REPORT 1234

ON THE KERNEL FUNCTION OF THE INTEGRAL EQUATION RELATING THE LIFT AND DOWNWASH DISTRIBUTIONS OF OSCILLATING FINITE WINGS IN SUBSONIC FLOW

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ON THE KERNEL FUNCTION OF THE INTEGRAL EQUATION RELATING THE LIFT AND DOWNWASH DISTRIBUTIONS OF OSCILLATING FINITE WINGS IN SUBSONIC FLOW [1]

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SUMMARY

This report treats the kernel function of an integral equation that relates a known or prescribed downwash distribution to an unknown lift distribution for a harmonically oscillating finite wing in compressible subsonic flow. The kernel function is reduced to a form that can be accurately evaluated by separating the kernel function into two parts: a part in which the singularities are isolated and analytically expressed and a nonsingular part which may be tabulated. The form of the kernel function for the sonic case (Mach number of 1) is treated separately. In addition, results for the special cases of Mach number of 0 (incompressible case) and frequency of 0 (steady case) are given.

The derivation of the integral equation which involves this kernel function, originally performed elsewhere (see, for example, NACA Technical Memorandum 979), is reproduced as an appendix. Another appendix gives the reduction of the form of the kernel function obtained herein for the three-dimensional case to a known result of Possio for two-dimensional flow. A third appendix contains some remarks on the evaluation of the kernel function, and a fourth appendix presents an alternate form of expression for the kernel function.

INTRODUCTION

The analytical determination of air forces on oscillating wings in subsonic flow has been a continuing problem for the past 30 years. Throughout the first and greater part of this time, efforts were directed mainly toward the determination of forces on wings in incompressible flow. These efforts have led to important closed-form solutions for rigid wings in two-dimensional flow (ref. 1), to solutions in terms of series of Legendre functions for distorting wings of circular plan form (refs. 2 and 3), and to many approximate, yet useful, results for wings of elliptic, rectangular, and triangular plan form (see, for example, refs. 4 to 12).

Although these results for incompressible flow play a highly significant role in applications of unsteady aerodynamic theory, the advent of higher and higher speed aircraft during the last 15 years has brought a growing need for knowledge of the effect that the compressibility of air might have on unsteady air forces, or for analytically derived unsteady air forces based on a compressible medium. The transition to results for a compressible fluid from those for an incompressible fluid is not likely to be accomplished by applications of simple transformations or correction factors, such as the well-known Prandtl-Glauert factor for steady flow. This difficulty is associated with the fact that the time required for signals arising at one point in the medium to reach other points gives rise not only to changes in magnitudes of forces but also to additional phase lags between instantaneous positions, velocities, and accelerations of the wing and the corresponding instantaneous forces associated with these quantities. In order to obtain results for the compressible case, it therefore appears necessary to deal directly with the boundary-value problem for this case.

The boundary-value problem for a two-dimensional wing in compressible flow has been successfully attacked from two points of view. First, by consideration of an acceleration or pressure potential, Possio (ref. 13) reduced the problem to that of an integral equation relating a prescribed downwash distribution to an unknown lift distribution. The kernel of this integral equation, which is a rather abstruse function, was reduced to a form that, except at singular points, could be evaluated. Schwarz (ref. 14) later isolated and determined the analytic behavior of the singular points of Possio's results and made fairly extensive tables of the kernel function. These tabular values were used by various investigators (for examples, refs. 15 and 16) to obtain, by numerical procedures, initial tables of force and moment coefficients for oscillating wings in compressible subsonic flow.

The second successful approach to the solution of the boundary-value problem for a two-dimensional wing (see refs. 17 to 19) is achieved by a transformation to elliptic coordinates followed by a separation of variables that reduces the boundary-value problem from one in partial-differential equations to one in ordinary differential equations of the Mathieu type. The solutions turn out as infinite series in terms of Mathieu functions. Numerical results obtained recently by this procedure agree with results previously obtained by the numerical procedures using the kernel function (see, for example, ref. 20).

With regard to boundary-value problems for finite wings in compressible flow, it appears that the procedure of separation of variables could be a feasible approach only for wings of very special plan forms such as a circle or an ellipse. In any case, the development of the appropriate mathematical functions for a particular plan form would become

[1] Supersedes NACA TN 3131, 1954.

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highly involved. On the other hand, it appears that approximate procedures similar to those used for two-dimensional wings might afford an approach to solutions of these problems which, though laborious, might be handled by routine numerical methods.

The kernel function of the integral equation relating pressure and downwash for the three-dimensional case appears as an improper integral. The purpose of this report is to treat and discuss this kernel function. The improper integral is reduced to a form that can be accurately evaluated by numerical procedures. The form and order of all its singularities are determined and an expression for the kernel function is derived in which the singularities are isolated. Special forms of the kernel for the sonic case ($M=1$), the incompressible case ($M=0$), and the steady case ($k=0$) are presented. A series expansion in powers of the reduced-frequency parameter k is developed.

The availability of the kernel in a form which can be rapidly evaluated makes possible the use of numerical procedures, similar to those used in the two-dimensional case, to obtain aerodynamic forces for finite wings.

SYMBOLS

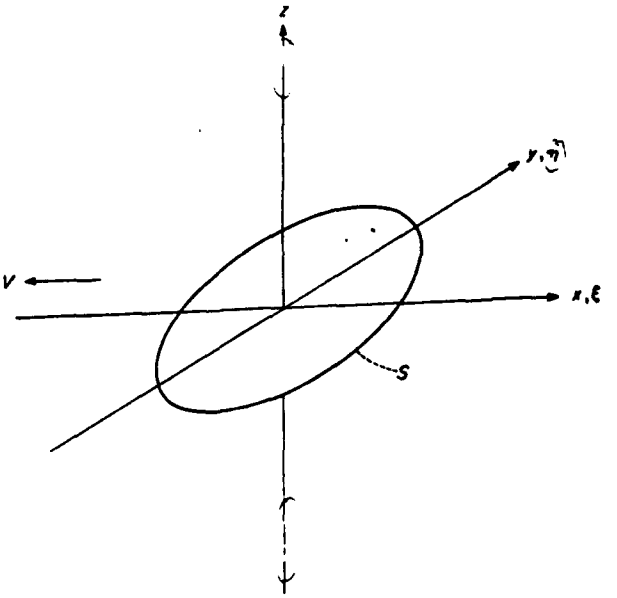
c	velocity of sound
$H_0^{(2)}, H_1^{(2)}$	Hankel functions of second kind of zero and first order, respectively
I_0, I_1	modified Bessel functions of first kind of zero and first order, respectively
J_0	Bessel function of first kind of zero order
K_0, K_1	modified Bessel functions of second kind of zero and first order, respectively
$K(x_0,y_0)$	kernel function of integral equation
$K'(x_0,y_0)$	singular part of $K(x_0,y_0)$
k	reduced-frequency parameter, $l\omega/V$
L_0, L_1	modified Struve functions of zero and first order, respectively
$L(\xi,\eta)$	unknown lift distribution
l	reference length
M	Mach number, V/c
p	pressure
$r=\beta\sqrt{y_0^2+z^2}$	
S	region of xy-plane occupied by wing
t	time
V	forward velocity of wing
$\overline{w}(x,y)$	amplitude function of prescribed downwash, $w(x,y,t)=e^{i\omega t}\overline{w}(x,y)$
x,y,z,ξ,η	Cartesian coordinates
$x_0=x-\xi$	
$y_0=y-\eta$	
$\beta=\sqrt{1-M^2}$	
γ	Euler's constant
$\epsilon=\sqrt{x_0^2+\beta^2y_0^2}$	
ϕ	velocity potential
ψ	acceleration potential
ρ	fluid density
ω	circular frequency of oscillation
$\overline{\omega}=\omega/V\beta^2$	

ANALYSIS

INTEGRAL EQUATION AND ORIGINAL FORM OF KERNEL FUNCTION

The main purpose of this analysis is to treat the kernel function of an integral equation that relates a known or prescribed downwash distribution to an unknown lift distribution for a harmonically oscillating finite wing in compressible subsonic flow. The integral equation referred to can be obtained by employing the Prandtl acceleration potential to treat linearized boundary-value problems for oscillating finite wings by means of doublet distributions. Derivation of this integral equation from the linearized boundary-value problem for a wing is a preliminary task that has been done elsewhere (see, for example, ref. 21), but it is reproduced herein as an appendix for the sake of completeness.

In keeping with the concepts of linear theory, the wing is considered a plane impenetrable surface S which lies nearly in the xy-plane as indicated in sketch 1:



Sketch 1.

The x,y,z coordinate system and the surface S are assumed to move in the negative x-direction at a uniform velocity V.

In terms of these coordinates, the integral equation may be formally written as

$$\overline{w}(x,y)=\frac{1}{4\pi}\iint_S L(\xi,\eta)\,K(x_0,y_0)\,d\xi\,d\eta \qquad (1)$$

where $\overline{w}(x,y)$ is the amplitude function of the prescribed downwash, $K(x_0,y_0)=K(x-\xi,\,y-\eta)$ is the kernel function and physically represents the contribution to downwash at a field point (x,y) due to a pulsating pressure doublet of unit strength located at any point (ξ,η), and $L(\xi,\eta)$ is the unknown lift distribution or local doublet strength.

The kernel function may be mathematically defined by the following improper integral expression (see eq. (A12), appendix A):

$$K(x_0,y_0)=\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-\frac{i\omega x_0}{V}}\int_{-\infty}^{x_0}\frac{e^{i\bar{\omega}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2+\beta^2z^2})}}{\sqrt{\lambda^2+\beta^2y_0^2+\beta^2z^2}}\,d\lambda \qquad (2)$$

where M is Mach number, $\beta=\sqrt{1-M^2}$, $\bar{\omega}=\omega/V\beta^2$, ω is the circular frequency of oscillation, V is the velocity, and λ is the variable of integration. Evaluation of this integral constitutes a main difficulty in obtaining aerodynamic coefficients for oscillating finite wings in compressible flow. The present analysis is therefore devoted to reducing it to a form that can be accurately evaluated by numerical procedures combined with the use of tables of certain tabulated functions. The form and order of all its singularities are determined, and an expression for the kernel function is derived in which the singularities are isolated.

REDUCTION OF THE KERNEL FUNCTION

In considering the reduction of the kernel function $K(x_0,y_0)$, the integral involved can, for convenience, be written as the sum of two integrals, namely

$$\int_{-\infty}^{x_0}\frac{e^{i\bar{\omega}(\lambda-M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda=\int_0^{\infty}\frac{e^{-i\bar{\omega}(\lambda+M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda+\int_0^{x_0}\frac{e^{i\bar{\omega}(\lambda-M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda \qquad (3)$$

Therefore,

$$K(x_0,y_0)=\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-\frac{i\omega x_0}{V}}(F)=\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-\frac{i\omega x_0}{V}}(F_1+F_2) \qquad (4)$$

where

$$F_1=\int_0^{\infty}\frac{e^{-i\bar{\omega}(\lambda+M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda \qquad (5)$$

and

$$F_2=\int_0^{x_0}\frac{e^{i\bar{\omega}(\lambda-M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda \qquad (6)$$

and where $r=\beta\sqrt{y_0^2+z^2}$.

The integrals F_1 and F_2 are treated separately in succeeding sections. The final forms are given in equations (15) and (19), respectively.

