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FINITE ELEMENT SOLUTION TO THE HELMHOLTZ EQUATION WITH HIGH WAVE NUMBER

PART I: THE h-VERSION OF THE FEM

by

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20. ABSTRACT

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Finite Element Solution to the Helmholtz Equation with High Wave Number Part I: The h-version of the FEM

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The h-version of the finite element method with piecewise linear approximation has been applied to solve a one-dimensional model problem for the Helmholtz equation. The main practical purpose of the investigation is to lay theoretical ground for safe "rules of the thumb" how to choose the meshwidth of the FE-model depending on the wavenumber. In this context we present new results for stability and error estimation of the FE-solution. Following the analysis, numerical results are discussed. In a second paper we will study the h-p-method for Helmholtz problems.

1 Introduction

Boundary value problems for the Helmholtz equation arise in a number of physical applications [DL], in particular in problems of wave scattering and fluid-solid-interaction [JF]. If we analyze the scattering from an elastic body embedded in a fluid, analytical solutions can be provided for regular shapes of the body (like, egs., a sphere or a cylinder [JF]). Numerical methods need to be applied if the body is of general shape. Here, the physically proper and numerically effective modeling of large exterior domains is the main difficulty. Most numerical solutions have been given starting from the Helmholtz integral equation applying boundary element methods. However, several difficulties are reported from practical applications and finite element techniques are used increasingly not only for the solid but also in the fluid domain (cf. [HH2], [Bu]). In this context, the numerical analysis of the finite element method applied to Helmholtz problems becomes of practical interest.

Analytical results for the finite element solution of two point value Helmholtz problems in one dimension with Robin boundary conditions are contained in [AKS] and [DSSS]. Proofs of existence-uniqueness are given for the exact and the finite element solution (h-version) and asymptotic error estimates are proved under the assumption that the stepwidth h is small s.t. the magnitude hk^2 (where k is the wavenumber) is small. This assumption is obviously a severe restriction for practical applications if the wavenumber is high. "Rules of the thumb" usid in engineering analysis of acoustic problems are given in the form $hk \equiv$ const (cf. [HH1, p.71]). Some initial experience from numerical experiments in fluid-structure-interaction had, however, shown that this rule failed in the very case of high wavenumbers.

Hence the following questions had been the starting point for the analysis presented in this paper:

- Are the restrictions imposed on h and k in [AKS] and [DSSS] indeed necessary for stability of the FE-solution (or due to technicalities in the proof)?
- What is the proper "rule of the thumb" for high wavenumbers?
- what are the numerical and "economical" effects of the h-p-version compared to the h-version?

As in [DSSS] and [AKS], we address these issues on a one dimensional model problem. A two-point boundary value problem for the Helmholtz equation with Dirichlet and Robin boundary conditions is considered. We start with a recollection of results for existence, uniqueness and stability of the exact solution in the strong sense. We then introduce a variational formulation for the problem, show existence-uniqueness for the weak solution and compute the Babuška-Brezzi stability constant. These results form the prerequisite for the main objective, the study of the finite element solution. The analytical results of this study are contained in section 3. We first (3.1.) formulate and prove a statement of existence-uniqueness for the FE-solution using the argument given by Douglas et al. in [DSSS]. The proof is outlined in detail in order to keep track of all restrictions that have to be made for h and k. The essence of the argument is that the FE-solution is quasioptimal (w.r. to k) provided the magnitude of hk^2 is sufficiently small. However, quasioptimality is more than what is needed in practical application where (1) stability of the discrete model and (2) error estimation at finite range are the main concern.

Of these two issues, we first address stability and show that on the finite-dimensional level the B-B-constant is the same as in the original problem provided that the magnitude of hk (!) is sufficiently small. We then turn to error estimation and show that the error is bounded if hk and h^2k^3 are appropriately constrained. Again this is proved by using an assumption on hk only. This error estimate is the quantitative equivalent to the observation (cf. [HH1]) that in general the error of the finite element solution is influenced by a phase lag between the exact and the finite element models.

In the numerical evaluation we present results from various computational experiments, applying and illustrating the main results of our study. We show, in particular, that the restriction of hk^2 is indeed necessary for quasioptimality of the finite element solution.

2 The Reduced Wave Equation in One Dimension

In this section we prove existence-uniqueness of the solution to the one-dimensional reduced wave equation with Dirichlet and nonreflecting boundary conditions. We analyze the cases $u \in H^2(0,1)$ and $u \in H^1(0,1)$ separately and show that different stability conditions apply for these two cases. The construction of the Green's function to the problem is essential to both proofs.

2.1 The boundary value problem

Let $\Omega = (0,1)$ and let on $\overline{\Omega}$ the BVP $Lu = -f$ be given:

$$u''(x) + k^2 u(x) = -f(x) \tag{2.1}$$

$$u(0) = 0 \tag{2.2}$$

$$u'(1) - iku(1) = 0 \tag{2.3}$$

where, for simplicity, $f(x) \in C^1(0,1)$ and $k \equiv$ const., $k \in \mathbb{R}, k > 0$.
Physically, if u is the variation of pressure in an acoustic medium at a fixed time, eq (2.1) is the equation of a plane wave with (nondimensional) wave number

$$k = \frac{\omega L}{c}$$

where ω is a given frequency, L is the measure of the domain and c is the speed of sound in the acoustic medium.
In $x = 0$ a Dirichlet boundary condition is given (prescribed pressure); the mixed boundary condition in $x = 1$ is the one-dimensional Robin condition.

Notation: As usual, we will denote by $L^2(0,1) := H^o(0,1)$ the space of all square-integrable complex-valued functions equipped with the inner product

$$(v,w) := \int_0^1 v(x)\bar{w}(x)dx$$

and the norm

$$\|w\| := \sqrt{(w,w)}.$$

By $H^s(0,1), s = 1 \vee 2$ we denote the Sobolev space

$$H^s = \left\{u | u \in L^2 \wedge \partial u_i \in L^2, i = 1 \ldots s\right\}$$

where ∂u_i are the derivatives of order i in the distribution sense. The norm of the space $H^1(0,1)$ is defined as

$$\|u\|_1 := \left(\|u\|^2 + \|u'\|^2\right)^{\frac{1}{2}}.$$

Functions from H^1 with Dirichlet boundary data can be measured equivalently by the H^1-seminorm

$$|u|_1 := \|u'\|.$$

Uniqueness of the solution in $H^2(0,1)$: The BVP (2.1-2.3) has unique solution in the space $H^2(0,1)$.
Indeed, suppose there exist two solutions u_1 and u_2 to the BVP (2.1 - 2.3), then $u(x) = u_1(x) - u_2(x) \neq 0$ is a solution of (2.1 – 2.3) with homogeneous data $f(x) \equiv 0$. Then $u(x)$ is a solution in the classical sense and the general solution of eqs (1) and (2) is $u(x) = C\sin(x)$. Substitution into (3) then gives

$$Ck(\cos k - i \sin k) = 0.$$

Since

$$|\cos k - i \sin k| = 1$$

we have $C = 0$ and thus

$$u(x) \equiv 0$$

which is a contradiction and uniqueness is proved.
The existence of the solution is concluded from the following construction.

