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One-Dimensional Quasilinear Heat Flow With Boundary Conditions Periodic in Time[1]

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If thermal conductivity and specific heat are taken as linear functions of temperature, a nonlinear heat-conduction equation results. For small nonlinearities an approximate first-order analytical solution may be obtained in certain cases. The present analysis deals with one-dimensional problems with periodic boundary conditions. Only the steady-state solution (i.e., one which is periodic in time) is considered. Solutions are obtained for the following cases: 1) Semi-infinite solid with sinusoidal boundary temperature, 2) thick slab with sinusoidal temperature at one boundary and constant temperature at the other, and 3) thick slab with prescribed heat flux (a constant term plus a sinusoidal term) at one boundary, constant temperature at the other. The effects of the nonlinearities are discussed; they are found to be surprisingly small.

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NOMENCLATURE

The following nomenclature is used in this paper:

a = Temperature coefficient for k in Equation [5]

B_o, B_1 = Amplitudes of constant and first harmonic terms of heat flux input, Equation [41]

b = Temperature coefficient for s in Equation [6]

c = Specific heat (for unit mass)

f = Time-independent term in temperature solution

k = Thermal conductivity

k_o = Value of k at reference temperature (assumed zero)

L = Thickness of slab

q'' = Heat flux

R_n = Amplitude of n^{th} harmonic of temperature

s = $c\rho$ = Volumetric specific heat

s_o = Value of s at reference temperature (assumed zero)

T_o = Amplitude of temperature input, Equation [17]

$T_o^* = B_1 / \sqrt{2}\, k_o \gamma_1$, = Amplitude of linear solution at boundary

T_L = Constant temperature at $x = L$.

t = Temperature

u = Transformed temperature, defined by Equations [3] and [7]

x = Distance coordinate

$\alpha = k / c\rho$ = Thermal diffusivity

$\gamma_n = \sqrt{n\omega/2\alpha_o}$

$\delta = (b-a)\, B_o / 2k_o \gamma_1$

ρ = Density

τ = Time

ϕ_n = Coefficient of cosine term of n^{th} temperature harmonic

χ_n = Phase angle of n^{th} harmonic in temperature solution

ψ_n = Coefficient of sine term of n^{th} temperature harmonic

ω = Angular frequency

INTRODUCTION

The general equation for heat conduction in an isotropic material, without heat sources or sinks, is

$$\operatorname{div}(k \operatorname{grad} t) = s \frac{\partial t}{\partial \tau} \qquad [1]$$

where $\underline{t}$ is temperature, τ is time, $\underline{k}$ is thermal conductivity, and $\underline{s} = \underline{c}\rho$ the volumetric specific heat, i.e., the product of specific heat and density. In the case where $\underline{k}$ and $\underline{s}$ are constants this becomes

$$\alpha \nabla^2 t = \frac{\partial t}{\partial \tau} \qquad [2]$$

where $\alpha \equiv \underline{k}/\underline{s}$ is a constant. This is a linear partial differential equation; solutions have been obtained by standard methods for many cases of practical interest (see for example[4] (1)). However, in the more general case, both $\underline{k}$ and $\underline{s}$ in Equation [1] are functions of temperature; hence the differential equation is nonlinear.

In the nonlinear case $\underline{k}$ and $\underline{s}$ may readily be combined into a single temperature-dependent quantity through the transformation used by van Dusen (2)

$$u = \frac{1}{k_0} \int_{t_0}^{t} k(t')\,dt' \qquad [3]$$

---

4 Numbers in parentheses refer to the Bibliography at the end of the paper.

where k_o is the value of k at temperature t_o. It follows that k grad t = k_o grad u and $(\partial t/\partial \tau) = (k_o/k)(\partial u/\partial \tau)$ and hence

$$\alpha(u)\nabla^2 u = \frac{\partial u}{\partial \tau} \qquad [4]$$

where $\alpha(u) = k(u)/s(u)$. For the time-independent case, $\nabla^2 u = 0$; thus, after the boundary conditions have been expressed in terms of u , the problem reduces to the familiar linear one.

In the transient case the scope of available analytical solutions is much more limited. Hopkins (3) has suggested the method of successive approximations and applied it to a number of examples. Another analytical technique has been the Boltzmann transformation (4), which converts the partial differential equation into an ordinary one; this is limited to the one-dimensional case with rather specialized initial condition and boundary conditions. Additional solutions for various cases are given by Crank (7) and by Friedmann (5,6) who includes an extensive bibliography. Much of the work in the field is based on numerical solutions; most of it limited to the one-dimensional problem.

