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The Complexity of PDL with Interleaving

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Abstract. To provide a logic for reasoning about concurrently executing programs, Abrahamson has defined an extension of propositional dynamic logic (PDL) by allowing interleaving as an operator for combining programs, in addition to the regular PDL operators union, concatenation, and star. We show that the satisfiability problem for interleaving PDL is complete for deterministic double-exponential time, and that this problem requires time double-exponential in $cn/\log n$ for some positive constant c. Moreover, this lower bound holds even when restricted to formulas where each program appearing in the formula has the form $a_1 \mid a_2 \mid \ldots \mid a_k$ where $\mid$ denotes the interleaving operator and where $a_1, \ldots, a_k$ are regular programs, i.e., programs built from atomic programs using only the regular operators. Another consequence of the method used to prove this result is that the equivalence problem for regular expressions with interleaving requires space $2^{cn/\log n}$ and that this lower bound holds even to decide whether $(E_1 \mid E_2 \mid \ldots \mid E_k) \cup F \equiv \Sigma^*$ where $E_1, \ldots, E_k, F$ are ordinary regular expressions; this improves a previous result of the authors. Moreover, the same lower bound holds for the containment problem for expressions of the form $E_1 \mid E_2 \mid \ldots \mid E_k$.

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1 Introduction

Propositional dynamic logic (PDL) was defined by Fischer and Ladner [4] as a formal system for reasoning about programs. It is a propositional version of a first-order dynamic logic introduced earlier by Pratt [15]. In PDL, the regular operators (union, concatenation, and Kleene star) are used as operators for constructing programs. If a and b are programs, then $a \cup b$ means to nondeterministically run either a or b, $a;b$ (e.g., concatenation of programs) means to run a followed by b, and a^* means to run a any finite number of times. The satisfiability problem for PDL is known to be complete for deterministic exponential time [4, 16]. The effect on complexity of using different formalisms for writing programs has been studied, for example, by Abrahamson [1] for programs with Boolean variables, and by Harel, Rosner, and Vardi [7] for programs specified by finite-state automata using various concurrency mechanisms such as existential branching, universal branching, and bounded cooperative (e.g., communicating) concurrency. A recent survey on logics of programs, including PDL, is given by Kozen and Tiuryn [11]. The survey by Harel [6] concentrates on complexity and decidability for variants of PDL.

To permit reasoning about concurrently executing programs, Abrahamson [1] has extended PDL by including interleaving as an operator on programs. For example, if a, b, and c are atomic programs, possible executions of the program $a^* \mid (b;c)$ are *abac*, *bcaaa*, and *baaaca*. A complete definition of interleaving PDL (IPDL) appears in Section 3. Building on Fischer and Ladner's [4] nondeterministic exponential time decision procedure for PDL, Abrahamson [1] shows that the satisfiability problem for IPDL can be decided in nondeterministic double-exponential time. Using a result of Pratt [17] and Harel and Sherman [8], the upper bound can be improved to deterministic double-exponential time. Our main result is that the satisfiability problem for IPDL is complete for deterministic double-exponential time (2-EXPTIME), and that a lower bound on time is double-exponential in $cn/\log n$ for some constant $c > 0$. Moreover, to prove the lower bound, we do not need the full power of IPDL which allows the interleaving operator to be arbitrarily nested with the other operators. The lower bound holds even when restricted to formulas where each program appearing in the formula has the form $a_1 \mid a_2 \mid \ldots \mid a_k$ where $\mid$ denotes the interleaving operator and where $a_1, \ldots, a_k$ are regular programs, i.e., programs built from atomic programs using only the regular operators – union, concatenation and star.

As noted above, Harel, Rosner and Vardi [7] have previously studied the complexity of PDL under various models of concurrency. Among the many results in [7], the one which is closest in spirit to our result is that PDL is complete for 2-EXPTIME if programs are specified by *concurrent automata*. There are, however, differences between the concurrent automata model and the interleaving model. One difference is that we use expressions while [7] uses automata (and it is known that automata can express certain languages much more succinctly that expressions [3]). Another difference is that the concurrent automata model corresponds to synchronous concurrent execution with communication, whereas the interleaving model corresponds more closely to asynchronous concurrent execution without communication.

