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Balanced Matrices

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Abstract

A $0, \pm 1$ matrix A is *balanced* if, in every submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This definition was introduced by Truemper and generalizes the notion of balanced $0,1$ matrix introduced by Berge. In this paper, we survey what is currently known about these matrices, including polyhedral results, structural theorems, recognition algorithms and the relation with some problems in logic.

1 Introduction

A $0,1$ matrix is *balanced* if it does not contain a square submatrix of odd order with two ones per row and per column. This notion was introduced by Berge [4]. A $0, \pm 1$ matrix A is *balanced* if, in every submatrix with two nonzero entries per row and per column, the sum of the entries is a multiple of four. This definition is due to Truemper [63]. The class of balanced $0, \pm 1$ matrices includes balanced $0,1$ matrices and totally unimodular $0, \pm 1$ matrices. (A matrix is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$. The fact that total unimodularity implies balancedness follows, for example, from Camion's theorem [13] which states that a $0, \pm 1$ matrix is totally unimodular if and only if, in every square submatrix with an *even number* of nonzero entries per row and per column, the sum of the entries is a multiple of four.)

In Section 2, we characterize balanced $0, \pm 1$ matrices in terms of "bicoloring". This extends the notion of graph bipartition to $0, \pm 1$ matrices. We then discuss integral polytopes associated with "generalized" set packing, partitioning and covering problems. These results extend the integrality of set packing, partitioning and covering polytopes when the constraint matrix is balanced. We then discuss classes of $0, \pm 1$ matrices with related polyhedral properties, such as perfect and ideal $0, \pm 1$ matrices. Finally we introduce the connection with propositional logic and nonlinear $0,1$ optimization.

In Section 3, we show how to sign a $0,1$ matrix into a $0, \pm 1$ balanced matrix (when such a signing exists, the matrix is called *balanceable*). It follows that, in order to understand the structure of balanced $0, \pm 1$ matrices, it is equivalent to study $0,1$ matrices that are balanceable. We then introduce a decomposition theorem for these matrices. The decomposition theorem can be used to obtain a polynomial algorithm to test membership in the class of balanced $0, \pm 1$ matrices. This is discussed in Section 4. Section 5 surveys special classes of balanced matrices while Section 6 states a coloring theorem for graphs whose clique-node matrix is balanced. Finally, in Section 7, we propose some conjectures and indicate some directions for further research.

2 Bicoloring, Logic and Integer Polyhedra

2.1 Bicoloring

Berge [4] introduced the following notion. A $0,1$ matrix is *bicolorable* if its columns can be partitioned into blue and red columns in such a way that every row with two or more $1's$ contains a 1 in a blue column and a 1 in a red column. This notion provides the following characterization of balanced $0,1$ matrices.

Theorem 2.1 (Berge [4]) *A $0,1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

A $0,1$ matrix A can be represented by a hypergraph (the columns of A represent nodes and the rows represent edges). Then the definition of balancedness for $0,1$ matrices is a natural extension of the property of not containing odd cycles for graphs, and the notion of bicoloring is a natural extension of bipartition in graphs. Berge's theorem can be viewed as an extension to hypergraphs of the fact that a graph is bipartite if and only if it contains no odd cycle. In fact, this is the motivation that led Berge to introduce the notion of balancedness. Several results on bipartite graphs generalize to balanced hypergraphs, such as König's bipartite matching theorem, as stated in the next theorem. In a hypergraph, a *matching* is a set of pairwise nonintersecting edges and a *transversal* is a node set intersecting all the edges.

Theorem 2.2 (Berge, Las Vergnas [9]) *In a balanced hypergraph, the maximum cardinality of a matching equals the minimum cardinality of a transversal.*

The next result generalizes a theorem of Gupta [43] on bipartite multigraphs.

Theorem 2.3 (Berge [6]) *In a balanced hypergraph, the minimum number of nodes in an edge equals the maximum cardinality of a family of disjoint transversals.*

Ghouila-Houri [40] introduced the notion of *equitable bicoloring* for a $0, \pm 1$ matrix A as follows. The columns of A are partitioned into blue columns and red columns in such a way that, for every row of A, the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

Theorem 2.4 (Ghouila-Houri [40]) *A $0, \pm 1$ matrix A is totally unimodular if and only if every submatrix of A has an equitable bicoloring.*

A $0, \pm 1$ matrix A is *bicolorable* if its columns can be partitioned into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. For a $0, 1$ matrix, this definition coincides with Berge's notion of bicoloring. Clearly, if a $0, \pm 1$ matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable.

Theorem 2.5 (Heller, Tompkins [45]) *Let A be a $0, \pm 1$ matrix with at most two nonzero entries per row. A is totally unimodular if and only if A is bicolorable.*

A consequence of Camion's theorem is that a $0, \pm 1$ matrix with at most two nonzero entries per row is balanced if and only if it is totally unimodular. So Theorem 2.5 shows that a $0, \pm 1$ matrix with at most two nonzero entries per row is balanced if and only if it is bicolorable. The following theorem extends Theorem 2.1 to $0, \pm 1$ matrices and Theorem 2.5 to matrices with more than two nonzero entries per row.

Theorem 2.6 (Conforti, Cornuéjols [20]) *A $0, \pm 1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

Cameron and Edmonds [12] observed that the following simple algorithm finds a valid bicoloring of a balanced matrix. They described their algorithm for $0, 1$ matrices, but it also works for $0, \pm 1$ matrices.

Algorithm
Input: A $0, \pm 1$ matrix A.
Output: A bicoloring of A or a proof that the matrix A is not balanced.

Stop if all columns are colored or if some row is improperly colored. Otherwise, color a new column red or blue as follows.

If no row of A forces the color of a column, arbitrarily color one of the uncolored columns.

If some row of A forces the color of a column, color this column accordingly.

When the algorithm fails to find a bicoloring, the sequence of forcings that resulted in an improperly colored row identifies a submatrix with two nonzeros per row and column which violates balancedness. However, this algorithm cannot be used as a recognition of balancedness for the following reason: When the matrix A is not balanced, the algorithm may still find a bicoloring if one exists.

2.2 Integrality of Packing, Partitioning and Covering Polytopes

A polytope is *integral* if all its extreme points have only integer-valued components. Given a $0,1$ matrix A, the *set packing polytope* is

$$P(A) = \{x : \ Ax \leq \mathbf{1}, \ \mathbf{0} \leq x \leq \mathbf{1}\}.$$

The integrality of the set packing polytope is related to the notion of perfect graph. A graph G is *perfect* if, for every node induced subgraph H of G, the chromatic number of H equals the size of its largest clique. The fundamental connection between the theory of perfect graphs and integer programming was established by Fulkerson [37], Lovász [53] and Chvátal [17]. The *clique-node matrix* C_G of a graph G is a $0,1$ matrix whose columns are indexed by the nodes of G and whose rows are the incidence vectors of the maximal cliques of G.

Theorem 2.7 (Lovász [53], Fulkerson [37], Chvátal [17]) *A graph G is perfect if and only if the set packing polytope $P(C_G)$ is integral.*

A $0,1$ matrix is *perfect* if $P(A)$ is integral. It follows from Theorem 2.7 that a $0,1$ matrix is perfect if and only if its rows of maximal support form the clique-node matrix of a perfect graph. Berge [5] showed that every balanced 0,1 matrix is perfect. In fact, the next theorem characterizes a balanced 0,1 matrix A in terms of the set packing polytope $P(A)$ as well as the set covering polytope $Q(A)$ and the set partitioning polytope $R(A)$:

$$Q(A) = \{x : \; Ax \geq \mathbf{1}, \; \mathbf{0} \leq x \leq \mathbf{1}\},$$
$$R(A) = \{x : \; Ax = \mathbf{1}, \; \mathbf{0} \leq x \leq \mathbf{1}\}.$$

Theorem 2.8 (Berge [5], Fulkerson, Hoffman, Oppenheim [38]) *Let M be a 0,1 matrix. Then the following statements are equivalent:*

(i) M is balanced.

(ii) For each submatrix A of M, the set covering polytope $Q(A)$ is integral.

(iii) For each submatrix A of M, the set packing polytope $P(A)$ is integral.

(iv) For each submatrix A of M, the set partitioning polytope $R(A)$ is integral.

Conforti and Cornuéjols [20] generalize this result to $0, \pm 1$ matrices. Given a $0, \pm 1$ matrix A, let $n(A)$ denote the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A.

Theorem 2.9 (Conforti, Cornuéjols [20]) *Let M be a $0, \pm 1$ matrix. Then the following statements are equivalent:*

(i) M is balanced.

(ii) For each submatrix A of M, the generalized set covering polytope $\{x : \; Ax \geq \mathbf{1} - n(A), \; \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.

(iii) For each submatrix A of M, the generalized set packing polytope $\{x : \; Ax \leq \mathbf{1} - n(A), \; \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.

(iv) For each submatrix A of M, the generalized set partitioning polytope $\{x : \; Ax = \mathbf{1} - n(A), \; \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.

A system of linear constraints is *totally dual integral* (TDI) if, for each integral objective function vector c, the dual linear program has an integral optimal solution (if an optimal solution exists). Edmonds and Giles [35] proved that, if a linear system $Ax \leq b$ is TDI and b is integral, then $\{x : Ax \leq b\}$ is an integral polyhedron.

Theorem 2.10 (Fulkerson, Hoffman, Oppenheim [38]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* 0,1 *matrix. Then the linear system*

$$\begin{cases} A_1 x \geq 1 \\ A_2 x \leq 1 \\ A_3 x = 1 \\ x \geq 0 \end{cases}$$

is TDI.