The case in which boundary conditions are periodic in time has not received as much attention. However, if the initial transient is neglected, the time-dependent steady state solution can readily be found, under physically reasonable restrictions. Vernotte (8,9) has treated a semi-infinite bar with sinusoidal heat flux input and has briefly discussed the case of square-wave heat flux.

The present paper, based on the fuller analysis by one of the authors (10), is an extension and amplification of Vernotte's work. It treats additional problems, presents more explicit results, and includes second harmonic terms. (However, it deals only with a one-dimensional solid rather than a bar with convection losses at the cylindrical surface.) Qualitative conclusions are discussed from two different view-points: a) the case in which the thermal properties are known and the temperature solution is desired (e.g. the periodic solar heating of the crust of the earth or the periodic heating of a cylinder wall in an internal combustion engine); b) the case in which the required temperatures are experimentally measured in order to determine the thermal properties.

PROBLEMS AND ASSUMPTIONS

Three problems will not be considered:

a) A one-dimensional, semi-infinite solid with the surface temperature sinusoidal in time.

b) A one-dimensional, thick slab with sinusoidal temperature variation at one boundary and constant temperature at the other. (The term "thick slab" implies a negligibly small periodic temperature variation near the second boundary.

c) A one-dimensional slab with the right boundary at constant temperature. At the left boundary there is a prescribed heat flux input, consisting of a constant term plus a sinusoidal term. This example may be useful in the experimental measurement of thermal conductivity.

Two basic assumptions are made throughout the analysis:

1) It is supposed that the periodic input has been applied long enough so that the transient has died out. In other words, only the steady-state, time-dependent solution is obtained; it is assumed that the solution is periodic with the same period as involved in the boundary conditions. Hence, no initial conditions appear in any of the cases considered.

2) It is assumed that the temperature-dependent properties are linear functions and that the variations are small. More specifically, the thermal conductivity k and the volumetric specific heat s are taken as

$$k = k_o \quad (1 + 2\,at) \qquad [5]$$

$$s = s \quad (1 + 2\,bt) \qquad [6]$$

where k_o, s_o, a and b are constants. Since the variation is assumed small, $2at \ll 1$ and $2bt \ll 1$. Hence terms independent of a and b are considered as zero-order terms, linear terms in these parameters are considered as first-order in smallness, quadratic terms in these parameters are taken as second order and assumed negligibly small.

METHOD OF ANALYSIS

The method of solution is that of "harmonic balance." This technique is the same as that employed by Vernotte (8,9) except that he deals with amplitude and phase angle for each frequency while the present analysis deals with amplitudes of sine and cosine terms. "Harmonic balance" is a familiar technique for ordinary nonlinear differential equations (e.g., references (11) through (14).

It will be convenient to treat the problem in terms of the transformed variable $\underline{u}$ defined by Equation [3]. Using Equation [5] for $\underline{k}$ in this transformation yields

$$u = t + at^2 = t\,(1 + at) \qquad [7]$$

If second-order terms are neglected (as stated in Assumption 2), $1 + \underline{a}\,\underline{u} \cong 1 + \underline{a}\,\underline{t}$ and the inverse transformation is

$$t \cong u\,(1 + au)^{-1} \cong u\,(1 - au) \qquad [8]$$

In the same way, Equations [5] and [6] become $\underline{k} \cong \underline{k}_o\,(1 + 2\,au)$ and $\underline{s} \cong \underline{s}_o\,(1 + 2\,bu)$. Substituting these expressions in the one-dimensional case of Equation [4] gives

$$\left[\frac{k_o\,(1+2au)}{s_o\,(1+2bu)}\right]\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} \qquad [9]$$

or

$$\frac{\partial^2 u}{\partial x^2} \cong \left[\frac{1+2\,(b-a)\,u}{\alpha_o}\right]\frac{\partial u}{\partial \tau} = \frac{1}{\alpha_o}\frac{\partial u}{\partial \tau} + \frac{(b-a)}{\alpha_o}\frac{\partial u^2}{\partial \tau} \qquad [10]$$

where $\alpha_o \equiv \underline{k}_o / \underline{s}_o$.