Evaluation of F_1.—The integral F_1 can be converted to a form that can be more easily handled by writing

$$F_1=\int_0^{\infty}e^{-i\bar{\omega}\lambda}\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2+r^2}}\,d\lambda$$

and introducing the following relation (see p. 416 of ref. 22)

$$\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2+r^2}}=\int_0^{\infty}J_0(T\lambda)\frac{e^{-r\sqrt{T^2-M^2\bar{\omega}^2}}}{\sqrt{T^2-M^2\bar{\omega}^2}}\,T\,dT$$

$$=\int_{M\bar{\omega}}^{\infty}J_0(T\lambda)\frac{e^{-r\sqrt{T^2-M^2\bar{\omega}^2}}}{\sqrt{T^2-M^2\bar{\omega}^2}}\,T\,dT-i\int_0^{M\bar{\omega}}J_0(T\lambda)\frac{e^{-ir\sqrt{M^2\bar{\omega}^2-T^2}}}{\sqrt{M^2\bar{\omega}^2-T^2}}\,T\,dT \qquad (7)$$

In the first integral of these last two integrals, make the substitution

$$\sqrt{T^2-M^2\bar{\omega}^2}=\tau$$

and in the second integral make the substitution

$$\sqrt{M^2\bar{\omega}^2-T^2}=\tau$$

Then

$$\frac{e^{-i\bar{\omega}M\sqrt{\lambda^2+r^2}}}{\sqrt{\lambda^2+r^2}}=\int_0^{\infty}e^{-\tau r}J_0(\lambda\sqrt{\tau^2+M^2\bar{\omega}^2})\,d\tau-i\int_0^{M\bar{\omega}}e^{-i\tau r}J_0(\lambda\sqrt{M^2\bar{\omega}^2-\tau^2})\,d\tau \qquad (8)$$

(It is of interest to note, in the expression on the left of eq. (8), that λ and r appear in the same manner. The roles of these two quantities could, therefore, be interchanged in the expression on the right.)

With use of equation (8), the equation for F_1 can be written as

$$F_1=\int_0^{\infty}e^{i\bar{\omega}\lambda}\,d\lambda\left[\int_0^{\infty}e^{-\tau r}J_0(\lambda\sqrt{\tau^2+M^2\bar{\omega}^2})\,d\tau-i\int_0^{M\bar{\omega}}e^{-i\tau r}J_0(\lambda\sqrt{M^2\bar{\omega}^2-\tau^2})\,d\tau\right] \qquad (9)$$

Changing the order of integration in each integral (which is a legitimate step because the integrands involved satisfy the continuity conditions required for such operations) leads to the following expression for F_1:

$$F_1=\int_0^{\infty}e^{-\tau r}\,d\tau\left[\int_0^{\infty}e^{-i\bar{\omega}\lambda}J_0(\lambda\sqrt{\tau^2+M^2\bar{\omega}^2})\,d\lambda\right]-i\int_0^{M\bar{\omega}}e^{-i\tau r}\,d\tau\left[\int_0^{\infty}e^{-i\bar{\omega}\lambda}J_0(\lambda\sqrt{M^2\bar{\omega}^2-\tau^2})\,d\lambda\right] \qquad (10)$$

The integrals within the brackets in equation (10) may be evaluated from tables of Fourier or Laplace transforms as (see, for example, pair no. 55 of appendix III of ref. 23)

$$\int_0^{\infty}e^{-i\bar{\omega}\lambda}J_0(\lambda\sqrt{\tau^2+M^2\bar{\omega}^2})\,d\lambda=\frac{1}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}$$

$$\int_0^{\infty}e^{-i\bar{\omega}\lambda}J_0(\lambda\sqrt{M^2\bar{\omega}^2-\tau^2})\,d\lambda=\frac{-i}{\sqrt{\tau^2+\beta^2\bar{\omega}^2}}$$

so that

$$F_1=\int_0^{\infty}\frac{e^{-\tau r}}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}\,d\tau-\int_0^{M\bar{\omega}}\frac{e^{-i\tau r}}{\sqrt{\tau^2+\beta^2\bar{\omega}^2}}\,d\tau \qquad (11)$$

The first integral in equation (11) can be written as

$$\int_0^{\infty}\frac{e^{-\tau r}}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}\,d\tau=\int_{\beta\bar{\omega}}^{\infty}\frac{e^{-\tau r}}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}\,d\tau-i\int_0^{\beta\bar{\omega}}\frac{e^{-\tau r}}{\sqrt{\beta^2\bar{\omega}^2-\tau^2}}\,d\tau$$

or

$$\int_0^{\infty}\frac{e^{-\tau r}}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}\,d\tau=\int_0^{\infty}e^{-\beta\bar{\omega}r\cosh\theta}\,d\theta-\frac{i}{2}\int_{-\pi/2}^{\pi/2}e^{-i\beta\bar{\omega}r\cos\theta}\,d\theta \qquad (11a)$$

The first integral on the right of equation (11a) is given on page 181 of reference 22 as

$$\int_0^{\infty} e^{-\bar{\beta\omega}r\cosh\theta}\,d\theta=K_0(\beta\bar{\omega}r)$$

where K_0 is the modified Bessel function of the second kind of zero order. The second integral on the right of equation (11a) is given on page 338 of reference 22 as

$$-\frac{i}{2}\int_{-\pi/2}^{\pi/2} e^{-\beta\bar{\omega}r\cos\theta}\,d\theta=-\frac{\pi i}{2}\left[I_0(\beta\bar{\omega}r)-L_0(\beta\bar{\omega}r)\right]$$

where I_0 is the modified Bessel function of the first kind of zero order and L_0 is the modified Struve function of zero order. Then, the first integral of equation (11) can be written as

$$\int_0^{\infty}\frac{e^{-i\tau}}{\sqrt{\tau^2-\beta^2\bar{\omega}^2}}\,d\tau=K_0(\beta\bar{\omega}r)-\frac{\pi i}{2}\left[I_0(\beta\bar{\omega}r)-L_0(\beta\bar{\omega}r)\right]$$

$$=K_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left[I_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)-L_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)\right] \tag{12}$$

Note that the end result indicated in equation (12) is independent of Mach number. The second integral in equation (11) may be written in another form as

$$\int_0^{M\bar{\omega}}\frac{e^{-i\tau}}{\sqrt{\tau^2+\beta^2\bar{\omega}^2}}\,d\tau=\int_0^{M/\beta}\frac{e^{-i\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)\tau}}{\sqrt{1+\tau^2}}\,d\tau \tag{13}$$

This integral has not been reduced to closed form; however, it is nonsingular and can be readily handled by numerical methods.

Combining equations (12) and (13) gives the following expression for F_1:

$$F_1=K_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left[I_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)-L_0\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)\right]-\int_0^{M/\beta}\frac{e^{-i\left(\frac{\omega}{\bar{V}}\sqrt{y_0^2+z^2}\right)\tau}}{\sqrt{1+\tau^2}}\,d\tau \tag{14}$$

By performing the differentiations indicated in equation (4), there is obtained for the first part of equation (4) the following expression:

$$\lim_{z\to 0}\frac{\partial^2 F_1}{\partial z^2}=\frac{\omega}{V|y_0|}\left\{-K_1\left(\frac{\omega}{\bar{V}}|y_0|\right)-\frac{\pi i}{2}\left[I_1\left(\frac{\omega}{\bar{V}}|y_0|\right)-L_1\left(\frac{\omega}{\bar{V}}|y_0|\right)\right]+\frac{i}{\beta}e^{-\frac{iM\omega|y_0|}{\beta\bar{V}}}-\frac{\omega}{\bar{V}}|y_0|\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-i\frac{\omega}{\bar{V}}|y_0|\tau}\,d\tau\right\} \tag{15}$$

All terms of this expression other than the integral may be evaluated at small intervals of y_0 from existing tables, except at $y_0=0$ where the function is singular. The integral is well behaved and can be accurately evaluated by numerical or approximate procedures. The type and order of the singularities at $y_0=0$ are discussed in a later section.

Evaluation of F_2.—In order to reduce the integral F_2, equation (6), it is convenient to make the substitution

$$\lambda=r\sinh\theta \tag{16}$$

so that

$$F_2=\int_0^{\sinh^{-1}\frac{x_0}{r}} e^{i\bar{\omega}r(\sinh\theta-M\cosh\theta)}\,d\theta \tag{17}$$

Noting that z appears only in r and performing the differentiations indicated in equation (4) yields

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_{z=0}=\frac{i\bar{\omega}\beta}{|y_0|}\int_0^{\sinh^{-1}\frac{x_0}{\beta|y_0|}}(\sinh\theta-M\cosh\theta)\,e^{i\bar{\omega}\beta|y_0|(\sinh\theta-M\cosh\theta)}\,d\theta-\frac{x_0}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}\,e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}$$

$$=-\beta^2\left\{\frac{x_0}{\beta^2y_0^2\sqrt{x_0^2+\beta^2y_0^2}}\,e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}-\frac{i\bar{\omega}}{M\beta|y_0|}\int_0^{\sinh^{-1}\frac{x_0}{\beta|y_0|}}\left[\beta^2\cosh\theta-(\cosh\theta-M\sinh\theta)\right]e^{i\bar{\omega}\beta|y_0|(\sinh\theta-M\cosh\theta)}\,d\theta\right\}$$

$$=-\beta^2\left\{\frac{x_0\,e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}}{\beta^2y_0^2\sqrt{x_0^2+\beta^2y_0^2}}+\frac{1}{M\beta^2y_0^2}\left[e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}-e^{-i\bar{\omega}M\beta|y_0|}\right]-\frac{i\bar{\omega}\beta}{M|y_0|}\int_0^{\sinh^{-1}\frac{x_0}{\beta|y_0|}}\cosh\theta\,e^{i\bar{\omega}\beta|y_0|(\sinh\theta-M\cosh\theta)}\,d\theta\right\} \tag{18}$$

or, by reverting completely to Cartesian coordinates through equation (16), there is obtained

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_{z=0}=-\left\{\frac{x_0\,e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}+\frac{1}{My_0^2}\left[e^{i\bar{\omega}\left(x_0-M\sqrt{x_0^2+\beta^2y_0^2}\right)}-e^{-i\bar{\omega}M\beta|y_0|}\right]-\frac{i\beta^2\bar{\omega}}{My_0^2}\int_0^{x_0}e^{i\bar{\omega}\left(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2}\right)}\,d\lambda\right\} \tag{19}$$

This expression vanishes, as it should, for $x_0=0$ and, like that in equation (15), has singularities at $y_0=0$ which, also, will be handled in a later section. The integral that remains, like the integral remaining in equation (15), is nonsingular and simple in form and can be readily evaluated by numerical procedures.

Expression for the kernel in terms of nondimensional length variables.—Equations (15) and (19) can now be combined to give a reduced form of the kernel function $K(x_0,y_0)$. However, in application, the variables x_0 and y_0 are employed, for convenience, in nondimensional form. This is accomplished by considering these variables in a new sense to mean that they have been referred to some chosen length l and by introducing the reduced-frequency parameter $k=l\omega/V$. The variables will be used in this new sense throughout the remainder of the report. The kernel can be written in terms of these nondimensional variables as

$$K(x_0,y_0)=e^{-ikx_0}\frac{\partial^2}{\partial z^2}(F_1+F_2)_{z=0}$$

$$=\frac{k^2}{i^2}e^{-ikx_0}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\frac{\pi i}{2k|y_0|}[I_1(k|y_0|)-L_1(k|y_0|)]+\right.$$

$$\frac{iMk|y_0|+\beta}{M\beta(ky_0)^2}e^{-\frac{ikM|y_0|}{\beta}}-\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}d\tau-$$

$$\frac{Mkx_0+\sqrt{(kx_0)^2+\beta^2(ky_0)^2}}{M(ky_0)^2\sqrt{(kx_0)^2+\beta^2(ky_0)^2}}e^{\frac{i}{\beta^2}[kx_0-M\sqrt{(kx_0)^2+\beta^2(ky_0)^2}]}+$$

$$\left.\frac{i}{M(ky_0)^2}\int_0^{kx_0}e^{\frac{i}{\beta^2}[\lambda-M\sqrt{\lambda^2+\beta^2(ky_0)^2}]}d\lambda\right\} \qquad (20)$$

An alternate and perhaps more desirable form of expression for the kernel function is given in appendix D.

