normal form generally leads to an explosive growth in the size of the formula. In the worst-case, this cannot be prevented. But we [PW93a] have described methods that are effective in dealing with these problems for the cases we encounter. One key idea to to recognize that we can transform $A \wedge \neg B$ to $A \wedge \neg(\text{gist } B \text{ given } A)$. Given several negated clauses, we simplify them all this way before choose one to negate and distribute.

Secondly, previous techniques for negating non-convex constraints, based on quasilinear constraints [AI91], were discovered to be incomplete in certain pathological cases [PW93a]. We [PW93a] describe a method that is exact and complete for all cases.

2.5 Simplifying arbitrary Presburger formulas

Utilizing the capabilities described above, we can simplify and/or verify arbitrary Presburger formulas. In general, this may be prohibitively expensive. There is a known lower bound of $2^{2^{\Omega(n)}}$ on the worst case nondeterministic time complexity, and a known upper bound of $2^{2^{2^{O(n)}}}$ on the deterministic time complexity, of Presburger formula verification. However, we have found that we are able to efficiently analyze many Presburger formulas that arise in practice.

For example, our current implementation requires 12 milliseconds on a Sun Sparc IPX to simplify

$$\begin{gathered} 1 \leq \mathrm{i} \leq 2\mathrm{n} \ \wedge \ 1 \leq \mathrm{i}'' \leq 2\mathrm{n} \wedge \mathrm{i} = \mathrm{i}'' \\ \wedge \neg(\ \exists \mathrm{i}', \mathrm{j}' \text{ s.t. } 1 \leq \mathrm{i}' \leq 2\mathrm{n} \wedge 1 \leq \mathrm{j}' \leq \mathrm{n} - 1 \wedge \mathrm{i} \leq \mathrm{i}' \wedge \mathrm{i}' = \mathrm{i}'' \wedge 2\mathrm{j}' = \mathrm{i}''\ ) \\ \wedge \neg(\ \exists \mathrm{i}', \mathrm{j}' \text{ s.t. } 1 \leq \mathrm{i}' \leq 2\mathrm{n} \wedge 1 \leq \mathrm{j}' \leq \mathrm{n} - 1 \wedge \mathrm{i} \leq \mathrm{i}' \wedge \mathrm{i}' = \mathrm{i}'' \wedge 2\mathrm{j}' + 1 = \mathrm{i}''\ ) \end{gathered}$$

to

$$(1 = \mathrm{i} = \mathrm{i}'' \leq \mathrm{n}) \vee (1 \leq \mathrm{i} = \mathrm{i}'' = 2\mathrm{n}) \vee (1 \leq \mathrm{i} = \mathrm{i}'' \leq 2 \wedge \mathrm{n} = 1)$$

Related work

Other researchers have proposed extensions to Fourier variable elimination as a decision method for array data dependence analysis [MHL91a, WT92, IJT91]. Lam et al. [MHL91a] extend Fourier variable elimination to integers by computing a sample solution, using branch and bound techniques if needed. Michael Wolfe and Chau-Wen Tseng [WT92] discuss how to recognize when Fourier variable elimination may produce a conservative result, but do not give a method to verify the existence of integer solutions. These methods are decision tests and cannot return symbolic answers.

Corinne Ancourt and François Irigoin [AI91] describe the use of Fourier variable elimination for quantified variable elimination. They use this to generate loop bounds that scan convex polyhedra. They extend Fourier variable elimination to integers by introducing floor and ceiling operators. Although this makes their

elimination exact, it may not be possible to eliminate additional variables from a set of constraints involving floor and ceiling operators. This limits their ability to check for the existence of integer solutions and remove redundant constraints.

Cooper [Coo72] describes a complete algorithm for verifying and/or simplifying Presburger formulas. His method for quantified variable elimination always introduces disjunctions, even if the result is convex. We have not yet performed a head-to-head comparison of the Omega test with Cooper's algorithm. However, we believe that the Omega test will prove better for quantified variable elimination when the result is convex and better for verification of a formula already in disjunctive normal form. Cooper's algorithm does not require formulas to be transformed into disjunctive normal form and may be better for formulas that would be expensive to put into disjunctive normal form (although our methods for handling negation address this as well).

The SUP-INF method [Ble75, Sho77] is a semi-decision procedure. It sometimes detects solutions when only real solutions exist and it cannot be used for symbolic quantified variable elimination.

H.P. Williams [Wil76] describes an extension of Fourier elimination to integers. His scheme leads to a much more explosive growth than our scheme. If the only constraints involving an eliminated variable x are $L \leq lx$ and $ux \leq U$, his scheme produces $\mathrm{lcm}(l, u)$ clauses, while ours produces

$$1 + \left\lceil \frac{(l-1)(u-1)}{\max(l,u)} \right\rceil$$

clauses. If there are p lower bounds $L_i \leq l_i x$ and q upper bounds $u_j x \leq U_j$, Williams' method produces a formula that, when converted into disjunctive normal form, contains

$$\prod_{1 \leq i \leq p \wedge 1 \leq j \leq q} \mathrm{lcm}(l_i, u_j)$$

clauses, while the number of clauses produced by our scheme is

$$1 + \min\left(\sum_{1 \leq i \leq p} \left\lceil \frac{(l_i - 1)(\max(u_j) - 1)}{\max(u_j)} \right\rceil, \sum_{1 \leq j \leq q} \left\lceil \frac{(\max(l_i) - 1)(u_j - 1)}{\max(l_i)} \right\rceil \right)$$

For example, if the l_i's are $\{1, 1, 1, 2, 3, 5\}$ and the u_j's are $\{1, 1, 3, 7\}$, Williams' m d produces

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clauses, while ours produces 12. It is almost certainly possible to improve Williams' method while using the same approach as Williams, but we know of no description of such an improvement.

Jean-Louis Lassez [LHM89, LL92, HLL92] gives an alternative to Fourier variable elimination for elimination of existentially quantified variables. However, his methods work over real variables, are optimized for dense constraints (constraints with few zero coefficients) and are inefficient when the final problem contains more than a few variables since they build a convex hull in the space of variables remaining after all quantified variables have been eliminated.

3 Constraint Based Dependence Analysis

Array dependence testing can be viewed as constraint analysis. Simply testing for the existence of a dependence (as in Figure 1) is equivalent to testing for solutions to a set of constraints.

We can also use constraint manipulation to obtain information about the possible differences in the values of the corresponding index variables at the times of the two accesses (this information can be used to test for the legality of some program transformations). To do so, we introduce variables corresponding to these differences, and existentially quantify and eliminate all other variables. Alternatively, we can choose to eliminate everything but the symbolic constants, and thus determine the conditions under which the dependence exists ([PW92]).

Program to be analyzed:

```
for j = 0 to 20 do
  for i = max(-j,-10) to 0 do
    for k = max(-j,-10)-i to -1 do
      for l = 0 to 5 do
        a(l, i,j) = ...  a(l, k,i+j) ...
```

Constraints before equality substitution:

$$\exists j_w, i_w, k_w, l_w, j_r, i_r, k_r, l_r \text{ s.t.}$$

$$\Delta i = i_r - i_w \wedge \Delta j = j_r - j_w$$
$$\Delta k = k_r - k_w \wedge \Delta l = l_r - l_w$$

$$l_w = l_r \wedge i_w = k_r \wedge j_w = j_r + i_r$$

$$0 \le j_w \le 20$$
$$-10, -j_w \le i_w \le 0$$
$$-j_w - i_w, -10 - i_w \le k_w \le -1$$
$$0 \le l_w \le 5$$

$$0 \le j_r \le 20$$
$$-10, -j_r \le i_r \le 0$$
$$-j_r - i_r, -10 - i_r \le k_r \le -1$$
$$0 \le l_r \le 5$$

Constraints after equality substitution:

$$\exists j_r, l_w \text{ s.t.}$$
$$0 \le l_w \le 5$$
$$0 \le j_r \le 20$$
$$3\Delta j + 2\Delta i + \Delta k \le j_r$$
$$\Delta j \le j_r \le 20 + \Delta j$$
$$2\Delta j + \Delta i \le j_r$$
$$2\Delta j + 2\Delta i + \Delta k \le 10$$
$$1 \le \Delta j + \Delta i + \Delta k$$
$$1 \le \Delta j + \Delta i \le 10$$
$$0 \le \Delta j \le 10$$
$$2\Delta j + \Delta i \le 10$$
$$\Delta l := 0$$

Constraints after eliminating l_w and j_r:

$$2\Delta j + \Delta i \le 10$$
$$0 \le \Delta j \le 10$$
$$3\Delta j + 2\Delta i + \Delta k \le 20$$
$$2\Delta j + 2\Delta i + \Delta k \le 10$$
$$1 \le \Delta j + \Delta i + \Delta k$$
$$1 \le \Delta j + \Delta i \le 10$$
$$\Delta l := 0$$

Figure 3: Constraint-based dependence analysis

Figure 3 shows a relatively complicated example of constraint-based dependence analysis, from one of the NASA NAS benchmarks. Note that our techniques for eliminating equalities let us reduce both the number of variables and the number of constraints before resorting to Fourier elimination.

If we extend our constraint manipulation system to handle negated conjunctions of linear constraints, we can include constraints that rule out the dependences that are "killed" by other writes to the array, producing array data flow information ([PW93a]). The analysis tells us the source of the value read at any particular point; standard array data dependence tests just tell us who had previously written to the memory location read at any particular point. We have also found that our use of constraints to represent dependences is useful for other forms of program analysis and transformation ([Pug91, PW93b, KP93]).

4 Experiences

One of the main drawbacks of Fourier's method of variable elimination is the huge number of constraints that can be generated by repeated elimination, many of which could be redundant. Other researchers have found Fourier's technique to be prohibitively expensive [HLL92, Imb93] and have proposed either alternative methods for projection [HLL92] or methods to avoid generating so many redundant constraints [Imb93].

Our experiences have been exactly the opposite. We have found Fourier's method to be efficient, and do not experience substantial increases in the number of constraints. Our empirical studies have shown that Fourier's method can be used in dependence analysis without a significant impact on total compile time [Pug92, PW93a]. The average time required for memory-based analysis (as in Figure 1) was well under 1 millisecond per pair of references, and the average time for array data flow analysis a few milliseconds. These time trials were measured on a set of benchmarks that includes some of the NASA NAS kernels and some code from the Perfect Club Benchmarks ([B+89]).

We believe this speed is the result of several attributes of the sets of constraints we produce for dependence

Averages	when	# vars	kind	# of constraints involving 1 var	2 vars	3+ vars	total
	initial	5.6	as given	2.9	3.3	1.4	7.6
			nonredundant	2.0	2.1	0.9	5.0
	final	2.4	as generated	1.8	0.5	0.1	2.4
			nonredundant	1.2	0.3	0.07	1.6

a worst-case (but noncontrived) example encountered in benchmarks	when	# vars	kind	# of constraints involving 1 var	2 vars	3+ vars	total
	initial	5	as given	6	5	4	15
			nonredundant	4	2	3	9
	final	3	as generated	2	3	3	8
			nonredundant	1	2	2	5

Figure 4: Characteristics of constraint sets used in dependence analysis

analysis. First, loop bounds and array subscripts are often either constant or a function of a single variable. If all loop bounds and array subscripts have this form, all of our constraints will involve only one or two variables. Variable elimination is much less expensive within this restricted domain (known as LI(2)), even if we use the general algorithm. The number of constraints generated is bounded by a subexponential (though more than polynomial) function, rather than the $2^{n/2}$ of the general case [Cha93, Nel78].

Second, our constraints contain many unit coefficients. When the non-zero coefficients in a sparse set of constraints are all ± 1, projection ends up producing many parallel constraints, which can then be eliminated by our simple test for redundant constraints. Variable elimination in a LI(2) problem with unit coefficients preserves unit coefficients (after dividing through by the GCD of the coefficients). Under such situations, there cannot be more than $O(n^2)$ non-parallel constraints over n variables, and our method needs no more then $O(n^3)$ time to eliminate as many variables as desired [Pug92].

Finally, our constraint sets contain numerous equality constraints. Since we use these constraints to eliminate variables without resorting to projection, they help to keep down the size of the constraint sets that we must manipulate with Fourier's technique.

4.1 Empirical studies of dependence analysis constraints

We instrumented our system to analyze the types of constraints we deal with during dependence analysis. For each application of the Omega test, we analyzed the constraints that remained (a) after our initial removal of equality constraints and (b) after we had either eliminated all but two variables or run out of quantified variables to eliminate. In doing this analysis, we computed real shadows, as opposed to integer shadows (because the integer shadow may not be a simple conjunct). However, we still performed a number of other operations to rule out non-integer solutions (such as normalizing $2x + 4y \geq 3$ to $x + 2y \geq 2$).

When analyzing a set of constraints, we counted the number of variables, and counted (separately) the number of constraints that involved 1, 2 or 3+ variables. We then eliminated *all* redundant constraints, and recounted.

We performed these tests over our dataflow benchmark set [PW93a], which includes some of the NASA NAS kernels and some code from the Perfect Club Benchmarks ([B+89]). In total, we considered 1144 sets of constraints, and obtained the results shown in Figure 4.

Note that our methods always check for parallel constraints and eliminate the redundant one immediately (e.g., given $x + y \leq 5$ and $x + y \leq 10$, the second is eliminated). This can be done in constant time per constraint (through the use of a hash table).

Quite surprisingly, in *none* of the 1144 cases did the number of constraints increase as variables were eliminated (this is without any redundant constraint elimination other than elimination of parallel redundant

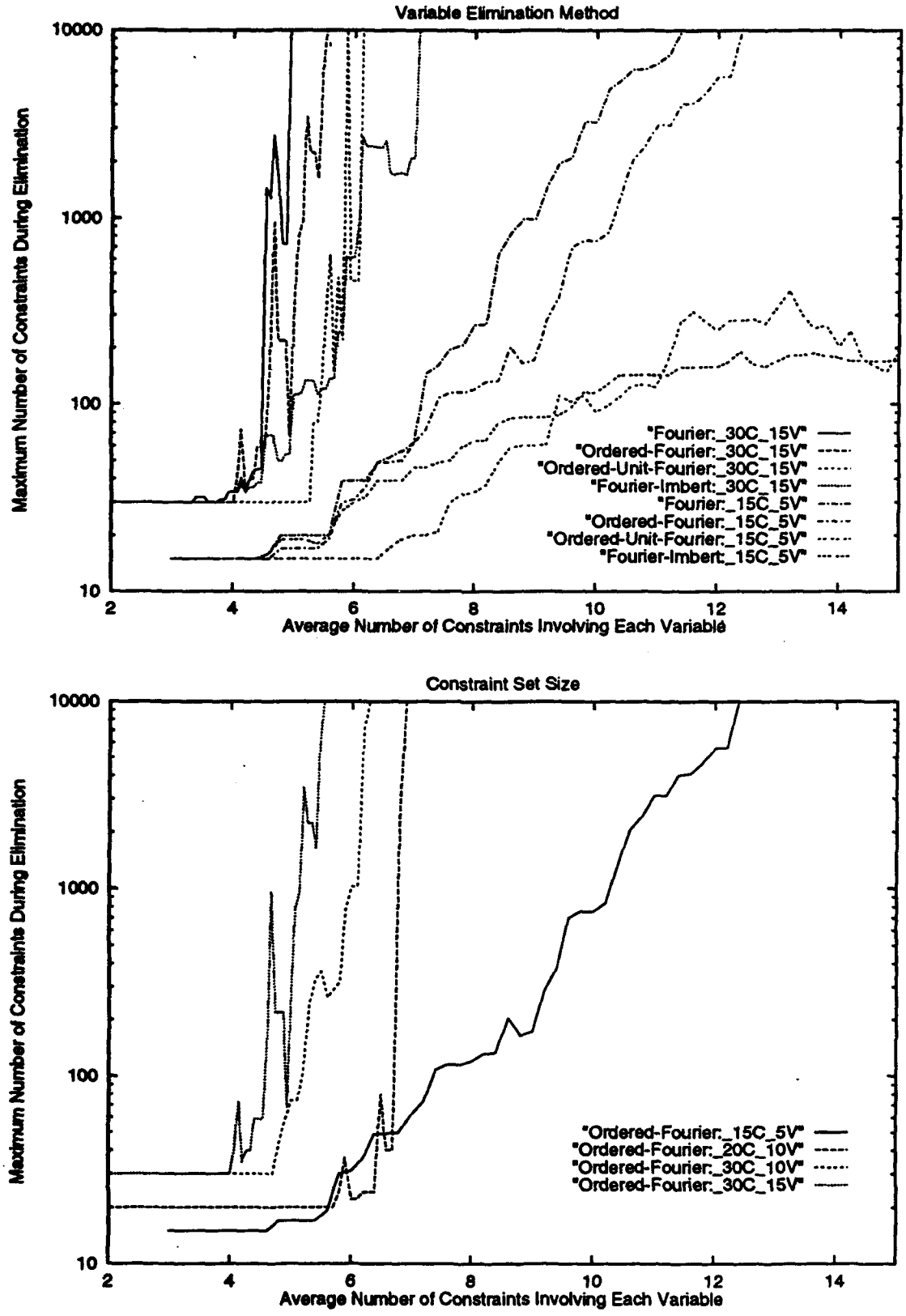

Figure 5: Factors that affect explosion of constraints

constraints).

4.2 Empirical studies of random constraints

To better understand the reasons for our good fortune in avoiding an explosion of constraints, we also studied the behavior of Fourier elimination, on sets of random constraints. Figure 5 shows the results of these studies.

In each experiment, we fixed the number of constraints and variables, added one random non-zero to each constraint. When then projected the constraints onto the first two variables, and recorded the maximum number of constraints encounted during the elimination. We then added an additional nonzero coefficient to the original set of constraints, and repeated the projection. We continued doing this until the problem had no non-zeros left. Each line represents the median of 5-21 experiement. The key gives the elimination method used, the number of initial constraints and the number of initial variables.

The top graph compares the effectiveness of several variations on Fourier's method:

Fourier Standard Fourier elimination of all variables in a random order

Ordered Fourier Standard Fourier elimination in which we choose to eliminate first the variable that produces the smallest increase in the number of constraints

Ordered Unit Fourier Ordered Fourier elimination combined with detection of parallel redundant constraints. For this test, values of the coefficients affect the performance; to show this technique at its best, we restrict the (non-zero) coefficients to ± 1.

Fourier-Imbert Fourier elimination combined with a partial application of Imbert's method [Imb93] of redundant constraint detection. We use Theorem 10 of [Imb93] to determine that some constraints are redundant. However, we do not use the more expensive comparison or matrical tests.

The lower four lines show the performance on sets of 15 constraints on 5 variables, as per the most extreme cases we encountered during dependence analysis. The upper four lines correspond to sets of 30 constraints on 15 variables.

Imbert's technique is clearly important for dense constraints, but until we approach seven constraints per variable, even standard Fourier elimination is well behaved for constraint sets of the sizes that we have encountered in our work with dependence analysis. At intermediate densities with unit coefficients, eliminating parallel constraints is more useful and important than computing historical subsets.

Note that the "worst case" example from Figure 4 started with 15 constraints over 5 variables and almost 6 constraints on each variable. As can be seen in Figure 5, this is just a little less complex than the point where Fourier elimination over unit coefficients starts to run into problems. We generally deal with constraint sets with fewer than three constraints per variable. In this region of the graph, the number of constraints does not grow with projection.

The graph on the bottom of Figure 5 shows the result of standard Fourier elimination with constraint sets of various sizes. Notice that, in all cases, the number of generated constraints does not become excessive as long as we start with an average of fewer than 4 constraints on each variable. Thus, we believe the Omega test could be useful for sparse problems that are significantly larger than those that arise in dependence analysis.

5 Conclusions

Other researchers [HLL92, Imb93] have been quite leary of Fourier variable elimination. These researchers have studied the effectiveness of Fourier variable elimination on sets of dense constraints. Our experience has lead us to believe that Fourier's method has quite different characteristics (and is quite efficient) when applied to sparse constraints. Furthermore, we believe that sparse constraints arise in many applications.

We have extended our work beyond Fourier variable elimination: first to handling variable elimination for integer variables, and then to simplifying arbitrary Presburger formulas. We hope these extensions may be of interest to a broader community.

6 Availability

Technical reports about the Omega test and an implementation of the Omega test are available via anonymous ftp from `ftp.cs.umd.edu:pub/omega` or the world wide web `http://www.cs.umd.edu/projects/omega`.

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Some Remarks on the Design of Constraint Satisfaction Problems

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Abstract

The development of a system based on constraint programming includes two main phases: first, the problem to be solved is formulated as a constraint satisfaction problem; then, the formulation is implemented in a constraint-programming language. Constraint-programming research has mainly concentrated on the second phase, by studying powerful declarative languages that automatically propagate the constraints in a program. Nevertheless, when developing a solution to a real-world problem, the cost due to the first one, design, is more relevant. This paper addresses the issue of effectively designing models of real-world constraint satisfaction problems.

1 Introduction

A constraint satisfaction problem (CSP) can be formulated as follows: given a set of variables and a set of constraints that limit the combination of values of the variables, find one assignment of values to the variables such that all the constraints are satisfied.

A large number of problems in many areas of computer science can be viewed as special cases of constraint satisfaction problems (for a survey see [Nad90]).

To solve more effectively this type of problems, several constraint-programming languages have been developed (for a survey see [Rot93]). The basic idea of these environments is to provide a declarative language where the programmer just defines variables and constraints, while the propagation engine of the language automatically computes the assignment of values that satisfies the constraints.

The author's Department at Bull developed a constraint-programming industrial environment, Charme [Opl89], and successively a number of real-world applications based on it [Cha94, DAn92, Gos93, MaT89, PMT92].

From these experiences, it emerged that the main cost of a constraint-based application is the design, rather than the implementation. It was also observed the lack of a methodology to define the model of the problem, possibly in tight collaboration with the end user.

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This paper discusses the limitations of the notion of CSP to represent real-world problems and extends it along the object-oriented paradigm to overcome those limitations; then, it outlines desirable features of a design methodology for constraint programming and sketches a design methodology that embodies such features for the extended notion of CSP's; finally, it revisits a classical example where the proposed methodology dramatically reduces the size of the model.

2. Constraint Satisfaction Problems

A *constraint satisfaction problem* is defined by a set $X_1,...,X_n$ of variables, each associated with a domain $D_1,...,D_n$ respectively, and a set $C_1,...,C_m$ of constraints, i.e., subsets of $D_1 x..x D_n$.

CSP's have extensively been studied to develop various types of consistency algorithms (for a survey, see [Kum92]). Nevertheless, when real-world problems are tackled, CSP's suffer from the following limitations:

① variables are semantically poor entities
② the only relations over variables are constraints
③ variables cannot be organized.

① It is the usual case that entities participating to a problem are characterized by more than one feature. For example, a task is characterized by its name, start time, duration, etc. Furthermore, several features can be initially unknown, such as the start time of a task and the machine assigned to it. This is not expressible in CSP's, because the only feature of a variable is its domain.

② As a consequence, all the relations of a CSP are over domains, i.e., they are constraints. On the contrary, when solving real-world problems it is necessary to define other relations (e.g. a given task employs a given resource) and confine the combinatorial component to parts of the problem.

③ When defining a problem, it is often useful to aggregate entities according to some criterion. This is not possible in CSP's, because variables are just gathered in a set, i.e. a flat structure, and they cannot be organized at any level of abstraction.

Each CSP can be graphically represented as a *constraint graph* in which nodes represent variables and edges represent constraints. In principle, constraint graphs could be employed to model CSP's. In practise, they are untractable for real-world problems, as mentioned in ③.

For instance, to represent a binary global constraint, the number of edges to be drawn grows as the square of the number of nodes, which is untractable when the size of the problem grows considerably. In general, it is possible to conclude that the constraint graph is not appropriate to model CSP's.

3. Enhancing Constraint Satisfaction Problems

To overcome the mentioned limitations, the definition of CSP is enhanced through concepts deriving from the object-oriented paradigm. The main difference is that here objects do not have methods (but just data members) since their state is updated by the constraints.

The solution that we propose relies on *abstraction*, i.e., recognizing similarities and concentrating on them. Abstractions are defined both for variables and constraints following the object-oriented paradigm. This leads to an enhanced model, called *object-oriented constraint satisfaction problem* (OOCSP).

An *attribute* is a feature of some type. Types are associated with domains. An *object* is a collection of attributes. Object attributes correspond to variables in CSP's. The set of attributes of an object defines the *structure* of the object. Objects sharing the same structure are grouped into *classes*. Classes are organized into a *hierarchy*. The structure of a lower class includes that of a higher class.

To distinguish associations over objects from those over classes, we call *associations* the former and *class associations* the latter. In particular, a (class) association is called (class) constraint if it is defined over (class) object attributes, otherwise it is called (class) relation. Constraints on object attributes have the same meaning as in CSP's, while constraints on classes induce constraints on objects.

A solution to an OOCSP is an assignment of domain values to object attributes such that all the constraints, including those induced, are satisfied. Each OOCSP can be graphically represented as an *object constraint graph*, as explained in the section 5.

4. Design

By design, we mean the creation of a *model* of the problem, as understood through analysis, consisting of abstractions and relationships that provide an architecture for implementation.

While it is quite well understood, for traditional software development, what the desirable features of a good design and its resulting model should be, this topic has not been much addressed in the field of constraint programming.

We would like the design of a constraint-based application to be

- problem driven
- methodological
- computer aided
- interactive.

Problem Driven. Design should concentrate on the essence of the problem in exam without influences from tangible components such as a target platform or language. It should be a straightforward activity for experts of the domain in exam, even with no computer skills.

Methodological. Design should follow a methodology, based on massive succesful experiences, aiming at identifying and ordering the main steps of the process.

Computer Aided. Design should be supported by tools that make it more effective and faster while possibly controlling it.

Interactive. Design should facilitate and stimulate the participation of domain experts

Desirable features of the model yielded by design are

- visual
- compact
- dynamic
- composable
- modular
- multi purpose
- reusable
- language independent
- executable.

Visual. Intuitive graphical formalisms should be employed to study and define the abstractions and relationships in the model.

Compact. The model should be compact and possibly provide different levels of abstractions.

Dynamic. The model should account for problems that change rapidly, for instance because their size grows.

Composable. It should be possible to compose models to define new bigger models with more functionalities.

Modular. The model should be decomposable into consistent views, each one focusing on a different aspect of the problem.

Multi purpose. It should be possible to employ the model to investigate different aspects of the problem, for instance by asking different questions.

Reusable. It should be possible to use the model as a starting point for new models of related problems.

Language independent. The model should be easily implemented in any constraint-programming language.

Executable. It should be possible to automatically generate code in a target constraint-programming language. In such a way, the model could be executed to provide immediate feedback on problem formulation.

5. Design Methodology

A methodology to design OOCSP's that embodies the features foreseen in the previous section is here sketched. It based on classical object-oriented design methodologies (for a survey see [Fow93]), and adapted to the constraint-programming domain. The methodology consists of

- a *notation* to represent the model
- a *process* to construct the model

The notation consists of graphical entities that are combined to generate object constraint graphs, the models of OOCSP's. Dotted boxes denote classes, while solid boxes denote objects. Edges denote associations and directed edges inheritance (Fig. 1). Classes and objects are represented on two different plans (classes in the upper level and objects in the lower one).

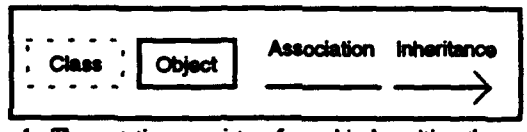

Fig. 1. The notation consists of graphical entities that are combined into object constraint graphs, the models of OOCSP's.

The process consists of the following four steps:

- identify classes and objects
- identify the semantics of these classes and objects
- identify the associations among these classes and objects
- identify the semantics of these associations.

The purpose of the first step is identifying classes and objects to establish the boundaries of the problem. The purpose of the second step is establishing the features of the abstractions identified at the previous step. The purpose of the third step is identifying the dependencies among abstractions. The fourth step formally specifies, through logical formulae, the meaning of the associations identified in the previous steps. This issue is futher discussed in the next section.

The process is incremental and iterative. It is incremental because when new classes, objects or associations are identified, existing classes, objects and associations can be refined and improved. It is iterative, because the definition of new classes, objects and associations often gives new insights on the problem that allow the user to simplify and generalize the design.

6. Executable Models

The models of OOCSP's can be directly executed: the edges of the object constraint graph are labeled with logical formulae defining the constraints of the problem at various levels of abstraction. Such formulae can be either executed by an interpreter of the object constraint graph, or preprocessed to a target constraint-programming language, not necessarily an object-orinted one. We abstract away from the syntactic details of such a language by taking a multi-sorted first-order logic in which the domain of interpretation for sorts has been fixed.

The logical formulae are obtained by connecting, through logical connectives, predicate symbols whose arguments are terms, i.e., constants, objects, classes, object attributes and class attributes as well as functions on terms. Formulae on classes induce formulae on objects, called *instances* of the given formula. They are obtained by replacing each class with its objects or the objects of its derived (through inheritance) classes, in all the possible combinations. A solution is an assignment of domain values to object attributes such that all the formulae, including those induced, are satisfied in the classical sense.

An object constraint graph can be preprocessed to an equivalent, i.e. with the same solutions, program written in any constraint-programming language. This is obtained by replacing each class association with its instances, so eliminating classes and hierachy. The resulting formulae on objects are the constraints of the program, where the object attributes are the variables.

7. The Bridge Problem

The Bridge Problem is a classical project-planning problem consisting of determining the starting dates of the tasks necessary to build a five-segment bridge. The project includes 46 tasks (*A1*, *P1*, etc.) that process 11 bridge components (abutment *Ab1*, pillar *Pl1*, etc.) and employ 7 resources (excavator *Ex*, concrete-mixer *CM*, etc.). The constraints of the problem include 77 disjunctive constraints (tasks *A1* and *A2* cannot overlap because they both employ the excavator, tasks *T2* and *T5* cannot overlap because they both employ the crane, etc.), 66 precedence constraints (execute task *T5* before task *V2*, execute task *M5* before task *T4*, etc.) and 25 specific constraints (the time between the completion of task *S1* and the completion of task *B1* is at most 4 days, the time between the completion of task *A4* and the completion of task *S4* is at most 3 days, etc.), for a total of 168 constraints. Our methodology is now employed to design the Bridge Problem.

Identify Classes and Objects. The basic class of the problem is *Task*. It has 14 subclasses (*Excavation*, *Foundation*, etc.), characterized by the fact that all the tasks in one of such classes employ the same resource. Each task of the problem is an object of the model.

Identify the Semantics of These Classes and Objects. Each task is characterized by six attributes: *name*, *start time*, *duration*, bridge *component* that it operates on, *resource* that it employs and set of tasks that come *before* it. The *start time* is the unknown to be determined: its domain is 0..200, meaning that the start time of each task is initially unknown, it will be automatically determined by the system, and it must be included between 0 and 200 days (an estimated upper bound).

Identify the Associations Among These Classes and Objects. The 77 disjunctive constraints and the 66 precedence constraints are expressed as just two class constraints, referred to as *Disjunction* and *Precedence* respectively. The 25 specific constraints are also expressed at class level and referred to as *K1-K5*.

Identify the Semantics of These Associations. The semantics of these associations is specified by the following formulae on classes:

Disjunction. IF (*Task1.resource* = *Task2.resource* AND *Task1.name* ≠ *Task2.name*) THEN (*Task1.start* + *Task1.duration* ≤ *Task2.start* OR *Task2.start* + *Task2.duration* ≤ *Task1.start*)

Precedence. IF (*Task1.name* ∈ *Task2.previous*) THEN (*Task1.start* + *Task1.duration* < *Task2.start*)

where *Task1* and *Task2* are two instances of the same class *Task*. The 60 specific constraints can also be expressed at class level:

K1. IF (*Formwork.part* = *Foundation.part*) THEN (*Foundation.start* + *Foundation.duration* - 4 ≤ *Formwork.start* + *Formwork.duration*)

K2. IF (*Excavation.part* = *Formwork.part*) THEN (*Formwork.start* - 3 ≤ *Excavation.start* + *Excavation.duration*)

K3. *Erection.start* ≤ *Formwork.start* - 6

K4. *Masonry.start* + *Masonry.duration* - 2 ≤ *Removal.start*

K5. *Delivery.start* = *Beginning.start* + 30

Model Complexity. The design of the Bridge Problem is complete (see Fig. 3). It consists of 1 base class, 14 derived classes, 46 objects and 7 class constraints. The fact that there are no constraints on objects means that the model is well conceived, because abstraction has been fully exploited to factorize common features. As a result, the 168 constraints of the problem are expressed with just 7 class constraints, a factor of 21 times. Other models can be obtained by defining resources and bridge components as objects, rather than task attributes. Those formulations are more complex in terms of constraints, but more explicit in terms of objects.

Preprocessing. The object constraint graph can be preprocessed to a target constraint language: each class constraint induces a set of object constraints, its instances, obtained by replacing class symbols by symbols of objects of that class or a derived class in all the possible combinations. For example, an instance of *Resource* is

IF (*A3.resource* = *A4.resource* AND *A3.name* ≠ *A4.name*)
THEN (*A3.start* + *A3.duration* ≤ *A4.start* OR *A4.start* + *A4.duration* ≤ *A3.start*)

where *A3* and *A4* are object symbols of the *Excavation* class, derived from *Task*; an instance of *K1* is

IF (*S2.part* = *B5.part*)
THEN (*B5.start* + *B5.duration* - 4 ≤ *S2.start* + *S2.duration*)

where *S2* is an object of the *Formwork* class and *B5* is an object of the *Foundation* class.

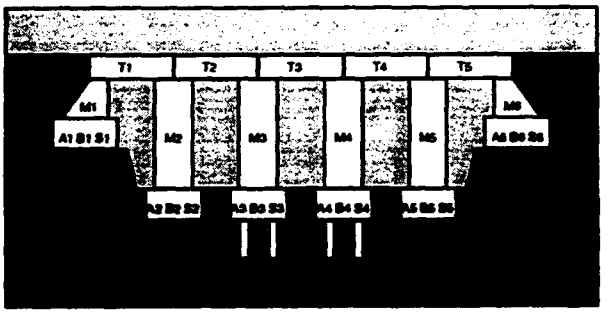

Fig. 2. The five-segments bridge.

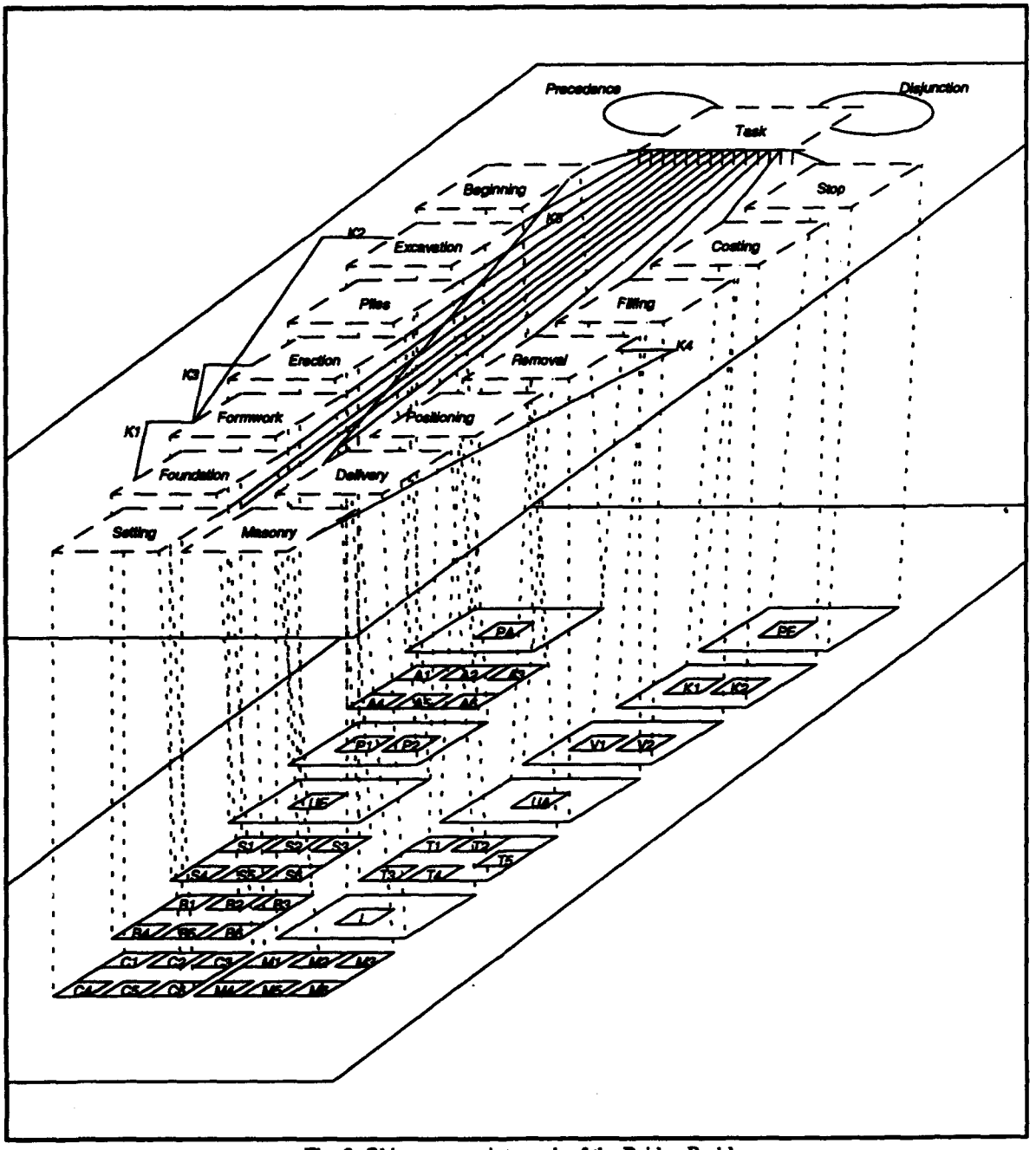

Fig. 3. Object constraint graph of the Bridge Problem.

8. Discussion

This work originates from the observation of the lack of a framework where

- CSP's are effectively designed through
- a visual methodology producing
- models directly executable by traditional constraint programming languages.

A design methodology for CSP's has been sketched and illustrated on an example. It is object oriented, to abstract variables and constraints, and visual, to be more intuitive and effective. The possibility of executing models produced through the methology has also been discussed.