As stated in Assumption 1, the solution will be assumed periodic and may thus be written as a Fourier series, i.e.

$$u(x,\tau) = f(x) + \phi_1(x)\cos\omega\tau + \psi_1(x)\sin\omega\tau$$

$$+ \phi_2(x)\cos 2\omega\tau + \psi_2(x)\sin 2\omega\tau \qquad [11]$$

(If higher terms are included in the series, they will be found to be at least second-order for the problems under consideration here.) Inserting Equation [11] in Equation [10] now yields the approximate expression

$$\frac{d^2 f}{dx^2} + \frac{d^2\phi_1}{dx^2} \cos \omega\tau + \frac{d^2\psi_1}{dx^2} \sin \omega\tau + \frac{d^2\phi_2}{dx^2} \cos 2\omega\tau + \frac{d^2\psi_2}{dx^2} \sin 2\omega\tau$$

$$\cong \frac{1}{\alpha_0}\left[-\phi_1\omega \sin \omega\tau + \psi_1\omega \cos \omega\tau - 2\phi_2\omega \sin 2\omega\tau + 2\psi_2\omega \cos 2\omega\tau\right]$$

$$+ \frac{(b-a)}{\alpha_0}\left[-2f\phi_1\omega \sin \omega\tau + 2f\psi_1\omega \cos \omega\tau - (\phi_1^2 - \psi_1^2) \sin 2\omega\tau + 2\phi_1\psi_1 \cos 2\omega\tau\right]$$

(Since the last bracket is multiplied by (b - a) and since ϕ_2 and ψ_2 may be shown to be first-order terms, their product has been omitted in the bracket.)

Equating coefficients of the time-independent terms and of the respective sine and cosine terms now gives the five ordinary differential equations

$$\frac{d^2 f}{dx^2} = 0 \qquad [12]$$

$$\frac{d^2\phi_1}{dx^2} - \frac{\omega}{\alpha_0}\psi_1 = \frac{2f(b-a)\omega}{\alpha_0}\psi_1 \qquad [13]$$

$$\frac{d^2\psi_1}{dx^2} + \frac{\omega}{\alpha_0}\phi_1 = -\frac{2f(b-a)\omega}{\alpha_0}\phi_1 \qquad [14]$$

$$\frac{d^2\phi_2}{dx^2} - \frac{2\omega}{\alpha_0}\psi_2 = \frac{2(b-a)\omega}{\alpha_0}\phi_1\psi_1 \qquad [15]$$

$$\frac{d^2\psi_2}{dx^2} + \frac{2\omega}{\alpha_0}\phi_2 = -\frac{(b-a)\omega}{\alpha_0}\left(\phi_1^2 - \psi_1^2\right) \qquad [16]$$

Solution of a specific problem now consists in expressing the boundary conditions in terms of u, comparing these expressions with Equation [11] to determine the boundary conditions on f, ϕ_1, ψ_1, ϕ_2, and ψ_2, solving the five ordinary differential equations (Equations [12] through [16]) subject to these conditions and, finally, expressing the solution in terms of t. The various steps will be explained in some detail in the first problem below, while the other cases will be outlined more briefly.

SOLUTIONS

a) Semi-Infinite Solid, Periodic Surface Temperature

This problem is mathematically defined by: 1) the one-dimensional form of Equation [1], 2) the expressions for k and s, Equations [5] and [6], and 3) the boundary conditions

$$t(0,\tau) = T_0 \sin \omega\tau \qquad [17]$$

$$|t(\infty,\tau)| \neq \infty \qquad [18]$$

The first two items are combined in Equation [10], which in turn led to Equations [12] through [16]. The third item, the boundary conditions, may be transformed by Equation [7] to become

$$u(0,\tau) = \frac{aT_0^2}{2} + T_0 \sin \omega\tau - \frac{aT_0^2}{2} \cos 2\omega\tau$$

$$|u(\infty,\tau)| \neq \infty$$

By comparing $\underline{u}(o, \tau)$ with Equation [11] , it follows that

$$f(o) = \frac{aT_o^2}{2} \qquad [19]$$

$$\phi_1(o) = 0 \quad , \qquad \psi_2(o) = T_o \qquad [20]$$

$$\phi_2(o) = -\frac{aT_o^2}{2} \quad , \qquad \psi_2(o) = 0 \qquad [21]$$

$\underline{u}(\infty, \tau)$ indicates that $\underline{f}$, ϕ_1 , ψ_1 , ϕ_2 and ψ_2 all remain finite at infinity.

