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# SYNERGISM OF MECHANICS, MATHEMATICS AND ANISOTROPIC ELASTIC MATERIALS

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## INTRODUCTION

Anisotropic elastic materials are interesting materials. A simple tensile stress applied to the material produces not only an extensional strain but also a shear strain. Likewise, a pure shear stress applied to the material produces a shear strain and an extensional (or compression) strain. Therefore a loading which is symmetric (or antisymmetric) with the $x_2$–axis, say, in general does not produce a deformation which is symmetric (or antisymmetric) with the $x_2$–axis. There are surprises in which anisotropic materials behave like isotropic materials. These will be pointed out in the paper.

In contrast to isotropic elastic materials which have two elastic constants, anisotropic elastic materials may have as many as twenty one elastic constants. When two–dimensional deformations are considered, the analysis still requires fifteen elastic constants. In view of this, there is a wide spread and justifiable misconception that the analysis of anisotropic elastic materials is much more complicated than that of isotropic elastic materials. This is not necessarily true if one employs the Stroh formalism. With the Stroh formalism the solutions to anisotropic elasticity problems are in most cases simpler than those for isotropic elasticity problems. The reason is simply that isotropic materials are more than a special case of anisotropic materials. They are mathematically degenerate materials.

Much progress has been made since Stroh's two pioneering papers appeared in 1958 and 1962 [1,2]. We will point out in the paper the integral formalism of Barnett–Lothe [3] which allows us to compute three Barnett–Lothe tensors S, H and L, which are real, without finding the Stroh eigenvalues p and the associated eigenvectors a, b, which are complex. We will also point out some identities which enable us to convert certain combinations of p, a and b to S, H, L and other real quantities. Owing to these identities, several existing complex form solutions are simplified to real form solutions and solutions are obtained for some heretofore unsolved problems. As a result, many physically interesting and unexpected phenomena, which have been shrouded in the complex form solutions, have been discovered recently. Most of the unexpected phenomena defy an intuitive explanation.

The Stroh formalism is not only mathematically elegant and technically powerful, but some of its mathematical quantities such as the eigenvalues p and the eigenvectors **a** and **b** have physical meanings. The mathematical structure of **S**, **H** and **L** provides us a rare insight into the relations between anisotropic and isotropic materials.

For isotopic materials the in–plane displacement and the antiplane displacement are uncoupled. The in–plane displacement ($u_1$, $u_2$) and the associated surface traction vector on any boundary $\Gamma$ are polarized on the ($x_1$, $x_2$) plane while the antiplane displacement $u_3$ and the associated surface traction are polarized along the $x_3$ axis. For general anisotropic materials under the assumption of two–dimensional deformations, the $u_3$ component is in general non–zero and cannot be uncoupled from the in–plane displacements $u_1$, $u_2$. This does not mean that there are no planes or axes on which the displacement and the surface traction are polarized. There are, as we will show, oblique planes and axes on which the displacement and the surface traction are polarized.

The synergism of mathematics and mechanics appears to work very well for anisotropic elastic materials. Examples presented in the paper illustrate that this is indeed the case.

**1. THE STROH FORMALISM.** In a fixed rectangular coordinate system $x_i$ ($i = 1, 2, 3$) let $u_i$, $\sigma_{ij}$ be, respectively, the displacement and stress in an anisotropic elastic material. The stress strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s} , \tag{1.1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{1.2}$$

where a comma stands for differentiation, repeated indices imply summation and $C_{ijks}$ are the elasticity constants which are assumed to possess the symmetry property

$$C_{ijks} = C_{jiks} = C_{ijsk} = C_{ksij} .$$

For two–dimensional deformations in which $u_i$ ($i = 1, 2, 3$) depends on $x_1$, $x_2$ only, a general solution to (1.2) is, in matrix notation,

$$\mathbf{u} = \mathbf{a}\, f(z), \quad z = x_1 + p x_2 . \tag{1.3}$$

In the above f is an arbitrary function of z, and p and **a** are determined by inserting (1.3) into (1.2). We have

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}\}\,\mathbf{a} = 0 \tag{1.4}$$

where the superscript T denotes the transpose and **Q**, **R**, **T** are 3x3 real matrices whose components are

$$Q_{ik} = C_{i1k1},\ R_{ik} = C_{i1k2},\ T_{ik} = C_{i2k2}. \tag{1.5}$$

We note that **Q** and **T** are symmetric and, subject to the positiveness of strain energy, positive definite. The stresses obtained by substituting (1.3) into (1.1) can be written in terms of the stress function $\phi$ as

$$\sigma_{i1} = -\phi_{i,2},\quad \sigma_{i2} = \phi_{i,1}, \tag{1.6}$$

in which

$$\phi = \mathbf{b}\, f(z), \tag{1.7}$$

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \tag{1.8}$$

The second equality in (1.8) follows from (1.4). It suffices therefore to consider the stress function $\phi$ because the stresses $\sigma_{ij}$ can be obtained by differentiation.

There are six eigenvalues p and six eigenvectors **a** from (1.4). Since p cannot be real if the strain energy is positive [4], there are three pairs of complex conjugates for p. If $p_\alpha$, $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ ($\alpha = 1, 2,..,6$) are the eigenvalues and the associated eigenvectors we let

$$\operatorname{Im} p_\alpha > 0,\ p_{\alpha+3} = \bar{p}_\alpha,\ \mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha,\ \mathbf{b}_{\alpha+3} = \bar{\mathbf{b}}_\alpha, \tag{1.9}$$

($\alpha = 1, 2, 3$), where Im stands for the imaginary part, the overbar denotes the complex conjugate and $\mathbf{b}_\alpha$ is related to $\mathbf{a}_\alpha$ through (1.8). Assuming that the $p_\alpha$ are distinct, the general solutions for **u** and $\phi$ obtained by superposing six solutions of the form (1.3) and (1.7) are

$$\begin{aligned} \mathbf{u} &= \sum_{\alpha=1}^{3}\left\{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\right\}, \\ \phi &= \sum_{\alpha=1}^{3}\left\{\mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\right\}. \end{aligned} \tag{1.10}$$

In (1.10) $f_1, f_2,..., f_6$ are arbitrary functions of their argument and

$$z_\alpha = x_1 + p_\alpha x_2.$$

The above formalism is due to Stroh [1,2]. In applications all we have to determine is the form of the arbitrary functions $f_\alpha$. What distinguishes the Stroh formalism from others is that there are relations between $a_\alpha$ and $b_\alpha$ which allow us to find the solution easily and/or to simplify the solution obtained. These relations and the Barnett–Lothe integral formalism are presented next.

