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After identifying typical situations, we present a theory of conditional information in the form of the new concept of "conditional events," compatible with all conditional probability quantifications. We specify applications of this theory to various problems in intelligent systems. The approach taken here to conditional events is through intervals of events.

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A THEORY OF CONDITIONAL INFORMATION FOR PROBABILISTIC INFERENCE IN INTELLIGENT SYSTEMS: I, INTERVAL OF EVENTS APPROACH

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ABSTRACT

This paper emphasizes the need to develop further probability theory at the service of probabilistic intelligent systems. In the field of probabilistic systems, the causal relationships among variables of interest are viewed as if-then (or production) rules whose certainty factors are quantified as conditional probabilities. With some additional assumptions about the variables of interest, such as conditional independence, standard probability theory can be applied to carry out the reasoning processes. In more general situations, in which all information (in the premises as well as the conclusions) is in unconditional and conditional form - or in only conditional form - current probabilistic machinery requires more development to cope with this new situation.

After identifying typical situations, as mentioned above, we present a theory of conditional information in the form of the new concept of "conditional events", compatible with all conditional probability quantifications. We specify applications of this theory to various problems in intelligent systems. The approach taken here to conditional events is through intervals of events.

Key Words

Bayesian methodology, conditional events, conditional event algebra, conditional probability, conditional random variables, conditionals, if-then statements, implications, intelligent systems, logic of conditionals, probabilistic inference, quantification of if-then rules

1. INTRODUCTION

One of the aims of this paper is to point out potential areas in the field of probabilistic intelligent systems, where the need to model uncertain if-then rules as mathematical entities is apparent.

The analysis carried out in this paper is essentially restricted to probability considerations. However, all of this can be extended to a more general setting, such as to apply, e.g., to conditional linguistic information. There, fuzzy set theory and fuzzy logic could be used as a vehicle to model vagueness of natural language concepts.

To facilitate the writing, we now specify the notation to be used throughout this paper, once and for all.

Basically, intelligent systems are concerned with reasoning with knowledge. As is well known, it is sufficient to use a probability space (Ω,A,P) to describe probabilistic knowledge, where (Ω,A) is a measurable space, i.e., $\Omega \in A \subseteq P(\Omega)$, A a sigma algebra or boolean algebra of subsets of Ω, $P(\Omega)$ denoting the power class, or collection of all subsets of Ω, and where $P:A \to [0,1]$ is a probability measure on A, $[0,1]$ denoting the unit interval $\{t \text{ real}: 0 \le t \le 1\}$). We will also use small letters at the beginning of the alphabet to denote elements of A; these are usually interpreted as events or subsets of Ω (but also as propositions or statements - see comments below). Set operations on Ω are denoted as: $\cdot$ - or a lack of the symbol, when no ambiguity precludes its omission - for "and" or conjunction or set intersection $\cap$; $\vee$ for "or" or disjunction or set union $\cup$; $()'$ for "not" or negation or set complement; $\leq$ for "contained in" or subset inclusion $\subseteq$. We may also interpret $\leq$ as the basic entailment or deduction relation among elements of A. The symbol $<$ stands for strict set inclusion. When there is no confusion, the same symbols, $\leq$ and $<$ will be employed to denote both the set relations as above and the usual numerical ordering relations of "less than or equal to" and "strictly less than", respectively. We will use throughout the paper $\emptyset$ to indicate the zero element or null or empty event in A and Ω to indicate the unity or universal element in A. The usual mathematical notation will also be used, such as $\in$ "in" or set membership; $\{x:R(x)\}$ for the set of all elements x such that predicate $R(x)$ holds; $f:G \to H$ for the function f mapping space G into H ; $\times$ for cartesian product, as $a \times b \times c = \{(r,s,t): r \in a,\ s \in b,$

$t \in c\}$ and $A_1 \times A_2 = \{(a_1, a_2): a_j \in A_j\}$. Superscripted notation using integers indicates repeated cartesian product as A^3 for $A \times A \times A$ and $\{a,b\}^n$ for $\{a,b\} \times \ldots \times \{a,b\}$ (n factors). Other notation will be introduced as needed.

A knowledge-base usually consists of facts (i.e., propositions or statements in a natural language) and rules (i.e., conditional or if-then statements). In view of the well-known Stone Representation Theorem (Mendelson, [1]), one can view a boolean (or sigma-) algebra A of propositions as being equivalent to one of events (or subsets) as above, where logical connections among propositions are identified as corresponding set operations.

By an if-then rule here, we mean "if b, then a", symbolized as $b \to a$. When such a rule is uncertain - as e.g. "When you have dried skin that, e.g., feels to the touch as wood, you have disease Z" - or not always true, one needs to quantify the strength of that rule, or at least our degree of belief in such a rule. In the probabilistic setting, there are several ways of doing this. If the arrow ($\to$) is interpreted as the *material conditional* of classical logic, i.e.,

$$b \to a = b' \vee a \ (= b' \vee ab) ,$$

then

$$P(b \to a) = P(b' \vee a) \ (= 1 - P(b) + P(ab))$$

is one such quantification. (See, e.g. Nilsson [2] .)

