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TIME-DEPENDENT LINEAR DAE'S WITH DISCONTINUOUS INPUTS[1]

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ABSTRACT. Existence and uniqueness results are proved for initial value problems associated with linear, time-varying, differential-algebraic equations. The right-hand sides are chosen in a space of distributions allowing for solutions exhibiting discontinuities as well as "impulses". This approach also provides a satisfactory answer to the problem of "inconsistent initial conditions" of crucial importance for the physical applications. Furthermore, our theoretical results yield an efficient numerical procedure for the calculation of the jump and impulse of a solution at a point of discontinuity. Numerical examples are given.

1. Introduction.

In this paper, we prove existence and uniqueness results for initial value problems associated with differential-algebraic equations (DAE's) in $\mathbf{R}^n$

$$A\dot{x} + Bx = b, \tag{1.1}$$

where A, B are smooth *time-varying* linear operators, and b belongs to a class of distributions with values in $\mathbf{R}^n$ containing the functions that are smooth in $(-\infty, 0]$ and $[0, \infty)$ and have a discontinuity at the origin. Such discontinuities on the right side occur frequently in physical problems modelled by DAE's. For instance, in electrical network problems a discontinuity of b may correspond to the operation of a switch at a given time.

The existence and uniqueness theory for problems of the form (1.1) with smooth b (and consistent initial conditions) is now well understood, see, e.g., [C87], [KuM92], [RR93a] and

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the references given there. But elementary examples show that the setting of distributions is indispensable for handling discontinuous right-hand sides. For example, if A is constant with $A^2 = 0$, $B = I$, and $b = b_0 H$ where $b_0 \in \mathbf{R}^n$ and H is the Heaviside function, then the solution of (1.1) (in this case unique: no initial condition needs to be or should be prescribed) is $x = b_0 H - Ab_0\delta$ and hence involves the Dirac delta distribution.

When A and B are constant and b exhibits jumps, Laplace transform methods are available to find the solutions of (1.1), but the problem appears to remain open for time dependent coefficients A, B. This is the case considered in this paper. Our results depend essentially upon our recent work [RR93a] on a reduction procedure that transforms the distribution solutions of (1.1) into the distribution solutions of an explicit ODE.

The "consistency" of initial conditions represents another topic of considerable theoretical and practical importance in the study of DAE's. As is well known, even for smooth b, (1.1) will not have a solution starting at arbitrary points $x_0 \in \mathbf{R}^n$. Rather, existence of a solution in the classical sense requires that x_0 satisfies certain constraints called the consistency conditions. On the other hand, suppose that the physical process modelled by (1.1) starts at time $t = 0$, and that for $t < 0$ the state variable $x(t)$ has evolved in a way totally unrelated with (1.1). If $\lim_{t \to 0-} x(t) = x_0$ exists this x_0 represents a natural data value for the initial condition at $t = 0$. But, since x_0 has no reason to be consistent with (1.1) at $t = 0$, the mathematical theory only provides that (1.1) has no solution for this choice of initial condition, which is, of course, a physically unacceptable statement.

It turns out that the consistency question is closely related to the problems addressed here. By viewing this question as that of extending a known state $x(t)$ for $t < 0$ to a solution of (1.1) for $t > 0$ via a solution of (1.1) in $(\mathcal{D}'(\mathbf{R}))^n$, we show that the ambiguity can be resolved: From $x_0 = \lim_{t \to 0-} x(t)$ we find that a unique, computable jump to a consistent value occurs at $t = 0$. Furthermore, for problems with index $\nu \geq 2$, the sudden transition between x_0 and the consistent initial value may also create a (computable) impulse; that is, a linear combination of δ and its derivatives. Further evidence that our solution is the correct one is provided by showing that it is the limit of the classical solutions of the problems obtained by smoothing out the right-hand side near $t = 0$. These results complement in various ways those already obtained for problems with constant coefficients in [VLK81], [Co82], or [G93].

Section 2 gives a brief review of the reduction procedure developed in [RR93a]. Initial value problems for (1.1) are then considered in Section 3 for right-hand sides in a class of distributions which is a close relative of the class C^n_{imp} of "impulsive-smooth" distributions introduced in [HS83]. The application of these results to the problem of inconsistent initial conditions is discussed in Section 4, and some straightforward generalizations are presented in Section 5. Finally, Section 6 presents some computational results that illustrate the resulting algorithms.

2. Reduction procedure for linear DAE's.

Let $A, B \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$ and set $A_0 = A$, $B_0 = B$. The reduction procedure developed in [RR93a] generates, under appropriate conditions, a new pair (A_{j+1}, B_{j+1}) from the pair of coefficient functions (A_j, B_j), $j \geq 0$. More precisely, set $r_{-1} = n$ and assume that $A_j, B_j \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{j-1}}))$ for some integer $0 \leq r_{j-1} \leq n$. Moreover, suppose that

$$\text{rank } A_j(t) = r_j, \quad \forall t \in \mathbf{R}, \tag{2.1}$$

where $0 \leq r_j \leq r_{j-1}$ is a fixed integer, and that

$$\text{rank } A_j(t) \oplus B_j(t) = r_{j-1}, \qquad \forall t \in \mathbf{R}, \tag{2.2}$$

where $A_j(t) \oplus B_j(t) \in \mathcal{L}(\mathbf{R}^{r_{j-1}} \times \mathbf{R}^{r_{j-1}}, \mathbf{R}^{r_{j-1}})$ is defined by $A_j(t) \oplus B_j(t)(u, v) = A_j(t)u + B_j(t)v$.

Under the conditions (2.1) and (2.2), it is shown in [RR93a] that the following mappings exist:

(i) $P_j \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{j-1}}))$ such that $P_j(t)$ is a projection onto rge $A_j(t)$, $\forall t \in \mathbf{R}$.

(ii) $C_j \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_j}, \mathbf{R}^{r_{j-1}})$ such that $C_j(t) \in GL(\mathbf{R}^{r_j}, \ker Q_j(t)B_j(t))$, $\forall t \in \mathbf{R}$, where $Q_j = I - P_j$.

(iii) $D_j \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{j-1}}, \mathbf{R}^{r_j}))$ such that $D_j(t) \in GL(\text{rge } A_j(t), \mathbf{R}^{r_j})$, $\forall t \in \mathbf{R}$.

With C_j and D_j as in (ii) and (iii) above, we define

$$A_{j+1} = D_j A_j C_j, \quad B_{j+1} = D_j(B_j C_j + A_j \dot{C}_j), \tag{2.3}$$

so that $A_{j+1}, B_{j+1} \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_j}))$.

If (2.1) and (2.2) hold for every index $j \geq 0$, the procedure can be continued indefinitely. At the same time, since the sequence r_j is non-increasing, there is a smallest integer $\nu \geq 0$ such that $r_\nu = r_{\nu-1}$. By (2.1), we then have $A_\nu(t) \in GL(\mathbf{R}^{r_{\nu-1}})$, $\forall t \in \mathbf{R}$, and further reductions produce pairs (A_j, B_j), equivalent to the pair (A_ν, B_ν) in a sense defined in [RR93a]. The integer $\nu \geq 0$ is called the index of the pair (A, B), and it can be shown that ν is independent of the specific choices of P_j, C_j and D_j, $0 \leq j \leq \nu - 1$ made during the process.

Remark 2.1. For *constant* A and B it can be shown that (A, B) has index ν for some $\nu \geq 0$ if and only if the matrix pencil $\lambda A + B$ is regular, and that ν is exactly the index of the matrix pencil $\lambda A + B$. $\Box$

From now on, when referring to the pair (A, B) with index $\nu \geq 0$, it will always be implicitly assumed that the reduction was possible up to and including step ν (and hence beyond); that is, for the time being, that (2.1) and (2.2) hold for $0 \leq j \leq \nu$ (and hence for $j \geq \nu + 1$).

Suppose now that the pair (A, B) has index ν, and consider the DAE (1.1) with $b \in (\mathcal{D}'(\mathbf{R}))^n$. The condition (2.2) is equivalent to the invertibility of $[A_j(t) \oplus B_j(t)]^T$ and hence to $\ker A_j^T(t) \cap \ker B_j^T(t) = \{0\}$, or, equivalently, to the invertibility of $A_j(t)A_j(t)^T + B_j(t)B_j(t)^T$. We now define sequences $u_0, \ldots, u_{\nu-1}$ and $b_0, \ldots, b_\nu$ of distributions as follows: Set $b_0 = b$ and, generally, if b_j, $0 \leq j \leq \nu$ is known, construct u_j by multiplying the distribution b_j by the C^∞ operator $B_j^T(A_j A_j^T + B_j B_j^T)^{-1}$; that is,

$$u_j = B_j^T(A_j A_j^T + B_j B_j^T)^{-1} b_j, \tag{2.4}$$

Moreover, for $0 \leq j \leq \nu - 1$, define

$$b_{j+1} = D_j(b_j - Bu_j - A\dot{u}_j), \tag{2.5}$$

and

$$\Gamma_{\nu-1} = C_0 C_1 \cdots C_{\nu-1} \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^{r_{\nu-1}}, \mathbf{R}^n)) \tag{2.6}$$

$$v_{\nu-1} = u_0 + C_0 u_1 + C_0 C_1 u_2 + \cdots + C_0 \cdots C_{\nu-2} u_{\nu-1} \in (\mathcal{D}'(\mathbf{R}))^n. \tag{2.7}$$

In [RR93a] it is shown that a distribution $x \in (\mathcal{D}'(\mathbf{R}))^n$ solves (1.1) if and only if x has the form

$$x = \Gamma_{\nu-1} x_\nu + v_{\nu-1}, \tag{2.8}$$

with $\Gamma_{\nu-1}$ and $v_{\nu-1}$ given by (2.6) and (2.7), respectively, and $x_\nu \in (\mathcal{D}'(\mathbf{R}))^{r_{\nu-1}}$ is a solution of the ODE

$$\dot{x}_\nu + A_\nu^{-1} B_\nu x_\nu = A_\nu^{-1} b_\nu. \tag{2.9}$$

Naturally, the equivalence between (1.1) and (2.9) via (2.8) is true, in particular, for classical solutions; that is, when (say) $b \in C^\infty(\mathbf{R}; \mathbf{R}^n)$. Then also u_j and b_j, defined by (2.4) and (2.5), respectively, are of class C^∞, and so is $v_{\nu-1}$ in (2.7). In this case, (2.8) also transforms initial value problems for (1.1) into initial value problems for (2.9). In fact, x solves (1.1) under the initial condition $x(t_0) = x_0$ for fixed $t_0 \in \mathbf{R}$ if and only if x_0 verifies

$$x_0 = \Gamma_{\nu-1}(t_0) x_{\nu 0} + v_{\nu-1}(t_0), \tag{2.10}$$

for some $x_{\nu 0} \in \mathbf{R}^{r_{\nu-1}}$ which, of course, is necessarily unique by the injectivity of $\Gamma_{\nu-1}(t_0)$. Such values x_0 are called *consistent* with the DAE (1.1) at t_0. Evidently, if x solves (1.1) then the values $x(t) \in \mathbf{R}^n$ are consistent with (1.1) at t, $\forall t \in \mathbf{R}^n$. Moreover, initial value problems for (1.1) with consistent initial values at the given point t_0 have a unique (C^∞) solution while initial value problems with non-consistent initial values have no (classical) solution.