The solution of Equation [12] is obviously

$$f(x) = \frac{aT_o^2}{2} \qquad [22]$$

Since $\underline{f}(\underline{x})$ itself is first-order in this case, the right hand sides of Equations [13] and [14] are clearly negligible. Hence these two equations with the boundary condition given by Equation [20] reduce to the well-known linear case; the solutions are

$$\phi_1(x) = -T_o e^{-\gamma_1 x} \sin \gamma_1 x \qquad [23]$$

$$\psi_1(x) = T_o e^{-\gamma_1 x} \cos \gamma_1 x \qquad [24]$$

where

$$\gamma_1 = \sqrt{\omega / 2\alpha_o} \qquad [25]$$

Inserting Equations [23] and [24] in [15] and [16] yields

$$\frac{d^2\phi_2}{dx^2} - \frac{2\omega}{\alpha_0}\psi_2 = -\frac{(b-a)\omega T_0^2}{\alpha_0} e^{-2\gamma_1 x} \sin 2\gamma_1 x$$

$$\frac{d^2\psi_2}{dx^2} + \frac{2\omega}{\alpha_0}\phi_2 = \frac{(b-a)\omega T_0^2}{\alpha_0} e^{-2\gamma_1 x} \cos 2\gamma_1 x$$

with boundary conditions given by Equation [21]. The solutions are

$$\phi_2(x) = -\frac{(2a-b)T_0^2}{2} e^{-\gamma_2 x} \cos \gamma_2 x - \frac{(b-a)T_0^2}{2} e^{-2\gamma_1 x} \cos 2\gamma_1 x \qquad [26]$$

$$\psi_2(x) = -\frac{(2a-b)T_0^2}{2} e^{-\gamma_2 x} \sin \gamma_2 x - \frac{(b-a)T_0^2}{2} e^{-2\gamma_1 x} \sin 2\gamma_1 x \qquad [27]$$

where

$$\gamma_2 = \sqrt{\omega/\alpha_0} = \sqrt{2}\,\gamma_1 \qquad [28]$$

The solution in terms of u is now obtained by inserting Equations [22], [23], [24], [26], and [27] in Equation [11] and finally converted back to t by the inverse transformation, Equation [8]. When second-order terms are neglected, the result is

$$t(x,\tau) = \frac{aT_0^2}{2} - \frac{aT_0^2}{2} e^{-2\gamma_1 x} + T_0 e^{-\gamma_1 x} \sin(\omega\tau - \gamma_1 x)$$

$$+ \frac{(2a-b)T_0^2}{2}\left[e^{-2\gamma_1 x}\cos(2\omega\tau - 2\gamma_1 x) - e^{-\gamma_2 x}\cos(2\omega\tau - \gamma_2 x)\right] \qquad [29]$$

It is of interest to consider the amplitude R_2 and the phase angle χ_2 of the second harmonic term. In reference (10) it is shown that these quantities are closely approximated by

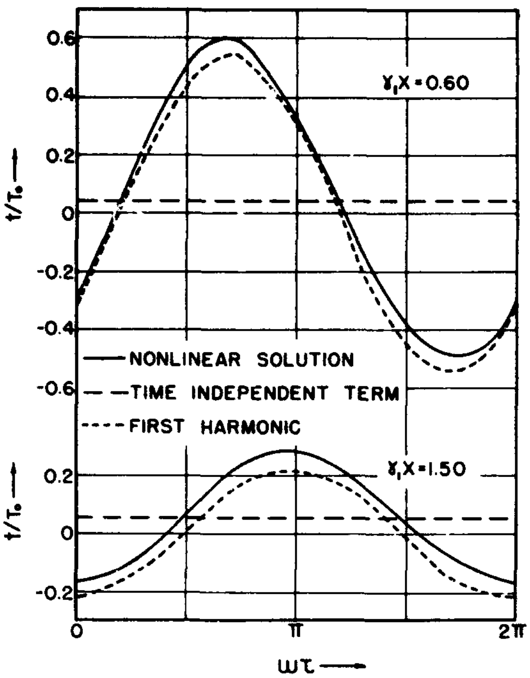

Fig. 1 Temperature solution for different positions in a semi-infinite solid with sinusoidal boundary temperature, given by Equation (29), for the case 2(a) T_o = 0.25, b = 0. The first harmonic is identical to the linear solution.