On the other hand, due to the causal relationship among variables of interest, in a domain of investigation, $b \to a$ is often quantified by *conditional probabilities* (see, e.g., Pearl [3])

$$P(b \to a) = P(a|b) = P_b(a) = P(ab)/P(b) ,$$

when $P(b) > 0$. This quantification concurs also with a number of logicians' and philosophers' thinking (including, e.g., Adams [4] and McGee [5]). If we insist on using the latter quantification methodology, then it is easy to see that in general,

$$b \to a \neq b' \vee a ,$$

since by inspection

$$P(a|b) \neq P(b' \vee a).$$

More generally, $b \to a$ *cannot* be an element of A , except for trivial cases such as $b = \Omega$. This fact is known as the triviality result (Lewis [6]; see also [7].)

In existing theories dealing with if-then rules, there apparently has been no need to model them as separate mathematical entities (Pearl [3]), since it suffices to specify the joint distribution of all variables involved in a bayesian network. While the subject of conditional statements in natural language and logic remains very much a topic of interest (see, e.g., Jackson [8], McGee [5] , and Traugott et al. [9]), there is little emphasis, however, on mathematical modeling of conditionals (or conditional events). Instead, the emphasis has been on logical aspects of them, keeping such conditionals as primitive or undefined concepts.

In the probability theory literature, there have been relatively few works concerning the modeling of conditional events (e.g., De Finetti [10] and Koopman [11]). However, these works have been largely forgotten, perhaps due to the common fact that any mathematical development which does not contribute to advances in applications may be considered unimportant. This is somewhat similar to the employment of the term "conditional random variables" by Wilks [12] to motivate conditional distributions, but where the concept itself is not presented clearly as a separate useful well-defined mathematical entity.

As we will see, the mathematical modeling of if-then rules, compatible with conditional probability evaluations, i.e.,

$$P(b \to a) = P(a|b)$$

is indispensable in extending probabilistic techniques to general intelligent systems. It should be noted that in probabilistic inference for systems, more than the standard probability calculus may be needed. Indeed, due to the nature of, say, expert systems, certain new logical tools have been added (see, e.g., Pearl [3]), namely the "logic of high probabilities" (see also Adams [13]). Our work in this paper can be viewed as a concretizing of the general theory of conditionals discussed by logicians and philosophers for use here in intelligent systems.

In Section 2, we will illustrate various problems of interest in intelligent systems in which mathematical modeling of conditional events is needed. In Section 3 we first discuss and contrast the use of conditional probabilities and probabilities of material conditionals and see that it is more useful to employ the former than the latter in evaluating if-then rules. This is followed by a basic introduction to the interval of events approach to conditional events.

In Part II of this work, a new approach to conditional events will be introduced which yields a more computationally complex structure, but which has certain advantages over the interval of events approach. Part III of this effort will be an overview and a mathematical appendix documenting the more technically detailed results.

2. CONDITIONAL INFERENCE IN PROBABILISTIC SYSTEMS

In order to motivate our development of conditional events and their associated logics - or "conditional event algebras" - we consider in this section eight typical types of problems concerning conditional information or related concepts. It is our belief that up till the present, there has been no real unifying rigorous framework extending standard probability theory for dealing with these problems; rather instead, apparently only informal or ad hoc procedures have sufficed.

Problem 1. A knowledge base of a diagnostic system consists of rules of the form $b_i \to a_i$, $i=1,..,n$. The strength of each rule is $P(a_i|b_i)$. Clearly, the unconditional forms a_i, b_i are in actuality special cases of conditionals relative to Ω, i.e., in symbols

$$a_i = (a_i|\Omega) \quad \text{and} \quad b_i = (b_i|\Omega) .$$

At this point we can even regard such "conditional events" as well-defined objects as in De Finetti [14], Gilio [15]. When the strength of the rule $b_i \to a_i$ is computed in the context of another event c, following the bayesian viewpoint, the prior probability P is replaced by the conditional probability P_c, so that these values are equal to $P_c(a_i|b_i)$, representing formally $P_c(b_i \to a_i)$, or hopefully, equivalently, $P((c \to b_i) \to (c \to a_i))$. In the more realistic situation, where each a_i and b_i are each conditioned on separate premises, say c_i and d_i, respectively, the rules should become formally $(b_i|d_i) \to (a_i|c_i)$. If we insist on using probability quantification of rules, we need to be able to define and compute the probabilities of the above formal expressions expressed as $P((b_i|d_i) \to (a_i|c_i))$. These formal rules are mentioned, e.g., in Goldszmidt & Pearl [16].

It is obvious that in order to solve the above problem, we need to define conditional events of the form $(a_i|b_i)$, as well as logical operations among them. The implication arrow ($\rightarrow$) among these objects could be defined based on logical considerations among conditional events. The problem of assigning probabilities to compounds of conditional statements in a natural language is also of concern to logicians (see, e.g., McGee [5]).