Constraint-based systems integrating either visual or object-oriented components are: Sketchpad [Sut63], a pioneering visual system developed at the beginning of the sixties allowing the user to build geometric objects from language primitives and certain constraints; ThingLab [Bor79], providing users with a set of tools to help them graphically represent simulations and "experiments" in a constraint-oriented environment; Socle [Har86], an hybrid system that contains a structured partitioning component and a constraint component; Garnet [MGV92], to create large-scale user interfaces combining pre-defined objects into collections; Kaleidoscope [FBB92], integrating the declarative nature of constraints with the imperative nature of object-oriented languages; Ilog-Solver [Pug92], a C++ library of classes defining variables, constraints and algorithms.

Future work concerns the development of a tool supporting the methodology [Lel93]. To facilitate the development of applications, it should be provided with a *library of models* for different domains. The library can be organized on different levels: the first level includes models for generic domains, such as project planning, scheduling, finance, resources management, etc.; the second level includes models for more specific domains, such as construction-project management and software-project management for the project planning domain; production scheduling and meeting scheduling for the scheduling domain, etc. To solve a problem, the user selects the appropriate model and customizes it by adding or deleting nodes and edges.

Acknowledgements

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Logic-Based Methods for Optimization *

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Abstract

This paper proposes a logic-based approach to optimization that combines solution methods from mathematical programming and logic programming. From mathematical programming it borrows strategies for exploiting structure that have logic-based analogs. From logic programming it borrows methods for extracting information that are unavailable in a traditional mathematical programming framework. Logic-based methods also provide a unified approach to solving optimization problems with both quantitative and logical constraints.

1 Introduction

The theory and practice of integer and mixed integer programming are based primarily on polyhedral methods. The thesis of this paper is that one can develop a parallel theory and practice using logic-based methods.

The basic idea is to replace the essential elements of optimization methods with logical analogs. The integer variables are regarded as atomic propositions, and inequality constraints involving them are rewritten as logical formulas. In a branch-and-cut scheme, discrete relaxations replace the tradition linear programming and Lagrangian relaxations, and they are solved by logic-based algorithms. Logical implications replace cutting planes. In particular, "prime" and other strong "logic cuts" replace facet-defining cuts. Separating cuts, Gomory cuts, etc., also have analogs. Much of the theory of cutting planes, duality, etc., has a logical counterpart.

This approach can combine some of the problem-solving wisdom accumulated by mathematical programmers with techniques and insights from constraint programming and logic programming. Most importantly, the optimization community's ways of exploiting structure (strong cutting planes, etc.) carry over into a logical context. They may also take on greater variety and adaptability when moved out of the polyhedral context. Strong cuts are traditionally found by studying the abstract polyhedral structure of a model. But strong logic cuts can often be found by using one's intuitions about the concrete application of a model, even in cases where the polyhedron is far too complex to analyze. There is also a much greater variety of problem relaxations in the logical context.

The logical tradition also makes a key contribution. Logic processing can make more effective use of cuts, once they are discovered, than the traditional mathematical programming methods. A branch-and-bound method typically solves a relaxation of the constraint set generated at a given node of the search tree and may thereby fail to recognize when it is infeasible. An appropriate constraint propagation or logical inference technique may detect infeasibility and avoid the generation of successor nodes. The rapid speedup of propositional satisfiability algorithms over the past few years makes logic processing of this sort increasingly attractive.

So the logic-based methods described here go beyond both mathematical programming and logic programming. They enrich logic programming with strategies for discovering structure that paral-

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lel those of mathematical programming. They enrich mathematical programming with methods for extracting information that are supplied by logic and constraint programming.

This paper is a condensation of a longer tutorial on logic-based methods [7]. Its main contributions are to show in general how solution strategies for integer and mixed integer programming can be given logical analogs, and to outline a research program in this direction. To do this it draws on a number of results established elsewhere [6, 8, 9] and presents at least two new results, those of logical duality and the logical analysis of nonbipartite matching problems.

2 Historical Context

If logic-based methods for optimization are so attractive, why have they not gained acceptance already? Actually there is nothing new about them. Hammer and Rudeanu wrote a classic 1968 treatise [5] on boolean methods in operations research. Granot and Hammer [4] showed in 1971 how boolean methods might be used to solve integer programming problems.

Although boolean methods have seen applications (logical reduction techniques, solution of certain combinatorial problems), they have not been accepted as a general-purpose approach to optimization. There seem to be two main reasons for this. One is that they have not been demonstrated to be more effective than branch-and-cut. So there has been no apparent advantage in converting a problem to logical form.

A second reason is that the conversion to a logical problem is itself hard. The most straightforward way to convert an inequality constraint to logical form, for instance, is to write it as an equivalent set of logical clauses. But the number of clauses can grow exponentially with the number of variables in the inequality. Consider for instance the following constraint from a problem in Nemhauser and Wolsey ([11], p. 465).

$$\begin{aligned} &300x_3 + 300x_4 + 285x_5 + 285x_6 + 265x_8 \\ &+265x_9 + 230x_{12} + 230x_{13} + 190x_{14} \\ &+200x_{22} + 400x_{23} + 200x_{24} + 400x_{25} \\ &+200x_{26} + 400x_{27} + 200x_{28} + 400x_{29} \\ &+200x_{30} + 400x_{31} \leq 2700. \end{aligned} \quad (1)$$

Barth [1] reports that this constraint expands to 117,520 nonredundant logical clauses, using the method of Granot and Hammer [4].

So for several years prospects for logic-based methods, as a general approach to optimization, looked bleak. But several factors have recently converged to make them much more attractive. As noted earlier, satisfiability algorithms, a key element of logic-based methods, have improved dramatically. Also it is foolish to expand an inequality constraint into its full logical equivalent. This is analogous to generating all possible cutting planes for an integer programming problem, which is never done. Practical algorithms generate a few "separating cuts," and a closely analogous approach is available in the logical context.

Further, there is a growing trend toward the merger of quantitative and logical elements into a single model, and logic-based methods are a natural approach to solving such models. Purely mathematical models (integer programming, etc.) are often unsuitable for messy problems without much mathematical structure, whereas pure logic models (PROLOG programs, etc.) do not capture the mathematical structure that does exist and are consequently hard to solve. Historically, solution techniques for the two types of models have been unrelated. A technique that solves both opens the door to a wider variety of tractable models.

Logic-based optimization also serves a heuristic function of providing a whole new perspective on optimization problems. In fact, it is in some ways more natural to view a pure integer programming problem as a logical inference problem rather than a polyhedral problem. Similarly, the integer variables of a mixed integer problem can be viewed as artificial devices that can just as well be eliminated.

3 Integer Programming as Logical Inference

A 0-1 inequality $bx \geq \beta$ can be viewed as a logical proposition that is true when the inequality is satisfied. The variables x_j are viewed as atomic propositions that are true when $x_j = 1$ and false when $x_j = 0$. A system of 0-1 inequalities $Ax \geq a$ implies $bx \geq \beta$ when all 0-1 solutions of the former

satisfy the latter. Any logical proposition, inequality or otherwise, implied by $Ax \geq a$ is a *logic cut*. The following are obvious but fundamental.

Theorem 1 *An inequality is a valid cut (in the polyhedral sense) for a system of inequalities if and only if it is a logic cut.*

Theorem 2 *Consider an integer programming problem*

$$\begin{aligned} \min \quad & cx \\ s.t. \quad & Ax \geq a \\ & x_j \in \{0,1\}, \quad all\, j. \end{aligned} \tag{2}$$

The optimal value of the objective function is the largest β for which $cx \geq \beta$ is a logic cut.

This fact can be framed as a duality relationship. The following is the *logical dual* of integer programming problem (2).

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & Ax \geq a \text{ implies } cx \geq \beta \end{aligned} \tag{3}$$

The optimal value β in (3) is equal to the optimal value of (2). There is a close connection with linear programming duality, which is obtained by replacing $x_j \in \{0,1\}$ with $0 \leq x_j \leq 1$ in (2) and 'implies' with 'implies as a nonnegative linear combination' in (3).

4 A Generic Branch-and-Cut Algorithm

Figure 1 contains a rudimentary logic-based branch-and-cut algorithm (essentially a specialized A*
search) that solves the integer programming problem (2). It combines three strategies that have proved much more effective in combination than when used separately: an enumeration tree, generation of valid separating cuts, and solution of relaxations of the problem.

Note that the problem is not solved subject to the original constraint set $Ax \geq a$ but to a set S of logic cuts (perhaps logical clauses) for these constraints. The cuts are generated only as needed.

Nodes of the search tree are obtained by branching on the cases $x_j = 1, x_j = 0$. At each node, an optimization problem with a relaxed constraint set is solved to obtain a lower bound on the optimal value of the original problem. If this bound is already greater than the value of a feasible solution obtained earlier, there is no point in generating successor nodes. Classically the relaxations are usually linear (replace $x_j \in \{0,1\}$ with $0 \leq x_j \leq 1$) or Lagrangean, but a wide variety of discrete relaxations are possible in the logical setting. The simplest is to minimize cx subject to each clause in S separately (a trivial problem) and pick the best bound so obtained.

Finally, logic processing is applied to make explicit some constraints that were only explicit. Traditionally this has been achieved by generating valid inequalities (cuts) with coefficients chosen so that the linear relaxation is as tight as possible, preferably a facet of the convex hull of 0-1 solutions. In a logic-based setting, logic processing can be applied either in the form of a satisfiability algorithm or a cut generation algorithm, or both. The former would normally be an incomplete procedure, such as unit resolution (which happens to be equivalent in deductive power to solving the traditional linear relaxation). The latter would generate *separating* logic cuts, which are those that are violated by the solution just obtained for the current relaxation. Coefficients are no longer relevant, but the logic cuts should be strong (i.e., exclude as many 0-1 solutions as possible).

5 Strong Cuts

The logical analog of a facet-defining cut is a *prime cut*, which is defined with respect to a class C of logical propositions. A prime cut for a system $Ax \geq a$ of inequalities is a logic cut F that is equivalent to any cut in C that is implied by $Ax \geq a$ and implies F. It is a *prime inequality* if C is the set of all inequalities (with integer coefficients and right-hand side).

Useful logic cuts in practice need not and ordinarily would not be prime cuts. But an investigation of of how prime cuts can in principle be generated provides insight into the nature of strong logic cuts.

A fundamental result of integer programming,

Figure 1:

```
Logic-Based Branch-and-Cut Algorithm.
   Set UB=∞.
   Execute Branch(∅,0).
End.

Procedure Branch(S,k)
   If k = 0 then
      the optimal solution is the best found so far
      (infeasible if none found); stop.
   Apply a partial or complete satisfiability algorithm to S.
   If no contradiction is found then
      Find the minimum LB of cx subject to a relaxation of S.
      If LB<UB then:
         Generate separating logic cuts.
         Branch:
            Pick a literal L containing a variable
               that occurs in S.
            Perform Branch(S ∪ {L}, k + 1).
            Perform Branch(S ∪ {¬L}, k + 1).
End.
```

due to Chvátal [3], says that a finite procedure generates all facet-defining inequalities (the strongest cutting planes) for a 0-1 system $Ax \geq a$. A parallel result can be proved for logic-based programming [6]. Let a *clausal* inequality have the form $ax \geq 1 + n(a)$, where each $a_j \in \{0, 1, -1\}$ and $n(a)$ is the sum of the negative components of a. For instance, the inequality $x_1 + (1 - x_2) \geq 1$, or $x_1 - x_2 \geq 0$, represents the logical clause $x_1 \vee \neg x_2$. A *resolvent* of two clausal inequalities is simply the clausal inequality that represents the resolvent of the corresponding clauses. Let a *diagonal sum* be defined as illustrated by the following example.

$$\begin{aligned} x_1 + 5x_2 + 3x_3 + x_4 &\geq 4 \\ 2x_1 + 4x_2 + 3x_3 + x_4 &\geq 4 \\ 2x_1 + 5x_2 + 2x_3 + x_4 &\geq 4 \\ 2x_1 + 5x_2 + 3x_3 &\geq 4 \\ 2x_1 + 5x_2 + 3x_3 + x_4 &\geq 5 \end{aligned}$$

The fifth inequality is the diagonal sum of the first four. Note that the first four inequalities are identical except that the diagonal term is reduced by one. Also the right-hand side of the sum is increased by one.

A resolvent can be "generated" from a set S of inequalities if it is a resolvent of two clausal inequalities, each of which is implied by a single inequality of S. A diagonal sum is "generated" in a similar sense. Finally, let a set T of inequalities be *monotone* when T contains all clausal inequalities, and for any given inequality $ax \geq \beta + n(a)$ in T, T contains all inequalities $a'x \geq \beta' + n(a')$ such that $|a'| \leq |a|$ and $0 \leq \beta' \leq \beta$.

Theorem 3 *Let T be a monotone set of inequalities, and let S contain all resolvents and diagonal sums in T in that can be recursively generated from a feasible 0-1 system $Ax \geq a$, up to equivalence. Then every prime inequality for $Ax \geq a$ with respect to T is equivalent to some inequality in S.*

The *rank* of a logic cut (analogous to the Chvátal rank of a polyhedral cut) is the minimum number of iterations of this recursive procedure required to generate the cut.

6 Example: Matching Problems

Logic cuts can be stronger and therefore more useful than facet-defining cuts.[1] A good illustration

[1] This section represents joint work with Ajai Kapoor.

of this is a nonbipartite matching problem. The augmenting paths traditionally used in the best matching algorithms [11] in effect rely on logic cuts that strictly imply the less useful facet-defining inequalities (odd-set constraints) for the problem.

A matching problem is defined on an undirected graph (V, E) for which each edge in E is given a weight. The edges connect vertices that may be matched or paired, and a *matching* pairs some or all of the vertices. A matching can therefore be regarded as a set of edges, at most one of which touches any given vertex. The *weighted matching problem* is to find a maximum weight matching; i.e., matching that maximizes the total weight of the edges used in the matching.

The matching problem can be written,

$$\max \sum_{e \in E} x_e \tag{4}$$

$$\text{s.t.} \sum_{e \in \delta(v)} x_e \leq 1, \text{ for } v \in V \tag{5}$$

$$x_e \in \{0,1\}, \ e \in E,$$

where $\delta(v)$ is the set of edges incident to v. x_e is 1 when e is part of the matching and 0 otherwise.

The convex hull of possible matchings has a particularly simple description. It is based on the fact that a matching for a graph (U, E) with an odd number of vertices can have at most $\frac{|U|}{2}$ edges. So the following *odd set constraints* are valid:

$$\sum_{e \in E(U)} x_e \leq \frac{|U|}{2}, \text{ all } U \subset V \text{ with } |U| \geq 3 \text{ and odd}, \tag{6}$$

where $E(U)$ contains the edges in the subgraph of (V, E) induced by U. In fact (5)-(6) define the convex hull of matchings.

For the purposes of logical analysis it is conveninent to reverse the sense of the matching constraints (5) and odd set constraints (6) by replacing variables x_e with $y_e = 1 - x_e$, so that $y_e = 1$ when edge e is absent from the matching.

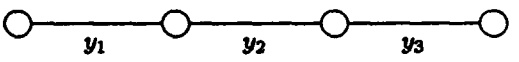
$$\sum_{e \in \delta(v)} y_e \geq |\delta(v)| - 1, \text{ for } v \in V \tag{7}$$

$$\sum_{e \in E(U)} y_e \geq |E(U)| - \frac{|U|}{2},$$

$$\text{all } U \subset V \text{ with } |U| \geq 3 \text{ and odd}. \tag{8}$$

The following is proved in [7].

Figure 2: A very small matching problem.



Theorem 4 *An odd set constraint (7) for a matching problem is a logic cut of rank at most*

$$|E(U)| - \frac{|U|}{2} - 1.$$

Odd set constraints are strictly implied by *augmenting path cuts*. Consider a matching problem on the simple graph of Fig. 6. The odd set constraints (facet-defining cuts) are simply the matching constraints $y_1 + y_2 \geq 1$ and $y_2 + y_3 \geq 1$. They are strictly implied by the augmenting path cut $y_1 + 2y_2 + y_3 \geq 2$, which says that either edge 2 is not in the matching or else edges 1 and 3 are not in the matching.

In general a path of odd length whose edges correspond to $y_{j_1}, \ldots, y_{j_m}$ defines an augmenting path cut,

$$\tfrac{m-1}{2} y_{j_1} + \tfrac{m+1}{2} y_{j_2} + \tfrac{m-1}{2} y_{j_3} + \ldots + \tfrac{m+1}{2} y_{j_{m-1}} + \tfrac{m-1}{2} y_{j_m} \geq \tfrac{(m-1)(m+1)}{2},$$

which says that if the $(m+1)/2$ odd segments belong to a matching, then none of the $(m-1)/2$ even segments may belong to it, and vice-versa.

7 Mixed Integer Programming

Consider a general mixed integer programming (MIP) problem,

$$\min \ cx + dy \tag{9}$$

$$\text{s.t.} \ Ax + By \geq a$$

$$y_j \in \{0,1\}, \ j = 1, \ldots, n,$$

A 0-1 point y is *feasible* if (x, y) is feasible for some x. Each 0-1 value of y is associated with a polyhedron $\Pi(y)$ in x-space, namely the set of points

satisfying (9) when y is so fixed. The feasible region can therefore be regarded as the union of $\Pi(y)$ over all feasible y.

To write an MIP in logical form, regard the y_j's as atomic propositions.

$$\begin{aligned} \min \quad & cx + dy && (10)\\ \text{s.t.} \quad & y \in Y \\ & x \in \bigcup_{y \in Y} \Pi(y), \end{aligned}$$

Here $y \in Y$ represents a set of logical propositions. (10) is actually more general than (9), due to a theorem of Jeroslow [9, 10]. It states that (10) can be written in the form (9) if and only if the polyhedra $\Pi(y)$ all have the same recession cone.

An MIP in form (10) can be solved by a branch-and cut algorithm that enumerates linear programming constraint sets defining $\Pi(y)$'s, where the enumeration is controlled by the logical propositions $y \in Y$. The enumeration can be markedly accelerated by the use of an expanded sense of logic cuts that obtain in an MIP setting, namely a *nonvalid* logic cut. These may cut off feasible solutions but do not change the optimal solution.

8 An MIP Example

Suppose one wants to decide which of three processing units to install in the processing network of Fig. 3. The units are represented as boxes. Naturally one must install unit 3 if the network is to process anything, and one must install units 1 or 2. Let's suppose in addition that units 1 and 2 should not both be installed. There is a variable cost associated with the flow through each unit, a fixed cost with building the unit, and revenue with the finished product. If x_j's represent flows as indicated in Fig. 13 and y_j's are 0-1 variables indicating which units are installed, the problem has the following MIP model.

$$\begin{aligned} \min \quad & 3x_3 + 2.8x_5 - 9x_7 + 2x_1 + \\ & \quad z_1 + z_2 + z_3 && (11)\\ \text{s.t.} \quad & x_1 - x_2 - x_4 = 0 && (12)\\ & x_6 - x_3 - x_5 = 0 && (13)\\ & x_3 - 0.9x_2 = 0 && (14)\\ & x_5 - 0.85x_4 = 0 && (15)\\ & x_7 - 0.75x_6 = 0 && (16)\\ & x_7 \le 10 && (17)\\ & x_3 - 30y_1 \le 0 && (18)\\ & x_5 - 30y_2 \le 0 && (19)\\ & x_7 - 50y_3 \le 0 && (20)\\ & y_1 + y_2 \le 1 && (21)\\ & z_1 = 14y_1 && (22)\\ & z_2 = 12y_2 && (23)\\ & z_3 = 10y_3 && (24)\\ & x_j \ge 0, \text{ all } j \\ & y_1, y_2, y_3 \in \{0, 1\}. \end{aligned}$$

Constraints (12)-(13) are flow balance constraints. (14)-(16) specify yields from the processing units. (17) bounds the output. (18)-(20) are "Big M" constraints that prohibit flow through a unit unless it is built. (22)-(24) define the fixed costs.

A conventional branch-and-bound tree for this problem appears in Fig. 4. Note that the optimal solution is to build none of the units.

9 Logic-Based Solution of an MIP

I will now illustrate how logic-based branch-and-bound can solve an MIP problem in logical form. It is convenient to suppose that the objective function of (10) is simply cx. This can be done by introducing a continuous variable z_j for each $d_j \neq 0$, letting the z_j have coefficient 1 in the objective function, and augmenting $\Pi(y)$ with the constraint $z_j = d_j$ whenever $y_j = 1$. The generic algorithm appears in Fig. 5.

The example of the previous section is put in logical form as follows. Note first that the objective function is already of the form cx. The set S of logical constraints is simply $\{\neg y_1 \vee \neg y_2\}$, which corresponds to constraint (21). The linear constraint set $\Pi(y)$ consists of constraints (12)-(17), nonnegativity constraints, and the following:

$$\begin{array}{llll} x_3 = 0 & \text{if } y_1 = 0, & z_1 = 14 & \text{if } y_1 = 1 \\ x_5 = 0 & \text{if } y_2 = 0, & z_2 = 14 & \text{if } y_2 = 1 \\ x_7 = 0 & \text{if } y_3 = 0, & z_3 = 14 & \text{if } y_3 = 1 \end{array} \qquad (25)$$

Figure 3: A simple processing network.

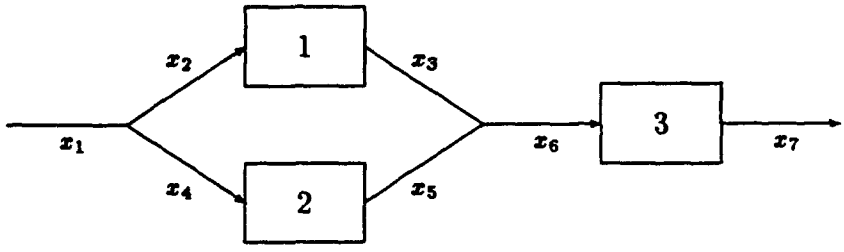


Figure 4: Branch-and-bound solution of a small mixed integer programming problem.

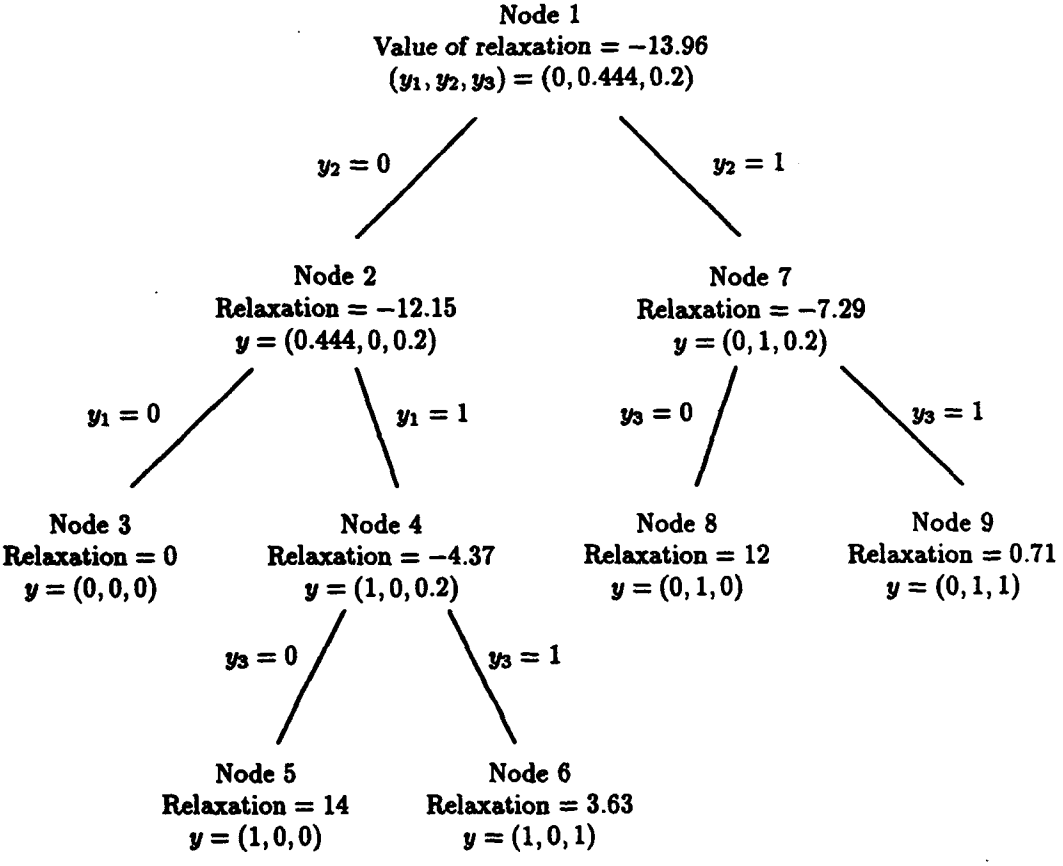


Figure 5:

```
Logic-Based Branch-and-Cut Algorithm for MIP.
   Set UB=∞, y = (u,...,u) (where u =undetermined).
   Execute Branch(∅, y, 0).
End.

Procedure Branch(S, k)
   If k = 0 then
      the optimal solution is the best found so far
      (infeasible if none found); stop.
   Apply a partial or complete satisfiability algorithm to
      S, fixing some variables in y if possible.
   If no contradiction is found, then
      Find the minimum LB of cx subject to x ∈ Π(y).
      If LB<UB then:
         Generate separating logic cuts.
         Branch:
            Pick a literal L containing a variable
               that occurs in S.
            Perform Branch(S ∪ {L}, y, k + 1).
            Perform Branch(S ∪ {¬L}, y, k + 1).
End.
```

Note that $\Pi(y)$ is defined even when some components of y are undetermined ($y_j = u$).

Before solving this example, it is useful to introduce in the next section some additional logic cuts.

10 Nonvalid Logic Cuts

In the context of mixed integer programming it is useful to define a more general sense of logic cut. Let the *graph* G for a mixed integer optimization problem (10) be the set

$$\{(cx + dy, x, y) \mid y \in Y, x \in \bigcup_{y \in Y} \Pi(y)\}.$$

The *epigraph* E is

$$\{(z, x, y) \mid (z', x, y) \in G \text{ for some } z' \leq z\}.$$

The *projection* of the epigraph onto the space of continuous variables is

$$\{(z, x) \mid (z, x, y) \in E \text{ for some } y\}.$$

A *logic cut* in the extended sense is a constraint $y \in T$ that, when added to the constraint set of (10), results in the same projected epigraph. The cut is *valid* if

$$\bigcup_{y \in Y} \Pi(y) = \bigcup_{y \in Y \cap T} \Pi(y).$$

A cut can be nonvalid (i.e., cut off feasible values of y), but it never changes the value of the optimal solution.

Some nonvalid logic cuts can be generated for the example of the previous section as follows. Note that it makes no sense to consider a solution in which a unit is installed but carries no flow. Yet such solutions can and do occur in the branch-and-bound tree. Nodes 5 and 8 of Fig. 4 have LP solutions in which the installed unit carries no flow. Computational experience [8, 12] suggests that such superfluous nodes can be very numerous in a branch-and-bound tree.

This situation can be prevented by adding constraints that allow a unit to be installed only if a downstream unit is installed:

$$\neg y_1 \vee y_3 \tag{26}$$

$$\neg y_2 \vee y_3 \tag{27}$$

and only if at least one upstream unit is installed:

$$y_1 \vee y_2 \vee \neg y_3. \tag{28}$$

These are nonvalid logic cuts because they cut off feasible values of (y_1, y_2, y_3). It is shown in [8] that they essentially exhaust the nonvalid logic cuts for such a problem.

Figure. 2 displays the search tree for a logic-based solution of the example that uses (26)-(28). Note that the tree is smaller than the branch-and-bound tree of Fig. 4. The superfluous nodes 5 and 8, as well as other nodes, have been deleted.

Logic-based methods have been applied to MIP models of chemical processing network design problems [8]. They solve larger problems substantially more rapidly than a state-of-the-art MIP solver with preprocessor (OSL), and in some cases solve problems that OSL cannot solve. Logic cuts are also being applied to truss structure design problems with discrete bar sizes [2].

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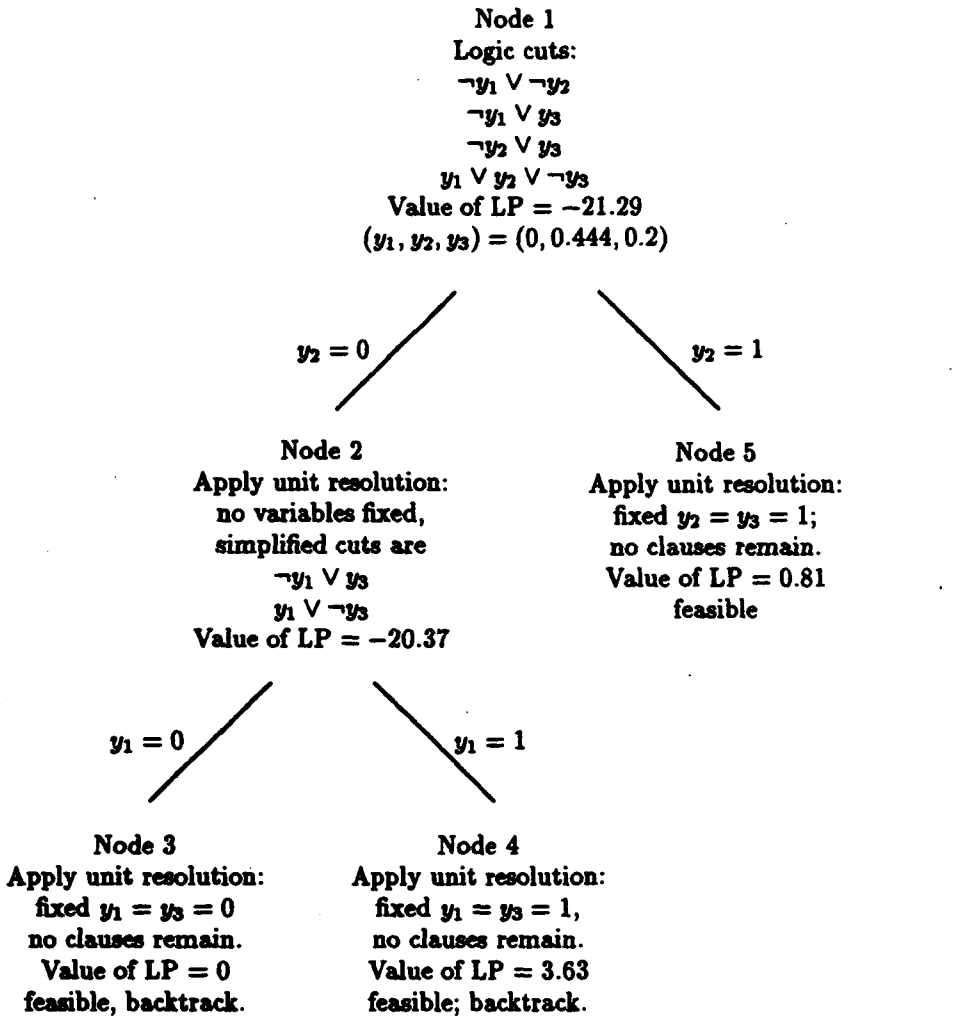
Figure 6: Logic-based solution of the problem with nonvalid logic cuts.



Specification and Verification of Constraint-Based Dynamic Systems

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Abstract

Constraint satisfaction can be seen as a dynamic process that approaches the solution set of the constraints asymptotically [8]. Constraint programming is seen as creating a dynamic system with the desired property. We have developed a semantic model for dynamic systems, Constraint Nets, which serves as a useful abstract target machine for constraint programming languages, providing both semantics and pragmatics. Generalizing, here we view a constraint-based dynamic system as a dynamic system which approaches the solution set of the constraints infinitely often. Most robotic systems are constraint-based dynamic systems with tasks specified as constraints. In this paper, we further explore the specification and verification of constraint-based dynamic systems. We first develop generalized ∀-automata for the specification and verification of general (hybrid) dynamic systems, then explicate the relationship between constraint-based dynamic systems and their desired behavior specifications.

1 Motivation and Introduction

We have previously proposed viewing constraints as relations and constraint satisfaction as a dynamic process of approaching the solution set of the constraints asymptotically [8]. Under this view, constraint programming is the creation of a dynamic system with the desired property. We have developed a semantic model for dynamic systems, Constraint Nets, which serves as a useful abstract target machine for constraint programming languages, providing both semantics and pragmatics. Properties of various discrete and continuous constraint methods for constraint programming were also examined [8].

Generalizing, here we consider a constraint-based dynamic system as a dynamic system which approaches the solution set of the constraints infinitely often. One of the motivations for this view is to design and analyze a robotic system composed of a controller that is coupled to a plant and an environment. The desired behavior of the controller may be specified as a set of constraints, which, in general, vary with time. Thus, the controller should be synthesized so as to solve the constraints on-line. Consider a tracking system where the target may move from time to time. A well-designed tracking control system has to ensure that the target can be tracked down infinitely often.

Here we start with general concepts of dynamic systems using abstract notions of time, domains and traces. With this abstraction, hybrid as well as discrete and continuous dynamic systems can be studied in a unitary framework. The behavior of a dynamic system is then defined as the set of possible traces produced by the system.

In order to specify desired behaviors of a dynamic system, we develop a formal specification language, a generalized version of ∀-automata [3]. In order to verify that a dynamic system satisfies its behavior specification, we develop a formal model checking method with generalized Liapunov functions.

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A constraint-based dynamic system is a special type of dynamic system. We explore the properties of constraint-based dynamic systems and constraint-based behavior specifications, then relate system verification to control synthesis.

The rest of the paper is organized as follows. Section 2 briefly presents concepts of general dynamic systems and constraint net modeling. Section 3 develops generalized ∀-automata for specifying and verifying desired behaviors of dynamic systems. Section 4 characterizes constraint-based dynamic systems and their behavior specifications. Section 5 concludes the paper and points out related work.

2 General Dynamic Systems

In this section, we first introduce some basic concepts in general dynamic systems: time, domains and traces, then present a formal model for general dynamic systems.

2.1 Concepts in dynamic systems

In order to model dynamic systems in a unitary framework, we present abstract notions of time structures, domains and traces. Both time structures and domains are defined on metric spaces.

Let $\mathcal{R}^+$ be the set of nonnegative real numbers. A *metric space* is a pair $\langle X, d\rangle$ where X is a set and $d : X \times X \rightarrow \mathcal{R}^+$ is a *metric* defined on X, satisfying the following axioms for all $x, y, z \in X$:

1. $d(x, y) = d(y, x)$.
2. $d(x, y) + d(y, z) \geq d(x, z)$.
3. $d(x, y) = 0$ iff $x = y$.

In a metric space $\langle X, d\rangle$, $d(x, y)$ is called "the distance between x and y". We will use X to denote the metric space $\langle X, d\rangle$ if no ambiguity arises.

A *time structure* is a metric space $\langle \mathcal{T}, d\rangle$ where $\mathcal{T}$ is a totally ordered set with a least element 0, d is a metric satisfying $\forall t_0 \leq t_1 \leq t_2 : d(t_0, t_2) = d(t_0, t_1) + d(t_1, t_2)$. We will use $\mathcal{T}$ to denote the time structure $\langle \mathcal{T}, d\rangle$ if no ambiguity arises. A discrete or continuous time structure can be defined according to the topology of its metric space. For example, the set of natural numbers can define a discrete time structure, a left closed interval of real numbers can define a continuous time structure.

A *domain* is a metric space $\langle A, d\rangle$. Let $v : T \rightarrow A$ be a function from a total order T to a domain A. A point $a^* \in A$ is a *limit* of v, iff $\forall \epsilon \exists t_0 \forall t \geq t_0 : d(v(t), a^*) < \epsilon$. Any limit is unique if it exists. We will use $\lim v$ to denote the limit of v if it exists and $\perp_A$ if it does not. Clearly, if T has a greatest element t_0, $\lim v = v(t_0)$.

A trace $v : \mathcal{T} \rightarrow A$ is a function from a time structure $\mathcal{T}$ to a domain A. Let $T \subseteq \mathcal{T}$ be a downward closed subset of $\mathcal{T}$, i.e. $t \in T$ implies $\forall t' \leq t : t' \in T$. We use $\lim v_{|T}$ to denote the limit of v on the total order T. For simplicity in representation, we introduce the following notions: given a time structure $\langle \mathcal{T}, d\rangle$ and a real number $\tau \in \mathcal{R}^+$,

- $pre(t) = \{t' \in \mathcal{T} | t' < t\}$, and $v(pre(t)) = \lim v_{|pre(t)}$;
- $t - \tau = \{t' \in \mathcal{T} | t' < t, d(t, t') \geq \tau\}$, and $v(t - \tau) = \lim v_{|t-\tau}$;
- $t + \tau = \{t' \in \mathcal{T} | t' > t, d(t, t') \leq \tau\}$.

Clearly, $pre(t) = t - 0$. If $\mathcal{T}$ is discrete, $v(pre(t))$ is the value of the previous time point and $pre(t) = t - \tau$ whenever τ is small enough.

A time structure $\mathcal{T}$ is *infinite* iff $\forall m > 0, \exists t_0 \in \mathcal{T}, \forall t \geq t_0 : d(0, t) \geq m$. We will restrict ourselves to infinite time structures. This is not a real restriction, since any time structure $\mathcal{T}$ can be extended to an infinite one $\mathcal{T}' \supset \mathcal{T}$ by letting $v(t) = \lim v_{|\mathcal{T}}$ for all $t \notin \mathcal{T}$.

2.2 Constraint Nets: a model for dynamic systems

We have developed a semantic model, Constraint Nets, for general (hybrid) dynamic systems [10]. We have used the Constraint Net model as an abstract target machine for constraint programming languages [8], while constraint programming is considered as designing a dynamic system that approaches the solution set of the given constraints asymptotically.

Intuitively, a constraint net consists of a finite set of locations, a finite set of transductions, each with a finite set of input ports and an output port, and a finite set of connections between locations and ports of transductions. A location can be regarded as a wire, a channel, a variable, or a memory location, whose values may change over time. A transduction is a mapping from input traces to output traces, with the causal restriction, viz. the output value at any time is determined by the input values up to that time. For example, a temporal integration with an initial value is a typical transduction on a continuous time structure and any state automaton with an initial state defines a transduction on a discrete time structure.

A location l is the *output location* of a transduction F, iff l connects to the output port of F; l is an *input location* of F, iff l connects to an input port of F. Let CN be a constraint net. A location l is an *output location* of CN if l is an output location of some transduction in CN otherwise it is an *input location* of CN. The set of input locations of CN is denoted by $I(CN)$, the set of output locations of CN is denoted by $O(CN)$; CN is *closed* if $I(CN) = \emptyset$ otherwise it is *open*.