Note that this expression for $K(x_0,y_0)$ can be considered as a function of only three parameters, namely, $k|y_0|$, kx_0, and M. To be more specific, the first two terms are functions only of $k|y_0|$; the next two terms are functions of $k|y_0|$ and M; and the last two terms are functions of $k|y_0|$, kx_0, and M.

Equation (20) constitutes the principal result of this report. Some partial checks as to its correctness are: (1) For $k=0$, it reduces, as discussed subsequently, to the downwash of a pressure doublet in steady flow and (2) an integration with regard to the y-direction between the limits $-\infty$ to $+\infty$ yields Possio's result for the two-dimensional case. This integration is carried out in appendix B. Other special forms of the kernel function for $M=1$, $M=0$, and $k=0$ are derived in subsequent sections. A power series expansion of the kernel which is applicable for certain ranges of the parameters $k|y_0|$, kx_0, and M is presented. In the section immediately following, the orders and types of the singularities of the kernel function are discussed.

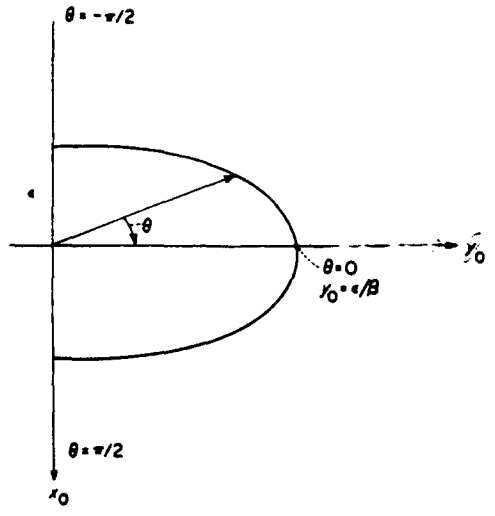
DISCUSSION OF THE SINGULARITIES OF THE KERNEL FUNCTION

As previously indicated, the kernel function becomes singular or indeterminate at $y_0=0$. The forms that the kernel function takes when it becomes singular are of particular importance in applications to lifting surface theory. It is therefore desirable to extract and treat the singularities separately.

This extraction can be conveniently made by considering the value of $K(x_0,y_0)$, equation (20), at points on the semicircumference of a small ellipse (see sketch 2), the polar equation of which may be written as

$$\left.\begin{aligned}x_0&=\epsilon\sin\theta\\ y_0&=\frac{\epsilon}{\beta}\cos\theta\end{aligned}\right\} \qquad (21)$$

where, because of the symmetry of $K(x_0,y_0)$ with respect to y_0, only the limits $-\pi/2\leqq\theta\leqq\pi/2$ need be examined. Note that in these equations values of θ in the range $-\pi/2\leqq\theta<0$ correspond to field points ahead of or upstream from the doublet position and values of θ in the range $0<\theta\leqq\pi/2$, to field points behind or downstream from the doublet position. In particular, $\theta=\pi/2$ corresponds to points directly behind or in the wake of the doublet.



Sketch 2.

After substituting these expressions for x_0 and y_0 into equation (20), the results may be written as

$$K(\epsilon,\theta)=\frac{\beta^2e^{-ik\epsilon\sin\theta}}{i^2\epsilon^2\cos^2\theta}\left\{-\frac{k\epsilon\cos\theta}{\beta}K_1\left(\frac{k\epsilon\cos\theta}{\beta}\right)-\right.$$

$$\frac{i\pi k\epsilon\cos\theta}{2\beta}\left[I_1\left(\frac{k\epsilon\cos\theta}{\beta}\right)-L_1\left(\frac{k\epsilon\cos\theta}{\beta}\right)\right]+$$

$$\frac{e^{\frac{ikM\epsilon\cos\theta}{\beta^2}}}{M}-\frac{e^{\frac{ik\epsilon(\sin\theta-M)}{\beta^2}}}{M}+\frac{ik\epsilon\cos\theta}{\beta^2}e^{-\frac{ikM\epsilon\cos\theta}{\beta^2}}$$

$$\sin\theta\,e^{\frac{ik\epsilon(\sin\theta-M)}{\beta^2}}-\frac{k^2\epsilon^2\cos^2\theta}{\beta^2}\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-\left(\frac{ik\epsilon\cos\theta}{\beta}\right)\tau}d\tau+$$

$$\left.\frac{ik}{M}\int_0^{\epsilon\sin\theta}e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\epsilon^2\cos^2\theta})}d\lambda\right\} \qquad (22)$$

With the use of the following series expressions for $K_1(z)$ and $[I_1(z)-L_1(z)]$ (which can be obtained from ref. 22—for K_1, see p. 80; for I_1, see p. 77; and for L_1, see p. 329):

$$K_1(z)=\left(\gamma+\log\frac{z}{2}\right)\left(\frac{z}{2}+\frac{z^3}{16}+\frac{z^5}{384}+\cdots\right)+$$

$$\frac{1}{z}-\left(\frac{z}{4}+\frac{5z^3}{64}+\frac{5z^5}{1152}+\cdots\right) \qquad (23)$$

where γ is Euler's constant ($\gamma=0.5772157$), and

$$[I_1(z)-L_1(z)]=\frac{z}{2}-\frac{2z^3}{3\pi}+\frac{z^3}{16}-\frac{2z^4}{45\pi}+\frac{z^5}{384}+\cdots \qquad (24)$$

it is found that for vanishingly small values of ϵ the limiting value of the expression for $K(\epsilon,\theta)$ in equation (22) is for $M<1$

$$K(\epsilon,\theta)\approx\frac{e^{-ik\epsilon\sin\theta}}{i^2}\left\{\frac{-\beta^2}{\epsilon^2(1-\sin\theta)}+\frac{ik}{\epsilon}-\frac{k^2}{2}\log\frac{k\epsilon(1-\sin\theta)}{2(1-M)}-\right.$$

$$\left.\frac{k^2}{2}\left[\gamma-\frac{1}{2}-\frac{1}{\beta^2}\left(M-\sin\theta-\frac{i\pi\beta^2}{2}\right)\right]+O(\epsilon^2)\right\} \qquad (25)$$

where $0(\epsilon^n)$ represents terms of order ϵ^n for $n \geqq 1$. Expressed in terms of x_0 and y_0, equation (25) becomes

$$K(x_0,y_0) \approx \frac{e^{-ikx_0}}{l^2}\left\{\frac{-(x_0+\sqrt{x_0^2+\beta^2y_0^2})}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}+\frac{ik}{\sqrt{x_0^2+\beta^2y_0^2}}-\right.$$
$$\frac{k^2}{2}\log\frac{k(\sqrt{x_0^2+\beta^2y_0^2}-x_0)}{2(1-M)}-\frac{k^2}{2}\left[\gamma-\frac{1}{2}-\right.$$
$$\left.\left.\frac{1}{\beta^2}\left(M-\frac{x_0}{\sqrt{x_0^2+\beta^2y_0^2}}-\frac{i\pi\beta^2}{2}\right)\right]+0(\epsilon^n)\right\} \quad (26)$$

Examination of equation (25) shows that the kernel function $K(\epsilon,\theta)$ has singularities with respect to $\epsilon=\sqrt{x_0^2+\beta^2y_0^2}$ as follows:

$$-\frac{f_1(\theta)}{\epsilon^2};\quad \frac{ik}{\epsilon};\quad -\frac{k^2}{2}[f_2(\theta)+\log\epsilon] \quad (27)$$

where, from equation (25),

$$\left.\begin{aligned} f_1(\theta)&=\frac{\beta^2}{1-\sin\theta}=\frac{\beta^2(1+\sin\theta)}{\cos^2\theta}\\ f_2(\theta)&=\log\frac{k(1-\sin\theta)}{2(1-M)}=\log\frac{k\cos^2\theta}{2(1-M)(1+\sin\theta)}\end{aligned}\right\} \quad (28)$$

Although of no particular significance in applications, it is of interest to note that the quantities f_1 and f_2 each have minimum values $\left(|f_1|_{min}=\frac{\beta^2}{2}\right.$ and $\left.|f_2|_{min}=\log\frac{k}{1-M}\right)$ at $\theta=-\pi/2$, which corresponds to points directly ahead of the doublet position; and, as θ increases from $-\pi/2$ to $+\pi/2$, the values of these quantities continuously increase from these minimum values to infinite quantities as follows:

$$\left.\begin{aligned} \left|f_1\left(\frac{\pi}{2}\right)\right|&=\lim_{\delta\to0}\left|\frac{\beta^2\left[1+\sin\left(\frac{\pi}{2}-\delta\right)\right]}{\cos^2\left(\frac{\pi}{2}-\delta\right)}\right|=\lim_{\delta\to0}\left|\frac{2\beta^2}{\delta^2}\right|\\ \left|f_2\left(\frac{\pi}{2}\right)\right|&=\lim_{\delta\to0}\left|\log\frac{k\cos^2\left(\frac{\pi}{2}-\delta\right)}{2(1-M)\left[1+\sin\left(\frac{\pi}{2}-\delta\right)\right]}\right|\\ &=\lim_{\delta\to0}\left|\log\frac{k\delta^2}{4(1-M)}\right|\end{aligned}\right\} \quad (29)$$

Thus $K(x_0,y_0)$ is singular for $\theta=\pi/2$ even when the distance ϵ from the doublet is not necessarily of zero order. This implies that the doublet produces a wake of discontinuous downwash that extends downstream from the doublet position to infinity.

With knowledge of the singularities involved in the kernel function $K(x_0,y_0)$, an expression can be written in which the kernel is separated into a singular part and a nonsingular part (as was done by Schwarz, ref. 14, for the two-dimensional case) as follows

$$K(x_0,y_0)=[K(x_0,y_0)-K'(x_0,y_0)]+K'(x_0,y_0) \quad (30)$$

where $K(x_0,y_0)$ is defined in equation (20) or (22) and

$$K'(x_0,y_0)=\frac{e^{-ikx_0}}{l^2}\left[-\frac{\sqrt{x_0^2+\beta^2y_0^2}+x_0}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}+\frac{ik}{\sqrt{x_0^2+\beta^2y_0^2}}-\right.$$
$$\left.\frac{k^2}{2\beta^2}\frac{x_0-M\sqrt{x_0^2+\beta^2y_0^2}}{\sqrt{x_0^2+\beta^2y_0^2}}-\frac{k^2}{2}\log\frac{k(\sqrt{x_0^2+\beta^2y_0^2}-x_0)}{2(1-M)}\right] \quad (31)$$

or in terms of ϵ and θ, introduced by equations (21),

$$K'(\epsilon,\theta)=\frac{e^{-ik\epsilon\sin\theta}}{l^2}\left[-\frac{\beta^2}{\epsilon^2(1-\sin\theta)}+\frac{ik}{\epsilon}-\frac{k^2}{2\beta^2}(\sin\theta-M)-\right.$$
$$\left.\frac{k^2}{2}\log\frac{k\epsilon(1-\sin\theta)}{2(1-M)}\right] \quad (32)$$

The term $[K(x_0,y_0)-K'(x_0,y_0)]$ in equation (30) is a continuous function for all values of k, x_0, and y_0 and for values of M in the range of $0 \leqq M \leqq 1$. The term $K'(x_0,y_0)$ is discontinuous at the doublet position ($x_0=0$, $y_0=0$) and at all points in the wake ($x_0>0, y_0=0$). It is to be noted, however, that each term of $K'(x_0,y_0)$ possesses a simple indefinite integral with respect to y_0 or with respect to $\eta=y-y_0$, a fact that may be useful in some numerical applications. The manner in which these integrals are to be evaluated is indicated in a subsequent section that deals with steady flow. The limiting values at $y_0=0$ of $[K(x_0,y_0)-K'(x_0,y_0)]$ for both subsonic and sonic flow are given in appendix C together with some remarks on evaluation of the kernel function.