Problem 2. A common inference rule in intelligent systems is *modus ponens*. In a classical two-valued logic setting, given $b \rightarrow a = b' \vee a$, we deduce a if the evidence is b, since $b \rightarrow a$ and b conjoined entail a, where the entailment relation is $\leqslant$. Equivalently, we have

$$(b' \vee a) \cdot b = ab \leqslant a \ .$$

But, under the probability quantification $b \rightarrow a$ (as in $P(b \rightarrow a) = P(a|b)$) is not equal to $b' \vee a$. Hence, not only objects like $b \rightarrow a = (a|b)$ need to be defined, but also the conjunction with $b = (b|\Omega)$, so that the analogue of modus ponens holds:

$$(a|b) \cdot (b|\Omega) = (ab|\Omega) \leqslant (a|\Omega) \ .$$

Moreover, when new evidence c no longer matches exactly premise b (i.e., no longer "fires" inference rule $b \rightarrow a$; in real-world applications such partial matching happens quite frequently), we also need to be able to determine the more general conjunction between $(a|b)$ and $(c|\Omega)$. In the same vein, even more generally, the evaluation of probabilities of expressions such as $P((a|b) \vee (c|d))$ are also of interest, such as in the truth or probability evaluation of the natural language expression (referring to a hand of cards) "If I pick all picture cards, I will get a king, or, if I pick all black cards, I will get a Queen or King".

Problem 3. Let (Ω, A, P) be a probability space and consider events a,b,c,d,.. in A. Suppose event e in A is known partially in the sense that according to source j, e lies in *event interval*

$$[a_j, b_j] = \{x \in A: a_j \leqslant x \leqslant b_j\} \quad ; \ a_j \leqslant e_j \leqslant b_j \ , \ j=1,..,r.$$

For example, let

e = "John purchased x items of type A, B, C today" (x,A,B,C unknown)

a_j = "John always buys at least items $A_1,..,A_m$", according to source j ,

b_j = "John always buys at most items $A_1,..,A_m,A_{m+1},..,A_n$", according to source j also, j=1,2.

We wish to estimate the probability of e relative to each source j, as well as obtain the probability of the logical conjunction, disjunction, or any other appropriate compound of these event intervals. Certainly, one could simply estimate P(e) for source j,

$$Est(P(e)) = (1/2)(P(a_j) + P(b_j)) \ , \ j=1,2.$$

But, this subsumes equal weight assignments to $P(a_j)$ and $P(b_j)$. On the other hand, we might agree that we do not know what appropriate weights to assign these values (perhaps the interval of possible resulting values should not be considered to have uniform weighting, but possibly some biased-type emphasizing upper bounds as more important). Instead, we could attempt to determine the weights adaptively by beginning with some reasonable prior $w_{o,j}$ assigned to $P(b_j)$ and $1-w_{o,j}$ assigned to $P(a_j)$, yielding back the weighted average, say

$$Est_1(P(e_j)) = (1-w_{o,j}) \cdot P(a_j) + w_{o,j} \cdot P(a_j) \ .$$

By succesively resubstituting these estimates back as weights and computing the resulting weighted averages, it follows that the resulting sequence of values always converges to the same value, independent of intial estimate $w_{o,j}$, and the resulting limit is

$$Est_\infty(P(e_j)) = P(a_j | b'_j \vee a_j) \ , \ j=1,2.$$

(See [7], pp. 151,152 and the related discussion later in Section 3 (iv).) In turn, if we seek to find the estimate of e relative to the conjunction of the event intervals $[a_1,b_1]$ and $[a_2,b_2]$, do we consider the class intersection

$$[a_1,b_1] \cap [a_2,b_2] = \begin{cases} [a_1 \vee a_2 \ , \ b_1 \cdot b_2], \text{ if } a_1 \vee a_2 \leq b_1 \cdot b_2 \\ \emptyset \ ; \text{ if otherwise} \end{cases} ,$$

or perhaps, the "functional image extension" of event intersections or conjunctions (for definition of functional image extension in a rigorous setting, see Section 3(iii); here, the term refers to component-wise application of operations)

$$[a_1,b_1] \cdot [a_2,b_2] = [a_1 \cdot a_2 \ , \ b_1 \cdot b_2 \ ] \ ?$$

Which is more natural ?

In any case, the above problem is equivalent to assuming, in effect, interval $[a_j, b_j]$ is identifiable as "conditional event" $(a_j | b'_j \vee a_j)$ and the choice of particular conditional event conjunction operation to be made for $(a_1|b'_1 \vee a_1)\cdot(a_2|b'_2 \vee a_2)$, and subsequently, for $P((a_1|b'_1 \vee a_1)\cdot(a_2|b'_2 \vee a_2))$. (All of this will be shown later.)

More generally, the problem of representing, logically manipulating, and evaluating probabilistically, event intervals in the above sense can be seen to be the same as identifying conditional events as intervals of events and the determination of the most appropriate conditional event algebra.