Semantically, a transduction F denotes an equation $l_0 = F(l_1, \ldots, l_n)$ where l_0 is the output location of F and $\langle l_1, \ldots, l_n \rangle$ is the tuple of input locations of F. A constraint net CN denotes a set of equations, each corresponds to a transduction in CN. The semantics of CN is a 'solution' of the set of equations [10], which is a set of pairs of input and output traces satisfying the equations. Let $Lc = I(CN) \cup O(CN)$ and $\{A_l\}_{l \in Lc}$ be a set of domains in CN. A *state* s of CN is a mapping from the set of locations to their corresponding domains: i.e. $s \in \times_{Lc} A_l$. Therefore, the semantics of CN is also a set of state traces with domain $\times_{Lc} A_l$.

We have modeled two types of constraint solvers, state transition systems and state integration systems, in constraint nets. The former models discrete dynamic processes and the latter models continuous dynamic processes [8]. Hybrid dynamic systems, with both discrete and continuous components, can also be modeled in constraint nets [10, 9]. The *behavior* of a dynamic system is defined as a set of possible input/output traces produced by the system, in our case, the semantics of the constraint net which models the system.

We illustrate the constraint net modeling with two simple examples. The first is a 'standard' example of *Cat and Mouse* modified from [1]. Suppose a cat and a mouse start running from initial positions X_c and X_m respectively, $X_c > X_m > 0$, with constant velocities $V_c < V_m < 0$. Both of them will stop running when the cat catches the mouse, or the mouse runs into the hole in the wall at 0. The behavior of this system is modeled by the following equations CM_1:

$$\begin{aligned} x_c &= \int(X_c)(V_c \cdot c), \\ x_m &= \int(X_m)(V_m \cdot c), \\ c &= (x_c > x_m) \wedge (x_m > 0) \end{aligned}$$

where $\int(X)$ is a temporal integration with initial state X. At any time, c is 1 if the running condition $(x_c > x_m) \wedge (x_m > 0)$ is satisfied and 0 otherwise. Let $\mathcal{R}$ be the set of real numbers and $\mathcal{B} = \{0, 1\}$. This is a closed system. The state of this system is $\langle x_c, x_m, c \rangle \in \mathcal{R} \times \mathcal{R} \times \mathcal{B}$, with its initial state $\langle X_c, X_m, 1 \rangle$. If the cat catches the mouse before the mouse runs into the hole in the wall at 0, i.e. $0 \leq x_c \leq x_m$, the cat wins; if the mouse runs into the hole before the cat, i.e. $x_m \leq 0 \leq x_c$, the mouse wins.

Consider another *Cat and Mouse* problem, where the controller of the cat is synthesized from its constraint specification, i.e. $x_c = x_m$. Suppose the plant of the cat obeys the dynamics $u = \dot{x}_c$ where u is the control input, i.e. the velocity of the cat is controlled. One possible design for the cat controller uses the gradient descent method [8] on the energy function $(x_m - x_c)^2$ to synthesize the feedback control law $u = k \cdot (x_m - x_c), k > 0$ where the distance between the cat and the mouse $x_m - x_c$ can be sensed by the cat. The cat can be modeled as an open constraint net with two equations CM_2:

$$x_c = \int(X_c)u, \quad u = k \cdot (x_m - x_c).$$

Will the cat catch the mouse?

3 Generalized ∀-Automata

While modeling focuses on the underlying structure of a system, the organization and coordination of components or subsystems, the overall behavior of the modeled system is not explicitly expressed. However, for many situations, it is important to specify some global properties and guarantee that these properties hold in the proposed design.

We advocate a formal approach to specifying desired behaviors and to verifying the relationship between a dynamic system and its behavior specification. A trace $v : \mathcal{T} \rightarrow A$ is a generalization of a sequence. In fact, when $\mathcal{T}$ is the set of natural numbers, v is an infinite sequence. A set of sequences defines a conventional formal language. If we take the abstract behavior of a system as a language, a specification can be represented as an automaton, and verification checks the inclusion relation between the language of the system and the language accepted by the automaton.

There is always a trade-off between the power of representation, i.e., the class of languages the type of automaton can accept, and the power of analysis, i.e. the computability of checking the acceptance of traces. We would like the type of automaton to be powerful enough to state certain temporal and real-time properties, yet simple enough to have formal, semi-automatic or automatic verifications. We generalize ∀-automata [3] and Liapunov functions for our purposes.

∀-automata are non-deterministic finite state automata over infinite sequences. These automata were proposed as a formalism for the specification and verification of temporal properties of concurrent programs. It has been shown that ∀-automata have the same expressive power as Buchi automata [6] and the extended temporal logic (ETL) [7], which are strictly more powerful than the linear propositional temporal logic [6, 7]. More importantly, there is a formal verification method [3].

In this section, we generalize ∀-automata to specify languages composed of traces on continuous as well as discrete time structures, and modify the formal verification method [3] by generalizing Liapunov functions [8] and the method of continuous induction [2].

3.1 Behavior Specification

Let an *assertion* be a logical formula defined on states of a dynamic system, i.e. any assertion α on a given state s, denoted $\alpha(s)$, will be evaluated to either *true*, $s \models \alpha$, or *false*, $s \not\models \alpha$.

A ∀-*automaton* $\mathcal{A}$ is a quintuple $\langle Q, R, S, e, c \rangle$ where Q is a finite set of *automaton-states*, $R \subseteq Q$ is a set of *recurrent states* and $S \subseteq Q$ is a set of *stable states*. With each $q \in Q$, we associate an assertion $e(q)$, which characterizes the *entry condition* under which the automaton may start its activity in q. With each pair $q, q' \in Q$, we associate an assertion $c(q, q')$, which characterizes the *transition condition* under which the automaton may move from q to q'. R and S are the generalization of *accepting* states to the case of infinite inputs. We denote by $B = Q - (R \cup S)$ the set of *non-accepting (bad)* states.

A ∀-automaton is called *complete* iff the following requirements are met:

- $\bigvee_{q \in Q} e(q)$ is valid.
- For every $q \in Q$, $\bigvee_{q' \in Q} c(q, q')$ is valid.

We will restrict ourselves to complete automata. This is not a real restriction, since any automaton can be transformed to a complete automaton by introducing an additional error state $q_E \in B$, with the corresponding entry condition and transition conditions [3].

Let $\mathcal{T}$ be a time structure and $v : \mathcal{T} \rightarrow A$ be a trace. A *run* of $\mathcal{A}$ over v is a trace $r : \mathcal{T} \rightarrow Q$ satisfying

1. *Initiality*: $v(0) \models e(r(0))$;
2. *Consecution*:
   - inductivity: $\forall t > 0, \exists q \in Q$ and $\delta > 0, \forall 0 < \tau \leq \delta : r(t - \tau) = q$ and $v(t) \models c(r(t - \tau), r(t))$ and
   - continuity: $\forall t, \exists q \in Q$ and $\delta > 0, \forall t' \in t + \delta : r(t') = q$ and $v(t') \models c(r(t), r(t'))$.

It is easy to check that when $\mathcal{T}$ is discrete, the two conditions in *Consecution* are reduced to one, i.e. $\forall t > 0, v(t) \models c(r(pre(t)), r(t))$; and if, in addition, $\mathcal{A}$ is complete, every trace has a run. However, if $\mathcal{T}$ is not discrete, even if $\mathcal{A}$ is complete, not every trace has a run. For example, a trace with infinite transitions

among Q within a finite interval has no run. A trace v is *specifiable* by $\mathcal{A}$ iff there is a run of $\mathcal{A}$ over v. The behavior of a system is *specifiable* by $\mathcal{A}$, iff every trace of the behavior is specifiable.

If r is a run, let $Inf(r)$ be the set of automaton-states appearing infinitely many times in r, i.e. $Inf(r) = \{q | \forall t \exists t_0 \geq t, r(t_0) = q\}$. Clearly, if T has a greatest element t_0, $Inf(r) = r(t_0)$. A run r is defined to be *accepting* iff:

1. $Inf(r) \cap R \neq \emptyset$, i.e. *some* of the states appearing infinitely many times in r belong to R, or
2. $Inf(r) \subseteq S$, i.e. *all* the states appearing infinitely many times in r belong to S.

A $\forall$-automaton $\mathcal{A}$ *accepts* a trace v, written $v \models \mathcal{A}$, iff *all* possible runs of $\mathcal{A}$ over v are accepting. A $\forall$-automaton $\mathcal{A}$ *accepts* a dynamic system S, written $S \models \mathcal{A}$, iff for every trace v of the behavior of S, $v \models \mathcal{A}$.

One of the advantages of using automata as a specification language is the graphical representation. It is useful and illuminating to represent $\forall$-automata by diagrams. The basic conventions for such representations are the following:

- The automaton-states are depicted by nodes in a directed graph.
- Each initial state is marked by a small arrow, called the *entry arc*, pointing to it.
- Arcs, drawn as arrows, connect some of the states.
- Each recurrent state is depicted by a diamond shape inscribed within a circle.
- Each stable state is depicted by a square inscribed within a circle.

Nodes and arcs are labeled by assertions. A node or an arc that is left unlabeled is considered to be labeled with *true*. The labels define the entry conditions and the transition conditions of the associated automaton as follows:

- Let $q \in Q$ be a node in the diagram. If q is labeled by ψ and the entry arc is labeled by φ, the entry condition $e(q)$ is given by: $e(q) = \varphi \wedge \psi$. If there is no entry arc, $e(q) = false$.
- Let q, q' be two nodes in the diagram. If q' is labeled by ϕ, and arcs from q to q' are labeled by $\varphi_i, i = 1..n$, the transition condition $c(q, q')$ is given by: $c(q, q') = (\varphi_1 \vee \ldots \vee \varphi_n) \wedge \psi$. If there is no arc from q to q', $c(q, q') = false$.

A diagram representing an incomplete automaton is interpreted as a complete automaton by introducing an error state and associated entry and transition conditions.

This type of automaton is powerful enough to specify various qualitative behaviors. Some typical desired behaviors are shown in Fig. 1. Figure 1(a) accepts a trace which satisfies $\neg G$ only finitely many times, Figure 1(b) accepts a trace which never satisfies B, and Figure 1(c) accepts a trace which will satisfy S in the finite future whenever it satisfies R. For the *Cat and Mouse* examples, we can have the formal

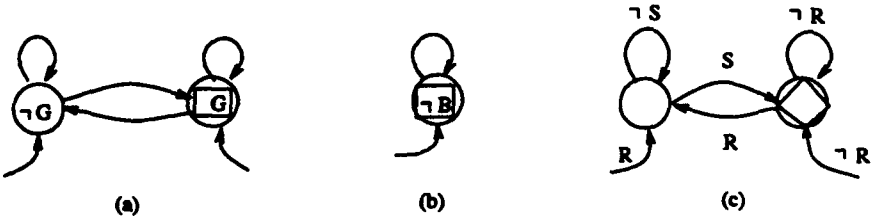

Figure 1: $\forall$-automata: (a) goal achievement or reachability (b) safety (c) bounded response

behavior specifications shown in Fig. 2.

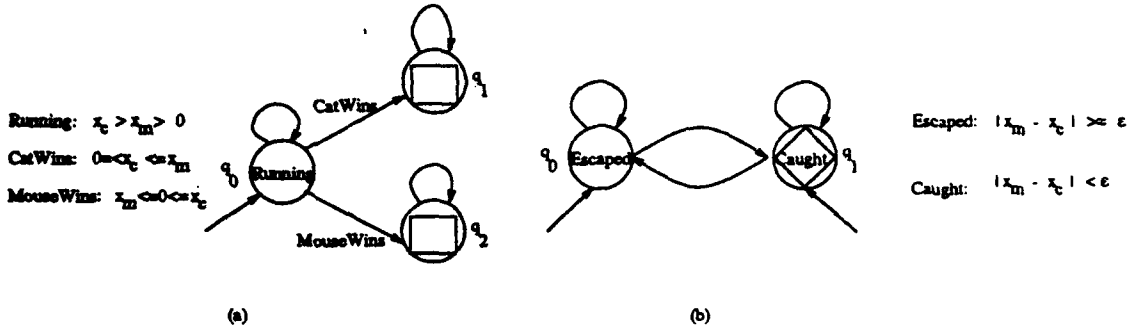

Figure 2: (a) Either the cat wins or the mouse wins (b) The cat catches the mouse persistently

3.2 System Verification

Given a constraint net model of a discrete- or continuous-time dynamic system, and a ∀-automaton specification of a desired behavior, a formal method is developed here for verifying that the constraint net exhibits its desired behavior.

Let CN be a constraint net model of a dynamic system, whose behavior is a set of traces. Let $\{\varphi\}CN\{\psi\}$ denote the validity of the consecutive condition: for every trace v of the behavior of CN,

- $\forall t > 0, \exists \delta > 0, \forall 0 < \tau \leq \delta : v(t-\tau) \models \varphi$ implies $v(t) \models \psi$ and
- $\forall t, \exists \delta > 0, \forall t' \in t + \delta : v(t) \models \varphi$ implies $v(t') \models \psi$.

Clearly, if $\mathcal{T}$ is discrete, these two conditions are reduced to one: $\forall t > 0, v(pre(t)) \models \varphi$ implies $v(t) \models \psi$.

Let CN be a constraint net with the set of locations Lc and the set of states $\times_{Lc} A_l$, Θ be an assertion indicating the initial state of CN, and $\mathcal{A} = \langle Q, R, S, e, c \rangle$ be a ∀-automaton. A set of assertions $\{\alpha_q\}_{q \in Q}$ is called a set of *invariants* for CN and $\mathcal{A}$ iff

- *Initiality*: $\forall q \in Q.\Theta \wedge e(q) \rightarrow \alpha_q$.
- *Consecution*: $\forall q, q' \in Q.\{\alpha_q\}CN\{c(q,q') \rightarrow \alpha_{q'}\}$.

Given that $\{\alpha_q\}_{q \in Q}$ is a set of invariants for CN and $\mathcal{A}$, a set of partial functions $\{\rho_q\}_{q \in Q} : \times_{Lc} A_l \rightarrow \mathcal{R}^+$ is called a set of *Liapunov* functions for CN and $\mathcal{A}$ iff the following conditions are satisfied:

- *Definedness*: $\forall q \in Q : \ \alpha_q \rightarrow \exists w.\rho_q = w$.
- *Non-increase*: $\forall q \in Q, q' \in S$:

$$\{\alpha_q \wedge \rho_q = w\}CN\{c(q,q') \rightarrow \rho_{q'} \leq w\}.$$

- *Decrease*: Let t_c denote the current time, $\exists \epsilon > 0, \forall q \in Q, q' \in B$:

$$\{\alpha_q \wedge \rho_q = w \wedge t_c = t\}CN\{c(q,q') \rightarrow \frac{\rho_{q'} - w}{d(t_c, t)} \leq -\epsilon\}$$

Let the time structure be infinite with either discrete or continuous topology. We conclude that if the behavior of a constraint net CN is specifiable by a ∀-automaton $\mathcal{A}$ and the following requirements are satisfied the validity of $\mathcal{A}$ over CN is proved:

(I) Associate with each automaton-state $q \in Q$ an assertion α_q, such that $\{\alpha_q\}_{q \in Q}$ is a set of invariants for CN and $\mathcal{A}$.

(L) Associate with each automaton-state $q \in Q$ a partial function $\rho_q : \times_{Lc} A_l \rightarrow \mathcal{R}^+$, such that $\{\rho_q\}_{q \in Q}$ is a set of Liapunov functions for CN and $\mathcal{A}$.

As in [3], the verification rules (I) and (L) are sound and complete, i.e. $\mathcal{A}$ accepts CN iff there exist a set of invariants and Liapunov functions.

Theorem 1 *Let the time structure be infinite with either discrete or continuous topology. If the behavior of a constraint net CN is specifiable by a $\forall$-automaton $\mathcal{A}$, verification rules (I) and (L) are sound and complete.*

Proof: (Sketch) Apply the method of continuous induction [2]. The detailed proof is shown in Appendix A of the extended version of this paper. □

We illustrate this verification method by the *Cat and Mouse* examples. Consider the first *Cat and Mouse* example adopted from [1]. We show that the constraint net model CM_1 in section 2 satisfies the behavior specification in Fig. 2(a).

First of all, the $\forall$-automaton in Fig. 2(a) is not complete. To make it complete, add an 'error' state $q_E \in B$, with $e(q_E) = false$, $c(q_E, q_E) = true$ and $c(q_0, q_E) = x_c < 0$. It is easy to see that CM_1 is specifiable by the complete $\forall$-automaton.

Secondly, associate with q_0, q_1, q_2, q_E assertions *Running*, *CatWins*, *MouseWins* and *false* respectively. Note that

$$\{x_c > x_m > 0\}CM_1\{x_c < 0 \rightarrow false\}$$

since x_c is continuous. (Imagine that if the cat jumps, it may not catch the mouse but hit the wall instead. Fortunately, this is not the case here.) Therefore, the set of assertions is a set of invariants.

Thirdly, associate with q_0, q_1, q_2, q_E the same function: $\rho : \mathcal{R} \times \mathcal{R} \times \mathcal{B} \rightarrow \mathcal{R}^+$, such that $\rho(x_c, x_m, 0) = 0$ and $\rho(x_c, x_m, 1) = -(\frac{x_m}{V_m} + \frac{x_c}{V_c})$. Clearly, ρ is decreasing at q_0 with rate 2 and q_E can never be reached. Therefore, it is a Liapunov function.

If we remove the square □ from node q_2 in Fig. 2(a), i.e. $q_2 \in B$, the modified behavior specification states that "the cat always wins". Clearly, not every trace of the behavior of CM_1 satisfies this specification. However, if the initial state satisfies $\frac{X_c}{V_c} > \frac{X_m}{V_m}$, in addition to $X_c > X_m > 0$, we can prove that "the cat always wins".

To see this, let $\Delta = \frac{X_c}{V_c} - \frac{X_m}{V_m}$ and let *Inv* denote $\frac{x_c}{V_c} - \frac{x_m}{V_m} = \Delta$. Associate with q_0, q_1, q_2, q_E assertions $Running \wedge Inv$, *CatWins*, *false* and *false* respectively. Note that

$$\{Running \wedge Inv\}CM_1\{Running \rightarrow Running \wedge Inv\}$$

since the derivative of $\frac{x_c}{V_c} - \frac{x_m}{V_m}$ is 0 given that *Running* is satisfied; and

$$\{Running \wedge Inv\}CM_1\{MouseWin \rightarrow false\}$$

since x_m is continuous. Therefore, the set of assertions is a set of invariants.

Associate with q_0, q_1, q_2, q_E the same function: $\rho : \mathcal{R} \times \mathcal{R} \times \mathcal{B} \rightarrow \mathcal{R}^+$, such that $\rho(x_c, x_m, 0) = 0$ and $\rho(x_c, x_m, 1) = -(\frac{x_m}{V_m} + \frac{x_c}{V_c})$. Again, it is a Liapunov function.

Consider the second *Cat and Mouse* example, in which the motion of the mouse is unknown, but the cat tries to catch the mouse anyhow. Clearly, not every trace of the behavior of the constraint net CM_2 satisfies the behavior specification in Fig. 2(b). For example, if $\dot{x}_c = \dot{x}_m$ all the time, the distance between the cat and the mouse will be constant and the cat can never catch the mouse. However, suppose the mouse is short-sighted, i.e. it can only see the cat if their distance $|x_m - x_c| < \delta < \epsilon$, and when it does not see the cat, it will stop running within time τ.

The short-sighted property of the mouse is equivalent to adding the following assumption to CM_2:

$$\{|x_m - x_c| \geq \delta \wedge \dot{x}_m = 0\}CM_2\{|x_m - x_c| \geq \delta \rightarrow \dot{x}_m = 0\}$$

i.e. the mouse will not run if it does not see the cat. The maximum running time property of the mouse is equivalent to adding the following assumption to CM_2: let l_t be the time left for the mouse to run when it does not see the cat,

$$\{|x_m - x_c| < \delta\}CM_2\{|x_m - x_c| \geq \delta \wedge \dot{x}_m \neq 0 \rightarrow l_t \leq \tau\}$$

and

$$\{|x_m - x_c| \geq \delta \wedge \dot{x}_m \neq 0 \wedge l_t = l \wedge t_c = t\}CM_2\{|x_m - x_c| \geq \delta \wedge \dot{x}_m \neq 0 \rightarrow l_t \leq l - d(t_c, t)\}.$$

We show that no matter how fast the mouse may run, the cat tracks down the mouse infinitely often (including the case in which the mouse is caught permanently).

In order to prove this claim, we may decompose the automaton-state q_0 in Fig. 2(b) into two automaton-states q_{00} and q_{01} as shown in Fig. 3. This automaton is not complete. To make it complete, add an 'error' state $q_E \in B$ with $e(q_E) = false$, $c(q_E, q_E) = true$ and $c(q_{00}, q_E) = Escape$.

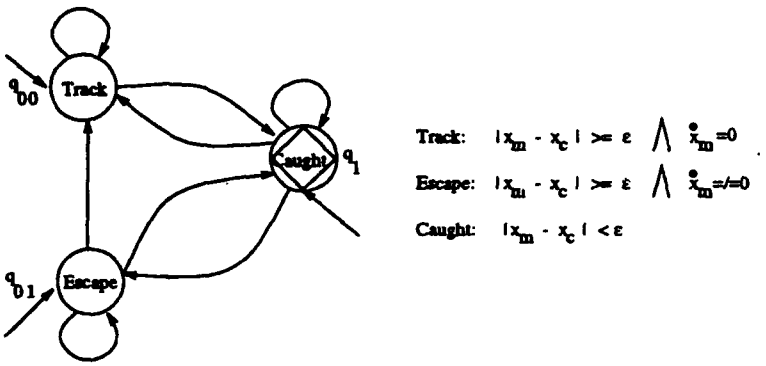

Figure 3: A refinement of the cat-mouse specification

Associate with automaton-states q_{00}, q_{01}, q_1, q_E assertions $Track, Escape, Caught$ and $false$ respectively. Note that

$$\{Track\}CM_2\{Escape \rightarrow false\}$$

because of the short-sighted property of the mouse. Therefore, the set of assertions is a set of invariants.

Let $d_m \in \mathcal{R}^+$ be the maximum distance between the cat and the mouse. Associate with automaton-states q_{00}, q_{01}, q_1, q_E functions $\rho_{q_{00}}, \rho_{q_{01}}, \rho_{q_1}$ and ρ_{q_E} respectively, where

$$\rho_{q_{00}} = (x_m - x_c)^2, \ \rho_{q_{01}} = d_m^2 + l_t, \ \rho_{q_1} = \rho_{q_E} = d_m^2 + \tau.$$

The feedback control law of the cat guarantees that $\rho_{q_{00}}$ decreases at q_{00} at a rate not less than $2k\epsilon^2$. The maximum running time property of the mouse guarantees that $\rho_{q_{01}}$ decrease at q_{01} at a rate not less than 1. Therefore, the set of functions is a set of Liapunov functions.

4 Constraint-Based Dynamic Systems

In this section, we first explore the relationship between a constraint solver and its desired behavior specification, then define constraint-based dynamic systems as a generalization of constraint solvers.

4.1 Dynamic process and constraint solver

Constraint satisfaction can be seen as a dynamic process that approaches the solution set of the constraints asymptotically, and a constraint solver, modeled by a constraint net, exhibits this required behavior [8]. Here we briefly introduce some related concepts.

Let $\langle A, d \rangle$ be a domain. Given a point $a \in A$ and a subset $A^* \subset A$, the *distance* between a and A^* is defined as $d(a, A^*) = \inf_{a^* \in A^*}\{d(a, a^*)\}$. For any $\epsilon > 0$, the *ϵ-neighborhood* of A^* is defined as $N^\epsilon(A^*) = \{a | d(a, A^*) < \epsilon\}$; it is *strict* if it is a strict superset of A^*. For a function $v : T \rightarrow A$ from a total order T, v *approaches* A^* iff $\forall \epsilon \exists t_0 \forall t \geq t_0 : d(v(t), A^*) < \epsilon$.

Let S be a domain indicating a state space and T be a time structure which can be either discrete or continuous. A *dynamic process* p is a function $p : S \rightarrow S^T$ with $p(s)(0) = s, \forall s \in S$. For any subset $S^* \subset S$, let $\phi_p(S^*) = \{p(s)(t) | s \in S^*, t \in T\}$. S^* is an *equilibrium* of p iff $\phi_p(S^*) = S^*$. S^* is a *stable equilibrium* of p iff S^* is an equilibrium and $\forall \epsilon \exists \delta : \phi_p(N^\delta(S^*)) \subseteq N^\epsilon(S^*)$. S^* is an *attractor* of p iff there exists a strict ϵ-neighborhood $N^\epsilon(S^*)$ such that $\forall s \in N^\epsilon(S^*)$, $p(s)$ approaches S^*; S^* is an *attractor in the large* iff $\forall s \in S$,

$p(s)$ approaches S^*. If S^* is an attractor (in the large) and S^* is a stable equilibrium, S^* is an *asymptotically* stable equilibrium (in the large).

Let $C = \{C_i\}_{i \in I}$ be a set of constraints, whose solution $sol(C) = \{s | \forall i \in I : C_i(s)\}$ is a subset of a state space S. A *constraint solver* for C is a constraint net CS whose semantics is a dynamic process $p : S \rightarrow S^T$ with $sol(C)$ as an asymptotically stable equilibrium. CS *solves* C *globally* iff $sol(C)$ is an asymptotically stable equilibrium in the large.

We have discussed two types of constraint solvers: state transition systems and state integration systems. Various discrete and continuous constraint methods have been presented, and also analyzed using Liapunov functions [8].

4.2 Constraint-based computation and control

Given a set of constraints C, let C^ϵ denote the assertion which is true on the ϵ-neighborhood of its solution set, $N^\epsilon(sol(C)) \subset S$, and let $\mathcal{A}(C^\epsilon; \Box)$ denote the $\forall$-automaton in Fig. 4(a). Clearly, CS solves C iff there exists an initial condition $\Theta \supset sol(C)$ such that $\forall \epsilon : CS(\Theta) \models \mathcal{A}(C^\epsilon; \Box)$; CS solves C globally when $\Theta = S$. We call $\mathcal{A}(C^\epsilon; \Box)$ an *open specification* of the set of constraints C. Note that it is important to have

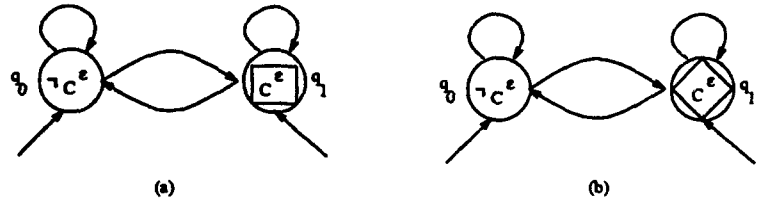

Figure 4: Specification for (a) Constraint solver (b) Constraint-based dynamic system

open specifications, otherwise, if we replace C^ϵ with $sol(C)$, a constraint solver for C may never satisfy the specification, since it may take *infinite* time to approach $sol(C)$.

However, requiring the integration of a controller with its environment to be a constraint solver is still too stringent for a control problem, with disturbance and uncertainty in its environment. If we consider the solution set of a set of constraints as the 'goal' for the controller to achieve, one relaxed requirement for the controller is to make the system stable at the goal. In other words, if the system diverges from the goal by some disturbance, the controller should always be able to regulate the system back to its goal. We call a system CB *constraint-based* w.r.t. a set of constraints C, iff $\forall \epsilon : CB \models \mathcal{A}(C^\epsilon; \Diamond)$ where $\mathcal{A}(C^\epsilon; \Diamond)$ denotes the $\forall$-automaton in Fig. 4(b). In other words, a dynamic system is constraint-based iff it approaches the solution set of the constraints infinitely often.

We may relax this condition further and define constraint-based systems with errors. We call a system CB *constraint-based* w.r.t. a set of constraints C with error δ, iff $\forall \epsilon > \delta : CB \models \mathcal{A}(C^\epsilon; \Diamond)$; δ is called the *steady-state error* of the system. Normally, steady-state errors are caused by uncertainty and disturbance of the environment. For example, the second cat-mouse system CM_2 is a constraint-based system with steady-state error δ, which is the radius of the mouse sensing range.

If $\mathcal{A}(C^\epsilon; \Box)$ is considered as an open specification of a constraint-based *computation* for a closed system, $\mathcal{A}(C^\epsilon; \Diamond)$ can been seen as an *open specification* of a constraint-based *control* for an open or embedded system.

We have developed a systematic approach to control synthesis from requirement specifications [8]. In particular, requirement specifications impose constraints over a system's global behavior and controllers can be synthesized as embedded constraint solvers which solve constraints over time. By exploring a relation between constraint satisfaction and dynamic systems via constraint methods, discrete/continuous constraint solvers or constraint-based controllers are derived.

We have developed here a behavior specification language and a formal verification method for dynamic systems. With this approach, control synthesis and system verification are coupled via requirement specifications and Liapunov functions. If we consider a Liapunov function for a set of constraints as a measurement of the degree of satisfaction, this function can be used for both control synthesis and system verification.

5 Conclusion and Related Work

We have presented a formal specification language, called generalized ∀-automata, for desired behaviors of dynamic systems. We have also presented a formal verification method, using generalized Liapunov functions, for checking that a dynamic system exhibits its desired behavior. A constraint-based dynamic system can be modeled by a constraint net, whose desired behavior can be specified by a ∀-automaton. The Liapunov functions for a given constraint specification can be used for both control synthesis and system verification.

Some related work has been done recently along these lines. Nerode and Kohn have proposed the notion of open specification for control systems [4]. Saraswat et al. have developed a family of timed concurrent constraint languages for modeling and specification of discrete dynamic systems [5]. Problems on specification and verification of hybrid dynamic systems have become a new challenge to traditional control system design and traditional programming methodologies [1]. Our work is unique in that we distinguish the executable (modeling) and logical (requirement) specifications, and develop the model checking technique based on properties of dynamic systems.

Acknowledgement

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GSAT and Dynamic Backtracking

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Abstract

There has been substantial recent interest in two new families of search techniques. One family consists of nonsystematic methods such as GSAT; the other contains systematic approaches that use a polynomial amount of justification information to prune the search space. This paper introduces a new technique that combines these two approaches. The algorithm allows substantial freedom of movement in the search space but enough information is retained to ensure the systematicity of the resulting analysis. Bounds are given for the size of the justification database and conditions are presented that guarantee that this database will be polynomial in the size of the problem in question.

1 INTRODUCTION

The past few years have seen rapid progress in the development of algorithms for solving constraint-satisfaction problems, or CSPs. CSPs arise naturally in subfields of AI from planning to vision, and examples include propositional theorem proving, map coloring and scheduling problems. The problems are difficult because they involve search; there is never a guarantee that (for example) a successful coloring of a portion of a large map can be extended to a coloring of the map in its entirety.

The algorithms developed recently have been of two types. *Systematic* algorithms determine whether a solution exists by searching the entire space. *Local* algorithms use hill-climbing techniques to find a solution quickly but are *nonsystematic* in that they search the entire space in only a probabilistic sense.

The empirical effectiveness of these nonsystematic algorithms appears to be a result of their ability to follow local gradients in the search space. Traditional systematic procedures explore the space in a fixed order that is independent of local gradients; the fixed order makes following local gradients impossible but is needed to ensure that no node is examined twice and that the search remains systematic.

Dynamic backtracking [6] attempts to overcome this problem by retaining specific information about those portions of the search space that have been eliminated and then following local gradients in the remainder. Unlike previous algorithms that recorded such elimination information, such as dependency-directed backtracking [15], dynamic backtracking is selective about the information it caches so that only a polynomial amount of memory is required. These earlier techniques cached a new result with every backtrack, using an amount of memory that was linear in the run time and thus exponential in the size of the problem being solved.

Unfortunately, neither dynamic nor dependency-directed backtracking (or any other known similar method) is truly effective at local maneuvering within the search space, since the basic underlying methodology remains simple chronological backtracking. New techniques are included to make the search more efficient, but an exponential number of nodes in the search space must still be examined before early choices can be retracted. No existing search technique is able to both move freely within the search space and keep track of what has been searched and what hasn't.

The second class of algorithms developed recently presume that freedom of movement is of greater importance than systematicity. Algorithms in this class achieve their freedom of movement by abandoning the conventional description of the search space as a tree of partial solutions, instead thinking of it as a space of total assignments of values to variables. Motion is permitted between any two assignments that differ on a single value, and a hill-climbing procedure is employed to try to minimise the number of constraints violated by the overall assignment. The best-known algorithms in this class are min-conflicts [11] and GSAT [14].

Min-conflicts has been applied to the scheduling domain specifically and used to schedule tasks on the Hubble space telescope. GSAT is restricted to Boolean satisfiability problems (where every variable is assigned simply true or false), and has led to remarkable progress in the solution of randomly generated problems of this type; its performance is reported [12, 13, 14] as surpassing that of other techniques such as simulated annealing [8] and systematic techniques based on the Davis-Putnam procedure [4].

GSAT is not a panacea, however; there are many problems on which it performs fairly poorly. If a problem has no solution, for example, GSAT will never be able to report this with confidence. Even if a solution does exist, there appear to be at least two possible difficulties that GSAT may encounter.

First, the GSAT search space may contain so many local minima that it is not clear how GSAT can move so as to reduce the number of constraints violated by a given assignment. As an example, consider the CSP of generating crossword puzzles by filling words from a fixed dictionary into an empty frame [7]. The constraints indicate that there must be no conflict in each of the squares; thus two words that begin on the same square must also begin with the same letter. In this domain, getting "close" is not necessarily any indication that the problem is nearly solved, since correcting a conflict at a single square may involve modifying much of the current solution. Konolige has recently reported that GSAT specifically has difficulty solving problems of this sort [9].

Second, GSAT does no forward propagation. In the crossword domain once again, selecting one word may well force the selection of a variety of subsequent words. In a Boolean satisfiability problem, assigning one variable the value true may cause an immediate cascade of values to be assigned to other variables via a technique known as *unit resolution*. It seems plausible that forward propagation will be more common on realistic problems than on randomly generated ones; the most difficult random problems appear to be tangles of closely related individual variables while naturally occurring problems tend to be tangles of sequences of related variables. Furthermore, it appears that GSAT's performance degrades (relative to systematic approaches) as these sequences of variables arise [3].

Our aim in this paper is to describe a new search procedure that appears to combine the benefits of both of the earlier approaches; in some very loose sense, it can be thought of as a systematic version of GSAT.

The next three sections summarize the original dynamic backtracking algorithm [6], presenting it from the perspective of local search. The termination proof is omitted here but can be found in earlier papers [6, 10]. Section 5 present a modification of dynamic backtracking called *partial-order dynamic backtracking*, or PDB. This algorithm builds on work of McAllester's [10]. Partial-order dynamic backtracking provides greater flexibility in the allowed set of search directions while preserving systematicity and polynomial worst case space usage. Section 6 presents a new variant of dynamic backtracking that is still more flexible in the allowed set of search directions. While this final procedure is still systematic, it can use exponential space in the worst case. Section 7 presents some empirical results comparing PDB with other well known algorithms on a class of "local" randomly generated 3-SAT problems. Concluding remarks are contained in Section 8, and proofs appear in the full paper.

2 CONSTRAINTS AND NOGOODS

We begin with a slightly nonstandard definition of a CSP.

Definition 2.1 *By a* constraint satisfaction problem (I, V, κ) *we will mean a finite set I of variables; for each $x \in I$, there is a finite set V_x of possible values for the variable x. κ is a set of constraints each of the form $\neg[(x_1 = v_1) \wedge \cdots \wedge (x_k = v_k)]$ where each x_j is a variable in I and each v_j is an element of V_{x_j}. A* solution *to the* CSP *is an assignment P of values to variables that satisfies every constraint. For each variable x we require that $P(x) \in V_x$ and for each constraint $\neg[(x_1 = v_1) \wedge \cdots \wedge (x_k = v_k)]$ we require that $P(x_i) \neq v_i$ for some x_i.*

By the size of a constraint-satisfaction problem (I, V, κ), we will mean the product of the domain sizes of the various variables, $\prod_x |V_x|$.

The technical convenience of the above definition of a constraint will be clear shortly. For the moment, we merely note that the above description is clearly equivalent to the conventional one; rather than represent the constraints in terms of allowed value combinations for various variables, we write axioms that disallow specific value combinations one at a time. The size of a CSP is the number of possible assignments of values to variables.

Systematic algorithms attempting to find a solution to a CSP typically work with partial solutions that are then discovered to be inextensible or to violate the given constraints; when this happens, a backtrack occurs and the partial solution under consideration is modified. Such a procedure will, of course, need to record information that guarantees that the same partial solution not be considered again as the search proceeds. This information might be recorded in the structure of the search itself; depth-first search with chronological backtracking is an example. More sophisticated methods maintain a database of some form indicating explicitly which choices have been elimi-

nated and which have not. In this paper, we will use a database consisting of a set of *nogoods* [5].

Definition 2.2 *A* nogood *is an expression of the form*

$$(x_1 = v_1) \wedge \cdots \wedge (x_k = v_k) \rightarrow x \neq v \qquad (1)$$

A nogood can be used to represent a constraint as an implication; (1) is logically equivalent to the constraint

$$\neg[(x_1 = v_1) \wedge \cdots \wedge (x_k = v_k) \wedge (x = v)]$$

There are clearly many different ways of representing a given constraint as a nogood.

One special nogood is the *empty* nogood, which is tautologically false. We will denote the empty nogood by $\perp$; if $\perp$ can be derived from the given set of constraints, it follows that no solution exists for the problem being attempted.