In closing this section we note that, in most applications, $f_\alpha$ has the same function form so that we may write

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha),$$

$$f_{\alpha+3}(z_\alpha) = \bar{q}_\alpha \bar{f}(\bar{z}_\alpha), \quad \alpha = 1, 2, 3,$$

where $q_\alpha$ are arbitrary constants. The second equation is for obtaining real solutions for $u$ and $\phi$. Equations (1.10) can then be written as

$$u = 2\,\mathrm{Re}\sum_{\alpha=1}^{3} a_\alpha q_\alpha f(z_\alpha), \quad \phi = 2\,\mathrm{Re}\sum_{\alpha=1}^{3} b_\alpha q_\alpha f(z_\alpha). \tag{1.11}$$

**2. THE BARNETT–LOTHE TENSORS.** The two equations in (1.8) can be rewritten as

$$\begin{bmatrix} -R^T & I \\ -Q & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = p \begin{bmatrix} T & 0 \\ R & I \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where I is the 3x3 identity matrix. Multiplying both sides by the matrix

$$\begin{bmatrix} T^{-1} & 0 \\ -RT^{-1} & I \end{bmatrix}$$

leads to the standard eigenrelation [5,6]

$$N\xi = p\xi, \tag{2.1}$$

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \tag{2.2}$$

$$N_1 = -T^{-1}R^T, \quad N_2 = T^{-1}, \quad N_3 = RT^{-1}R^T - Q. \tag{2.3}$$

It is clear that $N_2$ and $N_3$ are symmetric and $N_2$ is positive definite. It can be shown that $-N_3$ is positive semi–definite [7]. Moreover, $-N_1$ and $-N_3$ have the structure

$$-N_1 = \begin{bmatrix} * & 1 & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \quad -N_3 = \begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix}, \tag{2.4}$$

in which the * denotes a possibly non–zero element. These * elements have surprisingly simple expressions in terms of elastic compliances [7]. The structure of $N_1$, $N_3$ shown in (2.4) plays important roles in solving problems and interpreting the final solutions.

The vector $\xi = (\mathbf{a}, \mathbf{b})$ in (2.2) is the right eigenvector of N. It can be shown that $(\mathbf{b}, \mathbf{a})$ is the left eigenvector. The left and right eigenvectors associated with different eigenvalues are orthogonal to each other. The orthogonality relations can be normalized such that

$$\mathbf{a}_\alpha \cdot \mathbf{b}_\beta + \mathbf{b}_\alpha \cdot \mathbf{a}_\beta = \delta_{\alpha\beta} \tag{2.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Introducing the 3x3 matrices A and B by

$$A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{2.6}$$

and employing (1.9), the orthogonality relations (2.5) take the form

$$\begin{bmatrix} B^T & A^T \\ \overline{B}^T & \overline{A}^T \end{bmatrix} \begin{bmatrix} A & \overline{A} \\ B & \overline{B} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{2.7}$$

The two 6x6 matrices on the left are the inverses of each other and their product can be interchanged. The interchanged product is

$$\begin{aligned} AB^T + \overline{AB}^T &= I = BA^T + \overline{BA}^T, \\ AA^T + \overline{AA}^T &= 0 = BB^T + \overline{BB}^T. \end{aligned} \tag{2.8}$$

Equations (2.8) tell us that the real part of $AB^T$ is I/2 and that $AA^T$ and $BB^T$ are purely imaginary. Hence the three matrices S, H, L, defined by

$$S = i(2AB^T - I), \quad H = 2iAA^T, \quad L = -2iBB^T, \tag{2.9}$$

are real. It is clear that $\mathbf{H}$ and $\mathbf{L}$ are symmetric. It can be shown that they are positive definite, the products

$$\mathbf{SH},\quad \mathbf{LS},\quad \mathbf{H}^{-1}\mathbf{S},\quad \mathbf{SL}^{-1}$$

are antisymmetric, and the relation

$$\mathbf{HL} - \mathbf{SS} = \mathbf{I} \tag{2.10}$$

holds [3,6].

The formulation presented so far assumes that the eigenvalues p are distinct. If $p_1 = p_2$, say, and $\mathbf{a}_1 = \mathbf{a}_2$, the solution (1.10) is not general. The matrices **A** and **B** would be singular and the orthogonality relation (2.7) is not valid. Anisotropic materials for which $p_1 = p_2$ and $\mathbf{a}_1 = \mathbf{a}_2$ are called **degenerate** materials. They are degenerate in the mathematical sense, not necessarily in the physical sense. Isotropic materials are a special case of degenerate materials for which $p_1 = p_2 = p_3 = i$ and $\mathbf{a}_1 = \mathbf{a}_2 \neq \mathbf{a}_3$. In many applications however the final solution depends only on the three real matrices **S**, **H**, **L** defined in (2.9). Barnett and Lothe [3] devised an integral formalism of these three real matrices which circumvented the need of determining the eigenvalues and the eigenvectors. Thus the problem of degenerate materials disappears. The integral formalism is as follows. Define the three real matrices

$$Q_{ik}(\theta) = C_{ijks}n_jn_s,\quad R_{ik}(\theta) = C_{ijks}n_jm_s,\quad T_{ik}(\theta) = C_{ijks}m_jm_s, \tag{2.11}$$

in which $\theta$ is a real parameter and

$$n_i = [\cos\theta, \sin\theta, 0],\quad m_i = [-\sin\theta, \cos\theta, 0].$$

Equations (2.11) reduce to (1.5) when $\theta = 0$. Next consider the incomplete integrals

$$\mathbf{S}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}_1(\omega)d\omega,\quad \mathbf{H}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}_2(\omega)d\omega,$$

$$\mathbf{L}(\theta) = \frac{1}{\pi}\int_0^\theta -\mathbf{N}_3(\omega)d\omega, \tag{2.12}$$

where

$$\mathbf{N}_1(\theta) = -\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta),\quad \mathbf{N}_2(\theta) = \mathbf{T}^{-1}(\theta),$$

$$\mathbf{N}_3(\theta) = \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta) - \mathbf{Q}(\theta).$$