Problem 4. Given a probability space (Ω, A, P) and a random variable (rv) $Y:(\Omega,A) \to (\mathbb{R},\mathcal{B})$ (the last measurable space is the real numbers with the usual borel field of associated subsets of Ω). Suppose that two experiments are made with experiment j corresponding to a fixed $a_j < b_j \in \mathcal{B}$, so that

$$\begin{cases} \text{success occurs if } Y \text{ is in } a_j \,, \\ \text{failure occurs if } Y \text{ is in } a'_j \cdot b_j, \\ \text{uncertain outcome occurs, if } Y \text{ is in } b'_j \,, \; j=1,2. \end{cases}$$

Using either an iterated weighting argument as in Problem 3, or a fair betting argument - where say 1 unit of payoff is assigned to success, 0 for failure, and $p(a_j|b_j)$ for the uncertain outcome (see, e.g., McGee [5]) - or a coherency argument in the sense of De Finetti as employed by Gilio et al [15],[17] or Coletti et al. [18], one obtains

$$E(\text{success level for expt } j) = 1\cdot P(a_j) + 0\cdot P(a'_j b_j) + p(a_j|b_j)\cdot p(b'_j)$$
$$= p(a_j|b_j) \;, \; j=1,2 \;.$$

In turn, how does one compute, e.g., the conjunction

$$E(\text{success level for expt 1 and 2}) \quad ?$$

This problem is related to obtaining $P((a_1|b_1)\cdot(a_2|b_2))$, provided one can compute the conjunction of such conditional events.

Problem 5. Let P be a prior probability measure on the measurable space (Ω, A). If the new information is in the form of an element $b \in A$, the usual bayesian updating procedure consists of replacing P by P_b which is, again, a probability measure on A (or equivalently, on the trace of A on b, $A \cdot b = \{xb : x \in A\}$) defined by

$P_b(a) = P(a|b) = P(ab)/P(b)$, for all $a \in A$ with $p(b) > 0$. Suppose, instead of b, we learn a new rule of the form $c \to b$. If $c \to b = c' \vee b$, then the bayesian updating probability measure will be $P_{c' \vee b}$. It is not clear what the updating probabilities should be in the case where conditional probability evaluations are used (in view again of the Lewis triviality result [6]). If a mathematical object $c \to b$, denoted as $(b|c)$, can be found as an event in some other boolean (or sigma-) algebra besides A, then we can extend bayesian updating to the case of conditional information in a fully satisfactory way.

Problem 6. The issue in Problem 5 also appears in the analysis of evidence, based upon the Dempster-Shafer theory of belief functions (Shafer [19]). Specifically, from the knowledge of P(a), one can construct the mass assignment function m on Ω (assumed finite for simplicity) by

$$m(b) = \begin{cases} P(a), & \text{if } b = a, \\ 1-P(a), & \text{if } b = \Omega, \\ 0, & \text{if otherwise}, \end{cases}$$

for any $b \subseteq \Omega$.

The above construction can be carried out when we learn a,b, and $P(a|b)$, if the object $(a|b)$ can be identified as some subset of a set bigger than Ω.

Problem 7. Let R be a relational database with entries of three types: 1 to affirm the occurrence or satisfaction of a given combination of attributes; 0 to indicate the negation of occurrence of that combination of attributes; and u to indicate "nulls", i.e., missing or unknown data relative to that situation. Can a consistent calculus of operations be developed - compatible with probability evaluations - for representing and logically manipulating such databases? Date [20], in effect, proposed the use of Lukasiewicz logic (without explicitly recognizing it) as a way for logically combining such entries. However, other three-valued logics are possibly justified, depending on the rationale chosen. (See, e.g., Rescher [21] for background on this area.) Moreover, it has recently been established ([7], chapter 3) that each three-valued logic corresponds to the semantics of a "conditional event algebra" (this term will be clarified later). This fact, together with the interpretation of each database as actually being the three-valued indicator function of a uniquely determined conditional event, points to the use of conditional event algebra as a guideline for developing calculi of logical operators for databases.

Problem 8. The following problem may have only theoretical interest, but the hope is that it will open up a wide range of applications due to its fundamental thrust: The idea of a "conditional random variable" is not a standard one in the literature, with only a handful of individuals even using the term informally. (See Wilks [12] again as the key example of this.) But, as appealing intuitively the concept may be, a rigorous definition is yet to be made. (In [7], a preliminary attempt was made in developing this concept, but a number of ad hoc assumptions had to be made, in effect, to take into account the non-booleanness of the interval approach, which at the time was the only available approach. All of these difficulties disappear with the new boolean "α-form" approach taken in Section 4.)

Since events can be identified as special random variables, via their ordinary indicator functions, it is anticipated that once the concept of conditional events is well-established, a rigorous definition of a conditional random variable can be obtained. Also, if this idea can be made rigorous, we can procede to determine induced conditional probability measures and their joint distributions, using standard measure theory.

In examining the above problems, we realize the need to develop further probability theory, especially in defining conditional events and their logical operations. For Problems 5,6, and 8, we also need a boolean (or sigma)algebra form for conditional events.

The detailed solutions to a number of the problems will be given in Section 5, while sketches of directions of action will be indicated for the more obvious ones, once the key tools are provided in Sections 3 and 4.