The typical way in which new nogoods are obtained is by resolving together old ones. As an example, suppose we have derived the following:

$$\begin{aligned} (x = a) \wedge (y = b) &\rightarrow u \neq v_1 \\ (x = a) \wedge (z = c) &\rightarrow u \neq v_2 \\ (y = b) &\rightarrow u \neq v_3 \end{aligned}$$

where v_1, v_2 and v_3 are the only values in the domain of u. It follows that we can combine these nogoods to conclude that there is no solution with

$$(x = a) \wedge (y = b) \wedge (z = c) \qquad (2)$$

Moving z to the conclusion of (2) gives us

$$(x = a) \wedge (y = b) \rightarrow z \neq c$$

In general, suppose we have a collection of nogoods of the form

$$x_{i1} = v_{i1} \wedge \cdots \wedge x_{in_i} = v_{in_i} \rightarrow x \neq v_i$$

as i varies, where the same variable appears in the conclusions of all the nogoods. Suppose further that the antecedents all agree as to the value of the x_i's, so that any time x_i appears in the antecedent of one of the nogoods, it is in a term $x_i = v_i$ for a fixed v_i. If the nogoods collectively eliminate all of the possible values for x, we can conclude that $\bigwedge_j (x_j = v_j)$ is inconsistent; moving one specific x_k to the conclusion gives us

$$\bigwedge_{j \neq k} (x_j = v_j) \rightarrow x_k \neq v_k \qquad (3)$$

As before, note the freedom in our choice of variable appearing in the conclusion of the nogood. Since the next step in our search algorithm will presumably satisfy (3) by changing the value for x_k, the selection of consequent variable corresponds to the choice of variable to "flip" in the terms used by GSAT or other hill-climbing algorithms.

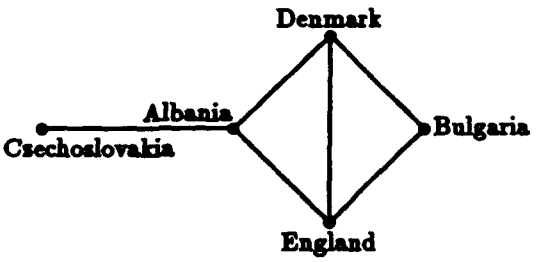

Figure 1: A small map-coloring problem

As we have remarked, dynamic backtracking accumulates information in a set of nogoods. To see how this is done, consider the map coloring problem in Figure 1, repeated from [6]. The map consists of five countries: Albania, Bulgaria, Czechoslovakia, Denmark and England. We assume – wrongly – that the countries border each other as shown in the figure, where countries are denoted by nodes and border one another if and only if there is an arc connecting them.

In coloring the map, we can use the three colors red, green and blue. We will typically abbreviate the colors and country names to single letters in the obvious way. The following table gives a trace of how a conventional dependency-directed backtracking scheme might attack this problem; each row shows a state of the procedure in the middle of a backtrack step, after a new nogood has been identified but before colors are erased to reflect the new conclusion. The coloring that is about to be removed appears in boldface. The "drop" column will be discussed shortly.

A	B	C	D	E	add		drop
r	g	**r**			$A = r \rightarrow C \neq r$	1	
r	g	b	**r**		$A = r \rightarrow D \neq r$	2	
r	g	b	**g**		$B = g \rightarrow D \neq g$	3	
r	g	b	b	**r**	$A = r \rightarrow E \neq r$	4	
r	g	b	b	**g**	$B = g \rightarrow E \neq g$	5	
r	g	b	b	**b**	$D = b \rightarrow E \neq b$	6	
r	g	b	**b**		$(A = r) \wedge (B = g) \rightarrow D \neq b$	7	6
r	**g**	b			$A = r \rightarrow B \neq g$	8	3, 5, 7

We begin by coloring Albania red and Bulgaria green, and then try to color Czechoslovakia red as well. Since this violates the constraint that Albania and Czechoslovakia be different colors, nogood (1) in the above table is produced.

We change Czechoslovakia's color to blue and then turn to Denmark. Since Denmark cannot be colored red or green, nogoods (2) and (3) appear; the only remaining color for Denmark is blue.

Unfortunately, having colored Denmark blue, we cannot color England. The three nogoods generated are (4), (5) and (6), and we can resolve these together because the three conclusions eliminate all of the possible colors for England. The result is that there is no solu-

tion with $(A = r) \wedge (B = g) \wedge (D = b)$, which we rewrite as (7) above. This can in turn be resolved with (2) and (3) to get (8), correctly indicating that the color of red for Albania is inconsistent with the choice of green for Bulgaria. The analysis can continue at this point to gradually determine that Bulgaria has to be red, Denmark can be green or blue, and England must then be the color not chosen for Denmark.

As we mentioned in the introduction, the problem with this approach is that the set Γ of nogoods grows monotonically, with a new nogood being added at every step. The number of nogoods stored therefore grows linearly with the run time and thus (presumably) exponentially with the size of the problem. A related problem is that it may become increasingly difficult to extend the partial solution P without violating one of the nogoods in Γ.

Dynamic backtracking deals with this by discarding nogoods when they become "irrelevant" in the sense that their antecedents no longer match the partial solution in question. In the example above, nogoods can be eliminated as indicated in the final column of the trace. When we derive (7), we remove (6) because Denmark is no longer colored blue. When we derive (8), we remove all of the nogoods with $B = g$ in their antecedents. Thus the only information we retain is that Albania's red color precludes red for Czechoslovakia, Denmark and England (1, 2 and 4) and also green for Bulgaria (3).

3 DYNAMIC BACKTRACKING

Dynamic backtracking uses the set of nogoods to both record information about the portion of the search space that has been eliminated and to record the current partial assignment being considered by the procedure. The current partial assignment is encoded in the antecedents of the current nogood set. More formally:

Definition 3.1 *An* acceptable next assignment *for a nogood set Γ is an assignment P satisfying every nogood in Γ and every antecedent of every such nogood. We will call a set of nogoods Γ* acceptable *if no two nogoods in Γ have the same conclusion and either $\bot \in \Gamma$ or there exists an acceptable next assignment for Γ.*

If Γ is acceptable, the antecedents of the nogoods in Γ induce a partial assignment of values to variables; any acceptable next assignment must be an extension of this partial assignment. In the above table, for example, nogoods (1) through (6) encode the partial assignment given by $A = r$, $B = g$, and $D = b$. Nogoods (1) though (7) fail to encode a partial assignment because the seventh nogood is inconsistent with the partial assignment encoded in nogoods (1) through (6). This is why the sixth nogood is removed when the seventh nogood is added.

Procedure 3.2 (Dynamic backtracking) To solve a CSP:

$P :=$ any complete assignment of values to variables
$\Gamma := \emptyset$
until *either* P *is a solution or* $\bot \in \Gamma$:
    $\gamma :=$ any constraint violated by P
    $\Gamma :=$ simp$(\Gamma \cup \gamma)$
    $P :=$ any acceptable next assignment for Γ

To simplify the discussion we assume a fixed total order on the variables. Versions of dynamic backtracking with dynamic rearrangement of the variable order can be found elsewhere [6, 10]. Whenever a new nogood is added, the fixed variable ordering is used to select the variable that appears in the conclusion of the nogood – the latest variable always appears in the conclusion. The subroutine simp closes the set of nogoods under the resolution inference rule discussed in the previous section and removes all nogoods which have an antecedent $x = v$ such that $x \neq v$ appears in the conclusion of some other nogood. Without giving a detailed analysis, we note that simplification ensures that Γ remains acceptable. To prove termination we introduce the following notation:

Definition 3.3 *For any acceptable Γ and variable x, we define the* live domain of x *to be those values v such that $x \neq v$ does not appear in the conclusion of any nogood in Γ. We will denote the size of the live domain of x by $|x|_\Gamma$, and will denote by $m(\Gamma)$ the tuple $\langle |x_1|_\Gamma, \ldots, |x_n|_\Gamma \rangle$ where $x_1, \ldots, x_n$ are the variables in the* CSP *in their specified order.*

Given an acceptable Γ, we define the size of Γ *to be*

$$\text{size}(\Gamma) = \prod_x |V_x| - \sum_x \left[(|V_x| - |x|_\Gamma) \prod_{x_i > x} |V_{x_i}| \right]$$

Informally, the size of Γ is the size of the remaining search space given the live domains for the variables and assuming that all information about x_i will be lost when we change the value for any variable $x_j < x_i$.

Lemma 3.4 *Suppose that Γ and Γ' are such that $m(\Gamma)$ is lexicographically less than $m(\Gamma')$. Then* $\text{size}(\Gamma) < \text{size}(\Gamma')$.

The termination proof (which we do not repeat here) is based on the observation that every simplification lexicographically reduces $m(\Gamma)$. Assuming that $\Gamma = \emptyset$ initially, since $\text{size}(\emptyset) = \prod_x |V_x|$ it follows that the running time of dynamic backtracking is bounded by the size of the problem being solved.

Proposition 3.5 *Any acceptable set of nogoods can be stored in $o(n^2 v)$ space where n is the number of*

variables and v is the maximum domain size of any single variable.

It is worth considering the behavior of Procedure 3.2 when applied to a CSP that is the union of two disjoint CSPs that do not share variables or constraints. If each of the two subproblems is unsatisfiable and the variable ordering interleaves the variables of the two subproblems, a classical backtracking search will take time proportional to the product of the times required to search each assignment space separately.[1] In contrast, Procedure 3.2 works on the two problems independently, and the time taken to solve the union of problems is therefore the sum of the times needed for the individual subproblems. It follows that Procedure 3.2 is fundamentally different from classical backtracking or backjumping procedures; Procedure 3.2 is in fact what has been called a *polynomial space aggressive backtracking procedure* [10].

4 DYNAMIC BACKTRACKING AS LOCAL SEARCH

Before proceeding, let us highlight the obvious similarities between Procedure 3.2 and Selman's description of GSAT [14]:

Procedure 4.1 (GSAT) To solve a CSP:

```
for i := 1 to MAX-TRIES
    P := a randomly generated truth assignment
    for j := 1 to MAX-FLIPS
        if P is a solution, then return it
        else flip any variable in P that results in
                the greatest decrease in the number
                of unsatisfied clauses
        end if
    end for
end for
return failure
```

The inner loop of the above procedure makes a local move in the search space in a direction consistent with the goal of satisfying a maximum number of clauses; we will say that GSAT follows the local gradient of a "maxsat" objective function. But local search can get stuck in local minima; the outer loop provides a partial escape by giving the procedure several independent chances to find a solution.

Like GSAT, dynamic backtracking examines a sequence of total assignments. Initially, dynamic backtracking has considerable freedom in selecting the next assignment; in many cases, it can update the total assignment in a manner identical to GSAT. The nogood set ultimately both constrains the allowed directions of motion and forces the procedure to search systematically. Dynamic backtracking cannot get stuck in local minima.

Both systematicity and the ability to follow local gradients are desirable. The observations of the previous paragraphs, however, indicate that these two properties are in conflict – systematic enumeration of the search space appears incompatible with gradient descent. To better understand the interaction of systematicity and local gradients, we need to examine more closely the structure of the nogoods used in dynamic backtracking.

We have already discussed the fact that a single constraint can be represented as a nogood in a variety of ways. For example, the constraint $\neg(A = r \wedge B = g)$ can be represented either as $A = r \rightarrow B \neq g$ or as $B = g \rightarrow A \neq r$. Although these nogoods capture the same information, they behave differently in the dynamic backtracking procedure because they encode different partial truth assignments and represent different choices of variable ordering. In particular, the set of acceptable next assignments for $A = r \rightarrow B \neq g$ is quite different from the set of acceptable next assignments for $B = g \rightarrow A \neq r$. In the former case an acceptable assignment must satisfy $A = r$; in the latter case, $B = g$ must hold. Intuitively, the former nogood corresponds to changing the value of B while the latter nogood corresponds to changing that of A. The manner in which we represent the constraint $\neg(A = r \wedge B = g)$ influences the direction in which the search is allowed to proceed. In Procedure 3.2, the choice of representation is forced by the need to respect the fixed variable ordering and to change the latest variable in the constraint.[2] Similar restrictions exist in the original presentation of dynamic backtracking itself [8].

5 PARTIAL-ORDER DYNAMIC BACKTRACKING

Partial-order dynamic backtracking [10] replaces the fixed variable order with a *partial* order that is dynamically modified during the search. When a new nogood is added, this partial ordering need not fix a unique representation – there can be considerable choice in the selection of the variable to appear in the conclusion of the nogood. This leads to freedom in the selection of the variable whose value is to be changed, thereby allowing greater flexibility in the directions that the procedure can take while traversing the search space. The locally optimal gradient followed by GSAT can be adhered to more often. The partial order on variables

[1] This observation remains true even if backjumping techniques are used.

[2] Note, however, that there is still considerable freedom in the choice of the constraint itself. A total assignment usually violates many different constraints.

is represented by a set of ordering constraints called *safety conditions*.

Definition 5.1 *A* safety condition *is an assertion of the form $x < y$ where x and y are variables. Given a set S of safety conditions, we will denote by $\leq_S$ the transitive closure of $<$, saying that S is* acyclic *if $\leq_S$ is antisymmetric. We will write $x <_S y$ to mean that $x \leq_S y$ and $y \not\leq_S x$.*

In other words, $x \leq y$ if there is some (possibly empty) sequence of safety conditions

$$x < z_1 < \ldots < z_n < y$$

The requirement of antisymmetry means simply that there are no two distinct x and y for which $x \leq y$ and $y \leq x$; in other words, $\leq_S$ has no "loops" and is a partial order on the variables. In this section, we restrict our attention to acyclic sets of safety conditions.

Definition 5.2 *For a nogood γ, we will denote by S_γ the set of all safety conditions $x < y$ such that x is in the antecedent of γ and y is the variable in its conclusion.*

Informally, we require variables in the antecedent of nogoods to precede the variables in their conclusions, since the antecedent variables have been used to constrain the live domains of the conclusions.

The state of the partial order dynamic backtracking procedure is represented by a pair $\langle \Gamma, S \rangle$ consisting of a set of nogoods and a set of safety conditions. In many cases, we will be interested in only the ordering information about variables that can precede a fixed variable x. To discard the rest of the ordering information, we discard all of the safety conditions involving any variable y that follows x, and then record only that y does indeed follow x. Somewhat more formally:

Definition 5.3 *For any set S of safety conditions and variable x, we define the* weakening *of S at x, to be denoted $W(S, x)$, to be the set of safety conditions given by removing from S all safety conditions of the form $z < y$ where $x <_S y$ and then adding the safety condition $x < y$ for all such y.*

The set $W(S, x)$ is a weakening of S in the sense that every total ordering consistent with S is also consistent with $W(S, x)$. However $W(S, x)$ usually admits more total orderings than S does; for example, if S specifies a total order then $W(S, x)$ allows any order which agrees with S up to and including the variable x. In general, we have the following:

Lemma 5.4 *For any set S of safety conditions, variable x, and total order $<$ consistent with the safety conditions in $W(S, x)$, there exists a total order consistent with S that agrees with $<$ through x.*

Procedure 5.5 To solve a CSP:

$P :=$ any complete assignment of values to variables
$\Gamma := \emptyset$
$S := \emptyset$
until either P is a solution or $\bot \in \Gamma$:
    $\gamma :=$ a constraint violated by P
    $\langle \Gamma, S \rangle := \mathtt{simp}(\Gamma, S, \gamma)$
    $P :=$ any acceptable next assignment for Γ

Procedure 5.6 To compute $\mathtt{simp}(\Gamma, S, \gamma)$:

select the conclusion x of γ so that $S \cup S_\gamma$ is acyclic
$\Gamma := \Gamma \cup \{\gamma\}$
$S := W(S \cup S_\gamma, x)$
remove from Γ each nogood with x in its antecedent
if the conclusions of nogoods in Γ rule out all
        possible values for x **then**
  $\rho :=$ the result of resolving all nogoods in Γ with x
        in their conclusion
  $\langle \Gamma, S \rangle := \mathtt{simp}(\Gamma, S, \rho)$
end if
return $\langle \Gamma, S \rangle$

The above simplification procedure maintains the invariant that Γ be acceptable and S be acyclic; in addition, the time needed for a single call to simp appears to grow significantly sublinearly with the size of the problem in question (see Section 7).

Theorem 5.7 *Procedure 5.5 terminates. The number of calls to* simp *is bounded by the size of the problem being solved.*

As an example, suppose that we return to our map-coloring problem. We begin by coloring all of the countries red except Bulgaria, which is green. The table on the next page shows the total assignment that existed at the moment each new nogood was generated.

The initial coloring violates a variety of constraints; suppose that we choose to work on one with Albania in its conclusion because Albania is involved in three violated constraints. We choose $C = r \rightarrow A \neq r$ specifically, and add it as (1) below.

We next modify Albania to be blue. The only constraint violated is that Denmark and England be different colors, so we add (2) to Γ. This suggests that we change the color for England; we try green, but this conflicts with Bulgaria. If we write the new nogood as $E = g \rightarrow B \neq g$, we will change Bulgaria to blue and be done. In the table above, however, we make the less optimal choice (3), changing the coloring for England again.

We are now forced to color England blue. This conflicts with Albania, and we continue to leave England in the conclusion of the nogood as we add (4). This nogood resolves with (2) and (3) to produce (5), where

we have once again made the worst choice and put D in the conclusion. We add this nogood to Γ and remove nogood (2), which is the only nogood with D in its antecedent. In (6) we add a safety condition indicating that D must continue to precede E. (This safety condition has been present since nogood (2) was discovered, but we have not indicated it explicitly until the original nogood was dropped from the database.)

A	B	C	D	E	add		drop
r	g	r	r	r	$C = r \rightarrow A \neq r$	1	
b	g	r	r	r	$D = r \rightarrow E \neq r$	2	
b	g	r	r	g	$B = g \rightarrow E \neq g$	3	
b	g	r	r	b	$A = b \rightarrow E \neq b$	4	
					$(A = b) \wedge (B = g) \rightarrow D \neq r$	5	2
					$D < E$	6	
b	g	r	g	r	$B = g \rightarrow D \neq g$	7	
b	g	r	b	r	$A = b \rightarrow D \neq b$	8	
					$A = b \rightarrow B \neq g$	9	3, 5, 7
					$B < E$	10	6
					$B < D$	11	

We next change Denmark to green; England is forced to be red once again. But now Bulgaria and Denmark are both green; we have to write this new nogood (7) with Denmark in the conclusion because of the ordering implied by nogood (5) above. Changing Denmark to blue conflicts with Albania (8), which we have to write as $A = b \rightarrow D \neq b$. This new nogood resolves with (5) and (7) to produce (9).

We drop (3), (5) and (7) because they involve $B = g$, and introduce the two safety conditions (10) and (11). Since E follows B, we drop the safety condition $E < D$. At this point, we are finally forced to change the color for Bulgaria and the search continues.

It is important to note that the added flexibility of PDB over dynamic backtracking arises from the flexibility in the first step of the simplification procedure where the conclusion of the new nogood is selected. This selection corresponds to a selection of a variable whose value is to be changed.

As with the procedure in the previous section, when given a CSP that is a union of disjoint CSPs the above procedure will treat the two subproblems independently. The total running time remains the sum of the times required for the subproblems.

6 ARBITRARY MOVEMENT

Partial-order dynamic backtracking still does not provide total freedom in the choice of direction through the search space. When a new nogood is discovered, the existing partial order constrains how we are to interpret that nogood – roughly speaking, we are forced to change the value of late variables before changing the values of their predecessors. The use of a partial order makes this constraint looser than previously, but it is still present. In this section, we allow cycles in the nogoods and safety conditions, thereby permitting arbitrary choice in the selection of the variable appearing in the conclusion of a new nogood.

The basic idea is the following: Suppose that we have introduced a loop into the variable ordering, perhaps by including the pair of nogoods $x \rightarrow \neg y$ and $y \rightarrow x$. Rather than rewrite one of these nogoods so that the same variable appears in the conclusion of both, we will view the (x, y) combination as a single variable that takes a value in the product set $V_x \times V_y$.

If x and y are variables that have been "combined" in this way, we can rewrite a nogood with (for example) x in its antecedent and y in its conclusion so that both x and y are in the conclusion. As an example, we can rewrite

$$x = v_x \wedge z = v_z \rightarrow y \neq v_y \tag{4}$$

as

$$z = v_z \rightarrow (x, y) \neq (v_x, v_y) \tag{5}$$

which is logically equivalent. We can view this as eliminating a particular value for the pair of variables (x, y).

Definition 6.1 *Let S be a set of safety conditions (possibly not acyclic). We will write $x \equiv_S y$ if $x \leq_S y$ and $y \leq_S x$. The equivalence class of x under $\equiv$ will be denoted $\langle x \rangle_S$. If γ is a nogood whose conclusion involves the variable x, we will denote by γ_S the result of moving to the conclusion of γ all terms involving members of $\langle x \rangle_S$. If Γ is a set of nogoods, we will denote by Γ_S is the set of nogoods of the form γ_S for $\gamma \in \Gamma$.*

It is not difficult to show that for any set S of safety conditions, the relation $\equiv_S$ is an equivalence relation. As an example of rewriting a nogood in the presence of ordering cycles, suppose that γ is the nogood (4) and let S be such that $\langle y \rangle_S = \{x, y\}$; now γ_S is given by (5).

Placing more than one literal in the conclusions of nogoods forces us to reconsider the notion of an acceptable next assignment:

Definition 6.2 *A* cyclically acceptable next assignment *for a nogood set Γ under a set S of safety conditions is a total assignment P of values to variables satisfying every nogood in Γ_S and every antecedent of every such nogood.*

We now define a third dynamic backtracking procedure. Note that $W(S, x)$ remains well defined even if S is not acyclic, since $W(S, x)$ drops ordering constraints only on variables y such that $x <_S y$.

Procedure 6.3 To solve a CSP:

$P :=$ any complete assignment of values to variables
$\Gamma := \emptyset$
$S := \emptyset$
until either P is a solution or $\bot \in \Gamma$:
    $\gamma :=$ a constraint violated by P
    $\langle \Gamma, S \rangle := \mathtt{simp}(\Gamma, S, \gamma)$
    $P :=$ any cyclically acceptable next assignment
        for Γ under S

Procedure 6.4 To compute $\mathtt{simp}(\Gamma, S, \gamma)$:

select a conclusion x for γ (now unconstrained)
$\Gamma := \Gamma \cup \{\gamma\}$
$S := W(S \cup S_\gamma, x)$
remove from Γ each nogood α with an element of $\langle x \rangle_S$
    in the antecedent of α_S
if the conclusions of nogoods in Γ_S rule out all
    possible values for the variables in $\langle x \rangle_S$ then
  $\rho :=$ the result of resolving all nogoods in Γ_S whose
    conclusions involve variables in $\langle x \rangle_S$
  $\langle \Gamma, S \rangle := \mathtt{simp}(\Gamma, S, \rho)$
  end if
return $\langle \Gamma, S \rangle$

If the conclusion is selected so that S remains acyclic, the above procedure is identical to the one in the previous section.

Proposition 6.5 *Suppose that we are working on a problem with n variables, that the size of the largest domain of any variable is v, and that we have constructed Γ and S using repeated applications of* simp. *If the largest equivalence class $\langle x \rangle_S$ contains d elements, the space required to store Γ is $o(n^2 v^d)$.*

If we have an equivalence class of d variables each of which has v possible values then the number of possible values of the "combined variable" is v^d. The above procedure can now generate a distinct nogood to eliminate each of the v^d possible values, and the space requirements of the procedure can therefore grow exponentially in the size of the equivalence classes. The time required to find a cyclically allowed next assignment can also grow exponentially in the size of the equivalence classes. We can address these difficulties by selecting in advance a bound for the largest allowed size of any equivalence class. In any event, termination is still guaranteed:

Theorem 6.6 *Procedure 6.3 terminates. The number of calls to* simp *is bounded by the size of the problem being solved.*

Selecting a variable to place in the conclusion of a new nogood corresponds to choosing the variable whose value is to be changed on the next iteration and is analogous to selecting the variable to flip in GSAT. Since the choice of conclusion is unconstrained in the above procedure, the procedure has tremendous flexibility in the way it traverses the search space. Like the procedures in the previous sections, Procedure 6.3 continues to solve combinations of independent subproblems in time bounded by the sum of the times needed to solve the subproblems individually.

Here are these ideas in use on a Boolean CSP with the constraints $a \rightarrow b$, $b \rightarrow c$ and $c \rightarrow \neg b$. As before, we present a trace and then explain it:

a	b	c	add to Γ		remove from Γ
t	f	f	$a \rightarrow b$	1	
t	t	f	$b \rightarrow c$	2	
t	t	t	$c \rightarrow \neg b$	3	
			$\neg a$	4	1
			$a < b$	5	

The first three nogoods are simply the three constraints appearing in the problem. Although the orderings of the second and third nogoods conflict, we choose to write them in the given form in any case.

Since this puts b and c into an equivalence class, we do not drop nogood (2) at this point. Instead, we interpret nogood (1) as requiring that the value taken by (b, c) be either (t, t) or (t, f); (2) disallows (t, f) and (3) disallows (t, t). It follows that the three nogoods can be resolved together to obtain the new nogood given simply by $\neg a$. We add this as (4) above, dropping nogood (1) because its antecedent is falsified.

7 EXPERIMENTAL RESULTS

In this section, we present preliminary results regarding the implemented effectiveness of the procedure we have described. The implementation is based on the somewhat restricted Procedure 5.5 as opposed to the more general Procedure 6.3. We compared a search engine based on this procedure with two others, TABLEAU [2] and WSAT, or "walk-sat" [13]. TABLEAU is an efficient implementation of the Davis-Putnam algorithm and is systematic; WSAT is a modification to GSAT and is not. We used WSAT instead of GSAT because WSAT is more effective on a fairly wide range of problem distributions [13].

The experimental data was not collected using the random 3-SAT problems that have been the target of much recent investigation, since there is growing evidence that these problems are not representative of the difficulties encountered in practice [3]. Instead, we generated our problems so that the clauses they contain involve groups of locally connected variables as opposed to variables selected at random.

Somewhat more specifically, we filled an $n \times n$ square grid with variables, and then required that the three variables appearing in any single clause be neighbors

in this grid. LISP code generating these examples appears in the appendix. We believe that the qualitative properties of the results reported here hold for a wide class of distributions where variables are given spatial locations and clauses are required to be local.

The experiments were performed at the crossover point where approximately half of the instances generated could be expected to be satisfiable, since this appears to be where the most difficult problems lie [2]. Note that not all instances at the crossover point are hard; as an example, the local variable interactions in these problems can lead to short resolution proofs that no solution exists in unsatisfiable cases. This is in sharp contrast with random 3-SAT problems (where no short proofs appear to exist in general, and it can even be shown that proof lengths are growing exponentially on average [1]). Realistic problems may often have short proof paths: A particular scheduling problem may be unsatisfiable simply because there is no way to schedule a specific resource as opposed to because of global issues involving the problem in its entirety. Satisfiability problems arising in VLSI circuit design can also be expected to have locality properties similar to those we have described.

The problems involved 25, 100, 225, 400 and 625 variables. For each size, we generated 100 satisfiable and 100 unsatisfiable instances and then executed the three procedures to measure their performance. (WSAT was not tested on the unsatisfiable instances.) For WSAT, we measured the number of times specific variable values were flipped. For PDB, we measured the number of top-level calls to Procedure 5.6. For TABLEAU, we measured the number of choice nodes expanded. WSAT and PDB were limited to 100,000 flips; TABLEAU was limited to a running time of 150 seconds.

The results for the satisfiable problems were as follows. For TABLEAU, we give the node count for successful runs only; we also indicate parenthetically what fraction of the problems were solved given the computational resource limitations. (WSAT and PDB successfully solved all instances.)

Variables	PDB	WSAT	TABLEAU
25	35	89	9 (1.0)
100	210	877	255 (1.0)
225	434	1626	504 (.98)
400	731	2737	856 (.70)
625	816	3121	502 (.68)

For the unsatisfiable instances, the results were:

Variables	PDB	TABLEAU
25	122	8 (1.0)
100	509	1779 (1.0)
225	988	5682 (.38)
400	1090	558 (.11)
625	1204	114 (.06)

The times required for PDB and WSAT appear to be growing comparably, although only PDB is able to solve the unsatisfiable instances. The eventual *decrease* in the average time needed by TABLEAU is because it is only managing to solve the easiest instances in each class. This causes TABLEAU to become almost completely ineffective in the unsatisfiable case and only partially effective in the satisfiable case. Even where it does succeed on large problems, TABLEAU's run time is greater than that of the other two methods.

Finally, we collected data on the time needed for each top-level call to simp in partial-order dynamic backtracking. As a function of the number of variables in the problem, this was:

Number of variables	PDB (msec)	WSAT (msec)
25	3.9	0.5
100	5.3	0.3
225	6.7	0.6
400	7.0	0.7
625	8.4	1.4

All times were measured on a Sparc 10/40 running unoptimised Allegro Common Lisp. An efficient C implementation could expect to improve either method by approximately an order of magnitude. As mentioned in Section 5, the time per flip is growing sublinearly with the number of variables in question.

8 CONCLUSION AND FUTURE WORK

Our aim in this paper has been to make a primarily theoretical contribution, describing a new class of constraint-satisfaction algorithms that appear to combine many of the advantages of previous systematic and nonsystematic approaches. Since our focus has been on a description of the algorithms, there is obviously much that remains to be done.

First, of course, the procedures must be tested on a variety of problems, both synthetic and naturally occurring; the results reported in Section 7 only scratch the surface. It is especially important that realistic problems be included in any experimental evaluation of these ideas, since these problems are likely to have performance profiles substantially different from those of randomly generated problems [3]. The experiments of the previous section need to be extended to include unit resolution, and we need to determine the frequency with which exponential space is needed in practice by the full procedure 6.3.

Finally, we have left completely untouched the question of how the flexibility of Procedure 6.3 is to be exploited. Given a group of violated constraints, which should we pick to add to Γ? Which variable should be in the conclusion of the constraint? These choices

correspond to choice of backtrack strategy in a more conventional setting, and it will be important to understand them in this setting as well.

A Experimental code

Here is the code used to generate instances of the class of problems on which our ideas were tested. The two arguments to the procedure are the size s of the variable grid and the number c of clauses to be "centered" on any single variable.

For each grid variable x we generated $\lfloor c \rfloor$ or $\lfloor c \rfloor + 1$ clauses at random subject to the constraint that the variables in each clause form a right triangle with horizontal and vertical sides of length 1 and where x is the vertex opposite the hypotenuse. There are four such triangles for a given x. There are eight assignments of values to variable for each triangle giving 32 possible clauses. Variables at the edge of the grid usually generate fewer than c clauses so the boundary of the grid is relatively unconstrained.

```
(defun make-problem (s c &aux result xx yy)
  (dotimes (x s result)
    (dotimes (y s)
      (dotimes (i (+ (floor c)
                     (if (> (random 1.0)
                            (rem c 1.0))
                         0 1)))
        (setq xx (+ x -1 (* 2 (random 2)))
              yy (+ y -1 (* 2 (random 2))))
        (when (and (< -1 xx s) (< -1 yy s))
          (push (new-clause x y xx yy s)
                result))))))

(defun new-clause (x y xx yy s)
  (mapcar
   #'(lambda (a b &aux (v (+ 1 (* s a) b)))
       (if (zerop (random 2)) v (- v))))
   (list x xx x) (list y y yy))
```

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Foundations of Indefinite Constraint Databases

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Abstract

We lay the foundations of a theory of constraint databases with indefinite information based on the relational model. We develop the scheme of indefinite $\mathcal{L}$-constraint databases where $\mathcal{L}$, the parameter, is a first-order constraint language. This scheme extends the proposal of Kanellakis, Kuper and Revesz to include indefinite information in the style of Imielinski and Lipski. We propose declarative and procedural query languages for the new scheme and study the semantics of query evaluation.

1 Introduction

In this paper we lay the foundations of a theory of *indefinite constraint databases* based on the relational model [Mai83]. As a starting point of our investigation, we take the model of constraint databases proposed in [KKR90]. This model is useful for the representation of *unrestricted* (i.e., finite or infinite) *definite* information. However, indefinite information is also important in many applications e.g., planning and scheduling, medical expert systems, geographical information systems and natural language processing systems. Motivated by these practical considerations, we develop the model of *indefinite constraint databases* which allows the representation of *definite, indefinite, finite* and *infinite* information in a single unifying framework.

Our contributions to the theory of constraint databases can be summarized as follows:

- We develop the scheme of *indefinite $\mathcal{L}$-constraint databases* where $\mathcal{L}$, the parameter, is a first-order *constraint language*. This parameterized model extends the scheme of [KKR90] to include indefinite information in the style of [IL84, Gra89] (section 3).
- We propose *modal relational calculus* with $\mathcal{L}$-constraints as a declarative query languages for indefinite $\mathcal{L}$-constraint databases (section 4). We also propose a procedural query language: the *modal $\mathcal{L}$-constraint algebra* (section 5).
- We show that expressions of modal relational calculus with $\mathcal{L}$-constraints can be evaluated *bottom-up in closed form* on indefinite $\mathcal{L}$-constraint databases. This is a direct consequence of the fact that every expression of modal relational calculus with $\mathcal{L}$-constraints has an equivalent expression in modal $\mathcal{L}$-constraint algebra (section 7). This result could be the first step in developing optimization techniques for $\mathcal{L}$-constraint databases and indefinite $\mathcal{L}$-constraint databases.

This paper is organized as follows. The next section presents some examples of constraint languages and defines the relevant abstract concepts. In section 3 we present the scheme of indefinite $\mathcal{L}$-constraint databases.

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In sections 4 and 5 we discuss the modal relational calculus with $\mathcal{L}$-constraints and the modal $\mathcal{L}$-constraint algebra. In section 6 we present several results concerning algebraic query evaluation in $\mathcal{L}$-constraint databases and indefinite $\mathcal{L}$-constraint databases. In section 7 we discuss the translation of expressions of modal relational calculus with $\mathcal{L}$-constraints into expressions of modal $\mathcal{L}$-constraint algebra. Finally, section 8 presents related work.

2 Constraint Languages

In this paper we consider many-sorted languages, structures and theories [End72]. Every language $\mathcal{L}$ will be interpreted over a *fixed* structure, called the *intended structure*, which will usually be denoted by $\mathbf{M}_{\mathcal{L}}$. If $\mathbf{M}$ is a structure then $Th(\mathbf{M})$ will denote the theory of $\mathbf{M}$ i.e., the set of sentences which are true in $\mathbf{M}$. For every language $\mathcal{L}$, we will distinguish a class of quantifier free formulas called *$\mathcal{L}$-constraints*. The atomic formulas of $\mathcal{L}$ will be included in the class of $\mathcal{L}$-constraints. There will also be two distinguished $\mathcal{L}$-constraints *true* and *false* with obvious semantics. Similar assumptions have been made in [Mah93] in the contex of the CLP scheme. A set of $\mathcal{L}$-constraints will be the algebraic counterpart of the logical conjunction of its members. Thus we will freely mix the terms "set of $\mathcal{L}$-constraints" and "conjunction of $\mathcal{L}$-constraints". We will assume that the reader is familiar with the notions of *solution, consistency* and *equivalence* of sets of constraints [Mah93].

Let us now give some examples of constraint languages.

Example 2.1 The language *ECL* (*E*quality *C*onstraint *L*anguage) with predicate symbols $=$, $\neq$ and an infinite number of constants has been defined in [KKR90]. The intended structure for this language interprets $=$ as equality, $\neq$ as non-equality and constants as "themselves". An *ECL-constraint* is an ECL formula of the form $x_1 = x_2$ or $x_1 \neq x_2$ where x_1, x_2 are variables or constants. ECL has been used by [KKR90] for the development of an extended relational model based on ECL-constraints.

We now present a language for expressing *temporal constraints*.

Example 2.2 The language *dePCL* (*d*ense *P*oint *C*onstraint *L*anguage) allows us to make stamements about points in dense time. dePCL is a first-order language with equality and the following set of non-logical symbols: the set of rational numerals, function symbol $-$ of arity 2 and predicate symbol $<$ of arity 2. The *terms* and *atomic formulas* of dePCL are defined as follows. Constants and variables are terms. If t_1 and t_2 are variables or constants then $t_1 - t_2$ is a term. An *atomic formula* of dePCL is a formula of the form $t \sim c$ or $c \sim t$ where $\sim$ is $<$ or $=$ and t is a term.

The intended structure for dePCL is $\mathbf{Q}$. $\mathbf{Q}$ interprets each rational numeral by its corresponding rational number, function symbol $-$ by the subtraction operation over the rationals and $<$ by the relation "less than". The theory $Th(\mathbf{Q})$ is a subtheory of real addition with order [Rab77].

A *dePCL-constraint* is a dePCL formula of the form $t \sim c$ where t is a term, c is a constant and $\sim$ is $=$, $<$, $>$, $\leq$ or $\geq$. For example, the formulas $p_1 < p_2$, $p_3 - p_4 \geq 15$, $p_3 = 5/4$ are dePCL-constraints.

Example 2.3 Let us also consider the many-sorted language ECL+dePCL which is the union of ECL and dePCL. The sorts of ECL+dePCL are $\mathcal{D}$ (for the infinite set of constants of ECL) and Q (for the rational numerals of dePCL). The symbols of ECL+dePCL are interpreted by the many-sorted structure which is the union of the intended structures for ECL and dePCL.

Let us now define the concept of *variable elimination*.[1]

Definition 2.1 Let $\mathcal{L}$ be a many-sorted first-order language. The class of $\mathcal{L}$-constraints *admits variable elimination* iff for every boolean combination ϕ of $\mathcal{L}$-constraints in variables $\overline{x}$, and every vector of variables $\overline{z} \subseteq \overline{x}$, there exists a disjunction ϕ' of conjunctions of $\mathcal{L}$-constraints in variables $\overline{x} \setminus \overline{z}$ such that

[1] **Notation: The vector of symbols $(o_1, \ldots, o_n)$ will be denoted by $\overline{o}$. The natural number n will be called the *size* of $\overline{o}$ and will be denoted by $|\overline{o}|$. This notation will be used for vectors of variables but also for vectors of domain elements. Variables will be denoted by x, y, z, t etc. and vectors of variables by $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ etc. If $\overline{x}$ and $\overline{y}$ are vectors of variables then $\overline{x} \setminus \overline{y}$ will denote the vector obtained from $\overline{x}$ by deleting the variables in $\overline{y}$. If $\overline{x}$ is a vector of variables then $\overline{x}^0$ will be a vector of constants of the same size.**

1. If $\overline{x}^0$ is a solution of ϕ then $\overline{x}^0 \setminus \overline{z}^0$ is a solution of ϕ'.

2. If $\overline{x}^0 \setminus \overline{z}^0$ is a solution of ϕ' then this solution can be extended to a solution $\overline{x}^0$ of ϕ.

Some people might find the above definition overly strong. But requiring ϕ' to be just a boolean combination of $\mathcal{L}$-constraints would turn out to be unsatisfactory for the database models discussed in section 3. The reason is very simple: when we eliminate variables, we would have to deal with negations of $\mathcal{L}$-constraints. Similar arguments and definitions appear in [Stu91].