So Theorem 2.10 and the Edmonds-Giles theorem imply Theorem 2.8. Note that the total dual integrality of the set packing problem when the constraint matrix is a balanced 0,1 matrix also follows from the perfect graph theorem of Lovász [53].

Theorem 2.11 (Conforti, Cornuéjols [20]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* $0, \pm 1$ *matrix. Then the linear system*

$$\begin{cases} A_1 x \geq 1 - n(A_1) \\ A_2 x \leq 1 - n(A_2) \\ A_3 x = 1 - n(A_3) \\ 0 \leq x \leq 1 \end{cases}$$

is TDI.

It may be worth noting that this theorem does not hold when the upper bound $x \leq 1$ is dropped from the linear system, see [20]. In fact, the resulting polyhedron may not be integral. For comparison, we state a result that follows from the Hoffman-Kruskal theorem [46].

Theorem 2.12 (Hoffman, Kruskal [46]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a totally unimodular matrix and* $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *an integral vector of appropriate dimensions. Then the linear system*

$$\begin{cases} A_1x & \geq & b_1 \\ A_2x & \leq & b_2 \\ A_3x & = & b_3 \\ x & \geq & \mathbf{0} \end{cases}$$

is TDI.

2.3 Related Classes of $0, \pm 1$ Matrices

In this section, we first introduce a family of integral polytopes obtained by spanning the spectrum from totally unimodular to balanced $0, \pm 1$ matrices. In the second part of the section we consider two natural extensions of the concept of balancedness, namely perfection and idealness.

The matrix A is *minimally non-totally unimodular* if it is not totally unimodular, but every proper submatrix has that property.

Theorem 2.13 (Camion [15] and Gomory (cited in [15])) *Let A be a* $0, \pm 1$ *minimally non-totally unimodular matrix. Then A is square,* $det(A) = \pm 2$ *and* A^{-1} *has only* $\pm\frac{1}{2}$ *entries. Furthermore, each row and column of A has an even number of nonzeros.*

Let $\mathcal{H}$ be the class of minimally non-totally unimodular matrices. Recent results of Truemper [65] (see also [66]), give a simple construction and several characterizations of all matrices in $\mathcal{H}$. For a $0, \pm 1$ matrix A, denote by $t(A)$ the column vector whose i^{th} component is the number of nonzeroes in row i. Finally, let $\mathcal{J}$ be the family of matrices that can be obtained from the identity matrix by changing some $+1's$ into $-1's$.

Theorem 2.14 (Conforti, Cornuéjols, Truemper [24]) *The following two statements are equivalent for a* $0, \pm 1$ *matrix A and a nonnegative integral vector c.*

(i) A does not contain a submatrix $A' \in \mathcal{H}$ such that $t(A') \leq 2c'$, where c' is the subvector of c corresponding to the rows of A'.

(ii) The polytope $\{(x,s) : \; Bx + Js = b - n(B), \; \mathbf{0} \leq x \leq \mathbf{1}, \; s \geq \mathbf{0}\}$ is integral for all column submatrices B of A, all $J \in \mathcal{J}$ and all integral vectors b such that $\mathbf{0} \leq b \leq c$.

We make the following remarks about this theorem.

- When $2c \geq t(A)$, Theorem 2.14 gives a characterization of totally unimodular matrices which can be deduced from the Hoffman-Kruskal theorem (Theorem 2.12).
- It is easy to see that A is balanced if and only if A does not contain a submatrix $A' \in \mathcal{H}$ with $t(A') \leq \mathbf{2}$. So, when $c = \mathbf{1}$ in Theorem 2.14, we get a variation of Theorem 2.9.
- When A is a $0,1$ matrix, Theorem 2.14 reduces to a result of Truemper and Chandrasekaran [67].

Now we consider two extensions of the concept of balanced $0, \pm 1$ matrix. A $0, \pm 1$ matrix A is *ideal* if the generalized set covering polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A), \; \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral. A generalized set covering inequality $ax \geq 1 - n(a)$ is *dominated* by $bx \geq 1 - n(b)$, if $\{k : b_k = 1\} \subseteq \{k : a_k = 1\}$ and $\{k : b_k = -1\} \subseteq \{k : a_k = -1\}$. A *prime implication* of $Q(A)$ is a generalized set covering inequality $ax \geq 1 - n(a)$ which is satisfied by all the $0,1$ vectors in $Q(A)$ but is not dominated by any other generalized set covering inequality valid for $Q(A)$. A *row monotonization* of A is any $0,1$ matrix obtained from a row submatrix of A by multiplying some of its columns by -1. A row monotonization of A is *maximal* if it is not a proper submatrix of any row monotonization of A. Little is known about ideal $0, \pm 1$ matrices but ideal $0,1$ matrices have been studied [51], [52], [59], [56], [30].

Theorem 2.15 (Hooker [49]) *Let A be a $0, \pm 1$ matrix such that the generalized set covering polytope $Q(A)$ contains all of its prime implications. Then A is ideal if and only if all the maximal row monotonizations of A are ideal $0,1$ matrices.*

A $0,\pm 1$ matrix A is *perfect* if the generalized set packing polytope $P(A) = \{x : Ax \leq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral. For $0,1$ matrices, the concept of perfection is well studied (through Theorem 2.7 and the extensive literature on perfect graphs), but very little is known about perfect $0,\pm 1$ matrices. Therefore it seems natural to relate the notion of perfection for $0,\pm 1$ matrices to that for $0,1$ matrices.

We say that a polytope Q contained in the unit hypercube $[0,1]^n$ is *irreducible* if, for each j, both polytopes $Q \cap \{x_j = 0\}$ and $Q \cap \{x_j = 1\}$ are nonempty. A generalized set packing inequality $ax \leq 1 - n(a)$ is *dominated* by $bx \leq 1 - n(b)$, if $\{k : a_k = 1\} \subseteq \{k : b_k = 1\}$ and $\{k : a_k = -1\} \subseteq \{k : b_k = -1\}$. Given a $0,\pm 1$ matrix A, the *completion* of A is the matrix A^* obtained by adding to A all row vectors a that induce a generalized set packing inequality $ax \leq 1 - n(a)$ which is valid for $P(A)$ and not dominated by any other inequality in A^*. A $0,1$ matrix B obtained from A^* by multiplying through some columns by -1 and replacing all negative entries of the resulting matrix by 0 is called a *monotone completion* of A.

Theorem 2.16 (Conforti, Cornuéjols, De Francesco [21]) *Let A be a $0,\pm 1$ matrix such that the generalized set packing polytope $P(A)$ is irreducible. Then A is perfect if and only if all the monotone completions of A are perfect $0,1$ matrices.*

2.4 Propositional Logic

In propositional logic, *atomic propositions* $x_1, \ldots, x_j, \ldots, x_n$ can be either *true* or *false*. A *truth assignment* is an assignment of "true" or "false" to every atomic proposition. A *literal* is an atomic proposition x_j or its negation $\neg x_j$. A *clause* is a disjunction of literals and is *satisfied* by a given truth assignment if at least one of its literals is true.

A survey of the connections between propositional logic and integer programming can be found in [48]. The following formulation appears in Dantzig [34].

A truth assignment satisfies the set S of clauses

$$\bigvee_{j \in P_i} x_j \vee \Big(\bigvee_{j \in N_i} \neg x_j \Big) \quad \textit{for all } i \in S$$

if and only if the corresponding 0,1 vector satisfies the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} x_j \geq 1 - |N_i| \quad \textit{for all } i \in S.$$

The above system of inequalities is of the form

$$Ax \geq \mathbf{1} - n(A). \tag{1}$$

We consider three classical problems in logic. Given a set S of clauses, the ***satisfiability problem*** (SAT) consists of finding a truth assignment that satisfies all the clauses in S or showing that none exists. Equivalently, SAT consists of finding a 0,1 solution x to (1) or showing that none exists.

Given a set S of clauses and a weight vector w whose components are indexed by the clauses in S, the ***weighted maximum satisfiability problem*** (MAXSAT) consists of finding a truth assignment that maximizes the total weight of the satisfied clauses. MAXSAT can be formulated as the integer program

$$\begin{array}{ll} Min & \sum_{i=1}^{m} w_i s_i \\ & Ax + s \geq \mathbf{1} - n(A) \\ & x \in \{0,1\}^n, s \in \{0,1\}^m. \end{array}$$

Given a set S of clauses (the premises) and a clause C (the conclusion), *logical inference* in propositional logic consists of deciding whether every truth assignment that satisfies all the clauses in S also satisfies the conclusion C.

To the clause C, using transformation (1), we associate an inequality

$$cx \geq 1 - n(c),$$

where c is a $0, \pm 1$ vector. Therefore C cannot be deduced from S if and only if the integer program

$$min\ \{cx : Ax \geq \mathbf{1} - n(A),\ x \in \{0,1\}^n\} \tag{2}$$

has a solution with value $-n(c)$.

These three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [11], [16], [64]. MAXSAT remains NP-hard for Horn clauses with at most two literals [39]. A set S of clauses is ***balanced*** if the corresponding $0, \pm 1$ matrix A defined in (1) is balanced. Similarly, a set of clauses is ***ideal*** if A is ideal. If S is ideal, SAT, MAXSAT and logical inference can be solved by linear programming. The following theorem is an immediate consequence of Theorem 2.9.

Theorem 2.17 *Let S be a balanced set of clauses. Then SAT, MAXSAT and logical inference can be solved in polynomial time by linear programming.*

This has consequences for probabilistic logic as defined by Nilsson [55]. Being able to solve MAXSAT in polynomial time provides a polynomial time separation algorithm for probabilistic logic via the ellipsoid method, as observed by Georgakopoulos, Kavvadias and Papadimitriou [39]. Hence probabilistic logic is solvable in polynomial time for ideal sets of clauses.