3. CONDITIONAL INFORMATION AS CONDITIONAL EVENTS FROM THE EVENT INTERVAL VIEWPOINT

As in the Introduction, we use a boolean or sigma algebra A of subsets of a set Ω to denote propositions of interest in some knowledge domain. A probabilistic knowledge is described by a probability measure P on (Ω,A). Since

$$P(b' \vee a) = 1-P(b)+P(ab) = P(a|b) + P(b')\cdot P(a'|b) \geq P(a|b) ,$$

with strict inequality holding in general, unless the trivial cases hold

$$P(b) = 1 \quad \text{or} \quad P(ab) = P(b) \ ,$$

the quantification of rules by the material conditional is not compatible with conditional probability. (The above elementary-appearing relation surprisingly is not well-known. See, e.g., [7] for a history of it.)

In eliciting an expert's knowledge to construct rule-based systems, the expert's degree of belief in each rule is often interpreted as a conditional probability. This is in agreement with the thesis concerning common-sense reasoning; this view of assigning numerical values to conditional statements is also common in statistical applications. For example, one would be hard-pressed not to agree that a reasonable interpretation of "Birds fly" is that among the population of birds (subsuming a reasonable experience with seeing or reading about different birds), only a small percentage cannot fly. This is, of course, the standard empirical estimate of the conditional probability of the statement above.

In all of the following, subsequent rules of the form "if b then a" will be needed compatibly with conditional probability evaluations, i.e., formally, as stated previously,

$$P(b \rightarrow a) = P(a|b) = P_b(a) = P(ab)/P(b) \ ; \ P(b) > 0 \ .$$

Also, as mentioned before, if such an object $b \rightarrow a$ existed, by the Lewis triviality result, it cannot be an element in A. (For various proofs, see here Appendix, Theorem 1 and the proof following Theorem 3, or see [7], chapter 1.)

The above negative result, however, *does not rule out the possibility of modeling* $b \rightarrow a$ *in a well-defined way, compatible with all conditional probability evaluations !* It merely states that if we are going to search for an object representing $b \rightarrow a$, we must look "outside" of A itself. Being outside of A does not necessarily mean $b \rightarrow a$ is not an event, i.e., it does not imply that $b \rightarrow a$ is not an element of some other larger boolean algebra. This observation is important for mathematical investigations of this problem, since it seems that because of the triviality result, logicians and philosophers, as well (Adams [4], Jackson [8], Harper et al. [22], Sanford [23]) leave conditional statements as primitives in a natural language or formal language setting, rather than seraching for a concrete counterpart of boolean algebra.

Lewis' triviality result appears to be indeed the key stumbling block to progress in developing conditional events, compatible with conditional probability. Its impact on the field of logic and philosophy has been enormous. (See again the above cited references). However, it is the opinion of these authors that the reaction to Lewis' significant result has been perhaps too strong, resulting in a lack of attempts at changing his hypothesis through simple natural modifications to the underlying algebraic structure. Lewis himself proposed a basic way to avoid the problem through the use of "imaging", whereby a probability measure P relative to a set a, instead of increasing by the normalization factor (1/P(a)) over a in forming the conditional P_a, now increases over a by reassignment of additional probability measure outside of a, i.e., in a', to within a , preserving the structure of a probability measure, as does P_a. (See the excellent exposition by Gärdenfors [24].)

On the probability and statistics side, Boole appeared to be the first to consider the concept of conditional events under the form of a "division" of events [25]. Boole's book is definitely the thought-provoking source for development of a rigorous theory of probability. However, Kolmogorov did not find it useful to rigorize Boole's division of events. Perhaps, as far as statistics and probability applications are concerned, such a concept shed no actual new light. (See also the series of papers by Koopman [11],[26], e.g.) Or, perhaps, Boole's work on this topic was not rigorous enough. Jevons [27] and Schröder [28] who were among the main codifiers of modern boolean algebra, omitted the concept entirely, though Hailperin [29] over a hundred years later showed Boole's ideas could be made rigorous. However, in effect, Hailperin - as Boole- did not go beyond certain inverse and constrained probability problems in applying the idea. In fact, neither ever developed a calculus for combining conditional events nontrivially, i.e., for the case of differing premises. Nevertheless, Boole's basic idea that an object should exist which, when evaluated by probability measures, yielded back conditional probabilities, is a sound one, and even the existence of these objects in a boolean setting will be demonstrated in Section 4.

In his theory of subjective probabilities, De Finetti [10],[14] identified (a|b) as an object standing alone - but compatible with conditional probability P(a|b) - which has three truth values, namely the same as stated in Problem 4, but without the third value (denoted, by convention here as u) being evaluated. The usefulness of this identification seems to lie in the possibility of using the quantity $(a|b)$ explicitly (as opposed to a primitive form extensively in logic studies, as mentioned earlier). This is exemplified by, e.g., Lindley [30], and later Goodman et al. [31], in approaching probability through a scoring characterization, extending De Finetti's coherency principle. (See also Gilio et al. [17] for additional results.) Mention should also be made of those who, albeit briefly, considered Boole's division of events. These include MacFarlane [32], Whitehead [33], C.I. Lewis [34].