It is an interesting fact that the condition (2.1) is essentially superfluous if A and B are *analytic* (see [RR93a]), partly because in that case (2.1) automatically holds with $r_j = \max_{t \in \mathbf{R}} \operatorname{rank} A_j(t)$, except perhaps at points of a subset $\mathcal{S}_j$ consisting only of isolated points in $\mathbf{R}$. For $t \in \mathcal{S}_j$, we have dim rge $A_j(t) < r_j$, but it turns out that an "extended range" of $A_j(t)$, denoted by ext rge $A_j(t)$, can be defined with the properties that ext rge $A_j(t) \supset$ rge $A_j(t)$ and dim ext rge $A_j(t) = r_j$, $\forall t \in \mathbf{R}$ (and hence ext rge $A_j(t) =$ rge $A_j(t)$, $\forall t \in \mathbf{R} \setminus \mathcal{S}_j$). This allows for the construction of parametrized families P_j, C_j and D_j as before, except that "ext rge $A_j(t)$" now replaces "rge $A_j(t)$" everywhere. Thus, assuming only that (2.2) holds for all indices j, we can still construct a reduction (A_{j+1}, B_{j+1}) of (A_j, B_j), and the index ν of (A, B) is defined as before. But now, we have only $A_\nu(t) \in GL(\mathbf{R}^{r_{\nu-1}})$ for $t \in \mathbf{R} \setminus \mathcal{S}_\nu$, and (1.1) reduces to (2.9) via (2.8) only if $A_\nu(t) \in GL(\mathbf{R}^{r_{\nu-1}})$ for every $t \in \mathbf{R}$. Thus, in this case, the invertibility of $A_\nu(t)$ for all t is no longer guaranteed and must be assumed independently. As a result, future reference to pairs (A, B) of index ν "with invertible $A_\nu(t)$ for every $t \in \mathbf{R}$" should not be viewed as a redundancy but, rather, as a reminder that condition (2.1) can be dropped if A and B are analytic but that then invertibility of $A_\nu(t)$ is no longer guaranteed to hold for all t.

All indicated results extend verbatim to the case when $\mathbf{R}$ is replaced by an arbitrary interval $\mathcal{J}$ where distributions in $\mathcal{J}$ are now understood to be distributions in $\mathring{\mathcal{J}}$. If $\mathcal{J}$ contains one of its endpoints, initial value problems for (1.1) with a consistent initial value at that endpoint can be considered in the classical setting.

3. Initial value problems with discontinuous right sides.

In [HS83], Hautus and Silverman introduced the class $\mathcal{C}_{\text{imp}}$ of "impulsive-smooth" distributions in $[0, \infty)$. We first need a straightforward variant of this concept for distributions in $\mathbf{R}$. Throughout the remainder of this presentation, $\mathbf{R}^*$ denotes $\mathbf{R} \setminus \{0\}$.

Definition 3.1. The distribution $x \in \mathcal{D}'(\mathbf{R})$ is said to be impulsive-smooth, $x \in \mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$ *for short, if there are functions* $\varphi, \psi \in C^\infty(\mathbf{R})$ *such that* $x - \varphi H - \psi(1-H)$ *is a distribution with support* $\{0\}$ *where* H *denotes the Heaviside function.*

If $x \in \mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$ and $\varphi_1, \varphi_2, \psi_1, \psi_2 \in C^\infty(\mathbf{R})$ are such that $x - \varphi_i H - \psi_i(1-H)$ is a distribution with support $\{0\}$, $i = 1, 2$, then $(\varphi_1 - \varphi_2)H + (\psi_1 - \psi_2)(1-H)$ is a distribution with support $\{0\}$ and hence must be a linear combination of the Dirac δ and its derivatives. But, since it is also a function, it must be 0; that is, $\varphi_1 H + \psi_1(1-H) = \varphi_2 H + \psi_2(1-H)$. This shows that $x - \varphi_i H - \psi_i(1-H)$ is identical for $i = 1$ and $i = 2$, and hence can be called the impulsive part of x, denoted by x_{imp}.

Therefore, for given $x \in \mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$ the difference $x - x_{\mathrm{imp}}$ has the form $\varphi H + \psi(1-H)$ with $\varphi, \psi \in C^\infty(\mathbf{R})$. Of course, φ and ψ are not uniquely determined by this condition, but $\psi_{|(-\infty,0]}$ and $\varphi_{|[0,\infty)}$ are. Thus, there is no ambiguity in setting

$$x_- = \psi_{|(-\infty,0]}, \qquad x_+ = \varphi_{|[0,\infty)}.$$

With this definition, $x_- \in C^\infty((-\infty,0])$ and $x_+ \in C^\infty([0,\infty))$, and extending x_- by 0 for $t > 0$ and x_+ by 0 for $t < 0$, we may write

$$x = x_- + x_+ + x_{\mathrm{imp}}, \tag{3.1}$$

where each of the three terms on the right side is uniquely determined by x. Conversely, given $x_- \in C^\infty((-\infty,0])$ and $x_+ \in C^\infty([0,\infty))$ and a distribution x_{imp} with support $\{0\}$, equation (3.1) defines an element x of $\mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$.

Remark 3.1. Despite the terminology "impulsive-smooth", it should be kept in mind that for $x \in \mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$, $x - x_{\mathrm{imp}}$ is not a smooth function in $\mathbf{R}$ since it may have a jump at 0. But its restrictions x_- and x_+ to $(-\infty,0)$ and $(0,\infty)$, respectively, extend as smooth functions in $(-\infty,0]$ and $[0,\infty)$, respectively. □

Three trivial but essential properties of impulsive-smooth distributions are the following: (i) Every $x \in \mathcal{C}_{\mathrm{imp}}(\mathbf{R}^*)$ may be assigned a value at every point $t \neq 0$, namely $x(t) = x_-(t)$

if $t < 0$ and $x(t) = x_+(t)$ if $t > 0$. (ii) The derivative and the primitives (in the sense of distributions) of $x \in C_{imp}(\mathbf{R}^*)$ are themselves in $C_{imp}(\mathbf{R}^*)$. (iii) $C_{imp}(\mathbf{R}^*)$ is both a vector space over $\mathbf{R}$ and a $C^\infty(\mathbf{R})$-module. In fact, if $x \in C_{imp}(\mathbf{R}^*)$ and (3.1) is used, then we have

$$\text{(3.2)} \qquad \alpha x = \alpha x_- + \alpha x_+ + \sum_{i=0}^{k} \Big(\sum_{j=0}^{k-i} (-1)^j \binom{j+i}{j} \alpha^{(j)}(0)\lambda_{i+j} \Big) \delta^{(i)}, \quad \forall \alpha \in C^\infty(\mathbf{R}),$$

whenever $x_{imp} = \sum_{i=0}^{k} \lambda_i \delta^{(i)}, \quad \lambda_i \in \mathbf{R}, \quad 0 \le i \le k$. The above properties, including (3.1), have an immediate generalization to elements of $C^n_{imp}(\mathbf{R}^*) \equiv (C_{imp}(\mathbf{R}^*))^n$. In particular, for $x \in C^n_{imp}(\mathbf{R}^*)$, and $M \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m))$, we have $Mx \in C^m_{imp}(\mathbf{R}^*)$ and

$$\text{(3.3)} \qquad Mx = Mx_- + Mx_+ + \sum_{i=0}^{k} \Big(\sum_{j=0}^{k-i} (-1)^j \binom{j+i}{j} M^{(j)}(0)\lambda_{i+j} \Big) \delta^{(i)},$$

whenever

$$\text{(3.4)} \qquad x_{imp} = \sum_{i=0}^{k} \lambda_i \delta^{(i)}, \quad \lambda_i \in \mathbf{R}^n, \quad 0 \le i \le k.$$

Definition 3.2. *Let $x \in C^n_{imp}(\mathbf{R}^*)$, so that there exists a unique decomposition (3.4). The impulse order of x, denoted by iord(x), is defined as follows:*

(i) If $\lambda_i = 0, \ 0 \le i \le k$ and $x_+(0) = x_-(0)$, and hence $x \in C^0(\mathbf{R}; \mathbf{R}^n)$), then set

$$\text{iord}(x) = -m - 2,$$

where $0 \le m \le \infty$ is the largest integer such that $x \in C^m(\mathbf{R}; \mathbf{R}^n)$.

(ii) If $\lambda_i = 0, 0 \le i \le k$ and $x_+(0) \ne x_-(0)$ (and hence x has a discontinuity at the origin), then set

$$\text{iord}(x) = -1.$$

(iii) If $\lambda_i \ne 0$ for some $0 \le i \le k$ then set

$$\text{iord}(x) = \max\{i : 0 \le i \le k, \ \lambda_i \ne 0\}.$$

Remark 3.2: Let $M \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m))$. By (3.3) we have $\mathrm{iord}(Mx) \leq \mathrm{iord}(x)$ and equality holds if $n = m$ and $M(0)$ is invertible. □

The following lemma provides some more precise preliminary results about the primitives in the sense of distributions of the elements of $\mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$.

Lemma 3.1. *(i) Let $f \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ have impulse order $k \in \mathbf{Z} \cup \{-\infty\}$ and let $y \in (\mathcal{D}'(\mathbf{R}))^n$ be such that $\dot{y} = f$. Then, $y \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ and y has impulse order $k - 1$.*

(ii) Let $f \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ and let $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}^\bullet$ be given. Then, there is a unique $y \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ such that $\dot{y} = f$ and exactly one of the following conditions holds

$$(a)\ \ y(t_0) = x_0, \quad (b_+)\ \ y_+(0) = x_0, \quad (b_-)\ \ y_-(0) = x_0. \tag{3.5}$$

(iii) Let the sequence $f^\ell \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$, $\ell \geq 1$, and $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}^\bullet$ be given. Suppose that there are an open interval I_{t_0} about t_0 and some $f \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ such that

$$f^\ell{}_{|I_{t_0}} = f_{|I_{t_0}}, \quad \forall \ell \geq 1, \tag{3.6}$$

as distributions in I_{t_0} (i.e. as functions if $0 \notin I_{t_0}$) and that

$$\lim_{\ell \to \infty} f^\ell = f \quad in \ \ (\mathcal{D}'(\mathbf{R}))^n. \tag{3.7}$$

Let $y^\ell \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ and $y \in \mathcal{C}^n_{\mathrm{imp}}(\mathbf{R}^\bullet)$ be such that $\dot{y}^\ell = f^\ell$, $y^\ell(t_0) = x_0$ and $\dot{y} = f$, $y(t_0) = x_0$ (see (ii) above). Then, we have

$$\lim_{\ell \to \infty} y^\ell = y \quad in \ \ (\mathcal{D}'(\mathbf{R}))^n. \tag{3.8}$$

(A similar result holds if $t_0 = 0$ and y^ℓ, y are characterized by $\dot{y}^\ell = f^\ell$, $y^\ell_\pm(0) = x_0$ and $\dot{y} = f$, $y_\pm(0) = x_0$.)