Remark 2.18 *Let S be an ideal set of clauses. If every clause of S contains more than one literal then, for every atomic proposition x_j, there exist at least two truth assignments satisfying S, one in which x_j is true and one in which x_j is false.*

Proof: Since the point $x_j = 1/2$, $j = 1, \ldots, n$ belongs to the polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ and $Q(A)$ is an integral polytope, then the above point can be expressed as a convex combination of $0, 1$ vectors in $Q(A)$. Clearly, for every index j, there exists in the convex combination a $0, 1$ vector with $x_j = 0$ and another with $x_j = 1$. □

A consequence of Remark 2.18 is that, for an ideal set of clauses, SAT can be solved more efficiently than by general linear programming.

Theorem 2.19 (Conforti, Cornuéjols [19]) *Let S be an ideal set of clauses. Then S is satisfiable if and only if a recursive application of the following procedure stops with an empty set of clauses.*

Recursive Step

If $S = \emptyset$ then S is satisfiable.

If S contains a clause C with a single literal (unit clause), *set the corresponding atomic proposition x_j so that C is satisfied. Eliminate from S all clauses that become satisfied and remove x_j from all the other clauses. If a clause becomes empty, then S is not satisfiable* (unit resolution).

If every clause in S contains at least two literals, choose any atomic proposition x_j appearing in a clause of S and add to S an arbitrary clause x_j or $\neg x_j$.

The above algorithm for SAT can also be used to solve the logical inference problem when S is an ideal set of clauses, see [19]. For balanced (or ideal) sets of clauses, it is an open problem to solve MAXSAT in polynomial time by a direct method, without appealing to polynomial time algorithms for general linear programming.

2.5 Nonlinear 0, 1 Optimization

Consider the nonlinear 0, 1 maximization problem

$$max_{x \in \{0,1\}^n} \quad \sum_k a_k \prod_{j \in T_k} x_j \prod_{j \in R_k} (1 - x_j),$$

where, w.l.o.g., all ordered pairs (T_k, R_k) are distinct and $T_k \cap R_k = \emptyset$. This is an NP-hard problem. A standard linearization of this problem was proposed by Fortet [36]:

$$\begin{array}{rcll} max \ \sum a_k y_k & & & \\ y_k - x_j & \leq & 0 & for\ all\ k\ s.t.\ a_k > 0,\ for\ all\ j \in T_k \\ y_k + x_j & \leq & 1 & for\ all\ k\ s.t.\ a_k > 0,\ for\ all\ j \in R_k \\ y_k - \sum_{j \in T_k} x_j + \sum_{j \in R_k} x_j & \geq & 1 - |T_k| & for\ all\ k\ s.t.\ a_k < 0 \\ y_k,\ x_j & \in & \{0,1\} & for\ all\ k\ and\ j. \end{array}$$

When the constraint matrix is balanced, this integer program can be solved as a linear program, as a consequence of Theorem 2.11. Therefore, in this case, the nonlinear 0, 1 maximization problem can be solved in polynomial time. The relevance of balancedness in this context was pointed out by Crama [31].

3 Decomposition Theorems

In this section, we state a decomposition theorem for balanced $0, \pm 1$ matrices due to Conforti, Cornuéjols and Rao [23] and Conforti, Cornuéjols, Kapoor and Vušković [22]. Section 3.1 discusses the problem of changing the sign of some of the entries of a $0, 1$ matrix so that the resulting $0, \pm 1$ matrix becomes balanced. In Section 3.2, we present the main theorem. Section 3.3 contains an outline of its proof.

3.1 Signing $0, 1$ Matrices to Be Balanced

In this section, we consider the following question: given a $0, 1$ matrix A, is it possible to turn some of the 1's into -1's in order to obtain a balanced $0, \pm 1$ matrix? It turns out, as we will see shortly, that if such a signing exists, it is unique up to multiplication of rows and columns by -1. Furthermore, there is a very simple algorithm to perform the signing. As a consequence, in order to understand the structure of balanced $0, \pm 1$ matrices, it will be sufficient to concentrate on the zero-nonzero pattern.

Given a $0, \pm 1$ matrix A, the *signed bipartite graph representation of A* is a bipartite graph G together with an assignment of weights $+1$ or -1 to the edges of G, defined as follows. The nodes of G correspond to the rows and columns of A, ij is an edge of G if and only if the entry a_{ij} of A is nonzero and the weight of edge ij is the value of a_{ij}. We say that G is balanced if A is. A *hole* in a graph is a chordless cycle of length four or greater. Thus, a signed bipartite graph G is balanced if and only if, for every hole H of G, the sum of the weights of the edges in H is a multiple of four. (Beware that the material presented in this paper is unrelated to the notion of balanced signed graphs introduced in [44], [1] in connection with a problem in attitudinal psychology. There, a signed graph is balanced if every cycle contains an even number of edges with weight -1.)

A bipartite graph G is *balanceable* if there exists a signing of its edges so that the resulting signed graph is balanced. Equivalently, a $0, 1$ matrix is *balanceable* if it is possible to turn some of the 1's into -1's and obtain a balanced $0, \pm 1$ matrix.

Since a cut and a cycle in a graph have even intersection, it follows that if a signed bipartite graph G is balanced, then the signed bipartite graph G' obtained by switching signs on the edges of a cut, is also balanced. For every edge uv of a spanning tree there is a cut containing uv and no other edge of the tree. These cuts are known as *fundamental cuts* and every cut is the symmetric difference of fundamental cuts. Thus, if G is a balanceable bipartite graph, its signing into a balanced signed bipartite graph is unique up to the (arbitrary) signing of a spanning tree of G. This was already observed by Camion [14] in the context of $0,1$ matrices that can be signed to be totally unimodular. It follows that a bipartite graph G is balanceable if and only if the following signing algorithm produces a balanced signed bipartite graph:

Signing Algorithm

Input: A bipartite graph G.

Output: A signing of G which is balanced if and only if G is balanceable.

Choose a spanning tree of G and sign its edges arbitrarily. Then recursively choose an unsigned edge uv which closes a hole H of G with previously signed edges, and sign uv so that the sum of the weights of the edges in H is a multiple of four.

Note that the recursive step of the signing algorithm can be performed efficiently. Indeed, the unsigned edge uv can be chosen to close the smallest length hole with signed edges. Such a hole H is also a hole in G, else a chord of H in G contradicts the choice of uv.

It follows from this signing algorithm, and the uniqueness of the signing (up to the signing of a spanning tree), that the problem of recognizing whether a bipartite graph is balanceable is equivalent to the problem of recognizing whether a signed bipartite graph is balanced.

Figure 1 shows two classes of bipartite graphs which are important in this study. In all figures, solid lines represent edges and dashed lines represent chordless paths of length at least one. The black and white nodes are on opposite sides of the bipartition.

Let G be a bipartite graph. Let u, v be two nonadjacent nodes in opposite sides of the bipartition. A ***3-path configuration connecting u and v*** is defined by three chordless paths P_1, P_2, P_3 connecting u and v, having no common intermediate nodes, such that the subgraph induced by the nodes of these three paths contains no edge other than those of the paths. See Figure 1. Since paths P_1, P_2, P_3 of a 3-path configuration are of length one or three modulo four, the sum of the weights of the edges in each path is also one or three modulo four. It follows that two of the three paths induce a hole of weight two modulo four. So a 3-path configuration is not balanceable.

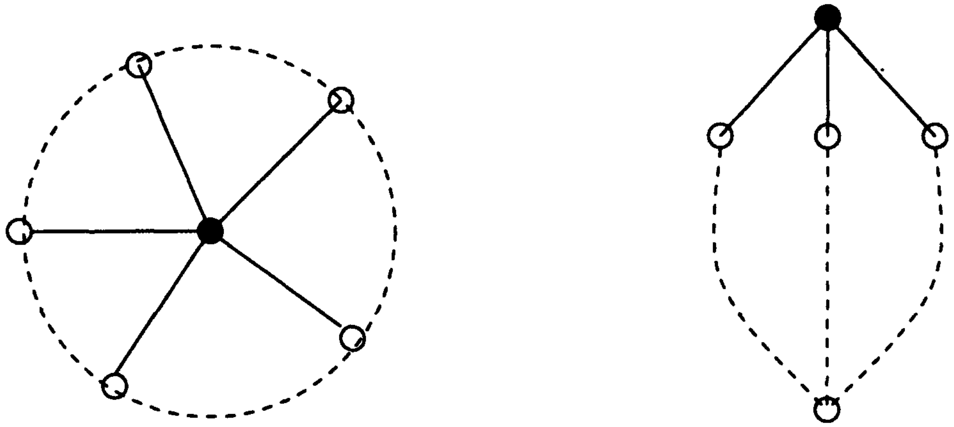


Figure 1: Odd wheel and 3-path configuration

A *wheel* is defined by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, x_2, \ldots, x_n$. If n is even, the wheel is an *even wheel*, otherwise it is an *odd wheel*, see Figure 1. An edge xx_i is a *spoke*. A subpath of H connecting x_i and x_j is called a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. Consider a balanceable wheel. By the signing algorithm, all spokes of the wheel can be assumed to be signed positive. This implies that the sum of the weights of the edges in each sector is two modulo four. Hence the wheel must be an even wheel.

So, bipartite graphs that are balanceable contain neither odd wheels nor 3-path configurations as node induced subgraphs. The following important theorem of Truemper states that the converse is also true.

Theorem 3.1 (Truemper [63]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-path configuration as a node induced subgraph.*

3.2 Decomposition Theorem

In this section we give the main decomposition theorem for balanceable bipartite graphs. The theorem states that if a balanceable bipartite graph does not belong to a restricted class, called basic, then it has one of three cutsets.