Inverse Operator: The Green's function of the BVP (2.1-2.3) is:

$$G(x,s) = \frac{1}{k}\begin{cases} \sin kx e^{iks}; & 0 \le x \le s \\ \sin ks e^{ikx}; & s \le x \le 1 \end{cases} \tag{2.4}$$

The solution $u(x)$ of (1)-(3) exists for all $k > 0$ and can be written as

$$\begin{aligned} u(x) &= \int_0^1 G(x,s)f(s)ds \\ &= \frac{1}{k}\left(\sin kx \int_x^1 \cos ks f(s)ds + \cos kx \int_0^x \sin ks f(s)ds \right. \\ &+ \left. i \sin kx \int_0^1 \sin ks f(s)ds\right). \end{aligned} \tag{2.5}$$

Furthermore, integrating in eq (2.5) by parts we see that $u \in C^2(0,1)$.

Using the Green's function we now establish bounds of the exact solution and it's derivatives by the data f.

Lemma 1 *Let $u \in H^2(0,1)$ be the solution to the BVP (2.1-2.3) for given data $f \in L^2(0,1) = H^o(0,1)$. Then*

$$\|u\| \le \frac{1}{k}\|f\| \tag{2.6}$$

$$\|u'\| \le \|f\| \tag{2.7}$$

$$\|u''\| \le (1+k)\|f\|. \tag{2.8}$$

Proof: Estimates (2.6) and (2.7) follow immediately from eq (2.5).

The estimate for the second derivative is obtained from

$$\|u''\|^2 = \int_0^1 (u'')^2 = \int (f - k^2 u)^2 = \int_0^1 f^2 - 2k^2 \int_0^1 fu + k^4 \|u\|^2$$

$$\le \|f\|^2 + 2k^2 \|f\| \, \|u\| + k^4 \|u\|^2$$

where the Cauchy-Schwarz inequality has been applied. With eq (2.6) we then get

$$\|u''\|^2 \le \|f\|^2 + 2k^2 \|f\| \frac{1}{k}\|f\| + k^2 \|f\|^2 = (1+k)^2 \|f\|^2 \tag{2.9}$$

which is the desired result.

Remark 1: It can be easily seen that all aforementioned results are valid also for the adjoint problem (2.1), (2.2) and

$$u'(1) + iku(1) = 0.$$

2.2 Variational formulation and weak solution

A variational formulation of the BVP (2.1-2.3) can be obtained formally by multiplying eq (1) with the complex conjugate of a suitably choosen testing function v. Then, after partial integration and substitution of eqs (2,3) we arrive at the variational problem

$$\mathcal{B}(u,v) = \int_0^1 \left(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\right) dx - iku(1)\bar{v}(1) = \mathcal{F}(v) \tag{2.10}$$

where

$$\mathcal{F}(v) = \int_0^1 f(x)\bar{v}(x)dx. \tag{2.11}$$

If, in general, there are given Hilbert spaces V^1 and V^2 and $u \in V^1$ (the *trial space*), $v \in V^2$ (the *test space*) then $\mathcal{B}$ is a sesquilinear form

$$\mathcal{B} : V^1 \times V^2 \to \mathbb{C},$$

$\mathcal{F}$ is a functional

$$\mathcal{F} : V^2 \to \mathbb{C}$$

and a function $u \in V^1$ is called a weak solution of (2.10) if

$$\mathcal{B}(u,v) = \mathcal{F}(v) \tag{2.12}$$

for all $v \in V^2$.

In our case, the natural choices for the trial and test spaces are

$$V^1 = V^2 = V = \left\{v \in H^1(0,1) \wedge v(0) = 0\right\} \tag{2.13}$$

For test functions $v \in H^1(0,1)$, the problem (2.10) is well defined if the data f lies at least in the dual space

$$H^{-1}(0,1) := \left\{ f \;\middle|\; |f|_{-1} := \sup_{v \in H^1} \frac{|\int_0^1 fv|}{|v|_1} < \infty \right\}.$$

Note that the variational problem (2.10) is equivalent to the BVP (2.1-2.3) in the sense that for sufficiently smooth data any weak solution of (2.10) is a "strong" solution of (2.1-2.3).

Continuity of the form B: Applying Poincaré's inequality we obtain elementarily the continuity estimate

$$|\mathcal{B}(u,v)| \leq C_o(k)|u|_1|v|_1$$

with $C_o = 1 + k + k^2$.

Existence-uniqueness of the weak solution: We first show uniqueness. Suppose again that there exist two solutions $u_1, u_2 \in \tilde{U}$ to the variational problem (2.10). Then $u = u_1 - u_2 \neq 0$ is a homogeneous solution; i.e. $u \in V$ and eq (2.10) holds with $\mathcal{F}(v) = 0$. In particular we have for $v = u$:

$$\mathcal{B}(u,u) = \int_0^1 \left(u'(x)\bar{u}'(x) - k^2 u(x)\bar{u}(x)\right) dx - iku(1)\bar{u}(1) = 0.$$

Since the right-hand side is real this equation can be true only if

$$u(1) = 0.$$

Then it follows from eq (2.3) that

$$\forall v \in V: \quad \int_0^1 u'\bar{v}'dx = k^2 \int_0^1 u\bar{v}dx$$

and hence for $v = x$:

$$0 = u(1) - u(0) = \int_0^1 u'dx = k^2 \int_0^1 uxdx.$$

Assume now $\int_0^1 ux^n dx = 0$ for some natural n, then partial integration yields

$$0 = -\frac{1}{n+1}\int_0^1 u'x^{n+1}dx = \frac{k^2}{(n+1)(n+2)}\int_0^1 ux^{n+2}dx.$$

It follows by incuction that

$$0 = \int_0^1 ux^s dx \ , s = 1,3,5,\ldots$$

Since, as a consequence from Müntz's theorem [A, p. 45], the set

$$\text{span}\,\{x^s \,|s = 1,3,5,\ldots\}$$

is dense in $L^2(0,1)$ we conclude that $u \equiv 0$. This is a contradiction to the assumption and uniqueness is proved.