$N_i(\theta)$ reduce to $N_i$ in (2.3) when $\theta = 0$. When $\theta = \pi$ we have the complete integrals $S(\pi)$, $H(\pi)$, $L(\pi)$. Barnett and Lothe proved that S, H, L of (2.9) are identical to the complete integrals, i.e.,

$$S = S(\pi), \quad H = H(\pi), \quad L = L(\pi). \tag{2.13}$$

Thus S, H, L are called the Barnett–Lothe tensors and $S(\theta)$, $H(\theta)$, $L(\theta)$ the associated tensors. In the sequel, dependence of $S(\theta)$, $H(\theta)$, $L(\theta)$ on $\theta$ will be given explicitly unless $\theta = \pi$, and dependence of $N_i(\theta)$ on $\theta$ will be given explicitly unless $\theta = 0$.

As we see from the integrals in (2.12), there is no need to determine the eigenvalues p and the associated eigenvectors **a** and **b**. This is a remarkable result which has been widely used in the analysis of anisotropic elasticity. It should be pointed out that there are cases in which the final solution cannot be presented entirely in terms of Barnett–Lothe tensors and their associated tensors. In that case we have to modify the general solution (1.10) for degenerate materials [8,9].

For isotropic elastic materials use of (2.12) leads to

$$S = s\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \frac{1}{\mu}\begin{bmatrix} \frac{1-s^2}{\kappa} & 0 & 0 \\ 0 & \frac{1-s^2}{\kappa} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \mu\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.14}$$

where $\mu$ is the shear modulus,

$$\kappa = \frac{1}{1-\nu}, \quad s = \frac{1-2\nu}{2(1-\nu)},$$

and $\nu$ is the Poisson ratio. For general anisotropic materials the structure of S, H, L is more complicated. For orthotropic materials and for monoclinic materials with the plane of symmetry at $x_3 = 0$, explicit expressions of S, H, L are obtained in [10,11]. We will show later that, if a proper oblique coordinate system is chosen as the natural base of the tensors S, H, L, the tensor components $S^i{}_j$, $H^{ij}$ and $L_{ij}$ for general anisotropic materials have the exact expressions as that shown in (2.14) for isotropic materials.

## 3. PHYSICAL MEANINGS OF THE EIGENVECTORS a AND b.

Let $\mathbf{a}'$, $\mathbf{a}''$ be the real and imaginary parts of **a**,

$$\mathbf{a} = \mathbf{a}' + i\mathbf{a}''.$$

A complex vector is also called a bivector [12,13]. The real vectors $\mathbf{a}'$

and a" span a plane. If $\hat{\mathbf{a}}$ is obtained by multiplying $\mathbf{a}$ by a complex factor $e^{i\psi}$ where $\psi$ is real,

$$\hat{\mathbf{a}} = e^{i\psi}\mathbf{a} = \hat{\mathbf{a}}' + i\hat{\mathbf{a}}'',$$

in which

$$\begin{aligned} \hat{\mathbf{a}}' &= \mathbf{a}'\cos\psi - \mathbf{a}''\sin\psi, \\ \hat{\mathbf{a}}'' &= \mathbf{a}'\sin\psi + \mathbf{a}''\cos\psi. \end{aligned} \tag{3.1}$$

Thus the real and imaginary parts of $\hat{\mathbf{a}}$ lie on the plane spanned by a' and a". Therefore the plane is called the polarization plane of **a**, or simply the plane **a**, which is invariant with the multiplication factor on **a**. As $\psi$ varies (3.1) show that $\hat{\mathbf{a}}'$ and $\hat{\mathbf{a}}''$ trace an ellipse. A pair of diameters in an ellipse is said to be conjugate if all chords parallel to one diameter are bisected by the other diameter. Therefore the tangent at the extremity of one diameter is parallel to the other diameter. It can be shown that $\hat{\mathbf{a}}'$ and $\hat{\mathbf{a}}''$ form a pair of conjugate radii. One could choose a $\psi$ such that $\hat{\mathbf{a}}'$ and $\hat{\mathbf{a}}''$ are orthogonal and hence are the principal radii of the ellipse [14].

It is clear that the bivector **a** and its complex conjugate $\bar{\mathbf{a}}$ define the same polarization plane.

Consider now the solution (1.3). The displacement **u** is a bivector **a** multiplied by f(z). Regardless of the position $(x_1, x_2)$, f(z) is a complex factor of the form $\rho e^{i\psi}$ where $\rho$ is real. Whether we take the real or imaginary part of **a** f(z), **u** is polarized on the plane **a** for all $(x_1, x_2)$. Likewise, the stress function $\boldsymbol{\phi}$ of (1.7) is polarized on the plane **b**. If $\mathbf{t}_\Gamma$ is the surface traction vector on a curved boundary Γ, it can be shown from (1.6) that

$$\mathbf{t}_\Gamma = \frac{\partial\boldsymbol{\phi}}{\partial\eta} \tag{3.2}$$

where $\eta$ is the arclength of Γ measured in the direction such that the material is located on the right hand side of Γ. Equations $(1.6)_1$ and $(1.6)_2$ are special cases of (3.2) when Γ is the surface $x_1$ = constant and $x_2$ = constant, respectively. Since $\boldsymbol{\phi}$ is polarized on the plane **b**, (3.2) tells us that the surface traction $\mathbf{t}_\Gamma$ is polarized on the plane **b**.