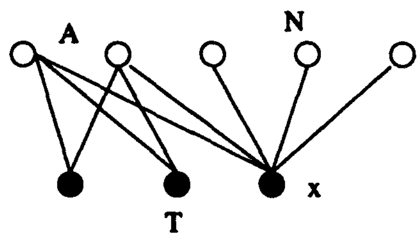


Figure 2: Extended star

Cutsets

A set S of nodes (edges) of a connected graph G is a *node (edge) cutset* if the subgraph of G obtained by removing the nodes (edges) in S, is disconnected.

A *biclique* is a complete bipartite graph with at least one node from each side of the bipartition and it is denoted by K_{BD} where B and D are the node sets in each side of the bipartition.

For a node x, let $N(x)$ denote the set of all neighbors of x. In a bipartite graph, an *extended star* is defined by disjoint subsets T, A, N of $V(G)$ and a node $x \in T$ such that

(i) $A \cup N \subseteq N(x)$,

(ii) node set $T \cup A$ induces a biclique (with T on one side of the bipartition and A on the other),

(iii) if $|T| \geq 2$, then $|A| \geq 2$.

This concept was introduced in [23] and is illustrated in Figure 2. An *extended star cutset* is one where $T \cup A \cup N$ is a node cutset. Since the nodes in $T \cup A$ induce a biclique, an extended star cutset with $N = \emptyset$ is called a *biclique cutset.* An extended star cutset having $T = \{x\}$ is called a *star cutset.* Note that a star cutset is a special case of a biclique cutset.

Let K_{BD} be a biclique with the property that its edge set $E(K_{BD})$ is an edge cutset of the connected bipartite graph G and no connected component of $G \setminus E(K_{BD})$ contains both a node of B and a node of D. Let G_B be the union of the components of $G \setminus E(K_{BD})$ containing a node of B. Similarly,

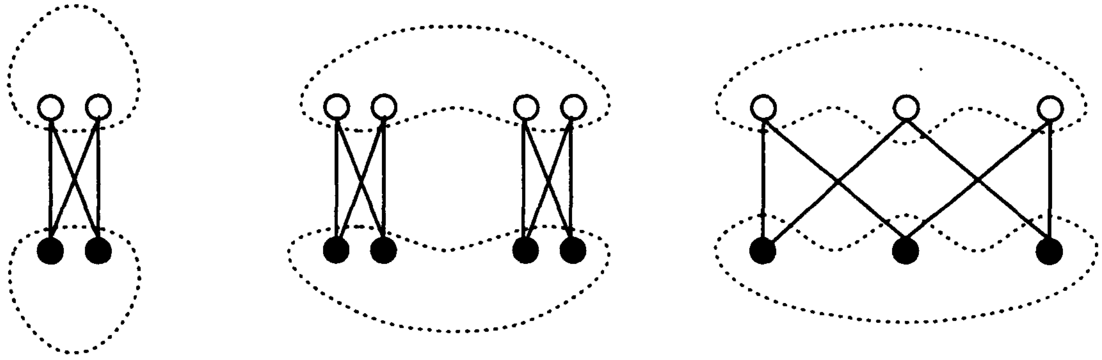


Figure 3: A 1-join, a 2-join and a 6-join

let G_D be the union of the components of $G \setminus E(K_{BD})$ containing a node of D. The set $E(K_{BD})$ is a *1-join* if the graphs G_B and G_D each contains at least two nodes. This concept was introduced by Cunningham and Edmonds [33].

Let K_{BD} and K_{EF} be two bicliques of a connected bipartite graph G, where B, D, E, F are disjoint node sets with the property that $E(K_{BD}) \cup E(K_{EF})$ is an edge cutset and every component of $G \setminus E(K_{BD}) \cup E(K_{EF})$ either contains a node of B and a node of E but no node of $D \cup F$, or contains a node of D and a node of F but no node of $B \cup E$. Let G_{BE} be the union of the components of $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of B and a node of E. Similarly, let G_{DF} be the union of the components in $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of D and a node of F. The set $E(K_{BD}) \cup E(K_{EF})$ is a *2-join* if neither of the graphs G_{BE} and G_{DF} is a chordless path with no intermediate nodes in $B \cup D \cup E \cup F$. This concept was introduced by Cornuéjols and Cunningham [29].

In a connected bipartite graph G, let $A_i, i = 1, \ldots, 6$, be disjoint nonempty node sets such that, for each i, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. Assume that $E(A)$ is an edge cutset but that no subset of its edges forms a 1-join or a 2-join. Furthermore assume that no connected component of $G \setminus E(A)$ contains a node in $A_1 \cup A_3 \cup A_5$ and a node in $A_2 \cup A_4 \cup A_6$. Let G_{135} be the union of the components of $G \setminus E(A)$ containing a node in $A_1 \cup A_3 \cup A_5$ and G_{246} be

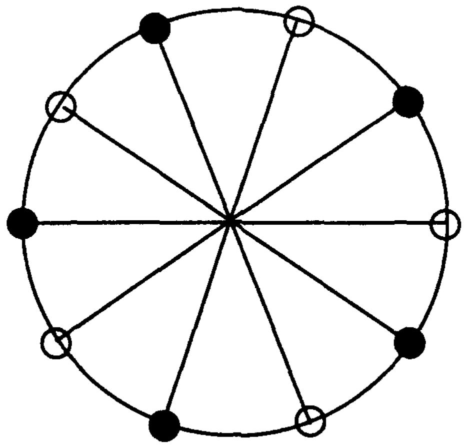


Figure 4: R_{10}

the union of components containing a node in $A_2 \cup A_4 \cup A_6$. The set $E(A)$ constitutes a *6-join* if the graphs G_{135} and G_{246} contain at least four nodes each. This concept was introduced in [22].

Two Basic Classes

A bipartite graph is *restricted balanceable* if its edges can be signed so that the sum of the weights in each *cycle* is a multiple of four. This class of bipartite graphs is well studied in the literature, see [18], [60], [68], [26]. We discuss it in a later section. R_{10} is the bipartite graph on ten nodes defined by the cycle $x_1, \ldots, x_{10}, x_1$ of length ten with chords $x_i x_{i+5}$, $1 \le i \le 5$, see Figure 4.

We can now state the decomposition theorem for balanceable bipartite graphs:

Theorem 3.2 *A balanceable bipartite graph that is not restricted balanceable is either R_{10} or contains a 2-join, a 6-join or an extended star cutset.*

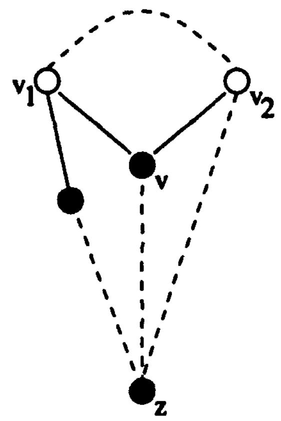


Figure 5: Parachute

3.3 Outline of the Proof

The key idea in the proof of Theorem 3.2 is that if a balanceable bipartite graph G is not basic, then G contains one of several node induced subgraphs, which force a decomposition of G with one of the cutsets described in Section 3.2.

Parachutes

A *parachute* is defined by four chordless paths of positive lengths, $T = v_1, \ldots, v_2$; $P_1 = v_1, \ldots, z$; $P_2 = v_2, \ldots, z$; $M = v, \ldots, z$, where v_1, v_2, v, z are distinct nodes, and two edges vv_1 and vv_2. No other edges exist in the parachute, except the ones mentioned above. Furthermore $|E(P_1)| + |E(P_2)| \geq 3$. See Figure 5.

Note that if G is balanceable then nodes v, z belong to the same side of the bipartition, else the parachute contains a 3-path configuration connecting v and z or an odd wheel (H, v) with three spokes.

Connected Squares and Goggles

Connected squares are defined by four chordless paths $P_1 = a, \ldots, b$; $P_2 = c, \ldots, d$; $P_3 = e, \ldots, f$; $P_4 = g, \ldots, h$, where nodes a and c are adjacent to both e and g and b and d are adjacent to both f and h, as in Figure 6. No other adjacency exists in the connected squares. Note that nodes a and b

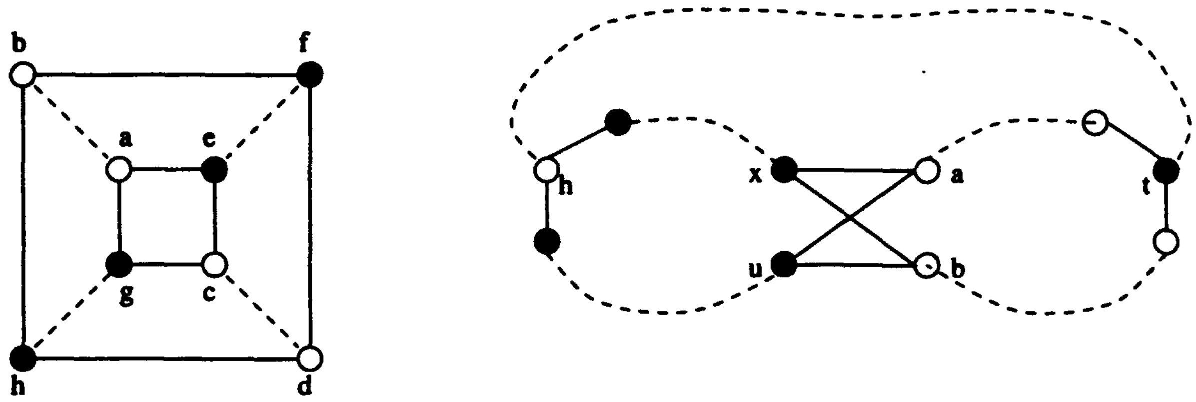


Figure 6: Connected squares and goggles

belong to the same side of the bipartition, else the connected squares contain a 3-path configuration connecting a and b or, if $|E(P_1)| = 1$, an odd wheel with center a. Therefore the nodes a, b, c, d are in one side of the bipartition and e, f, g, h are in the other.