Later, apparently not being aware of the above individuals' work, Schay [35] proposed to define rigorously the concept of a conditional event, equivalent to De Finetti's definition. But, the novel contribution of Schay is that he was the first to propose a full algebraic structure for conditional events. That is, he considered logical operations among them all, not just for the ones having a common antecedent - which is, in reality, no different than the classical situation for conditional probability. (In fact, Schay even developed an extension of the Stone Representation Theorem relative to his conditional event algebras.)

Dubois and Prade [36] in their work on the theory of possibilities for intelligent systems searched for a three-valued logic related to conditional information. In the same year (1987)- some 19 years following Schay's contribution, and 12 years after Adams proposed his conditional event algebra as "quasi-operations" [4]), Calabrese independently also proposed a similar algebra of conditional events, aiming toward computer-oriented applications [37] In view of the new field of AI, Calabrese's work brought some attention to this revival of the subject, followed immediately by Goodman [38] and Goodman & Nguyen [39]. The reaction of the AI community to the foundational work was somewhat positive, but cautious, as exemplified by the work of Dubois & Prade [40], [41], Weber [42], Spies [43], Walker [44] and Nguyen & Rogers [45]. The state of the art of the interval derivation of conditional events and their algebraic structures has been presented by Goodman et al. [7]. (See also Goodman [46] for a detailed history of the development of conditional events and conditional event algebra, up to 1991.)

In the following, we briefly review the essentials of the interval-oriented theory. This will be used, in conjunction with the completely new developments of Section 4, to treat the eight problems discussed in Section 2.

The material presented below can be put in a more general setting as an abstract boolean algebra. However, for definitiveness, we choose the familiar framework of probability theory, in which the boolean algebra is taken to be a sigma-algebra A of ordinary subsets of a set Ω.

The following definition is the basis of the interval approach to conditional events, summarized in [7]:

(i) Definition of conditional events.

For $a,b \in A$, by conditional event "if b,then a" or "a given b", denoted as $(a|b)$, we mean

$$(a|b) = \{x \in A : ab \leqslant x \leqslant b' \vee a\} ,$$

in symbols the (event) interval $[ab,b' \vee a]$. Thus, immediately,

$$(a|b) = (c|d) \quad \text{iff} \quad ab = cd \ \& \ b = d .$$

Remarks.

(a) For the justification of this definition, see [7] or Theorems 4 and 5 of the Appendix here. As mentioned before, an alternative definition - which is not equivalent in general, but does possess a number of common properties - will be presented in Section 4. A third type of approach to conditioning, through use of an extended numerical division operator acting upon ordinary set indicator functions is given in [47] and summarized in [48], and will not be discussed any further here.

(b) There are several equivalent representations of conditional events of the type defined here for any $a,b \in A$:

$$(a|b) = (a,b) \qquad \text{(ordered pair)},$$

$$(a|b) = Ab' \vee ab \qquad \text{(coset form)}$$
$$= \{xb' \vee ab : x \in A\} ,$$

$$(a|b) = \{x \in A : xb = ab\} \text{ (inverse conjunction or "division")},$$

the last being perhaps the most natural or intuitive, as including all those elements in A whose intersection with b is the same as a intersecting b.

(c) The set of all intervals in A is precisely the same as the set of all conditional events of this type for A. Indeed, for any $a,b \in A$,

$$[a,b] = (ab|b' \vee a)$$

(d) Note that $[a,b] \notin A$, but rather, $[a,b] \in P(A)$. Therefore, there is no contradiction with the Lewis triviality result here. It is intuitive that unconditional propositions are special cases of conditional propositions or events. In other words, A should be viewed as a subset of the set of all intervals $[a,b] \in P(A)$, denoted as $(A|A)$, the conditional event space generated by A . This can be done by identifying $[a,a]$ with a . Indeed, compatible with the above comments, note already the reduction of the definition in (i) for $b = \Omega$:

$$(a|\Omega) = \{a\} = [a,a] .$$

(e) Note also the special cases for any $a,b \in A$:

$$(a|a) = (\Omega|a) = [a,\Omega] = A \vee a = \{x \in A: x \geq a\} \text{ (principal filter gen. by a)},$$

$$(\emptyset|b) = (b'|b) = [\emptyset,b'] = Ab' = \{x \in A: x \leq b'\} \text{ (principal ideal gen. by b')}.$$

Also, note

$$(a|b) = (ab|b).$$

(ii) Three-valued logics

Consider the trivial sigma-algebra $V = \{\Omega,\emptyset\}$. Its conditional event space is

$$(V|V) = \{(\Omega|\Omega),(\emptyset|\Omega), (\emptyset|\emptyset)\} .$$

From the logical viewpoint, the truth space of two-valued logics is $\{0,1\}$, representing true and false values. If we view Ω as 1 and $\emptyset$ as 0 , then $(V|V)$ consists of three truth values, true, false, and "undefined", corresponding to $(\Omega|\Omega)$, $(\emptyset|\Omega)$, $(\emptyset|\emptyset)$, respectively. Thus, in this simplest case, it is revealed that conditional events have three truth values.

