AD-A277 453

DOCUMENTATION PAGE

FINAL/01 JUL 92 TO 30 JUN 93

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A STUDY OF WEAK SOLUTIONS AND THEIR REGULARIZATIONS BY NUMERICAL METHODS (U)

2304/A3
AFOSR-91-0309

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AFOSR-91-0309

94-09332

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Consider the incompressible Euler equations with vortex sheet initial data. For this initial value problem, there are a number of outstanding conjectures: (1) This initial value problem does not have a unique weak or measure-valued solution, (2) A selection principle is required to pick out a unique solution, (3) The linit of vanishing viscosity (in the Navier Stokes equations) provides the correct selection principle, and (4) Different regularizations, such as adding viscosity or smoothing the initial vortex sheet, may converge to different limits as the regularization tends to zero.

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Summary of Results

Consider the incompressible Euler equations with vortex sheet initial data. For this initial value problem, there are a number of outstanding conjectures:

(1) This initial value problem does not have a unique weak or measure-valued solution.

(2) A selection principle is required to pick out a unique solution.

(3) The limit of vanishing viscosity (in the Navier Stokes equations) provides the correct selection principle.

(4) Different regularizations, such as adding viscosity or smoothing the initial vortex sheet, may converge to different limits as the regularization tends to zero.

These issues are central to understanding the strengths and limitations of using vortex sheets to model fluids in aerodynamics, for example. Unfortunately, while there are some heuristic arguments which justify some of these claims, there are no explicit examples or rigorous proofs which substantiate them.

Given the incomplete status of the theory for the Euler equations with vortex sheet initial data, one can proceed in two reasonable ways. One can continue to work directly with the Euler equations, or one can first study a simpler model problem which is closely related, easier to analyze, and suggests answers to the fundamental open questions about the Euler equations.

In research funded by this grant, and done in collaboration with Andrew Majda, Princeton University, and Yuxi Zheng, Courant Institute and Indiana University, we considered a closely related model problem from plasma physics, namely, the one component 1-D Vlasov-Poisson equations (1CVPE) for a collisionless plasma of electrons in a uniform background of ions. We considered initial data for this system of equations which consisted of a measure supported over a curve in phase space, the analogue of vortex sheet initial data, which we call electron sheet initial data. We show that the 1CVPE with electron sheet initial data has many properties which are direct analogues of the 2-D vorticity equation with vortex sheet initial data. Furthermore, since this problem is simpler analytically and easier to solve numerically, we can make a number of definitive statements about this model problem pertaining to the unresolved issues (1)-(4) proposed about the Euler equations. Among these statements are the following ones:

(1#) The initial value problem for the 1CVPE, with electron sheet initial data, does not have unique solutions. We demonstrate this by explicitly constructing an uncountable number of weak solutions to the 1CVPE with the same initial electron sheet.

(2#) Different regularizations of the 1CVPE, with electron sheet initial data, can converge to different solutions. This result is demonstrated numerically by using a carefully designed particle code. In particular, we show that different regularizations of the initial data can lead to different solutions, and the "viscous" limit (from the Fokker-Planc equation) can be different from the limit obtained by regularizing the initial data.

We consider our research on the 1CVPE, as a model for vortex sheets, essentially complete. This work will produce at least 2 major publications and one invited article for a conference proceedings which I will either be the author or share authorship.

In [1] I detail many of the strong analogies between the Euler equations with vortex sheet initial data and the 1CVPE with electron sheet initial data, and demonstrate the performance of the particle method. This paper is included with this research summary.

In [2] I discuss the details of the particle method for the one component 1-D Vlasov-Poisson and Fokker-Planc equations. I present the rapid solver which I developed to make the particle method efficient, explain the particular way to discretize electron sheet initial data, demonstrate the performance of the method on a number of very interesting exact solutions of the 1CVPE, and make a number of conjectures about the properties of solutions to the 1CVPE.

In [3] we present a detailed justification of statements (1#) and (2#) among other results, and show how our study of the 1CVPE suggests a strategy for resolving issues (1)-(4) for the Euler equations with vortex sheet initial data.