The following definition will be useful in the forthcoming sections.

Definition 2.2 Let $\mathcal{L}$ be a many-sorted first-order language. The class of $\mathcal{L}$-constraints is *weakly closed under negation* if the negation of every $\mathcal{L}$-constraint is equivalent to a disjunction of $\mathcal{L}$-constraints.

In the rest of this paper we will only be interested in constraints which admit variable elimination and are weakly closed under negation. Many interesting classes of constraints fall under this category. The following proposition shows that this is also the case for the constraint classes defined in this section.

Proposition 2.1 *The classes of ECL-constraints, dePCL-constraints and ECL+dePCL-constraints admit variable elimination and are weakly closed under negation.*

3 Indefinite Constraint Databases

We will now extend the $\mathcal{L}$-constraint database model of [KKR90] to account for indefinite information in the style of [IL84, Gra89]. For the rest of this section, let $\mathcal{L}$ be a many-sorted language and $\mathbf{M}_\mathcal{L}$ be the *intended $\mathcal{L}$-structure*. Let us also assume that the class of $\mathcal{L}$-constraints admits variable elimination and is weakly closed under negation.

For each sort $s \in sorts(\mathcal{L})$, let U_s be a countably infinite set of *attributes* of sort s. The set of all attributes, denoted by $\mathcal{U}$, is $\bigcup_{s \in sorts(\mathcal{L})} U_s$. The sort of attribute A will be denoted by $sort(A)$. With each $A \in \mathcal{U}$ we associate a set of values $dom(A) = dom(s, \mathbf{M}_\mathcal{L})$ called the *domain* of A.[2] A *relation scheme* R is a finite subset of $\mathcal{U}$.

We will first define $\mathbf{M}_\mathcal{L}$-relations which are unrestricted (i.e., finite or infinite) standard relations. $\mathbf{M}_\mathcal{L}$-relations are a theoretical device for giving semantics to indefinite $\mathcal{L}$-constraint relations.

Definition 3.1 Let R be a relation scheme. An *$\mathbf{M}_\mathcal{L}$-relational tuple* t over scheme R is a mapping from R to $\bigcup_{s \in sorts(\mathcal{L})} dom(s, \mathbf{M}_\mathcal{L})$ such that $t(A) \in dom(sort(A), \mathbf{M}_\mathcal{L})$. An *$\mathbf{M}_\mathcal{L}$-relation* r over scheme R is an *unrestricted* set of $\mathbf{M}_\mathcal{L}$-relational tuples over R.

For every $s \in sorts(\mathcal{L})$, we now assume the existence of two disjoint countably infinite sets of *variables*: the set of *u-variables* $UVAR^s_\mathcal{L}$ and the set of *e-variables* $EVAR^s_\mathcal{L}$. Let $UVAR_\mathcal{L}$ and $EVAR_\mathcal{L}$ denote $\bigcup_{s \in sorts(\mathcal{L})} UVAR^s_\mathcal{L}$ and $\bigcup_{s \in sorts(\mathcal{L})} EVAR^s_\mathcal{L}$ respectively. The intersection of the sets $UVAR_\mathcal{L}$ and $EVAR_\mathcal{L}$ with the domains of attributes is empty.

Notation 3.1 U-variables will be denoted by letters of the English alphabet, usually x, y, z, t, possibly subscripted. E-variables will be denoted by letters of the Greek alphabet, usually $\omega, \lambda, \zeta, \nu$, possibly subscripted.

Definition 3.2 Let R be a relation scheme. An *indefinite $\mathcal{L}$-constraint tuple* t over scheme R is a mapping from $R \cup \{CON\}$ to $UVAR_\mathcal{L} \cup WFF(\mathcal{L})$ such that (i) $t(A) \in UVAR^{sort(A)}_\mathcal{L}$ for each $A \in R$, (ii) $t(A_i)$ is different than $t(A_j)$ for all distinct A_i, $A_j \in R$, (iii) $t(CON)$ is a conjunction of $\mathcal{L}$-constraints and (iv) the free variables of $t(CON)$ are included in $\{t(A) : A \in R\} \cup EVAR_\mathcal{L}$. $t(CON)$ is called the *local condition* of the tuple t while $t(R)$ is called the *proper part* of t.

Definition 3.3 Let R be a relation scheme. An *indefinite $\mathcal{L}$-constraint relation* over scheme R is a *finite* set of indefinite $\mathcal{L}$-constraint tuples over R. Each indefinite $\mathcal{L}$-constraint relation r is associated with a boolean combination of $\mathcal{L}$-constraints $G(r)$, called the *global condition* of r.

[2] If s is a sort and $\mathbf{M}$ is a structure then $dom(s, \mathbf{M})$ denotes the domain of s in structure $\mathbf{M}$.

Similarly we can define database schemes, $\mathbf{M}_{\mathcal{L}}$-relational databases and indefinite $\mathcal{L}$-constraint databases [Kou94a]. Database schemes and databases will usually be denoted by $\widetilde{R}$ and $\widetilde{r}$ respectively.

The above definitions extend the model of [KKR90] by introducing *e-variables* which have the semantics of marked nulls of [IL84]. As in [Gra89], the possible values of the e-variables can be constrained by a *global condition*.

Example 3.1 BOOKED is an indefinite ECL+dePCL-constraint relation giving the times that rooms are booked. The first tuple says that room WP212 is booked from 1:00 to 7:00. For room WP219 the information is indefinite: it is booked from 1:00 until some time between 5:00 and 8:00. This indefinite information is captured by the *e-variable* ω and its global condition $5 \leq \omega \leq 8$. *E-variables* can be understood as being existentially quantified and their scope is the entire database. They represent values that exist but are not known precisely [IL84, Gra89]. All we know about these values is captured by the global condition. *U-variables* (e.g., x_1, x_2, t_1, t_2) can be understood as being universally quantified and their scope is the tuple in which they appear [KKR90].

BOOKED

Room	Time	CON
x_1	t_1	$x_1 = WP212,\ 1 \leq t_1 < 7$
x_2	t_2	$x_2 = WP219,\ 1 \leq t_2 < \omega$

$$G(BOOKED):\ 5 \leq \omega \leq 8$$

3.1 Semantics

Let us first define two special kinds of valuations. An *e-valuation* in $\mathbf{M}_{\mathcal{L}}$ is a valuation whose domain is restricted to the set $EVAR_{\mathcal{L}}$. Similarly, a *u-valuation* in $\mathbf{M}_{\mathcal{L}}$ is a valuation whose domain is restricted to the set $UVAR_{\mathcal{L}}$. The symbols $Val^{e}_{\mathbf{M}_{\mathcal{L}}}$ and $Val^{u}_{\mathbf{M}_{\mathcal{L}}}$ will denote the set of e-valuations and u-valuations in $\mathbf{M}_{\mathcal{L}}$ respectively. The result of applying an e-valuation v to an indefinite $\mathcal{L}$-constraint relation r over R will be denoted by $v(r)$. $v(r)$ is an $\mathcal{L}$-constraint relation over R obtained from r by substituting each e-variable ω of r by the constant symbol whose denotation in structure $\mathbf{M}_{\mathcal{L}}$ is $v(\omega)$. The result of applying a u-valuation of $\mathbf{M}_{\mathcal{L}}$ to the proper part of a tuple can be defined as follows. If t is an $\mathcal{L}$-constraint tuple on scheme R and u is a u-valuation in $\mathbf{M}_{\mathcal{L}}$ then $u(t)$ is an $\mathbf{M}_{\mathcal{L}}$-tuple over R such that for each $A \in R$, $u(t)(A) = u(t(A))$.

The *semantics* of an $\mathcal{L}$-constraint relation is given by the function *points* [KKR90]. *points* takes as argument an $\mathcal{L}$-constraint relation r over R and returns the $\mathbf{M}_{\mathcal{L}}$-relation over R which is finitely represented by r:

$$points(r) = \{u(t):\ t \in r,\ u \in Val^{u}_{\mathbf{M}_{\mathcal{L}}} \text{ and } \mathbf{M}_{\mathcal{L}} \models t(CON)[u]\}.$$

The *semantics* of an indefinite $\mathcal{L}$-constraint relation r over scheme R is defined to be the following set of $\mathbf{M}_{\mathcal{L}}$-relations:

$$sem(r) = \{points(v(r)):\ \text{there exists } v \in Val^{e}_{\mathbf{M}_{\mathcal{L}}} \text{ s.t. } \mathbf{M}_{\mathcal{L}} \models G(r)[v]\}.$$

The function *rep* will also be useful in the rest of this paper. If r is an indefinite $\mathcal{L}$-constraint relation over scheme R then *rep* gives the set of $\mathcal{L}$-constraint relations represented by r:

$$rep(r) = \{v(r):\ \text{there exists } v \in Val^{e}_{\mathbf{M}_{\mathcal{L}}} \text{ s.t. } \mathbf{M}_{\mathcal{L}} \models G(r)[v]\}$$

The functions *points*, *sem* and *rep* can be extended to databases in the obvious way.[3]

[3]The above definitions imply that indefinite $\mathcal{L}$-constraint relations are interpreted in a *closed-world* fashion. They are assumed to represent all facts relevant to an application domain. However the exact value of any attribute of these facts may not be known precisely.

4 Declarative Query Languages

[KKR90] proposed *relational calculus with $\mathcal{L}$-constraints* as a declarative query language for $\mathcal{L}$-constraint databases. In this section we propose *modal relational calculus with $\mathcal{L}$-constraints* as a declarative query language for indefinite $\mathcal{L}$-constraint databases. Similar query languages have been investigated in [Lip79, Lev84, Rei88].

Definition 4.1 Let $\widetilde{R}$ be a database scheme and $R(C_1, \ldots, C_m)$ be a relation scheme. An expression over $\widetilde{R}$ in *modal relational calculus with $\mathcal{L}$-constraints* is $\{R(C_1, \ldots, C_m), x_1/s_1, \ldots, x_m/s_m \ : \ OP\ \phi(x_1, \ldots, x_m)\}$ where $s_i \in sorts(\mathcal{L})$ is the sort of C_i, OP is an *optional* modal operator $\Diamond$ or $\Box$, ϕ is a well-formed formula of relational calculus with $\mathcal{L}$-constraints and $x_1, \ldots, x_m$ are the only free variables of ϕ. If an expression does not contain a modal operator then it will be called *pure*, otherwise it will be called *modal*.

Let us now define the *value* of expressions in modal relational calculus.

Definition 4.2 Let f be the pure expression $\{R(C_1, \ldots, C_m), x_1/s_1, \ldots, x_m/s_m \ : \ \phi(x_1, \ldots, x_m)\}$ over $\widetilde{R}$ in modal relational calculus with $\mathcal{L}$-constraints. If $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database over $\widetilde{R}$ then the *value* of f on input database $\widetilde{r}$, denoted by $f(\widetilde{r})$, is the following set of $\mathbf{M}_{\mathcal{L}}$-relations:

$$\{\ \{(a_1, \ldots, a_m) \in dom(s_1) \times \cdots \times dom(s_m) : \ (\mathbf{M}_{\mathcal{L}}, Dom, \widetilde{r}') \models \phi(a_1, \ldots, a_m)\} \ : \widetilde{r}' \in sem(\widetilde{r})\}$$

The above definition is somewhat problematic. The value of a pure expression over an indefinite $\mathcal{L}$-constraint database is defined to be an unrestricted set whose elements are unrestricted sets of tuples! Can we guarantee *closure* as required by the constraint query language principles laid out in [KKR90]? In other words, given a pure expression f of modal relational calculus with $\mathcal{L}$-constraints, and an indefinite $\mathcal{L}$-constraint database $\widetilde{r}$, is it possible to find an indefinite $\mathcal{L}$-constraint relation which finitely represents $f(\widetilde{r})$? In section 7, we show that this closure property can indeed be guaranteed.

Example 4.1 The query "Find all rooms that are booked at 6:00" over the database of example 3.1 can be expressed as $\{BOOKED\_AT\_6(Room), x/\mathcal{D} : BOOKED(x, 6)\}$. If this query is evaluated using the method of section 7, the answer will be the following relation:

BOOKED_AT_6

Room	CON
x_1	$x_1 = WP212$
x_2	$x_2 = WP219,\ \omega > 6$

This answer is *conditional*. Room WP212 is booked on time 6. However, room WP219 is booked on time 6 *only under the condition* that ω is greater than 6.

Definition 4.3 Let f be the modal expression $\{R(C_1, \ldots, C_m), x_1/s_1, \ldots, x_m/s_m \ : \ \Box\ \phi(x_1, \ldots, x_m)\}$ over $\widetilde{R}$ in modal relational calculus with $\mathcal{L}$-constraints. If $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database over $\widetilde{R}$ then the *value* of f on input database $\widetilde{r}$, denoted by $f(\widetilde{r})$, is the following set containing a *single* $\mathbf{M}_{\mathcal{L}}$-relation:

$$\{\ \{(a_1, \ldots, a_m) \in dom(s_1) \times \cdots \times dom(s_m) : \text{ for every } \mathbf{M}_{\mathcal{L}}\text{-relational database } \widetilde{r}' \in sem(\widetilde{r}) \\ (\mathbf{M}_{\mathcal{L}}, Dom, \widetilde{r}') \models \phi(a_1, \ldots, a_m)\}\ \}$$

The value of a $\Diamond$-expression is defined in the same way but now the quantification over $\mathbf{M}_{\mathcal{L}}$-relational databases in $sem(\widetilde{r})$ is existential. Section 7 demonstrates that expressions of modal relational calculus with $\mathcal{L}$-constraints can also be evaluated bottom-up in closed form. In summary, for every expression f (pure or modal) in modal relational calculus with $\mathcal{L}$-constraints and indefinite $\mathcal{L}$-constraint database $\widetilde{r}$, it is possible to find an indefinite $\mathcal{L}$-constraint relation which finitely represents $f(\widetilde{r})$.

Example 4.2 The query "Find all rooms that are possibly booked at 6:00" over the database of example 3.1 can be expressed as $\{POSS\_BOOKED\_AT\_6(Room), x/\mathcal{D} : \Diamond BOOKED(x, 6)\}$. If this query is evaluated using the method of section 7, the answer will be the following relation:

POSS_BOOKED_AT_6

Room	CON
x_1	$x_1 = WP212$
x_2	$x_2 = WP219$

The above answer is *unconditional*. It is possible that both rooms WP212 and WP219 are booked on time 6.

The next lemma demonstrates an intuitive property of modal relational calculus with $\mathcal{L}$-constraints. If S is a set of sets then $\bigcap S$ (resp. $\bigcup S$) denotes the set $\{\cap_{s \in S} s\}$ (resp. $\{\cup_{s \in S} s\}$).

Lemma 4.1 *Let f be a $\Box$-expression (resp. $\Diamond$-expression) over $\tilde{R}$ in modal relational calculus with $\mathcal{L}$-constraints. Let f' be the pure expression which corresponds to f. Then for all indefinite $\mathcal{L}$-constraint databases $\tilde{r}$ over $\tilde{R}$, $f(\tilde{r}) = \bigcap f'(\tilde{r})$ (resp. $f(\tilde{r}) = \bigcup f'(\tilde{r})$).*

5 Procedural Query Languages

In this section, we briefly sketch three procedural query languages, one for each of the models discussed in section 3: the *$M_{\mathcal{L}}$-relational algebra*, the *$\mathcal{L}$-constraint algebra* and the *modal $\mathcal{L}$-constraint algebra*. The $M_{\mathcal{L}}$-relational algebra is a procedural query language for $M_{\mathcal{L}}$-relational databases. It is interesting only from a theoretical point of view because $M_{\mathcal{L}}$-relations are unrestricted. The operations of $M_{\mathcal{L}}$-relational algebra can be defined verbatim as in the case of finite relations [Kan90].

The operations of the $\mathcal{L}$-constraint algebra are extensions of similar operations of standard relational algebra [Kan90]. The $\mathcal{L}$-constraint algebra has not been presented in [KKR90] where the model of $\mathcal{L}$-constraint databases was originally defined. However it can be easily developed given the algebraic languages defined for the models of [KSW90, Kou93]; these models are essentially instances of the scheme of $\mathcal{L}$-constraint databases. Detailed definitions can be found in [Kou94a].

The operations of the *modal $\mathcal{L}$-constraint algebra* take as input one (or two) indefinite $\mathcal{L}$-constraint relations associated with a common global condition and return an indefinite $\mathcal{L}$-constraint relation associated with the *same* global condition. The modal $\mathcal{L}$-constraint algebra contains an operation for every $\mathcal{L}$-constraint algebra operation. The definitions of these operations were originally given in [Kou93] for the special case of indefinite dePCL-constraint relations.[4] These operations treat e-variables as uninterpreted parameters thus they are defined exactly as the $\mathcal{L}$-constraint algebra operations. Similar operations were defined in [IL84, Gra89] for the special case of conditional tables.

The modal algebra also includes two additional operations $POSS$ and $CERT$, which take a more active stand towards e-variables. Given an indefinite $\mathcal{L}$-constraint relation r, the expression $POSS(r)$ evaluates to an $\mathcal{L}$-constraint relation which finitely represents the set of all tuples contained in *any* relation of $sem(r)$. The expression $CERT(r)$ evaluates to an $\mathcal{L}$-constraint relation which finitely represents the set of all tuples contained in *every* relation of $sem(r)$.

Possibility. Let r be an indefinite $\mathcal{L}$-constraint relation on scheme R. Then $POSS(r)$ is an *$\mathcal{L}$-constraint relation* defined as follows:

1. $sch(POSS(r)) = sch(r)$
2. $POSS(r) = \{poss(t) : t \in r\}$.

For each tuple t on scheme R, $poss(t)$ is a tuple on scheme R such that $poss(t)(R) = t(R)$ and $poss(t)(CON) = \psi$ where ψ is obtained by eliminating all e-variables from the boolean combination of $\mathcal{L}$-constraints $G(r) \wedge t(CON)$. The expression $poss(t)(CON)$ is well-defined since the class of $\mathcal{L}$-constraints admits variable elimination.

Certainty. Let r be an indefinite $\mathcal{L}$-constraint relation on scheme R. Then $CERT(r)$ is an $\mathcal{L}$-constraint relation defined as follows:

1. $sch(CERT(r)) = sch(r)$

[4] [Kou93] uses the term *temporal tables* for indefinite dePCL-constraint relations.

2. $CERT(r) = \{cert(t) : t \in r^{\downarrow}\}^{\uparrow}$.

For each tuple t on scheme R, $cert(t)$ is a tuple on scheme R such that $cert(t)(R) = t(R)$ and $cert(t)(CON) = \neg\psi$ where ψ is obtained by eliminating all e-variables from the boolean combination of $\mathcal{L}$-constraints $G(r) \wedge \neg t(CON)$. The expression $cert(t)(CON)$ is well-defined since the class of $\mathcal{L}$-constraints admits variable elimination.

The operation $r^{\downarrow}$ has the effect of *denormalizing* $\mathcal{L}$-constraint relation r. This is achieved by collecting all tuples $\{t_1, \ldots, t_{|r|}\}$ of r into a single tuple t' on scheme R such that $t'(R) = (x_1, \ldots, x_{|R|})$ and $t'(CON) = t'_1(CON) \vee \cdots \vee t'_{|r|}(CON)$. In the new tuple t' u-variables have been standardized apart: $x_1, \ldots, x_{|R|}$ are brand new u-variables, and for $1 \leq i \leq |r|$, $t'_i(CON)$ is the same as $t_i(CON)$ except that $t(X)$ has been substituted by $t'(X)$ for each $X \in R$.

The operation $r^{\uparrow}$ has the effect of *normalizing* the local conditions of a relation r in order to obtain a true $\mathcal{L}$-constraint relation. This is done by the following three steps:

- Application of De Morgan's laws to transform the negated parts of each local condition of r into a disjunction whose disjuncts are $\mathcal{L}$-constraints. This operation is well-defined since the class of $\mathcal{L}$-constraints is weakly closed under negation.
- Application of the law of associativity of conjunction with respect to disjunction to transform each local condition of r into a disjunction of conjunctions of $\mathcal{L}$-constraints.
- Splitting of disjuncts into different tuples.

Let us now define modal $\mathcal{L}$-constraint algebra expressions.

Definition 5.1 A *pure expression* over scheme $\widetilde{R}$ in modal $\mathcal{L}$-constraint algebra is any well-formed expression built from constant $\mathcal{L}$-constraint relations, relation schemes from $\widetilde{R}$ and the above operators excluding $POSS$ and $CERT$. A *modal $\mathcal{L}$-constraint algebra expression* is a pure expression, or an expression of the form $CERT(g)$ or $POSS(g)$ where g is a pure expression. Expressions of the form $CERT(g)$ or $POSS(g)$ are called *$CERT$-expressions* or *$POSS$-expressions* respectively.

Modal $\mathcal{L}$-constraint algebra expressions define functions from indefinite $\mathcal{L}$-constraint databases to indefinite $\mathcal{L}$-constraint relations. The result of applying an expression e to an indefinite $\mathcal{L}$-constraint database $\widetilde{r}$ is defined as for the $\mathcal{L}$-constraint algebra. Let us simply stress that $G(e(\widetilde{r})) = G(\widetilde{r})$ for all indefinite $\mathcal{L}$-constraint databases $\widetilde{r}$ and expressions e over $\widetilde{R}$.

The following lemma gives an intuitive property of $POSS$ and $CERT$.

Lemma 5.1 *Let e be a pure expression over scheme $\widetilde{R}$ in modal $\mathcal{L}$-constraint algebra. Then for all indefinite $\mathcal{L}$-constraint databases $\widetilde{r}$ over $\widetilde{R}$*

$$sem(CERT(e(\widetilde{r}))) = \bigcap sem(e(\widetilde{r})) \text{ and } sem(POSS(e(\widetilde{r}))) = \bigcup sem(e(\widetilde{r})).$$

6 On the Semantics of Algebraic Query Evaluation

Let $\widetilde{r}$ be an $\mathcal{L}$-constraint database, e an $\mathcal{L}$-constraint algebra expression and $e1$ its corresponding $\mathbf{M}_{\mathcal{L}}$-relational algebra expression. Recall that an $\mathcal{L}$-constraint relation $\widetilde{r}$ is a finite representation of the unrestricted set of tuples $points(\widetilde{r})$. The following theorem shows that the operations of $\mathcal{L}$-constraint algebra "behave" according to our intuitions: when we evaluate e on $\widetilde{r}$, we essentially evaluate $e1$ on the unrestricted relation $points(\widetilde{r})$.

Theorem 6.1 *Let e be an $\mathcal{L}$-constraint algebra expression over $\widetilde{R}$ and $e1$ be its corresponding $\mathbf{M}_{\mathcal{L}}$-relational algebra expression. If $\widetilde{r}$ is an $\mathcal{L}$-constraint database over scheme $\widetilde{R}$, then $points(e(\widetilde{r})) = e1(points(\widetilde{r}))$.*

Let us now assume that $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database, e is a pure expression of modal $\mathcal{L}$-constraint algebra and $e1$ is its corresponding expression in $\mathcal{L}$-constraint algebra. Recall that the semantic function $sem(\widetilde{r})$ returns all the "possible worlds" represented by $\widetilde{r}$. When we evaluate e on indefinite

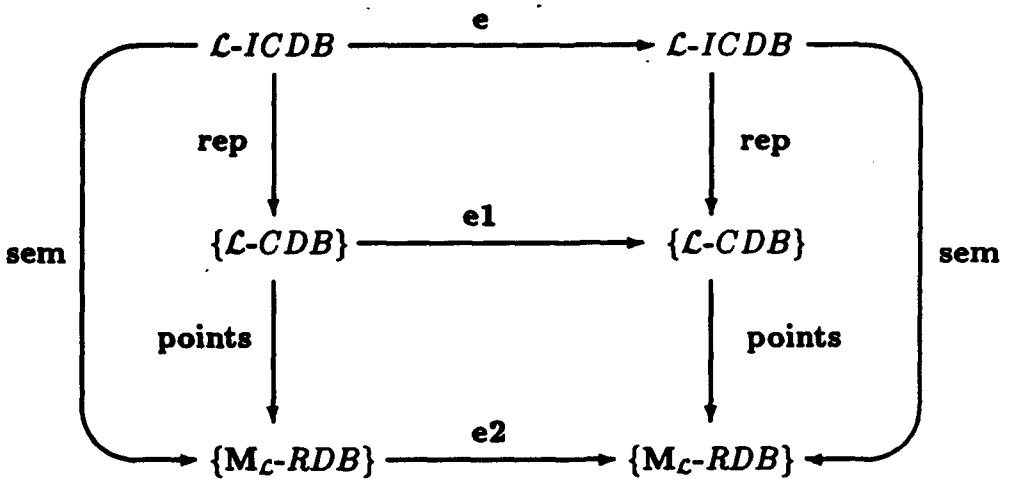

Figure 1: Relating the three algebras

$\mathcal{L}$-constraint database $\widetilde{r}$ using the operations defined above, we essentially evaluate the corresponding $M_{\mathcal{L}}$-relational algebra expression on each possible world in $sem(\widetilde{r})$. As discussed in [Imi89], an extension of an $M_{\mathcal{L}}$-relational algebra expression $e1$ to an expression e for an $\mathcal{L}$-constraint representation of indefinite information can claim to be "faithful to the underlying semantics" if and only if for every indefinite $\mathcal{L}$-constraint database $\widetilde{r}$,

$$sem(e(\widetilde{r})) = e1(sem(\widetilde{r})) = \{e1(r_1) : r_1 \in sem(\widetilde{r})\}.$$

Equivalently, one would like to guarantee that there is always an indefinite $\mathcal{L}$-constraint database $\widetilde{r}'$ such that $sem(\widetilde{r}') = e1(sem(\widetilde{r}))$. The following theorem demonstrates that $\mathcal{L}$-constraint databases satisfy this form of algebraic closure.

Theorem 6.2 *Let e be a pure expression of modal $\mathcal{L}$-constraint algebra over $\widetilde{R}$, $e1$ its corresponding $\mathcal{L}$-constraint algebra expression and $e2$ its corresponding $M_{\mathcal{L}}$-relational algebra expression. If $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database over scheme $\widetilde{R}$ then $rep(e(\widetilde{r}))$ is equivalent to $e1(rep(\widetilde{r}))$ and $sem(e(\widetilde{r})) = e2(sem(\widetilde{r}))$.*

The above theorems are summarized graphically in the commutative diagram of figure 1 where $M_{\mathcal{L}}$-RDB denotes the set of all $M_{\mathcal{L}}$-relational databases, $\mathcal{L}$-CDB denotes the set of all $\mathcal{L}$-constraint databases and $\mathcal{L}$-$ICDB$ denotes the set of all indefinite $\mathcal{L}$-constraint databases. Since the above results have been proved in our general framework, special cases of constraint databases [KKR90, KSW90, Kou93] can simply refer to these theorems to demonstrate the "correctness" of the operations of their algebraic query languages.

7 Translating Calculus Expressions into Algebraic Expressions

In this section we show that expressions of modal relational calculus with $\mathcal{L}$-constraints have equivalent expressions in modal $\mathcal{L}$-constraint algebra. Thus we can evaluate a calculus expression by evaluating an equivalent algebraic expression. As we have seen in section 5, algebraic query evaluation can be done bottom-up and the answer is obtained in closed form. Therefore calculus expressions can also be evaluated bottom-up in closed form on indefinite $\mathcal{L}$-constraint databases. [Kou94a] gives an alternative proof of this result by employing quantifier elimination techniques as suggested in [KKR90].

We start by considering the simpler case of $\mathcal{L}$-constraint databases. [KKR90] has showed that, for several languages $\mathcal{L}$, expressions of relational calculus with $\mathcal{L}$-constraints can be evaluated bottom-up in closed form on $\mathcal{L}$-constraint databases. The following theorem generalizes this result in the abstract setting of this paper.

Theorem 7.1 *For every expression f over $\widetilde{R}$ in relational calculus with $\mathcal{L}$-constraints there exists an $\mathcal{L}$-constraint algebra expression e over $\widetilde{R}$ such that the following property holds. If $\widetilde{r}$ is an $\mathcal{L}$-constraint database over $\widetilde{R}$ then $f(\widetilde{r}) = points(e(\widetilde{r}))$.*

Let us note here that the analogous proofs of [KKR90] rely on quantifier elimination methods which achieve good data complexity lower bounds but do not seem to have practical implementations. In contrast, the above theorem provides a translation of calculus expressions into algebraic expressions. We believe that this translation can be the first step in optimizing the evaluation of expressions in relational calculus with $\mathcal{L}$-constraints.

Let us now turn to modal relational calculus with $\mathcal{L}$-constraints and modal $\mathcal{L}$-constraint algebra.

Lemma 7.1 *Let e be a pure expression over $\widetilde{R}$ in modal $\mathcal{L}$-constraint algebra and e' be its corresponding $\mathcal{L}$-constraint algebra expression. Then $sem(e(\widetilde{r})) = \{points(e'(\widetilde{r}')) : \widetilde{r}' \in rep(\widetilde{r})\}$ for all indefinite $\mathcal{L}$-constraint databases $\widetilde{r}$ over $\widetilde{R}$.*

The following theorem demonstrates that pure expressions of modal relational calculus with $\mathcal{L}$-constraints over indefinite $\mathcal{L}$-constraint databases can also be evaluated bottom-up in closed form.

Theorem 7.2 *For every pure expression f over $\widetilde{R}$ in relational calculus with $\mathcal{L}$-constraints there exists a pure expression e over $\widetilde{R}$ in modal $\mathcal{L}$-constraint algebra such that the following property holds. If $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database over $\widetilde{R}$ then $f(\widetilde{r}) = sem(e(\widetilde{r}))$.*

xample 7.1 The algebraic expression equivalent to the calculus expression of example 4.1 is

$$\pi_{Room}(\sigma_{Time=6}(BOOKED)).$$

Finally we turn to modal expressions.

Theorem 7.3 *Let f be a $\Box$-expression (resp. $\Diamond$-expression) over $\widetilde{R}$ in modal relational calculus with $\mathcal{L}$-constraints. Then there exists a $CERT$-expression (resp. $POSS$-expression) e over $\widetilde{R}$ in modal $\mathcal{L}$-constraint algebra such that the following property holds. If $\widetilde{r}$ is an indefinite $\mathcal{L}$-constraint database over $\widetilde{R}$ then $f(\widetilde{r}) = sem(e(\widetilde{r}))$.*

Example 7.2 The algebraic expression equivalent to the calculus expression of example 4.2 is

$$POSS(\pi_{Room}(\sigma_{Time=6}(BOOKED))).$$

8 Related Work

The results of this study are extended in [Kou94b, Kou94a] where we concentrate on *temporal constraint databases* (with or without indefinite information). In particular, we study the complexity of query evaluation in $\mathcal{L}$-constraint databases and indefinite $\mathcal{L}$-constraint databases where $\mathcal{L}$ ranges over several temporal constraint languages (including dePCL). Our analysis shows that the worst-case data/combined complexity of query evaluation *does not change* when we move from queries in relational calculus over relational databases, to queries in relational calculus with temporal constraints over temporal constraint databases. This fact remains true even if we consider indefinite relational databases vs. indefinite temporal constraint databases. Unfortunately, the presence of indefinite information makes query evaluation intractable in many cases. Our analysis complements the results of [Rev90, CM93] and extends the results of [KKR90, vdM92].

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Global Consistency for Continuous Constraints[1]

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Abstract

This paper provides a framework for solving general constraint satisfaction problems (CSPs) with continuous variables. Constraints are represented by a hierarchical binary decomposition of the space of feasible values. We propose algorithms for path- and higher degrees of consistency based on logical operations defined on the constraint representation mentioned above and we demonstrate that this algorithms terminate in polynomial time. We show that, in analogy to convex temporal problems and discrete row-convex problems, convexity properties of the solution spaces can be exploited to compute minimal and decomposable networks using path consistency algorithms. Based on this properties, we also show that a certain class of non binary CSPs can be solved using strong 5-consistency.

1 Introduction

In the general case, constraint satisfaction problems (CSPs) are NP-complete. Trying to solve them by search algorithms, even if theoretically feasible, often results in prohibitive computational cost. One approach to overcome this complexity consists of pre-processing the initial problem using *propagation algorithms*. These algorithms establish various degrees of local consistency which narrow the initial feasible domain of the variables, thus reducing the subsequent search effort. Traditional consistency techniques and propagation algorithms — such as the Waltz propagation algorithm— provide relatively poor results when applied to continuous CSPs: they ensure neither completeness nor convergence in the general case (a good insight of the problems encountered can be found in [1]). However, Faltings [5] has shown that some undesirable features of propagation algorithms with interval labels must be attributed to the inadequacy of the propagation rule and to a lack of precision in the solution space description. He has also demonstrated that the problem with local propagation could be resolved by using *total constraints* on pairs of variables. Lhomme [11] has identified similar problems and proposed an interval propagation formalism based on bound propagation.

Van Beek's work on temporal reasoning [15] using Helly's theorem has shown the importance of path-consistency for achieving globally consistent labellings. In certain cases, path-consistency algorithms are difficult to implement in continuous domains because they require intersection and composition operations on constraints. We propose a constraint representation by recursive decomposition similar to the one described by Tanimoto in [14] which allows to implement these operations. This allows us to apply Helly's theorem to general continuous constraint satisfaction problems. The results obtained for temporal CSPs could therefore be generalized to less specific classes of continuous CSPs.

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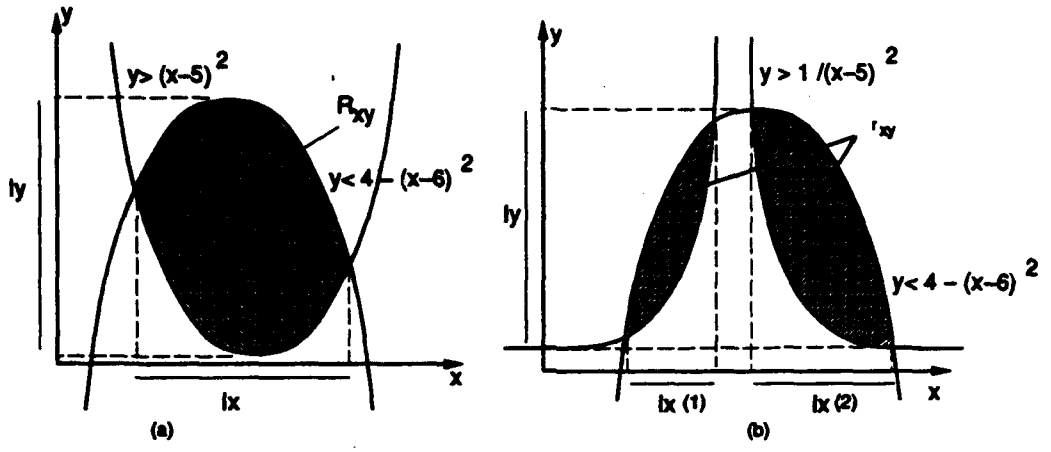

Figure 1: *Figure (a) illustrates a binary relation R_{xy} given intensionally by the two inequalities $y > (x-5)^2$ and $y < 4-(x-6)^2$. R_{xy} determines the region r_{xy} and is both y- and x-convex: the projection of r_{xy} respectively over the x and y axes yields single bounded intervals (resp. I_x and I_y). In Figure (b), the relation R_{xy} is given intensionally by the constraints $y > 1/(x-5)^2$ and $y < 4-(x-6)^2$. In this last case, the relation is only y-convex since its projection over the x axis yields two distinct intervals I_{x1} and I_{x2}.*

In the following, a continuous CSP (CCSP),$(P = (V, D, R))$, is defined as a set V of *variables* $x_1, x_2, \ldots x_n$, taking their values respectively in a set D of continuous *domains* $D_1, D_2, \ldots, D_n$ and constrained by a set of *relations* $R_1, \ldots, R_m$. A *domain* is an interval of $\mathcal{R}$. A *relation* is defined intensionally by a set of algebraic equalities and inequalities (see figure 1). A relation R_{ij} is a total constraint: it takes into account the whole set of algebraic constraints involving the variables i and j. Each variable has a *label* defining the set of possible consistent values. The label L_x of a variable x is represented as a set of intervals $\{I_{x,1} = [x_{min,1} \ldots x_{max,1}], \ldots\}$.

2 Constraint and Label Representation

Constraints on continuous variables are most naturally represented by algebraic or transcendental equations and inequalities. However, as Faltings [5] has shown, this leads to incomplete local propagation when there are several simultaneous constraints between the same variables. More importantly, making a network path-consistent requires computing the intersection and union of constraints, operations which cannot be performed on (in)equalities. It is therefore necessary to explicitly represent and manipulate the sets of feasible value combinations.

Providing each variable with an interval label implicitly represents feasible regions by enclosing rectangles or hypercubes. As shown in Figure 2, this is not powerful enough for region intersection operations. To define a more precise and yet efficient representation, we observe that most applications satisfy the following two assumptions:

- each variable takes its values in a bounded domain (bounded interval)
- there often exists a maximum precision with which results can be used.

Provided that these two assumptions are verified, a relation $R_{x_1 \ldots x_k}$ can be approximated by carrying out a hierarchical binary decomposition of its solution space into 2^k-trees (quadtrees for binary relations,octrees for ternary ones etc...)(see Figure 3). A similar representation

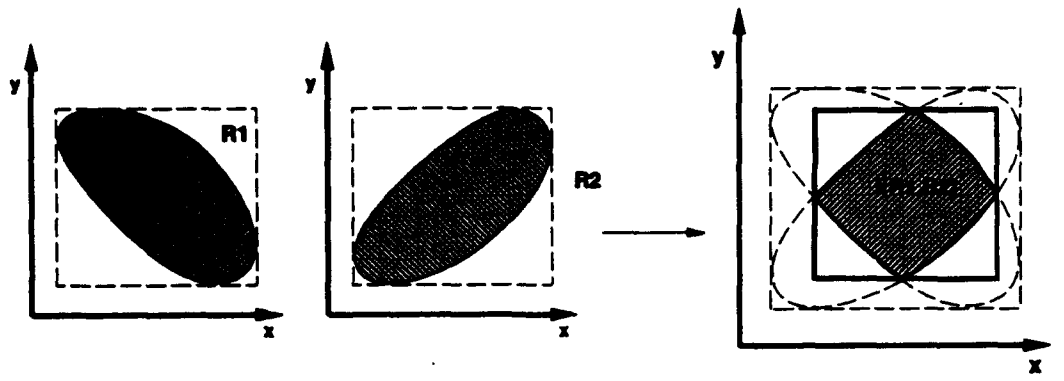

Figure 2: *The enclosing rectangle of an intersection of regions R_1 and R_2 is in general different from the intersection of the enclosing rectangles of R_1 and R_2.*

has recently been proposed by Tanimoto for representing spatial constraints [14]. When a relation is determined by *inequalities*, it can be approximated by a 2^k-tree where each node represents a k-dimensional cubic sub-region of the original domain (i.e. the domain over which the decomposition is carried out). A node has one of three possible states:

- *white*: if the region it defines is completely legal
- *gray*: if the region is partially legal and partially illegal
- *black*: if the region is completely illegal

When a black or white node is identified, the recursive division stops. Each gray k-dimensional cube is decomposed into 2^k smaller ones whose sides are half times the length. Unless the boundaries of a region are parallel to the coordinates axes, infinitely many levels of representation are required to accurately represent a region. However, since the maximum precision is fixed, any gray node with a smaller size than the maximum granularity can be declared black and the decomposition stops.