The general solution (1.10) or (1.11) implies that there are three polarization planes $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ for the displacement **u** and three polarization planes $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ for the surface traction $\mathbf{t}_\Gamma$. For monoclinic materials with the plane of symmetry at $x_3 = 0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$ all define the same plane, namely, the $(x_1, x_2)$ plane. As to $\mathbf{a}_3$ and $\mathbf{b}_3$, their real and imaginary parts are parallel. The polarization planes degenerate into lines parallel to the $x_3$–axis. The displacement associated with $\mathbf{a}_3$ and the surface traction $\mathbf{t}_\Gamma$ associated with $\mathbf{b}_3$ are in the $x_3$ direction.

In summary, there are three independent (or three one–component) solutions for general anisotropic materials. The displacement of a one–component solution is polarized on the plane $\mathbf{a}$ while the surface traction on any boundary is polarized on the plane $\mathbf{b}$. To satisfy a prescribed boundary condition, all three one–component solutions are in general needed. In surface waves, there are one–component surface waves [15, 16] and two–component surface waves [17, 18]. For Green's functions for the infinite space due to a line force and a line dislocation, there are one–component Green's functions. The latter will be discussed in Section 5.

**4. THE S TENSOR.** Of the three Barnett–Lothe tensors, the tensor S is the most interesting one. By writing S as

$$\mathbf{S} = \mathbf{L}^{-1}(\mathbf{LS}), \tag{4.1}$$

S is the product of the symmetric positive definite tensor $\mathbf{L}^{-1}$ and the antisymmetric tensor LS. It has the property that

$$\mathrm{tr}\,\mathbf{S} = 0, \quad \det \mathbf{S} = 0.$$

Therefore the eigenvalues of S are 0 and $\mp is$ where

$$s = \left\{-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{S}^2)\right\}^{1/2}. \tag{4.2}$$

Denoting the associated eigenvectors by $\mathbf{e}_3$ and $\mathbf{e}_1 \pm i\mathbf{e}_2$ where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are real, we have

$$\mathbf{S}(\mathbf{e}_1 \pm i\mathbf{e}_2) = \mp is(\mathbf{e}_1 \pm i\mathbf{e}_2), \quad \mathbf{S}\mathbf{e}_3 = 0. \tag{4.3}$$

Thus $\mathbf{e}_3$ is the **right null vector** of S and $\mathbf{e}_1 \pm i\mathbf{e}_2$ are the right eigenvectors. The new right eigenvectors $\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2$ obtained by multiplying $\mathbf{e}_1 \pm i\mathbf{e}_2$ by a complex factor span the same plane as $\mathbf{e}_1 \pm i\mathbf{e}_2$. Therefore the plane spanned by $(\mathbf{e}_1, \mathbf{e}_2)$ is called the **right eigenplane**.

Let $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ be the reciprocal of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ so that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i{}_j. \tag{4.4}$$

Consider the following tensor components of **S**, **H**, **L** :

$$\mathbf{S} = S^i{}_j\, \mathbf{e}_i \otimes \mathbf{e}^j, \quad \mathbf{H} = H^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{L} = L_{ij}\, \mathbf{e}^i \otimes \mathbf{e}^j. \tag{4.5}$$

Using (2.10) and the fact that **SH**, **LS** are antisymmetric, the matrices formed by $S^i{}_j$, $H^{ij}$, $L_{ij}$ can be shown to have the structure given in (2.14) where s, $\mu$, $\kappa$ are now independent constants [19]. Thus as far as the Barnett–Lothe tensors are concerned, anisotropic materials are identical to isotropic materials if we choose an oblique coordinate system represented by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. For isotropic materials $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors in the direction of the $x_1$, $x_2$, $x_3$ axis, respectively.

It should be pointed out that ($\mathbf{e}^1$, $\mathbf{e}^2$) and $\mathbf{e}^3$ are, respectively, the **left eigenplane** and the **left null vector** of **S**. Do $\mathbf{e}_i$, $\mathbf{e}^j$ have physical interpretations? They do. They are explained in the next Section.

**5. GREEN'S FUNCTIONS FOR LINE FORCES AND LINE DISLOCATIONS IN THE INFINITE SPACE.** There are several interesting properties associated with Green's functions for the infinite space due to a line force $\boldsymbol{f}$ and a line dislocation with Burgers vector $\boldsymbol{b}$ applied along the $x_3$ axis. The basic solution is obtained from (1.11) by choosing the function $f(z_\alpha)$ such that

$$\mathbf{u} = \frac{1}{\pi}\,\mathrm{Im}\sum_{\alpha=1}^{3} \mathbf{a}_\alpha q_\alpha \ln z_\alpha, \quad \boldsymbol{\phi} = \frac{1}{\pi}\,\mathrm{Im}\sum_{\alpha=1}^{3} \mathbf{b}_\alpha q_\alpha \ln z_\alpha. \tag{5.1}$$

Since $\ln z_\alpha$ is a multi–valued function we introduce a cut along the negative $x_1$–axis. In the polar coordinate system

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \tag{5.2}$$

the solution (5.1) applies to

$$-\pi < \theta < \pi, \qquad r > 0.$$

Therefore

$$\ln z_\alpha = \ln r \pm i\pi \quad \text{at} \quad \theta = \pm\pi, \ \text{for } \alpha = 1, 2, 3. \tag{5.3}$$

Equations (5.1) represent three one–component Green's functions. For each $\alpha$, **u** is polarized on the plane $\mathbf{a}_\alpha$ and the surface traction $\mathbf{t}_\Gamma$ is polarized on the plane $\mathbf{b}_\alpha$. The discontinuities in **u** and $\boldsymbol{\phi}$ across $\theta = \pm\pi$ are, respectively, the line dislocation $\boldsymbol{b}^\alpha$ and the line force

$f^\alpha$ for the one–component Green's function. Hence by (5.3),

$$b^\alpha = 2\,\mathrm{Re}(\mathbf{a}_\alpha q_\alpha), \quad f^\alpha = 2\,\mathrm{Re}(\mathbf{b}_\alpha q_\alpha), \tag{5.4}$$

which show that $b^\alpha$ is on the plane $\mathbf{a}_\alpha$ and $f^\alpha$ is on the plane $\mathbf{b}_\alpha$. We therefore have the result that the one–component Green's function has $\mathbf{u}$ and $b^\alpha$ polarized on the plane $\mathbf{a}_\alpha$ and has $f^\alpha$ and the surface traction $\mathbf{t}_\Gamma$ polarized on the plane $\mathbf{b}_\alpha$.