For the proof of existence we observe that for the form $\mathcal{B}$ a Gårdings inequality

$$Re(\mathcal{B}(u,u)) + C\|u\|^2 \geq \|u\|_1^2 \tag{2.14}$$

(where $\|u\|_1 = (\|u\|^2 + \|u'\|^2)^{1/2}$ is the H^1-norm) holds for $C = C(k) = 1 + k^2$.
We then have (see, e.g., [J, p. 194]) the alternative statement: either there exists a non-trivial solution of the homogeneous problem $Lu = 0$ with Dirichlet data 0 or a solution of $Lu = f$ with Dirichlet data 0 exists for every sufficiently regular f. Since uniqueness has been proved existence follows. The proof is completed.

Remark 2: As in the strong case we remark that existence-uniqueness holds obviously also for the adjoint form

$$\mathcal{B}^*(u,v) = \int_0^1 \left(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\right)dx + iku(1)\bar{v}(1).$$

Remark 3: All statements made so far hold also for the Dirichlet condition $u(0) = g$. In that case, the set of admissible functions u is given by

$$\tilde{U} = \left\{\tilde{u} \in H^1 \wedge \tilde{u} = g\right\},$$

this set is related to the trial space by the bijective map

$$\tilde{u} = u + u^*$$

where $u \in V$ and u^* is an arbitrarily fixed element in $\tilde{U}$ (see, e.g., [SB, pp 16/17] for further detail).

Stability in H^1-norm and Babuška-Brezzi-constant: In Lemma 1, stability of the exact solution has been proved for the "strong" case $f \in H^0(0,1); u \in H^2(0,1)$. However, recent computations [D] indicated that the stability estimates of Lemma 1 do not hold for the weak case $f \in H^{-1}(0,1); u \in H^1(0,1)$. Indeed we will show the following

Theorem 1 *Let $V \subset H^1(0,1)$ be the Hilbert space defined in eq(2.13). Then for the sesquilinear form $\mathcal{B} : V \times V \to \mathbf{C}$ defined in eq(2.10) the Babuška-Brezzi stability constant*

$$\gamma := \inf_{u \in V} \sup_{v \in V} \frac{|\mathcal{B}(u,v)|}{|u|_1 |v|_1}$$

is of order $\frac{1}{k}$; more precisely, there exist positive constants C_1, C_2 not depending on k s.t.

$$\frac{C_1}{k} \leq \gamma \leq \frac{C_2}{k}. \tag{2.15}$$

Proof: Let us first proof the left inequality of (2.15). We will show that for any given $u \in V$ there exists an element $v \in V$ s.t.

$$|\mathcal{B}(u,v)| \geq \frac{C}{k} \|u'\| \, \|v'\|. \tag{2.16}$$

Let $u \in V$ be given. Define $v := u + z$ where z is a solution of the problem

$$\forall w \in V : \quad \mathcal{B}(w,z) = k^2(w,u). \tag{2.17}$$

The solution z exists and is uniquely defined. Furthermore, $z \in H^2(0,1)$ and is a solution of the BVP

$$\begin{aligned} z'' + k^2 z &= -k^2 u \\ z(0) &= 0 \\ z'(1) &= ikz(1). \end{aligned}$$

Then z is the Green's function transform of $f(s) = k^2 u(s)$.

The proof proceeds as follows: With $v = u + z$ we have

$$\begin{aligned} |\mathcal{B}(u,v)| &\geq \operatorname{Re} \mathcal{B}(u,v) \\ &= \operatorname{Re} \left(\mathcal{B}(u,u) + \mathcal{B}(u,z)\right) \\ &= \operatorname{Re} \left(\mathcal{B}(u,u) + \mathcal{B}(u,z) + k^2(u,u) - k^2(u,u)\right) \\ &= \operatorname{Re} \mathcal{B}(u,u) + k^2\|u\|^2 = \|u'\|^2. \end{aligned}$$

Then, if we show that

$$\|u'\| \geq \frac{C}{k} \|v'\| \tag{2.18}$$

we have proved ineq. (2.16) and the inf-sup-condition follows.

To obtain ineq. (2.18) we integrate by parts the function z:

$$\begin{aligned} z(x) &= k^2 \int_0^1 G(x,s) u(s) ds \\ &= k^2 \left(H(x,1)u(1) - \int_0^1 H(x,s) u'(s) ds \right) \end{aligned}$$

where

$$H(x,s) := \int_0^s G(x,t)dt$$

and the integral on the r.h.s. is well defined since $u \in H^1(0,1)$.
We now differentiate

$$z'(x) = k^2 \left(H_x(x,1)u(1) - \int_0^1 H_x(x,s)u'(s)ds \right)$$

where the function H_x is readily computed as

$$H_x(x,s) = -\frac{i}{k} \begin{cases} \cos kx e^{iks} & 0 \le x \le s \\ \cos ks e^{ikx} & s \le x \le 1 \end{cases}$$

Then

$$\begin{aligned} |z'(x)| &\le k^2 \left(|H_x(x,1)||u(1)| + \int_0^1 |H_x(x,s)u'(s)|ds \right) \\ &\le k^2 \left(|H_x(x,1)| + \|H_x\| \right) \|u'\| \le 2k\|u'\| \end{aligned}$$

since obviously $|H_x(x,1)| \le \frac{1}{k}, \|H_x\| \le \frac{1}{k}$.
Hence

$$\|v'\| = \|u' + z'\| \le \|u'\| + \|z'\| \le (1+2k)\|u'\|$$

or

$$\|u'\| \ge \frac{1}{1+2k}\|v'\| \ge \frac{C}{k}\|v'\|$$

for $k > 1$ and the first part of the proof is completed.