In work also funded by this grant, my graduate student, John Hamilton, and I have already taken the first steps for carrying out the numerical part of a follow-up project with the Euler equations. I believe that vortex methods are the best way to carry out this study; these methods are also the ones most like the particle method used to solve the 1CVPE. Anticipating the need for extremely high resolution calculations with a large number of vortex blobs, we will need a method for rapidly evaluating the vortex interactions. We have been working on a modification of the Rokhlin-Greengard fast vortex algorithm to compute solutions of the Euler equations which are periodic in x and singular like vortex sheets. Our preliminary results on a Sun workstation show that this is possible. We have also discovered a way to do vortex sheet calculations using a vortex method with initial blobs which are not initially uniformly spaced in the circulation variable. This procedure gives the ability to locate lots of initial vortex blobs in places where we expect a complicated solution, and fewer vortex blobs where the vortex sheet is not very complicated. We are currently preparing these details for publication.

Publications

[1] G. Majda, *"On singular solutions of the Vlasov-Poisson equations"*. Invited presentation, to appear in the Proceedings of the NATO Advanced Research Workshop on Vortex Flows and Related Numerical Methods, G.H. Cottet, ed., Grenoble - St. Pierre de Chartreuse, France, June 15–19, 1992.

[2] G. Majda, *"Particle approximation of weak solutions to the Vlasov-Poisson and Fokker-Planc equations"*. In preparation.

[3] A. Majda, G. Majda and Y. Zheng, *"1-D Vlasov-Poisson equations"*. In preparation.

On Singular Solutions of the Vlasov-Poisson Equations*

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A vortex sheet is, loosely speaking, a surface in a fluid such that the normal velocity of the fluid is continuous along the surface but the tangential velocity of the fluid is discontinuous across the surface. Many theoretical, numerical and analytical investigations have been done to try to understand the properties of solutions to the incompressible Euler Equations with non-smooth (vortex sheet) initial data, see [1], [3], [4], [5], [6], [8], [9], [10], [13], [14], [15], [16], [17] and [18] for example. Despite this research, many fundamental open mathematical problems about the nature of these solutions still exist. In order to get a handle on some of these open problems, Andrew Majda proposed that one should study a simpler problem, namely, the one-component Vlasov Poisson Equations (VPE) from plasma physics. Hopefully, insight gained by studying this model problem will provide new insights into the original problems about incompressible fluids. In this paper I will present this system of equations and some connections between the VPE and the vorticity equation for a 2-D incompressible fluid with vortex sheet initial data.

The work which I will present constitutes one small part of an on-going collaboration with Andrew Majda and Yuxi Zheng. See [11], [12] and [19] for a complete description of the results.

In this paper we consider the single component 1–D Vlasov Poisson Equations (VPE) for a collisionless plasma of electrons in a uniform background of ions. Assume that the problem is periodic in x and that the initial electron density is a Dirac delta function supported over a curve in $x-v$ space (phase space). Let $f(x,v,t)$ denote the density of electrons, $E(x,t)$ the electric field, $\rho(x,t)$ the charge, $\vec{x} = (x,v)$ and $\vec{c}$ a curve in $x-v$ space. In dimensionless variables, the problem is defined by the system of equations

(1) $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E(x,t)\frac{\partial f}{\partial v} = 0$ for $t > 0$ and $x \in [0,1]$,

(2) $-\phi_{xx} = \rho(x,t) = 1 - \int_{-\infty}^{\infty} f(x,v,t)\,dv, \quad E = -\phi_x$,
with the initial condition

(3) $f(x,v,0) = g(x,v)\ \delta_{\vec{c}}$,
the periodic boundary conditions

(4) $E(0,t) = E(1,t),\ \ f(0,v,t) = f(1,v,t)$,
and the zero-mean electric field condition

(5) $\int_0^1 E(x,t)\,dx = 0$.

*This paper was presented in an invited lecture at the NATO Advanced Research Workshop on *Vortex Flows and Related Numerical Methods*, Grenoble-St.Pierre de Chartreuse, June 15–19, 1992. It will appear in the conference precedings edited by G.H. Cottet.