The proof of our lower bound rests on showing how regular expressions with interleaving can succinctly encode Turing machine computations. Using the same encoding, we improve a result of [12]. The *Non-Empty Complement* (NEC) problem for a class of expressions

is the problem of deciding, for a given expression E over alphabet Σ, whether E does not describe all words in Σ^*. It is shown in [12] that the NEC problem is exponential-space-complete for regular expressions with interleaving, and that this problem requires space $2^{c\sqrt{n}}$ for some constant $c > 0$. Here, we improve this result in two ways: first, the lower bound is improved to $2^{cn/\log n}$; second, the lower bound holds even for expressions of the form $(E_1 \,|\, E_2 \,|\, \ldots \,|\, E_k) \cup F$ where $E_1, \ldots, E_k, F$ are ordinary regular expressions. The best known upper bound is space $2^{O(n)}$, so there is still a gap. Also open is the computational complexity of the NEC and equivalence problems for expressions of the form $E_1 \,|\, E_2 \,|\, \ldots \,|\, E_k$. We do show, however, that the containment problem for expressions of this form is exponential-space-complete.

2 Encoding Turing Machine Computations by Regular Expressions with Interleaving

We assume familiarity with regular expressions and time and space complexity; see, e.g., [9] or [20] if needed.

The *interleaving* of words x and y, denoted $x|y$, is the set of all words of the form

$$x_1 y_1 x_2 y_2 \ldots x_k y_k$$

where $x = x_1 x_2 \ldots x_k$ and $y = y_1 y_2 \ldots y_k$ and where the words x_i and y_i, $1 \leq i \leq k$, can be of arbitrary length (including the empty word). If X and Y are sets of words, then $X \,|\, Y$ is the union of the sets $x|y$ over all $x \in X$ and $y \in Y$. An *interleaving expression* is a regular expression which can contain the interleaving operator, in addition to the usual operators union, concatenation and star. The language $L(E)$ described by an interleaving expression E is defined recursively in the obvious way; in particular, $L(E_1 \,|\, E_2) = L(E_1) \,|\, L(E_2)$. By a *regular expression* we mean a regular expression as usually defined, containing only union, concatenation and star. Say that an interleaving expression E is a *top-level concurrent expression* if

$$E = E_1 \,|\, E_2 \,|\, \ldots \,|\, E_k$$

for some $k \geq 1$ and some regular expressions $E_1, \ldots, E_k$.

We define below a particular encoding of a Turing machine computation as a word over a finite alphabet. We then show how to construct, for any nondeterministic Turing machine M with space bound $2^{p(n)}$ for some polynomial $p(n)$ and any input x, an interleaving expression I such that $L(I)$ contains precisely the words which do not encode accepting computations of M on input x. Moreover, I has the form $E \cup F$ where E is a top-level concurrent expression and F is a regular expression, and the length of I is $O(p(n) \log n)$ where n is the length of x (and where the constant factor implicit in the O-notation depends on M).

Before getting into the details, it is useful to explain the main idea by a simple example. A key part of the construction is a top-level concurrent expression which can identify identical subwords in a long word, provided that the long word has a particular restricted format. We illustrate how this is done. Let D be a finite alphabet, and let $b, c_0, c_1, \ldots, c_{m-1}$ be symbols not in D. If u is a word with length divisible by m, say $u = u_0 u_1 \ldots u_{z-1}$ where m divides z, let $h(u)$ be the word obtained by placing the symbol $c_{i \bmod m}$ before u_i for all

i. Words in the restricted format are those in $R = h((D^m b^m)^*)$. View a word in R as a concatenation of blocks, where a *block* is any subword in $h(D^m b^m)$; i.e., a block has the form

$$c_0\, d_0\, c_1\, d_1\, \ldots\, c_{m-1}\, d_{m-1}\, c_0\, b\, c_1\, b\, \ldots\, c_{m-1}\, b$$

for some $d_0, \ldots, d_{m-1} \in D$. Let P be the set of words $w \in R$ such that (at least) two blocks of w are identical. We claim that the following top-level concurrent expression A of length $O(m)$, when restricted to words in R, describes precisely the words having two identical blocks, i.e., $L(A) \cap R = P$.

$$A = A_0 \,|\, A_1 \,|\, \ldots \,|\, A_m$$

where, for $0 \le k \le m-1$,

$$A_k = \bigcup_{d \in D} c_k \cdot d \cdot c_k \cdot b \cdot c_k \cdot d \cdot c_k \cdot b$$

and

$$A_m = (c_0 \cdot D \cdot c_1 \cdot D \cdots c_{m-1} \cdot D \cdot c_0 \cdot b \cdot c_1 \cdot b \cdots c_{m-1} \cdot b)^*.$$