To obtain a one–component Green's function we may assume an arbitrary complex constant $q_\alpha$. Equations (5.4) then provide $b^\alpha$ and $f^\alpha$ required for the one–component Green's function. Alternately we may prescribe an $f^\alpha$ which lies on the plane $\mathbf{b}_\alpha$. Equation $(5.4)_2$ can be solved for $q_\alpha$ and $(5.4)_1$ gives the associated $b^\alpha$. To solve $(5.4)_2$ for $q_\alpha$, let the real and imaginary parts of $\mathbf{b}_\alpha$ and $q_\alpha$ be written as

$$\mathbf{b}_\alpha = \mathbf{b}'_\alpha + i\mathbf{b}''_\alpha, \quad q_\alpha = q'_\alpha + iq''_\alpha.$$

We then have

$$f^\alpha = 2(\mathbf{b}'_\alpha q'_\alpha - \mathbf{b}''_\alpha q''_\alpha)$$

from which $q'_\alpha$ and $q''_\alpha$ can be determined.

When $f$ and $b$ are prescribed arbitrarily, we need all three one–component Green's functions for the solution. Making use of (2.6), (5.1) are rewritten as

$$\mathbf{u} = \frac{1}{\pi}\,\mathrm{Im}\{\mathbf{A}\langle \ln z\rangle \mathbf{q}\}, \quad \boldsymbol{\phi} = \frac{1}{\pi}\,\mathrm{Im}\{\mathbf{B}\langle \ln z\rangle \mathbf{q}\}, \tag{5.5}$$

in which

$$\mathbf{q}^T = [q_1, q_2, q_3]$$

and

$$\langle \ln z\rangle = \mathrm{diag}[\ln z_1, \ln z_2, \ln z_3]$$

is a diagonal matrix. Equations (5.5) must satisfy the conditions

$$\mathbf{u}(\pi) - \mathbf{u}(-\pi) = b,$$

$$\boldsymbol{\phi}(\pi) - \boldsymbol{\phi}(-\pi) = f,$$

which lead to

$$2\,\mathrm{Re}(\mathbf{A}\mathbf{q}) = \boldsymbol{b}, \quad 2\,\mathrm{Re}(\mathbf{B}\mathbf{q}) = \boldsymbol{f}. \tag{5.6}$$

This can be written as

$$\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \overline{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

It follows from (2.7) that

$$\begin{bmatrix} \mathbf{q} \\ \overline{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Hence

$$\mathbf{q} = \mathbf{A}^T \boldsymbol{f} + \mathbf{B}^T \boldsymbol{b}, \tag{5.7a}$$

or

$$q_\alpha = \mathbf{a}_\alpha \cdot \boldsymbol{f} + \mathbf{b}_\alpha \cdot \boldsymbol{b}. \tag{5.7b}$$

Inserting (5.7b) into (5.4) gives us $\boldsymbol{b}^\alpha$ and $\boldsymbol{f}^\alpha$ in terms of $\boldsymbol{b}$ and $\boldsymbol{f}$.

We show next that the solution (5.5) together with (5.7a) can be rewritten into a real form. Equations (2.9) are identities which convert certain combinations of complex quantities involving A, B to real quantities S, H and L. The following identities are useful for problems related to line forces and line dislocations [20].

$$\begin{aligned}
2\,\mathbf{A}\langle \ln z \rangle \mathbf{A}^T &= -i[(\ln r)\mathbf{I} + \pi\mathbf{S}(\theta)]\mathbf{H} + \pi\mathbf{H}(\theta)[\mathbf{I} - i\mathbf{S}^T], \\
2\,\mathbf{B}\langle \ln z \rangle \mathbf{A}^T &= [(\ln r)\mathbf{I} + \pi\mathbf{S}^T(\theta)][\mathbf{I} - i\mathbf{S}^T] + i\pi\mathbf{L}(\theta), \\
2\,\mathbf{A}\langle \ln z \rangle \mathbf{B}^T &= [(\ln r)\mathbf{I} + \pi\mathbf{S}(\theta)][\mathbf{I} - i\mathbf{S}] + i\pi\mathbf{H}(\theta)\mathbf{L}, \\
2\,\mathbf{B}\langle \ln z \rangle \mathbf{B}^T &= i[(\ln r)\mathbf{I} + \pi\mathbf{S}^T(\theta)]\mathbf{L} - \pi\mathbf{L}(\theta)[\mathbf{I} - i\mathbf{S}].
\end{aligned} \tag{5.8}$$

These identities allow us to convert the complex expressions on the left to real quantities shown on the right which are obtainable directly in terms of elasticity constants through (2.12) and (2.13). With the identities (5.8), the solution (5.5) together with (5.7a) is converted into a real form as

$$
\begin{aligned}
2\mathbf{u} &= -\frac{1}{\pi}(\ln r)\mathbf{h} - \mathbf{S}(\theta)\mathbf{h} + \mathbf{H}(\theta)\mathbf{g}, \\
2\boldsymbol{\phi} &= \frac{1}{\pi}(\ln r)\mathbf{g} + \mathbf{L}(\theta)\mathbf{h} + \mathbf{S}^{T}(\theta)\mathbf{g},
\end{aligned}
\tag{5.9}
$$

where

$$
\mathbf{g} = \mathbf{L}\boldsymbol{b} - \mathbf{S}^{T}\boldsymbol{f}, \quad \mathbf{h} = \mathbf{S}\boldsymbol{b} + \mathbf{H}\boldsymbol{f}. \tag{5.10}
$$

From (3.2), the surface traction $\mathbf{t}_{\theta}$ on any radial plane $\theta$ = constant is in the direction of g which is invariant with the choice of the radial plane. The infinite displacement $\mathbf{u}_0$ at $r = 0$ is in the direction of **h**. Moreover, the relation [14]

$$
\mathbf{g}\cdot\mathbf{h} = \boldsymbol{f}\cdot\boldsymbol{b}
$$

is easily established using (2.10) and the anti–symmetric property of **LS** and **SH**.