Initial data with the form (3) is called **electron sheet initial data**, and the solution of problem (1)–(5) is called an **electron sheet**, to emphasize the close connection with vortex sheets which we now explain.

I briefly highlight some of the features which problem (1)–(5) has in common with the two dimensional vorticity form of the Euler equations with vortex sheet initial data. The reader should consult [4] or [5] for a precise definition of the latter problem.

Let $\vec{v} = (v, -E(x,t))^t$ denote a "velocity" field, $\text{div}_{\overline{x}} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial v})$, and introduce the particle trajectory equations

$$\frac{d}{dt}\begin{pmatrix} X \\ V \end{pmatrix} = \vec{v}(X, V) = \begin{pmatrix} V \\ -E(X,t) \end{pmatrix}.$$

Now just make a correspondence between the electron density f and the vorticity for the 2-D Euler equations. Equation (1) can be interpreted as a transport equation for f by a divergence free velocity field $\vec{v}$ since $\text{div}_{\overline{x}}(v, -E(x,t))^t = 0$. Furthermore, f is constant along the particle trajectory equations. The 2-D vorticity equation has a similar interpretation with the electron density replaced by the vorticity. Using this correspondence, it easily follows that the electron density f is the analogue of the vorticity, and the initial condition (3) is the direct analogue of vortex sheet initial data.

To demonstrate another analogy, we recall that for an incompressible fluid the velocity field is recovered from the vorticity field through the Biot-Savart Law. This law establishes a global relationship between the velocity field and vorticity. The elliptic equation (2) implies that there is a global relationship between the electric field and the electron density. This, in turn, establishes a global relationship between the velocity field for the VPE and the electron density. Other similarities than the ones presented here are mentioned later in this paper and in the papers [11], [12] and [19].

We believe that given the close relationship between problem (1)–(5) and the vorticity form of the Euler Equations with vortex sheet initial data, one will gain important insights about the open problems for the latter problem by thoroughly understanding the properties of the solutions of problem (1)–(5). Furthermore, the electron sheet problem for the VPE has several important advantages over the corresponding vortex sheet problem, and we list two of them.

(1) One can construct explicit solutions for problem (1)–(5) for a perturbed uniform electron sheet, the analogue of a perturbed vortex sheet. (We describe one solution below.) These solutions can form an algebraic singularity at a finite time, just like the predicted behavior of perturbed vortex sheets. The perturbed vortex sheet problem has no similar explicit solutions.

(2) For the VPE, the global relationship between the electric field and the electron density involves an integral with a discontinuous kernel. The Biot-Savart Law for an incompressible fluid contains a kernel with a stronger

singularity. Consequently, one believes that a mathematical analysis of problem (1)–(5) should be easier than the corresponding vortex sheet problem.

I now present an explicit solution, derived by Dziurzynski [7], corresponding to the case of a sinusoidally perturbed uniform electron sheet. A **uniform electron sheet** is a time-independent, exact weak solution of problem (1)–(5) given by

$$\begin{aligned} Z(\alpha) &= (x(\alpha), v(\alpha)) = (\alpha, 0) = \vec{c}(0) \quad \text{for} \quad 0 \leq \alpha \leq 1, \\ f(Z(\alpha)) &= \delta_{\vec{c}(0)}, \\ E(x(\alpha)) &\equiv 0. \end{aligned}$$

Now consider problem (1)–(5) with initial data corresponding to a sinusoidally perturbed uniform electron sheet. Let $\varepsilon > 0$ denote a parameter and $0 \leq \alpha \leq 1$. This initial data is given by

$$\begin{aligned} Z(\alpha,0) &= (x(\alpha,0),\, v(\alpha,0)) = (\alpha, \varepsilon \sin(2\pi j\alpha)) = \vec{c}(0) \\ f(Z(\alpha,0)) &= \left(1 + (\varepsilon 2\pi j)^2 \cos^2(2\pi j\alpha)\right)^{-\frac{1}{2}} \delta_{\vec{c}(0)}. \end{aligned} \tag{6}$$