Now, in the literature of three-valued logics (Rescher [21]), it is amazing that only truth tables are given, as opposed to both truth tables and extended boolean operators, as holds in the two-valued classical case for truth tables and ordinary boolean operators. It is shown (see [7], chapter 3) that the well-known bijective correspondence between semantics (truth tables for logical connectives) and syntactics (boolean operators or boolean poynomials) in two-valued logic can now be extended to the three-valued logic case. This is true not only for V , but also for general A . (See Appendix, Theorem 6 for some details; for more background, see the previous reference to [7].)

(iii) Logical operations among conditional events

In view of the results in (ii), if we choose Lukasiewicz 3-valued logic (see again Rescher [21]), then the algebraic operations among conditional events are exactly the operations among intervals of A in the following order-preserving fashion for $\cdot$ and v, and order switching for $()'$:

$$[a,b]\cdot[c,d] = [ac,bd] \quad ,$$

$$[a,b]v[c,d] = [avc,bvd],$$

$$[a,b]' = [b',a'] \quad ,$$

where by abuse of notation, $\cdot, v, ()'$ now stand for the *functional image extensions* of their counterparts among elements in A. By this, we mean that given any $f:A^2 \to A$, for any $A,B \in P(A)$, define

$$f(A,B) = \{f(a,b): a\epsilon A,\ b\epsilon B\} \quad ,$$

which clearly extends f to $f:P(A)^2 \to P(A)$. Specializing this to f being v or $\cdot$ or $()'$ leads to the above results.

Of course, $()'$ has several properties that are similar to a negation, including

$$[a,b]'' = [a,b] \qquad \text{(involution)}$$

$$([a,b]\cdot[c,d])' = [a,b]' \ v\ [c,d]' \quad \text{(deMorgan)} \ ;$$

but note

$$[a,b]\cdot[a,b]' = [ab', a'b] \neq \emptyset \ (=[\emptyset,\emptyset]),$$

$$[a,b]v[a,b]' = [a\,v\,b', b\,v\,a'] \neq \Omega \ (=[\Omega,\Omega]).$$

However, $(A|A)$ does not have a true complement, since for any $a \leq b\ \epsilon A$, there is no $[c,d], c,d\ \epsilon\ A$, such that

$$[a,b]\cdot[c,d] = \emptyset \ \text{ and } \ [a,b]v[c,d] = \Omega \ .$$

If we define

$$[a,b]^* = [b',b'] = b' \quad ,$$

then it can be verified that $()^*$ is a pseudocomplementation. Moreover, $()^*$ satisfies the following Stone identity

$$[a,b]^* \ v\ [a,b]^{**} = [\Omega,\Omega] = \Omega \quad ,$$

so that the bounded distributive lattice $(A|A)$ is also a Stone algebra. (For background, see Grätzer [49].) This algebraic structure turns out to be the logic of "rough sets". (See Pawlak [50] and Pomykala and Pomykala [51].)

For ease of calculations, we list the following conditional event counterparts of the interval formulas:

$$\left.\begin{aligned} (a|b)\cdot(c|d) &= (abcd \mid a'b \vee c'd \vee bd), \\ (a|b)\vee(c|d) &= (ab \vee cd \mid ab \vee cd \vee bd), \\ (a|b)' &= (a'|b). \end{aligned}\right\} \tag{1}$$

Note also, the special cases

$$(a|b)\cdot b = ab \quad \text{(modus ponens form)}, \tag{2}$$

$$(ac|b) = (a|cb)\cdot(c|b) \quad \text{(chaining form)}. \tag{3}$$

In addition, one implication operation among conditional events can be taken as the functional image extension of the material conditional:

$$(c|d) \Rightarrow (a|b) = \{y' \vee x : y \in (c|d),\ x \in (a|b)\}$$

$$= (c|d)' \vee (a|b) = (c'd \vee ab \mid c'd \vee ab \vee bd).$$

Finally, the partial order, denoted also as $\leq$, on the lattice $(A|A)$ is given by

$$(a|b) \leq (c|d) \quad \text{iff } (a|b) = (a|b)\cdot(c|d).$$

Note that $\leq$ also takes the following forms:

$$\begin{aligned} (a|b) \leq (c|d) \quad &\text{iff } (c|d) = (a|b) \vee (c|d) \\ &\text{iff } ab \leq cd \ \ \&\ \ b' \vee a \leq d' \vee c \\ &\text{iff } ab \leq cd \ \ \&\ \ c'd \leq a'b. \end{aligned} \tag{4}$$

Remarks.

1. Note that because of the failure of $(A|A)$ to have a true complement, though it is a Stone algebra, it is *not* a boolean algebra.