It is easy to see that every word $w \in P$ belongs to $L(A)$: if we imagine that w is scanned from left to right, the two occurrences of the repeated block, say $h(d_0 d_1 \ldots d_{m-1} b^m)$, are "parsed" to $A_0, A_1, \ldots, A_{m-1}$ where the d in the union for A_k matches d_k; the other blocks are parsed to A_m. In the other direction, suppose that $w \in L(A) \cap R$, so $w \in L(w_0 \,|\, w_1 \,|\, \ldots \,|\, w_{m-1})$ where $w_k \in L(A_k)$ for all k. A key observation is that each block of w must be either parsed entirely to w_m, or parsed entirely to $w_0, \ldots, w_{m-1}$. If the observation does not hold, consider the first block for which it fails. If we start by parsing this block to w_0, then w_m cannot be used later in parsing this block since the part of w_m that has not been used yet begins with $c_0 d$ for some $d \in D$. In the other case, suppose we start by parsing this block to w_m, but switch to w_k when parsing the subword $c_k d$, where $k \ge 1$ and $d \in D$. Later we have to parse the subword $c_{k-1} b$. We cannot parse this subword to w_m since the part of w_m that has not been used yet begins with c_k. We cannot parse this subword to w_{k-1} since the part of w_{k-1} that has not been used yet begins with $c_{k-1} d$ for some $d \in D$. Given this observation, it is easy to see that w must have two identical blocks, namely, the two blocks that are parsed entirely to $w_0, w_1, \ldots, w_{m-1}$.

We now return to the details. Let M be a nondeterministic Turing machine with space bound $2^{p(n)}$, where $p(n)$ is a polynomial and $p(n) \ge n$. Fix an input x and let n be the length of x. Let $l = \lceil \log_2 p(n) \rceil$ and $m = 2^l$, and note that $p(n) \le m \le 2p(n)$. Let $s = 2^m$, so s is at least as large as the space bound $2^{p(n)}$. Let Q be the set of states of M and let T be the tape symbols. An ID of M is a word of length $s+1$ in T^*QTT^*. The meaning of the ID $\alpha q \sigma \beta$ where $\alpha, \beta \in T^*$, $\sigma \in T$, and $q \in Q$, is that $\alpha\sigma\beta$ is written on the tape and M is in state q with the head scanning σ. It will be useful to use a redundant representation of an ID. If

$$a_i = a_{i,0} a_{i,1} \ldots a_{i,s}$$

is the ith ID in a computation, this is represented by the word

$$b_{i,1} b_{i,2} \ldots b_{i,s-1}$$

where $b_{i,j} = (a_{i,j-1}, a_{i,j}, a_{i,j+1})$ for $1 \le j \le s-1$. Let $\Delta = (Q \cup T)^3$ be the alphabet of symbols used in the redundant representation.

As in [5], we use "marked binary numbers" to index the symbols of an ID. A *marked binary number* is a word over the alphabet $\{0, \underline{0}, 1, \underline{1}\}$ in the language described by the expression $(0 \cup 1)^* \underline{1}\, \underline{0}^* \cup \underline{0}^*$; i.e., the rightmost (lowest order) 1 is marked, as well as all 0's to the right of this 1; and in the representation of 0, all 0's are marked. For $0 \leq k \leq m-1$, let $[k]$ denote the length-l marked binary representation of k. Call these the *low-level* numbers. For $0 \leq j \leq s-1$, let $[[j]]$ denote the length-m marked binary representation of j. Call these the *high-level* numbers. It is useful to use different symbols for the digits in the two types of numbers, say, $\{0, 1, \underline{0}, \underline{1}\}$ for the low-level and $\{0', 1', \underline{0}', \underline{1}'\}$ for the high-level. The marking allows the successor relation to be tested locally as follows. Define $\mathrm{succ}(0) = \mathrm{succ}(\underline{0}) = \{0, \underline{1}\}$ and $\mathrm{succ}(1) = \mathrm{succ}(\underline{1}) = \{1, \underline{0}\}$. If $y_1 \ldots y_l = [k]$, and $z = z_1 \ldots z_l$ is a marked binary number of length l, then $z = [k+1 \bmod m]$ iff $z_i \in \mathrm{succ}(y_i)$ for $1 \leq i \leq l$. Similarly, the successor relation for the high-level numbers can be checked locally.

The idea is that we use high-level numbers to number the symbols of ID's obtaining some word a', and then use the low-level numbers to number the symbols of a'. The low-level numbering is done as follows. If w is a word with length divisible by m, say $w = \sigma_0 \sigma_1 \ldots \sigma_{z-1}$ where m divides z, let $g(w)$ be the word obtained from w by placing the word $2[i \bmod m]3$ before σ_i for all i. I.e.,

$$g(w) = 2[0]3\,\sigma_0\, 2[1]3\,\sigma_1 \ \ldots\ 2[m-1]3\,\sigma_{m-1}\, 2[0]3\,\sigma_m \ \ldots\ 2[m-1]3\,\sigma_{z-1}\,.$$

(The word $2[k]3$ plays the role of the symbol c_k in the simple example above.)