Goggles are defined by a cycle $C = h, P, x, a, Q, t, R, b, u, S, h$, with two chords ua and xb, and chordless paths P, Q, R, S of length greater that one, and a chordless path $T = h, \ldots, t$ of length at least one, such that no intermediate node of T belongs to C. No other edge exists, connecting nodes of the goggles, see Figure 6.

Connected 6-Holes

A *triad* consists of three internally node-disjoint paths $t, \ldots, u$; $t, \ldots, v$ and $t, \ldots, w$, where t, u, v, w are distinct nodes and u, v, w belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes u, v and w are called the *attachments* of the triad.

A *fan* consists of a chordless path $x, \ldots, y$ together with a node z adjacent to at least one node of the path, where x, y and z are distinct nodes all belonging to the same side of the bipartition. Nodes x, y and z are called the *attachments* of the fan.

A *connected 6-hole* Σ is a graph induced by two disjoint node sets $T(\Sigma)$ and $B(\Sigma)$ such that each induces either a triad or a fan, the attachments of

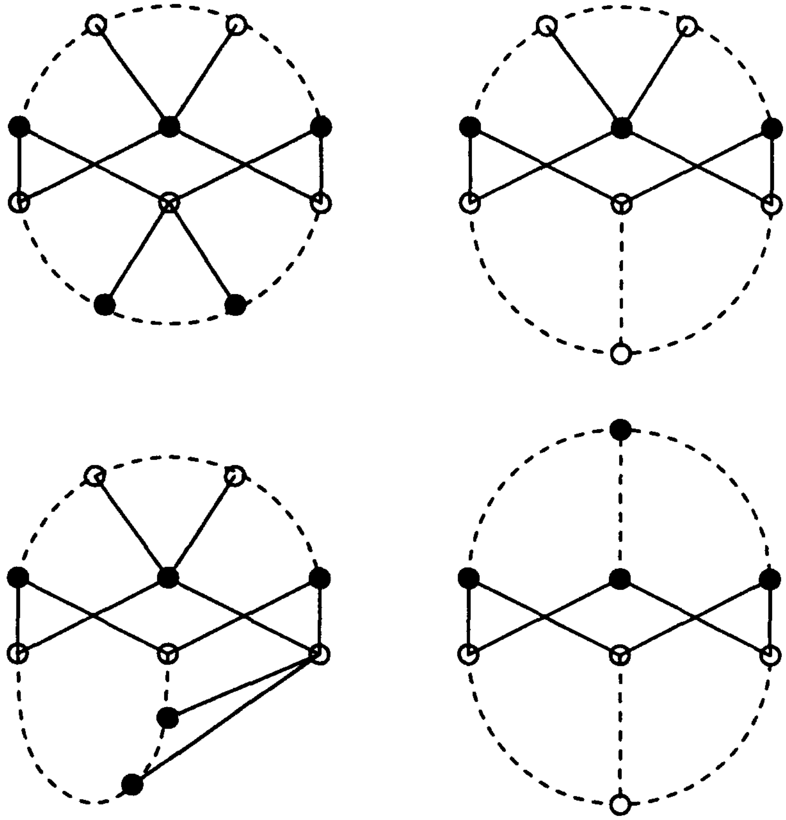


Figure 7: The four types of connected 6-holes

$T(\Sigma)$ and $B(\Sigma)$ induce a 6-hole and there are no other adjacencies between the nodes of $T(\Sigma)$ and $B(\Sigma)$. Figure 7 depicts the four types of connected 6-holes.

The following theorem proved by Conforti, Cornuéjols and Rao [23] concerns the class of balanceable bipartite graphs that do not contain a connected 6-hole or R_{10} as a node induced subgraph.

Theorem 3.3 *A balanceable bipartite graph not containing R_{10} or a connected 6-hole as a node induced subgraph either is restricted balanceable or contains a 2-join or an extended star cutset.*

We now discuss the proof of this theorem. A bipartite graph is *strongly balanceable* if its edges can be signed so that each cycle whose sum of weights is congruent to 2 *mod* 4 has at least two chords. It follows from the definition that every restricted balanceable bipartite graph is strongly balanceable. Conforti and Rao [26] prove the following:

Theorem 3.4 *A strongly balanceable bipartite graph either is restricted balanceable or contains a 1-join.*

Let K_{BD} and K_{EF} be two bicliques on disjoint node sets such that the node set $B \cup E$ induces another biclique K_{BE}. If K_{BE} is a biclique articulation of G and the removal of the edges $E(K_{BD}) \cup E(K_{EF})$ disconnects G, then $E(K_{BD}) \cup E(K_{EF})$ is a *strong 2-join.* Part II of [23] proves the following:

Theorem 3.5 *A balanceable bipartite graph that is not strongly balanceable either contains a parachute or a wheel as a node induced subgraph or has a strong 2-join.*

Part III of [23] disposes of the parachutes through the following variant of Theorem 3.5:

Theorem 3.6 *A balanceable bipartite graph that is not strongly balanceable, that contains no wheel, no R_{10} and no connected 6-hole as a node induced subgraph, either contains an extended star cutset or contains connected squares or goggles as a node induced subgraph.*

Part IV contains a decomposition result for connected squares:

Theorem 3.7 *A balanceable bipartite graph that contains connected squares but no wheel as a node induced subgraph, has a biclique articulation or a 2-join.*

Part V decomposes goggles:

Theorem 3.8 *A balanceable bipartite graph that contains goggles but no wheel, no R_{10} and no connected 6-hole as a node induced subgraph, has an extended star cutset or a 2-join.*

So the theorems contained in Parts II-V give a decompositon theorem for balanceable bipartite graphs that do not contain R_{10}, connected 6-holes or wheels as node induced subgraphs. Part VI gives a decomposition when wheels are present as node induced subgraphs:

Theorem 3.9 *A balanceable bipartite graph containing a wheel but no connected 6-hole as a node induced subgraph, has an extended star cutset.*

Since a graph that has a 1-join has a biclique articulation, Theorems 3.4 and 3.6-3.9 prove Theorem 3.3.

So it remains to find a decomposition of balanceable bipartite graphs that contain R_{10} or connected 6-holes as node induced subgraphs. This is accomplished by Conforti, Cornuéjols, Kapoor and Vušković in [22].

Theorem 3.10 *A balanceable bipartite graph containing R_{10} as a proper node induced subgraph has a biclique articulation.*

Theorem 3.11 *A balanceable bipartite graph that contains a connected 6-hole as a node induced subgraph, has an extended star cutset or a 6-join.*

Now the proof of Theorem 3.2 follows from Theorems 3.3, 3.10 and 3.11.

4 Recognition Algorithm

Conforti, Cornuéjols, Kapoor and Vušković [22] give a polynomial time algorithm to check whether a $0, \pm 1$ matrix is balanced. It generalizes the algorithm of Conforti, Cornuéjols and Rao [23] to check whether a $0, 1$ matrix is balanced. Note that, together with the signing algorithm described in Section 3.1, the algorithm to check whether a $0, \pm 1$ matrix is balanced tests whether a $0, 1$ matrix is balanceable. We describe the recognition algorithm using the bipartite representation introduced in Section 3.

4.1 Balancedness Preserving Decompositions

Let G be a connected signed bipartite graph. The removal of a node or edge cutset disconnects G into two or more connected components. From these components we construct *blocks* by adding some new nodes and signed edges. We say that a decomposition is *balancedness preserving* when it has the following property: all the blocks are balanced if and only if G itself is balanced. The central idea in the algorithm is to decompose G using balancedness preserving decompositions into a polynomial number of basic blocks that can be checked for balancedness in polynomial time.

For the 2-join and 6-join, the blocks can be defined so that the decompositions are balancedness preserving. For the extended star cutset this is not immediately possible.

2-Join Decomposition

Let $E(K_{BD}) \cup E(K_{EF})$ be a 2-join of G and let G_{BE} and G_{DF} be the graphs defined in Section 3.2. We construct the block G_1 from G_{BE} as follows.

- Add two nodes d and f, connected respectively to all nodes in B and to all nodes in E.
- Let P be a shortest path in G_{DF} connecting a node in D to a node in F. If the weight of P is 0 or 2 *mod* 4, nodes d and f are connected by a path of length 2 in G_1. If the weight of P is 0 *mod* 4, one edge of Q is signed $+1$ and the other -1, and if the weight of P is 2 *mod* 4, both edges of Q are signed $+1$. Similarly if the weight of P is 1 or 3 *mod* 4, nodes d and f are connected by a path of length 3 with edges signed so that Q and P have the same weight modulo 4. Let d' and f' be the endnodes of P in D and F respectively. Sign the edges between node d and the nodes in B exactly the same as the corresponding edges between d' and the nodes of B in G. Similarly sign the edges between f and E exactly the same as the corresponding edges between f' and the nodes in E.

The block G_2 is defined similarly.