Proof: (i) Any two primitives (in the sense of distributions) of an element of $(\mathcal{D}'(\mathbf{R}))^n$ differ from a constant vector of $\mathbf{R}^n$, and addition of a constant vector does not affect membership to $C^n_{\mathrm{imp}}(\mathbf{R}^*)$ nor the impulse order. Thus, it suffices to show that f has one primitive in $C^n_{\mathrm{imp}}(\mathbf{R}^*)$ with impulse order $k-1$.

Write $f = f_+ + f_- + f_{\mathrm{imp}}$ with $f_{\mathrm{imp}} = \sum_{i=0}^k \mu_i \delta^{(i)}$, $\mu_i \in \mathbf{R}^n$, $\mu_k \neq 0$ and choose $t_0 \in \mathbf{R}$. Since the function $f_+ + f_-$ is locally integrable, set $\tilde{y} = \int_{t_0}^t (f_+ + f_-)(s)ds$, so that $\tilde{y} \in C^\infty((-\infty,0];\mathbf{R}^n) \cap C^\infty([0,\infty);\mathbf{R}^n) \cap C^0(\mathbf{R};\mathbf{R}^n) \cap C^n_{\mathrm{imp}}(\mathbf{R}^*)$ and $\tilde{y}$ is a primitive of $f_+ + f_-$ in the sense of distributions. Furthermore, it is obvious that $\tilde{y} \in C^{m+1}(\mathbf{R};\mathbf{R}^n)$ whenever $f_+ + f_- \in C^m(\mathbf{R};\mathbf{R}^n)$, $0 \leq m \leq \infty$. Thus, $\mathrm{iord}(\tilde{y}) = \mathrm{iord}(f_+ + f_-) - 1 \leq -2$. In particular, $\tilde{y}$ is a primitive of f with impulse order $k-1$ when $k < 0$ since $f = f_+ + f_-$ in this case.

Suppose now that $k \geq 0$ and set $y_{\mathrm{imp}} = \sum_{i=1}^k \mu_i \delta^{i-1}$, so that $\mu_0 H + y_{\mathrm{imp}}$ is a primitive of f_{imp}. Evidently, $\mu_0 H + y_{\mathrm{imp}} \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ and $\mathrm{iord}(\mu_0 H + y_{\mathrm{imp}}) = k - 1 \geq -1$ since $k \geq 0$ and $\mu_k \neq 0$. Thus, $y = \tilde{y} + \mu_0 H + y_{\mathrm{imp}} \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ is a primitive of f and $\mathrm{iord}(y) = k-1$ since $\mathrm{iord}(\tilde{y}) \leq -2 < -1 \leq \mathrm{iord}(\mu_0 H + y_{\mathrm{imp}})$.

(ii) The primitive of f obtained in (i) verifies $y_- = \tilde{y}_-$ and $y_+ = \tilde{y}_+ + \mu_0$. Since $\tilde{y}$ is continuous and $\tilde{y}(t_0) = 0$, this yields $y_-(t_0) = 0$ and $y_+(t_0) = \mu_0$. Every other primitive of f (still denoted by y for simplicity of notation) is uniquely characterized by a vector $\lambda_0 \in \mathbf{R}^n$ and verifies $y_-(t_0) = \lambda_0, y_+(t_0) = \mu_0 + \lambda_0$. As a result, $\lambda_0 = x_0$ (resp. $\lambda_0 = x_0 - \mu_0$) is the only possible choice yielding $y_-(t_0) = x_0$ (resp. $y_+(t_0) = x_0$). Letting $t_0 = 0$, we obtain existence and uniqueness of $y \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ such that $\dot{y} = f$ and either (3.5)(b$_+$) or (3.5)(b$_-$) holds. Next, letting $t_0 \neq 0$ and observing that $y(t_0) = y_-(t_0)$ if $t_0 < 0$ and $y(t_0) = y_+(t_0)$ if $t_0 > 0$, we obtain existence and uniqueness of $y \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ such that $\dot{y} = f$ and $y(t_0) = x_0$.

(iii) To begin with, let us briefly recall how primitives of distributions are defined: Let $\theta \in \mathcal{D}(\mathbf{R})$ be such that $\int_{\mathbf{R}} \theta = 1$. For $\varphi \in (\mathcal{D}(\mathbf{R}))^n$, there is a unique $\psi \in (\mathcal{D}(\mathbf{R}))^n$ such that $\dot{\psi} = \varphi - \theta \int_{\mathbf{R}} \varphi$ and the correspondence $\varphi \mapsto \psi$ is continuous for the usual topology of $(\mathcal{D}(\mathbf{R}))^n$. Note also that $\mathrm{supp}\, \psi \subset \mathrm{supp}\, \varphi \cup \mathrm{supp}\, \theta$. Given $T = (T_1, \cdots, T_n) \in (\mathcal{D}'(\mathbf{R}))^n$, the formula

$$\langle S, \varphi \rangle = -\langle T, \psi \rangle + c \cdot \int_{\mathbf{R}} \varphi, \tag{3.9}$$

with $c \in \mathbf{R}^n$ and the dot denoting the usual inner product of $\mathbf{R}^n$, defines S as a distribution with values in $\mathbf{R}^n$ and shows that $\dot{S} = T$, and all the primitives of T are of the form (3.9) for some $c \in \mathbf{R}^n$.

In general, the formula (3.9) does not permit us to assign a value $S(t)$ to S for any $t \in \mathbf{R}$. But suppose that there are $t_0 \in \mathbf{R}$ and an open interval I_{t_0} about t_0 such that $T_{|I_{t_0}}$ is (say) a C^∞ function, whence $\langle T, \varphi\rangle = \int_{I_{t_0}} T \cdot \varphi = \int_{\mathbf{R}} T \cdot \varphi$, for all $\varphi \in (\mathcal{D}(I_{t_0}))^n$. We may choose θ such that supp $\theta \subset I_{t_0}$ and then, for $t \in I_{t_0}$, we may define $S_0(t) = \int_{t_0}^t T(s)ds$; that is, $S = (S_{01}, \ldots, S_{0n})$ with $S_{0i}(t) = \int_{t_0}^t T_i(s)ds$. In (3.9) let $c = (c_1, \ldots, c_n)$ with

$$c_i = \int_{\mathbf{R}} S_{0i}(t)\theta(t)dt, \quad 1 \le i \le n. \tag{3.10}$$

This makes sense since supp $\theta \subset I_{t_0}$ and $S_{0i}(t)$ is defined for $t \in I_{t_0}$. Let $\varphi \in (\mathcal{D}(I_{t_0}))^n$, whence $\psi \in (\mathcal{D}(I_{t_0}))^n$ since supp θ, supp $\varphi \subset I_{t_0}$. As $T = \dot{S}_0$ in $(\mathcal{D}'(I_{t_0}))^n$, relation (3.9) reads

$$\begin{aligned}
\langle S, \varphi\rangle &= -\langle \dot{S}_0, \psi\rangle + c \cdot \int_{\mathbf{R}} \varphi \\
&= \langle S_0, \dot{\psi}\rangle + c \cdot \int_{\mathbf{R}} \varphi = \langle S_0, \varphi - \theta \int_{\mathbf{R}} \varphi\rangle + c \cdot \int_{\mathbf{R}} \varphi \\
&= \langle S_0, \varphi\rangle - \langle S_0, \theta \int_{\mathbf{R}} \varphi\rangle + c \cdot \int_{\mathbf{R}} \varphi.
\end{aligned}$$

But

$$\begin{aligned}
\langle S_0, \theta \int_{\mathbf{R}} \varphi\rangle &= \int_{I_{t_0}} S_0 \cdot \theta \int_{\mathbf{R}} \varphi \\
&= \int_{\mathbf{R}} S_0 \cdot \theta \int_{\mathbf{R}} \varphi = \sum_{i=1}^n \left(\int_{\mathbf{R}} S_{0i}\theta\right) \int_{\mathbf{R}} \varphi_i = c \cdot \int_{\mathbf{R}} \varphi,
\end{aligned}$$

by definition of c in (3.10). Thus, $\langle S, \varphi\rangle = \langle S_0, \varphi\rangle$, for all $\varphi \in (\mathcal{D}(I_{t_0}))^n$, i.e., $S_{|I_{t_0}} = S_0$. Because S_0 is a function and vanishes at t_0, it follows that S in (3.9) may be referred to as the primitive of T vanishing at t_0 when c is chosen as in (3.10) (and θ verifies supp $\theta \subset I_{t_0}$). The independence of this definition from the choice of θ is easily seen: If $S, \hat{S}$ corresponds to two such choices, we have $\tilde{S} = S + c, c \in \mathbf{R}^n$ since both S and $\tilde{S}$ are primitives of T, and $c = 0$ from $\tilde{S}_{|I_{t_0}} = S_0 = S_{|I_{t_0}}$. As a result, given $x_0 \in \mathbf{R}^n$, $S + x_0$ may be referred to as the primitive of T verifying $S(t_0) = x_0$.

Now, with T as above, let $T^\ell \in (\mathcal{D}'(\mathbf{R}))^n$, $\ell \geq 1$, be a sequence such that $T^\ell_{|I_{t_0}} = T_{|I_{t_0}}$. This assumption ensures that T^ℓ, $\ell \geq 1$, as well as T define the same vector c in (3.10). With this choice of c, the distribution $S^\ell \in (\mathcal{D}'(\mathbf{R}))^n$ obtained by replacing T by T^ℓ in (3.9) is the primitive of T^ℓ vanishing at t_0, and, under the assumption $\lim_{\ell\to\infty} T^\ell = T$ in $(\mathcal{D}'(\mathbf{R}))^n$, it is then obvious that for $\varphi \in (\mathcal{D}(\mathbf{R}))^n$ we have $\lim_{\ell\to\infty} \langle S^\ell, \varphi\rangle = \langle S, \varphi\rangle$, i.e. $\lim_{\ell\to\infty} S^\ell = S$ in $(\mathcal{D}'(\mathbf{R}))^n$. In turn, this implies that $\lim_{\ell\to\infty} S^\ell + x_0 = S + x_0$ for $x_0 \in \mathbf{R}^n$.

It should be clear that part (iii) of the lemma follows from the above considerations with $T = f$ and $T^\ell = f^\ell$, so that $S + x_0 = y$ and $S^\ell + x_0 = y^\ell$. The proof that a result similar to (3.8) holds when $t_0 = 0$ and y^ℓ, y are characterized by $\dot{y}^\ell = f^\ell$. $y^\ell_\pm(0) = x_0$ and $\dot{y} = f, y_\pm(0) = x_0$, easily follows from the above considerations with $T = f_+ + f_-$, $T^\ell = f^\ell_+ + f^\ell_-$, and the remark that $f^\ell_{\text{imp}} = f_{\text{imp}}$, for all $\ell \geq 1$, since f^ℓ and f coincide as distributions in some open interval about the origin by hypothesis. Details are left to the reader. □

Remark 3.3: Because of condition (3.6), Lemma 3.1 (iii) gives an unusual result about continuous dependence for initial value problems. The incorrectness of this result under condition (3.7) alone can be seen even in the case when $n = 1$ and $f, f^\ell \in C^\infty(\mathbf{R})$. In fact, by the theory of Fourier series, the sequences $\ell^\alpha \cos \ell t$ and $\ell^\alpha \sin \ell t$ tend to 0 in $\mathcal{D}'(\mathbf{R})$ for every $\alpha \in \mathbf{R}$. In particular, if $f = 0$, $f^\ell(t) = \ell \sin \ell t$, we have $\lim_{\ell\to\infty} f^\ell = f$ in $\mathcal{D}'(\mathbf{R})$ as in (3.7). Choosing $t_0 = 0$, we find $y = 0$, $y^\ell(t) = 1 - \cos \ell t$ in Lemma 3.1 (iii). But then, $\lim_{\ell\to\infty} y^\ell = 1 \neq y$ in $\mathcal{D}'(\mathbf{R})$ and (3.8) fails to hold. □

Lemma 3.1 has a direct application to initial value problems for the ODE

$$\dot{x} + Mx = f, \tag{3.11}$$

considered in the following theorem:

Theorem 3.1. *Let $M \in C^\infty(\mathbf{R};\mathcal{L}(\mathbf{R}^n))$ and let $f \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ have impulse order $k \in \mathbf{Z} \cup \{-\infty\}$. Then*

(i) The solutions $x \in (\mathcal{D}'(\mathbf{R}))^n$ of the ODE (3.11) belong to $C^n_{\mathrm{imp}}(\mathbf{R}^*)$ and have impulse order $k-1$.