An accepting computation of M on input x is represented by

$$a = g(a')$$

where

$$\begin{aligned} a' \;=\; & [[0]]\,\#^m\,[[1]]\,b^m_{0,1}\,[[2]]\,b^m_{0,2}\ \ldots\ [[s-1]]\,b^m_{0,s-1} \\ & [[0]]\,\#^m\,[[1]]\,b^m_{1,1}\,[[2]]\,b^m_{1,2}\ \ldots\ [[s-1]]\,b^m_{1,s-1} \\ & \qquad \ldots \\ & [[0]]\,\#^m\,[[1]]\,b^m_{t,1}\,[[2]]\,b^m_{t,2}\ \ldots\ [[s-1]]\,b^m_{t,s-1}\,[[0]]\,\#^m \end{aligned} \tag{1}$$

where, for $0 \leq i \leq t$, the word $b_{i,1} b_{i,2} \ldots b_{i,s-1}$ is the redundant representation of the ith ID in the computation of M on input x, and the accepting state appears in $b_{t,1} b_{t,2} \ldots b_{t,s-1}$.

A word a has the *correct framework* if $a = g(a')$ for some a' of the form (1) where the $b_{i,j}$ can be any symbols of Δ which are locally consistent within the same ID, i.e., if $b_{i,j} = (\sigma_1, \sigma_2, \sigma_3)$, then $b_{i,j+1} = (\sigma_2, \sigma_3, \sigma_4)$ for some σ_4.

There is a regular expression F of length $O(p(n) \log n)$ which describes all words which (1) do not have the correct framework, or (2) do not contain the accepting state, or (3) $b_{0,1} \ldots b_{0,s-1}$ does not represent the initial ID on input x. The construction of F is fairly straightforward, although tedious, using standard methods as in, for example, [5, 13, 18, 19]. F is written as a union of "mistakes" which cause a word to violate (1), (2), or (3). The expression has length $O(ml)$, i.e., length $O(p(n) \log n)$, since each type of mistake involves making local checks in a where the region of locality has length $O(ml)$. For example, letting $D = \{0', 1', \underline{0}', \underline{1}'\}$ be the digits used in high-level numbers and $\Sigma = \Delta \cup D \cup \{0, 1, \underline{0}, \underline{1}, 2, 3, \#\}$

be the entire alphabet, the following expression describes the mistake that the high-level numbers are not incremented correctly:

$$\bigcup_{d \in D} \Sigma^* \cdot d \cdot \Sigma^{2m(l+3)-1} \cdot (D - \mathrm{succ}(d)) \cdot \Sigma^* \ .$$

As one more example, we write an expression of length $O(n)$ which (given that a has the correct framework) describes the mistake that $b_{0,1} \ldots b_{0,s-1}$ is not the redundant representation of the initial ID, $q_0 x B^{s-n}$, where B is the blank tape symbol. Let $c_1, c_2, \ldots, c_{n+1} \in \Delta$ be such that the redundant representation of the initial ID is $c_1 c_2 \ldots c_{n+1}(B,B,B)^{s-n-2}$. Let $G = \{0, 1, \underline{0}, \underline{1}, 2, 3\}$, $\bar{D} = D \cup G$, $\bar{\Delta} = \Delta \cup G$, and $\bar{\#} = \{\#\} \cup G$. Let S^+ abbreviate SS^*. The expression is

$$\bar{D}^+ \cdot \bar{\#}^+ \cdot \bar{D}^+ \cdot ((\Delta - \{c_1\}) \cup c_1 \cdot \bar{\Delta}^+ \cdot \bar{D}^+ \cdot ((\Delta - \{c_2\}) \cup c_2 \cdot \bar{\Delta}^+ \cdot \bar{D}^+ \cdot (\ldots \\ \ldots ((\Delta - \{c_{n+1}\}) \cup c_{n+1} \cdot \bar{\Delta}^+ \cdot \bar{D}^+ \cdot (\bar{\Delta} \cup \bar{D})^* \cdot (\Delta - \{(B,B,B)\}))))\ldots) \cdot \Sigma^* \ .$$

The rest of the construction of F is left to the interested reader.