TREATMENT OF THE SONIC CASE

Because of its special nature, the borderline case, $M=1$, between subsonic and supersonic flow deserves and requires separate treatment.

As $M\to1$, the expression for the kernel function given in equation (20) becomes indeterminate. It is possible, however, to obtain conditional limiting values for the kernel by considering the integral F, equation (4), and breaking it into two integrals, F_1 and F_2, as was done for the general case.

With regard to F_1, its limiting value and the value of its derivatives with respect to z at $z=0$ can be shown to be zero as $M\to1$. From the form of F_1 given by equation (14),

$$\lim_{M\to1}F_1=\lim_{M\to1}\left\{K_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)-\right.\right.$$
$$\left.\left.L_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)\right]-\int_0^{M/\beta}\frac{e^{-i\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)\tau}}{\sqrt{1+\tau^2}}d\tau\right\}$$
$$=K_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left[I_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)-\right.$$
$$\left.L_0\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)\right]-\int_0^\infty\frac{\cos\left[\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)\tau\right]}{\sqrt{1+\tau^2}}d\tau+$$
$$i\int_0^\infty\frac{\sin\left[\left(\frac{\omega}{V}\sqrt{y_0^2+z^2}\right)\tau\right]}{\sqrt{1+\tau^2}}d\tau \quad (33)$$

But since (see ref. 22, p. 172)

$$-\int_0^\infty\frac{\cos\zeta\tau}{\sqrt{1+\tau^2}}d\tau=-K_0(\zeta) \quad (34)$$

and (see ref. 22, p. 332)

$$i\int_0^\infty \frac{\sin \zeta\tau}{\sqrt{1+\tau^2}}\,d\tau=\frac{\pi i}{2}[I_0(\zeta)-L_0(\zeta)] \tag{35}$$

it may be concluded from equation (33) that

$$\lim_{M\to 1} F_1=\lim_{M\to 1}\left(\frac{\partial^2 F_1}{\partial z^2}\right)=0 \tag{36}$$

The total contribution to $K(x_0,y_0)$ at $M=1$, therefore, arises from the limit of F_2, equation (6), as $M\to 1$. The limiting form of F_2 may be written in terms of nondimensional coordinates as

$$\lim_{M\to 1} F_2=\lim_{M\to 1}\int_0^{x_0}\frac{e^{\frac{ik}{\beta^2}\left[\lambda-M\sqrt{\lambda^2+\beta^2(y_0^2+z^2)}\right]}}{\sqrt{\lambda^2+\beta^2(y_0^2+z^2)}}\,d\lambda \tag{37}$$

In approaching the limit $M=1$ (from the subsonic side) in equation (37), it is convenient to replace M by

$$M=1-\epsilon$$

where ϵ is infinitesimally small so that

$$\beta^2=(1-M)\ (1+M)=\epsilon(2-\epsilon)\approx 2\epsilon$$

With this approximation, equation (37) may be written as

$$\lim_{M\to 1} F_2=\lim_{\epsilon\to 0}\int_0^{x_0}\frac{e^{\frac{ik}{2\epsilon}\left\{\lambda-|\lambda|(1-\epsilon)\left[1+\frac{\epsilon(y_0^2+z^2)}{\lambda^2}\right]\right\}}}{\sqrt{\lambda^2+2\epsilon(y_0^2+z^2)}}\,d\lambda$$

$$=\int_0^{x_0}\frac{e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2+z^2}{\lambda}\right)}}{\lambda}\,d\lambda \qquad (\text{for } x_0>0) \tag{38}$$

From physical considerations, the right side of equation (38) is to be considered zero for $x_0\leqq 0$. This is in keeping with results that would be obtained if the limit under consideration were sought from theory of supersonic flow, $M>1$.

The integral in equation (38) cannot be completely expressed in terms of known functions. Furthermore, since it is singular at its lower limit, further treatment is required to reduce it to a form such that its derivatives with respect to z can be numerically evaluated. For this purpose the integral may be written as two integrals, namely

$$(F_2)_{M=1}=F_2'+F_2'' \tag{39}$$

where

$$F_2'=\int_0^{\sqrt{y_0^2+z^2}}\frac{e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2+z^2}{\lambda}\right)}}{\lambda}\,d\lambda \tag{40}$$

and

$$F_2''=\int_{\sqrt{y_0^2+z^2}}^{x_0}\frac{e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2+z^2}{\lambda}\right)}}{\lambda}\,d\lambda \tag{41}$$

The limits of integration in equation (40) are so chosen that the integral in this equation can be reduced to a known form by making the substitution

$$\lambda=\sqrt{\tau^2+(y_0^2+z^2)}-\tau \quad \text{or} \quad \tau=\frac{1}{2}\left(\frac{y_0^2+z^2}{\lambda}-\lambda\right)$$

Thus,

$$F_2'=\int_0^\infty \frac{e^{-ik\tau}}{\sqrt{\tau^2+(y_0^2+z^2)}}\,d\tau=\int_0^\infty \frac{e^{-ik\sqrt{y_0^2+z^2}\,\tau}}{\sqrt{1+\tau^2}}\,d\tau \tag{42}$$

Equation (42) may be written in terms of the integrals involved in F_1 (see eqs. (34) and (35)), namely,

$$F_2'=K_0\left(k\sqrt{y_0^2+z^2}\right)-\frac{\pi i}{2}\left[I_0\left(k\sqrt{y_0^2+z^2}\right)-L_0\left(k\sqrt{y_0^2+z^2}\right)\right] \tag{43}$$

Differentiating this result twice with respect to z and then setting $z=0$ gives

$$\left(\frac{\partial^2 F_2'}{\partial z^2}\right)_{z=0}=\frac{k^2}{l^2}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\right.$$

$$\left.\frac{\pi i}{2k|y_0|}\left[I_1(k|y_0|)-L_1(k|y_0|)-\frac{2}{\pi}\right]\right\} \tag{44}$$

Differentiating equation (41) twice with respect to z and setting $z=0$ gives

$$\left(\frac{\partial^2 F_2''}{\partial z^2}\right)_{z=0}=-\frac{k^2}{l^2}\left[\frac{1}{k^2y_0^2}+\frac{i}{k}\int_{|y_0|}^{x_0}\frac{e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}}{\lambda^2}\,d\lambda\right] \tag{45}$$

After performing an integration by parts and collecting terms, equation (45) may be written as

$$\left(\frac{\partial^2 F_2''}{\partial z^2}\right)_{z=0}=\frac{k^2}{l^2}\left[\frac{1}{k^2y_0^2}-\frac{2}{k^2y_0^2}e^{\frac{i}{2}\left(kx_0-\frac{k^2y_0^2}{kx_0}\right)}+\right.$$

$$\left.\frac{i}{k^2y_0^2}\int_{k|y_0|}^{kx_0}e^{\frac{i}{2}\left(\lambda-\frac{k^2y_0^2}{\lambda}\right)}\,d\lambda\right] \tag{46}$$

Equations (44) and (46) are combined to give $\left(\frac{\partial^2 F_2}{\partial z}\right)_{z=0}$. Then, in accordance with equation (4), there is obtained for $K(x_0,y_0)_{M=1}$:

For $x_0>0$,

$$K(x_0,y_0)_{M=1}=\frac{k^2}{l^2}e^{-ikx_0}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\frac{\pi i}{2k|y_0|}\left[I_1(k|y_0|)-\right.\right.$$

$$\left.L_1(k|y_0|)-\frac{2}{\pi}\right]+\frac{1}{k^2y_0^2}-\frac{2}{k^2y_0^2}e^{\frac{i}{2}\left(kx_0-\frac{k^2y_0^2}{kx_0}\right)}+$$

$$\left.\frac{i}{k^2y_0^2}\int_{k|y_0|}^{kx_0}e^{\frac{i}{2}\left(\lambda-\frac{k^2y_0^2}{\lambda}\right)}\,d\lambda\right\} \tag{47a}$$

and, for $x_0\leqq 0$,

$$K(x_0,y_0)_{M=1}=0 \tag{47b}$$

The integral appearing in equation (47a) is finite and proper and can be evaluated by numerical procedures.

TREATMENT OF THE STEADY AND INCOMPRESSIBLE CASES

It is of interest to consider the form of the kernel function given in equation (20) for some particular values of M and k. In the following sections a discussion is given for the steady case ($k=0$) and the incompressible case ($M=0$). The two-dimensional case is handled in appendix B.

Reduction of the kernel for the case of steady flow.—In order to obtain the reduction of the kernel for the case of steady flow, consider the expanded form given by equation (26). As $k\to 0$, there results the following expression

$$K(x_0,y_0)_{k=0}=-\frac{1}{l^2}\left(\frac{1}{y_0^2}+\frac{x_0}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}\right) \tag{48}$$

which represents the downwash of a pressure doublet for steady flow. This result serves as a partial check as to the correctness of the expression for $K(x_0,y_0)$ given by equation (20).

By replacing y_0 in equation (48) by $y-\eta$ and integrating from -1 to 1 with respect to η, there is obtained

$$\oint_{-1}^{1}K(x_0,y_0)\,d\eta=-\frac{1}{l}\left[\frac{x_0+\sqrt{x_0^2+\beta^2(y-1)^2}}{x_0(y-1)}-\frac{x_0+\sqrt{x_0^2+\beta^2(y+1)^2}}{x_0(y+1)}\right] \tag{49}$$

where the symbol $\oint$ indicates that a principal value or finite part of the improper integral must be taken. (See, for example, ref. 24 for a discussion of finite parts of such integrals.) This result corresponds to the downwash produced by a simple horseshoe vortex two units wide. An equivalent expression for incompressible flow is given, for example, in reference 25, where in contrast to the present notation, x_0 has been chosen as positive forward.

Reduction of the kernel for $M=0$.—In order to effect the reduction of the kernel for the incompressible case, the expressions for F_1, equation (15), and F_2, equation (18), will be examined for the limit $M\to 0$:

From equation (15)

$$\lim_{\substack{M\to 0\\ z\to 0}}\frac{\partial^2F_1}{\partial z^2}=\frac{k}{y_0}\left\{-K_1(k|y_0|)-\frac{\pi i}{2}\left[I_1(k|y_0|)-L_1(k|y_0|)\right]+i\right\} \tag{50}$$

and from equation (18)

$$\lim_{\substack{M\to 0\\ z\to 0}}\frac{\partial^2F_2}{\partial z^2}=\frac{ik}{|y_0|}\int_0^{\sinh^{-1}\frac{x_0}{|y_0|}}\sinh\theta\,e^{ik|y_0|\sinh\theta}\,d\theta-\frac{x_0}{y_0^2\sqrt{x_0^2+y_0^2}}\,e^{ikx_0} \tag{51}$$

Integrating by parts yields

$$\lim_{\substack{M\to 0\\ z\to 0}}\frac{\partial^2F_2}{\partial z^2}=\frac{ik}{y_0^2}-\frac{ik}{y_0}+\frac{k}{y_0^2}\int_0^{x_0}\sqrt{y_0^2+\lambda^2}\,e^{ik\lambda}\,d\lambda-\frac{x_0}{y_0^2\sqrt{x_0^2+y_0^2}}\,e^{ikx_0} \tag{52}$$

Combining the results from F_1 and F_2 gives for the kernel function

$$K(x_0,y_0)_{M=0}=\frac{e^{-ikx_0}}{l^2}\left\{-\frac{k}{|y_0|}K_1(k|y_0|)-\frac{i\pi k}{2|y_0|}\left[I_1(k|y_0|)-L_1(k|y_0|)\right]-\frac{x_0}{y_0^2\sqrt{x_0^2+y_0^2}}\,e^{ikx_0}+\frac{ik\sqrt{x_0^2+y_0^2}}{y_0^2}\,e^{ikx_0}+\frac{k^2}{y_0^2}\int_0^{x_0}\sqrt{\lambda^2+y_0^2}\,e^{ik\lambda}\,d\lambda\right\} \tag{53}$$

By setting $x_0=0$ in equation (53), a form is obtained which can be shown to agree with results derived by Küssner for the case $M=0$, $x_0=0$ (ref. (26)).