Equalities In the case of equality constraints, a strict application of the binary decomposition into 2^k-tree described before would amount to pursuing the decomposition to infinity since an infinite degree of precision is required to represent solutions which are points. We can avoid this problem by exploiting the fact that many practical applications require a limited degree of precision and it is then admissible to treat equalities with a certain error range. Presently, our system translates strict equalities $f(x_1, \ldots x_k) = C$ into a weaker form, $f(x_1, \ldots x_k) = C \pm \varepsilon/2$, where ε is the maximum precision fixed, as defined for inequalities. This amounts to replacing each equality by two inequalities.

3 Consistency algorithms using 2^k-trees

Path consistency algorithms, such as PC-1 [13] and PC-2 [12] require the application of the following update rule defined on constraints:

$$C'_{ij} = C_{ij} \oplus \prod^{k} (C_{ik} \otimes C_{kj}) \qquad (1)$$

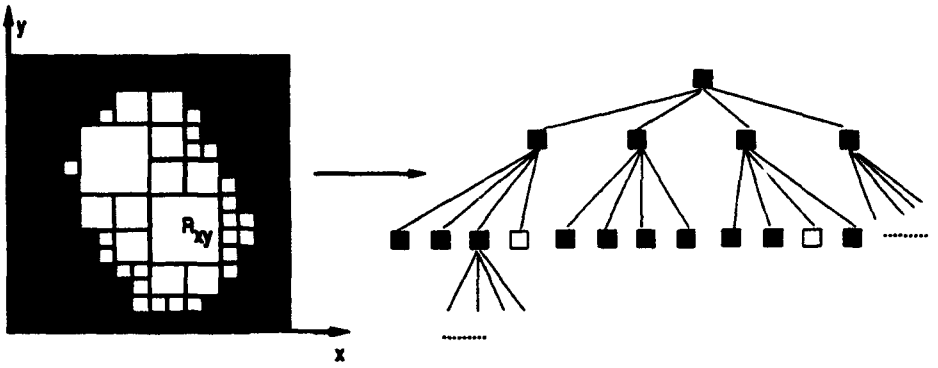

Figure 3: *A binary relation R_{xy} can be approximated by carrying out a hierarchical binary decomposition of its solution space into a quadtree*

This relaxation operation uses two binary operators (intersection and composition, denoted respectively by $\oplus$ and $\otimes$) and a unary one (projection, denoted by $\prod$), which can be defined on 2^k-trees. Since all variables are decomposed within the same interval (see figure 4), intersection is simply the logical intersection of the corresponding quadtrees and can be carried out efficiently: given an ordering *white* < *gray* < *black* the intersection operator can be defined as $color(node_1 \oplus node_2) = Max(color(node_1), color(node_2)))$.

Information on a k-dimensional node can be simply derived by *composing* its facets ((k-1)-dimensional nodes) $(color(node_1 \otimes node_2) = Max(color(node_1), color(node_2)))$, and vice versa, information on a (k-1)-dimensional node can be obtained by *projecting* the k-dimensional node over one of its facets ($color(\prod^*(node_1)) = Min(color(node_i))$, where $node_i$ are the nodes having $node_1$ as facet).

The operators required for path consistency algorithms (and their generalization for higher degrees of consistency) can therefore be implemented as straightforward logical rather than numerical operations.

N-ary CSPs In many realistic problems, the constraints are not binary, but n-ary. However, each n-ary constraint can be reduced to a set of ternary constraints without loss of information. An n-ary algebraic relation, $C(x_1, \ldots x_n)$, can be transformed into a set of ternary algebraic expressions by:

i. replacing iteratively in C each sub-expression $< x_i$ operator $x_j >$ by a new variable x_{n+1}

ii. adding a ternary equality constraint $x_{n+1} =< x_i$ operator $x_j >$

The process stops when C itself becomes ternary. This transformation is only based on symbolic manipulations and consequently, no information is lost in the solution space description. For example, the 5-ary CSP with one constraint ,$(x-y)^2 + \frac{(z+t)}{u} > 2$, can be translated into a ternary one with three constraints: $w_1^2 + (w_2/u) > 2$, $w_1 = x - y$, $w_2 = z + t$. Hence, addressing n-ary continuous CSPs amounts to giving the ternary counterparts of the algorithms and representation used for solving binary continuous CSPs.

Constructing 2^k-tree representations A total binary constraint R_{xy} is given intensionally by a set of algebraic equations $(C_1 \ldots C_l)$. The quadtree approximation T_{xy} of a binary relation R_{xy} can be obtained as follows:

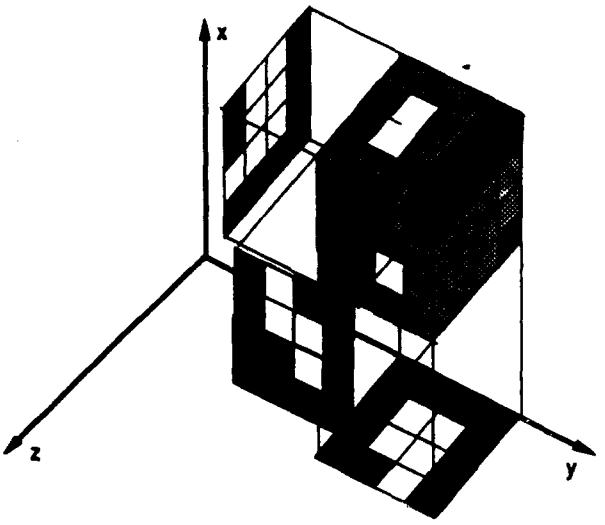

Figure 4: *In a binary CCSP with three variables (x, y, z), the octree obtained by composing the quadtrees T_{xy}, T_{xz} and T_{yz} will give all the consistent instantiations according to the chosen granularity (white cubes)*

For each $C_i \in (C_1 \ldots C_l)$ Do

1. build a quadtree representation T^i_{xy} for the basic constraint C_i
2. $T_{xy} = T_{xy} \oplus T^i_{xy}$

Constructing the quadtree representation of an individual algebraic constraint, require a procedure for determining the color of each sub-region (rectangle) created by the recursive decomposition.When the constraint curve determines a *transversal segment* within the considered rectangle, testing for the rectangle color amounts to finding an intersection of its boundaries with the curve. This test requires iterative numerical analysis in the general case. On the other hand, when the curve is closed more complicated treatment must be accommodated. Computing octrees for representing ternary relations can be carried out similarly to the case of binary relations.

4 Global consistency in constraint networks

A *minimal* network is *globally consistent* (all the constraints are as explicit as possible in the network) while a *decomposable* network allows for a *backtrack-free* search of the solution (the search process can generally be carried out in linear time). In this work we show that certain convexity properties of the solution space allows for computing minimal and decomposable network in polynomial time for a continuous CSPs.

Encouraging results have been obtained for continuous CSPs in the domain of temporal reasoning: Dechter, Meiri and Pearl [2] have shown that for simple temporal problems (STP), where labels have to be convex intervals (i.e: disjunctive constraints are not allowed), the minimal constraint network can be constructed in polynomial time by ensuring path consistency. Similar results have been obtained by Van Beek [15] on a subset of the Allen's interval algebra excluding the binary relation $\neq$. Recently,Van Beek [16] has generalized the convexity property to the case of discrete CSPs: a property of discrete constraints — called *row-convexity* —

has been identified that guarantees the minimality and the decomposability of the constraint network when path consistency is ensured.

Although the convexity properties exploited in temporal and row-convex discrete problems derive mainly from results in the continuous domain (see Helly's theorem for convex sets [15]), no framework has been defined to exploit them in the case of general continuous CSPs. This is because the restriction imposed by the convexity condition on algebraic continuous solution spaces is too strong. In this work, we show that the *arcwise connectivity* property (a weaker condition) is sufficient for generalizing the results obtained in simple temporal [2] and row-convex discrete [16] domains to continuous CSPs.

In simple temporal problems (STPs) constraints take the form of bounded differences $b_1 < x_i - x_j < b_2$ where $[b_1 b_2]$ has to be a *single* interval. This condition amounts to saying that each variable takes its value within a single interval (convex interval). Path consistent STPs can be solved by backtrack-free search. The key observation is that this solution requires the convexity property *only* for each individual variable domain. Hence, generalizing to non-temporal continuous CSPs would amount to imposing convexity conditions only on the *projections* of the solution space over the different axes involved (the convexity condition is required only on projected intervals).

Consequently, for generalizing the results obtained for STPs, it is sufficient that the solution space verifies the *arcwise connectivity* property.

The *arcwise connectivity* requirement is clearly weaker than convexity: a k-ary relation, defined on a set of k variables $V = x_1, \ldots, x_k$ and determining a convex region has convex projections for each variable x_i of V. However, the converse is not true, a region may have convex projections for each involved variable x without being convex.

4.1 Convex binary CCSPs

Let first describe how convexity properties can be exploited in the case of binary constraints. The case of n-ary constraints will be dealt with later on. We define:

Definition 1 : ***x-Convexity property***

i. Let r be a bi-dimensional region defined by a set of algebraic constraints on two variables x_i and x_j. r is said to be x_k-convex in the domain D_{x_k} if its projection over the x_k axis yields a convex interval ($k \in \{i, j\}$).

ii. A binary relation R_{x_i,x_j} is x_k-convex in the domain D_{x_k} if it determines a x_k-convex region in D_{x_k}.

Definition 2 : ***Convex constraint network***
A constraint network representing a CCSP (V,D,R) is convex if for all relation R_{x_i,x_j} in R, Rx_i, x_j is x_k-convex for each k in $\{i, j\}$.

Continuous constraint satisfaction problems (CCSPs) having convex constraint network representations are the generalized counterparts of simple temporal problems (STP) as defined in [2]. Since an *arcwise connected* region in R^k is a single closed and bounded sub-region of R^k, the x-convexity property is verified for each relation determining an arcwise connected set.

Note finally that CCSPs including disjunctive or non-linear constraints may admit no convex constraint network representation since these type of constraints often create splits in the solution space.

Now, we are in position to extend the main theorem of Van Beek (theorem 1 of [16]) to the case of CCSPs. We first have to extend the lemma on which his proofs are based. This can be done as follows:

Definition 3 :
Let $r^1_{x_i,x_j}$, and $r^2_{x_i,x_j}$ two x_i-convex bi-dimensional regions. The x_i-intersection of r^1 and r^2 is defined as the intersection of their projection over the x_i axis.

Lemma 1 *Let F be a finite collection of x-convex regions in R^2. If F is such that every pair of regions have a non null x-intersection, then the x-intersection of all these regions is not null (i.e: there exist at least one value v for x so that each region $r_{x,y}$ contains a point (v, y_i), where y_i is a possible value for y)*

Proof. This lemma is a direct application of Helly's theorem to the case of R^2.

We can generalize the theorem as follows:

Theorem 1 *A binary constraint network which is convex and path-consistent is minimal and decomposable*

Proof. Analogous to the one given in [16].

Theorem 4 in [16] generalizes then to the case of x-convex relations:

Theorem 2 *Let N be a path consistent binary constraint network. If there exists an ordering of the variables $x_1, \ldots x_n$ such that each relation of N R_{x_i,x_j}, $1 \leq j \leq i$, is x_i-convex, then a consistent instantiation can be found without backtracking.*

Proof: The proof derives from the generalization of the backtrack-free instantiation algorithm proposed by Van Beek in [16].

4.2 Convex n-ary CCSPs

As stated before, generalizing to n-ary CCSPs the results described before for binary CCSPs amounts to giving the ternary counterparts of theorems 1 and 2.

Global consistency for ternary CCSPs The x-convexity property generalizes straightforwardly to the case of non binary CCSPs. In the case of ternary constraints, the generalization of lemma 1 can be used to prove the decomposability of the constraint network only if each pair of ternary relations have a non null x-intersection. Two ternary relations $R_{i_1,j_1,k}$ and $R_{i_2,j_2,k}$ have a non null k-intersection when each subset of *five* variables (i_1, i_2, j_1, j_2, k) are consistently labelled. In the particular case where each pair of ternary constraints have two variables in common,(i.e: $i_1 = i_2$ or j_1 or j_2), the number of variables that must be consistently labelled reduces to four and strong 4-consistency is sufficient for the network to be decomposable. Hence, theorem 1 generalizes to ternary constraints as follows:

Theorem 3 *A ternary constraint network which is convex and strong 5-consistent is minimal and decomposable. Furthermore, in the particular case where each pair of relations share two variables, strong 4-consistency is enough to ensure that a convex ternary constraint network is minimal and decomposable.*

Since the translation of an n-ary network into a ternary one is done at the cost of increasing the number of variables, the practicality of 5-consistency for n-ary CCSPs is still an open question. This result is mainly intended to provide a theoretical bound for solving certain classes of n-ary CCSPs in a complexity better than exponential.

4.3 Non-convex CCSPs

A general CCSP may admit no convex constraint network representation. Moreover, even if the initial problem is convex, consistency algorithms may not preserve this property since intersecting two non convex — even if arcwise connected— regions may result in an arbitrary number of distinct arcwise connected sub-regions. We can distinguish three classes of CCSPs:

i. CCSPs where all the relations determine convex regions

ii. CCSP where each relation determining a non arcwise connected solution space is constituted by a set of convex regions.

iii. CCSPs where there exist non convex regions

In case *i.*, since the intersection of two convex regions is necessarily a compact region, consistency algorithms will preserve the convexity of the constraint network representation. Hence, problems of this first category can be solved, with no further search, using partial consistency algorithms (as stated in theorems 1 and 3). In case *ii.*, the problem can be decomposed into convex sub-problems (one for each possible combination of convex sub-region). Each sub-problem is of type *i.*. Solution to the whole problem can be determined by solving individually each sub-problem and then combining their solutions. Even if the complexity is, in this case, exponential in the number of disjoint convex sub-regions, the computational effort can be bounded *a priori* since consistency algorithms cannot create new case splits in the individual sub-problems. In the last case finally, the splitting problem (similar to the one described in [10]) may occur and the complexity is difficult to estimate. In the best case, the consistency algorithm may create a convex constraint network from a set of non-convex relations. In the worst case however, the intersection of each pair of non convex regions may result in an undetermined number of disjoint new sub-regions which can in turn split again. Practical solutions (such as stopping the splitting process when the maximum precision is reached) can be used to bound the combinatorial explosion, but in general the complexity remains exponential for CCSPs of type *iii.*

5 Complexity of Consistency algorithms

The complexity of the intersection, composition and projection operators on 2^k-trees can be roughly estimated in terms of the number of nodes generated by each operation. $O(2^{k*s/\epsilon})$ (where s is the maximum domain size and ϵ the tightest interval size accepted for variables) gives a rough approximation of the complexity. This measure assume that, in the worst case, a 2^k-tree resulting from a given operation is complete. A more realistic measure can be done in terms of the number of gray nodes generated, since the recursive quartering stops as soon as a node color is set to white or black. We can show that this measure is function of the boundaries size of the solution space. Furthermore, 2^k-tree structures are by nature well-adapted to parallel processing. Parallel implementation of the intersection, composition and projection are likely to reduce significantly the complexity. Important improvements in time and space complexity can also be achieved by storing and processing only the white nodes (see linear quadtrees in [9]).

Convex Binary CCSPs The algorithm PC-2 can be implemented using eq. 1 by way of the `revise` function. According to the definitions of $\oplus$ and $\otimes$ for 2^k-trees, the relaxation operation described by eq. 1 is monotonic. Moreover, since the region decomposition into 2^k-trees discretizes the solution space, showing that PC-1 (and hence PC-2) terminates and computes a path-consistent network using the relaxation operation $T'_{ij} = T_{ij} \oplus T_{ik} \otimes T_{kj}$, can be done similarly to the case of discrete-domains CSPs (see [13]). The worst case running time of PC-2 occurs when each revision step suppresses only one node from the considered relation (i.e. the node becomes black), hence:

Theorem 4 *PC-2 computes the path consistent network representation of binary CCSPs, (V, D, R), in $O(2^{(2*s/\varepsilon)}n^3)$ where s is the largest interval size in D and ε the tightest interval size accepted for variables of V.*

According to theorem 1, when the path consistent network computed by PC-2 is convex, it is also minimal and decomposable. Similarly, we can demonstrate that strong 5-consistency can be ensured for a ternary CCSP in $O(2^{(3*s/\varepsilon)}n^5)$.

Non convex CCSPs During the construction and propagation of 2^k-trees, the case when a single region is split into several can be reliably detected. At this point the algorithm branches and explore both regions separately (a new CCSP is generated). The pathological case where infinite number of sub-regions are generated is avoided in practice, since the regions smaller than the maximum precision are not explored. However, the complexity is clearly exponential in the worst case.

6 Conclusion

In this paper we present a generalization of the results obtained for convex temporal problems and discrete row-convex problems to more general classes of continuous CSPs (called convex CCSPs). One of its main contribution is to show that *arcwise connectivity* properties of continuous solutions spaces can be exploited to compute solutions to CCSPs in polynomial time complexity. This paper also presents a recursive decomposition scheme that solves the problem of representing general regions. The 2^k-tree decomposition amounts to performing the stable binary-search method which guarantees convergence according to numerical analysis results. The cycling problems, generally posed by fixed point iteration methods (such as those observed by Davis for the Waltz algorithm [1]) are consequently avoided. Finally, we show that solving non convex CCSPs remains inherently costly, but decomposition methods can be proposed and might be of practical interest for many particular applications.

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Study of symmetry in Constraint Satisfaction Problems*

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Abstract. Constraint satisfaction problems (CSP's) involve finding values for variables subject to constraints on which combinations of values are permitted. Symmetrical values of a CSP variable are in a sense redundant. Their removal will simplify the problem space. In this paper we give the principle of symmetry and show that the concept of interchangeability introduced by Freuder, is a particular case of symmetry. Some symmetries can be computed efficiently thanks to the structure of the problem (neighborhood interchangeability is a kind of these symmetries). Therefore we show how such symmetries can be used by existing constraint propagation algorithms and introduce a backtrack procedure exploiting symmetries. Both theoritical analysis and expiriments indicate that our proposed approach is an improvment of neighborhood interchangeability use, and has very good behavior for pigeon-hole problems.

1 Introduction

The finite domain constraint satisfaction problem (CSP)[2] is well known in Artificial Intelligence. It has been investigated in the past by a number of researchers in different contexts; and steal a well-studied research area of recent years (refer to Kumar [10]). A CSP involves, (1) a (finite) set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, (2) a finite set $D = \{D_1, D_2, \ldots, D_n\}$ of discret domain values in which D_i is the finite discrete domain associated whith the variable v_i; to avoid confusions between values of different domains, d_i will denote the fact that it belonges to the domain D_i, (3) a finite set $C = \{c_1, c_2, \ldots, c_n\}$ of constraints, a k-ary constraint c_i is defined on a subset $V_k \subseteq V$ of variables which we denote $var(c_i)$, (4) and a finite set $R = \{R_1, R_2, \ldots, R_n\}$ of relations corresponding to the constants in C, R_i represents the list of tuples form in which the tuples of values satisfying the constraint c_i are enumerated. Thus, a CSP can be seen as a quadriplet $\mathcal{P}(V, D, C, R)$.

A value assignment is a mapping which specifies a value for each variable: formally a value assignment I can be seen as: $I : V \longrightarrow \cup_{i \in [1,n]} D_i$ such that $I[v_i] \in D_i$, $\forall i \in [1, n]$. A value assignment satisfies a constraint if it gives a combination of values to variables that is permitted by the constraint; otherwise it falsifies it. Thus a constraint satisfaction problem is the task of finding one or all value assignments for the constraints network such that all the constraints are satisfied together.

As beeing expected, various techniques for solving CSP's have been developed; these include backtraking, arc consistancy (Waltz [13], Mackworth [11]), path consistan-

[2] Through out this paper, we use CSP to refer to the finite domain constraint satisfaction problem.

cy (Freuder [6]

On other hand, symmetries for boolean constraints are well studied in (Benhamou and Sais [2,3]). They showed that it is a real improvment for efficiency of several automated deduction algorithms. In this paper we develop the concept of symmetry for CSP's. Symmetrical domain values will be in a sense redundant. Their removal will simplify the problem search space. On other hand the set of solution of a CSP can be represented in a more compact way using symmetry. Indeed only non-symmetrical solutions are computed (basical solutions) from which we process the other solutions whitout duplication of efforts. The paper is organized as following :

Two levels of *semantic symmetry* are defined in Section 2. Section 3 discusses *syntactical symmetry* which is a form of semantic symmetry that can be computed efficiently using only the structure of the considered problem. In other words, syntactical symmetry is considered as a suffisient condition to hold semantic symmetry (Neighborhood interchangeability (Freuder [7]) is a case of syntactical symmetry). Section 4 explains how symmetrical values can be used in various algorithms such as propagation methods and propose a backtrack procedure taking advantage of symmetrical values. In section 5 we evaluate the proposed techniques by experimental results. Section 6 concludes the work.

For simplicity we studie binary CSP's, which involve only constraints between two variables. However, symmetry remains available for non-binary CSP's; and non-binary CSP's can be transformed into binary ones (Rossi, Dhar and Petri [12]).

2 Semantic symmetry

We are interested by two problems in CSP's : the problem of finding a solution (test of satisfiability) and the problem of findind all the solutions of the CSP. Thus two levels of semantic symmetry are difined whith respect to the two previous problems.

Definition 1 Symmetry for satisfiability. Two domain values b_i and c_i for a CSP variable $v_i \in V$ are symmetrical for satisfiability (notation $b_i \approx c_i$) iff the following assertions are equivalent :

1. There is a solution of the CSP which contains the value b_i;
2. There is a solution of the CSP which contains the value c_i.

Domain values can be not only symmetrical for satisfiability (definition 1) but symmetrical for the set of all solutions as well. Thus, if $sol(\mathcal{P})$ denotes the set of solutions of the CSP $\mathcal{P}$, then we define a second level of semantic symmetry as follow :

Definition 2 Symmetry for all solutions. Two domain values b_i and c_i for a CSP variable $v_i \in V$ are symmetrical for $sol(\mathcal{P})$ (notation $b_i \simeq c_i$) if and only if each solution of the CSP containing the value b_i can be mapped into a solution containing the value c_i and vice-versa.

Remark. Symmetrical values for all solutions (definition 2) are also symmetrical values for satisfiability (definition 1).

Example 1 Graph coloring problem.
The problem consists in coloring the vertices so that no two vertices which are joined by an edge have the same color. The available colors (domain values) at each vertex are shown (figure 1).

The red_1 and $white_1$ colors for vertex v_1 are two symmetrical domain values. Indeed, solutions in which one of them participates, can

be obtained from the solutions in which the other value appears by permuting the values *red* and *white* for the variables v_1, v_2 and v_3.

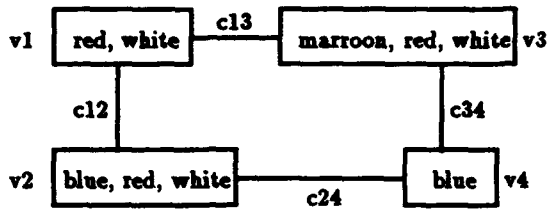

Fig. 1. The graph coloring problem.

In other hand, Freuder introduced in ([7]) the notion of interchangeability, where two domain values are interchangeable in some enveronment, if they can be substituted for each other without any effects to the environment. Let us summarize the main definition.

Definition 3. Two domain values b_i and c_i for a CSP variable $v_i \in V$ are fully interchangeable iff (1) every solution to the CSP which contains b_i remains a solution when c_i is substituted for b_i, (2) every solution to the CSP which contains c_i remains a solution when b_i is substituted for c_i.

Remark. Interchangeable values are particular symmetrical values for all solutions in which the mapping consists to permute the interchangeable values and still identity for the other values.

In the previous examples, values red_1 and $white_1$ are not interchangeable. Thus, the principle of symmetry seems to be more general than the notion of interchangeability. Therefore, eliminating symmetrical values can prune more great deal of effort from a backtrack search tree if such values are processed efficiently. We study in the next section syntactical symmetry of domain values which is a sufficient condition to hold semantic symmetry (definition 2) and give an efficient method for search of such symmetries.

3 Syntactical symmetry

Identifiying semantic symmetries as difined in (definitions 1 and 2) is straightforward time consoming, as this requires solving the problem. This section studies a family of symmetries (syntactical symmetries) which are more tractable computationally, thanks to the structures of the considered problem.

A permutation σ of domain values of a binary CSP $\mathcal{P} = (V, D, C, R)$ can be seen as: $\sigma : \cup_{i\in[1,n]} D_i \longrightarrow \cup_{i\in[1,n]} D_i$, such that $\sigma(d_i) \in D_i$, $\forall i \in [1, n]$ and $\forall d_i \in D_i$. The permutation σ have no influence on the sets $\{V, D, C\}$ of the CSP $\mathcal{P}$. However, it induces a permutation σ_t on the tuples in each relation $R_{ij} \in R$ and then a permutation σ_R [3] on the relations themselves. Therefore a syntactical symmetry of a CSP $\mathcal{P} = (V, D, C, R)$ is a permutation of domain values which leaves the CSP $\mathcal{P}$ invariant (i.e. $\sigma_R(R_i) = R_i, \forall R_i \in R$). Formally:

Definition 4 Syntactical symmetry.
A permutation σ is a syntactical symmetry of the CSP $\mathcal{P} = (V, D, C, R)$ iff $[\forall R_{ij} \in R$, $\ll d_i, d_j \gg \in tuples(R_{ij}) \Rightarrow$
$\ll \sigma(d_i), \sigma(d_j) \gg \in tuples(R_{ij})]$.

Remark. A syntactical symmetry of a CSP is a domain value permutation σ such that $\sigma_R(R_i) = R_i$, $\forall R_i \in R$.

Example 2 Pigeon-hole problem. The problem consists in putting n pigeons in $n-1$ holes such that each hole holds at most one pigeon. Take for instance 4 pigeons and 3 holes. The pigeons are represented by the set of variables the holes by the domain values, as it was shown

[3] Both σ_t resp. σ_R are natural generalizations for σ to tuples resp. relations.

in the constraint graph of figure 2, the constraint c_{13} is given in its microstructure form showing the permitted tuples in the relation R_{13}.

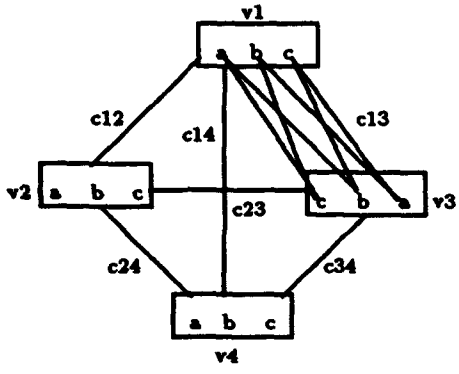

Fig. 2. Pigeon-hole problem for 4 pigeons and 3 holes.

The permutation σ defined as: $\sigma(a_i) = (b_i)$, $\sigma(b_i) = (c_i)$, $\sigma(c_i) = (a_i)$, $\forall i \in [1,4]$ keeps the CSP invariant (i.e. $\sigma_R(R_i) = R_i$, $\forall i \in [1,4]$). Thus, it is a symmetry of the CSP.

Definition 5. Two domain values b_i and c_i for a CSP variable $v_i \in V$ are syntactically symmetrical (notation $b_i \sim c_i$) if there exists a syntactical symmetry σ of the CSP $\mathcal{P}$ such that $\sigma(b_i) = c_i$.

Remark. The relation ($\sim$) is a relation of equivalence.

In the previous example domain values a_1 and b_1 of the variable v_1 are syntactically symmetrical.

Definition 6. A set $\{a_1^1, a_1^2, \ldots, a_1^n\}$ of domain values form a cycle of symmetry in $\mathcal{P}$, if there exists a syntactical symmetry σ of $\mathcal{P}$ such that $\sigma(a_1^1) = a_1^2, \sigma(a_1^2) = a_1^3, \ldots, \sigma(a_1^{n-1}) = a_1^n, \sigma(a_1^n) = a_1^1$

Example 3. The sets of values $\{a_i, b_i, c_i\}$, $i \in [1,4]$ of the previous example forme four cycles of symmetry.

All values in a cycle of symmetry are symmetrical two by two. Therefore, our method of search of symmetry will process a symmetry which gives for each domain, classes ($cl(d_i)$ denotes the class of d_i) of values which are symmetrical together. Each classe will be identified by a cycle of symmetry. Before, describing the search method of symmetry, we will prove that syntactical symmetry is a sufficient condition for semantic symmetry.

Theorem 7. *If b_i and c_i are two syntactical symmetrical values of a CSP variable $v_i \in V$ ($b_i \sim c_i$) then b_i and c_i are semantic symmetrical values for all solutions of the CSP ($b_i \simeq c_i$).*

Proof. Cf. ([1]).

Remark. Syntactical symmetrical values are also semantic symmetrical values.

Symmetry expresses an important property that we use to make prune the search tree. Indeed if d_i participates in no soliution of the CSP $\mathcal{P}$ and $d_i \sim \dot{d}_i$, then $\dot{d}_i$ will participate in no solution too. Thus, we prune the sub-tree which corresponds to its assignment. Therefore, if there are n symmetrical domain values in $cl(d)$, then we can cut $n-1$ branches in the search tree if one of the domain values has already been identified that it paticipates in no solutions.

See that neighborhood interchangeability is a very particular syntactical symmetry which permuts the interchangeable values and still identity for the other values. Such symmetries can not exists frequentlly. Our approache is more general and will get more use. Bellow we give the search method for syntactical symmetry.

3.1 Search method for symmetry

To be syntactically symmetrical, values need to satisfy some necessary conditions:

Proposition 8. *Let $\lambda_{R_{ij}}(d_i)$ be the number of occurences of the value $d_i \in D_i$ in the relation R_{ij} and tuples($R_{ij}^{d_i}$) the set of tuples of R_{ij} in which d_i appears, then to be syntactically symmetrical, values b_i and c_i must satisfy the following conditions:*

1. $\lambda_{R_{ij}}(b_i) = \lambda_{R_{ij}}(c_i)$, $\forall R_{ij} \in R$;
2. *for each* $d_j \in tuples(R_{ij}^{b_i})$, $\exists \dot{d_j} \in tuples(R_{ij}^{c_i})$ *such that* $\lambda_{R_{ij}}(d_j) = \lambda_{R_{ij}}(\dot{d_j})$, $\forall R_{ij} \in R$.

Proof. Cf. ([1]).

The search method consits in two steps: (I) to partition each domain *w.r.t* the previous necessary conditions into primary classes of values which will be condidates for symmetry. (II) process a permutation σ from the primary classes which keeps the CSP invariant. We develope the step (II) which will give the complexity of the search method.

procedure ***symmetry***($D_i \in D$)
Repeat for each $R_{ij} \in R$:
  Repeat for each $d_i \in D_i$,
  such that $\ll d_i, d_j \gg \in tuples(R_{ij})$:
  choose $\sigma(d_i) \in cl(d_i)$ and $\sigma(d_j) \in cl(d_j)$,
  such that $\ll \sigma(d_i), \sigma(d_j) \gg \in tuples(R_{ij})$

Fig.3. The search *symmetry* algorithm

The classes of symmetrical values are the different sycles of σ. A complexity bound for this algorithm can be found by assigning a worst case bound to each repeat loop. Given m relations, at most a values in each domain variable, we have the bound (the factors correspond to the repeat loops and the choose operation in topdown order): $\mathcal{O}(m * a * a^2) = \mathcal{O}(m.a^3)$. Bellow we show how several methods can be augmented with the advantage of symmetry.

4 Adaptation of various Constraints Propagation Algorithms

Now we are in the position to show how these domain symmetrical values can be used to increase efficiency of various existing algorithms. We give a few modifications of the key procedures and show the advantages of the use of symmetry techniques for certain problem types. We focus on binary CSPs.

4.1 Constraint filtering algorithms

The critical and most time consuming task in network consistency procedures is to check if all values of a particular variable domain can potentially be a member of a solution. These checks are done repetitively for singular variables *w.r.t* singular constrains. In the case of binary constraints, the procedure *revise*(D_i,D_j) is usually used. It removes all values of D_i for which no value of the domain D_j can be found such that the binary constraint c_{ij} between variables v_i and v_j is satisfied. It is abvious that the worst-case complexity of revise is $\mathcal{O}(a^2)$ where a is the maximum domain size.

The procedure *revise* is applied on different constraints seperatelly, then symmetrical domain values must be computed *w.r.t* a given constraint c. The main idea is that domain values can be symmetrical *w.r.t* a constraint c, but not symmetrical *w.r.t* other constraints. So it is important to caracterize symmetrical values for each constraint of the network independently.

We use the expression $cl(d)_c^v$ (d is a domain value of the CSP variable $v \in var(c)$) to denote the equivalence class of symmetrical values *w.r.t* to the constraint c in which d appears; formally: $cl(d)_c^v = \{\dot{d} \in D_v : \dot{d} \sim d\}$.

Figure 4 shows the procedure *revise* augmented by the advantage of symmetry.

```
procedure revise^SV(var D_i:domain,D_j:domain)
begin
    Δ_i = {}
    repeat
        x := an element of D_i
        Δ_j = D_j
        repeat
            y := an element of Δ_j
            if ≪ x, y ≫ ∈ tuples(R_ij) then
              begin
              Δ_i := Δ_i ⋃ {cl(x)^{v_i}_{c_ij} ∩ D_i}
              Δ_j ={ }
              end
            else Δ_j := Δ_j-{cl(y)^{v_j}_{c_ij}}
        until(Δ_j ={ })
        D_i := D_i - {cl(x)^{v_i}_{c_ij}}
    until (D_i ={ })
    D_i := Δ_i
end
```

Fig.4. The $revise^{SV}$ algorithm.

The main difference between the *classical revise* and $revise^{SV}$ is that the former checks in the worst case all tuples $D_i \times D_j$ and the later treats groups of symmetrical values equally. A symmetry σ defined on a constraint c partionnes the domain D_i into subsets of domain values which are symmetrical together. If Π^v_c is the set of symmetrical subsets domain values of the variable $v \in var(c)$, *w.r.t* the constraint c, if we assume that the sets Π^v_c for all constraint c and all variables $v \in var(c)$ are of size $\acute{a}$ $(1 \leq \acute{a} \leq a)$, then a worst case bound of the algorithm $revise^{SV}$ is $\mathcal{O}(\acute{a}^2)$.

4.2 Backtrack search

In the following, we want to envolve a tree search scheme where symmetrical search branches are recognized by use of symmetrical values. The algorithm is basically the same as classical backtrack tree search as discribed, for instance in (Fox and Nadel [5]).

But first we have to give some notations we need for the development of the search procedure. Each output of a traditional backtrack procedure is an assignment tuple representing a solution for the given CSP. Because we want to handle groups of symmetrical values, we have to modify the form of the output. Instead of single assignment values, sets are used. As it was done in ([9]), assignment tuples are chifted to assignment bundles.

Definition 9 Assignment Bundle. Let V be the set of n variables of the CSP $\mathcal{P}$. An n-tuple Δ where the ith element $(1 \leq i \leq n)$ is a non-vacuous subset of the domain D_i is called an assignment bundle.

Definition 10 Solution Bundle. Let $sol(\mathcal{P})$ be the set of all solution of the CSP $\mathcal{P}$. An assignment bundle $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ on the variables V of the CSP is said to be a solution bundle, if and only if $\Delta_1 \times \Delta_2 \ldots \times \Delta_n \subseteq sol(\mathcal{P})$.

Solutions bundle represent then groups of paths throught the search tree, which are solutions of the CSP. The terms of local and global consistency (see, for instance Dechter[4]) can be extended to assignment bundles.

Definition 11.
- An assignment bundle Δ^p on the variables $V_p \subseteq V$ is said to be locally consistent, if every assignment tuple extractable from Δ^p is locally[4], consitent;
- An assignment bundle Δ^p on the variable $V_p \subseteq V$ is said to be globally consistent, if there exists an extention assignment bundle Δ^e on the variables $(V - V_p)$ such that $\Delta^p \cup \Delta^e$ is a solution bundle;
- An assignment bundle Δ^p is said to be inconsistent, if every assignment tuple extractable from Δ^p is inconsistent (i.e., no tuple in Δ^p can be extended to a solution).

Now we modify the classical backtrack search such that for each pass a bundle assignment is computed. Solutions bundle regroup

[4] I.e., all the constraints of the subnetwork defined by the variables V_p are satisfied.

sets of symmetrical solutions. The following theorem gives the fundamental basis for the utilisation of symmetrical values.

Theorem 12. *Let Δ^p be an assignment bundle on the variables $V_P \subseteq V$ which is either globally consistent or inconsistent. Let v be a variable of $V - V_p$, $\delta_v \subseteq D_v$ and C^k all binary constraints from v to variables of $(V - V_p)$, such that the two following two conditions hold :*

1. *$\Delta^p + \delta_v$ is locally consistent;*
2. *$\forall d_1, d_2 \in \delta_v : \forall c \in C^k, d_1 \sim d_2$.*
   Then $\Delta^p + \delta_v$ is either globally consistent or inconsistent.

Proof. Cf. ([1])

```
procedure backtrack^SV (k:integer,B:assign-bundle);
begin
    revise^SV(D_k, D_p), for 1 ≤ p < k;
    {or do some kind of look ahead filtring}
    d_k := D_k;
    repeat
    x := an element of D_k;
    C^f := all constraints on v_k to future variables;
    B[k] := (⋂_{c∈C^f} cl(x)_c^{v_k}) ∩ d_k;
    d_k := d_k − B[k]
    if k = n then write(B)
    else backtracking^SV(k + 1, B);
    until (d_k = {});
end.
```

Fig.5. The backtrack algorithm.