Theorem 4.1 *Let G be a signed bipartite graph with a 2-join $E(K_{BD}) \cup E(K_{EF})$ where K_{BD} and K_{EF} are balanced and neither $D \cup F$ nor $B \cup E$ induces a biclique. Then G is balanced if and only if both blocks G_1 and G_2 are balanced.*

6-Join Decomposition

Let G be a signed bipartite graph and let $A_1, \ldots, A_6$ be disjoint nonempty node sets such that the edges of the graph A induced by $\cup_{i=1}^{6} A_i$ form a 6-join. Let G_{135} and G_{246} be the graphs defined in Section 3.2. We construct the block G_1 from G_{135} as follows:

- Add a node a_2 adjacent to all the nodes in A_1 and A_3, a node a_4 adjacent to all the nodes in A_3 and A_5 and a node a_6 adjacent to all the nodes in A_5 and A_1.
- Pick any three nodes $a'_2 \in A_2$, $a'_4 \in A_4$ and $a'_6 \in A_6$ and, in G_1, sign the edges incident with a_2, a_4 and a_6 according to the signs of the corresponding edges of G incident with a'_2, a'_4 and a'_6.

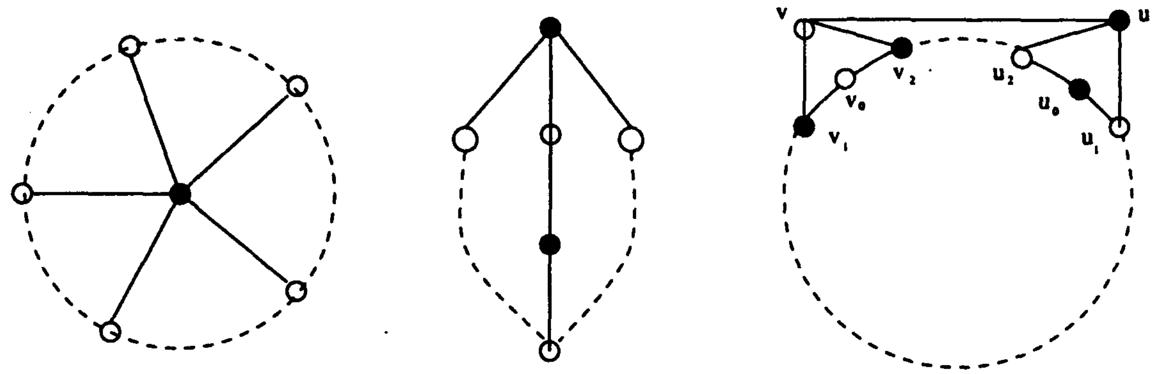
The block G_2 is defined similarly.

Theorem 4.2 *Let G be a signed bipartite graph with a 6-join $E(A)$ such that A is balanced. Then G is balanced if and only if both blocks G_1 and G_2 are balanced.*

4.2 Extended Star Cutset Decomposition

Consider the following way of defining the blocks for the extended star decomposition of a connected signed bipartite graph G. Let S be an extended star cutset of G and $G'_1, \ldots, G'_k$ the connected components of $G \setminus S$. Define the blocks to be $G_1, \ldots, G_k$ where G_i is the subgraph of G induced by $V(G'_i) \cup S$ with all edges keeping the same weight as in G.

The extended star decomposition defined in this way is not balancedness preserving. Consider, for example, a signed odd wheel (H, x) where H is an *unquad hole* (a hole of weight congruent to 2 *mod* 4). If we decompose (H, x) by the extended star cutset $\{x\} \cup N(x)$, then it is possible that all of the blocks are balanced, whereas (H, x) itself is not since H is an unquad hole. Two other classes of bipartite graphs that can present a similar problem when decomposing with an extented star cutset are tents and short 3-path configurations, see Figure 8. A *tent*, denoted by $\tau(H, u, v)$, is a bipartite graph induced by a hole H and two adjacent nodes $u, v \notin V(H)$ each having two neighbors on H, say u_1, u_2 and v_1, v_2 respectively, with the property that



(a) Odd wheel (b) Short 3PC (c) Tent

Figure 8: Odd wheel, short 3-path configuration and tent

u_1, u_2, v_2, v_1 appear in this order on H. A *short 3-path configuration* is a 3-path configuration in which one of the paths contains three edges.

To overcome the fact that our extended star decomposition is not balancedness preserving, we proceed in the following way. We transform the input graph G into a graph G' that contains a polynomial number of connected components, each of which is a node induced subgraph of G, and which has the property that if G is not balanced, then G' contains an unquad hole that will either never be broken by any of the decompositions we use, or else be detected while performing the decomposition. We call this process a *cleaning procedure*. To do this, we have to study the structure of signed bipartite graphs that are not balanced, in particular the structure of a smallest (in the number of edges) unquad hole. For such a hole we prove the following theorem.

Theorem 4.3 *In a non-balanced signed bipartite graph, a smallest unquad hole H^* contains two edges x_1x_2 and y_1y_2 such that:*

- *The set $N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)$ contains all nodes with an odd number (greater than 1) of neighbors in H.*
- *For every tent $\tau(H^*, u, v)$, u or v is contained in $N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)$.*

Let x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 be subpaths of H^*. The above theorem shows that if we remove from G the nodes $N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2) \setminus \{x_0, x_1, x_2, x_3, y_0, y_2, y_3, y_4\}$, then H^* will be *clean* (i.e. it will not be contained in any odd wheel or tent). If H^* is contained in a short 3-path configuration, this can be detected during the decomposition (before it is broken). It turns out that, by this process, all the problems are eliminated. So the cleaning procedure consists of enumerating all possible pairs of chordless paths of length 3, and in each case, generating the subgraph of G as described above. The number of subgraphs thus generated is polynomial and, if G is not balanced, then at least one of these subgraphs contains a clean unquad hole.

4.3 Algorithm Outline

The recognition algorithm takes a signed bipartite graph as input and recognizes whether or not it is balanced. The algorithm consists of four phases:

- **Preprocessing:** The cleaning procedure is applied to the input graph.
- **Extended Stars:** Extended star decompositions are performed, until no block contains an extended star cutset.
- **6-joins:** 6-join decompositions are performed until no block contains a 6-join.
- **2-joins:** Finally, 2-join decompositions are performed until no block contains a 2-join.

The 2-join and 6-join decompositions cannot create any new extended star cutset, except in one case which can be dealt with easily. Also a 2-join decomposition does not create any new 6-joins. So, when the algorithm terminates, none of the blocks have an extended star cutset, a 2-join or a 6-join. By the decomposition theorem (Theorem 3.2), if the original signed bipartite graph is balanced, the blocks must be copies of R_{10} or restricted balanced (i.e. the weight of every cycle is a multiple of four). R_{10} is a graph with only ten nodes and so it can be checked in constant time. Checking whether a signed bipartite graph is restricted balanced can be done using the following algorithm of Conforti and Rao [26]:

Construct a spanning forest in the bipartite graph and check if there exists a cycle of weight 2 mod 4 which is either fundamental or is the symmetric difference of fundamental cycles. If no such cycle exists, the signed bipartite graph is restricted balanced.

A different algorithm for this recognition problem, due to Yannakakis [68], has linear time complexity and will be mentioned in Section 5.1.

The preprocessing phase and the decomposition phases using 2-joins and 6-joins are easily shown to be polynomial. For the extended star decomposition phase, it is shown that each bipartite graph which is decomposed has a path of length three which is not present in any of the blocks. This bounds the number of such decompositions by a polynomial in the size of the graph. Thus the entire algorithm is polynomial. See [22] for details.

4.4 Two Related Recognition Problems

The algorithm presented in the previous section recognizes in polynomial time whether a signed bipartite graph contains an unquad hole. Interestingly Kapoor [50] has shown that it is NP-complete to recognize whether a signed bipartite graph contains an unquad hole going through a prespecified node.

Theorem 4.4 (Kapoor [50]) *Given a bipartite graph G and a node v of G, it is NP-complete to check if G has an unquad hole which contains node v.*

One can also ask the following question: given a signed bipartite graph, does it contain a *quad* hole (i.e. a hole of weight 0 *mod* 4)? A linear algorithm for this recognition problem is given by Conforti, Cornuéjols and Vušković [25].

A signed bipartite graph is *unbalanced* if it does not contain a quad hole. Bipartite graphs which can be signed to be unbalanced are called *unbalanceable.* If a bipartite graph is unbalanceable, there is a simple algorithm to perform the signing (similar to the signing algorithm of Section 3.1). The class of unbalanced signed bipartite graphs is structurally much simpler than the class of balanced signed bipartite graphs, one of the reasons being the following property: in a signed bipartite graph G, all holes of G are unquad if and only if all cycles of G are unquad. The recognition algorithm in [25] is based on the following decomposition theorem.

Theorem 4.5 (Conforti, Cornuéjols, Vušković [25]) *An unbalanceable bipartite graph is either a hole or it contains a one node or a two node cutset.*

5 Classes of Balanceable Matrices

In this section we survey decomposition theorems for several classes of balanceable matrices. We relate these decompositions to Theorem 3.2.

5.1 Seymour's Decomposition of Totally Unimodular Matrices

Seymour [58] gave an important decomposition theorem for 0,1 matrices that can be signed to be totally unimodular. The decompositions involved in his theorem are 1-separations, 2-separations and 3-separations. A 0,1 matrix B has a *k-separation* if its rows and columns can be partitioned so that

$$B = \begin{pmatrix} A^1 & D^2 \\ D^1 & A^2 \end{pmatrix}$$

where $r(D^1)+r(D^2) = k-1$ and the number of rows plus number of columns of A^i is at least k, for $i = 1,2$. ($r(M)$ denotes the $GF(2)$-rank of the 0,1 matrix M).

The basic matrices used in Seymour's decomposition theorem are

$$R_{10} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad R'_{10} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

graphic matrices and their transpose. A 0,1 matrix M is *graphic* if there exists a tree T such that the rows of M are indexed by the edges of T and the columns of M are incidence vectors of paths of T. The transpose of a graphic matrix is said to be *cographic.*

Theorem 5.1 (Seymour [58]) *A 0,1 matrix that can be signed to be totally unimodular is either R_{10}, R'_{10}, graphic, cographic, or it contains a 1-, 2- or 3-separation.*

For a 1-separation $r(D^1)+r(D^2)=0$. Thus both D^1 and D^2 are matrices all of whose entries are 0. The bipartite graph corresponding to the matrix B is disconnected.