A SERIES EXPANSION WITH RESPECT TO "k"

An approximation for the function

$$\left|K(x_0,y_0)-K'(x_0,y_0)\right|$$

for small values of k can be obtained by making use of the series expansions for K_1 (eq. (23)) and for (I_1-L_1) (eq. (24)) and expanding all other terms of $K(x_0,y_0)$ (eq. (20)) into a power series in terms of k. After performing these expansions and collecting terms with respect to powers of k, there is obtained for $M<1$

$$\begin{aligned}K(x_0,y_0)\approx\frac{e^{-ikx_0}}{l^2\beta^2}\Bigg\{&-\frac{\beta^2(\sqrt{x_0^2+\beta^2y_0^2}+x_0)}{y_0^2\sqrt{x_0^2+\beta^2y_0^2}}+\frac{ik\beta^2}{\sqrt{x_0^2+\beta^2y_0^2}}+\\&\frac{k^2}{2}\left[M-\frac{x_0}{\sqrt{x_0^2+\beta^2y_0^2}}-\beta^2\left(\gamma-\frac{1}{2}\right)-\frac{i\pi\beta^2}{2}-\beta^2\log\frac{k(\sqrt{x_0^2+\beta^2y_0^2}-x_0)}{2(1-M)}\right]+\\&\frac{ik^3}{6\beta^2}\left[2M^3x_0+\frac{(1-3M^2)x_0^2+(2-3M^2)\beta^2y_0^2}{\sqrt{x_0^2+\beta^2y_0^2}}\right]+\\&\frac{k^4}{192\beta^4}\Bigg[(12M\beta^2-20M^3\beta^2+15\beta^6-12\beta^6\gamma)y_0^2-32M^3x_0^2+\\&\frac{4(3M^4+6M^2-1)x_0^3+12\beta^2(M^4+2M^2-1)x_0y_0^2}{\sqrt{x_0^2+\beta^2y_0^2}}-\\&12\beta^6y_0^2\log\frac{k(\sqrt{x_0^2+\beta^2y_0^2}-x_0)}{2(1-M)}-i\pi6\beta^6y_0^2\Bigg]+\\&\frac{ik^5}{360\beta^6}\Bigg[(15M^4+10M^2-1)x_0^2\sqrt{x_0^2+\beta^2y_0^2}-\\&4M^3(5+M^2)x_0^3+\frac{3\beta^4y_0^4}{\sqrt{x_0^2+\beta^2y_0^2}}-12M^5\beta^2x_0y_0^2-\\&5\beta^4(3M^2-1)y_0^2\sqrt{x_0^2+\beta^2y_0^2}\Bigg]\ldots\Bigg\}\end{aligned} \tag{54}$$

For values of the parameters that satisfy the following inequalities

$$\left.\begin{aligned}&\frac{ky_0}{\beta}<1\\&\frac{k}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})<1\end{aligned}\right\} \tag{55}$$

equation (54) yields results that are correct to within about 2 percent.

Correspondingly for $M=1$, equation (47) can be expanded to obtain

$$K(x_0, y_0)_{M=1} \approx \frac{e^{-ikx_0}}{\beta^2}\left\{-\frac{2e^{\frac{ikx_0}{2}}}{y_0^2}+ik\left[\frac{x_0}{y_0^2}+\frac{e^{\frac{ikx_0}{2}}}{x_0}\right]+\right.$$

$$\frac{k^2}{2}\left[1-\gamma-\log\frac{k|y_0|}{2}+\log\frac{x_0}{|y_0|}-\frac{x_0^2}{2y_0^4}-\frac{\pi i}{2}+\frac{y_0^2 e^{\frac{ikx_0}{2}}}{2x_0^2}\right]+$$

$$\frac{ik^3}{6}\left[\frac{3x_0}{2}-\frac{x_0^3}{4y_0^2}+\frac{3y_0^2}{4x_0}-\frac{y_0^4 e^{\frac{ikx_0}{2}}}{4x_0^3}\right]+$$

$$\frac{k^4}{96}\left[9y_0^2-6\gamma y_0^2-6y_0^2\log\frac{k|y_0|}{2}-3\pi i y_0^2+\frac{x_0^4}{2y_0^2}-\right.$$

$$\left.3x_0^2+6y_0^2\log\frac{x_0}{|y_0|}+\frac{y_0^4}{x_0^2}-\frac{y_0^6 e^{\frac{ikx_0}{2}}}{2x_0^4}\right]+$$

$$\left.\frac{ik^5}{96}\left[\frac{x_0^5}{20y_0^2}-\frac{x_0^3}{3}+\frac{3x_0y_0^2}{2}+\frac{y_0^4}{x_0}-\frac{y_0^6}{12x_0^2}+\frac{y_0^8 e^{\frac{ikx_0}{2}}}{20x_0^5}\right]\right\} \quad (56)$$

For values of the parameters that satisfy the following inequality:

$$kx_0-\frac{k^2y_0^2}{kx_0}<2 \quad (57)$$

equation (56) yields results that are correct to within about 2 percent.

CONCLUDING REMARKS

The main purpose of this report was to present the kernel function of the integral equation relating the downwash to the lift distribution in a form that can be computed. This purpose has been achieved by the presentation of the kernel in a form given in equation (20). This equation has been converted to a form more suitable for calculation by isolating the singularities as shown in equations (30) and (31). The special case of $M=1$ is given in equations (47). The forms of the kernel function for other limiting cases, namely $k=0$ and $M=0$, are given in equations (48) and (53), respectively.

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APPENDIX A

DERIVATION OF THE INTEGRAL EQUATION THAT RELATES THE DOWNWASH AND LIFT FOR A FINITE WING BASED ON REFERENCE 21

In keeping with the concepts of linear theory, the wing is considered as a nearly plane impenetrable surface. Let this surface S lie nearly in the xy-plane, as indicated in sketch 1 of the body of the report, and let it and the x, y, z coordinate system to which it is referred be assumed to move at a uniform speed V in the negative x-direction. At the same time, let each point of the wing be assumed to undergo harmonic translations of small amplitude $Z_m(x,y,t)$ at circular frequency ω and let c represent velocity of sound in the medium.

The problem for an oscillating wing consists in solving the wave equation subject to certain boundary conditions. The wave equation in rectangular coordinates is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\left(V\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)^2\psi=0 \tag{A1}$$

The independent variable ψ in equation (A1) is regarded herein as an acceleration potential; as such it is directly proportional to a perturbation pressure field and is related to a velocity potential ϕ as follows:

$$\psi=\frac{\partial\phi}{\partial t}+V\frac{\partial\phi}{\partial x} \tag{A2}$$

In order to complete the boundary-value problem for the wing, it is desirable to calculate the downwash $w(x,y,z,t)=\frac{\partial\phi}{\partial z}$ associated with ψ. Assuming this downwash to be harmonic with regard to time implies that both potentials ϕ and ψ are harmonic with regard to time and can be written, therefore, as

$$\left.\begin{aligned}\phi(x,y,z,t)&=e^{i\omega t}\,\bar{\phi}(x,y,z)\\ \psi(x,y,z,t)&=e^{i\omega t}\,\bar{\psi}(x,y,z)\end{aligned}\right\} \tag{A3}$$

With these expressions for ϕ and ψ, equation (A2) becomes independent of time and reduces to an ordinary equation with one independent variable, namely

$$\bar{\psi}=i\omega\bar{\phi}+V\frac{d\bar{\phi}}{dx} \tag{A4}$$

This equation can be integrated with respect to x to give

$$\bar{\phi}=\frac{e^{-\frac{i\omega x}{V}}}{V}\int_{-\infty}^{x}\bar{\psi}(\lambda,y,z)e^{\frac{i\omega\lambda}{V}}\,d\lambda \tag{A5}$$

where the lower limit of integration is chosen, for later convenience, so as to satisfy the condition that ϕ vanish as $x\to-\infty$.

The boundary-value problem for the wing may now be expressed mathematically as follows: Under the assumption of harmonic motion the differential equation, equation (A1), becomes

$$\frac{\partial^2\bar{\psi}}{\partial x^2}+\frac{\partial^2\bar{\psi}}{\partial y^2}+\frac{\partial^2\bar{\psi}}{\partial z^2}-\frac{1}{c^2}\left(V\frac{\partial}{\partial x}+i\omega\right)^2\bar{\psi}=0 \tag{A6}$$

In order to insure tangential flow at the wing surface, the potential must satisfy the downwash condition

$$\bar{w}(x,y)=\left(\frac{\partial\phi}{\partial z}\right)_{z=0}=\left(V\frac{\partial}{\partial x}+i\omega\right)\bar{Z}_m(x,y) \tag{A7}$$

where $\bar{w}$ and $\bar{Z}_m$ are amplitudes of velocity and displacements, respectively, and are assumed to be known from the motion of the wing. At $z=0$, the pressure

$$p=-\rho(\psi)_{z=0} \tag{A8}$$

must be zero at all points (x,y) off the wing. At all points on the wing ψ is allowed to be discontinuous and the value of p at a given point is determined by the magnitude of the discontinuity in ψ at the point. In the neighborhood of the trailing edge, p must go to zero, corresponding to the Kutta condition.

One other condition, that ϕ vanish far ahead of the wing, is inherently satisfied by the relation between ϕ and ψ given in equation (A5).

The potential ψ_0 at point (x, y, z) due to a harmonically pulsating doublet located in the xy-plane at $(\xi, \eta, 0)$ that satisfies equation (A6) is

$$\psi_0=A\frac{\partial}{\partial z}\frac{e^{i\omega\left[t+\frac{M}{c\beta^2}(x-\xi)-\frac{R'}{c\beta^2}\right]}}{R'} \tag{A9}$$

where

$$R'=\sqrt{(x-\xi)^2+\beta^2(y-\eta)^2+\beta^2z^2}$$

and the factor A is a strength and dimensionality factor that makes possible different uses and interpretations of the potential ψ_0. If ψ_0 is considered as an acceleration potential and substituted into equation (A5), there is obtained a corresponding velocity potential ϕ_0 which may be written as

$$\phi_0=A\frac{\partial}{\partial z}e^{-\frac{i\omega(x-\xi)}{V}}\int_{-\infty}^{\lambda=x-\xi}\frac{e^{i\omega\left(t+\frac{\lambda}{V}+\frac{M\lambda}{c\beta^2}-\frac{R}{c\beta^2}\right)}}{R}\,d\lambda \tag{A10}$$

where

$$R=\sqrt{\lambda^2+\beta^2(y-\eta)^2+\beta^2z^2}$$

The downwash $\frac{\partial\phi_0}{\partial z}$ associated with ψ_0 may be written as

$$\frac{\partial\phi_0}{\partial z}=A\frac{\partial^2}{\partial z^2}e^{-\frac{i\omega x_0}{V}}\int_{-\infty}^{x_0}\frac{e^{\overline{\omega}(\lambda-M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda \qquad \text{(A11)}$$

where $x_0=x-\xi$, $\omega=\omega/V\beta^2$, and $r=\beta\sqrt{(y-\eta)^2+z^2}$. With the use of this equation and the concept of solving linear boundary-value problems by means of superposition of elementary solutions to the governing differential equation, the boundary-value problem under discussion can be written as an integral equation, namely

$$\overline{w}(x,y)=\lim_{z\to 0}A\iint_S L(\xi,\eta)\,e^{-\frac{i\omega x_0}{V}}\,d\xi\,d\eta\,\frac{\partial^2}{\partial z^2}\int_{-\infty}^{x_0}\frac{e^{\overline{\omega}(\lambda-M\sqrt{\lambda^2+r^2})}}{\sqrt{\lambda^2+r^2}}\,d\lambda \qquad \text{(A12)}$$

where S represents the surface of the wing and $L(\xi, \eta)$ represents an unknown lift distribution or doublet strength on S. Equation (A12) may be seen to correspond essentially to equations (1) and (2).