The more interesting part of the construction is an expression E which describes the mistake that the symbols $b_{i,j}$ for $i \geq 1$ do not correspond to a computation of M. For $b \in \Delta$ where b contains at most one occurrence of a state symbol, let $N(b)$ be the set of triples which could occur in the next ID at the same position as b. We construct a top-level concurrent expression E such that, when restricted to words a having the correct framework, $a \in L(E)$ iff $b_{i+1,j} \notin N(b_{i,j})$ for some i and j. The expression will identify pairs $(b_{i,j}, b_{i+1,j})$ using the fact that the same high-level number $[[j]]$ precedes both $b_{i,j}$ and $b_{i+1,j}$, and that there is exactly one occurrence of $g(\#^m)$ between them. Recall that $D = \{0', 1', \underline{0}', \underline{1}'\}$.

$$E = E_0 \mid E_1 \mid \ldots \mid E_m$$

where, for $0 \leq k \leq m-1$,

$$\begin{aligned} E_k \ = \ & \bigcup_{d \in D} \bigcup_{b \in \Delta} (2[k]3 \cdot D \cdot 2[k]3 \cdot \#)^* \\ & \cdot (2[k]3 \cdot d \cdot 2[k]3 \cdot b \cdot 2[k]3 \cdot D \cdot 2[k]3 \cdot \# \cdot 2[k]3 \cdot d \cdot 2[k]3 \cdot (\Delta - N(b))) \\ & \cdot (2[k]3 \cdot D \cdot 2[k]3 \cdot \#)^* \end{aligned}$$

and

$$E_m = (2[0]3 \cdot D \cdot 2[1]3 \cdot D \cdots 2[m-1]3 \cdot D \cdot 2[0]3 \cdot \Delta \cdot 2[1]3 \cdot \Delta \cdots 2[m-1]3 \cdot \Delta)^*.$$

We now argue that E has the required property. Assuming that a has the correct framework, let a *block* of a be any subword of the form

$$2[0]3\, d_0\, 2[1]3\, d_1 \ldots 2[m-1]3\, d_{m-1}\, 2[0]3\, \sigma\, 2[1]3\, \sigma \ldots 2[m-1]3\, \sigma$$

for some $d_0, \ldots, d_{m-1} \in D$ and $\sigma \in \Delta \cup \{\#\}$. In other notation, a block has the form $g([[j]]\, \sigma^m)$ for some j and σ. A *#-block* is a block where $\sigma = \#$.

The easier direction is the case where a is such that $b_{i+1,j} \notin N(b_{i,j})$ for some i and j. Imagining that a is scanned from left to right, we describe how a is "parsed" to $E_0, E_1, \ldots, E_m$. In the expression E_k, we refer to the first (resp., last) occurrence of the

subexpression $(2[k]3 \cdot D \cdot 2[k]3 \cdot \#)^*$ as the first (resp., last) part of E_k, and we refer to the rest of E_k as the middle part. Each non-#-block is parsed to E_m and each #-block is parsed to the first parts of $E_0, \ldots, E_{m-1}$, until we reach the block $g([[j]]\, b^m_{i,j})$. At this point, $g([[j]])$ is parsed to the middle parts of $E_0, \ldots, E_{m-1}$, where the d in each such expression matches the corresponding digit of $[[j]]$. Then $g(b^m_{i,j})$ is parsed to the middle parts of $E_0, \ldots, E_{m-1}$, where b matches $b_{i,j}$. The following non-#-blocks are parsed to E_m, the next #-block is parsed to the middle parts of $E_0, \ldots, E_{m-1}$, and the following non-#-blocks are parsed to E_m up to the block $g([[j]]\, b_{i+1,j})$. This block is parsed to the middle parts of $E_0, \ldots, E_{m-1}$ (so the middle parts of $E_0, \ldots, E_{m-1}$ are now used up). Each remaining non-#-block is parsed to E_m and each remaining #-block is parsed to the last parts of $E_0, \ldots, E_{m-1}$.

In the other direction, if $a \in L(E)$, it can be seen that this is the only way a parse can proceed. Let $a \in L(w_0 \,|\, w_1 \,|\, \ldots \,|\, w_m)$ where $w_k \in L(E_k)$. A first observation is that each subword $2[k]3\,\sigma$, where $\sigma \in D \cup \Delta \cup \{\#\}$, must be parsed entirely to a single word, either w_k or w_m. A second key observation is that each block must be either parsed entirely to w_m, or parsed entirely to $w_0, \ldots, w_{m-1}$. Given the first observation, the argument for the second observation is exactly as in the simple example above, and we do not repeat it. Consider now the first block of a which is parsed to the middle parts of $E_0, \ldots, E_{m-1}$. (There must be such a block since the middle parts of $E_0, \ldots, E_{m-1}$ must be used.) Say that this block is $g([[j]]\, b^m_{i,j})$. This determines a $d = d_k$ and a $b = b_k$ in the two unions for each E_k $(0 \le k \le m-1)$ where $d_0 d_1 \ldots d_{m-1} = [[j]]$ and $b_k = b_{i,j}$ for all k. The following blocks, up to the next #-block (call this #-block β) must then be parsed to E_m, and β must be parsed to the middle parts of $E_0, \ldots, E_{m-1}$. Now some block γ between β and the next #-block after β must be parsed to the middle parts of $E_0, \ldots, E_{m-1}$, for otherwise there will be no way to parse the next #-block after β. Since the d_k's determine $[[j]]$, we must have $\gamma = g([[j]]\, b_{i+1,j})$. Since $b_k = b_{i,j}$, we must have $b_{i+1,j} \notin N(b_{i,j})$.