For the 2-separation $r(D^1)+r(D^2)=1$, thus w.l.o.g. D^2 has rank zero and is identically zero. Since $r(D^1)=1$, after permutation of rows and columns, $D^1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where $\mathbf{1}$ denotes a matrix all of whose entries are 1 and $\mathbf{0}$ is a matrix all of whose entries are 0. The 2-separation in the bipartite graph representation of B corresponds to a 1-join.

For the 3-separation $r(D^1)+r(D^2)=2$. If both D^1 and D^2 have rank 1 then, after permutation of rows and columns,

$$D^1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad D^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

This 3-separation in the bipartite graph representation of B corresponds to a 2-join. Now w.l.o.g. we assume $r(D^1)=2$ and $r(D^2)=0$. Up to permutation of rows and columns D^1 is of the form

$$D^1 = \begin{pmatrix} P & N \\ M & Q \end{pmatrix}$$

where N is a 2×2 nonsingular matrix (over $GF(2)$). Again, up to permutation of rows and columns, there are exactly two possible cases for N:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We first examine the structure of D^1 when N is of the first kind. Given N, P and Q, the matrix M is completely determined by the formula $M = QN^{-1}P$, because $r(D^1)=2$. So, the bipartite graph representation of D^1 has node sets C_1, C_2 and C_3 corresponding to columns of $(P\ N)$ of the type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, and the node sets R_1, R_2 and R_3 corresponding to rows of $\begin{pmatrix} N \\ Q \end{pmatrix}$ of the type $(1\ 1)$, $(1\ 0)$ and $(0\ 1)$ respectively. Either or both node sets C_1 and R_1 may be empty. When all the node sets are nonempty the 3-separation is a 6-join. When one of C_1 or R_1 is empty it is called a *4-join* and when both are empty it is a 2-join.

When N is of the second type, node sets C_2 and R_3 may be empty. When neither is empty, we get a 6-join in the bipartite graph representation. When one is empty, we get a 4-join and when both are empty, a 3-join. Note that, when a bipartite graph contains a 1-join, a 3-join or a 4-join, it also contains an extended star cutset. So the only 1-, 2-, and 3-separations which do not induce an extended star cutset are the 2-join and the 6-join. By noting that R'_{10} contains an extended star cutset, Seymour's theorem 5.1 implies a result resembling Theorem 3.2:

Corollary 5.2 *A* $0,1$ *matrix that can be signed to be totally unimodular is either* R_{10}*, graphic, cographic, or its bipartite representation contains an extended star cutset, a 2-join or a 6-join.*

5.2 More Decomposition Theorems

A signed bipartite graph is *strongly balanced* if every cycle of weight 2 *mod* 4 has at least two chords. Strongly balanced $0, \pm 1$ matrices are defined accordingly. It follows from the definition that restricted balanced $0, \pm 1$ matrices are strongly balanced, and it can be shown that strongly balanced $0, \pm 1$ matrices are totally unimodular, see [26]. Strongly balanceable $0,1$ matrices can be signed to be strongly balanced with the signing algorithm described in Section 3.1. Conforti and Rao [26] have shown that a strongly balanceable $0,1$ matrix that is not restricted balanceable has a 2-separation (the bipartite graph representation has a 1-join).

Theorem 5.3 (Conforti, Rao [26]) *A strongly balanceable bipartite graph either is restricted balanceable or contains a 1-join.*

Crama, Hammer and Ibaraki [32] say that a $0, \pm 1$ matrix A is *strongly unimodular* if every basis of (A, I) can be put in triangular form by permutation of rows and columns.

Theorem 5.4 (Crama, Hammer, Ibaraki [32]) *A* $0, \pm 1$ *matrix is strongly unimodular if and only if it is strongly balanced.*

Yannakakis [68] has shown that a restricted balanceable $0,1$ matrix having both a row and a column with more than two nonzero entries has a very special 3-separation: the bipartite graph representation has a 2-join consisting of two single edges. A bipartite graph is *2-bipartite* if all the nodes in one side of the bipartition have degree at most 2.

Theorem 5.5 (Yannakakis [68]) *A restricted balanceable bipartite graph either is 2-bipartite or contains a cutnode or contains a 2-join consisting of two edges.*

Based on this theorem, Yannakakis designed a linear time algorithm for checking whether a $0, \pm 1$ matrix is restricted balanced. A different algorithm for this recognition problem was stated in an earlier section of this survey.

A bipartite graph is *linear* if it does not contain a cycle of length 4. Note that an extended star cutset in a linear bipartite graph is always a star cutset, due to Condition (iii) in the definition of extended star cutsets. Conforti and Rao [27] proved the following theorem for linear balanced bipartite graphs:

Theorem 5.6 (Conforti, Rao [27]) *A linear balanced bipartite graph either is restricted balanced or contains a star cutset.*

5.3 Totally Balanced $0, 1$ Matrices

A bipartite graph is *totally balanced* if every hole has length 4. Totally balanced bipartite graphs arise in location theory and were the first balanced graphs to be the object of an extensive study. Several authors (Golumbic and Goss [42], Anstee and Farber [2] and Hoffman, Kolen and Sakarovitch [47] among others) have given properties of these graphs.

An edge uv is *bisimplicial* if either u or v has degree 1 or the node set $N(u) \cup N(v)$ induces a biclique. Note that if uv is a bisimplicial edge and nodes u and v have degree at least 2, then G has a strong 2-join formed by the edges adjacent to exactly one node in the set $\{u, v\}$. The 2-join is strong since $N(u) \cup N(v) \setminus \{u, v\}$ induces a biclique. The following theorem of Golumbic and Goss [42] characterizes totally balanced bipartite graphs.

Theorem 5.7 (Golumbic, Goss, [42]) *A totally balanced bipartite graph has a bisimplicial edge.*

Since wheels and parachutes contain holes of length greater than 4, neither of these two graphs can occur as a node induced subgraph of a totally balanced bipartite graph. Furthermore if uv is a bisimplicial edge such that the degree of both u and v is greater that 2, then nodes u, v together with their neighbors induce a strong 2-join. So the above theorem is related to Theorem 3.5.

A $0,1$ matrix A is in *standard greedy form* if it contains no 2×2 submatrix of the form $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, where the order of the rows and columns in the submatrix is the same as in the matrix A. This name comes from the fact that the linear program

$$\begin{array}{rcl} max \sum y_i & & \\ yA & \leq & c \\ 0 \leq y & \leq & p \end{array} \qquad (3)$$

can be solved by a greedy algorithm. Namely, given $y_1, \ldots, y_{k-1}$ such that $\sum_{i=1}^{k-1} a_{ij} y_i \leq c_j$, $j = 1, \ldots, n$ and $0 \leq y_i \leq p_i$, $i = 1, \ldots, k-1$, set y_k to the largest value such that $\sum_{i=1}^{k} a_{ij} y_i \leq c_j$, $j = 1, \ldots, n$ and $0 \leq y_k \leq p_k$. The resulting greedy solution is an optimum solution to this linear program. What does this have to do with totally balanced matrices? The answer is in the next theorem.

Theorem 5.8 (Hoffman, Kolen, Sakarovitch [47]) *A $0,1$ matrix is totally balanced if and only if its rows and columns can be permuted in standard greedy form.*

This transformation can be performed in time $O(nm^2)$ [47].

Totally balanced $0,1$ matrices come up in various ways in the context of facility location problems on trees. For example, the *covering problem*

$$\begin{array}{rcl} min \sum_1^n c_j x_j + \sum_1^m p_i z_i & & \\ \sum_j a_{ij} x_j + z_i & \geq & 1, \quad i = 1, \ldots, m \\ x_j, z_i & \in & \{0,1\} \end{array} \qquad (4)$$

can be interpreted as follows: c_j is the set up cost of establishing a facility at site j, p_i is the penalty if client i is not served by any facility, and $a_{ij} = 1$ if a facility at site j can serve client i, 0 otherwise.

When the underlying network is a tree and the facilities and clients are located at nodes of the tree, it is customary to assume that a facility at site j can serve all the clients in a *neighborhood subtree* of j, namely, all the clients within distance r_j from node j.

An *intersection matrix* of the set $\{S_1, \ldots, S_m\}$ versus $\{R_1, \ldots, R_n\}$, where S_i, $i = 1, \ldots, m$, and R_j, $j = 1, \ldots, n$, are subsets of a given set, is defined to be the $m \times n$ 0,1 matrix $A = (a_{ij})$ where $a_{ij} = 1$ if and only if $S_i \cap R_j \neq \emptyset$.

Theorem 5.9 (Giles [41]) *The intersection matrix of neighborhood subtrees versus nodes of a tree is totally balanced.*

It follows that the above location problem on trees (4) can be solved as a linear program (by Theorem 2.8 and the fact that totally balanced matrices are balanced). In fact, by using the standard greedy form of the neighborhood subtrees versus nodes matrix, and by noting that (4) is the dual of (3), the greedy solution described earlier for (3) can be used, in conjunction with complementary slackness, to obtain an elegant solution of the covering problem. The above theorem of Giles has been generalized as follows.

Theorem 5.10 (Tamir [61]) *The intersection matrix of neighborhood subtrees versus neighborhood subtrees of a tree is totally balanced.*

Other classes of totally balanced 0,1 matrices arising from location problems on trees can be found in [62].