To proof the second inequality it is sufficient to find some function $z_o(x) \in V$ for which

$$\forall v: \quad \frac{|B(z_o,v)|}{|z_o|_1} \le \frac{C}{k}|v|_1.$$

Consider the function

$$z_o(x) = \varphi(x)\frac{\sin kx}{k}$$

where $\varphi \in C^\infty(0,1)$ does not depend on k and is choosen s.t.

$$z_o(0) = z_o(1) = z_o'(0) = z_o'(1) = 0 \tag{2.19}$$

and that for some $\alpha > 0$, not depending on k ,

$$|z_o|_1 \ge \alpha$$

(take, e.g. $\varphi(x) = x(x-1)^2$).
Then

$$\forall v \in V: \quad \frac{|B(z_o,v)|}{|z_o|_1} \le \frac{1}{\alpha}|B(z_o,v)|$$

and with eqs (2.19) we obtain by partial integration

$$\forall v \in V: \quad B(z_o,v) = -\int_0^1 (z_o'' + k^2 z_o)\bar{v}.$$

Direct computation shows that

$$z_o'' + k^2 z_o = \varphi'' \frac{\sin kx}{k} + 2\varphi'(x)\cos kx.$$

Define

$$u(x) := \int_0^x \left(z_o''(s) + k^2 z_o(s) \right) ds,$$

then

$$|\mathcal{B}(z_o, v)| = \left| u(1)\bar{v}(1) - \int_0^1 u(x)\bar{v}'(x)\, dx \right| \leq (|u(1)| + \|u\|)|v|_1.$$

On the other hand, from the definition of u we obtain integrating by parts

$$\begin{aligned} |u(x)| &= \left| \int_0^x \left(\varphi''(s) \frac{\sin ks}{k} ds + 2\varphi' \cos ks \right) ds \right| \\ &= \left| 2\varphi'(x) \frac{\sin kx}{k} - \int_0^x \left(\varphi''(s) \frac{\sin ks}{k} \right) ds \right| \end{aligned}$$

Hence

$$\begin{aligned} |u(1)| &\leq \frac{1}{k} \|\varphi''\|_\infty \\ \|u\| &\leq \frac{1}{k} \left(\|\varphi''\|_\infty + 2\|\varphi'\|_\infty \right) \end{aligned}$$

so there exists a constant C s.t.

$$(|u(1)| + \|u\|) \leq \frac{C}{k}.$$

Consequently,

$$\forall v \in V: \quad |\mathcal{B}(z_o, v)| \leq \frac{C}{k} |v|_1$$

and the proof is completed.

From general theory [BA, p.112] now follows

Corollary 1 *Let $u \in H^1(0,1)$ be a solution of the variational problem (2.10) with given data f. Then the stability estimate*

$$|u|_1 \leq Ck|f|_{-1}$$

holds where C is a generic constant not depending on k.

3 Finite Element Analysis of the Wave Equation

Following preliminary definitions we state approximability of the exact solution as a direct conclusion from the approximation properties of the finite element space and stability of the exact solution (3.1). We apply an asymptotic approach ([AKS], [DSSS]) to obtain statements of stability and quasioptimal error estimates for the model problem (3.2).
In the second part of our analysis we study the properties of the finite element solution in the preasymptotic range. In a preliminary subsection (3.3) we analyze the discrete model on uniform mesh and construct a discrete Green's function. We then show the inf-sup condition (3.4) and state a preasymptotic error estimate (3.5). The section is concluded with some comments (3.6).

3.1 Approximability of the exact solution in the finite element space

Preliminaries: To solve the problem numerically with a finite element method, the interval $\overline{\Omega} = [0,1]$ is divided into n *finite elements* $[x_{j-1}, x_j]$

$$\bar{\Omega} = \bigcup_{j=1}^{n} [x_{j-1}, x_j] \ , \ 0 = x_o < x_1 < \ldots < x_n = 1. \tag{3.1}$$

By (3.1) a discrete subset

$$X_h = \{x_j, j = 0, 1, \ldots, n\} \subset [0,1] \tag{3.2}$$

(*finite element mesh*) is given on Ω . Any function defined on X_h is called a *mesh function* and will be referred to by subscript h.
We will denote by $h_j = (x_j - x_{j-1})$ the *size* of the finite element # j and define the *stepsize* h of the mesh X_h by

$$h = \max_j h_j. \tag{3.3}$$

In the following we will seek approximate solutions of the variational problem (2.10) using the Galerkin finite element method with piecewise linear test functions.

More precisely, we define the subspace

$$S_h(0,1) \subset H^1(0,1)$$

as the set of all functions $u \in H^1(0,1)$ such that the restriction of u to any element $[x_{i-1}, x_i]$ is a linear function. On each finite element $[x_{j-1}, x_j]$, a function $v \in S_h(0,1)$ is written by means of the *nodal shape functions* [SB, p.38] $N_1^{(j)}(x), N_2^{(j)}(x)$ as

$$v(x) = v_{j-1} N_1^{(j)}(x) + v_j N_2^{(j)}(x)$$

where v_{j-1}, v_j are the nodal values of v in x_{j-1}, x_j, respectively.

An admissible function $u_{fe}(x,h) \in H^1(\Omega)$ will be called a *finite element solution* to the variational problem (2.10) if

1. $u_{fe} \in S_h(0,1)$;

2. $u_{fe}(0) = g$;

3. u_{fe} is a solution to the variational problem (2.10) for all test functions $v \in V_h$

where

$$V_h := S_h[0,1[= \{v \in S_h(0,1) \wedge v(0) = 0\}.$$

Remark 4: As in the continuous (cf. Remark 2) case, test and trial space are identical in the discrete case:

$$V^1 = V^2 = S_h[0,1[.$$

Approximation properties of $S_h[0,1[$: It is a well known fact that, for any function $u \in H^1(\Omega)$, in one dimension the best approximation on given mesh X_h is obtained by the linear interpolant u_I of u. Furthermore, if $u \in H^2(\Omega)$, the following statements hold:

Lemma 2 *Let $u \in H^2(0,1)$ and $u_I \in S_h(0,1)$ be the piecewise linear interpolant of u. Then*

$$\inf_{v \in S_h} \|u - v\| \leq \|u - u_I\| \leq \left(\frac{h}{\pi}\right)^2 \|u''\| \tag{3.4}$$

$$\inf_{v \in S_h} |u - v|_1 = |u - u_I|_1 \leq \left(\frac{h}{\pi}\right) \|u''\| \tag{3.5}$$

$$\|u - u_I\| \leq \left(\frac{h}{\pi}\right) |u - v|_1 \tag{3.6}$$

Proof: see, e.g., [SF, p. 45].

A statement of approximability is now immediately obtained:

Theorem 2 *Let $u \in H^2(\Omega)$ be the solution of the variational problem (2.10) - or, equaivalently, of the BVP (2.1-2.3) - for given data $f \in L^2(\Omega)$.*
Then the for the error of the best approximation in H^1-seminorm there holds

$$|u - u_I|_1 \leq \frac{h}{\pi}(1 + k)\|f\|.$$

Proof: The statement follows directly from Lemmas 1 and 2.