This completes the construction of $I = E \cup F$ such that $L(I) \neq \Sigma^*$ iff M accepts x.

Let EXPSPACE denote the class of languages which can be recognized in space $2^{p(n)}$ for some polynomial $p(n)$. Recall that the *non-empty complement* (NEC) problem for a class of expressions is the problem of deciding, given an expression E over alphabet Σ, whether $L(E) \neq \Sigma^*$. In [12] we observe that the NEC problem for interleaving expressions can be solved in space $2^{O(n)}$. From the above construction, we get the following.

Theorem 2.1 *The non-empty complement problem for expressions of the form $E \cup F$, where E is a top-level concurrent expression and F is a regular expression, is EXPSPACE-complete. There is a constant $c > 0$ such that no Turing machine with space bound $2^{cn/\log n}$ can solve this problem.*

Remark. By using the "shuffle resistant" code of Warmuth and Haussler [21] (see also Proposition 3.1 of [12]), this theorem remains true for expressions over a binary alphabet $\Sigma = \{0, 1\}$.

Although it is an open question whether the NEC or equivalence problems are EXPSPACE-complete for top-level concurrent expressions, it follows from the above that the containment problem (i.e., deciding for given expressions R_1 and R_2 whether $L(R_1) \subseteq L(R_2)$) is EXPSPACE-complete for top-level concurrent expressions.

Theorem 2.2 *The containment problem for top-level concurrent expressions is EXPSPACE-complete. There is a constant $c > 0$ such that no Turing machine with space bound $2^{cn/\log n}$ can solve this problem.*

Proof. The exponential-space upper bound for the containment problem follows easily (like the exponential-space upper bound for the NEC problem in [12]) by converting the input expressions R_1 and R_2 to equivalent (and exponentially larger) nondeterministic finite-state automata.

To prove EXPSPACE-hardness, let F and $E = E_0 \mid E_1 \mid \ldots \mid E_m$ be the expressions constructed above, where F and the E_i's are regular expressions. Let b and c be symbols not in Σ. Letting

$$\begin{aligned} R_1 &= b^{m+1} \cdot c^{m+1} \cdot \Sigma^* \\ R_2 &= (b \cdot E_0 \cup c) \mid (b \cdot E_1 \cup c) \mid \ldots \mid (b \cdot E_m \cup c) \mid (b^{m+1} \cdot F \cup c^{m+1}), \end{aligned}$$

it is easy to see that $L(E \cup F) = \Sigma^*$ iff $L(R_1) \subseteq L(R_2)$. ∎

Remark. A similarity between the interleaving and the intersection operators is that the NEC and containment problems for regular expressions extended by interleaving have the same complexity as the NEC and containment problems for regular expressions extended by intersection: all of these problems are EXPSPACE-complete. For expressions with intersection, this was first proved by Hunt [10] (see also Fürer [5]). For expressions with interleaving, this was proved by the authors in [12]. In fact, the proof in [12] proceeds by giving a reduction from the NEC problem for expressions with intersection to the NEC problem for expressions with interleaving, by showing how interleaving can "simulate" intersection under certain conditions. In contrast, for expressions in the restricted forms used in Theorems 2.1 and 2.2, replacing interleaving by intersection lowers the complexity of the problem: the NEC and containment problems for expressions of the form

$$(E_1 \cap E_2 \cap \ldots \cap E_k) \cup F,$$

where $E_1, \ldots, E_k, F$ are regular expressions, can be solved in polynomial space. Since these problems contain the NEC problem for regular expressions as a special case (where $k = 1$ and $F = \emptyset$), these problems are PSPACE-complete, since the NEC problem for regular expressions is PSPACE-complete [13].

3 PDL with Interleaving

We review Abrahamsons's [1] definition of PDL with interleaving added as a program constructor. We call this logic *interleaving PDL* to avoid confusion with other definitions of concurrent PDL in the literature, e.g., [7, 14].

We begin with a set Φ_0 of atomic formulas which represent propositional variables and a set Ψ_0 of atomic programs which represent indivisible program steps. Syntactically, if p and q are formulas and a and b are programs, then $p \vee q$ and $\neg p$ are formulas, $\langle a \rangle p$ is a formula meaning "it is possible to run a to reach a state in which p is true," $a \cup b$ is a program meaning "run either a or b," $a; b$ is a program meaning "run a followed by b," a^* is

a program meaning "run a any finite number of times," $p?$ is a program meaning "continue iff p is true," and $a|b$ is a program meaning "run a and b concurrently."