(ii) For given $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}^*$, the ODE (3.11) together with one of the initial conditions

$$(a)\ x(t_0) = x_0, \quad (b_+)\ x_+(0) = x_0, \quad (b_-)\ x_-(0) = x_0, \tag{3.12}$$

has a unique solution $x \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$, but in the cases (3.12)(b_+) and (3.12)(b_-) the solutions corresponding to $x_+(0) = x_0$ and $x_-(0) = x_0$ need not be the same.

(iii) Let the sequence $f^\ell \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$, $\ell \geq 1$, and $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}^*$ be given. Suppose that there is an open interval I_{t_0} about t_0 such that

$$f^\ell{}_{|I_{t_0}} = f_{|I_{t_0}}, \quad \forall\, \ell \geq 1, \tag{3.13}$$

as distributions in I_{t_0}, (i.e., as functions if $0 \notin I_{t_0}$) and

$$\lim_{\ell\to\infty} f^\ell = f \quad in \quad (\mathcal{D}'(\mathbf{R}))^n. \tag{3.14}$$

Let x (resp. x^ℓ) $\in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ denote the unique solution of the ODE (3.11) (resp. (3.11) with f replaced by f^ℓ) verifying $x(t_0) = x_0$ (resp. $x^\ell(t_0) = x_0$) whose existence is ensured by part (ii) of the theorem. Then

$$\lim_{\ell\to\infty} x^\ell = x \quad in \quad (\mathcal{D}'(\mathbf{R}))^n. \tag{3.15}$$

(A similar result holds if $t_0 = 0$ and the initial condition for x^ℓ, x is chosen as $x^\ell_\pm(0) = x_0$ and $x_\pm(0) = x_0$, respectively.)

Proof: (i) Fix $t_0 \in \mathbf{R}$ and denote by $U \in C^\infty(\mathbf{R};\mathcal{L}(\mathbf{R}^n))$ the solution of the initial value problem $\dot{U} + MU = 0$, $U(t_0) = I$. It is well known that $U(t) \in GL(\mathbf{R}^n)$ for all $t \in \mathbf{R}^n$.

Then, $x \in (\mathcal{D}'(\mathbf{R}))^n$ solves (3.11) if and only if $y = U^{-1}x \in (\mathcal{D}'(\mathbf{R}))^n$ solves the equation $\dot{y} = U^{-1}f$. Since $f \in C^n_{\text{imp}}(\mathbf{R}^*)$, we have $U^{-1}f \in C^n_{\text{imp}}(\mathbf{R}^*)$ and $\text{iord}(U^{-1}f) = \text{iord}(f) = k$ from Remark 3.2. Next, by Lemma 3.1 (i) the solutions of $\dot{y} = U^{-1}f$ are in $C^n_{\text{imp}}(\mathbf{R}^*)$ and $\text{iord}(x) = \text{iord}(y) = k - 1$ by another application of Remark 3.2.

(ii) Since $U(t_0) = I$, we have $x(t_0) = x_0$ (resp. $t_0 = 0$ and $x_\pm(0) = x_0$) if and only if $y(t_0) = x_0$ (resp. $t_0 = 0$ and $y_\pm(0) = x_0$). Existence and uniqueness of x thus follows from existence and uniqueness of y ensured by Lemma 3.1 (ii).

(iii) From conditions (3.13) and (3.14) and using the continuity of multiplication of distributions by C^∞ matrix-valued functions, we infer that $U^{-1}f^\ell_{|_{I_{t_0}}} = U^{-1}f_{|_{I_{t_0}}}$ and $\lim_{\ell\to\infty} U^{-1}f^\ell = U^{-1}f$ in $(\mathcal{D}'(\mathbf{R}))^n$. Denoting by y^ℓ the solution of $\dot{y}^\ell = U^{-1}f^\ell$, $y^\ell(0) = x_0$, we find that $\lim_{\ell\to\infty} y^\ell = y$ in $(\mathcal{D}'(\mathbf{R}))^n$ by Lemma 3.1 (iii). Thus, $\lim_{\ell\to\infty} x^\ell = x$ in $(\mathcal{D}'(\mathbf{R}))^n$ since $x^\ell = Uy^\ell$, $x = Uy$ and multiplication by U is continuous. □

Remark 3.4: Let $f_{\text{imp}} = \sum_{i=0}^k \mu_i\delta^{(i)}$ with $\mu_k \neq 0$. From Theorem 3.1, we have $x_{\text{imp}} = \sum_{i=0}^{k-1} \lambda_i\delta^{(i)}$ for every solution $x \in C^n_{\text{imp}}(\mathbf{R}^*)$ of $\dot{x} + Mx = f$. Comparing impulsive parts and using (3.3) - (3.4) yields

$$\begin{cases} \lambda_{k-1} = \mu_k, \\ \lambda_{i-1} + \displaystyle\sum_{j=0}^{k-i-1} (-1)^j \binom{j+i}{j} M^{(j)}(0)\lambda_{i+j} = \mu_i, \ 1 \leq i \leq k-1, \\ x_+(0) - x_-(0) + \displaystyle\sum_{i=0}^{k-1} (-1)^j M^{(j)}(0)\lambda_j = \mu_0. \end{cases}$$

By inverting these formulas, we find $\lambda_0, \cdots, \lambda_{k-1}$ (depending only upon $\mu_1, \cdots, \mu_k$) as well as $x_+(0) - x_-(0)$. Thus, both x_{imp} and $x_+(0) - x_-(0)$ are calculable and depend solely upon f_{imp}. □

We now focus on initial value problems for the DAE

$$A\dot{x} + Bx = b, \tag{3.16}$$

where $A, B \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$ and $b \in C^n_{\text{imp}}(\mathbf{R}^*)$. Under the assumption that the pair (A, B)

has index $\nu \geq 0$ in $\mathbf{R}$ and that $A_\nu(t)$ is invertible for every $t \in \mathbf{R}$ (see Section 2), the DAE's

$$(3.17_-) \qquad A(t)\dot{x}_- + B(t)x_- = b_-(t) \qquad \text{in } (-\infty, 0),$$

and

$$(3.17_+) \qquad A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \qquad \text{in } (0, \infty),$$

have coefficients and right-hand sides of class C^∞ in $(-\infty, 0]$ and $[0, \infty)$, respectively. As a consequence, it makes sense to speak of values $x_0 \in \mathbf{R}^n$ which are consistent with (3.17_-) (resp. (3.17_+)) at a point $t_0 \leq 0$ (resp. $t_0 \geq 0$) in the sense of Section 2.

Theorem 3.2. *Let the pair (A, B), $A, B \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$ have index $\nu \geq 0$, and with the notation of Section 2, suppose that $A_\nu(t)$ is invertible for every $t \in \mathbf{R}$. If $b \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ has impulse order $k \in \mathbf{Z} \cup \{-\infty\}$, then, the solutions $x \in (\mathcal{D}'(\mathbf{R}))^n$ of the DAE (3.16) are in $C^n_{\mathrm{imp}}(\mathbf{R}^*)$ and have impulse order at most $k + \nu - 1$. Moreover,*

(i) if $t_0 < 0$ (resp. $t_0 > 0$) and $x_0 \in \mathbf{R}^n$ is consistent with the DAE (3.17_-) (resp. (3.17_+)) at t_0, the initial value problem

$$(3.18) \qquad A\dot{x} + Bx = b, \qquad x(t_0) = x_0,$$

has a unique solution $x \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$. Furthermore, if I_{t_0} is an open interval about t_0 and $b^\ell \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ is a sequence such that $b^\ell_{|I_{t_0}} = b_{|I_{t_0}}$, for all $\ell \geq 1$, as distributions in I_{t_0} and $\lim_{\ell \to \infty} b^\ell = b$ in $(\mathcal{D}'(\mathbf{R}))^n$, then x_0 is consistent with all the DAE's obtained by replacing b by b^ℓ in (3.17_-) (resp. (3.17_+)), and denoting by $x^\ell \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ the unique solution of the initial value problem

$$(3.19) \qquad A\dot{x}^\ell + Bx^\ell = b^\ell, \quad x^\ell(t_0) = x_0,$$

we have

$$(3.20) \qquad \lim_{\ell \to \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

(ii) *If* $t_0 = 0$ *and* $x_0 \in \mathbf{R}^n$ *is consistent with the DAE* (3.17$_-$) *(resp.* (3.17$_+$)*) at* $t_0 = 0$, *the initial value problem*

$$(3.21_\pm) \qquad A\dot{x} + Bx = b, \qquad x_-(0) = x_0 \text{ (resp. } x_+(0) = x_0\text{)},$$

has a unique solution $x \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$. *Furthermore, if* I_0 *is an open interval about* 0 *and* $b^\ell \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ *is a sequence such that* $b^\ell_{|I_0} = b_{|I_0}$, *for all* $\ell \geq 1$, *as distributions in* I_0 *and* $\lim_{\ell \to \infty} b^\ell = b$ *in* $(\mathcal{D}'(\mathbf{R}))^n$, *then* x_0 *is consistent with all the DAE's obtained by replacing* b *by* b^ℓ *in* (3.17$_-$) *(resp.* (3.17$_+$)*) and denoting by* $x^\ell \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ *the unique solution of the initial value problem*

$$(3.22) \qquad A\dot{x}^\ell + Bx^\ell = b^\ell, x^\ell_-(0) = x_0 \quad \text{(resp. } x^\ell_+(0) = x_0\text{)},$$

we have

$$(3.23) \qquad \lim_{\ell \to \infty} x^\ell = x \quad \text{in } (\mathcal{D}'(\mathbf{R}))^n.$$

Proof: In this proof, we use the notation of Section 2 without further mention. From the reduction procedure we know that every solution $x \in (\mathcal{D}'(\mathbf{R}))^n$ of the DAE (3.16) has the form $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$ where $\Gamma_{\nu-1}$ and $v_{\nu-1}$ are given by (2.6) and (2.7), respectively, and x_ν solves the ODE

$$(3.24) \qquad \dot{x}_\nu + A_\nu^{-1}B_\nu x_\nu = A_\nu^{-1}b_\nu.$$

The key point here is the simple fact that the distributions u_j and b_j of Section 2 belong to $C^{r_{j-1}}_{\mathrm{imp}}(\mathbf{R}^*)$ and have impulse order at most $k+j$. Indeed, recall that $b_0 = b$ and $r_{-1} = n$, whence, because of $u_0 = B^T(AA^T + BB^T)^{-1}b$, we have $u_0 \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ and $\mathrm{iord}(u_0) \leq \mathrm{iord}(b) = k$ by Remark 3.2. Therefore, $\dot{u}_0 \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ has impulse order at most $k+1$ which, in turn, implies that $b_1 = D(b - Bu_0 - A\dot{u}_0) \in C^{r_0}_{\mathrm{imp}}(\mathbf{R}^*)$ has impulse order at most $k+1$. Obviously, the statement about the sequences $u_0, \ldots, u_{\nu-1}, b_0, \ldots, b_\nu$ now follows inductively by the same argument.