If the distribution function $L(\xi, \eta)$ in equation (A12) is determined in accordance with the boundary conditions discussed in the preceding paragraph, equation (A12) can be considered as a complete solution to the boundary-value problem for an oscillating finite wing in compressible flow. It is also to be noted that equation (A12) can be considered to represent a solution to the so-called "indirect" problem, that is, that of finding the downwash distribution associated with a given lift distribution.

APPENDIX B

REDUCTION OF THE KERNEL FUNCTION FOR THREE-DIMENSIONAL FLOW TO THAT FOR TWO-DIMENSIONAL FLOW

The purpose of this appendix is to show that integration of the kernel function $K(x_0,y_0)$ from $-\infty$ to $+\infty$ with respect to $\eta=y-y_0$ leads to a known result for two-dimensional flow. The kernel is first modified to a form that, for the present case, is easier to handle. Then, after performing an integration by parts on the modified kernel, the form of the kernel for the two-dimensional case is given (eq. (B18)). In addition, the special cases of $M=1$ (eq. (B23)) and $M=0$ (eq. (B30)) are also shown.

The integration under consideration with respect to η is equivalent to an integration with respect to y_0, namely

$$l\int_{-\infty}^{\infty}K(x_0,y-\eta)\,d\eta=l\int_{-\infty}^{\infty}K(x_0,y_0)\,dy_0 \qquad \text{(B1)}$$

It is remarked in advance that since z has been made zero in the expression for $K(x_0,y_0)$, equation (20), it is necessary to employ the concept of "finite parts of infinite integrals" when integrating this function across the singularities at $y_0=0$. Use of this concept gives the same results that could be obtained by the more arduous task of performing the integrations before setting z equal to zero.

Modification of the kernel.—In order to effect the desired modification of the expression for $K(x_0,y_0)$ given by equation (20), consider the first integral of the expression, namely

$$-k^2\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau \qquad \text{(B2)}$$

This integral can be written as

$$\lim_{\delta\to 0}-k^2\int_0^{\infty}\sqrt{1+\tau^2}\,e^{-\tau(\delta+ik|y_0|)}\,d\tau+k^2\int_{M/\beta}^{\infty}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau \qquad \text{(B3)}$$

but according to page 331 of reference 22

$$\int_0^{\infty}\sqrt{1+\tau^2}\,e^{-\tau(\delta+ik|y_0|)}\,d\tau=\frac{\pi}{2(\delta+ik|y_0|)}\,[H_1(\delta+ik|y_0|)-Y_1(\delta+ik|y_0|)] \qquad \text{(B4)}$$

where H_1 is the unmodified Struve function of first order and Y_1 is the Bessel function of the second kind of first order. In the limit as $\delta\to 0$ these expressions have the following values: For the first expression in the bracket (see ref. 22, p. 329)

$$\lim_{\delta\to 0}H_1(\delta+ik|y_0|)=H_1(ik|y_0|)=-L_1(k|y_0|) \qquad \text{(B5)}$$

and for the second expression (see ref. 22, pp. 77 and 78)

$$\lim_{\delta\to 0}Y_1(\delta+ik|y_0|)=-iH_1^{(1)}(ik|y_0|)+iJ_1(ik|y_0|)=\frac{2i}{\pi}K_1(k|y_0|)-I_1(k|y_0|) \qquad \text{(B6)}$$

where $H_1^{(1)}$ denotes the Hankel function of the first kind of first order. With the use of equations (B3) to (B6), expression (B2) can be written as

$$-k^2\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau=k^2\int_{M/\beta}^{\infty}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau+\frac{k}{|y_0|}\left\{K_1(k|y_0|)+\frac{\pi i}{2}[I_1(k|y_0|)-L_1(k|y_0|)]\right\} \qquad \text{(B7)}$$

Substituting this result into equation (20) of the text gives the modified form of $K(x_0,y_0)$ sought, namely

$$K(x_0,y_0)=\frac{e^{-ikx_0}}{l^2}\left[\frac{ik}{\beta|y_0|}e^{-\frac{ik|y_0|}{\beta}}+\frac{1}{My_0^2}e^{-\frac{ik|y_0|}{\beta}}-\frac{Mx_0+\sqrt{x_0^2+\beta^2y_0^2}}{My_0^2\sqrt{x_0^2+\beta^2y_0^2}}e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+k^2\int_{M/\beta}^{\infty}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau+\frac{ik}{My_0^2}\int_0^{x_0}e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}\,d\lambda\right] \qquad \text{(B8)}$$

Integration of modified kernel.—Since the expression for $K(x_0,y_0)$ is symmetrical with respect to y_0, that is, $K(x_0,-y_0)=$

$K(x_0,+y_0)$, the integration under consideration can be expressed as

$$l\int_{-\infty}^{\infty} K(x_0,|y_0|)\,dy_0=2l\int_0^{\infty} K(x_0,y_0)\,dy_0 \tag{B9}$$

where, on the right, the absolute-value signs on y_0 can be dropped.

After performing an integration by parts by letting

$$dv=2\frac{e^{-ikx_0}}{l}\frac{dy_0}{y_0^2};\quad v=-2\frac{e^{-ikx_0}}{l}\frac{1}{y_0} \tag{B10}$$

and

$$u=\frac{iky_0}{\beta}e^{-\frac{ikMx_0}{\beta}}+\frac{1}{M}e^{-\frac{ikMx_0}{\beta}}-\frac{Mx_0+\sqrt{x_0^2+\beta^2y_0^2}}{M\sqrt{x_0^2+\beta^2y_0^2}}e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+$$

$$k^2y_0^2\int_{M/\beta}^{\infty}\sqrt{1+\tau^2}e^{-ikx_0\tau}\,d\tau+\frac{ik}{M}\int_0^{x_0}e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}\,d\lambda \tag{B11}$$

or

$$du=\left\{\left[\frac{\beta^2y_0x_0}{(x_0^2+\beta^2y_0^2)^{3/2}}+\frac{ikMx_0y_0}{x_0^2+\beta^2y_0^2}+\frac{iky_0}{\sqrt{x_0^2+\beta^2y_0^2}}\right]e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+\right.$$

$$\left.k^2y_0\int_{M/\beta}^{\infty}\frac{e^{-ikx_0\tau}}{\sqrt{1+\tau^2}}\,d\tau+k^2y_0\int_0^{x_0}\frac{e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}}{\sqrt{\lambda^2+\beta^2y_0^2}}\,d\lambda\right\}dy_0 \tag{B12}$$

there is obtained for $uv\Big]_0^{\infty}$

$$uv\Big]_0^{\infty}=2\frac{e^{-ikx_0}}{l}\left\{-\frac{1}{y_0}\left[\frac{iky_0}{\beta}e^{-\frac{ikMx_0}{\beta}}+\frac{1}{M}e^{-\frac{ikMx_0}{\beta}}-\right.\right.$$

$$\left.\left.\left(\frac{x_0}{\sqrt{x_0^2+\beta^2y_0^2}}+\frac{1}{M}\right)e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+\right.\right.$$

$$\left.\left.k^2y_0^2\int_{M/\beta}^{\infty}\sqrt{1+\tau^2}\,e^{-ikx_0\tau}\,d\tau+\frac{ik}{M}\int_0^{x_0}e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}\,d\lambda\right]_{y_0=0}^{y_0=\infty}\right\} \tag{B13}$$

This expression vanishes at its upper limit $y_0=\infty$ and is singular at its lower limit $y_0=0$. However, by not making $z\to 0$ in the derivation of $K(x_0,y_0)$ until after this stage is reached, this singular value is canceled by other terms that have otherwise been dropped. Thus, the expression (B13) may be considered to be zero, which is the value of its finite part. The integration under consideration is then reduced to the value of $-\int_0^{\infty} v\,du$ which is

$$-\int_0^{\infty}v\,du=2\frac{e^{-ikx_0}}{l}\int_0^{\infty}\left\{\left[\left[\frac{\beta^2x_0}{(x_0^2+\beta^2y_0^2)^{3/2}}+\frac{ikMx_0}{x_0^2+\beta^2y_0^2}+\right.\right.\right.$$

$$\left.\left.\left.\frac{ik}{\sqrt{x_0^2+\beta^2y_0^2}}\right]e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+k^2\int_{M/\beta}^{\infty}\frac{e^{-ikx_0\tau}}{\sqrt{1+\tau^2}}\,d\tau+\right.\right.$$

$$\left.\left.k^2\int_0^{x_0}\frac{e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}}{\sqrt{\lambda^2+\beta^2y_0^2}}\,d\lambda\right\}dy_0\right.$$

$$=2\frac{e^{-ikx_0}}{l}\int_0^{\infty}\left[\left(\frac{ikM}{x_0}+\frac{ik}{\sqrt{x_0^2+\beta^2y_0^2}}\right)e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+\right.$$

$$\left.k^2\int_{M/\beta}^{\infty}\frac{e^{-ikx_0\tau}}{\sqrt{1+\tau^2}}\,d\tau+k^2\int_0^{x_0}\frac{e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}}{\sqrt{\lambda^2+\beta^2y_0^2}}\,d\lambda\right]dy_0 \tag{B14}$$

The terms of this expression are treated separately in the next three equations:

First (see ref. 22, p. 180)

$$2\int_0^{\infty}\left(\frac{ikM}{x_0}+\frac{ik}{\sqrt{x_0^2+\beta^2y_0^2}}\right)e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\,dy_0=\frac{2ik}{\beta}e^{\frac{ikx_0}{\beta^2}}\int_0^{\infty}\left(\frac{|x_0|}{x_0}M\cosh\theta+1\right)e^{-\frac{ikM}{\beta^2}|x_0|\cosh\theta}\,d\theta$$

$$=-\frac{i\pi k}{\beta}e^{\frac{ikx_0}{\beta^2}}\left[M\frac{|x_0|}{x_0}H_1^{(2)}\left(\frac{kM|x_0|}{\beta^2}\right)+iH_0^{(2)}\left(\frac{kM|x_0|}{\beta^2}\right)\right] \tag{B15}$$

second

$$2k^2\int_0^{\infty}dy_0\int_{M/\beta}^{\infty}\frac{e^{-ikx_0\tau}}{\sqrt{1+\tau^2}}\,d\tau=2k^2\int_{M/\beta}^{\infty}\frac{d\tau}{\sqrt{1+\tau^2}}\int_0^{\infty}e^{-ikx_0\tau}\,dy_0$$

$$=-2ik\int_{M/\beta}^{\infty}\frac{d\tau}{\tau\sqrt{1+\tau^2}}$$

$$=-2ik\log\frac{1+\beta}{M} \tag{B16}$$

and third (see ref. 22, p. 180)

$$2k^2\int_0^{\infty}dy_0\int_0^{x_0}\frac{e^{\frac{ik}{\beta^2}(\lambda-M\sqrt{\lambda^2+\beta^2y_0^2})}}{\sqrt{\lambda^2+\beta^2y_0^2}}\,d\lambda=2k^2\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}d\lambda\int_0^{\infty}\frac{e^{-\frac{ik}{\beta^2}M\sqrt{\lambda^2+\beta^2y_0^2}}}{\sqrt{\lambda^2+\beta^2y_0^2}}\,dy_0=\frac{2k^2}{\beta}\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}d\lambda\int_0^{\infty}e^{-\frac{ikM}{\beta^2}|\lambda|\cosh\theta}\,d\theta=-\frac{\pi ik^2}{\beta}\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}H_0^{(2)}\left(\frac{kM}{\beta^2}|\lambda|\right)d\lambda \tag{B17}$$

Substituting the results in equations (B15) to (B17) into equation (B14) gives

$$l\int_{-\infty}^{\infty} K(x_0,y_0)dy_0=-\frac{\pi k}{l\beta}e^{-ikx_0}\left\{e^{\frac{ikx_0}{\beta^2}}\left[\frac{iM|x_0|}{x_0}H_1^{(2)}\left(\frac{kM|x_0|}{\beta^2}\right)-H_0^{(2)}\left(\frac{kM|x_0|}{\beta^2}\right)\right]+\frac{2i}{\pi}\beta\log\frac{1+\beta}{M}+ik\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}H_0^{(2)}\left(\frac{kM|\lambda|}{\beta^2}\right)d\lambda\right\} \quad \text{(B18)}$$

This result is a form of the expression for the kernel function of Possio's integral equation relating pressure and downwash for a two-dimensional oscillating wing in subsonic compressible flow. It checks the results given, for example, in reference 27.