The exact weak solution of (1)–(5) with this initial data is

$$\begin{cases} Z(\alpha,t) = (x(\alpha,t),\, v(\alpha,t)) \\ \qquad = (\alpha + \varepsilon \sin(2\pi j\alpha) \sin t,\, \varepsilon \sin(2\pi j\alpha) \cos t) = \vec{c}(t) \\ f(Z(\alpha,t)) = \left|\frac{dZ(\alpha,t)}{d\alpha}\right|^{-1} \delta_{\vec{c}(t)} \\ E(x(\alpha,t)) = \varepsilon \sin(2\pi j\alpha) \sin t. \end{cases} \tag{7}$$

Dziurzynski [7] shows that if $|2\pi j\varepsilon| < 1$, then solution (7) is defined for all $t \geq 0$. If $|2\pi j\varepsilon| > 1$, then there exists a time $t^* < \infty$ such that solution (7) is defined for the finite time interval $[0, t^*]$ and the charge $\rho(x(\alpha, t^*))$ forms an algebraic singularity.

This exact solution illustrates another striking similarity between the vortex sheet problem and the electron sheet problem. The algebraic singularity in the charge for $|2\pi j\delta| > 1$ is the analogue of the singularity observed in the vortex sheet case before rollup, see [9], [10], [13] and [17].

Just as in the vortex sheet problem, several questions are natural to ask including the following:

(1) Does the analytic formula continue to describe the electron sheet after the singularity time?

(2) What are the qualitative properties of the solution past the singularity time?

(3) What are the mathematical properties of the solution past the singularity time?

In order to answer these questions and many others, I developed a particle method to solve (1)–(5) based on the method analyzed by Cottet and Raviart in [2]. I needed to modify this method to make it efficient and to be able to discretize singular data like electron sheets. These details are described in [11].

To answer questions (1)–(3), I computed the solution of problem (1)–(5) with initial data of the form (6) for $\varepsilon = 2.0$ and $j = 1$. The singularity time for the solution is $t^* = .079$ to 3 decimal places. I computed the solution with a time step $\Delta t = .0158$, and used enough particles to converge to a solution with 3 decimal place accuracy both before and after the singularity time.

In Figures 1a and 1b I plotted the electron sheet $(Z(\alpha, t))$ and the electric field $E(x(\alpha, t))$, respectively, for 0, 3 and 5 time steps. We note that the singularity time occurs at the fifth time step, so Figures 1a and 1b show graphs of the solution before and at the time of singularity formation. In Figures 2a and 2b I plotted the electron sheet and electric field for 6, 8 and 10 time steps, so these figures show graphs of the solution past the singularity time. These numerical results indicate the following:

(i) The analytic formula (7) does not describe the solution past the singularity time.

(ii) The singularity time is precisely the time when the electron sheet ceases to be the graph of a single-valued function.

(iii) The electron sheet and electric field become multivalued functions past the singularity time.

(iv) The electric field loses smoothness past the singularity time.

Again, we point out the analogy between electron sheets governed by the VPE and vortex sheets. For vortex sheets, the singularity occurs just before the sheet begins to roll up. The folded, multi-valued electron sheet is the analogue of a rolled-up vortex sheet. Furthermore, the electron sheet forms a singularity in the charge just before it begins to fold.

In this short paper I have outlined a number of analogies between the Euler equations with vortex sheet initial data and the VPE with electron sheet initial data. These analogies indicate that problem (1)–(5) should be a very useful model for trying to understand the open questions about vortex sheets. The discussion presented here is just the starting point for a much more detailed study of problem (1)–(5). Complete details can be found in the papers [11], [12] and [19].

Acknowledgment. I would like to thank Andrew Majda, Yuxi Zheng and Ed Overman for many helpful discussions about this work.

My research was initially supported by the National Science Foundation under grant DMS–8903484, and is currently supported by the Air Force Office of Scientific Research under grant AFOSR–91–0309. The numerical computations are supported by the Ohio State Supercomputer Center under grant PAS330.