Remark 5: The practical conclusion from this theorem is that, for any given f, we can control the approximation error by bounding appropriately the magnitude of hk. More precisely, for any given data $f \in L^2(0,1)$ and error bound $\varepsilon > 0$ there exists $\delta > 0$ s.t. for $hk < \delta$

$$\inf_{v \in S_h} |u - v|_1 = |u - u_I|_1 < \varepsilon.$$

Since the wave number k is related to the (nondimensional) wavelength λ by

$$k = \frac{2\pi}{\lambda}$$

the product kh is a measure of the number of elements per wavelength.Hence the "rules of the thumb" recommending a certain number of mesh points per wave length *do apply*

effectively for approximability (of the exact solution) by piecewise linears.

The essential question is, however, if this rule does also apply for the finite element solution. Since the approximation property has been established, the answer to this question would now be given by an stability estimate of the form

$$|u - u_{fe}|_1 \leq C_s \inf_{v \in S_h} |u - v|_1$$

where the stability constant C_s, in general, depends on k. Before turning to this estimate we verify that the variational problem (2.10) is well-defined also an the finite level.

3.2 Asymptotic stability and quasioptimal error estimates

A proof of existence-uniqueness in terms of asymptotic error estimates of the FE-solution for the one-dimensional Helmholtz equation with non-reflecting boundary conditions has recently been given by Douglas et al. [DSSS]. Since we are interested in the dependence of error estimates on h and k, specifically in the case of large k, we outline here the argument from [DSSS] (slightly modified to account for the Dirichlet condition at $x = 0$) in detail to keep close track of the constants involved in the estimates.

Theorem 3 *Let $u \in H^2(0,1)$ be the exact and $u_{fe} \in S_h(0,1)$ be the finite element solutions of the BVP (2.1-2.3), respectively.*
The finite element solution is then uniquely determind by any data $f \in L^2(0,1)$ and, furthermore, the following error estimates hold

$$\|u - u_{fe}\| \leq C_1 C_2 (1+k)^2 h^2 \|f\| \tag{3.7}$$

$$\|u' - u'_{fe}\| \leq C_2 (1+k) h \|f\| \tag{3.8}$$

with

$$C_1 := \frac{2}{(1 - 2(1+k)\frac{k^2h^2}{\pi^2})\pi}$$

and

$$C_2 := \frac{2\left(1 + \left(\frac{hk}{2\pi}\right)^2\right)^{\frac{1}{2}}}{\pi\left(\frac{1}{2} - 6C_1^2k^2h^2(1+k)^2\right)^{\frac{1}{2}}}$$

provided that the stepwidth h and the wavenumber k are such that the denominators of the constants are positive.

Proof: Denote $e := u - u_{fe}$. Then e lies in the Hilbert space $V \subset H^1(0,1)$ and, consequently (cf. remark 3), there exists $z \in V$ s.t.

$$\forall v \in V: \quad \mathcal{B}(v,z) = (v,e).$$

In particular, $\mathcal{B}(e,z) = (e,e)$ for $v = e$.

Further the error is $\mathcal{B}$-orthogonal to the discrete test space $V_h := S_h[0,1[$:

$$\forall w \in V_h: \quad \mathcal{B}(e,w) = 0.$$

Then, for all $w \in V_h$,

$$\begin{aligned} \|e\|^2 = (e,e) &= \mathcal{B}(e, z-w) \\ &= \int e'\,(\overline{z-w})' - k^2 \int e\,(\overline{z-w}) - ike(1)\,(\overline{z(1)-w(1)}) \\ &\leq \|(z-w)'\|\,\|e'\| + k^2\|z-w\|\,\|e\| + k|z(1)-w(1)||e(1)| \end{aligned}$$

Apply the inequality $|v(1)| \leq \sqrt{2}\|v\|^{\frac{1}{2}}\|v'\|^{\frac{1}{2}}$ which is true for all $v \in V$ to obtain

$$\begin{aligned} k|z(1)-w(1)||e(1)| &\leq 2k\|(z-w)'\|^{\frac{1}{2}}\|e'\|^{\frac{1}{2}}\|z-w\|^{\frac{1}{2}}\|e\|^{\frac{1}{2}} \\ &\leq k^2\|z-w\|\,\|e\| + \|(z-w)'\|\,\|e'\| \end{aligned} \tag{3.9}$$

where the inequality $2ab \leq a^2 + b^2$ has been applied.
This gives, for all $w \in V_h$,

$$\|e\|^2 \leq 2\left(\|(z-w)'\|\,\|e'\| + k^2\|z-w\|\,\|e\|\right).$$

In particular we may apply Lemmas 1 and 2 for $w = z_I \in V_h$ (the piecewise linear interpolant of z) to obtain

$$\begin{aligned} \|e\|^2 &\leq \left(\|(z-z_I)'\|\,\|e'\| + k^2\|z-z_I\|\,\|e\|\right) \\ &\leq 2\left((1+k)\frac{h}{\pi}\|e'\|\,\|e\| + k^2\frac{h^2}{\pi^2}(1+k)\,\|e\|^2\right). \end{aligned}$$

Divide both sides of the inequality above by the common factor $\|e\|$, then

$$\|e\| \leq C_1\,(1+k)\,h\,\|e'\| \tag{3.10}$$

holds with

$$C_1 := \frac{2}{(1-2(1+k)\frac{k^2h^2}{\pi^2})\,\pi}.$$

under the assumption that k, h are such that the denominator of C_1 is positive.

Next, from $\mathcal{B}$-orthogonality of the error to elements from V_h we have

$$\mathcal{B}(e,e) = \mathcal{B}(e, u - u_{fe}) = \mathcal{B}(e,u)$$

and hence

$$\forall v \in V_h: \quad \mathcal{B}(e,e) = \mathcal{B}(e, u-v).$$

Thus, for all $v \in V_h$

$$\int e'\overline{e}' - k^2\int e\overline{e} - ik|e(1)|^2 = \int e'(\overline{u-v})' - k^2\int e(\overline{u-v}) - ike(1)(\bar{u}(1)-\bar{v}(1))$$

and therefore

$$\begin{aligned} \|e'\|^2 &\leq k^2\|e\|^2 + k|e(1)|^2 + \|e'\|\,\|(u-v)'\| + k^2\|e\|\,\|u-v\| + k|e(1)|\,|u(1)-v(1)| \\ &\leq k^2\|e\|^2 + 2k\|e'\|\,\|e\| + 2\|e'\|\,\|(u-v)'\| + 2k^2\|e\|\,\|u-v\| \end{aligned}$$

where the terms in $x = 1$ have been estimated as in (3.9). We now use the ε-inequality to get the estimates

$$\begin{aligned} 2k\,\|e'\|\,\|e\| &\leq \frac{1}{4}\,\|e'\|^2 + 4k^2\|e\|^2 \\ 2\,\|e'\|\,\|(u-v)'\| &\leq \frac{1}{4}\,\|e'\|^2 + 4\|(u-v)'\|^2 \\ 2k^2\,\|e\|\,\|u-v\| &\leq k^2\,\|e\|^2 + k^2\|u-v\|^2. \end{aligned}$$