Since $b_\nu \in C^{r_\nu - 1}_{\text{imp}}(\mathbf{R}^*)$ has impulse order at most $k+\nu-1$, the same is true of $A_\nu^{-1}b_\nu$. Therefore, by Lemma 3.1, the solutions of (3.24) are in $C^{r_\nu - 1}_{\text{imp}}(\mathbf{R}^*)$ and have impulse order at most $k+\nu-1$. This implies that the solutions $x = \Gamma_{\nu-1}x_\nu + v_\nu$ of (3.16) are in $C^n_{\text{imp}}(\mathbf{R}^*)$. Morover, iord $(\Gamma_{\nu-1}x_\nu) \le k+\nu-1$ since $\Gamma_{\nu-1}$ is C^∞, and iord $(v_\nu) \le k+\nu-1$, because the C_j's are C^∞ and iord $(u_j) \le k+j$ for $0 \le j \le \nu-1$. Thus x has impulse order at most $k+\nu-1$.

If now $t_0 < 0$ (resp. $t_0 > 0$) and $x_0 \in \mathbf{R}^n$ is consistent with the DAE (3.17_-) (resp. (3.17_+)), there is a unique $x_{0\nu} \in \mathbf{R}^{r_\nu - 1}$ such that $x_0 = \Gamma_{\nu-1}(t_0)x_{\nu 0} + v_{\nu-1}(t_0)$, (note that $v_{\nu-1}(t_0)$ makes sense since $t_0 \neq 0$). Hence, the solution x of (3.18) is obtained as $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$ where, in line with Theorem 3.1, $x_\nu \in C^{r_\nu - 1}_{\text{imp}}(\mathbf{R}^*)$ is the unique solution of

$$\dot{x}_\nu + A_\nu^{-1}B_\nu x_\nu = b_\nu, \qquad x_\nu(t_0) = x_{\nu 0}, \tag{3.25}$$

and no other initial values can be substituted for $x_{\nu 0}$ because $\Gamma_{\nu-1}(t_0)$ is one-to-one.

For the "furthermore" part in (i) of the theorem, observe first that consistency of a value $x_0 \in \mathbf{R}^n$ with a (linear) DAE at a point t_0 depends only upon the coefficients and the right-hand side of the DAE in an arbitrarily small neighborhood of t_0. As a result, the hypothesis $b^\ell{}_{|I_{t_0}} = b_{|I_{t_0}}$ ensures that x_0 remains consistent with the DAE obtained by replacing b by b^ℓ in (3.17_-) (resp. (3.17_+).

For fixed $\ell \ge 1$, denote by $u_j^\ell, 0 \le j \le \nu-1$ and b_j^ℓ, $0 \le j \le \nu$, the sequences corresponding to u_j, b_j in the procedure of Section 2 after replacing b by b^ℓ, and let $v_{\nu-1}^\ell$ be defined by (2.7) with $u_0, \ldots, u_{\nu-1}$ replaced by $u_0^\ell, \ldots, u_{\nu-1}^\ell$, respectively. With $\Gamma_{\nu-1}$ as in (2.6), we find that the solution x^ℓ of (3.19) has the form

$$x^\ell = \Gamma_{\nu-1}x_\nu^\ell + v_{\nu-1}^\ell, \tag{3.26}$$

where $x_\nu^\ell \in C^{r_\nu - 1}_{\text{imp}}(\mathbf{R}^*)$ solves the initial value problem

$$\dot{x}_\nu^\ell + A_\nu^{-1}B_\nu x_\nu = A_\nu^{-1}b_\nu^\ell, \quad x_\nu^\ell(t_0) = x_0. \tag{3.27}$$

From the hypothesis $b^\ell_{|_{I_{t_0}}} = b_{|_{I_{t_0}}}$, it follows at once that $u^\ell_{j|_{I_{t_0}}} = u_{j|_{I_{t_0}}}$ and $b^\ell_{j|_{I_{t_0}}} = b_{j|_{I_{t_0}}}$ for all the indices ℓ, j of interest.In particular, $b^\ell_{\nu|_{I_{t_0}}} = b_{\nu|_{I_{t_0}}}$ and hence

$$(3.28) \qquad A_\nu^{-1} b^\ell_{\nu|_{I_{t_0}}} = A_\nu^{-1} b_{\nu|_{I_{t_0}}}, \quad \forall \ell \geq 1.$$

Next, the hypothesis $\lim_{\ell\to\infty} b^\ell = b$ in $(\mathcal{D}'(\mathbf{R}))^n$ and continuity of the multiplication of distributions by C^∞ matrix-valued functions yield $\lim_{\ell\to\infty} u^\ell_j = u_j$ in $(\mathcal{D}'(\mathbf{R}))r_{j-1}$, $0 \leq j \leq \nu - 1$ and $\lim_{\ell\to\infty} b^\ell_j = b_j$ in $(\mathcal{D}'(\mathbf{R}))^{r_{j-1}}$, $0 \leq j \leq \nu$. In particular,

$$(3.29) \qquad \lim_{\ell\to\infty} A_\nu^{-1} b^\ell_\nu = A_\nu^{-1} b_\nu \quad \text{in} \quad (\mathcal{D}'(\mathbf{R}))^{r_{\nu-1}}$$

and

$$(3.30) \qquad \lim_{\ell\to\infty} v^\ell_{\nu-1} = v_{\nu-1} \quad \text{in} \quad (\mathcal{D}'(\mathbf{R}))^n.$$

Since x_ν and x^ℓ_ν solve the initial value problems (3.25) and (3.27), respectively, it follows from (3.28) and (3.29) and Theorem 3.1 (iii) that $\lim_{\ell\to\infty} x^\ell_\nu = x_\nu$ in $(\mathcal{D}'(\mathbf{R}))^{r_{\nu-1}}$. Together with (3.26) and (3.30), this implies that $\lim_{\ell\to\infty} x^\ell = \Gamma_{\nu-1}x_\nu + v_{\nu-1} = x$ in $(\mathcal{D}'(\mathbf{R}))^n$. This completes the proof of part (i) of the theorem.

Finally, for the proof of (ii), if $t_0 = 0$ and $x_0 \in \mathbf{R}^n$ is consistent with the DAE (3.17_-) (resp. (3.17_+) at $t_0 = 0$, then the solution x of $(3.21_\pm)$ is obtained in the form $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$ where, by Theorem 3.1, x_ν is the unique solution of

$$\dot{x}_\nu + A_\nu^{-1} B_\nu x_\nu = b_\nu, \qquad x_{\nu-}(0) = x_{\nu 0} \text{ (resp. } x_{\nu+}(0) = x_{\nu 0}),$$

and $x_{\nu 0} \in \mathbf{R}^{r_{\nu-1}}$ is (by injectivity of $\Gamma_{\nu-1}(0)$) the unique solution of the equation $x_0 = \Gamma_{\nu-1}(0)x_{\nu 0} + v_-(0)$ (resp. $x_0 = \Gamma_{\nu-1}(0)x_{\nu 0} + v_+(0)$). The proof of the remaining statement is identical to the proof of the "furthermore" part in (i) of the theorem. □

Remark 3.5: Let "[ ]" stand for "jump at 0". Then, since every solution x of the DAE (3.16) has the form $x = \Gamma_{\nu-1}x_\nu + v_{\nu-1}$ with x_ν solving the ODE (3.24), we have

$[x] = \Gamma_{\nu-1}(0)[x_\nu] + [v_{\nu-1}]$ and $x_{\text{imp}} = (\Gamma_{\nu-1}x_\nu)_{\text{imp}} + v_{\nu\ \text{imp}}$. On the other hand, $\Gamma_{\nu-1}$ and $v_{\nu-1}$ are obtained through an explicit procedure. As a result, $[x]$ and x_{imp} can be calculated if $[x_\nu]$ and $x_{\nu\ \text{imp}}$ are known (using (3.3) - (3.4) for the term $(\Gamma_{\nu-1}x_\nu)_{\text{imp}}$). But from Remark 3.4, $[x_\nu]$ and $x_{\nu\ \text{imp}}$ can be evaluated from $b_{\nu\ \text{imp}}$, and $b_{\nu\ \text{imp}}$ is calculable since b_ν is known explicitly. Thus, both $[x]$ and x_{imp} are calculable, at least in principle. □

4. Inconsistent initial values.

Let $A, B \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$ and $b_+ \in C^\infty([0,\infty); \mathbf{R}^n)$ be given. As noted in the Introduction, the problem of solving the initial value problem

$$A(t)\dot{x} + B(t)x = b_+(t), \quad \text{in } (0,\infty), \tag{4.1}$$

$$x(0) = x_0, \tag{4.2}$$

for arbitrary $x_0 \in \mathbf{R}^n$ that is not necessarily consistent with the DAE (4.1) at $t_0 = 0$, often arises when a known function x_- on $(-\infty, 0]$ verifies $x_-(0) = x_0$ and is to be extended into a solution of the DAE in (4.1). A general approach is suggested by the following observation:

Lemma 4.1. *Let $x_- \in C^\infty((-\infty,0]; \mathbf{R}^n)$ be given, and suppose that $x_0 = x_-(0)$ is consistent with the DAE (4.1) at $t_0 = 0$. Assume further that the pair (A,B) has index $\nu \geq 0$ in $\mathbf{R}$ and that, in the notation of Section 2, $A_\nu(t)$ is invertible for every $t \in \mathbf{R}$. Let $x_+ \in C^\infty([0,\infty); \mathbf{R}^n)$ be the unique solution of (4.1) and (4.2). Set*

$$b(t) = \begin{cases} A(t)\dot{x}_-(t) + B(t)x_-(t) & \text{if } t < 0, \\ b_+(t) & \text{if } t > 0, \end{cases} \tag{4.3}$$

so that $b \in C^n_{\text{imp}}(\mathbf{R}^*)$ (and $b_{\text{imp}} = 0$). Then, the function

$$x(t) = \begin{cases} x_-(t) & \text{if } t < 0, \\ x_+(t) & \text{if } t > 0, \end{cases} \tag{4.4}$$

verifies $x \in C^n_{\rm imp}(\mathbf{R}^*) \cap (C^0(\mathbf{R}))^n$ *and is the unique solution of both initial value problems*

$$A\dot{\xi} + B\xi = b \quad \text{in } \mathbf{R}, \quad \xi_-(0) = x_0, \tag{4.5}$$

$$A\dot{\xi} + B\xi = b \quad \text{in } \mathbf{R}, \quad \xi_+(0) = x_0. \tag{4.6}$$