A *model* is a triple $\mathcal{M} = (W, \pi, \tau)$ where W is a set of *states*; for each atomic formula p, $\pi(p) \subseteq W$ is the set of states in which p is true; and for each atomic program a, $\tau(a) \subseteq W \times W$ is the set of state transitions of a. π is extended to all formulas and τ is extended to all programs. In general, τ is a set of *computation sequences*, i.e., a subset of $(W \times W)^*$. To extend τ, the sets $\tau(a \cup b)$, $\tau(a;b)$, $\tau(a^*)$, and $\tau(a|b)$ are obtained from $\tau(a)$ and $\tau(b)$ by union, concatenation, star, and interleaving, respectively. $\tau(p?)$ is the set of all (u,u) such that $u \in \pi(p)$. Note that $\tau(a)$ can contain computation sequences such as $(u,v)(w,z)$ which do not make sense if $v \neq w$ and if a is run alone. We must include such sequences since, if a is interleaved with b, the program b could make the transition from v to w. A computation sequence σ is *legal* if, whenever $(u,v)(w,z)$ is a subword of σ, then $v = w$. To extend π, $\pi(p \vee q) = \pi(p) \cup \pi(q)$ and $\pi(\neg p) = W - \pi(p)$. Finally, $\pi(\langle a \rangle p)$ is the set of states u such that either there exists a state z and a legal computation sequence $\sigma \in \tau(a)$ such that $z \in \pi(p)$ and such that σ has the form $\sigma = (u, \ )\ldots(\ , z)$, or $\epsilon \in \tau(a)$ and $u \in \pi(p)$.

A formula is *test-free* if it contains no occurrence of "?".

A formula φ is a *top-level concurrent formula* if each program appearing in φ has the form $a_1 \,|\, a_2 \,|\, \ldots \,|\, a_k$ for some $k \geq 1$ where $a_1, \ldots, a_k$ are regular programs, i.e., these programs contain no occurrences of the interleaving operator.

A formula φ is *satisfiable* if there is a model $\mathcal{M}$ and a state u such that $u \in \pi(\varphi)$.

Pratt [17] and Harel and Sherman [8] show that the satisfiability problem for PDL (without interleaving) can be decided in deterministic exponential time even if programs are described by nondeterministic finite-state automata (NFA's) instead of regular expressions. By a straightforward cross-product construction (see, e.g., [12, §3]), any regular expression with interleaving can be converted to an exponentially larger NFA. It follows that the satisfiability problem for interleaving PDL belongs to 2-EXPTIME (the class of languages which can be recognized by deterministic Turing machines in time double-exponential in $p(n)$ for some polynomial $p(n)$). This gives the upper bound part of the following theorem.

Theorem 3.1 *The satisfiability problem for interleaving PDL is complete for 2-EXPTIME, even when restricted to top-level concurrent formulas which are test-free. There is a constant $c > 0$ such that no deterministic Turing machine with time bound $2^{2^{cn/\log n}}$ can solve this problem.*

Proof. To prove 2-EXPTIME-hardness, we use expressions similar to the ones constructed in the previous section. But since a model of PDL is a directed graph rather than just a sequence, we can simulate a $2^{p(n)}$ space-bounded alternating Turing machine (ATM) rather than a $2^{p(n)}$ space-bounded nondeterministic Turing machine. This idea was first used by Fischer and Ladner [4] (with a linear rather than an exponential space bound), and has been used in many other papers on the complexity of propositional program logics. Familiarity with the ATM model is assumed [2]. Recall that every language in 2-EXPTIME is accepted by some ATM with space bound $2^{p(n)}$ for some polynomial $p(n)$. Let M be such an ATM. We can assume that M begins in an existential state, existential and universal states alternate at each step, and M has exactly two possible moves at each step.

The expression I of the previous section is modified to describe all strings which do not represent valid computation paths of M on input x. A valid computation path is a sequence

of ID's which begins with the initial ID on input x, ends with an accepting ID, and such that each ID follows from the previous one by the rules of M. Instead of the single inter-ID marker #, we use four markers $\#_c^r$ for $c \in \{1,2\}$ and $r \in \{e,u\}$. The subscript c indicates whether the first or second choice is taken in going from the ID preceding the marker to the ID following the marker, and the superscript r indicates whether the ID preceding the marker is existential or universal.

Therefore, the sequence of marker superscripts in (the representation of) a valid computation path must be $ueueue\ldots$ (it starts with u because a marker precedes the first ID in our construction). The expression F for framework errors contains additional expressions for strings not of this form. Let F' denote F including these additions.