Introducing these estimates into the inequality leads to

$$\forall v \in V_h: \quad \|e'\|^2 \leq 6k^2\|e\|^2 + \frac{1}{2}\|e'\|^2 + 4\|(u-v)'\|^2 + k^2\|u-v\|^2. \tag{3.11}$$

Then, using the intermediary result (3.10) and the approximation results from Lemma 2 for $v = u_I$, we get

$$\frac{1}{2}\|e'\|^2 \leq 6k^2(1+k)^2C_1^2h^2\|e'\|^2 + 4\left(\frac{h}{\pi}\right)^2(1+k)^2\|f\|^2 + k^2(1+k)^2\left(\frac{h}{\pi}\right)^4\|f\|^2.$$

and hence

$$\left(\frac{1}{2} - 6k^2(1+k)^2C_1^2h^2)^{\frac{1}{2}}\right)\|e'\| \leq \left(\frac{2}{\pi}\right)\left(1+\left(\frac{hk}{2\pi}\right)^2\right)^{\frac{1}{2}} h(1+k)\|f\|$$

and the statement of the theorem follows . The proof is completed.

Remark 6: For the denominator of C_2 to be positive, the magnitudes of $(hk)^2$, h^2k^3 and h^2k^4 need to be small. The term $(hk/2\pi)^2$ in the numerator can then be omitted.

Let us state as a corollary:

Corollary 2 *With the assumptions of the theorem, the estimate*

$$|u_{ex} - u_{fe}|_1 \leq C_s \inf_{v\in V_h} |u_{ex} - v|_1 \tag{3.12}$$

holds for

$$C_s := \frac{2\left(1+\left(\frac{hk}{2\pi}\right)^2\right)^{\frac{1}{2}}}{\left(\frac{1}{2} - 6C_1^2k^2h^2(1+k)^2\right)^{\frac{1}{2}}}$$

with

$$C_1 := \frac{2}{(1-2(1+k)\frac{k^2h^2}{\pi^2})\,\pi}.$$

Proof: Introduce eq (3.6) from Lemma 2 to (3.11).

Remark 7: Note that, except for the final estimates in terms of $\|f\|$, the proof of the theorem is valid also with the weaker assumptions $u \in H^1(0,1)$, $f \in H^{-1}(0,1)$. In particular we obtain the statement of quasioptimality from the corollary. However, the assumption that k^2h be small is essential.

Hence if h and k fulfill the assumptions of the theorem, the finite element solution behaves effectively like the best approximation (i.e. C_3 can be replaced by some absolute constant not depending on k) and the "rules of the thumb" apply for the FE-solution.

The assumptions of the theorem imply, however, that the magnitudes of hk, $hk^{\frac{3}{2}}$ and hk^2 have to be bounded by sufficiently small magnitudes (cf. also assumptions in [DSSS, p. 177]). The theorem (and the corallary as well) then states that, with these restrictions to h and k, the error of the fe-solution is quasioptimal. While this is the desired result it is achieved at high cost if k is large and the stepsize h must be choosen s.t. the magnitude hk^2 is small.

At this state of our investigation, it is not clear whether the assumptions of the theorem are due to technicalities of the proof or really inherent to the problem considered.

The second and more important question is whether the assumptions of the theorem are indeed necessary to bound the discretization error by some finite magnitude (like, egs., a given tolerance for the relative error). The following simple computation indicates that this is not the case for high k. Indeed, let

$$hk^2 \leq \alpha$$

for some $\alpha > 0$. Then $h \leq \alpha/k^2$ and

$$|u - u_{fe}|_1 \leq C_2(1+k)\frac{1}{k^2}\|f\|$$

hence the error estimates of the theorem *tend towards 0* (while they have only to be bounded for practical purposes) as k is increased.

We will state stability under weaker assumptions and give more appropriate error estimates in a preasymptotic analysis using a discrete Green's function approach on uniform mesh.

3.3 Preasymptotic analysis: Preliminaries

Global FE-equations and discrete fundamental system: Let in the following the FE-mesh be uniform with $h = \frac{1}{n}$. After assembling the local equations (2.10) and multiplying the whole set by h, we arrive at a set of linear equations for the mesh-function $u_h = u_{fe}|_{X_h}$:

$$L_h u_h = r_h \tag{3.13}$$

where the discrete operator L_h can be written as a $n \times n$-tridiagonal matrix

$$L_h = \begin{bmatrix} 2S(l) & R(l) & & & \\ R(l) & 2S(l) & R(l) & & \\ & & \ddots & & \\ & R(l) & 2S(l) & R(l) & \\ & & R(l) & S(l) - il & \end{bmatrix} \tag{3.14}$$

with

$$R(l) = -1 - \frac{l^2}{6}\ ,\ S(l) = 1 - \frac{l^2}{3}$$

and

$$l = hk.$$

The right hand side r_h is a mesh function obtained from

$$\forall j: \quad r_j = h\left(\int_{x_{j-1}}^{x_j} f(x)\, N_2^{(j)}(x)\, dx + \int_{x_j}^{x_{j+1}} f(x)\, N_1^{(j)}(x)\, dx\right). \tag{3.15}$$

Remark 8: As noted before, the product $t = kh$ is a measure of the number of elements per wavelength (of the exact solution). In particular, if the stepwidth is such that $t = \frac{\pi}{l}$ for integer l then exactly l elements are placed one one half-wave of the exact solution.