I am a visiting member of the Institute for Advanced Study in Princeton, New Jersey, and a visitor in the Program in Computational and Applied Mathematics at Princeton University, for the 1992–93 academic year. I would like to thank the people at the I.A.S. for their hospitality. I would also like to thank the members of the PCAM for their hospitality and financial support under grants NSFDMS9001805.A02 and ONRN00014–89–J–1044.P00005.

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Figure Captions

Figure 1a Electron sheet at 0(solid), 3 (long dashes) and 5 (short dashes) time steps.

Figure 1b Electric field at 0 (solid), 3 (long dashes) and 5 (short dashes) time steps.

Figure 2a Electron sheet at 6 (solid), 8 (long dashes) and 10 (short dashes) time steps.

Figure 2b Electric field at 6 (solid), 8 (long dashes) and 10 (short dashes) time steps.

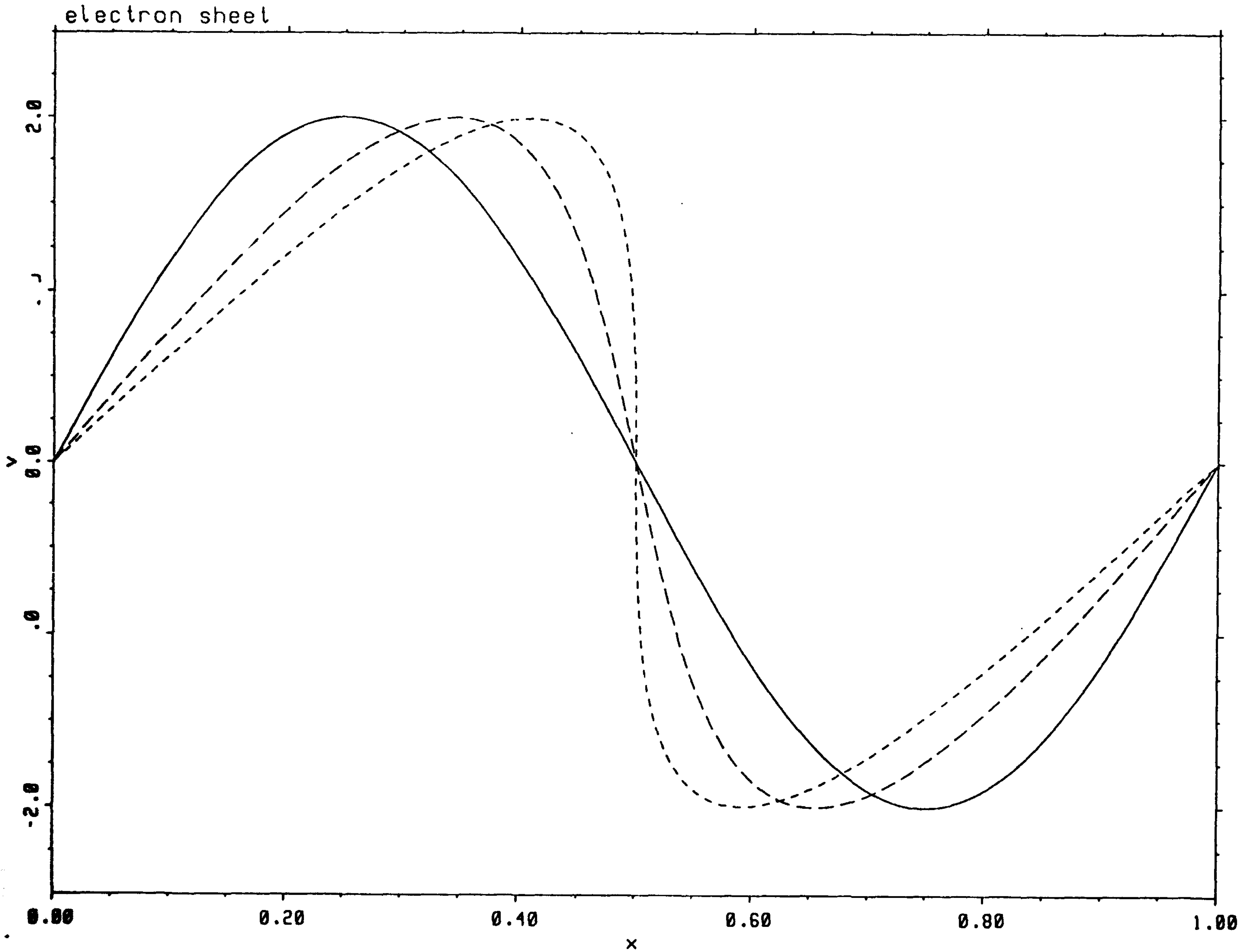
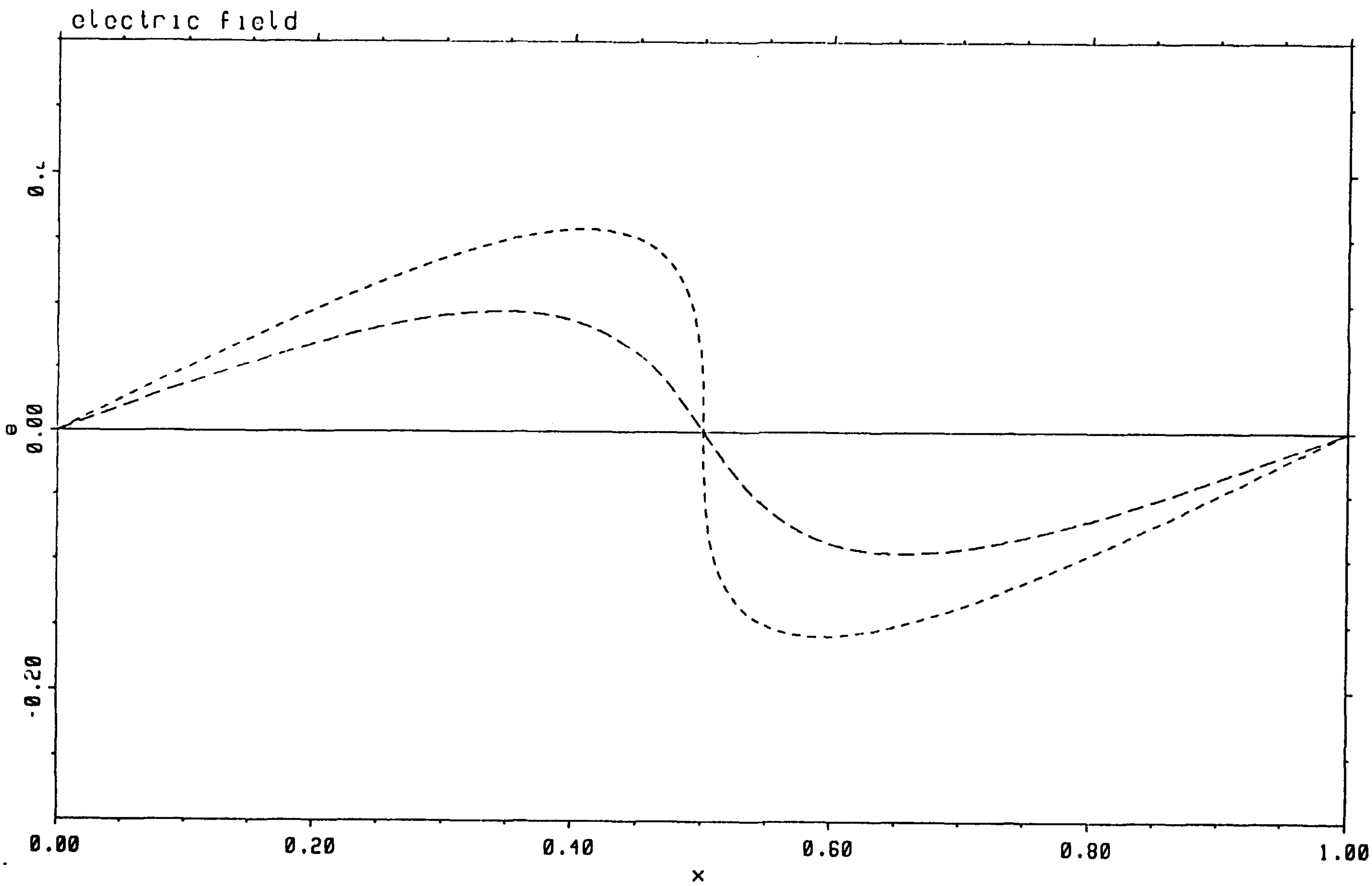