Proof: It is obvious that $x \in C^n_{\rm imp}(\mathbf{R}^*)$. In particular, the derivative $\dot{x} \in (\mathcal{D}'(\mathbf{R}))^n$ is the function given by $\dot{x}_-(t)$ for $t < 0$ and by $\dot{x}_+(t)$ for $t > 0$, whence $A\dot{x} + Bx = b$ in $\mathbf{R}$ in the sense of distributions. By definition of x_- and b, $x_0 = x_-(0)$ is consistent with the DAE (3.17_-) at $t_0 = 0$, and by hypothesis x_0 is also consistent with the DAE (3.17_+) at $t_0 = 0$. It then follows from Theorem 3.2 that (4.5) and (4.6) each have a unique solution in $C^n_{\rm imp}(\mathbf{R}^*)$. In both cases, this solution is x since $x_-(0) = x_+(0) = x_0$ by continuity of x at 0. □

Lemma 4.1 suggests that we should solve (4.1) for inconsistent x_0 by making use of the extension b of b_+ in (4.3). This approach is taken in the following result:

Theorem 4.1. *Let* $x_- \in C^\infty((-\infty,0];\mathbf{R}^n)$ *and* $b_+ \in C^\infty([0,\infty);\mathbf{R}^n)$ *be given. Suppose that the pair* (A,B) *has index* $\nu \geq 0$ *and that, in the notation of Section 2,* $A_\nu(t)$ *is invertible for every* $t \in \mathbf{R}$. *Then, for* $b \in C^n_{\rm imp}(\mathbf{R}^*)$ *defined by (4.3), there exists a unique distribution* $x \in (\mathcal{D}'(\mathbf{R}))^n$ *which solves*

$$A\dot{x} + Bx = b \quad \text{in } \mathbf{R}, \quad x_{|(-\infty,0)} = x_-. \tag{4.7}$$

Morever, we have

(i) $x \in C^n_{\rm imp}(\mathbf{R}^*)$ *and* x *has impulse order at most* $\nu - 2$.

(ii) $x_+ = x_{|(0,\infty)}$ *solves the DAE*

$$A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \quad \text{in } (0,\infty). \tag{4.8}$$

(iii) If $x_0 \equiv x_-(0)$ *is consistent with the DAE (4.8) at* $t_0 = 0$, *then* $x \in (C^0(\mathbf{R}))^n$ *and* x_+ *is the classical solution of the initial value problem*

$$A(t)\dot{x}_+ + B(t)x_+ = b_+(t) \quad \text{in } (0,\infty), \qquad x_+(0) = x_0. \tag{4.9}$$

(iv) Irrespective of the consistency of $x_0 \equiv x_-(0)$ with the DAE (4.8) at $t_0 = 0$, the distribution $\xi = x_+ + x_{\mathrm{imp}}$, with x_+ extended by 0 in $(-\infty, 0)$, is the unique solution in $C^n_{\mathrm{imp}}(\mathbf{R}^*)$ of the initial value problem

$$A\dot{\xi} + B\xi = b_+ + A(0)x_0\delta \quad \text{in } \mathbf{R}, \quad \xi_-(0) = 0, \tag{4.10}$$

where b_+ is extended by 0 in $(-\infty, 0)$. In particular, iord $(\xi) \leq \max(-1, \nu - 2)$ and hence iord $(\xi) \leq \nu - 2$ for $\nu \geq 1$ (in contrast to iord $(\xi) \leq \nu - 1$ obtained by a direct application of Theorem 3.2 to (4.10)).

Proof: Set $x_0 = x_-(0)$ and $b_-(t) = A(t)\dot{x}_-(t) + B(t)x_-(t)$ for $t \leq 0$, so that by definition x_- solves the DAE (3.17$_-$) and x_0 is consistent with (3.17$_-$) at $t_0 = 0$. Thus, by Theorem 3.2 there is a unique solution $y \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ of the initial value problem

$$A\dot{y} + By = b \quad \text{in } \mathbf{R}, \quad y_-(0) = x_0. \tag{4.11}$$

As a result, y_- is another solution of (3.17$_-$) which, just as x_-, verifies $y_-(0) = x_0$. This implies that $y_- = x_-$ and hence that $x = y$ solves (4.7). Conversely, if $y \in (\mathcal{D}'(\mathbf{R}))^n$ solves (4.7), then $y \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ by Theorem 3.2, and the equality $y_{|(-\infty,0)} = x_-$ as distributions in $(-\infty, 0)$ implies at once that $y_- = x_-$. Thus, by continuity, $y_-(0) = x_-(0) = x_0$, and y solves (4.11). Uniqueness of the solution of (4.7) then follows from the unique solvability of (4.11).

We now pass to the proof of the statements (i) - (iv). Part (i) follows from Theorem 3.2 and the fact that the right side b in (4.7) and (4.11) (which, as was just seen, have the same solution) is given by (4.3), and hence has impulse order $k \leq -1$. Property (ii) is a trivial consequence of (i) and the fact that x solves (4.7). For the proof of (iii) note that if $x_0 \equiv x_-(0)$ is also consistent with the DAE (4.8) at $t_0 = 0$, then Theorem 3.1 ensures that the unique solution $y = x$ of (4.11) (and hence also of (4.7)) is given by (4.4) and solves (4.6). This shows that $x \in (C^0(\mathbf{R}))^n$ and that x_+ solves (4.9).

For the proof of (iv), set $\xi = x_+ + x_{\mathrm{imp}} \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$, so that $\xi = x - x_-$ with x_-

extended by 0 in $(0, \infty)$. Since iord $(x) \leq \nu - 2$ and iord$(x_-) \leq -1$, we find that iord$(\xi) \leq \max(-1, \nu - 2)$. Moreover, viewing x_- as a function of $t \in \mathbf{R}$, we have $A\dot{x}_- + Bx_- = b_- - Ax_0\delta$, where

$$b_-(t) = \begin{cases} A(t)\dot{x}_-(t) + B(t)x_-(t) & \text{if } t < 0, \\ 0 \quad \text{if } t > 0. \end{cases}$$

From the above discussion and the definition of b in (4.3), it follows at once that $A\dot{\xi} + B\xi = A\dot{x} + Bx - A\dot{x}_- - Bx_- = b_+ + Ax_0\delta$. Moreover, we have $Ax_0\delta = A(0)x_0\delta$ and $\xi_-(0) = x_-(0) - x_-(0) = 0$ whence ξ solves (4.10) and thus coincides with the unique solution of that problem. Note here that the consistency of $0 \in \mathbf{R}^n$ with the DAE $A(t)\dot{\xi}_- + B(t)\xi_- = (b_+)_-(t) = 0$ in $(-\infty, 0)$ follows from the fact that $\xi_-(t) = 0$ is a solution. □

Theorem 4.1 justifies the choice of the solution ξ of (4.10) (or of its positive part ξ_+) to represent the solution x of (4.1) when x_0 is not consistent with (4.1) at $t_0 = 0$. Further justification will be provided by Theorem 4.2 below. For the time being observe that the characterization (4.10) of $\xi = x_+ + x_{\text{imp}}$ shows that the extension x of x_- as a solution of (4.7) depends only upon $x_0 \equiv x_-(0)$ and b_+ and hence is independent of $x_-(t)$ for $t < 0$. For the case when A and B are constant, the characterization (4.10) is exactly that of [VLK81],[Co82], [G93], but the equivalent characterization (4.7) is not explicitly noticed in these papers.

It is noteworthy that for index 1 problems, ξ solving (4.10) has impulse order at most -1, and hence $\xi_{\text{imp}} = 0$. In other words, ξ is a function with a possible discontinuity at the origin. A simple formula can be given for the jump $\xi_+(0)$ of ξ (recall $\xi_- = 0$) without using the more cumbersome general procedure outlined in Remark 3.5. Indeed, the relation $A\dot{\xi} + B\xi = b_+ + A(0)x_0\delta$ reads $A\xi_+(0)\delta + A(d\xi/dt) + B\xi = b_+ + A(0)x_0\delta$, where $d\xi/dt$ denotes the usual derivative of ξ at points of $\mathbf{R}^*$. Clearly, this requires that $A(0)\xi_+(0) = A(0)x_0$ and that $A(t)(d\xi/dt)(t) + B(t)\xi(t) = b_+(t)$, $\forall t \in \mathbf{R}^*$. In particular, for $t > 0$, we must have $Q_0(t)B(t)\xi_+(t) = Q_0(t)b_+(t)$ where $Q_0 \in C^\infty(\mathbf{R}; \mathcal{L}(\mathbf{R}^n))$ is as in Section 2; that is, $Q_0(t)$ projects onto a complement of rge $A(t)$, (or of ext rge $A(t)$ in the analytic case). By continuity, we obtain $Q_0(0)B(0)\xi_+(0) = Q_0(0)b_+(0)$. Thus, $\xi_+(0)$

solves the system

$$(A(0) + Q_0(0)B(0))\xi_+(0) = A(0)x_0 + Q_0(0)b_+(0). \tag{4.12}$$

Conversely, if $\xi_+(0)$ solves (4.12) then $A(0)\xi_+(0) = A(0)x_0$ and because of rge $A(0) \cap$ rge $Q_0(0) = \{0\}$ we have $Q_0(0)B(0)\xi_+(0) = Q_0(0)b_+(0)$. It turns out (see [RR93a]) that invertibility of $A(t) + Q_0(t)B(t)$ for every $t \in \mathbf{R}$ is implied by the index 1 assumption, and hence that $\xi_+(0)$ is given by

$$\xi_+(0) = [A(0) + Q_0(0)B(0)]^{-1}(A(0)x_0 + Q_0(0)b_+(0)).$$

Note that $\xi_+(0) = x_0$ if and only if $Q_0(0)B_+(0)x_0 = Q_0(0)b_+(0)$ (see (4.12)), a condition that is easily seen to be equivalent to the consistency of x_0 with the DAE (3.17_+) at $t_0 = 0$.