The expression E for computation errors is modified to take into account the subscript c in the unique occurrence of a #-block between the ID containing $b_{i,j}$ and the ID containing $b_{i+1,j}$. More precisely, for a triple $b \in \Delta$, let $N_1(b)$ (resp., $N_2(b)$) be the set of triples which could occur in the next ID at the same position as b, assuming that the first (resp., second) move is taken. Abbreviating $\# = \{\#_1^e, \#_1^u, \#_2^e, \#_2^u\}$ and $\#_c = \{\#_c^e, \#_c^u\}$, the modified E_k is

$$\begin{aligned} E_k' \;=\; \bigcup_{d\in D}\bigcup_{b\in\Delta}\bigcup_{c\in\{1,2\}} & (2[k]3\cdot D\cdot 2[k]3\cdot\#)^* \\ & \cdot(2[k]3\cdot d\cdot 2[k]3\cdot b\cdot 2[k]3\cdot D\cdot 2[k]3\cdot \#_c\cdot 2[k]3\cdot d\cdot 2[k]3\cdot(\Delta - N_c(b))) \\ & \cdot(2[k]3\cdot D\cdot 2[k]3\cdot\#)^*\,. \end{aligned}$$

Let E' denote the expression E after these modifications, i.e.,

$$E' = E_0' \mid \ldots \mid E_{m-1}' \mid E_m\,.$$

Now an interleaving PDL formula φ is constructed so that φ is satisfiable iff M accepts x. The length of φ is $O(p(n)\log n)$. The set of atomic programs is the alphabet Σ used in $E' \cup F'$, and there is one atomic formula P. A model of PDL is essentially a directed multigraph (i.e., there can be multiple edges between two nodes), where each edge (transition) is labeled with an element of Σ, and each node (state) is labeled either P or $\neg P$ depending on whether P is true or false at that state. The graph has a distinguished state u, the state where φ is true. Since all that matters about a model is its reachability structure with respect to labeled paths, it is useful to imagine that the graph has been "unwound" into a directed tree rooted at u with all paths directed away from u. The idea is that the portion of the tree where P is true should contain an accepting computation tree of M on input x. That is, (1) if ρ is a directed path which starts at u and terminates at the point where P first becomes false, then ρ should represent a valid computation path of M (more precisely, the sequence of atomic programs labeling the path ρ should represent a valid computation path), and (2) for each universal ID in the computation tree, both successors of this ID should be in the computation tree.

As usual, let $[a]p$ abbreviate $\neg\langle a\rangle\neg p$; the intuitive meaning of $[a]p$ is "all ways of running a reach a state where p is true."

φ is a conjunction of several components described next. Translations into formal PDL are also given.

1. P is true at u.
   Translation: P.

2. For every state w reachable from u, it is possible to reach from w a state where P is false.
   Translation: $[\Sigma^*](\langle\Sigma^*\rangle\neg P)$.
   (Note: Using Σ^* to define "reachable", a state is always reachable from itself.)

3. There are no invalid computation paths starting at u. That is, it is impossible to run $E' \cup F'$ from u to reach a state where P is false.
   Translation: $\neg\langle E'\rangle\neg P \wedge \neg\langle F'\rangle\neg P$.
   (Note: We break this formula into a conjunction, one part for E' and one part for F', to obtain a top-level concurrent formula.)

4. Both possible moves must be taken after every universal ID. That is, for every state w reachable from u and every $c \in \{1,2\}$, if it is possible to run $g((\#_c^u)^m)$ from w then it is possible to run $g((\#_{3-c}^u)^m)$ from w.
   Translation: $\bigwedge_{c\in\{1,2\}} [\Sigma^*](((\langle g((\#_c^u)^m)\rangle\, true) \rightarrow (\langle g((\#_{3-c}^u)^m)\rangle\, true))$.

∎

Remarks.

(1) By the remark following Theorem 2.1, two atomic programs suffice to prove Theorem 3.1.

(2) The atomic formula P in the construction can be replaced by the formula $(\langle\Sigma\rangle\, true)$, which is true at a state w iff there is some transition out of state w. That is, computation paths terminate at states which have no outgoing transitions. So Theorem 3.1 remains true for formulas containing no atomic formulas.

(3) It follows from [8] that an upper bound on the complexity of IPDL is deterministic time $2^{2^{dn}}$ for some constant $d > 0$. There is a gap between this upper bound and the lower bound of Theorem 3.1.

(4) 2-EXPTIME-completeness holds also for deterministic PDL (DPDL) with interleaving. In DPDL, models (W, π, τ) are restricted to those such that, for every atomic program a, if $(u,v) \in \tau(a)$ and $(u,w) \in \tau(a)$ then $v = w$. (This does not have to hold for non-atomic a, however.) The 2-EXPTIME upper bound follows again from [8]. We need only deterministic atomic programs to prove the lower bound.

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