6 A Coloring Theorem on Graphs

Let G be a graph and q a positive integer no greater than its chromatic number $\chi(G)$. A *partial q-coloring* of G is a family of q pairwise disjoint stable sets, say $S_1, \ldots, S_q$. If $x \in S_i$, node x is said to have color i. Not all nodes of G need have a color. The partial q-coloring is *optimal* if the number of colored nodes is as large as possible.

A family of cliques $C_1, \ldots, C_r$ is said to be *associate* of a partial q-coloring $S_1, \ldots, S_q$ if these cliques are pairwise disjoint, each clique C_j intersects each S_i, and every node of G is either colored or belongs to one of the cliques (or both).

Theorem 6.1 (Berge [8]) *Let G be a graph and $q \leq \chi(G)$ a positive integer. If the clique-node matrix of G is balanced, then every optimal partial q-coloring has an associate family of cliques.*

In the special case where $q = 1$, an optimal partial q-coloring is a maximum stable set. So the existence of an associate family also follows from Lovász's perfect graph theorem [53]. Indeed, let $\alpha(G)$ denote the stability number of G. When G is perfect, a minimum cardinality clique partition of G, say $C_1, \ldots, C_{\alpha(G)}$, is an associate family of cliques for any stable set S_1 of cardinality $\alpha(G)$.

7 Some Conjectures and Open Questions

7.1 Eliminating Edges

The following conjecture extends a conjecture of Conforti and Rao [27] to $0, \pm 1$ matrices.

Conjecture 7.1 *In a balanced signed bipartite graph G, either every edge belongs to some R_{10}, or some edge can be removed from G so that the resulting signed bipartite graph is still balanced.*

The condition on R_{10} is necessary since removing any edge from R_{10} yields a wheel with three spokes or a 3-path configuration as a node induced subgraph.

The truth of the above conjecture would imply that given a $0, \pm 1$ balanced matrix we can sequentially turn the nonzero entries to zero in some specific order until every nonzero belongs to some R_{10} matrix, while maintaining balanced $0, \pm 1$ matrices in the intermediate steps.

For $0, 1$ matrices, the above conjecture reduces to the following:

Conjecture 7.2 (Conforti, Rao [27]) *Every balanced bipartite graph contains an edge which is not the unique chord of a cycle.*

It follows from the definition that restricted balanced signed bipartite graphs are exactly the ones such that the removal of *any* subset of edges leaves a restricted balanced signed bipartite graph.

Conjecture 7.1 holds for signed bipartite graphs that are strongly balanced since, by definition, the removal of any edge leaves a chord in every unquad cycle.

Theorem 5.7 shows that the graph obtained by eliminating a bisimplicial edge in a totally balanced bipartite graph is totally balanced. Hence Conjecture 7.2 holds for totally balanced bipartite graphs.

7.2 Strengthening the Decomposition Theorems

The extended star decomposition is not balancedness preserving. This heavily affects the running time of the recognition algorithm for balancedness. Therefore it is important to find strengthenings of Theorem 3.2 so that only operations that preserve balancedness are used. We have been unable to obtain these results even for linear balanced bipartite graphs [28].

Let H be a hole where nodes $u_1, \ldots, u_p, v_1, \ldots, v_q, w_1, \ldots, w_p, x_1, \ldots, x_q$ appear in this order when traversing H, but are not necessarily adjacent. Let $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ be two node sets having empty intersection with $V(H)$ and inducing a biclique K_{YZ}. Node y_i is connected with u_i and w_i for $1 \leq i \leq p$. Node z_i is connected with v_i and x_i for $1 \leq i \leq q$. Such a graph, denoted with W_{pq} is balanceable and contains no 2-join, no 6-join and no biclique cutset. But this is the only balanceable bipartite graph with this property that we know. This suggests the following conjecture:

Conjecture 7.3 *Every balanceable bipartite graph that is not W_{pq}, R_{10} or restricted balanceable, has a 2-join, a 6-join or a biclique cutset.*

Another direction in which the main theorem might be strengthened is as follows.

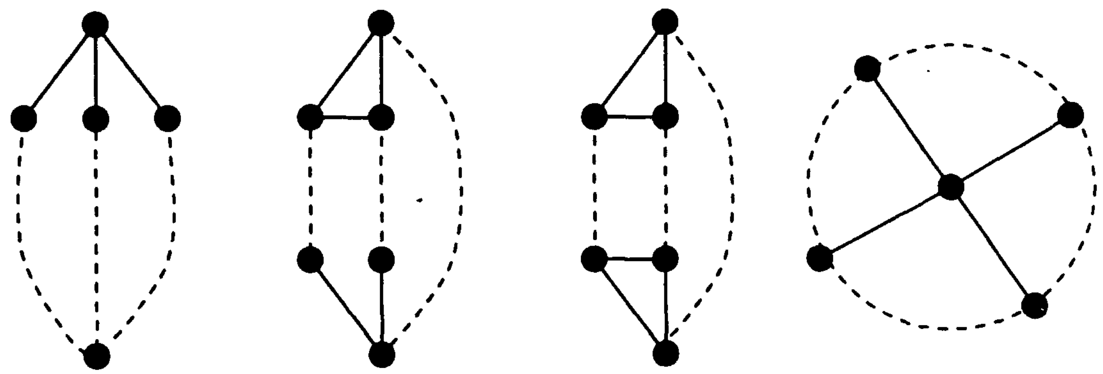
Conjecture 7.4 *Every balanceable bipartite graph which is not signable to be totally unimodular has an extended star cutset.*

In [23], it was shown that the bipartite representation of every balanced 0, 1 matrix which is not totally unimodular, has an extended star cutset.

7.3 Holes in Graphs

α-Balanced Graphs

Let G be a signed graph (not necessarily bipartite) and let α be a vector whose components are in one-to-one correspondence with the chordless cycles of G and take values in $\{0, 1, 2, 3\}$. G is said to be *α-balanced* if the sum of the weights on each chordless cycle H of G is congruent to α_H *mod* 4. In the special case where G is bipartite and $\alpha = \mathbf{0}$, this definition coincides with the notion of balanced signed bipartite graph, introduced earlier in this survey.



(a) (b) (c) (d)

Figure 9: 3-path configurations and wheel

A graph is *α-balanceable* if there is a signing of its edges such that the resulting signed graph is α-balanced. A *3-path configuration* is one of the three graphs represented in Figure 9 (a), (b) or (c). A *wheel* consists of a chordless cycle H and a node $v \notin V(H)$ with at least three neighbors on H, see Figure 9 (d).

Theorem 7.5 (Truemper [63]) *A graph G is α-balanceable if and only if*

- $\alpha_H \equiv |H| \bmod 2$ *for every chordless cycle H of G,*
- *every 3-path configuration and wheel of G is α-balanceable.*

Theorem 3.1 is the special case of this theorem where G is bipartite and $\alpha = \mathbf{0}$, while Theorem 4.5 provides an independent proof of Theorem 7.5 in the special case where G is bipartite and $\alpha = \mathbf{2}$. A difficult open problem is to extend the decomposition theorem 3.2 to α-balanceable graphs.

Odd Holes

A long standing open problem in graph theory is that of testing whether a graph contains an odd hole. No polynomial time algorithm is known for this problem. It was shown by Bienstock [10] that it is NP-complete to decide whether a graph has an odd (even respectively) hole containing a given node.

One might be lead to believe that testing whether a graph has an odd hole is also NP-complete. However, recall that testing whether a bipartite graph has a hole of length 2 *mod* 4 is polynomial time (Section 4.3) whereas testing whether a bipartite graph has a hole of length 2 *mod* 4 containing a given node is NP-complete (Theorem 4.4). This encourages us to beleive that deciding whether a graph has an odd hole can also be done in polynomial time.

A related open question is that of testing whether a graph or its complement has an odd hole. Berge [3] suggested that this question is in fact nothing but the recognition problem for perfect graphs. He made the following conjecture in 1961, when he introduced the concept of perfect graphs.

Conjecture 7.6 (Berge [3]) *A graph is perfect if and only if neither it nor its complement has an odd hole.*

This conjecture has been shown to hold for several classes of graphs and, for some of these classes, a polynomial time recognition algorithm is known as well. Such algorithms often rely on a decomposition theorem. So, a general algorithm for the recognition of perfect graphs may well require a decomposition with the flavor of Theorem 3.2. In particular, 2-join decompositions seem relevant for perfect graphs [29].

We know of two important classes of perfect $0, \pm 1$ matrices:

- the matrices obtained from perfect $0, 1$ matrices by switching signs in a subset of columns, and
- balanced $0, \pm 1$ matrices.

It is an open problem to construct all perfect $0, \pm 1$ matrices, starting from these two basic classes.

Even Holes

Another open problem is testing in polynomial time whether a graph contains an even hole. Even holes are related to β-perfect graphs introduced by Markossian, Gasparian and Reed [54]. A graph G is *β-perfect* if, for every node induced subgraph H of G, the chromatic number of H equals $max\{\delta(F) + 1 \ : \ F$ is a node induced subgraph of $H\}$, where $\delta(F)$ denotes the smallest node degree in F. No β-perfect graph contains an even hole, but the converse is not true. Also, the complement of a β-perfect graph need not β-perfect.

Theorem 7.7 (Markossian, Gasparian, Reed [54]) *A graph G and its complement $\bar{G}$ are both β-perfect if neither G nor $\bar{G}$ contains an even hole.*

Theorem 7.8 (Markossian, Gasparian, Reed [54]) *If G contains no even hole and no even cycle with precisely one chord, where this chord forms a triangle with two edges of the cycle, then G is β-perfect.*

A linear time algorithm to determine if either G or its complement $\bar{G}$ contains an even hole follows from their structural characterization of such graphs. It still remains open to refine Theorem 7.8 in order to determine exactly which graphs are β-perfect. Also open is the complexity of deciding if a given graph is β-perfect.

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