In Theorem 4.1, the function $b(t)$ may be approximated, in the sense of $(\mathcal{D}'(\mathbf{R}))^n$, by sequences of smooth functions $b^\ell \in C^\infty(\mathbf{R};\mathbf{R}^n)$. In practice, considering such a sequence amounts to viewing the transition from x_- to x_+ as the limiting case of a perhaps physically more realistic, situation where a rapid but not discontinuous modification of the input occurs in the vicinity of $t = 0$. In this setting, it is perfectly reasonable to assume that $b^\ell = b_-$, for all $\ell \geq 1$, in some interval $(-\infty, -a]$ for some $a > 0$ independent of ℓ. On the other hand, the function $x_{-|(-\infty,-a]}$ has a unique extension as a solution $x^\ell \in C^\infty(\mathbf{R};\mathbf{R}^n)$ of the DAE

$$A\dot{x}^\ell + Bx^\ell = b_\ell \quad \text{in } \mathbf{R}.$$

In fact, x^ℓ can be obtained as the solution of the initial value problem

$$A\dot{x}^\ell + Bx^\ell = b_\ell \quad \text{in } \mathbf{R}, \quad x^\ell(t_0) = x_-(t_0), \tag{4.13}$$

where $t_0 \leq -a$ is arbitrarily chosen. Evidently, it would be desirable that the sequence x^ℓ tends to the solution x of (4.7) in some sense. That this is indeed true, and more specifically that $\lim_{\ell\to\infty} x^\ell = x$ in $(\mathcal{D}'(\mathbf{R}))^n$ follows at once from Theorem 3.2 and the hypotheses $b^\ell = b$ in $(-\infty, -a]$, for all $\ell \geq 1$, and $\lim_{\ell\to\infty} b^\ell = b$ in $(\mathcal{D}'(\mathbf{R}))^n$ (just choose $t_0 < -a$ in (4.13)). We record this result in the following form:

Theorem 4.2. *Let $x_- \in C^\infty((-\infty, 0], \mathbf{R})$ and $b_+ \in C^\infty([0, \infty); \mathbf{R}^n)$ be given. Suppose that the pair (A, B) has index $\nu \geq 0$ and that, in the notation of Section 2, $A_\nu(t)$ is invertible for $t \in \mathbf{R}$. Let $b \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ be defined by (4.3) and let $b^\ell \in C^\infty(\mathbf{R}; \mathbf{R}^n)$, $\ell \geq 1$, be a sequence such that $b^\ell = b_-$ in $(-\infty, -a]$ for some $a \geq 0$ independent of ℓ and such that $\lim_{\ell\to\infty} b^\ell = b$ in $(\mathcal{D}'(\mathbf{R}))^n$. Denote by $x^\ell \in C^\infty(\mathbf{R}; \mathbf{R}^n)$ the unique extension of $x_{-|(-\infty,-a]}$ as a solution of the DAE*

$$A\dot{x}^\ell + Bx^\ell = b^\ell \quad in \ \mathbf{R},$$

and let $x \in C^n_{\mathrm{imp}}(\mathbf{R}^*)$ be the solution of (4.7). Then, we have

$$\lim_{\ell\to\infty} x^\ell = x \quad in \ \ (\mathcal{D}'(\mathbf{R}))^n.$$

5. Some generalizations.

Let $\mathcal{J} \subset \mathbf{R}$ be an open interval and let $\mathcal{S} = (a_i)_{i\in\mathbf{Z}}$ be a nondecreasing sequence of points of $\mathbf{R} \cup \{\pm\infty\}$ with $a_i < a_{i+1}$ if either a_i or a_{i+1} is real and $\lim_{i\to\pm\infty} a_i \notin \mathcal{J}$. Denote by $C_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S})$ the subspace of $\mathcal{D}'(\mathcal{J})$ of the distributions of the form $x = \tilde{x} + x_{\mathrm{imp}}$ where $\tilde{x}$ is a function such that $\tilde{x}_{|(a_i, a_{i+1})\cap\mathcal{J}} \in C^\infty([a_i, a_{i+1}] \cap \bar{\mathcal{J}})$, $\forall i \in \mathbf{Z}$, and x_{imp} is a distribution with support contained in $\mathcal{S} \cap \mathcal{J}$. Equivalently, if δ_{a_i} is the Dirac delta distribution at a_i, then x_{imp} is a finite or infinite linear combination of derivatives of δ_{a_i} with $a_i \in \mathcal{J}$. With this definition of $C_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S})$, we have $C_{\mathrm{imp}}(\mathcal{J}) = C^\infty(\bar{\mathcal{J}})$ if $\mathcal{J} \cap \mathcal{S} = \emptyset$.

With the definition $C^n_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S}) = (C_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S}))^n$, it should be evident how to formulate Theorems 3.1 and 3.2 for the case $b \in C^n_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S})$. Of course, the elements of $C^n_{\mathrm{imp}}(\mathcal{J} \setminus \mathcal{S})$ have an impulse order at each point $a_i \in \mathcal{J} \cap \mathcal{S}$. It may only be useful to note that for a given arbitrary sequence $\mu_{oi} \in \mathbf{R}^n$, a primitive of $\sum_{i=-\infty}^{\infty} \mu_{oi}\delta_{a_i}$ is $\sum_{i=-\infty}^{-1}(-\mu_{oi})(1 - H_{a_i}) + \sum_{i=0}^{\infty} \mu_{oi} H_{a_i}$ where $H_{a_i}(t) = H(t - a_i)$ if $a_i \in \mathbf{R}$, $H_{-\infty}(t) = 1$, $H_\infty(t) = 0$, and *not* $\sum_{i=-\infty}^{\infty} \mu_{oi} H_{a_i}$ which would not make sense when $\sum_{i=-\infty}^{-1} \mu_{oi}$ does not converge.

For $0 \in \mathcal{J}$ it should be equally obvious how solutions of the initial value problem

$$A\dot{x} + Bx = b_+ \quad \text{in } \mathcal{J}_+ = \mathcal{J} \cap (0, \infty), \qquad x(0) = x_0,$$

can be defined when $0 \notin S$, $b_+ \in C^n_{\rm imp}(\mathcal{J}_+ \setminus S)$, and $x_0 \in \mathbf{R}^n$ is not consistent with the DAE $A\dot{x} + Bx = b_+$ in $\mathcal{J}_+$ at $t = 0$. For problems with index $\nu \geq 2$ the solutions may exhibit a nonzero impulse at $t_0 = 0$. The case considered in Section 4 corresponds to $\mathcal{J} = \mathbf{R}$, $S = \{\infty\}$.

6. Numerical examples.

For index-one problems the computation of the jump (4.12) caused by inconsistent input can be easily incorporated into a numerical procedure for solving the initial value problems (4.1/2). We consider here a recently developed solution process, [RR93b], which is based on the reduction procedure of [RR93a] summarized in Section 2 above.

Suppose that the DAE (4.1) has index 1. For a given step $h > 0$ set $t_i = ih$, $i = 0, 1, \ldots$, and consider the explicit Euler approximation

$$(6.1) \qquad A(t_i)\frac{1}{h}(x_{i+1} - x_i) + B(t_i)x_i = b_+(t_i).$$

In [RR93b], (Theorem 3), it was shown that any solution $x_0, x_1, \ldots, x_m \in \mathbf{R}^n$ verifies for $i = 0, 1, \ldots, m-1$ the equations $Q(t_i)B(t_i)x_i = Q(t_i)b_+(t_i)$ and

$$(6.2) \qquad [A(t_i) + Q(t_{i+1})B(t_{i+1})]x_{i+1} = [A(t_i) - hB(t_i)]x_i + hb_+(t_i) + Q(t_{i+1})b_+(t_{i+1}).$$

Conversely, for sufficiently small h and any given $x_0 \in \mathbf{R}^n$ such that $Q(0)B(0)x_0 = Q(0)b_+(0)$, the solution $x_0, x_1, \ldots, x_m$ of (6.2) is unique and solves also (6.1). Smallness of h ensures that the operator $A(t_i) + Q(t_{i+1})B(t_{i+1})$ is invertible, given that invertibility of $A(t) + Q(t)B(t)$ for all t is equivalent to the index 1 assumption.

The difference scheme (6.2) has been used as the base method in an explicit extrapolation integrator, LTV1XE, for general index-one problems (4.1) - (4.2).

Now note that for $t_0 = 0$, $h = 0$, and with x_{i+1} replaced by $\xi_+(0)$ the difference equation (6.2) is identical with (4.12). Thus, the results of Section 4 ensure that for any given x_0 we only need to apply (6.2) with $h = 0$ to obtain the consistent starting point from which the solution process can then be started. This represents only a minor modification to the mentioned code LTV1XE. The resulting code accepts any given initial point and then computes the solution starting from the corresponding solution of (4.12).

As an example consider the index-one problem

$$(6.3) \qquad \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} + \begin{pmatrix} 1 & -(t+1) & t^2+2t \\ 0 & -1 & t-1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin t \end{pmatrix}$$

given in [CP88] which has the general solution

$$(6.4) \qquad x(t) = (\alpha t e^t + \beta e^{-t},\ \alpha e^t + t\sin t,\ \sin t)^T \in \mathbf{R}^3.$$

For several randomly selected points $(x_1, x_2, x_3)^T$ and starting times, Table 6.1 gives the corresponding consistent starting points computed by LTV1XE. It is readily checked that these consistent points verify (6.4) for suitable constants α and β.

point-type	t	x_1	x_2	x_3
given pnt.	0.0	0.0	1.0	1.0
consistent	0.0	0.0	1.0	0.0
given pnt.	1.0	1.0	1.0	1.0
consistent	1.0	1.0	0.84147098	0.84147098
given pnt.	1.0	-1.0	4.0	5.0
consistent	1.0	-1.0	-0.15852902	0.84147098
given pnt.	2.0	-5.0	2.0	3.0
consistent	2.0	-5.0	-2.1814051	0.90929743
given pnt.	-1.0	1.0	-1.0	2.0
consistent	-1.0	1.0	1.8414710	-0.84147098

Table 6.1: Consistent points for (6.3)

For the index-two case a code LTV2XE was developed which incorporates the reduction discussed in Section 2 for a given index-two problem (4.1) and then applies LTV1XE to the reduced index-one problem. The central part in the reduction is the computation of the mappings C and D. This can be implemented, in general, by using a singular value decomposition (SVD) to obtain a basis of rge A and the projection Q, and another SVD for generating a basis of $\ker QB$. But, it turns out that in many cases there are much simpler ways of generating these mappings. Thus LTV2XE assumes that subroutines are available not only for the coefficients A, B, b and their derivatives, but also for C and D. This allows us to bypass easily the costly general method for calculating these matrices whenever a simpler approach is feasible.

In this form LTV2XE will work as long as the coefficients of the problem are smooth. When the right side of the original equation has a jump, then, in general, the right side of the reduced equation exhibits not only a jump but also an impulse. Hence the earlier given simple jump computation (4.12) for index one problems is insufficient for the index-two case.

As an illustration consider the simple DAE

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ t \\ \tau(t) \end{pmatrix} \tag{6.5}$$

with the initial condition $x^0 = (0,0,1)^T$. When τ is a smooth function with $\tau(0) = 1$, $\dot{\tau}(0) = 0$ then the unique solution is

$$x_1(t) = \tau(t),\ x_2(t) = t - 1 + \exp(-t),\ x_3(t) = 1 - \dot{\tau}(t). \tag{6.6}$$

Suppose now that $\tau(t) = H_1(t)$ where H_1 is the Heaviside function with the step at $t = 1$. Then the solution has the same form as (6.6) but with $x_1(t) = H_1(t)$ and $x_3(t) = 1 - \delta_1(t)$. Thus at $t = 1$ we have a jump of size 1 in the first component and an impulse of size -1 in the third component. A graph of this solution does not show the impulse. But if we approximate the step of H_1 by a cubic spline; that is, if we consider (6.5) with

$$\tau(t) = \begin{cases} 0, & \text{for } 0 \le t < 1 - \epsilon, \\ \frac{1}{2} + \frac{1}{4}\,\sigma(t)(3 - \sigma(t)^2), & \text{for } 1 - \epsilon \le t \le 1 + \epsilon, \quad \sigma(t) = \frac{t-1}{\epsilon}\ , \\ 1, & \text{for } t > 1 + \epsilon, \end{cases} \tag{6.7}$$

with small $\epsilon > 0$, then (6.6) shows that $x_3(1) = 1 - 3/(4\epsilon)$. In other words, this solution approximates the impulse.