2. This structure is derived from our selection of Lukasiewicz three-valued logic. If another three-valued logic is chosen - such as Sobocinski's, the algebraic structure will differ. It turns out that, independently, Adams [4]

and Calabrese [37] proposed the same type of conditional event algebra, with both corresponding to Sobocinski logic; previously Schay [35] proposed a similar algebra, but disguised as fragments of two different algebras. In brief, this algebra takes the form

$$\left.\begin{aligned} (a|b)\ v_0(c|d) &= (ab \vee cd \mid b \vee d)\ , \\ (a|b)\ \cdot_0(c|d) &= (abd' \vee cdb' \vee abcd \mid b \vee d)\ , \\ (a|b)' &= (a'|b), \end{aligned}\right\}$$

and possesses the basic property that v here can be directly related to the class intersection of conditional events as intervals - a popular choice of operation in interval algebra (see e.g., Igoshin [52]):

$$(a|b) \cap (c|d) = \delta \cdot ((a|b)\ v_0(c|d))\ ,$$

where

$$\delta = \begin{cases} \Omega\ , & \text{if } (a|b) \cap (c|d) \neq \emptyset \text{ iff } ab \vee cd \leq (b' \vee a)\cdot(d' \vee c), \\ \emptyset\ , & \text{if otherwise}\ . \end{cases}$$

The above algebra also forms separate (but related) semi-lattices relative to • and v ([7], section 3.5). (See also [37] for other properties.)

3. For ease of reference, let us denote the order-preserving, functional image derived algebra as GNW and the above algebra as SAC, apropos to the authors' initials.

4. In any case, no choice of conditional event algebra for this interval setting can possibly lead to a boolean algebra, since it can be shown the only Stone algebra here is the GNW one, which is not boolean. (This follows from the use of the identifications of all conditional event algebras with three-valued logics, as in [7], Sections 3.4, 3.5 ; also this has been verified by personal communication with E.A. Walker, New Mexico State University.) This fact will be used as a basis for the development of another direction for conditional events in the next section.

(iv) Probabilities of conditionals

Unlike the assignment of probability values to conditional statements in natural language (Adams [4]), we are able now to define rigorously the probabilities of conditional events. Specifically, for any probability P on (Ω, A), we assign the probability of a conditional event to be the natural conditional probability

$$P((a|b)) = P(a|b) ,$$

when $P(b) > 0$. This assignment is well-defined in the sense that:

If $(a|b) = (c|d)$, then $P(a|b) = P(c|d)$ (for $P(b), P(d) > 0$).

This assignment is order-preserving. That is:

If $(a|b) \leq (c|d)$, then $P(a|b) \leq P(c|d)$.

Also, conversely, if for all probability measures P,

$$P(a|b) \leq P(c|d) ,$$

then one of the following possible cases must hold:

(I) $ab = \emptyset$, in which case $P(a|b) = 0$,

or

(II) $cd = d$, in which case $P(c|d) = 1$,

or

(III) $(a|b) \leq (c|d)$.

(See also Appendix, Theorem 2.)

This order-preserving property is in agreement with the assignment of truth values to conditional events as in the use of De Finetti's or Schay's three-valued indicator function, i.e., if $\phi(a|b): \Omega \to \{0, u, 1\}$ is that function with $0 < u < 1$ (see Appendix, Theorem 6 and definition prior to it), then

$$(\text{For all } \omega \in \Omega)(\phi(a|b)(\omega) \leq \phi(c|d)(\omega)) \text{ iff } (a|b) \leq (c|d).$$

Another rationale for probability assignment is the iterated weighting scheme, where one first interprets $P((a|b))$ in functional image form:

$$P((a|b)) = P[ab, b' \vee a] = \{P(x) : x \in [ab, b' \vee a]\}$$
$$\subseteq [P(ab), P(b' \vee a)] ,$$

with end-point values achieved. From this, a nominal value w_0 can be chosen to yield the weighted end-point combination

$$w_1 = (1-w_0) \cdot P(ab) + (1-w_0) \cdot P(b' \vee a) ,$$

and successively, the resulting values resubstituted as new weights

$$w_n = (1-w_{n-1})\cdot p(ab) + w_n \cdot p(b' \vee a) ,$$

whence

$$\lim_{n\to\infty} w_n = P(a|b) ,$$

no matter what w_0 is taken to be. (See again [7].)

With the probability assignment established, one notes that, e.g. eqs.(1)-(3) yield

$$P((a|b)\cdot(c|d)) = P(abcd \mid a'b \vee c'd \vee bd),$$

$$P((a|b)\vee(c|d)) = P(ab \vee cd \mid ab \vee cd \vee bd),$$

$$P((a|b)') = P(a'|b) = 1 - P(a|b) ,$$

$$P((a|b)\cdot b) = P(ab) = P(a|b)\cdot P(b) ,$$

etc.

Also note the property of the zero-type elements of $(A|A)$,

$$P(a|b) = 0 \quad \text{for all } P \quad \text{iff } (a|b) = (\emptyset|b) \quad \text{iff } ab = \emptyset ,$$

and the analogous property of the unity-type elements

$$P(a|b) = 1 \quad \text{for all } P \quad \text{iff } (a|b) = (b|b) \quad \text{iff } ab = b \quad (b \neq \emptyset).$$

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