The advantageous behavior of the procedure *backtracking*^{SV} is that symmetrical search branches are bundled and visited once. If a dead-end occurs, all the partial assignment extractable frome the derived assignment bundle are proven to be confliting.

5 experiments

Now we want to invistigate the indicated performance improvement of our augmented search technique by experimental analysis. We will test both interchangeability and symmetry and compare them on two kind of problems: (I) randomlly generated CSP's, we use the same test model as proposed in Freuder ([8]). (II) the pigeon-hole problem which is known to be hard, is solved using symmetries with a linear complexity, however interchangeability get no use for this problem.

5.1 The experiment model

Random CSP's are characterized by the following four parameters: (1) n, the number of variables. (2) a, the maximum domain size. (3) t, the constraint tightness which is the fraction of forbidden tuples to the number of possible tuples. (4) the constraint density which is a number between 0 and 1 given by d, indicates the fraction of additional constraints.

5.2 Results

Three *forward-checking* search procedures are compared: (1) (FC), the classical *forward-checking*. (2) $(FC - NI)$, *forward-checking* with the advantage of neighborhood interchangeability seen as particular syntactical symmetry. (3) $(FC - SV)$, the instance of the search scheme *backtracking*^{SV} (see, figure 5) where *forward-cheking* filtring is used. The indicator of the complexity is the number of checks. Of cours, the checks needed for the computation of neighborhood interchangeability resp. symmetry are added to the run time checks. The samples of each test are 30 randomly generated CSP's.

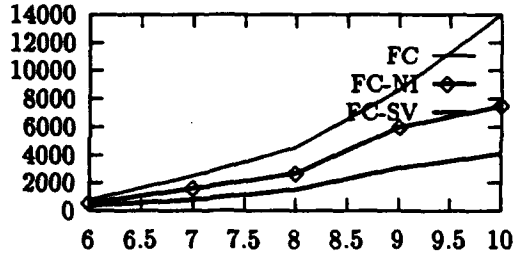

Fig.6.Symmetry effects w.r.t the number of variables

It can be seen in figure 6 that the effect of both symmetry and interchangeability grows if the problem increase. The variable size steps from 6 to 10, a is fixed on 5, t and d are from the interval [0.1-0.4] (the profitable ranges for the use of interchangeability as claimed in [8]). It can also be seen that $FC-SV$ definitily beats $FC-NI$ at these problems type.

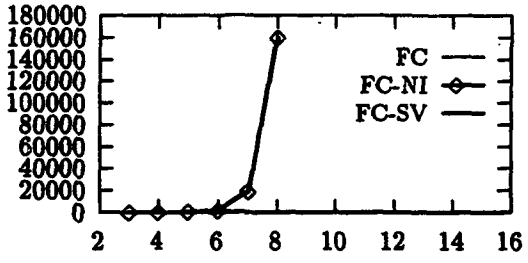

Fig.7. FC and FC-NI effects on Pigeon-hole problem.

Figure 8 shows that the complexity of $FC-SV$ for pigeon-hole problem seems to be linear, whereas both FC and $FC-NI$ (figure 7) cannot solve the problem when the number of pigeons is greater than eight, their complexities become quickly exponentiel when the number of pigeons is greater then 8. Thus, neighborhood interchangeability get no use for this problem.

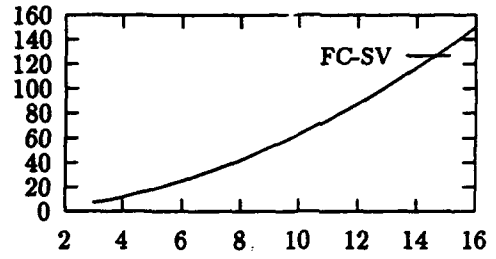

Fig.8. Symmetry effects on pigeon-hole problems.

6 Conclusion

We have developed the formal cocept of symmetry in constraint satisfaction problems, then various constraintes satisfaction algorithmes can be adapted to exploit such information. The principe of interchangebility is shown to be a particular case of symmetry. Further investigation will consist to extend symmetry to domain values of different variables and try to identify certain type of CSP's for which such symmetries get more use.

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Characterization of the set of models by means of symmetries

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Abstract

Many classes of *propositional calculus* problems display a large amount of symmetries, i.e. the set of clauses representing such problems remains invariant under certain permutations of variable names. In [2,1] we have shown how such symmetries can be detected and used to simplify satisfiability checking.

The problem of finding all models of a given CNF propositional theory is known to be hard. More generally, we need to explore a complete proof tree and, in some cases, the set of models is much too large to be represented explicitly.

In this paper, we show how symmetries can be used to represent a large set of models by a subset of characteristic models(non symmetric models). The other models can be obtained by applying the computed symmetries.

We present an algorithm for enumerating non symmetric models, and we show results obtained on some known problems, such as the pigeon-hole, queens and some other problems derived from mathematical theorems.

Key words. Theorem proving, propositional calculus, symmetries.

1 Introduction

Finding all satisfying models for a formula in conjonctive normal form (CNF), or even deciding whether a satisfying model exits (Sat), is known to be NP

hard. There are, however, a large classes of propositional problems which contains several symmetries. The principle of symmetry originaly suggested by Krishnamurty [8] can lead in many cases to a shorter proof of the problems. Indeed, for a set S of clauses with n propositional variables, there are 2^n possible interpretations (i.e. mapings from the variables to the set {True, False}). If S contains symmetries, then the interpretations can be partioned into equivalence classes. Satisfiability checking can be reduced to the problem of testing one interpretation from each such equivalence class. The number of such classes give us an estimate of the usefulness of a set of symmetries.

In [2,1] we have explained how symmetries are detected and used in some automated deduction methods such as Sl-Resolution algorithm and Davis and Putnam procedure. Good results have been obtained on some known hard propositional problems.

Enumerating all the models of CNF propositional theory is an interesting and difficult task[4]. The difficulty is that we generally need to explore a complete proof tree and in some cases the set of models is much too large to be represented explicitly. The interest of this, is that for certain kind of information the model-based representation is much more compact and enable much faster reasoning than the traditional representation using logical formulas[6,7].

In this paper, we show how symmetries can be used to represent the set of models by a subset of characteristic models(non symmetric models), from which all others can be generated. We present an algorithm for enumerating all non symmetric models, and we show results obtained on some known problems.

2 Preliminaries

We shall assume that the reader is familiar with the propositional calculus. For a propositional variable p there are two literals p the positive literal and $\neg p$ the negative one. A clause is a disjunction of literals such that no literal appears more than once, a clause containing no literals is called the empty clause. A set S of clauses is a conjunction of clauses. In other words we say that S is in the conjunctive normal form. A truth assignment to a set of clauses S is a map I from the set of variables occurring in S to the set {True, False}. The value of S under the truth assignment will be defined in the usual sense. We say that a set of clauses S is satisfiable if there exists some truth assignments in which S takes the value True; it is unsatisfiable otherwise. In the first case I is called a model of S. Also, if ℓ is true in a model of S, we say that ℓ has a model in S. We identify $\neg\ell$ to the opposite of ℓ.

3 Symmetries

We recall some definitions and property of symmetry, for more details see [2,1].

A bijective map $\sigma : V \rightarrow V$ is called a permutation of variables. If S is a set of clauses, c a clause of S and σ a permutation of variables occurring in S, then $\sigma(c)$ is the clause obtained by applying σ to each variable of c and $\sigma(S) = \{\sigma(c)/c \in S\}$.
In the following we define a permutation on literals.

Definition 3.1 *A set P of literals is called* complete *if* $\forall \ell \in P$, $\neg\ell \in P$

Definition 3.2 *Let P be a complete set of literals and S a set of clauses of which all literals are in P.*
A permutation σ defined on P ($\sigma : P \rightarrow P$) is called a symmetry *of S if it satisfies the following conditions :*

1. $\forall \ell \in P, \sigma(\neg\ell) = \neg\sigma(\ell)$
2. $\sigma(S) = S$

Definition 3.3 *Two literals (variables) ℓ and ℓ' are symmetric in S notation ($\ell \sim \ell'$) if there exists a symmetry σ of S such that $\sigma(\ell) = \ell'$. A tuple $(\ell_1, \ell_2, \ldots, \ell_n)$ of literals is called a cycle of symmetry in S if there exist a symmetry σ defined on S, such that $\sigma(\ell_1) = \ell_2, \ldots \sigma(\ell_{n-1}) = \ell_n, \sigma(\ell_n) = \ell_1$.*

Example 3.4 *Let S be the following set of clauses : $S = \{a \vee \neg b, c\}$*
and σ the map defined on the complete set P of literals occurring in S :
$\sigma(a) = \neg b$, $\sigma(\neg a) = b$, $\sigma(b) = \neg a$, $\sigma(\neg b) = a$, $\sigma(c) = c$ and $\sigma(\neg c) = \neg c$
σ is a symmetry of S , a and $\neg b$ are symmetric in S ($a \sim \neg b$).
$\sigma(S) = \{\neg b \vee a, c\} = S$.

Definition 3.5 *Let P be a complete set of literals, σ a symmetry, I a truth assignment of P and S a set of clauses then, $\sigma(I)$ is the truth assignment obtained by substituting every literal ℓ in I by $\sigma(\ell)$.*

Proposition 3.6 *I is a model of S iff $\sigma(I)$ is a model of S .*

Proof : cf.[2,1]

From the poposition above, one can define an equivalence relation on the set of interpretations by : $I_1 \sim I_2$ iff there exists a symmetry σ on S such that $I_1 = \sigma(I_2)$.
In the previous example, the set of possible interpretations can be partioned into six distinct classes : {[000],[110]}, {[001],[111]}, {[010]},{[011]}, {[100]} and {[101]}. Also we can easily distinguish two distinct classes of models : {[001],[111]} and {[101]} ([101] should be read as [a=True, b=False and c=True]).

Theorem 3.7 *Let ℓ and ℓ' be two literals of S.*
if $\ell \sim \ell'$ in S, then ℓ has a model in S iff ℓ' has a model in S .

Proof : direct consequence of proposition 3.6

Let us define $S_{\ell_1,\ell_2,\ldots,\ell_n}$ the set of clauses obtained after assigning to the literals $\ell_1, \ell_2, \ldots, \ell_n$ the value true, consequently,

Corollary 3.8 *Let $(\ell, \ell_1, \ell_2, \ldots, \ell_n)$ be a cycle of symetry of ℓ in S then, S is satisfiable iff S_ℓ or $S_{\neg\ell,\neg\ell_1,\neg\ell_2,\ldots,\neg\ell_n}$ is satisfiable.*

The previous theorem is very usefull to make prune the proof trees. Indeed, if ℓ has no model in S and $\ell \sim \ell'$, then ℓ' will have no model in S, thus we prune the branch which corresponds to the assignment of ℓ' in the proof tree. Therefore, if there are n symmetric literals we can cut $n-1$ branches.

In [2,1], we have explained how symmetries are detected and used in different automated deduction algorithms such as Sl-Resolution and the Davis and Putnam procedure. It should be noted that in our previous work [2,1], we search for symmetries at each level of the proof tree : on the set of clauses simplified by the current assignment, we call these symmetries local, in opposition to symmetries of the original set of clauses (global symmetries). Local symmetries must be very usefful, when the problem holds some symmetric kernels. This kind of symmetries can't be used, if we adress the problem of computing non symmetric solutions of the problem. In some other problems, the symmetries appear on the original problem (global), but they can disapear after assignment of some variables.

4 A characterization of the set of models

In the sequel, We show how the principle of symmetry can lead to a short representation of the set of models.

Definition 4.1 *Two models m_1 and m_2 of a set S of clauses are symmetric if there exists a symmetry σ of S such that $\sigma(m_1) = m_2$*

Remark 4.2 If $\ell \sim \ell'$ on S, then ℓ and ℓ' have the same number of models.

Definition 4.3 *Let S be a set of clauses and A a set of literals occurring in S which don't contain a literal and its opposite.*
We define $M(S, A)$ as a set of models of S which contain the literals of A. and $M(S, \emptyset)$ is the set of all models of S.

Definition 4.4 *Let S be a set of clauses and $A = \{\ell_1, \ell_2, \ldots, \ell_n\}$ a set of literals of S which don't contain a literal and its opposite.*

We define $M(S,\emptyset)/A$ as a partition of all the models of S on A :
$M(S,\emptyset)/A = \{M(S,\ell_1), M(S,\neg\ell_1\ell_2), M(S,\neg\ell_1\neg\ell_2\ell_3)\ldots$
$M(S,\neg\ell_1\neg\ell_2\ldots\neg\ell_{n-1}\ell_n), M(S,\neg\ell_1\neg\ell_2\ldots\neg\ell_n)\}$

Theorem 4.5 *Let S be a set of clauses and σ a symmetry on S such that $(\ell_1,\ell_2,\ell_3,\ldots,\ell_n)$ is a cycle of symmetry, then,*

1. *$\forall i$ such that $2 \leq i \leq n$,*
   $M(S,\neg\ell_1\ldots\neg\ell_{i-1}\ell_i) = \sigma^{i-1}(M(S,\ell_1\neg\ell_{n-(i-2)}\ldots\neg\ell_n))$ [1] and;
2. *The models $M(S,\ell_1)$ and $M(S,\neg\ell_1\ldots\neg\ell_n)$ are not symmetric.*

Proof :
1)- Obvious. By the definition of a cycle of symmetry, one can easily write $\sigma^{i-1}(M(S,\ell_1\neg\ell_{n-(i-2)}\ldots\neg\ell_n))$ as $M(S,\neg\ell_1\ldots\neg\ell_{i-1}\ell_i)$
2)- Suppose that there exists $m_1 \in M(S,\ell_1)$ and $m_2 \in M(S,\neg\ell_1\ldots\neg\ell_n)$ such that $\sigma^i(m_1) = m_2$ with $1 \leq i < n$.
$\ell_1 \in m_1$, then $\sigma^i(\ell_1) \in m_2$ and $\sigma^i(\ell_1) \in \{\ell_1,\ell_2,\ldots,\ell_n\}$. m_2 contain the literals $\neg\ell_1,\neg\ell_2,\ldots,\neg\ell_n$, one can see that m_2 contain a literal and its opposite (contradiction)□

This theorem show that, for a cycle of symmetry $\rho = (\ell_1,\ldots,\ell_n)$ on S, the set of models of S can be partitioned into three subsets :
$M_1 = M(S,\ell_1), M_2 = \cup_{i=2}^{n}\sigma^{i-1}(M(S,\ell_1\neg\ell_{n-(i-2)}\ldots\neg\ell_n))$ and
$M_3 = M(S,\neg\ell_1\ldots\neg\ell_n)$.
The models M_2 are included in M_1 up to symmetry. The models M_1 and M_3 are non symmetric.

4.1 Use of symmetries

We will show an algorithm *Find_Non_Symmetric_Models(S)*(Figure 1), for enumerating all non symmetric models of S, this algorithm uses as its basic subroutine *Solve(S)*. If S is satisfiable, then it returns a satisfying truth assignment; otherwise, it returns nil. The notation *Solve(S,ϕ)* is a shorthand for *Solve($S \cup \phi$)*, ϕ is the set of literals in the current assignment.
In order to find all the solution, we use the Davis and Putnam procedure without monotone (pure) literal rule.
Let $\phi = \{\ell_1\ldots\ell_k\}$ be the current set of literals assigned the value true. Suppose we have found all the models of $S \cup \phi$. Before searching for the models of $S \cup \{\ell_1\ldots\ell_{k-1},\neg\ell_k\}$, we search for a cycle of symmetry ρ of the literal ℓ_k on S(global symmetry), with the condition that $\{\ell_1,\ell_2\ldots\ell_{k-1}\}$ is invariant under the symmetry[2]. This additional condition allows us to avoid

[1] σ^i : application of σ i times
[2] A set of literals ϕ is invariant under a symmetry σ iff $\forall\ell \in \phi, \sigma(\ell) \in \phi$

symmetric models to the models found at the current level of the proof tree. Now, we search for the models of $S \cup \{\ell_1 \ldots \ell_{k-1} \neg \ell_k\} \cup \{\neg \ell, \forall \ell \in \rho\}$.
As it is shown in Figure 1, in subroutine *Find_Next_Model* we search for global symmetries to avoid symmetric models, and in *Solve* we search for local symmetries in case of contradiction.

```
Find_Non_Symmetric_Models(S)
{
φ ← ∅
model←Solve(S,φ)
while model≠ nil
do { print model
       model←Find_Next_Model(S, model)
       }
}
Find_Next_Model(S,{ℓ1, ℓ2 ... ℓn})
{
for i=n downto 1
do {
  /* global symmetry */
  compute a cycle of symmetry ψ of ℓi on S such that {ℓ1, ℓ2 ... ℓi−1} is invariant
  model←Solve(S,{ℓ1, ℓ2 ... ℓi−1¬ℓi} ∪ {¬ℓ, ∀ℓ ∈ ψ})
  if model≠nil then return(model)
  }
return(nil)
}
Solve(S,φ)
{
unit_propagate(S,φ) /* repeated application of unit-literal rule */
if contradiction discovered then return(nil)
else if all clauses are satisfied then return(φ)
       else {
          x ← some unvalued variable
          if Solve(S,φ ∪ {x})=nil
          then {
             compute a cycle of symmetry ψ of x on S ∪ φ /* local symmetry */
             return(Solve(S,φ ∪ {¬x} ∪ {¬ℓ, ∀ℓ ∈ ψ}))
          else return(Solve(S,φ ∪ {x}))
          }
}
```

Figure 1: Algorithms Find_Non_symmetric_Models

5 Results

We now present some results on the algorithm (Figure h and without symmetry. For each problem, we give the total number of models(NM) and

the number of non symmetric models(NSM). Also in the case of unsatisfiability we show also how symmetries affect the size of the proof tree.

5.1 Description of the benchmarks

- Queens. Placing N queens in $N \times N$ chessboard such that there is no couple of queens attacking each other. Notation Queen(N)

- Erdõs's theorem. Find the permutation σ of N first numbers such that for each 4-tuple $1 \leq i < j < k < l \leq$ N none of the two relations $\sigma(i) < \sigma(j) < \sigma(k) < \sigma(l)$ and $\sigma(l) < \sigma(k) < \sigma(j) < \sigma(i)$ is verified.
  This problem is modeled by creating for each couple (i,j) a variable $f_{i,j}$ which means $\sigma(i) < \sigma(j)$. The rules express the associativity of the relation $<$, and prohibit the misplaced 4-tuples.
  For $N \leq 9$ the problem admits solutions, beyond it doesn't.Notation Erdos(N)

- Pigeon Hole: Put n pigeon in $n-1$ pigeon-holes such that each pigeonhole holds at most one pigeon. The problem is unsatisfiable, for n pigeon and n holes the problem have $n!$ solutions. Notation Pigeon(P,H)

- Schur's lemma: How to distribute N counters numbered from 1 to N into 3 boxes A, B, C in accordance with the following rules:
  1) A box can't contain both the counters numbered i and $2 * i$
  2) A box can't contain the counters numbered i, j and $i + j$
  This problem is modeled simply by creating one variable by counter and by box. For $N \leq 13$ the problem admits solutions, beyond it doesn't. Notation Schur(N)

- Ramsey problem's: Color the edges of a complete graph on N vertices with k different colors such that no monochromatic triangle appears. Notation Ramsey(N,K)

Table 1 : Schur's Lemma and Ramsey's problem,etc.

Problems	SAT	Without symmetry			With symmetry		
		NM	Steps	Times	NSM	Steps	Times
Pigeon(10,10)	Y	10!	-	-	1	156	4.11"
Queen(4)	Y	2	76	0.53"	1	28	0.150"
Queen(6)	Y	4	1066	4.13"	2	278	2.150"
Queen(8)	Y	92	17304	1'21"	23	7321	30.53"
Erdős(9)	Y	1356	35732	3'57"	125	16328	2'.17"
Erdős(10)	N	0	2332	6.213"	0	1166	4.56"
Schur(13)	Y	18	2029	5.83"	1	148	1.96"
Schur(14)	N	0	1878	4.17"	0	374	1.517"
Ramsey(5,2)	Y	2	231	0.200"	1	43	0.01"

Table 2 : Pigeon-hole problems

Number of pigeons	Clauses	Variables	With symmetries	
			Steps	Times
14	1197	182	193	3.21"
16	1816	240	253	6.83"
18	2619	306	321	13.11"
20	3630	380	397	23.04"
22	4873	462	481	35.29"
24	6372	552	573	54.73"
26	8150	650	673	1'20"
28	10234	756	781	2'15"
30	12645	870	897	3'34"

6 Related Work

Krishnamurty[8] discuses the idea of using symmetries to reduce the length of resolution proofs, he uses a rule of symmetry to avoid repeated independent derivations of intermediate formulas that are permutations of others. His work does not adress the problem of detecting symmetries or of using them in search problems.

Benhamou and Sais[1,2] discusses the detection and the use of symmetries in automated deduction methods.

Freuder[5] discusses the elimination of interchangeable value in constraint satisfaction problem.

Also, a theoritical analysis of reasoning by symmetry in first-order logic have been presented in Crawford[3].

7 Conclusion

In this paper, we have shown how global symmetries can be used to obtain a new characterization of the set of models of a given CNF propositional theory. For some problems symmetries give us a way to represent large sets of models.
Also, the results obtained in this paper, shows the usefulness of global symmetries in case of checking satisfiability. There are, however, some problems which possesses abundant local symmetry. Consequently, in order to increase the tractable classes of problems by using symmetries, it is necessary to combine the two kinds of symmetries.
In special case of Horn formulas and 2-cnfs, computing all models, although counting is #P-complete, we intend to experiment our algorithm on this kind of formulas.

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Constraint-Generating Dependencies

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Abstract

Traditionally, dependency theory has been developed for uninterpreted data. Specifically, the only assumption that is made about the data domains is that data values can be compared for equality. However, data is often interpreted and there can be advantages in considering it as such, for instance obtaining more compact representations as done in constraint databases. This paper considers dependency theory in the context of interpreted data. Specifically, it studies *constraint-generating dependencies*. These are a generalization of equality-generating dependencies where equality requirements are replaced by constraints on an interpreted domain. The main technical results in the paper are decision procedures for the implication and consistency problems for constraint-generating dependencies. These decision procedures proceed by reducing the dependency problem to a decision problem for the constraint theory of interest, and are applicable as soon as the underlying constraint theory is decidable. Furthermore, complexity results for specific constraint domains can be transferred quite directly to the dependency problem.

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1 Introduction

Relational database theory is largely built upon the assumption of uninterpreted data. While this has advantages, mostly generality, it foregoes the possibility of exploiting the structure of specific data domains. The introduction of constraint databases [KKR90] was a break with this uninterpreted-data trend. Rather than defining the extension of relations by an explicit enumeration of tuples, a constraint database uses constraint expressions to implicitly specify sets of tuples. Of course, for this to be possible in a meaningful way, one needs to consider interpreted data, that is, data from a specific domain on which a basic set of predicates and functions is defined. A typical example of constraint expressions and domain are linear inequalities interpreted on the reals. The potential gains from this approach are in the compactness of the representation (a single constraint expression can represent many, even an infinite number of, explicit tuples) and in the efficiency of query evaluation (computing with constraint expressions amounts to manipulating many tuples simultaneously).

Related developments have concurrently been taking place in temporal databases. Indeed, time values are intrinsically interpreted and this can be exploited for finitely representing potentially infinite temporal extensions. For instance, in [KSW90] infinite temporal extensions are represented with the help of periodicity and inequality constraints, whereas in [*CI88*, *CI89*, *CI93*] and [Bau89, Bau92] deductive rules over the integers are used for the same purpose. Constraints have also been used recently for representing incomplete temporal information [vdM92, Kou92, Kou93].

If one surveys the existing work on databases with interpreted data and implicit representations, one finds contributions on the expressiveness of the various representation formalisms [Bau, BNW91, BCW93], on the complexity of query evaluation [Cho90, CM90, Rev90, vdM92], and on data structures and algorithms to be used in the representation of constraint expressions and in query evaluation [FVP92, Sri92, BJM93, BLL93, KRVV93]. However, much less has been done on extending other parts of traditional database theory, for instance schema design and dependency theory. It should be clear that dependency theory is of interest in this context. For instance, in [JS92], one finds a taxonomy of dependencies that are useful for temporal databases. In [GH83, GH86, IO93, ZO93], one finds a study of integrity constraints over databases with ordered domains, which can be viewed as constraint-generating dependencies.

One might think that the study of dependency theory has been close to exhaustive. While this is largely so for dependencies over uninterpreted data (that is, the context in which data values can only be compared for equality) [Tha91], the situation is quite different for dependencies over data domains with a richer structure. The subject of this paper is the theory of these interpreted dependencies.

Specifically, we study the class of *constraint-generating dependencies*. These are the generalization of equality-generating dependencies [BV84], allowing arbitrary constraints on the data domain to appear wherever the latter only allow equalities. For instance, a constraint-generating dependency over an ordered domain can specify that if the value of an attribute A in a tuple t_1 is less than the value of the same attribute in a tuple t_2, then an identical relation holds for the values of an attribute B. This type of dependency can express a wide variety of constraints on the data. For instance, most of the temporal dependencies appearing in the taxonomy of [JS92] are constraint-generating dependencies.

Our technical contributions address the implication and the consistency[1] problems for constraint-generating dependencies. The natural approach to these problems is to write the dependencies as logical formulas. Unfortunately, the resulting formulas are not just formulas in the theory of the data domain. Indeed, they also contain uninterpreted predicate symbols representing the relations and thus are not a priori decidable, even if the data domain theory is decidable.

To obtain decision procedures, we show that the predicate symbols can be eliminated. Since the predicate symbols are implicitly universally quantified, this can be viewed as a form of second-order quantifier elimination. It is based on the fact that it is sufficient to consider relations with a small finite number of tuples. This then allows quantifier elimination by explicit representation of the possible tuples. The fact that one only needs to consider a small finite number of tuples is analogous to the fact that the implication problem for functional dependencies can be decided over 2-tuple relations [Mai83]. Furthermore, for pure functional dependencies, our quantifier elimination procedures yields exactly the usual reduction to propositional logic. For more general constraint dependencies, it yields a formula in the theory of the data domain. Thus, if this theory is decidable, the implication and the consistency problems for constraint-dependencies are also decidable.

The complexity of the decision procedure depends on the specific data domain being considered and on the exact form of the constraint dependencies. We consider three typical constraint languages: equalities/inequalities, ordering constraints, and linear arithmetic constraints. We give a variety of complexity results for the implication problem of dependencies over these theories and show the impact of the form of the dependencies on tractability.

[1] Though consistency is always satisfied for equality-generating dependencies, more general constraints turn it into a nontrivial problem.

2 Constraint-Generating Dependencies

Consider a relational database where some attributes take their values in specific domains, such as the integers or the reals, on which a set of predicates and functions are defined. We call such attributes *interpreted*. For the simplicity of the presentation, let us assume that the database only contains one (universal) relation r and let us ignore the noninterpreted attributes. In this context, it is natural to generalize the notion of equality-generating dependency [BV84]. Rather than specifying the propagation of equality constraints, we write similar statements involving arbitrary constraints (i.e., arbitrary formulas in the theory of the data domain). Specifically, we define *constraint-generating k-dependencies* as follows (the constant k specifies the number of tuples the dependency refers to).

Definition 2.1 Given a relation r, a *constraint-generating k-dependency* over r (with $k \geq 1$) is a first-order formula of the form

$$(\forall t_1)\cdots(\forall t_k)\Big[\Big[r(t_1)\wedge\cdots\wedge r(t_k)\wedge C[t_1,\ldots,t_k]\Big] \Rightarrow C'[t_1,\ldots,t_k]\Big]$$

where $C[t_1,\ldots,t_k]$ and $C'[t_1,\ldots,t_k]$ denote arbitrary constraint formulas relating the values of various attributes in the tuples $t_1,\ldots,t_k$. There are no restrictions on these formulas, they can include all constructs of the constraint theory under consideration, including quantification on the constraint domain.

Constraint-generating 1-dependencies as well as constraint-generating 2-dependencies are the most common. Notice that functional dependencies are a special form of constraint-generating 2-dependencies. Constraint-generating dependencies can naturally express a variety of arithmetic integrity constraints. The following examples illustrate their definition and show some of their potential applications.

Example 2.1 In [JS92], an exhaustive taxonomy of dependencies that can be imposed on a temporal relation is given. Of the more than 30 types of dependencies that are defined there, all but 4 can be written as constraint-generating dependencies. These last 4 require a generalization of tuple-generating dependencies [BV84] (see Section 5).

For instance, let us consider a relation $r(tt, vt)$ with two temporal attributes: transaction time (tt) and valid time (vt). The property of r being "strongly retroactively bounded" with bound $c \geq 0$ is expressed as the constraint-generating 1-dependency

$$(\forall t_1)\Big[r(t_1)\Rightarrow [(t_1[tt]\leq t_1[vt]+c)\wedge(t_1[vt]\leq t_1[tt])]\Big].$$

The property of r being "globally nondecreasing" is expressed as the constraint-generating 2-dependency

$$(\forall t_1)(\forall t_2)\Big[[r(t_1)\wedge r(t_2)\wedge(t_1[tt]<t_2[tt])] \Rightarrow (t_1[vt]\leq t_2[vt])\Big]. \blacksquare$$

Example 2.2 Let us consider a relation $emp(name, boss, salary)$. Then the fact that an employee cannot make more than her boss is expressed as

$$(\forall t_1)(\forall t_2)\Big[[emp(t_1)\wedge emp(t_2)\wedge(t_1[boss]=t_2[name])] \Rightarrow (t_1[salary]\leq t_2[salary])\Big]. \blacksquare$$

3 Decision Problems for Constraint-Generating Dependencies

The basic decision problems for constraint-generating dependencies are:

- *implication:* does a finite set of dependencies D imply a dependency d_0?
- *consistency:* does a finite set of dependencies D have a non-trivial model, that is, is D true

in a nonempty relation?

The first problem is a classical problem of database theory. Its practical motivation comes from the need to detect *redundant* dependencies, that is, those that are implied by a given set of dependencies. The second problem has a trivial answer for uninterpreted dependencies: every set of equality- and tuple-generating dependencies has a 1-element model. However, even a single constraint-generating dependency may be inconsistent, as illustrated by $(\forall t)[r(t) \Rightarrow t[1] < t[1]]$. We only study the *implication* problem since the consistency problem is its dual: a set of dependencies D is inconsistent if and only if D implies a dependency of the form:

$$(\forall t)[r(t) \Rightarrow C]$$

where C is any unsatisfiable constraint (we assume the existence of at least one such unsatisfiable constraint formula).

The result we prove in this section is that the implication problem for constraint-generating dependencies reduces to the unsatisfiability problem for a formula in the underlying constraint theory. Specific dependencies and theories will be considered in Section 4, and the corresponding complexity results provided. The reduction proceeds in three steps. First, we prove that the implication problem is equivalent to the implication problem restricted to finite relations of bounded size. Second, we eliminate from the implication to be decided the second-order quantification (over relations). Third, we eliminate the first-order quantification (over tuples) from the dependencies themselves and replace it by quantification over the domain – a process that we call *symmetrization*. This gives us the desired result.

3.1 Statement of the Problem and Notation

Let r denote a relation with n interpreted attributes. Let $d_0, d_1, \ldots, d_m$ denote constraint-generating k-dependencies over the attributes of r. The value of k need not be the same for all d_i's. We denote by k_0 the value of k for d_0.

The *dependency implication problem* consists in deciding whether d_0 is implied by the set of dependencies $D = \{d_1, \ldots, d_m\}$ In other words, it consists in deciding whether d_0 is satisfied by every interpretation that satisfies D, which can be formulated as

$$(\forall r)\Big[r \models D \Rightarrow r \models d_0\Big], \qquad (1)$$

where D stands for $d_1 \wedge \cdots \wedge d_m$.

We equivalently write formula (1) as

$$(\forall r)\Big[D(r) \Rightarrow d_0(r)\Big]$$

when we wish to emphasize the fact that the dependencies apply to the tuples of r.

3.2 Towards a Decision Procedure

3.2.1 Reduction to k-tuple Relations

We first prove that, when dealing with constraint-generating k-dependencies, it is sufficient to consider relations of size[2] k.

Lemma 3.1 *Let d denote any constraint-generating k-dependency. If a relation r does not satisfy d, then there is a relation r' of size k that does not satisfy d. Furthermore, r' is obtained from r by removing and/or duplicating tuples.*

Proof: Let us assume that r does not satisfy the k-dependency d, which is of the form

$$(\forall t_1)\cdots(\forall t_k)\Big[\Big[r(t_1) \wedge \cdots \wedge r(t_k) \wedge C[t_1, \ldots, t_k]\Big] \Rightarrow C'[t_1, \ldots, t_k]\Big].$$

This means that there must exist k tuples $t_1 \ldots, t_k$ in r such that $C[t_1, \ldots, t_k]$ holds and $C'[t_1, \ldots, t_k]$ does not hold. Take r' to be the relation consisting of these tuples. Notice that these tuples are not necessarily distinct, but we do keep duplicate tuples in r' so that it is of size exactly k. Clearly r' does not satisfy d. ∎

[2] In what follows, we consider relations as multisets rather than sets. This has no impact on the implication problem, but simplifies our procedure.

Lemma 3.2 *If a relation r satisfies a set of constraint-generating k-dependencies $D = \{d_1, \ldots d_m\}$ and does not satisfy a constraint-generating k_0-dependency d_0, then there is a relation r' of size k_0 that satisfies D but does not satisfy d_0.*

Proof: Let us assume that r satisfies D and does not satisfy d_0. Since r does not satisfy d_0, we can conclude by Lemma 3.1 that there exists a relation r' of size k_0 that does not satisfy d_0. Since r satisfies D, this relation r' also satisfies D. Indeed, r' is obtained from r by eliminating and duplicating tuples from r (Lemma 3.1), and this cannot falsify the constraint-generating dependencies of D, which are universally quantified formulas over tuples. Therefore, there is a relation r' of size k_0 that satisfies D but not d_0. ∎

Theorem 3.3 *Consider an instance (D, d_0) of the dependency implication problem where d_0 is a constraint-generating k_0-dependency. The dependency d_0 is implied by D over all relations if and only if it is implied by D over relations of size k_0. In other words,*

$$(\forall r)\Big[r \models D \Rightarrow r \models d_0\Big]$$
$$\textit{if and only if}$$
$$(\forall r')\Big[|r'| = k_0 \Rightarrow \Big[r' \models D \Rightarrow r' \models d_0\Big]\Big].$$

Proof: One direction is trivial. For the other, assume that the implication is satisfied by all relations of size k_0. First, it is satisfied by all relations of size less than k_0 since such a relation can be transformed into a relation of size k_0 by duplicating tuples. Next, it must be satisfied by all relations of size greater than k_0. Indeed, let us assume that one such relation r does not satisfy the implication, that is, $r \models D$, but $r \not\models d_0$. Then, by Lemma 3.2, there must exist a relation of size k_0 that satisfies D but not d_0 – a contradiction. ∎

3.2.2 Second-order Quantifier Elimination

By Theorem 3.3, in order to decide the implication problem, we just need to be able to decide this problem over relations of size k for a given k. Deciding the implication (1) thus reduces to deciding

$$(\forall r')\Big[[|r'| = k \wedge D(r')] \Rightarrow d_0(r')\Big]. \qquad (2)$$

Let $r' = \{t_{x_1}, t_{x_2}, \ldots, t_{x_k}\}$ denote an arbitrary relation of size k where $t_{x_1}, t_{x_2} \ldots, t_{x_k}$ are arbitrary tuples. We can eliminate the (second-order) quantification over relations from the implication (2) and replace it with a quantification over tuples (that is, over vectors of elements of the domain). We get

$$(\forall t_{x_1}) \cdots (\forall t_{x_k})$$
$$\Big[D(\{t_{x_1}, \ldots, t_{x_k}\}) \Rightarrow d_0(\{t_{x_1}, \ldots, t_{x_k}\})\Big]. \quad (3)$$

3.2.3 Symmetrization

In this section, we simplify the formula (3), whose validity is equivalent to the constraint dependency implication problem, by eliminating the quantification over tuples that appears in the dependencies. We refer to this quantifier elimination procedure for dependencies as *symmetrization*. For the sake of clarity, we present the details of the symmetrization process for the case where $k = 2$. The process can be generalized directly to the more general case.

For the case where $k = 2$, the formula (3) to be decided is the following.

$$(\forall t_x)(\forall t_y)\Big[D(\{t_x, t_y\}) \Rightarrow d_0(\{t_x, t_y\})\Big].$$

We can simplify this formula further by eliminating the quantification over tuples that appears in the dependencies $d(\{t_x, t_y\})$ in $D \cup \{d_0\}$. Every such dependency $d(\{t_x, t_y\})$ can indeed be rewritten as a constraint formula $cf(d)$ in the following manner.

1. Let d be a 1-dependency, that is, d is of the form $(\forall t)\Big[[r'(t) \wedge C[t]] \Rightarrow C'[t]\Big]$. This dependency considered over $r' = \{t_x, t_y\}$ is equivalent to the constraint formula

   $$cf(d): \quad \Big[C[t_x] \Rightarrow C'[t_x]\Big] \wedge \Big[C[t_y] \Rightarrow C'[t_y]\Big],$$

   which is a conjunction of $k = 2$ constraint implications. Notice that the t_x and t_y appearing in this formula are just tuples of variables ranging over the domain of the constraint theory of interest.

2. Let d be a 2-dependency, that is, d is of the form

   $$(\forall t_1)(\forall t_2)\Big[[r'(t_1) \wedge r'(t_2) \wedge C[t_1, t_2]] \Rightarrow C'[t_1, t_2]\Big].$$

   This dependency considered over $r' = \{t_x, t_y\}$ is equivalent to the constraint formula

   $$cf(d): \quad \begin{bmatrix} C[t_x, t_y] \Rightarrow C'[t_x, t_y] \end{bmatrix} \wedge \begin{bmatrix} C[t_y, t_x] \Rightarrow C'[t_y, t_x] \end{bmatrix} \wedge \begin{bmatrix} C[t_x, t_x] \Rightarrow C'[t_x, t_x] \end{bmatrix} \wedge \begin{bmatrix} C[t_y, t_y] \Rightarrow C'[t_y, t_y] \end{bmatrix},$$

   which is a conjunction of $k^k = 4$ constraint implications.