Reduction of kernel for $M=1$.—The kernel function for $M=1$ may be written as (see eq. (47a))

$$K(x_0,y_0)_{M=1}=\frac{k^2}{l^2}e^{-ikx_0}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\frac{\pi i}{2k|y_0|}\left[I_1(k|y_0|)-L_1(k|y_0|)-\frac{2}{\pi}\right]+\frac{1}{k^2y_0^2}-\frac{2}{k^2y_0^2}e^{\frac{ik}{2}\left(x_0-\frac{y_0^2}{x_0}\right)}+\frac{i}{k^2y_0^2}\int_0^{kx_0}e^{\frac{i}{2}\left(\lambda-\frac{k^2y_0^2}{\lambda}\right)}d\lambda-\frac{i}{k^2y_0^2}\int_0^{k|y_0|}e^{\frac{i}{2}\left(\lambda-\frac{k^2y_0^2}{\lambda}\right)}d\lambda\right\} \quad \text{(B19)}$$

The second integral appearing in this equation can be shown to cancel several of the terms so that the kernel becomes

$$K(x_0,y_0)_{M=1}=-\frac{e^{-ikx_0}}{l^2}\left[\frac{2}{y_0^2}e^{\frac{ik}{2}\left(x_0-\frac{y_0^2}{x_0}\right)}-\frac{i}{y_0^2}\int_0^{kx_0}e^{\frac{i}{2}\left(\lambda-\frac{k^2y_0^2}{\lambda}\right)}d\lambda\right] \quad \text{(B20)}$$

so that the kernel for the sonic case in two-dimensional flow may be written as

$$l\int_{-\infty}^{\infty}K(x_0,y_0)_{M=1}dy_0=-\frac{e^{-ikx_0}}{l}\left(2e^{\frac{ikx_0}{2}}\int_{-\infty}^{\infty}\frac{e^{\frac{iky_0^2}{2x_0}}}{y_0^2}dy_0-i\int_0^{kx_0}e^{\frac{i\lambda}{2}}d\lambda\int_{-\infty}^{\infty}\frac{e^{\frac{ik^2y_0^2}{2\lambda}}}{y_0^2}dy_0\right) \quad \text{(B21)}$$

Integrating equation (B21) by parts with respect to y_0, retaining only finite parts of the integrated results, and making use of the relation

$$i\int_{-\infty}^{\infty}e^{-i\omega\tau^2}d\tau=2i\int_0^{\infty}e^{-i\omega\tau^2}d\tau=\frac{\sqrt{\pi i}}{a}$$

yields

$$l\int_{-\infty}^{\infty}K(x_0,y_0)_{M=1}dy_0=-\frac{e^{-ikx_0}}{l}\left\{2e^{\frac{ikx_0}{2}}\left(-\frac{1}{y_0}e^{\frac{iky_0^2}{2x_0}}\right]_{-\infty}^{\infty}-\frac{ik}{x_0}\int_{-\infty}^{\infty}e^{\frac{iky_0^2}{2x_0}}dy_0\right)-i\int_0^{kx_0}e^{\frac{i\lambda}{2}}d\lambda\left(-\frac{1}{y_0}e^{\frac{ik^2y_0^2}{2\lambda}}\right]_{-\infty}^{\infty}-\frac{ik^2}{\lambda}\int_{-\infty}^{\infty}e^{\frac{ik^2y_0^2}{2\lambda}}dy_0\right)\right\}$$

$$=-\frac{2k\sqrt{\pi}}{l}e^{-ikx_0}\left(-2\sqrt{\frac{i}{2kx_0}}e^{\frac{ikx_0}{2}}+i\sqrt{\frac{i}{2}}\int_0^{kx_0}\frac{e^{\frac{i\lambda}{2}}}{\sqrt{\lambda}}d\lambda\right) \quad \text{(B22)}$$

Finally, the kernel for the sonic case in two-dimensional flow may be written as

$$l\int_{-\infty}^{\infty}K(x_0,y_0)_{M=1}dy_0=\frac{4}{l}\sqrt{\frac{\pi i}{2}}e^{-ikx_0}\left(\frac{k}{\sqrt{kx_0}}e^{\frac{ikx_0}{2}}-ik\sqrt{\pi}\int_0^{\sqrt{\frac{kx_0}{\pi}}}e^{\frac{i\pi\lambda^2}{2}}d\lambda\right) \quad \text{(B23)}$$

It may be noted that the integrals in this equation are readily expressible in terms of Fresnel integrals

$$C(x)=\int_0^x\cos\frac{\pi}{2}t^2dt$$

and

$$S(x)=\int_0^x\sin\frac{\pi}{2}t^2dt$$

Reduction of kernel for $M=0$.—For $M=0$ it is convenient to modify the kernel function before integrating with respect to y_0. For this purpose use is made of the relation (see eq. (B7)):

$$-\frac{k}{|y_0|}K_1(k|y_0|)-\frac{i\pi k}{2|y_0|}\left[I_1(k|y_0|)-L_1(k|y_0|)\right]=k^2\int_0^{\infty}\sqrt{1+\tau^2}e^{-ik|y_0|\tau}d\tau=\frac{k^2}{y_0^2}\int_0^{\infty}\sqrt{y_0^2+\lambda^2}e^{-ik\lambda}d\lambda \quad \text{(B24)}$$

and the relation

$$\frac{k^2}{y_0^2}\int_0^{x_0}\sqrt{\lambda^2+y_0^2}e^{ik\lambda}d\lambda=\frac{k^2}{y_0^2}\int_{-x_0}^{0}\sqrt{y_0^2+\lambda^2}e^{-ik\lambda}d\lambda \quad \text{(B25)}$$

With these relations the expression for $K(x_0,y_0)_{M=0}$, equation (53), can be written as

$$K(x_0,y_0)_{M=0}=\frac{e^{-ikx_0}}{l^2}\left(-\frac{x_0}{y_0^2\sqrt{x_0^2+y_0^2}}e^{-ikx_0}+\frac{ik\sqrt{x_0^2+y_0^2}}{y_0^2}e^{ikx_0}+\frac{k^2}{y_0^2}\int_{-x_0}^{\infty}\sqrt{y_0^2+\lambda^2}e^{-ik\lambda}d\lambda\right) \quad \text{(B26)}$$

But

$$\frac{k^2}{y_0^2}\int_{-x_0}^{\infty}\sqrt{y_0^2+\lambda^2}\,e^{-ik\lambda}d\lambda=-\frac{ik\sqrt{x_0^2+y_0^2}}{y_0^2}e^{ikx_0}-\frac{ik}{y_0^2}\int_{-x_0}^{\infty}\frac{\lambda}{\sqrt{y_0^2+\lambda^2}}e^{-ik\lambda}d\lambda$$

$$=-\frac{ik\sqrt{x_0^2+y_0^2}}{y_0^2}e^{ikx_0}+\frac{x_0}{y_0^2\sqrt{x_0^2+y_0^2}}e^{ikx_0}-\int_{-x_0}^{\infty}\frac{e^{-ik\lambda}}{(y_0^2+\lambda^2)^{3/2}}d\lambda \qquad \text{(B27)}$$

therefore,

$$K(x_0,y_0)_{M=0}=-\frac{e^{-ikx_0}}{l^2}\int_{-x_0}^{\infty}\frac{e^{-ik\lambda}}{(y_0^2+\lambda^2)^{3/2}}d\lambda \qquad \text{(B28)}$$

Integrating with respect to y_0 gives

$$l\int_{-\infty}^{\infty}K(x_0,y_0)_{M=0}\,dy_0=-\frac{2}{l}e^{-ikx_0}\int_0^{\infty}\int_{-x_0}^{\infty}\frac{e^{-ik\lambda}}{(y_0^2+\lambda^2)^{3/2}}d\lambda\,dy_0$$

$$=-\frac{2}{l}e^{-ikx_0}\int_{-x_0}^{\infty}e^{-ik\lambda}\,d\lambda\int_0^{\infty}\frac{dy_0}{(y_0^2+\lambda^2)^{3/2}}$$

$$=-\frac{2}{l}e^{-ikx_0}\int_{-x_0}^{\infty}\frac{e^{-ik\lambda}}{\lambda^2}d\lambda$$

$$=-\frac{2}{l}e^{-ikx_0}\left(\int_{-\infty}^{\infty}\frac{e^{-ik\lambda}}{\lambda^2}d\lambda-\int_{x_0}^{\infty}\frac{e^{ik\lambda}}{\lambda^2}d\lambda\right) \qquad \text{(B29)}$$

Integrating each integral in equation (B29) and retaining only finite parts yields

$$l\int_{-\infty}^{\infty}K(x_0,y_0)_{M=0}\,dy_0=-\frac{2}{l}e^{-ikx_0}\left(-\frac{e^{ikx_0}}{x_0}-ik\int_{-\infty}^{\infty}\frac{e^{-ik\lambda}}{\lambda}d\lambda-ik\int_{x_0}^{\infty}\frac{e^{ik\lambda}}{\lambda}d\lambda\right)$$

$$=-\frac{2}{l}e^{-ikx_0}\left(-\frac{e^{ikx_0}}{x_0}-2k\int_0^{x_0}\frac{\sin k\lambda}{\lambda}d\lambda-ik\int_{x_0}^{\infty}\frac{\cos k\lambda}{\lambda}d\lambda+k\int_{x_0}^{\infty}\frac{\sin k\lambda}{\lambda}d\lambda\right)$$

$$=-\frac{4\pi k}{l}\left(-\frac{1}{2\pi kx_0}+\frac{i}{2\pi}e^{-ikx_0}\left\{\mathrm{Ci}\,(kx_0)+i\left[\mathrm{Si}\,(kx_0)+\frac{\pi}{2}\right]\right\}\right) \qquad \text{(B30)}$$

where $\mathrm{Ci}(kx_0)$ and $\mathrm{Si}(kx_0)$ denote, respectively, the "cosine-integral" and "sine-integral" functions defined as follows:

$$\mathrm{Ci}\,(x)=-\int_x^{\infty}\frac{\cos t}{t}\,dt$$

$$\mathrm{Si}\,(x)=\frac{\pi}{2}-\int_x^{\infty}\frac{\sin t}{t}\,dt$$

The results in the braces of equation (B30) check with results given for this case in reference 14.