For the general computation of the jump and impulse in the index-two case, suppose that for the given DAE (4.1) we have $b = \tilde{b} + [b]H_{t_0}$ where $\tilde{b}$ is the smooth part of the function and $[b]$ a jump at the time t_0. Then, we have $[u_0]H_{t_0} = B(t_0)^T(A(t_0)A(t_0)^T + B(t_0)B(t_0)^T)^{-1}[b]H_{t_0}$ which implies that $\dot{u}_0$ has the impulse $B^T(t_0)(A(t_0)A(t_0)^T + B(t_0)$ $B(t_0)^T)^{-1}[b]\delta_{t_0}$. Accordingly, the right side $b_1 = D(b - A\dot{u}_0 - Bu_0)$ has the impulse

$$\beta_1\delta_{t_0} = -DAB^T(AA^T + BB^T)^{-1}(t_0)[b]\delta_{t_0}. \tag{6.8}$$

Now let

$$A_1\,\dot{x}_1 + B_1\,x_1 = b_1, \quad b_1 = \tilde{b}_1 + [b_1]H_{t_0} + \beta_1\delta_{t_0} \tag{6.9}$$

be the reduced equation and consider its solution in the form $x_1 = \tilde{x}_1 + [x_1]H_{t_0} + \xi_1\delta_{t_0}$. By substituting this into (6.5) and comparing terms we obtain the conditions

$$A_1(t_0)\xi_1 = 0, \quad A_1(t_0)[x_1] + B_1(t_0)\xi_1 = \beta_1, \quad Q_1(t_0)B_1(t_0)[x_1] = Q_1(t_0)[b_1]$$

which can be combined into the two systems

$$(A_1(t_0) + Q_1(t_0)B_1(t_0))\xi_1 = Q_1(t_0)\beta_1 \tag{6.10a}$$

$$(A_1(t_0) + Q_1(t_0)B_1(t_0))[x_1] = Q_1(t_0)[b_1] + \beta_1 - B_1(t_0)\xi_1. \tag{6.10b}$$

Since the DAE is assumed to have index two, the matrix $A_1(t_0) + Q_1(t_0)B_1(t_0)$ is nonsingular and hence the two systems (6.10a/b) can be solved successively. A brief calculation shows that (6.10b) reduces to (4.12) exactly if $\beta_1 - B_1(t_0)\xi_1 = 0$.

The relations (6.8), (6.10a/b) were incorporated into LTV2XE to allow for the computation of the jump and impulse at any point t_0 where the right side b of the original DAE (4.1) has a jump.

As numerical example we consider the following index-two problem

$$(6.11a)\qquad \begin{cases} \dot{x}_1 + x_1 - x_2 - x_4 - x_5 = 0 \\ \dot{x}_2 + x_1 - x_2 + tx_3 - x_5 = 0 \\ \dot{x}_3 - tx_1 - x_3 - tx_4 = 0 \\ \dot{x}_4 + (t-1)x_2 + x_3 - tx_4 = 0 \\ t^2 x_1 + (1-t)^2 x_2 + (t-2)x_3 = \tau(t) \end{cases}$$

where $\tau(t) = 1$ for $t \leq 1$ and $\tau(t) = -1$ for $t > 1$. For the consistent starting point

$$(6.11b)\qquad x_1 = 0.5,\ x_2 = 0.0,\ x_3 = -0.5,\ x_4 = 0.0,\ x_5 = 0.0,$$

Table 6.2 shows all steps computed by LTV2XE for the problem (6.11a/b) for $0 \leq t \leq 1.2$. A relative tolerance of 10^{-5} and a maximal step of 0.1 was used.

The discontinuity at $t = 1$ causes a recalculation of the point obtained at $t = 1$ from which the solution proceeded. Clearly, in order to capture the discontinuity exactly at $t = 1$, this value has to be included in the list of required output-points of the code. This was indicated in the table by a dividing line. At any jump point the output of the code includes the values of the jump and the impulse of the solution. In this case, we found that at $t = 1$ the solution has the jump $(-2, -2, 0, 0, 2)^T H_1$ and the impulse $(0,0,0,0,2)^T \delta_1$. Of course, the jump is also clearly seen in Table 6.2.

In analogy with the simple problem (6.5) we approximate the step by

$$(6.11)\qquad \bar{\tau}(t) = \begin{cases} 1, & \text{for } 0 \leq t < 1-\epsilon, \\ \frac{1}{2}\sigma(t)^3 - \frac{3}{2}\sigma(t), & \text{for } 1-\epsilon \leq t \leq 1+\epsilon,\ \sigma(t) = \frac{t-1}{\epsilon}\ , \\ -1, & \text{for } t > 1+\epsilon, \end{cases}$$

t	x_1	x_2	x_3	$x4$	x_5
0.0000	0.5000	0.0000	-0.5000	0.0000	0.0000
0.5000(-1)	0.4724	-0.2763(-1)	-0.5250	0.2500(-1)	-0.1340
0.1500	0.3984	-0.1010	-0.5751	0.7502(-1)	-0.4573
0.2500	0.2941	-0.2033	-0.6263	0.1252	-0.8372
0.3475	0.1612	-0.3317	-0.6788	0.1752	-1.184
0.4475	0.3369(-2)	-0.4816	-0.7384	0.2297	-1.351
0.5406	-0.1384	-0.6120	-0.8014	0.2875	-1.193
0.6406	-0.2494	-0.7057	-0.8780	0.3635	-0.7034
0.7060	-0.2862	-0.7284	-0.9317	0.4246	-0.3105
0.7919	-0.2890	-0.7094	-1.003	0.5225	0.1445
0.8639	-0.2570	-0.6571	-1.060	0.6224	0.3976
0.9305	-0.2073	-0.5875	-1.105	0.7306	0.5071
0.9969	-0.1452	-0.5060	-1.141	0.8545	0.5046
1.000	-0.1421	-0.5020	-1.142	0.8608	0.5020
1.000	-2.142	-2.502	-1.142	0.8608	2.502
1.002	-2.137	-2.496	-1.146	0.8644	2.504
1.003	-2.134	-2.493	-1.149	0.8663	2.505
1.010	-2.111	-2.468	-1.167	0.8818	2.511
1.034	-2.037	-2.387	-1.224	0.9341	2.513
1.058	-1.962	-2.303	-1.281	0.9910	2.490
1.082	-1.887	-2.220	-1.334	1.052	2.444
1.105	-1.813	-2.136	-1.386	1.116	2.379
1.129	-1.740	-2.053	-1.435	1.184	2.296
1.151	-1.668	-1.971	-1.481	1.255	2.197
1.174	-1.597	-1.889	-1.523	1.330	2.086
1.196	-1.528	-1.810	-1.563	1.408	1.963
1.200	-1.516	-1.797	-1.569	1.422	1.941

Table 6.2: Solution of (6.11a/b)

with small $\epsilon > 0$, then we expect the solution to approximate the impulse $(0,0,0,0,2)^T\delta_1$. Figure 6.1 show the fifth components in the case of $\epsilon = 0.05$. For smaller values of ϵ the system becomes too stiff to capture the impulse.

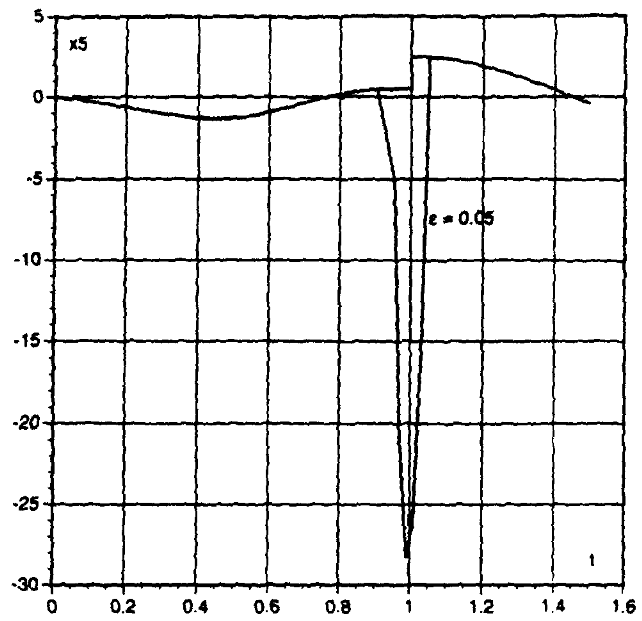

FIGURE 6.1

It may be noted that we did not succeed to compute the solutions of this problem either with DASSL (see e.g.[BCP89]) or RADAU5 (see [HW91]). Both codes failed at start-up.

References

[BP86] K. E. Brenan and L. R. Petzold, *The numerical solution of higher index differential/algebraic equations by implicit Runge Kutta methods*, SIAM J. Numer. Anal. 126 (1986), 976-996.

[BCP89] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential - Algebraic Equations*, North Holland, New York, NY, 1989.

[C87] S. L. Campbell, *A general form for solvable linear time varying singular systems of differential equations*, SIAM J. Math. Anal. 18 (1987), 1101-1115.

[CP88] K. D. Clark and L. R. Petzold, *Numerical solution of boundary value problems in differential/algebraic systems*, Preprint UCRL-98449, Lawrence Livermore National Lab., Univ. of Calif., (1988).

[Co82] D. Cobb, *On the solutions of linear differential equations with singular coefficients*, J. Diff. Equations 46 (1982), 311-323.

[G93] T. Geerts, *Solvability for singular systems*, Lin. Alg. Appl. **181** (1993), 111-130.

[HW91] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II*, Springer-Verlag, Heidelberg, Germany, 1991.

[HS83] M. L. T. Hautus and L. M. Silverman, *System structure and singular control*, Lin. Alg. Appl. **50** (1983), 369-402.

[KuM92] P. Kunkel and V. Mehrmann, *Canonical forms for linear differential algebraic equations with variable coefficients*, Internal Report 69, Institut für Geometrie und Praktische Mathematik, RWTH Aachen, Germany (1992).

[RR93a] P. J. Rabier and W. C. Rheinboldt, *Classical and generalized solutions of time-dependent linear DAE's*, Lin. Alg. Appl., submitted.

[RR93b] P. J. Rabier and W. C. Rheinboldt, *Finite difference methods for time dependent, linear differential algebraic equations*, Appl. Math. Letters, in press.

[S66] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

[VLK81] G. C. Verghese, B. C. Levy and T. Kailath, *A generalized state-space for singular systems*, IEEE Trans. Automat. Control **AC-26** (1981), 811-831.

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Existence and uniqueness results are proved for initial value problems associated with linear, time-varying, differential-algebraic equations. The right-hand sides are chosen in a space of distributions allowing for solutions exhibiting discontinuities as well as "impulses". This approach also provides a satisfactory answer to the problem of "inconsistent initial conditions" of crucial importance for the physical applications. Furthermore, our theoretical results yield an efficient numerical procedure for the calculation of the jump and impulse of a solution at a point of discontinuity. Numerical examples are given.

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