The rewriting of d as $cf(d)$ is what we call the *symmetrization* of d, for rather obvious reasons. It extends directly to any value of k. Notice that for a given k, any j-dependency d is rewritten as a constraint formula $cf(d)$, which is a conjunction of k^j constraint implications. Interestingly, in the case of functional dependencies, symmetrization is not needed. This is due to the fact that the underlying constraints are equalities, which are already symmetric. Hence, in that case as well as in any other case of symmetric constraints, symmetrization would produce several instances of the same constraint formulas.

Applying the symmetrization process to all the dependencies appearing in the formula (3), we get

$$(\forall t_{x_1}) \cdots (\forall t_{x_k}) \Big[cf(d_1) \wedge \cdots \wedge cf(d_m) \Rightarrow cf(d_0)\Big]. \qquad (4)$$

Notice that in formula (4), each tuple variable can be replaced by n domain variables, and thus the quantification over tuples can be replaced by a quantification over elements of the domain. For the sake of clarity, we simply denote by $(\forall *)$ the adequate quantification over elements of the domain (the *universal closure*). Formula (4) thus becomes

$$(\forall *)\Big[cf(d_1) \wedge \cdots \wedge cf(d_m) \Rightarrow cf(d_0)\Big], \qquad (5)$$

where each $cf(d)$ is a conjunction of k^j constraint implications if d is a j-dependency and d_0 is a k-dependency. Thus, we have reduced the implication problem to the validity of a universally quantified formula of the constraint theory.

Example 3.1 Let us consider the following constraint-generating 2-dependencies over a relation r with a single attribute.

$$d_1: \quad (\forall x)(\forall y)\Big[r(x) \wedge r(y) \Rightarrow x \leq y\Big]$$
$$d_2: \quad (\forall x)(\forall y)\Big[r(x) \wedge r(y) \Rightarrow x = y\Big]$$

Symmetrizing them produces the following constraint formulas.

$$cf(d_1): \quad x \leq y \wedge y \leq x \wedge x \leq x \wedge y \leq y$$
$$cf(d_2): \quad x = y \wedge y = x \wedge x = x \wedge y = y.$$

It is clear that these two constraint formulas are equivalent, as they should be. ∎

4 Complexity Results

In order to study the complexity of the implication problem for constraint-generating dependencies, first make the assumption that the constraint formulas appearing in these dependencies are *conjunctions of atomic constraints*. This assumption is satisfied by all the examples of interest. Without loss of generality, we also assume that the consequents of dependencies are atomic. We call such simpler dependencies *clausal constraint-generating dependencies*. Moreover, we assume that the constraint language is *closed under negation*.

Simple transformations demonstrate that for clausal dependencies the implication problem can be expressed as the unsatisfiability of a formula of the following form:

$$(\exists *)\Big[\bigwedge_i \Big(\bigvee_j (c_{ij})\Big)\Big].$$

where each c_{ij} is an atomic formula. When $|D| = m$ and d_0 is a k-dependency, the number of conjuncts in the formulas above is at most equal to $m \cdot k^k$ plus the number of constraints in d_0. Thus deciding the validity of the implication problem for k-dependencies (k fixed) can be done by checking the unsatisfiability of a fixed number of conjunctive normal form constraint formulas of length that is linear in the size of $D \cup \{d_0\}$. The opposite LOGSPACE reduction also exists.

Given the above reductions, we obtain several complexity results for the implication problem for specific constraint languages. Assuming that k is fixed, we have the following.

Theorem 4.1 *For constraints in the theory of equality and order over the integers or the reals, the implication problem for clausal constraint-generating k-dependencies is:*

- *in PTIME for dependencies with one atomic constraint (no constraints in the antecedent) [Ull89, page 892],*
- *co-NP-complete for dependencies with two or more atomic constraints,*

under the assumption that no domain constants appear in the dependencies.

It is interesting to note that in the second case equalities and inequalities suffice to obtain the co-NP lower bound. Notice that the corresponding propositional problem, 3SAT, requires *three* literals per clause. Also, for finite domains of size greater than 2 the implication problem is co-NP-complete even for dependencies with one atomic constraint.

We consider now *linear arithmetic constraints*, i.e., atomic constraints of the form

$$a_1x_1 + \cdots + a_kx_k \leq a$$

(domain constants are allowed here). We can use here the results about the complexity of linear programming [Sch86].

Theorem 4.2 *For linear arithmetic constraints the implication problem for clausal constraint-generating k-dependencies with one atomic constraint per dependency is:*

- *in PTIME for the reals,*
- *co-NP-complete for the integers.*

To obtain more tractable classes, we propose to restrict further the syntax of dependencies by typing. A clausal dependency is *typed* if each atomic constraint involves only the values of one given attribute in different tuples. The second dependency in Example 2.1 is typed, while the first one and the one in Example 2.2 are not.

We have then the following.

Theorem 4.3
The implication problem for typed clausal constraint-generating 2-dependencies is:

- *in PTIME for dependencies with at most two atomic constraints in the theory of equality over the integers or the reals,*
- *in PTIME for dependencies with at most two atomic constraints in the theory of order over the integers or the reals,*
- *co-NP-complete for dependencies with two or more atomic constraints in the theory of equality and order over the integers or the reals,*

under the assumption that no domain constants appear in the dependencies.

Note that the first result is different from the well-known result about linear-time implication for functional dependencies. Functional dependencies viewed as constraint-generating dependencies allow only equality constraints which are not closed under negation. Moreover, constraint-generating dependencies with two constraints in the body correspond to *unary* functional dependencies.

5 Conclusions and Related Work

A brief summary of this paper is that constraint-generating dependencies are an interesting concept, and that deciding implication of such dependencies is basically no harder than deciding the underlying constraint theory, which, a priori, was not obvious. We have only given a sample of complexity results for common constraint theories. It is clear that this is far from exhaustive and that, depending on the application, other constraint languages might also be relevant, for instance the *congruence constraints* that appear in [JS92].

Other forms of constraint dependencies can also be of interest. An obvious candidate is the concept of *tuple-generating* constraint dependency. Unfortunately, the implication problem for these dependencies is harder to decide and more closely linked to the underlying theory. Indeed, tuple-generating constraint dependencies can, for example, specify a dense domain. The obvious applications of constraint-generating dependencies are constraint database design theory and consistency checking.

As far as related work, we should first mention that Jensen and Snodgrass [JS92] induced us to think about constraint dependencies. We should note that the integrity constraints postulated there involve both typed and untyped constraint-generating dependencies, as well as tuple-generating ones.

Also, two recent papers on *implication constraints* by Ishakbeyoğlu, Ozsoyoğlu and Zhang [IO93, ZO93] present work fairly close to ours. However, there are several important differences. Foremost, they consider a fixed language of constraint formulas, namely equality ($=$), inequality ($\neq$), and order ($<, \leq$) constraints, while our results are applicable to any decidable constraint theory thanks to our general reduction strategy. Second, their complexity results are obtained in a slightly different model. They consider both the number of database literals and the arity of relations in a dependency as parts of the input, while we consider only the latter. We think that our model is more intuitive because it is difficult to come up with a meaningful dependency that references more than a few tuples in a relation. Our intractability results are stronger than theirs, while our positive characterizations of polynomial time decidable problems do not necessarily carry over to their framework. Also, in [IO93, ZO93], the tractable classes of dependencies are not defined syntactically but rather by the presence or absence of certain types of refutations.

Order dependencies, proposed by Ginsburg and Hull [GH83, GH86], are typed clausal 2-dependencies over the theory of equality and order (without $\neq$). The order is not required to be total. Ginsburg and Hull provided an axiomatization of such dependencies and proved that the implication problem is co-NP-complete for dependencies with at least three constraints. This does not subsume any of our results. They also provided a number of tractable dependency classes which are, again, different from ours.

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Constraint Objects

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1 Introduction

Object-oriented database (OODB) systems will, most probably, have a significant role to play in the next generation of commercial database systems. While OODB systems have a sophisticated collection of features for data modeling, current-day OODB systems provide little or no support for representing and manipulating partially specified information. Very often, however, the knowledge that we would like to represent in a database is incomplete.

For example, assume that an OODB is used to represent knowledge about plays and playwrights. If Shakespeare's year of birth were known to be 1564, this could be represented easily in the database. However, historians do not have complete information about playwrights such as Shakespeare; they have only estimates of his date of birth and when he wrote his various plays; occasionally these estimates are *refined* reflecting the results of new research. Partial information about Shakespeare's year of birth can be represented naturally as a conjunction of constraints, Shakespeare.Year_of_birth $\geq$ 1560 $\wedge$ Shakespeare.Year_of_birth $\leq$ 1570. As another example, an image from a weather satellite may allow a meteorologist to estimate the location of the eye of a hurricane only to within a small region, rather than know it precisely. Both these examples illustrate the use of *existential* constraints in the database; Shakespeare was born in *some* specific year within the range described, and the eye of the hurricane is at *some* specific location within the region.

Constraints can be used also to represent compactly (possibly infinite) sets of fully specified values. For example, electronic components typically have certain tolerances for voltage and frequency inputs, i.e., these components would work properly for all voltage and frequency inputs within the specified tolerances. A natural representation of such a set of acceptable voltage inputs is a conjunction of *universal* constraints, CDPlayer.Voltage $\geq$ 108 $\wedge$ CDPlayer.Voltage $\leq$ 117.

One of the contributions of this paper is to identify these two distinct uses of constraints in data models: to represent partially specified values and to compactly represent sets of fully specified values. The former use is related to the notion of store-as-constraint (see [Sar89], for instance), whereas the latter use is similar to the notion of a constraint fact as a finite presentation (see [BNW91, KKR90, KG94, Ram91, Rev93], for instance).

The technical contributions of this paper are as follows:

- We describe how an object-based data model can be enhanced with (existential) constraints to represent naturally partially specified information (Section 2). We refer to this as the Constraint Object Data Model (CODM).

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- We present a declarative, rule-based, language that can be used to reason with information represented in the CODM. We refer to this as the Constraint Object Query Language (COQL) (Section 3). COQL has a model-theoretic and an equivalent fixpoint semantics, based on the notions of constraint entailment and "truth in all possible worlds".

  One of the novel features of COQL is the notion of *monotonic refinement* of partial information in object-based databases.

- We present a novel polynomial-time algorithm for quantifier elimination for a restricted class of set constraints that uses $\in$ and $\subseteq$. We refer to this class as *set-order* constraints. The quantifier elimination algorithm can also be used to check satisfiability and entailment of conjunctions of set-order constraints in polynomial time.

Both the constraint object data model and the constraint object query language are easily extended to compactly represent sets of fully specified values using universal constraints, and manipulate such values using a declarative, rule-based language, following the approach of [KKR90, Ram91]. For reasons of space, we do not pursue this further in the paper.

Integration of constraints with objects has also been considered by Freeman-Benson and Borning ([FBB92a, FBB92b]). Their work differs from ours in that their languages, Kaleidoscope'90 and Kaleidoscope'91, are imperative languages using the Von Neumann memory model. We are interested in the incorporation of constraints into objects in a more declarative setting.

This paper is based on work in progress, and the various ideas are motivated primarily through examples.

2 Constraint Object Data Model

First, we need to understand the notions of a fact and an object. A *fact* is a tuple of typed attribute/value pairs; this is well-accepted in the literature. Unfortunately, there appears to be little consensus in the literature on answering the question "What is an object?". Hence, we (deliberately) use a simple and very general notion of an object, which is consistent with many of the object-based data models in the literature.

We treat an *object* as consisting of an object identifier (oid) and a tuple of typed attribute/value pairs. Thus, an object differs from a fact only in that it has an object identifier. An object identifier uniquely specifies an object, i.e., no two objects can have the same object identifier. Hence, an object identifier can be used to distinguish an object from other objects in the database, and can serve as a handle for updating the attribute values without changing the identity of the object. We do not make any assumptions about the domains of the attributes of objects (or of facts); these could be primitive types, tuple types, set types, user-defined classes, etc. Our view of an object thus far is fairly standard; for instance, it is consistent with the view of [AK89].

The Constraint Object Data Model (CODM) incorporates both facts and objects, but relaxes the restriction that the "value" of an attribute must be a constant of the appropriate type; domain-specific constraints can be used to represent *partial information* about the value of an attribute. Such attributes are referred to as *E-attributes*. (Attributes whose values are completely known can be modeled also as E-attributes with sufficiently tight constraints.) In the examples discussed in the paper, the domain of an E-attribute is either the integers or sets of objects.

Conceptually, all the constraints on E-attributes of facts and objects are maintained globally. This allows for specification of inter-object constraints, which are very useful in many situations. However, in many of the examples discussed in the paper it suffices to associate constraints with the objects whose E-attributes they constrain; when possible, we depict the constraints in this fashion for ease of understanding.

As is common in the literature, we assume that facts in the database are grouped together into relations, and objects in the database are grouped together into classes. Classes can be organized into an inheritance hierarchy; however, this is orthogonal to our discussion, and we do not deal with inheritance in this paper.

We motivate the modeling power of our constraint object data model using an example.

Example 2.1 (Playwrights and Plays)
There are two classes of objects in the database: *playwrights* and *plays*. Partial information is represented about the year of composition of the plays, the writers of the plays, and the year of birth of the playwrights.

playwrights			
Oid	Name	Year_of_birth	Constraints
oid1	Shakespeare	$Y1$	$Y1 \leq 1570 \wedge Y1 \geq 1560$
oid2	Fletcher	$Y2$	
oid3	Kalidasa	$Y3$	$Y3 \leq 1000$

The constraints associated with each object are *existential* constraints in that the value of the E-attribute is some unique value from the domain satisfying these constraints. Note that there is no information on Fletcher's year of birth, which is equivalent to stating that Fletcher could have been born in any year.

plays				
Oid	Name	Writers	Year_of_composition	Constraints
oid10	Othello	{ oid1 }	$Y10$	$(Y10 \leq 1605 \wedge Y10 \geq 1601) \vee (Y10 \leq 1598 \wedge Y10 \geq 1595)$
oid11	Macbeth	{ oid1 }	$Y11$	$Y11 \leq 1608 \wedge Y11 \geq 1604$
oid12	Henry VIII	S1	$Y12$	$Y12 \leq 1613 \wedge Y12 \geq 1608 \wedge$ oid2 $\in S1 \wedge S1 \subseteq$ { oid1, oid2 }
oid13	Meghdoot	{ oid3 }	$Y13$	$Y13 \leq 1050$

The form of the constraints allowed depends on the types of the E-attributes. The Year_of_birth and the Year_of_composition E-attributes are of type integer, and hence they are constrained using arithmetic constraints over integers. For example, the constraint on the Year_of_composition attribute of oid10 indicates that Othello was composed either between the years 1601 and 1605 or between 1595 and 1598.

Similarly, the Writers E-attribute of *plays* is of type set of playwrights and it is constrained using set constraints $\supseteq, \subseteq, \in$. For example, the constraint on the Writers attribute of oid12 indicates that either Fletcher is the sole writer of Henry VIII, or Fletcher and Shakespeare are joint writers of that play. Note that this represents partial information on the set of playwrights. □

A key feature of the Constraint Object Data Model is that the constraints that the CODM allows are *first-order*, i.e., the names and types of the attributes are fixed for each fact and object. The CODM does not permit the names and/or the types of attributes to be partially specified using constraints; only the values of these attributes can be partially specified using constraints.

In general, constraints can be incorporated into any *existing* data model (e.g., relational, nested relational, object-oriented) and the resulting constraint data model can be used to represent partially specified information, or compactly represent sets of fully specified values. We do not discuss this point further in this paper.

3 Constraint Object Query Language

We present the declarative Constraint Object Query Language (COQL) that can be used to reason with facts and objects in the constraint object data model. A COQL program is a collection of rules similar to Horn rules, where each rule has a body and a head. The body of a rule is a conjunction of literals and constraints, and the head of the rule can be either a positive literal or a constraint. COQL allows arbitrary constraints, not just conjunctions of primitive constraints, to occur in the bodies and heads of program rules. However, we do not allow any constraints in rule bodies that can manipulate the "ranges" of possible values of E-attributes; this can result in a non-monotonic behavior of the rules, which makes the semantics hard to define.

3.1 COQL: Inferring New Relationships

A COQL program can be used to infer new relationships (as facts) between existing objects and facts. Object-creating proposals (e.g., [KKS92]) also allow new relationships to be created as objects. For simplicity, we assume that COQL rules do not create new objects; this condition can be checked syntactically by having a safety requirement, that any object identifier appearing in the head of a COQL rule also appears in a body literal of that rule. Our results are orthogonal to object-creating proposals, and can be combined with them in a clean fashion.

We now present some simple queries to motivate the inferring of new relationships using COQL rules.

Example 3.1 (Selection)
Consider the database of plays and playwrights from Example 2.1. Suppose we want to know the names of all playwrights born before the year 1700. The following rule seems to express this intuition, using the dot notation for accessing object attributes:

q1 (P.Name) : − playwrights (P), P.Year_of_birth < 1700.

If the years of birth of all the playwrights in the database are completely specified, answering this query is straightforward. In the presence of partial information about the years of birth of the playwrights, there are two possible semantics that can be used to answer this query.

- Truth in *at least one possible world.*

  Under this semantics, a playwright "satisfies" the query if *at least one* assignment of fully specified values to the Year_of_birth attribute of the playwright, consistent with the object constraints, satisfies the query. All three playwrights, Shakespeare, Fletcher and Kalidasa would be retrieved as answers to the query under this semantics. Shakespeare could have been born in 1564, Fletcher in 1600 and Kalidasa in 975; these values are consistent with the constraints on the object attributes.

  To compute this answer set to the query, we need to check *satisfiability* of the conjunction of constraints present in the object and the constraints present in the query. For example, the conjunction of constraints oid3.Year_of_birth $\leq$ 1000 $\wedge$ oid3.Year_of_birth $<$ 1700 (where oid3 is the identifier of the object representing Kalidasa) is satisfiable in the domain of integers.

- Truth in *all possible worlds.*

  Under this semantics, a playwright "satisfies" the query if *every* assignment of a fully specified value to the Year_of_birth attribute of the playwright, consistent with the object constraints, satisfies the query. Only Shakespeare and Kalidasa would be retrieved as answers to the query under this semantics. Fletcher could have been born in 1800; this value is consistent with the constraints on the object attributes, while being inconsistent with the query constraints.

  To compute this answer set to the query, we need to check that the constraints present in the objects *entail* (i.e., imply) the query constraints. For example, the conjunction of object constraints oid1.Year_of_birth $\leq$ 1570 $\wedge$ oid1.Year_of_birth $\geq$ 1560 entails the (instantiated) query constraint oid1.Year_of_birth $<$ 1700 (where oid1 is the identifier of the object representing Shakespeare) in the domain of integers. However, the object constraints associated with Fletcher do not entail the (instantiated) query constraint oid2.Year_of_birth $<$ 1700 (where oid2 is the identifier of the object representing Fletcher) in the domain of integers.

These alternative semantics are closely related to the semantics of Imielinski et al. [INV91] for OR-objects; we do not elaborate on these relationships in the paper for lack of space. □

Example 3.2 (Equijoin)
Suppose we want to know the names of all plays written in the same year as Macbeth. The following rule seems to express this intuition:

q2 (P.Name) : – plays (P), plays (P1), P1.Name = "Macbeth",
P.Year_of_composition = P1.Year_of_composition.

If the years of composition of all the plays in the database are completely specified, answering this query is straightforward. In the presence of partial information, there are again two ways in which this query can be answered.

Under the "truth in at least one possible world" semantics, the play Othello would be an answer (since both Othello and Macbeth could have been composed in 1605, for example), as would the play Henry VIII (since both could have been composed in 1608, for example). These answers can be obtained by checking for *satisfiability* of the conjunction of object constraints with the query constraints. Meghdoot would not be an answer, however, since the conjunction of constraints in this case is unsatisfiable.

Note, however, that each answer to the query under the "truth in at least one possible world" semantics may hold in a separate possible world. While Othello and Henry VIII are both answers to the query ? q2 (Name), there is *no* possible world in which both of them could have been composed in the same year. One possible way of overcoming this problem is to give, along with each answer to a query, a description of the possible worlds in which that answer would hold. We do not investigate this issue here.

Under the "truth in all possible worlds" semantics, only the play Macbeth itself would be retrieved. The conjunction of object constraints oid11.Year_of_composition $\leq$ 1608 $\wedge$ oid11.Year_of_composition $\geq$ 1604 entails the (instantiated) query constraint oid11.Year_of_composition = oid11.Year_of_composition, where oid11 is the identifier for Macbeth. For all other plays, there are possible worlds in which they could have been composed in years other than Macbeth's year of composition; the check for entailment would fail. □

Example 3.3 (Set Constraints)
Suppose we want to know the names of all the plays written by Shakespeare. The following rule expresses the query:

q3 (P.Name) : – plays (P), playwrights (W), W.Name = "Shakespeare", P.Writers = S, W $\in$ S.

Under the "truth in at least one possible world" semantics, the play Henry VIII would be an answer (since Shakespeare could have written it together with Fletcher) as would Othello and Macbeth (since Shakespeare is known to have written these). The first answer can be obtained by checking the *satisfiability* of the conjunction of the object constraints oid2 $\in$ oid12.Writers $\wedge$ oid12.Writers $\subseteq$ { oid1, oid2 } with the (instantiated) query constraint oid1 $\in$ oid12.Writers.

Under the "truth in all possible worlds" semantics, however, Henry VIII would not be an answer. This is because the object constraints oid2 $\in$ oid12.Writers $\wedge$ oid12.Writers $\subseteq$ { oid1, oid2 } do not entail the (instantiated) query constraint oid1 $\in$ oid12.Writers. Othello and Macbeth would be the only answers in this case. □

3.2 COQL: Monotonically Refining Objects

COQL programs can be used also to *monotonically refine* objects, in response to additional information available about knowledge that we are trying to represent in the database. For example, suppose research determined that Shakespeare could have been born no later than 1565, then the object Shakespeare can be refined by conjoining the constraint Shakespeare.Year_of_birth $\leq$ 1565.

The notion of *declarative monotonic refinement* of partially specified objects is one of the novel contributions of this paper. Object refinement can be formalized in terms of a lattice structure describing the possible states of an object, with a given information theoretic ordering. The value $\perp$ corresponds to having no information about the attribute values of the object, and $\top$ corresponds to having inconsistent information about the object. Object refinement now can be thought of as moving up this information lattice.

Object refinement can be specified declaratively (under the "truth in all possible worlds" semantics, as discussed below), since the final state of the object does not depend on the specific order in which the various

refinements are performed. For example, suppose the "value" of the attribute Year_of_birth of Shakespeare is Shakespeare.Year_of_birth $\leq$ 1570 $\wedge$ Shakespeare.Year_of_birth $\geq$ 1560. Then, the final "value " of the attribute Year_of_birth of Shakespeare is independent of the order in which the refinements Shakespeare.Year_of_birth $\leq$ 1565 and Shakespeare.Year_of_birth $\geq$ 1562 are conjoined.

Refining partially specified facts in this fashion poses problems because facts do not have a notion of an identity, independent of the attribute values.

We give an example of declarative, rule-based, object attribute refinement next. The body of a refinement rule is similar to the body of a rule used to infer a new relationship, as described previously. The head of a refinement rule, on the other hand, is a constraint (not necessarily a conjunction of primitive constraints).

Example 3.4 (Refining Attributes of Objects)
The following refinement rule seems to express the intuition that a playwright could not write a play before birth:

W.Year_of_birth $\leq$ P.Year_of_composition : – playwrights (W), plays (P), W $\in$ P.Writers.

The right hand side (body) of the rule is the condition, and the left hand side (head) is the action of the rule. If the body is satisfied, then the instantiated head constraint is conjoined to the global constraints. (This is an example where the instantiated head constraint is an inter-object constraint, and hence cannot be associated solely with a single object.)

If the year of composition of Henry VIII were known to be 1612, then we could conjoin the constraint Fletcher.Year_of_birth $\leq$ 1612 to the global collection of constraints on E-attributes.

In the presence of partial information, we give a meaning to refinement rules based on the "truth in all possible worlds" semantics. In this case, we would conjoin the constraint Fletcher.Year_of_birth $\leq$ Henry VIII.Year_of_composition to the global collection of constraints. Conflicting refinements could, of course, result in an inconsistent constraint set. □

Rules that refine objects can be combined cleanly with rules that infer relationships between existing objects in COQL programs. For example, the rule in Example 3.4 can be combined with the rule in Example 3.1. In the resulting program, Fletcher also would be an answer to the query $q1$ under the "truth in all possible worlds" semantics.

Rules that refine objects can be used to create new objects as well, using any of the object-creating proposals. The advantage of our approach is that an object can be created multiple times, possibly with different values of the E-attributes; the result is to conjoin each of the constraints on the E-attributes. (If an object is created multiple times, with different values for a fully-specified attribute, the resultant set of constraints is inconsistent, as is natural.) This technique avoids the problem faced by many object-creating proposals (e.g., [KKS92]), of ensuring that the "same" object is not created multiple times.

If we adopted the "truth in at least one possible world" semantics for object refinement, object refinement becomes order dependent, and the program cannot be assigned a unique meaning. The following example illustrates this problem:

Example 3.5 (Order Dependence)
Consider a program with the following two refinement rules:[1]

W.Year_of_birth = 1560 : – playwrights (W), W.Year_of_birth $\leq$ 1565.
W.Year_of_birth = 1570 : – playwrights (W), W.Year_of_birth $\geq$ 1566.

In Example 2.1, Shakespeare's year of birth is known to be between 1560 and 1570. Under the "truth in at least one possible world" semantics, the order in which these two rules are applied could result in Shakespeare's

[1] Although the rules do not make intuitive sense, this example is purely for illustrating a point.

year of birth being refined to either 1560 or 1570. (Once one of the rules is applied, the other rule becomes inapplicable.) Under the truth in all possible worlds semantics, the object Shakespeare would not be refined.

Note that if Shakespeare's year of birth were initially specified as 1564, then the result of applying these refinement rules would make the object have inconsistent constraints, under the truth in all possible worlds semantics. This, however, would not be order dependent. □

4 Set-Order Constraints

In the examples discussed in the paper, we used order constraints (i.e., arithmetic constraints involving $<, \leq, =, \geq$ and $>$, but no arithmetic functions such as $+, -$ or $*$) and set constraints of a restricted form (i.e., those involving $\in, \subseteq$ and $\supseteq$, but not involving functions such as $\cup$ and $\cap$). Techniques for quantifier elimination, checking satisfaction and entailment for order constraints over various domains are known in the literature (see [Ull89], for instance).

We now briefly describe a polynomial-time quantifier elimination algorithm for a conjunction of a restricted form of set constraints, that we call *set-order* constraints. Satisfaction and entailment of conjunctions of set-order constraints can be solved (in polynomial-time) using the quantifier elimination algorithm.

4.1 Quantifier Elimination for Set-Order Constraints

We will use the symbols $\hat{X}, \hat{Y}, \hat{Z}$ to denote set variables that range over finite sets of elements of type D. A *set-order* constraint is of one of the following types:

$$c \in \hat{X}, \hat{X} \subseteq s, s \subseteq \hat{X}, \hat{X} \subseteq \hat{Y}$$

where c is a constant of type D, and s is a *set of constants* of type D.

Quantifier Elimination
Input: A conjunction Q of set-order constraints and a set variable $\hat{Y}$ to be eliminated.
Output: A conjunction Q' of set-order constraints, such that $\exists \hat{Y} Q$ and Q' are equivalent.
Algorithm: Do the following steps in order:

1. First rewrite every constraint of the form $c \in \hat{X}$ into $\{c\} \subseteq \hat{X}$.
2. For each set variable $\hat{X}$, take the union of all sets s, such that $s \subseteq \hat{X}$ is in the conjunction. Let the union be the set L_X. Delete all constraints of the form $s \subseteq \hat{X}$ from the conjunction, and add the constraint $L_X \subseteq \hat{X}$ to the conjunction.
3. For each set variable $\hat{X}$ take the intersection of all sets s, such that $\hat{X} \subseteq s$ is in the conjunction. Let the intersection be the set U_X. Delete all constraints of the form $\hat{X} \subseteq s$ from the conjunction and add the constraint $\hat{X} \subseteq U_X$.
4. For each pair of constraints of the form $N \subseteq \hat{Y}$ and $\hat{Y} \subseteq M$, where $\hat{Y}$ is the set variable to be eliminated, and N and M are either set variables or sets of constants, add the constraint $N \subseteq M$. After this is done for each such pair, delete all constraints in which $\hat{Y}$ occurs. Repeat steps 2 and 3.
5. Check each constraint of the form $s_1 \subseteq s_2$ where s_1 and s_2 are sets of constants from domain D. If they are all satisfied, delete all such constraints from the conjunction and return the conjunction of the remaining constraints. If any one of these constraints is not satisfied, then return FALSE.

Example 4.1 (Quantifier elimination)
Let Q be the following conjunction of set-order constraints:

$$3 \in \hat{Z}, \hat{Z} \subseteq \hat{X}, \hat{X} \subseteq \{3,4,8,9\}, \hat{X} \subseteq \hat{Y}, \hat{Y} \subseteq \{2,3,5,7,8\}.$$

From constraint Q we can eliminate set variable $\hat{Y}$ as follows:

Step 1 : Replace $3 \in \hat{Z}$ by $\{3\} \subseteq \hat{Z}$.

Step 2 : No change.

Step 3 : No change.

Step 4 : We get $\{3\} \subseteq \hat{Z}, \hat{Z} \subseteq \hat{X}, \hat{X} \subseteq \{3,4,8,9\}, \hat{X} \subseteq \{2,3,5,7,8\}$.

Step 2 : No change.

Step 3 : We get $\{3\} \subseteq \hat{Z}, \hat{Z} \subseteq \hat{X}, \hat{X} \subseteq \{3,8\}$.

Step 5 : No change. Hence, we return $\{3\} \subseteq \hat{Z}, \hat{Z} \subseteq \hat{X}, \hat{X} \subseteq \{3,8\}$.

Suppose now, that we also want to eliminate set variable $\hat{Z}$. This will be done by quantifier elimination algorithm as follows:

Steps 1–3 : No change.

Step 4 : We get $\{3\} \subseteq \hat{X}, \hat{X} \subseteq \{3,8\}$.

Step 5 : No change. Hence, we return $\{3\} \subseteq \hat{X}, \hat{X} \subseteq \{3,8\}$.

□

Theorem 4.1 *Let Q be a conjunction of set-order constraints and $\hat{Y}$ be a set variable. The quantifier elimination algorithm on input Q and $\hat{Y}$ will yield in PTIME, in the size of Q, a conjunction of set-order constraints Q' such that $\exists \hat{Y} Q$ and Q' are equivalent.*

Further, if Q' has n set variables, then the number of conjuncts in Q' is at most $n^2 + 2 * n$. □

The quantifier elimination algorithm can be used also to check for satisfiability of a conjunction of set-order constraints, by successively eliminating set variables until either there are no more set variables remaining (in which case the original conjunction is satisfiable) or the quantifier elimination algorithm returns FALSE (in which case the original conjunction is unsatisfiable). The bound on the maximum number of conjuncts after eliminating a set variable guarantees a polynomial-time algorithm for checking satisfiability.

Theorem 4.2 *Let Q be a conjunction of set-order constraints. Checking whether Q is satisfiable is in PTIME, in the size of Q.* □

The algorithm for checking satisfiability cannot be used for checking for entailment of conjunctions of set-order constraints (using the reduction from a check for entailment to a polynomial number of checks for satisfaction), since set-order constraints are not closed under negation (For example, $\hat{X} \not\subseteq \hat{Y}$ is not a set-order constraint.)

However, the quantifier elimination algorithm can be used directly as a basis for checking entailment of conjunctions of set-order constraints in PTIME, as follows.

Checking the entailment of a conjunction of set-order constraints Q_2 by a conjunction of set-order constraints Q_1 can be done by reduction to a number of entailment checks of each set-order constraint in Q_2 by the conjunction Q_1.

The following result shows how the quantifier elimination algorithm can be used to simplify checking the entailment of a set-order constraint by an arbitrarily large conjunction of set-order constraints.

Theorem 4.3 *Let Q be a conjunction of set-order constraints over the set variables $\hat{X}_1, \ldots, \hat{X}_m$. Let Q_1 be the result of elimination of variables $\hat{X}_3, \ldots, \hat{X}_m$ from Q. Let Q_2 be the result of elimination of variable $\hat{X}_2$ from Q_1. Then, (1) Q entails $\hat{X}_1 \subseteq \hat{X}_2$ if and only if Q_1 entails $\hat{X}_1 \subseteq \hat{X}_2$, (2) Q entails $\hat{X}_1 \subseteq s$ if and only if Q_2 entails $\hat{X}_1 \subseteq s$, and (3) Q entails $s \subseteq \hat{X}_1$ if and only if Q_2 entails $s \subseteq \hat{X}_1$.* □

The following two results show how to check whether a set-order constraint is entailed by a simple form of a conjunction of set-order constraints.

Theorem 4.4 *Let Q be a conjunction of set-order constraints over $\hat{X}$. Let U_X be the upper bound (possibly the set of all elements in domain D) on $\hat{X}$ and L_X be the lower bound (possibly the empty set) on $\hat{X}$.*

Then Q entails $\hat{X} \subseteq s$ if and only if $U_X \subseteq s$. Also, Q entails $s \subseteq \hat{X}$ if and only if $s \subseteq L_X$. □

Theorem 4.5 *Let Q be a conjunction of set-order constraints over $\hat{X}_1, \hat{X}_2$. Let U_{X1} be the upper bound (if any) on $\hat{X}_1$ and L_{X2} be the lower bound (if any) on $\hat{X}_2$. Then Q entails $\hat{X}_1 \subseteq \hat{X}_2$ if and only if (1) Q is unsatisfiable, or (2) $\hat{X}_1 \subseteq \hat{X}_2$ is in Q, or (3) $U_{X1} \subseteq L_{X2}$.* □

5 COQL: Model Theory and Fixpoint Semantics

COQL has a model-theoretic and an equivalent fixpoint semantics, based on the notions of constraint entailment and "truth in all possible worlds". The semantics of COQL is based on the notion of "truth in all possible worlds' for several reasons:

- Object refinement is order independent; this is a very desirable property.
- An answer to a query is unconditionally true.
- An answer to a query continues to be true, even after the database objects are monotonically refined.

We briefly describe the model-theoretic and fixpoint semantics here; details and the equivalence proof are omitted for reasons of space. Consider a COQL program P, and a collection of facts and objects I. We assume that all the variables in each rule body of P have been standardized apart, possibly by introducing equality constraints between some of the variables; this is important in checking for entailment.

5.1 Model-theoretic Semantics

An assignment of facts and objects to the body literals of a rule r of program P makes the body of r *true* if the constraints associated with the facts and the objects *entail* the (instantiated) constraints between the variables present in the body of rule r. A relationship inferring rule r is true in I if, for every assignment of facts and objects to the body literals of r that makes the body true, the instantiated head fact of rule r is *entailed* by (the constraints associated with) some fact f in I. An object refinement rule r is true in I if, for every assignment of facts and objects to the body literals of r that makes the body true, the instantiated head object o_r occurs in I, and the instantiated head constraint of the rule is *entailed* by the object constraints associated with o_r. The collection I of facts and objects is said to be a *model* of a COQL program if each program rule is true in I.

The model-theoretic semantics of COQL is a *least model* semantics, where model $M_1 \preceq$ model M_2, if for each fact (or object) f_1 in M_1, there is a fact (or object) f_2 in M_2, such that f_1 entails f_2. The existence of a least model is guaranteed since we can show that the "intersection" of COQL program models is also a COQL program model.

5.2 Fixpoint semantics

The fixpoint semantics is defined in terms of an immediate consequence operator, T_P. Given the collection I of facts and objects, we define $T_P(I)$ as follows. Let r be a rule. If there is an assignment of facts and objects from I to literals in the body of r such that the body is true, then the instantiated head fact (or head object) is in $T_P(I)$.

The fixpoint semantics of COQL is based on the least fixpoint of the T_P operator, which can be computed starting from the empty collection of facts and objects, as $T_P(\emptyset) \cup T_P(T_P(\emptyset)) \cup \ldots$. In computing the unions, all the constraints associated with an object o have to be conjoined together. The existence of the least fixpoint is guaranteed by the monotonicity of the T_P operator.

Theorem 5.1 *Consider a COQL program P. It has a least model semantics and a least fixpoint semantics, which coincide.* □

Note that in the absence of objects with object identifiers, the semantics of COQL is very similar to the standard semantics [vEK76], except that constraint entailment is used instead of constraint satisfaction; this is required by the notion of "truth in all possible worlds".

The techniques of [KKR90] can be used to show that if a COQL program and facts/objects use only arithmetic order constraints, the answer to a query can be computed in PTIME data complexity. The following result shows that a similar complexity is achieved for a restricted case of COQL programs with set-order constraints.

Theorem 5.2 *Consider a COQL program P with only refinement rules using set-order constraints, and a collection of objects I. Let the E-attributes of the objects in I be constrained using only set-order constraints. Computing the answer to a query can be done in PTIME in the size of I.* □

Our notion of partially specified objects is related to Saraswat's notion of store-as-constraint. The store can be viewed as a single object with a given collection of E-attributes. Consider this case; programs written in COQL and Saraswat's cc($\downarrow$, $\rightarrow$) refine the object/store, based on constraint entailment. However, the resulting semantics are quite different. Saraswat's semantics is an operational, indeterministic semantics, based on satisfying each goal in at most one possible way, whereas our semantics is a fixpoint semantics. We conjecture that a suitable combination of Magic Templates rewriting [Ram91] (which rewrites a program such that both goals and answers are computed in the fixpoint evaluation) and indeterminacy can be used to simulate the semantics of cc($\downarrow$, $\rightarrow$) using the semantics of COQL.

6 Conclusions and Future Work

We presented the Constraint Object Data Model, and the Constraint Object Query Language, which we believe go a long way in incorporating the ability to represent and manipulate partially specified information in object-based database systems.

There are many interesting directions to pursue. Determining classes of programs with tractable data complexity is extremely important. Optimizing COQL queries is another important direction of research. Stuckey and Sudarshan [SS94] present compilation techniques for query constraints in logic programs, essentially extending Magic sets to handle general query constraints, not just equality constraints on queries. It would be interesting to see how these techniques apply to COQL programs. Finally, many of our ideas and techniques seem applicable to temporal database languages. Exploring the interconnections is likely to be an interesting direction of research.

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