APPENDIX C

SOME REMARKS ON EVALUATION OF THE KERNEL FUNCTION

Exact expressions for the kernel function $K(x_0,y_0)$ are given in equation (20) for $0\leq M<1$ and in equation (47) for $M=1$. Corresponding approximate forms are given in equations (54) and (56).

Equations (20) and (47) are valid for any set of values of M, k, x_0, and y_0. To calculate the value of the kernel from these equations, it is necessary to evaluate numerically the integrals which appear. Values of the other terms can be obtained by making use of existing tables. Extensive tables of the Bessel functions K_1 and I_1 may be found in reference 28 and a table of the Struve function L_1 with second and fourth differences for interpolation purposes may be found in reference 29. Sample values of the kernel are given in table I.

For certain ranges of values of M, k, x_0, and y_0, as indicated by equations (55) and (57), the kernel can be evaluated by making use of the power series expansions given by equation (54) for $0\leq M<1$ and equation (56) for $M=1$.

The various expressions for $K(x_0,y_0)$ become singular when $y_0=y-\eta, x_0\geq 0$. In order to be able to evaluate the kernel in such circumstances, it has been separated into two parts as shown in equation (30). One of these is denoted by $K(x_0,y_0)-K'(x_0,y_0)$ and is not singular; the other is denoted by $K'(x_0,y_0)$ and contains all the singularities. Obtaining the value of $(K-K')$ from the form of the expression given in equation (30), however, may be troublesome. This particular value for $y_0=0$, $x_0>0$ can be obtained from the following limiting form:

$$\lim_{y_0\to 0}[K(x_0,y_0)-K'(x_0,y_0)]=\frac{e^{-ikx_0}}{l^2}\left\{\left(\frac{\beta^2}{2x_0^2}+\frac{ik\,(1+M)}{2x_0}\right)e^{\frac{ikx_0}{1+M}}-\frac{\beta^2}{2x_0^2}-\frac{ik}{x_0}+\frac{k^2}{2}\left[\frac{2+M}{1+M}-2\gamma-\log\left(\frac{kx_0}{1+M}\right)+\mathrm{Ci}\left(\frac{kx_0}{1+M}\right)+i\,\mathrm{Si}\left(\frac{kx_0}{1+M}\right)-\frac{\pi i}{2}\right]\right\} \qquad \text{(C1)}$$

where γ denotes Euler's constant ($\gamma=0.577216$) and Ci and Si denote cosine-integral and sine-integral functions, respectively. (These functions are tabulated in reference 30). For $M=1$, this expression reduces to

$$\lim_{M\to 1}[K(x_0,y_0)-K'(x_0,y_0)]=\frac{e^{-ikx_0}}{l^2}\left\{\frac{ik}{x_0}e^{\frac{ikx_0}{2}}-\frac{ik}{x_0}+\frac{k^2}{2}\left[\frac{3}{2}-2\gamma-\log\left(\frac{kx_0}{2}\right)+\mathrm{Ci}\left(\frac{kx_0}{2}\right)+i\,\mathrm{Si}\left(\frac{kx_0}{2}\right)-\frac{\pi i}{2}\right]\right\} \qquad \text{(C2)}$$

The kernel function is not singular for $x_0<0$. For $y_0=0$ and $x_0<0$ it may be written for $M<1$ as

$$\lim_{y_0\to 0} K(-x_0,y_0)=\frac{e^{ik|x_0|}}{l^2}\left\{e^{-\frac{ik}{\beta^2}(1+M)|x_0|}\left[\frac{-\beta^2}{2x_0^2}+\frac{ik\,(1-M)}{2|x_0|}\right]-\frac{k^2}{2}\left[\mathrm{Ci}\left(\frac{k|x_0|}{1-M}\right)-i\,\mathrm{Si}\left(\frac{k|x_0|}{1-M}\right)+\frac{\pi i}{2}\right]\right\} \quad \text{(C3)}$$

The expression for $K-K'$ for $x_0<0$, $y_0=0$ may also be useful. It is

$$\lim_{y_0\to 0}[K(-x_0,y_0)-K'(-x_0,y_0)]=\frac{e^{ik|x_0|}}{l^2}\left\{\frac{\beta^2}{2x_0^2}-\frac{ik}{|x_0|}-\left(\frac{\beta^2}{2x_0^2}-\frac{ik\,(1-M)}{2|x_0|}\right)e^{-\frac{ik|x_0|}{1-M}}-\frac{k^2}{2}\left[\frac{1}{1-M}-\log\left(\frac{k|x_0|}{1-M}\right)+\mathrm{Ci}\left(\frac{k|x_0|}{1-M}\right)-i\,\mathrm{Si}\left(\frac{k|x_0|}{1-M}\right)+\frac{\pi i}{2}\right]\right\} \quad \text{(C4)}$$

For $M=1$, $K(-x_0,y_0)=K(-x_0,y_0)-K'(-x_0,y_0)\equiv 0$.

Some results of evaluating the kernel and its nonsingular part are given as examples in table I. (In order to obtain these results the required integrations were performed numerically by manual computing methods.)

APPENDIX D

ALTERNATE FORM OF EQUATION (20)

Subsequent to the derivation of equation (20) as given in the text, it was found that the two integrals involved in this equation can be combined in a manner that leads to a more concise and, for many purposes, a more convenient form of expression for the kernel function. The purpose of this appendix is to derive this alternate form.

Consider first the integral

$$G_1=\frac{i}{M(ky_0)^2}\int_0^{kx_0} e^{\frac{i}{\beta^2}[\lambda-M\sqrt{\lambda^2+\beta^2(ky_0)^2}]}\,d\lambda \quad \text{(D1)}$$

and make the substitution

$$\frac{1}{\beta^2}\left(\lambda-M\sqrt{\lambda^2+\beta^2(ky_0)^2}\right)=-k|y_0|\tau \quad \text{(D2)}$$

or

$$\lambda=k|y_0|(M\sqrt{1+\tau^2}-\tau) \quad \text{(D3)}$$

This substitution gives for G_1

$$G_1=\frac{i}{Mk|y_0|}\int_{M/\beta}^{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}\,d\tau$$
$$=\frac{1}{Mk^2y_0^2}\left[e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}-e^{-\frac{ikM|y_0|}{\beta}}\right]+\frac{i}{k|y_0|}\int_{M/\beta}^{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau \quad \text{(D4)}$$

Consider now the integral

$$G_2=\int_0^{M/\beta}\sqrt{1+\tau^2}\,e^{-ik|y_0|\tau}\,d\tau \quad \text{(D5)}$$

and integrate by parts by letting

$$u=\sqrt{1+\tau^2}$$
$$dv=e^{-ik|y_0|\tau}\,d\tau$$

so that

$$du=\frac{\tau\,d\tau}{\sqrt{1+\tau^2}}$$
$$v=\frac{i}{k|y_0|}\,e^{-ik|y_0|\tau}$$

This integration gives for G_2:

$$G_2=\frac{i}{\beta k|y_0|}\,e^{-\frac{iMk|y_0|}{\beta}}-\frac{i}{k|y_0|}-\frac{i}{k|y_0|}\int_0^{M/\beta}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{-ik|y_0|\tau}d\tau \quad \text{(D6)}$$

Subtracting G_2 from G_1 (eqs. (D4) and (D6)) gives

$$G_1-G_2=\frac{i}{k|y_0|}-\frac{i}{\beta k|y_0|}\,e^{-\frac{iMk|y_0|}{\beta}}-\frac{1}{Mk^2y_0^2}\,e^{-\frac{iMk|y_0|}{\beta}}+\frac{1}{Mk^2y_0^2}\,e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+\frac{i}{k|y_0|}\int_0^{M/\beta}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{-ik|y_0|\tau}d\tau+\frac{i}{k|y_0|}\int_{M/\beta}^{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{-ik|y_0|\tau}d\tau$$
$$=\frac{i}{k|y_0|}-\frac{iMk|y_0|+\beta}{M\beta(ky_0)^2}\,e^{-\frac{iMk|y_0|}{\beta}}+\frac{1}{M(ky_0)^2}\,e^{\frac{ik}{\beta^2}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}+\frac{i}{k|y_0|}\int_0^{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{ik|y_0|\tau}d\tau \quad \text{(D7)}$$

Substituting this result into equation (20) of the text gives for $K(x_0,y_0)$

$$K(x_0,y_0)=\frac{k^2e^{-ikx_0}}{l^2}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\frac{\pi i}{2k|y_0|}[I_1(k|y_0|)-L_1(k|y_0|)]+\frac{i}{k|y_0|}-\frac{kx_0}{(ky_0)^2\sqrt{(kx_0)^2+\beta^2(ky_0)^2}}\,e^{\frac{ik}{\beta^2}[x_0-M\sqrt{(kx_0)^2+\beta^2(ky_0)^2}]}+\frac{i}{k|y_0|}\int_0^{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2+\beta^2y_0^2})}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{ik|y_0|\tau}d\tau\right\} \quad \text{(D8)}$$

The integral in this equation is in general more amenable to numerical evaluation than either of the two integrals appearing in equation (20). Furthermore, with this expression, it is not necessary to consider the incompressible case as a special case, since no trouble arises in setting $M=0$. Similarly, for the sonic case no trouble arises and this expression gives for $x_0>0$:

$$K(x_0,y_0)_{M\leq 1}=\frac{k^2e^{-ikx_0}}{l^2}\left\{-\frac{1}{k|y_0|}K_1(k|y_0|)-\frac{\pi i}{2k|y_0|}[I_1(k|y_0|)-L_1(k|y_0|)]+\frac{i}{k|y_0|}-\frac{1}{(ky_0)^2}\,e^{\frac{ik}{2}\left[kx_0-\frac{(ky_0)^2}{kx_0}\right]}+\frac{i}{k|y_0|}\int_0^{\frac{1}{2|y_0|}\left(x_0-\frac{y_0^2}{x_0}\right)}\frac{\tau}{\sqrt{1+\tau^2}}\,e^{ik|y_0|\tau}d\tau\right\} \quad \text{(D9)}$$

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TABLE I.—VALUES OF THE KERNEL AND ITS NONSINGULAR PART AT $M=0.7$

x_0	y_0	k	$K(x_0,y_0)$	$K(x_0,y_0)-K'(x_0,y_0)$
0	0.125	0.1	−63.827569+ 1.112400i	0.144529−.007824i
		.3	−63.801759+ 3.290793i	−.003441−.069879i
		.5	−63.513049+ 5.408465i	−.009423−.192655i
		.7	−63.127659+ 7.466762i	−.018114−.374807i
		1.0	−62.396691+ 10.445693i	−.035609−.756548i
1.5	.125	.1	−126.263912+ 19.142811i	.141754−.028841i
		.3	−114.855158+ 55.631898i	−.031317−.056060i
		.5	−92.964383+ 86.829346i	−.123447−.115703i
		.7	−62.878740+109.927026i	−.283001−.133318i
		1.0	−8.792808+125.223964i	−.581313+.022309i
0	6.0	.1	−.019271+ .016639i	−.000039−.006699i
		.3	+.007493+ .020950i	+.007793−.049064i
		.5	+.020861+ .001545i	.036165−.115145i
		.7	+.009570− .017888i	.095337−.181254i
		1.0	−.018833− .006290i	.305627−.239670i
1.5	6.0	.1	−.027209+ .020038i	−.00905 −.006215i
		.3	+.002452+ .028186i	−.005415−.041401i
		.5	+.021871+ .013305i	−.007432−.109920i
		.7	+.022588− .008980i	−.026790−.232786i
		1.0	−.004786− .022987i	−.190134−.523276i

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