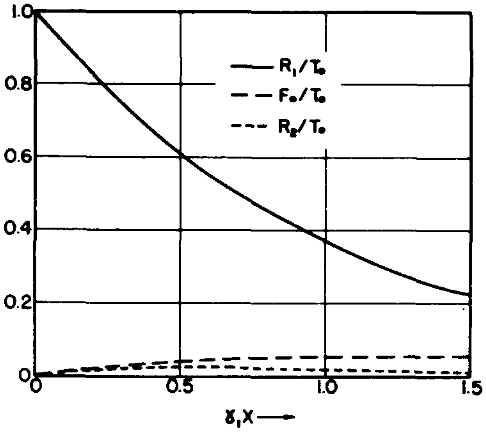

Fig. 2 Amplitudes of time-independent term F_O, first harmonic R_1, and second R_2 in a semi-infinite solid with sinusoidal boundary temperature for the case 2(a) T_O = 0.25, (b) = 0.

$$R_2 = .42\ (2a-b)\ T_o^2\ \gamma_1 x\ e^{-1.71\ \gamma_1 x} \qquad [30]$$

$$\chi_2 = -\frac{\pi}{4} - 1.71\ \gamma_1 x + .03\ (\gamma_1 x)^2 \qquad [31]$$

From equation [31] it readily follows that the maximum value of R_2 occurs at $\gamma_1 x$ = .59 and is given by

$$(R_2)_{max} = .09\ (2a-b) T_o^2 \qquad [32]$$

Figures 1 to 3 show various aspects of the nonlinear solution for the case in which $2aT_o$ = .25, i.e., the thermal conductivity varies by ± 25% about its mean value. The effects of the nonlinearity appear considerably smaller than might be expected from a preliminary estimate.

b) Thick Slab, Periodic Surface Temperature

The problem of the thick slab with periodic surface temperature is identical with the previous case except that Equation [18] is replaced by a boundary condition at the right hand surface $\underline{X} = \underline{L}$, i.e.,

$$t(L, \tau) = T_L \qquad [33]$$

where it will be assumed

$|\underline{T}_L| \leq |\underline{T}_o|$. By Equation [7],

this becomes

$$u(L, \tau) = T_L + aT_L^2 \qquad [34]$$

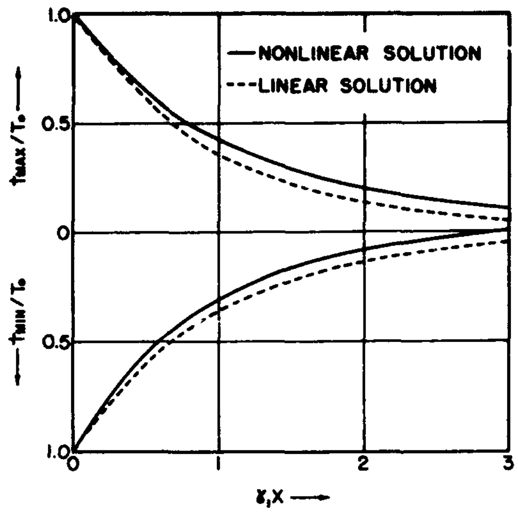

Fig. 3 Maximum and minimum temperatures in a semi-infinite solid with periodic boundary temperature for the case 2(a) $T_0 = 0.25$, (b) = 0. The linear case is shown for comparison.

Comparison with Equation [11] now shows

$$f(L) = T_L + a\,T_L^2 \qquad [35]$$

$$\phi_1(L) = \psi_1(L) = \phi_2(L) = \psi_2(L) = 0 \qquad [36]$$

The problem now reduces to solving Equations [12] through [16] with boundary conditions given by Equations [19], [20], [21] , [35] , and [36] .