For later use we introduce difference notation as follows: Given a mesh function $u = u_h$ defined on X_h we will denote left and right differences, respectively, by

$$d^i u := \frac{u(x_i) - u(x_{i-1})}{h_i}; \quad D^i u := \frac{u(x_{i+1}) - u(x_i)}{h_{i+1}}.$$

In the linear space of mesh functions, an inner product in L^2-analogy is defined by

$$(f_h, g_h)_h = h\sum_{j=1}^{n} f_j \bar{g}_j.$$

We will denote the discrete L^2-norm defined by this inner product by $\|\cdot\|$. The discrete analogon to the H^1-seminorm is given by

$$|u_h|_1^2 = h\sum_{i=1}^{n} \left|d^i u_h\right|^2.$$

Note that for any piecewise linear function u with nodal points on X_h

$$|u|_1 = \|u'\|^2 = |u_h|_1,$$

i.e. the discrete and exact H^1-norms are identical. We will use the discrete Dirac symbol defined as

$$\delta_{ij} := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

Discrete wavenumber and Green's function: The fundamental system of eq (3.13) is

$$F_h = \left\{e^{-ik'x}, e^{ik'x} | x \in \{j/n; j = 0, 1, \ldots, n\}\right\} \tag{3.16}$$

where k' is a parameter to be yet determined.
To this end, we solve any of the "interior" equations in the point $x_j = j/n$, $1 < j < n$:

$$R(t)e^{ik'(j-1)h} + 2S(t)e^{ik'jh} + R(t)e^{ik'(j+1)h} = 0. \tag{3.17}$$

With

$$\lambda = e^{ik'h}$$

eq (3.17) has the solutions

$$\lambda_{1,2} = -\frac{S(t)}{R(t)} \pm \sqrt{\frac{S^2(t)}{R^2(t)} - 1} = \begin{cases} (*) \quad \text{complex conjugate} & \text{if } \left|\frac{S(t)}{R(t)}\right| < 1 \\ (**) \quad \text{real} & \text{if } \left|\frac{S(t)}{R(t)}\right| \geq 1 \end{cases} \tag{3.18}$$

From the definition of λ we see that the *discrete wave number* k' is either real (in case (*)) or pure complex (case (**)). Physically, case (*) describes a propagating wave whereas case (**) describes a decaying wave [HI1]. For sufficiently small h (more precisely, for $h \leq \sqrt{12}/k$) one obtains always the complex conjugate solution of case (*).

The discrete wavenumber k' can be formally computed in terms of t: From eq (3.18), case (*), we get

$$\cos(k'h) = -\frac{S(t)}{R(t)}. \tag{3.19}$$

and hence

$$k' = \frac{1}{k} \arccos\left(-\frac{S(t)}{R(t)}\right). \tag{3.20}$$

Consider the Taylor expansion

$$\begin{aligned} k'h &= \arccos\left(-\frac{S(t)}{R(t)}\right) \\ &= kh - \frac{(kh)^3}{24} + \frac{3(kh)^5}{640} + O\left((kh)^7\right). \end{aligned}$$

Hence, for fixed k.

$$k' = k - \frac{k^3h^2}{24} + 0(k^5h^4) \tag{3.21}$$

Once the discrete wavenumber has been computed, a discrete Green's function $G_h(x,s)$; $x = x_h, s = s_h$ can be constructed. We give next a brief outline of this construction referring to [Sa] for details.

Similarly to the continuous case, we require that the r.h.s. of the linear system (3.13) is mappped to the discrete solution of this system by

$$u_h(x) = (G_h(x,s).r_h(s))_h.$$

We accordingly seek the discrete Green's function in the form

$$G_h(x,s) = \begin{cases} C_1 \sin k'x & x \leq s \\ C_2(A \sin k'x + \cos k'x) & s \leq x \leq 1 \end{cases} \tag{3.22}$$

where C_1, C_2 are functions of s and the constant A is determined from the discrete equation in the nodal point $x_n = 1$ as

$$A = \frac{\sin k' \cos k'(R(t)^2 \sin^2 k'h - t^2) - itR(t) \sin k'h}{R(t)^2 \cos^2 k' \sin^2 k'h + t^2 \sin^2 k'}.$$

Since $\alpha(x) := \sin k'x$ and $\beta(x) := A \sin k'x + \cos k'x$ are fundamental solutions of eq. (3.13), we can prove by discrete Green's formula ([Sa, pp. 120/121]) that

$$\Delta(x_j) = (d^j\alpha)\beta - \alpha(d^j\beta) = \text{const} = \frac{\sin k'h}{h}$$

Using this formula we find from the condition

$$D^i\left(d^i G_{\cdot j}\right) = -\frac{\delta_{ij}}{h}$$

the unknown coefficients C_1, C_2 to be

$$C_1(s) = \frac{\beta(s)}{\Delta(s)}$$

$$C_2(s) = \frac{\alpha(s)}{\Delta(s)}$$

Introducing these results into eq (3.22) the discrete Green's function is

$$G_h(x,s) = \frac{1}{h \sin k'h} \begin{cases} \sin k'x\,(A \sin k's + \cos k's) & x \le s \\ \sin k's(A \sin k'x + \cos k'x) & s \le x \le 1 \end{cases} \tag{3.23}$$

and the discrete solution $u_h(x_h) = h\sum_{j=1}^{n} G_h(x_h, s_j) r_h(s_j)$ becomes

$$u_h(x_l) = \frac{1}{h \sin k'h}\left(\cos k'hl \sum_{j=1}^{l} r_j \sin k'hj + \sin k'hl \sum_{j=l+1}^{n} r_j \cos k'hj + A \sin k'hl \sum_{j=1}^{n} r_j \cos k'hj\right). \tag{3.24}$$

for $0 \le l \le n$.

Remark 9: A straightforward asymptotic analysis of the discrete solution shows that, for $h \to 0$ the coefficient A converges to i and $u_h(x_h)$ converges to the exact solution $u(x)$ as given in the previous section.

Remark 10: Using eq (3.19) the constant A in the Green's functions (eq 3.22) can be simplified to

$$A = \frac{t^2 \sin k' \cos k' + i\sqrt{12}\sqrt{12 - t^2}}{12 - t^2 \cos^2 k'}. \tag{3.25}$$

Obviously $|A|$ is bounded independently of k for $t = hk \le \alpha < \sqrt{12}$.

3.4 The inf-sup-Stability Condition for the Finite Element Solution

In this subsection, we will compute the Babuška-Brezzi stability constant of finite element solutions on uniform mesh using the discrete Green's function. Existence-uniqueness of the FE-solution then follows under weaker (compared to the proof outlined in the previous subsection) assumptions on h and k.