We now present physical interpretations of $\mathbf{e}_i$ and $\mathbf{e}^i$. Using (4.5) and the discussions following (5.10), (5.10) can be written as

$$
\begin{aligned}
2\pi r\mathbf{t}_{\theta} &= \mathbf{g} = [L_{ij}(\mathbf{e}^j\cdot\boldsymbol{b}) - S^j{}_i(\mathbf{e}_j\cdot\boldsymbol{f})]\mathbf{e}^i, \\
-2\pi(\ln r)^{-1}\mathbf{u}_0 &= \mathbf{h} = [S^i{}_j(\mathbf{e}^j\cdot\boldsymbol{b}) + H^{ij}(\mathbf{e}_j\cdot\boldsymbol{f})]\mathbf{e}_i.
\end{aligned}
$$

With the structure of $S^i{}_j$, $H^{ij}$, $L_{ij}$ shown in (2.14) and using (4.4), it can be shown that if $\boldsymbol{b}$ is along $\mathbf{e}_3$ and $\boldsymbol{f}$ is along $\mathbf{e}^3$, $\mathbf{u}_0$ is in the direction of $\mathbf{e}_3$ and $\mathbf{t}_{\theta}$ in the direction of $\mathbf{e}^3$. On the other hand, if $\boldsymbol{b}$ is on the right eigenplane ($\mathbf{e}_1$, $\mathbf{e}_2$) and $\boldsymbol{f}$ is on the left eigenplane ($\mathbf{e}^1$, $\mathbf{e}^2$), $\mathbf{u}_0$ is polarized on the right eigenplane and $\mathbf{t}_{\theta}$ is polarized on the left eigenplane. More relations between $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}^1$,$\mathbf{e}^2$ and $\boldsymbol{b}$ and $\boldsymbol{f}$ can be found in [14].

**6. BIMATERIALS AND INTERFACE CRACKS.** Let $\theta = \theta_0$ be the interface between two materials in the bimaterial. The half–space $\theta_0 < \theta < \theta_0+\pi$ is occupied by material 1 and the other half–space $\theta_0-\pi < \theta < \theta_0$ is occupied by material 2. They are rigidly bonded together along $\theta = \theta_0$. For a line force $\boldsymbol{f}$ and a line dislocation $\boldsymbol{b}$ applied at the origin $r = 0$, (5.9) is a basic solution which applies to both materials. We may add constant terms to the right hand sides of (5.9) which produce a rigid body displacement but no stresses. Therefore consider the solution

$$
\begin{aligned}
2\,\mathbf{u}_1(r,\theta) &= -\frac{1}{\pi}(\ln r)\mathbf{h} - [\mathbf{S}_1(\theta)-\mathbf{S}_1(\theta_0)]\mathbf{h} + [\mathbf{H}_1(\theta)-\mathbf{H}_1(\theta_0)]\mathbf{g}, \\
2\,\boldsymbol{\phi}_1(r,\theta) &= \frac{1}{\pi}(\ln r)\mathbf{g} + [\mathbf{L}_1(\theta)-\mathbf{L}_1(\theta_0)]\mathbf{h} + [\mathbf{S}_1^{T}(\theta)-\mathbf{S}_1^{T}(\theta_0)]\mathbf{g},
\end{aligned}
\tag{6.1}
$$

for material 1 in $\theta_0 < \theta < \theta_0+\pi$. The subscript 1 denotes material 1. The solution for material 2 is obtained from (6.1) by replacing the subscript 1 by 2 while keeping the same constants **g** and **h**. It is readily shown that the continuity of **u** and $\boldsymbol{\phi}$ at $\theta = \theta_0$ is automatically satisfied. The discontinuity in **u** and $\boldsymbol{\phi}$ across $\theta = \theta_0 \pm \pi$, which should be equal to $b$ and $f$, respectively, leads to two equations for **g** and **h** which are independent of $\theta_0$ [21]. Therefore, the stresses obtained by substituting $\boldsymbol{\phi}_1$ of (6.1) and similar equation for $\boldsymbol{\phi}_2$ into (1.6) are independent of the location $\theta_0$ of the interface ! This unexpected phenomenon defies an intuitive explanation even for isotropic bimaterials.

One of the most studied problems in anisotropic elasticity is the problem of interface cracks in bimaterials [22–33]. Let $x_2 > 0$ be occupied by material 1 and $x_2 < 0$ be occupied by material 2. The interface crack of length $2a$ is located at

$$x_2 = 0, \quad |x_1| < a.$$

The bimaterial is subject to a uniform traction $\mathbf{t}_\Gamma$ and $-\mathbf{t}_\Gamma$ at the crack surfaces $x_2 = +0$ and $-0$, respectively. The stress singularities near a tip of the interface crack is proportional to $r^\delta$ where r is the radial distance from the crack tip and $\delta$ is a constant depending on the material property of the bimaterial. It is shown in [24] that there are three singularities given by

$$\delta = -\frac{1}{2}, \quad -\frac{1}{2} + i\gamma, \quad \text{and} \quad -\frac{1}{2} - i\gamma,$$

where

$$\gamma = \frac{1}{2\pi} \ln\frac{1+\beta}{1-\beta} = \frac{1}{\pi} \tanh^{-1}\beta,$$

$$\beta = \left[-\frac{1}{2}\,\mathrm{tr}(\hat{\mathbf{S}}^2)\right]^{1/2} < 1. \tag{6.2}$$

In the above

$$\hat{\mathbf{S}} = \mathbf{D}^{-1}\mathbf{W}, \tag{6.3}$$

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}, \quad \mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1},$$

in which **D** is symmetric, positive definite and **W** is anti–symmetric. Thus $\hat{\mathbf{S}}$ has the same properties as the **S** tensor. The eigenvalues of $\hat{\mathbf{S}}$ are $\mp i\beta$ and 0 and the associated right eigenvectors are denoted by $\mathbf{d}_1 \pm i\mathbf{d}_2$ and $\mathbf{d}_0$, respectively. The left eigenvectors can be shown to be $\mathbf{D}(\mathbf{d}_1 \mp i\mathbf{d}_2)$ and $\mathbf{D}\mathbf{d}_0$ [34]. Hence $\mathbf{d}_0$, $\mathbf{D}\mathbf{d}_0$ are, respectively, the right and left null vectors while the planes spanned by $(\mathbf{d}_1, \mathbf{d}_2)$ and $(\mathbf{D}\mathbf{d}_1, \mathbf{D}\mathbf{d}_2)$ are the right and left eigenplanes.