The time-independent part $\underline{f}(\underline{x})$ is easily found to be

$$f(x) = \frac{aT_o^2}{2} + \left(T_L + a\,T_L^2 - \frac{aT_o^2}{2}\right)\left(\frac{x}{L}\right) \qquad [37]$$

since the term "thick slab" will be taken to imply

$$\gamma_1 L = \sqrt{\omega/2\alpha_o}\; L >> 1 \qquad [38]$$

Hence terms in $\underline{e}^{-\gamma_1 x}$ or $\underline{e}^{-\gamma_2 x}$ may be assumed zero at the right boundary. Furthermore terms in $(\gamma_1 \underline{L})^{-1}$ may be taken as first-order in smallness.

Inserting Equation [37] in Equation [13] now yields

$$\frac{d^2\phi_1}{dx^2} - \frac{\omega}{\alpha_0}\psi_1 = 2\left(\frac{\omega}{\alpha_0}\right)(b-a)\,T_L\left(\frac{x\psi_1}{L}\right) \qquad [39]$$

will be approximately equal to the solution of the linear problem as given by Equation [24]. Thus the order of magnitude of the last factor in Equation [39] is

$$\left(\frac{x\psi_1}{L}\right) \sim \frac{T_0\, x\, e^{-\gamma_1 x}}{L}$$

The maximum value of the term on the right is $.37\, T_0\, (\gamma_1 L)^{-1}$; hence it is a first-order term. Since the right side of Equation [39] also contains a factor of $(b-a)T_L$, the product is negligible. Similarly the right hand side of Equation [14] vanishes.

Thus the remainder of the problem now becomes identical to the previous case and the final temperature solution is

$$\begin{aligned} t(x,\tau) = {} & T_L\left(1 + aT_L - \frac{aT_L x}{L}\right)\left(\frac{x}{L}\right) \\ & + \frac{aT_0^2}{2}\left(1 - \frac{x}{L} - e^{-2\gamma_1 x}\right) + T_0 e^{-\gamma_1 x}\sin(\omega\tau - \gamma_1 x) \\ & + \frac{(2a-b)T_0^2}{2}\left[e^{-2\gamma_1 x}\cos(2\omega\tau - 2\gamma_1 x) - e^{-\gamma_2 x}\cos(2\omega\tau - \gamma_2 x)\right] \end{aligned} \qquad [40]$$

c) Thick Slab, Periodic Heat Flux.

The boundary conditions for a thick slab with periodic heat flux will be taken as

$$q''(0,\tau) = -k\left(\frac{\partial t}{\partial x}\right)_{x=0} = B_0 + B_1 \sin\omega\tau \qquad [41]$$

$$t(L,\tau) = -\frac{B_0 L}{k_0} - \frac{a B_0^2 L^2}{k_0^2} \qquad [42]$$

(The second condition is chosen to make $\underline{u}$ vanish at the origin.) From Equation [3], it follows that $\underline{k}\,(\partial \underline{t}/\partial \underline{x}) = \underline{k}_0\,(\partial \underline{u}/\partial \underline{x})$; with this relation and Equation [7], the boundary conditions become

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = -\frac{B_0}{k_0} - \frac{B_1}{k_1}\sin\omega\tau \qquad [43]$$

$$u\,(L,\tau) = -B_0 L/k_0 \qquad [44]$$

Hence the boundary conditions on Equations [12] through [16] are

$$\left(\frac{df}{dx}\right)_0 = -\frac{B_0}{k_0}, \qquad f(L) = -\frac{B_0 L}{k_0} \qquad [45]$$

$$\left(\frac{d\phi_1}{dx}\right)_0 = 0, \quad \left(\frac{d\psi_1}{dx}\right)_0 = -\frac{B_1}{k_0}, \quad \phi_1(L) = 0 = \psi_1(L) \qquad [46]$$

$$\left(\frac{d\phi_2}{dx}\right)_0 = 0 = \left(\frac{d\psi_2}{dx}\right)_0, \qquad \phi_2(L) = 0 = \psi_2(L) \qquad [47]$$

The solution of Equation [12] is now

$$f(x) = -\frac{B_0\, x}{k_0} \qquad [48]$$

Substituting this result in Equation [13] yields

$$\frac{d^2\phi_1}{dx^2} - \frac{\omega}{\alpha_0}\psi_1 = -\frac{2(b-a)B_0\,\omega x}{k_0\,\alpha_0}\,\psi_1 \qquad [49]$$