Fig 1a.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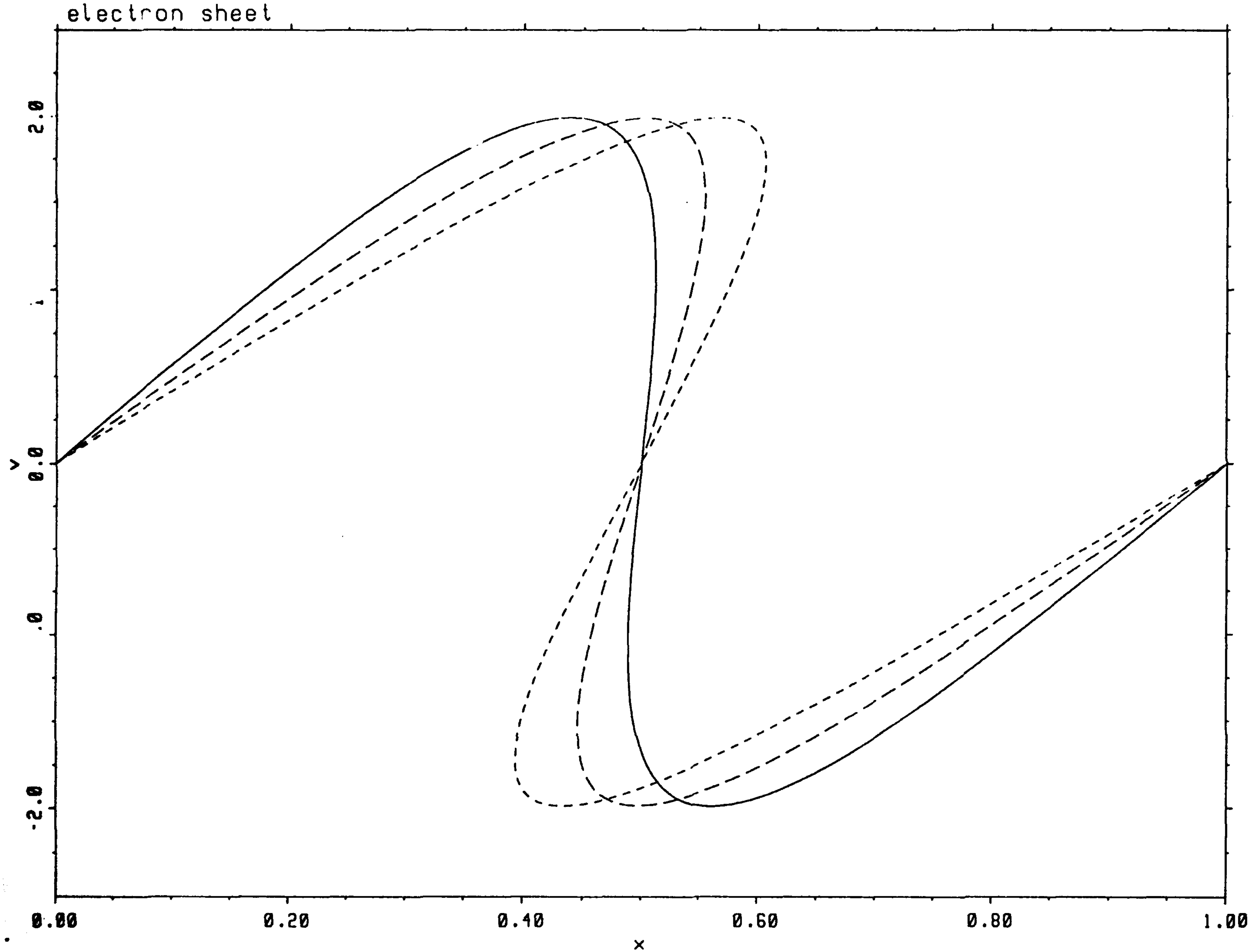