The two materials in the bimaterial are said to be **mismatched** when $\mathbf{W} \neq 0$. $\mathbf{W} = 0$ if and only if $\beta = 0$ (and hence $\gamma = 0$) [24, 25]. For mismatched bimaterials ($\gamma \neq 0$), the displacement at the crack surface is oscillatory. This leads to the physically unacceptable interpenetration of the crack surfaces.

When $\beta = 0$ the solution in materials 1 and 2 both have the expression

$$\begin{aligned} \mathbf{u} &= \mathrm{Re}\,\{\mathbf{A}\langle f(z)\rangle\mathbf{B}^{-1}\}\mathbf{t}_\Gamma, \\ \boldsymbol{\phi} &= \mathrm{Re}\,\{\mathbf{B}\langle f(z)\rangle\mathbf{B}^{-1}\}\mathbf{t}_\Gamma, \end{aligned} \tag{6.4}$$

in which

$$f(z) = \sqrt{z^2 - a^2} - z.$$

Of course A, B and z in material 1 and material 2 would be different. There is no oscillation in displacement and the stress is the square root singularities.

The following results are taken from [34]. When $\beta \neq 0$, the solution is still given by (6.4) if the applied traction $\mathbf{t}_\Gamma$ is the null vector of $\mathbf{W}$, i.e., if

$$\mathbf{W}\mathbf{t}_\Gamma = 0,$$

or, by (6.3),

$$\hat{\mathbf{S}}\mathbf{t}_\Gamma = 0.$$

Thus when the applied traction is in the direction of the right null vector $\mathbf{d}_0$, there is no oscillation in displacement. The crack surface opening

$$\Delta\mathbf{u} = \mathbf{u}(x_1, +0) - \mathbf{u}(x_1, -0), \quad |x_1| < a,$$

is in the direction of the left null vector $\mathbf{D}\mathbf{d}_0$ and the surface traction on the surface $x_2 = 0$ outside the crack is in the direction of the right null vector $\mathbf{d}_0$.

If the applied traction is not in the direction of $\mathbf{d}_0$, we decompose it into two components. One is along the right null vector $\mathbf{d}_0$ and the other is on the right eigenplane $(\mathbf{d}_1, \mathbf{d}_2)$. Explicit solutions associated with the one on the right eigenplane can be found in [34] in which the displacement is oscillatory. It suffices to mention that the crack surface opening $\Delta\mathbf{u}$ lies on the left eigenplane of $\hat{\mathbf{S}}$ while the surface traction along the surface $x_2 = 0$ lies on the right eigenplane of $\hat{\mathbf{S}}$.

## DISCUSSION

We have shown that, in many respects, anisotropic elastic materials have properties which are similar to, or generalization of, the properties of isotropic materials. Analogous to the antiplane deformations of isotropic materials, anisotropic materials have deformations which are polarized in one direction while the surface traction vector on any boundary is polarized on a different direction. Similar to the in–plane deformations of isotropic materials, anisotropic materials can have deformations which are polarized on one oblique plane while the surface traction vector on any boundary is polarized on another oblique plane.

Simple problems for which we thought we have understood them thoroughly still yield new information due to the simplification of the solutions by the Stroh formalism. For example, consider the Griffith crack of length $2a$ located at $x_2 = 0$, $|x_1| < a$ in the infinite anisotropic elastic medium. When the traction applied at the crack surfaces is in the direction of the $x_2$ axis, the crack opening is in general not symmetric with the $x_2$ axis as expected. However, the $x_1$ axis outside the crack remains a straight line (i. e., the $u_2$ component of the displacement along the $x_1$ axis vanishes). If the traction applied at the crack surface is the null vector of $SL^{-1}$, all three displacement components along the $x_1$ axis vanish. If the applied traction is in the direction of the vector formed from the second column of L, the hoop stress vector along the crack surface is independent of $x_1$ [21].

Other interesting properties worth mentioning are the physical implications of the eigenvalues p. For the Green's functions for a half–space subject to a singularity in the form of line forces and line dislocations, the solution can be obtained by a superposition of the Green's function due to the same singularity for the infinite space and several image singularities located outside of the space occupied by the material. The locations of the image singularities are determined exclusively by the eigenvalues p. Moreover, the locations of the image singularities are independent of the nature of the singularities concerned [35]. If the singularities are line forces and line dislocations, the image singularities are also line forces and line dislocations. For degenerate materials for which isotropic materials are a special case, two or more of the image singularities coalesce into one singularity, creating a new singularity in the form of a double force, a concentrated couple, and/or a higher order singularity which are well known for isotropic materials [36] but have not been satisfactorily explained in the past.

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## REFERENCES


[1] A. N. Stroh, "Dislocations and cracks in anisotropic elasticity," Phil Mag. 3, 625–646 (1958).

[2] A. N. Stroh, "Steady state problems in anisotropic elasticity," J. Math. Phys. 41, 77–103 (1962).

[3] D. M. Barnett and J. Lothe, "Synthesis of the sextic and the integral formalism for dislocations, Greens functions and surface waves in anisotropic elastic solids," Phys. Nor., 7, 13–19 (1973).

[4] J. D. Eshelby, W. T. Read, and W. Shockley, "Anisotropic elasticity with applications to dislocation theory," Acta Metall. 1, 251–259 (1953).