Since the coefficient on the right side of Equation [49] is first-order and since ψ_1 differs from the linear solution, $\psi_{1\ell}$, by first-order quantities, it is sufficiently accurate to use $\psi_{1\ell}$ in this term and obtain

$$\frac{d^2\phi_1}{dx^2} - \frac{\omega}{\alpha_0}\psi_1 = \frac{2(b-a)B_0\omega x}{k_0\alpha_0}\left[-\frac{B_1 e^{-\gamma_1 x}}{\sqrt{2}\,k_0\gamma_1}\cos\left(\gamma_1 x + \frac{\pi}{4}\right)\right] \qquad [50]$$

Equation [14] may be approximated in a similar way, i.e.,

$$\frac{d^2\psi_1}{dx^2} + \frac{\omega}{\alpha_0}\phi_1 = -\frac{2(b-a)B_0\omega x}{k_0\alpha_0}\left[\frac{B_1 e^{-\gamma_1 x}}{\sqrt{2}\,k_0\gamma_1}\sin\left(\gamma_1 x + \frac{\pi}{4}\right)\right] \qquad [51]$$

When Equations [50] and [51] are solved with boundary conditions given by [46] , the result is

$$\phi_1(x) = -T_0^* e^{-\gamma_1 x}\left(1+\frac{\delta}{2}\right)\Big\{\left[1+\delta(\gamma_1 x)+\delta(\gamma_1 x)^2\right]\sin\left(\gamma_1 x + \frac{\pi}{4} + \frac{\delta}{2}\right)$$

$$-\delta(\gamma_1 x)^2\cos\left(\gamma_1 x + \frac{\pi}{4} + \frac{\delta}{2}\right)\Big\} \qquad [52]$$

$$\psi_1(x) = T_0^* e^{-\gamma_1 x}\left(1+\frac{\delta}{2}\right)\Big\{\left[1+\delta(\gamma_1 x)+\delta(\gamma_1 x)^2\right]\cos\left(\gamma_1 x + \frac{\pi}{4} + \frac{\delta}{2}\right)$$

$$-\delta(\gamma_1 x)^2\sin\left(\gamma_1 x + \frac{\pi}{4} + \frac{\delta}{2}\right)\Big\} \qquad [53]$$

where

$$T_0^* \equiv B_1 \big/ \sqrt{2}\,k_0\gamma_1 \qquad [54]$$

$$\delta \equiv (b-a)B_0 \big/ 2\,k_0\gamma_1 \qquad [55]$$

T_0^* is the amplitude of the linear solution at the boundary while δ is a dimensionless parameter giving one measure of nonlinearity. ϕ_2 and ψ_2 may nowbe found as before.

In terms of t , the final result becomes

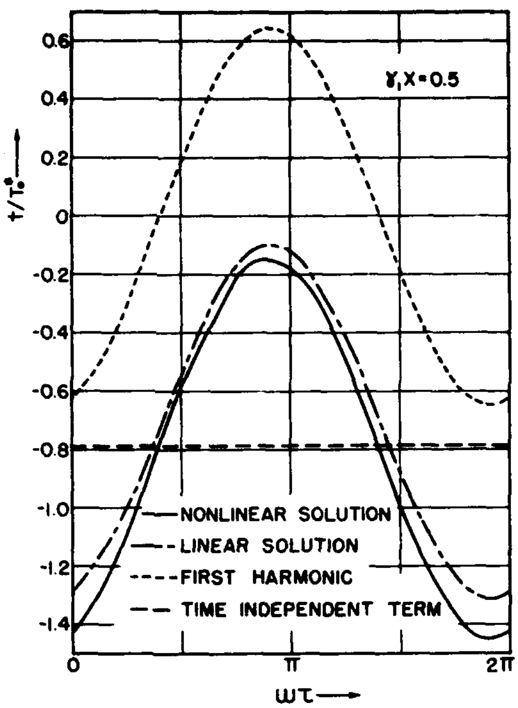

Fig. 4 Temperature solution, given by Equation (56), at γ, x = 0.5 in a thick slab with input flux consisting of a constant plus a sinusoidal term for the case 2(a) $T_0 = 0.25$, (b) = 0, $B_0 = B_1$. The linear solution is shown for comparison.