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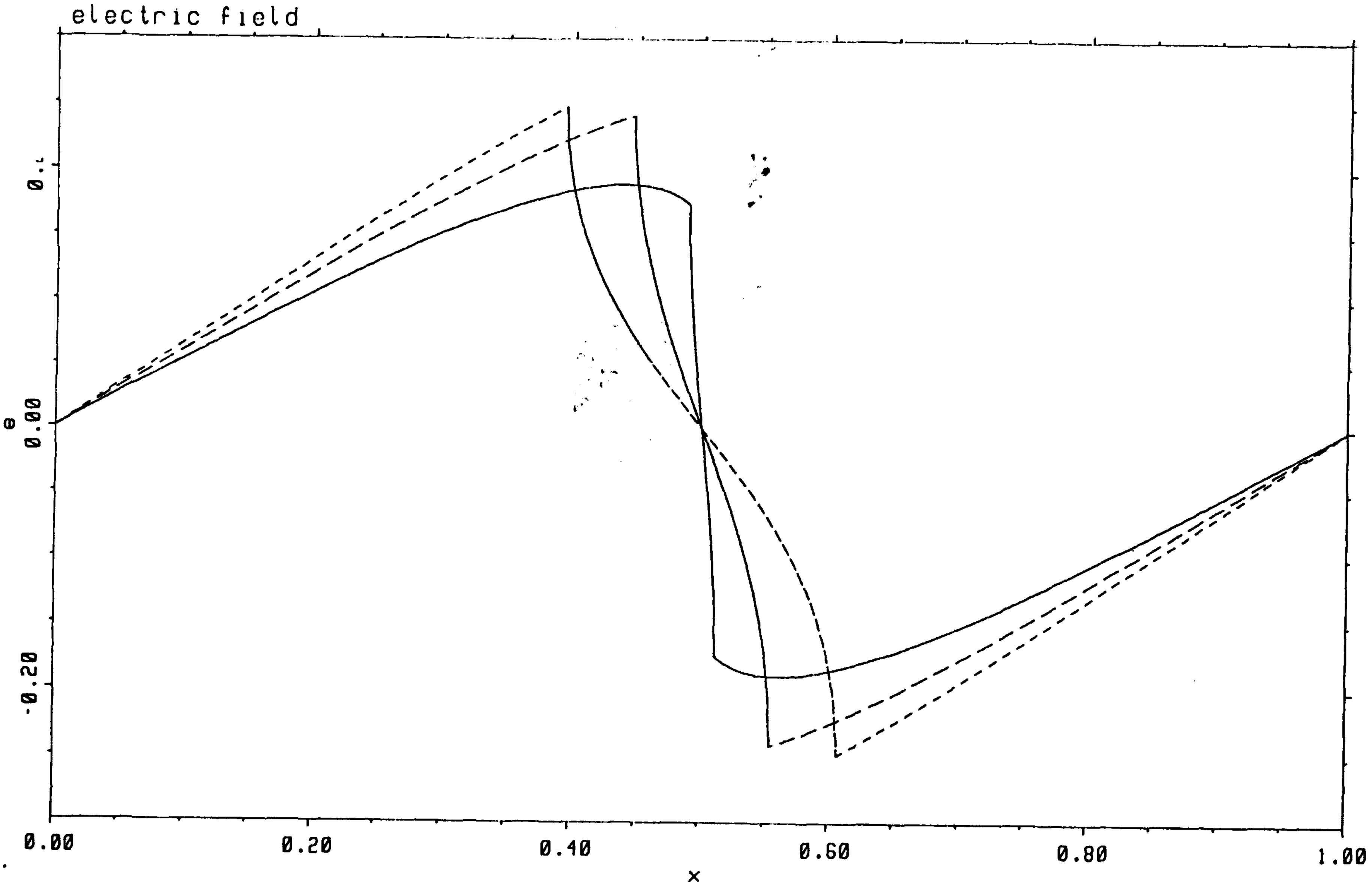

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