[5] K. Malen and J. Lothe, "Explicit expressions for dislocation derivatives," Phys. Status Solidi. 39, 287–296 (1970).

[6] P. Chadwick and G. D. Smith, "Foundations of the theory of surface waves in anisotropic elastic materials," Adv. Appl. Mech. 17, 303–376 (1977).

[7] T. C. T. Ting, "Some identities and the structure of $N_i$ in the Stroh formalism of anisotropic elasticity," Q. Appl. Math. 46, 109–120 (1988).

[8] T. C. T. Ting, "Effects of change of reference coordinates on the stress analyses of anisotropic elastic materials," Int. J. Solids Structures. 18, 139–152 (1982).

[9] T. C. T. Ting and Chyanbin Hwu, "Sextic formalism in anisotropic elasticity for almost non–semisimple matrix N," Int. J. Solids Structures. 24, 65–76 (1988).

[10] Changsong Dongye and T. C. T. Ting, "Explicit expressions of Barnett–Lothe tensors and their associated tensors for orthotropic materials," Q. Appl. Math. 47, 723–734 (1989).

[11] T. C. T. Ting, "Barnett–Lothe tensors and their associated tensors for monoclinic materials with the symmetry plane at $x_3 = 0$," J. Elasticity. 27, 143–165 (1992).

[12] J. W. Gibbs, "Elements of Vector Analysis," Dover, N.Y. (1961).

[13] Ph. Boulanger and M. Hayes, "Bivectors and inhomogeneous plane waves in anisotropic elastic bodies," in Modern Theory of

Anisotropic Elasticity and Applications. SIAM Proceedings Series. SIAM, Phila. 280–289 (1991).

[14] T. C. T. Ting, "The eigenvectors of the S matrix and their relations with line dislocations and forces in anisotropic elastic solids," in Micromechanics and Inhomogeneity, The Toshio Mura Anniversary Volume. Springer–Verlag, N.Y. 449–467 (1990).

[15] D. M. Barnett, P. Chadwick and J. Lothe, "The behavior of elastic surface waves polarized in a plane of material symmetry. I. Addendum," Proc. R. Soc. Lon. A **433**, 699–710 (1991).

[16] D. M. Barnett and P. Chadwick, "The existence of one–component surface waves and exceptional subsequent transonic states of type 2, 4 and E1 in anisotropic elastic media," in Modern Theory of Anisotropic Elasticity and Applications. SIAM Proceedings Series. SIAM, Phila. 199–214 (1991).

[17] D. M. Barnett, J. Lothe and S. A. Gundersen, "Zero curvature transonic states and free surface waves in anisotropic elastic media," Wave Motion. **12**, 341–360 (1990).

[18] S. A. Gundersen, L. Wang and J. Lothe, "Secluded supersonic surface waves," in Modern Theory of Anisotropic Elasticity and Applications. SIAM Proceedings Series. SIAM, Phila. 186–198 (1991).

[19] P. Chadwick and T. C. T. Ting, "On the structure and invariance of the Barnett–Lothe tensors," Q. Appl. Math. **45**, 419–427 (1987).

[20] T. C. T. Ting, "Line forces and dislocations in anisotropic elastic composite wedges and spaces," Phys. Status Solidi. B **146**, 81–90 (1988).

[21] T. C. T. Ting, "The Stroh formalism and certain invariances in two–dimensional anisotropic elasticity," in Modern Theory of Anisotropic Elasticity and Applications. SIAM Proceedings Series. SIAM, Phila. 3–32 (1991).

[22] D. L. Clements, "A crack between dissimilar anisotropic media," Int. J. Eng. Sci. **9**, 257–265 (1971).

[23] J. R. Willis, "Fracture mechanics of interface cracks," J. Mech. Phys. Solids. **19**, 353–368 (1971).

[24] T. C. T. Ting, "Explicit solution and invariance of the singularities at an interface crack in anisotropic composites," Int. J. Solids Structures. 22, 965–983 (1986).

[25] J. L. Bassani and J. Qu, "Finite cracks on bimaterial and bicrystal interfaces," J. Mech. Phys. Solids. 37, 435–453 (1989).

[26] V. K. Tewary, R. H. Wagoner and J. P. Hirth, "Elastic Green's function for a composite solid with a planar crack in the interface" J. Mater. Res. 4, 124–136 (1989).

[27] Zhigang Suo, "Singularities, interfaces and cracks in dissimilar anisotropic media," Proc. R. Soc. Lon. A 427, 331–358 (1990).

[28] K. C. Wu, "Stress intensity factor and energy release rate for interfacial cracks between dissimilar anisotropic materials," J. Appl. Mech. 57, 882–886 (1990).

[29] T. C. T. Ting, "Interface cracks in anisotropic bimaterials," J. Mech. Phys. Solids. 38, 505–513 (1990).

[30] Chyanbin Hwu, "Collinear cracks in anisotropic bodies," Int. J. Fracture. 54, 239–256 (1991).

[31] L. Ni and S. Nemat–Nasser, "Interface cracks in anisotropic dissimilar materials: an analytic solution," J. Mech. Phys. Solids. 39, 113–144 (1991).

[32] H. Gao, M. Abbudi and D. M. Barnett, "On interfacial crack–tip field in anisotropic elastic solids," J. Mech. Phys. Solids. 40, 393–416 (1992).

[33] Jianmin Qu and Qianqian Li, "Interfacial dislocation and its applications to interface cracks in anisotropic bimaterials," J. Elasticity. 26, 169–195 (1991).

[34] T. C. T. Ting, "Interface cracks in anisotropic elastic bimaterials – a decomposition principle," George Herrmann 70th Anniversary Issue, Int. J. Solids Structures. 29, 1989–2003 (1992).

[35] T. C. T. Ting, "Image singularities of Green's functions for anisotropic elastic half–spaces and bimaterials," Q. J. Mech. Appl. Math. 45, 119–139 (1992).

[36] J. Dundurs, "Elastic interaction of dislocations with inhomogeneities," in Mathematical Theory of Dislocations. T. Mura, ed. ASME, N. Y. 70–115 (1969).