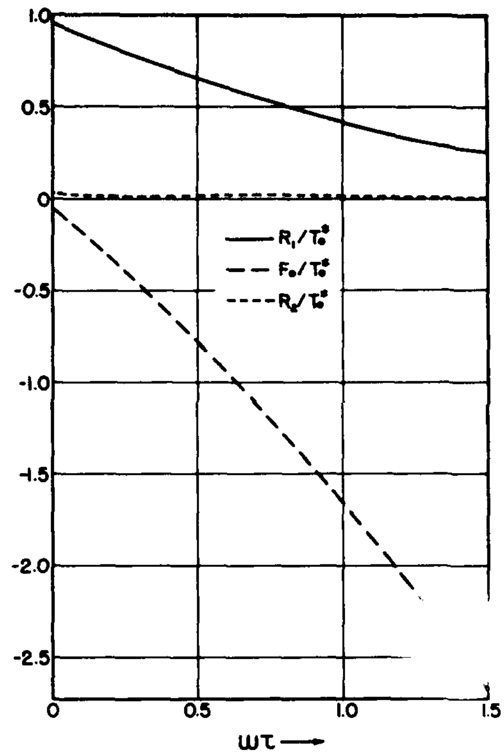

Fig. 5 Amplitudes of the time-independent term F_0, first harmonic R_1, and second harmonic R_2 in a thick slab with input flux consisting of a constant plus a sinusoidal term for the case 2(a) $T_0 = 0.25$, (b) = 0, $B_0 = B_1$.

$$t(x,\tau) = -\frac{B_0 x}{k_0} - a\left(\frac{B_0 x}{k_0}\right)^2 - \frac{a\,T_0^{*\,2}\,e^{-2\gamma_1 x}}{2}$$

$$+\,T_0^* e^{-\gamma_1 x}\left\{\left[1+\frac{\delta}{2}+\delta(\gamma_1 x)\right]\sin\left(\omega\tau-\gamma_1 x-\frac{\pi}{4}-\frac{\delta}{2}\right)+\sqrt{2}\,\delta(\gamma_1 x)\sin\left(\omega\tau-\gamma_1 x-\frac{\delta}{2}\right)\right\}$$

$$+\,T_0^*\left[\frac{(b-a)T_0^*}{\sqrt{2}}e^{-\gamma_2 x}\sin(2\omega\tau-\gamma_2 x)+\frac{(2a-b)T_0^*}{2}e^{-2\gamma_1 x}\sin(2\omega\tau-2\gamma_1 x)\right] \qquad [56]$$

Some aspects of this solution are shown in Figures 4 and 5.

CONCLUSIONS

a) The method presented here offers a rather general approach for finding the steady-state solution when boundary conditions are periodic and variations of the thermal parameters are small. However, the algebra may become quite complicated, particularly in two-dimensional and three-dimensional problems.

b) The most important qualitative effect appears to be the introduction of a second harmonic term; in the linear case, such a term does not appear at all. Both the time-independent term and the first harmonic may also be modified; in fact, in some cases, it appears that a time-independent term is introduced by nonlinearity. However, it must be recalled that the zero level of temperature is arbitrary; hence the time-independent term can be regarded as a quantitative change rather than a qualitative one.

c) The quantitative effects of the nonlinearities are surprisingly small; it appears that the thermal properties may safely be treated as constants in some cases where their variation is quite large. For example, when the thermal conductivity at the surface varies by ± 25% about its mean value, the maximum amplitude of the second harmonic may be only 2% or 3% of the amplitude of temperature variation at the surface. (The effect on the time-independent term is roughly twice as great; in the last problem this was accentuated by the constant term in the heat flux input.) The explanation seems to be that the conductivity varies more or less symmetrically about its mean value k_0 ; if th~~e variat~~ion in k were proportional to the square of the temperature, nonlinear effects might be considerably greater.

d) In general, variation in thermal conductivity k has more effect than variation in the volumetric specific heat

s. The time-independent part of the solution, if any, is unaffected by s.

e) Since the second harmonic is a characteristically nonlinear term and vanishes for the linear case, experimental measurement of the second harmonic amplitude seems to provide a sensitive method of determining the coefficients a and b, which describe the temperature variation of k and s, respectively. However, several possible drawbacks should be noted. First, it may be difficult to generate a sinusoidal temperature or heat flux input with sufficient accuracy. Secondly on the basis of the second harmonic alone it is difficult or impossible (depending on the boundary condition) to separate a and b. Finally the frequencies required would be quite low, thus perhaps complicating the experimental task of frequency analysis. Further study is being given to this problem.

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