Stability of the finite element solution and discrete B-B-constant: The stability investigation of the form $\mathcal{B}$ on the finite level is proceeded in close analogy to the infinite-dimensional case as considered in section 2. Namely, we will prove

Theorem 4 *Let $V_h := S_h[0,1[\in H^1(0,1)$ and $\mathcal{B} : V_h \times V_h \to \mathbf{C}$ be the sesquilinear form defined by equation (2.10).*
Then, if the stepwidth $h \leq \frac{1}{k}$ (or, respectively, $hk \leq 1$), the Babuška-Brezzi stability condition

$$\inf_{u \in V_h} \sup_{v \in V_h} \frac{|\mathcal{B}(u,v)|}{|u|_1 |v|_1} = \gamma_h > 0 \tag{3.26}$$

holds and there exist positive constants C_1 and C_2, not depending on k or h s.t.

$$\frac{C_1}{k} \leq \gamma_h \leq \frac{C_2}{k}.$$

Proof: The line of proof is similar to the infinite-dimensional case: we will show that for any given $u \in V_h$ there exists some $v \in V_h$ s.t.

$$|\mathcal{B}(u,v)| \geq \frac{C}{k} \|u'\| \, \|v'\|.$$

Let hence $u \in V_h$ be given and define $v := u + z$ where $z \in V_h$ is a solution of the variational problem

$$\forall w \in V_h : \quad \mathcal{B}(w,z) = k^2 (w,u). \tag{3.27}$$

Since V_h is a Hilbert space, the solution of (3.27) exists and is uniquely defined.
As in the continuous case, we will now prove that

$$|u|_1 \geq \frac{C}{k} |v|_1$$

using the Green's function representation of z:

$$z_i = z_h(x_i) = h \sum_{j=1}^{n} G_{ij} r_j \tag{3.28}$$

where

$$G_{ij} := G_h(x_i, s_j); \quad r_j := r_h(s_j).$$

Summation by parts in eq. (3.28) yields

$$z_i = H_{in} r_n - H_{i1} r_o - h \sum_{j=1}^{n} H_{ij} d^j r \tag{3.29}$$

with

$$D^j H_{i\cdot} = G_{ij}, \quad j = 1, \ldots n-1. \tag{3.30}$$

Since the mesh function H is defined by eq (3.30) up to a constant we are free to choose

$$H_{i1} = 0.$$

Let us now take the left differences of z_h in some fixed point $i = l$:

$$d^l z = d^l H_{\cdot n} r_n - h \sum_{j=1}^{n} d^l H_{\cdot j} d^j r. \tag{3.31}$$

Then, applying the Schwarz inequality, we obtain the estimate

$$\begin{aligned} |d^l z| &\leq |d^l H_{\cdot n}||r_n| + \|H_x\||r|_1 \\ &\leq \left(|d^l H_{\cdot n}| + \|H_x\|\right)|r|_1. \end{aligned} \tag{3.32}$$

The right hand side of the variational problem is by direct computation

$$r_j = \frac{1}{6}k^2h^2(u_{j-1} + 4u_j + u_{j+1}), \quad j = 1 \ldots n-1$$

hence

$$|r|_1 \leq Ch^2k^2|u|_1 \tag{3.33}$$

where C is a constant of order 1.
We now turn to estimation of the magnitude $|d^l H_{\cdot n}| + \|H_x\|$.
From eq (3.30) we obtain after summation over j:

$$H_{ij} - H_{i1} = h\sum_{l=1}^{j-1} D^l H_{i\cdot} = h\sum_{l=1}^{j-1} G_{il}$$

and consequently, since $H_{i1} = 0$.

$$H_{ij} = h\sum_{l=1}^{j-1} G_{il} \ . \tag{3.34}$$

Taking left differences we obtain

$$d^i H_{\cdot j} = h\sum_{l=1}^{j-1} d^i G_{\cdot l} \tag{3.35}$$

The derivatives (as left differences) of the discrete Greens function are

$$d^i G_{\cdot l} = \frac{1}{h\sin k'h}\begin{cases} d^i \sin k'x_h\,(A\sin k's_l + \cos k's_l) & x_h \leq s_l \\ \sin k's_l\,(Ad^i \sin k'x_h + d^i \cos k'x_h) & x_h \geq s_l \end{cases} . \tag{3.36}$$

We substitude

$$\begin{aligned} d^i \sin k'x_h &= \frac{2}{h}\cos\left(\frac{k'h}{2}(2i-1)\right)\sin\frac{k'h}{2} \\ d^i \cos k'x_h &= -\frac{2}{h}\sin\left(\frac{k'h}{2}(2i-1)\right)\sin\frac{k'h}{2} \end{aligned}$$

to obtain

$$d^i G_{\cdot l} = \frac{1}{h^2\cos\frac{k'h}{2}}\begin{cases} \cos\left(\frac{k'h}{2}(2i-1)\right)(A\sin k's_l + \cos k's_l) & i \leq l \\ \sin k's_l\left(A\cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right) & i \geq l \end{cases} . \tag{3.37}$$

Then

$$\begin{aligned}
\sum_{l=1}^{j-1} d^i G_{\cdot l} &= \frac{1}{h^2 \cos\frac{k'h}{2}} \left(\cos\left(\frac{k'h}{2}(2i-1)\right) \left(A \sum_{l=1}^{j\vee i} \sin k'hl + \sum_{l=1}^{j\vee i} \cos k'hl \right) \right. \\
&+ \left. \sum_{l=1}^{j\vee i} \sin k'hl \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right) \right) \right) \\
&= \frac{1}{h^2 \cos\frac{k'h}{2} \sin\frac{k'h}{2}} \left(\cos\left(\frac{k'h}{2}(2i-1)\right) \left(A \sin\frac{(i\vee j)k'h}{2} \sin\frac{(i\vee j+1)k'h}{2} + \sin\frac{jk'h}{2} \cos\frac{(j+1)k'h}{2} \right) \right. \\
&+ \left. \left(\sin\frac{(i\vee j)k'h}{2} \sin\frac{(i\vee j+1)k'h}{2} - \sin\frac{ik'h}{2} \sin\frac{(i+1)k'h}{2} \right) \right. \\
&\times \left. \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right) \right) \right) \\
&\leq \frac{D_1}{h^2 \sin k'h}
\end{aligned}$$

since $|A|$ and hence the expression in the brackets are bounded.
With the assumption that kh and hence $k'h$ is small there exists $D_2 > 0$ s.t.

$$\sin k'h = k'h \left(1 - \frac{k'^2h^2}{6} \pm \ldots \right) \geq D_2 k'h,$$

then

$$\begin{aligned}
\|H_x\| &= \left(h \sum_{j=1}^{n} \left| d^i H_{\cdot j} \right|^2 \right)^{\frac{1}{2}} \\
&= \left(h \sum_{j=1}^{n} \left| h \sum_{l=1}^{j-1} d^i G_{\cdot l} \right|^2 \right)^{\frac{1}{2}} \\
&= h^{\frac{3}{2}} \left(\sum_{j=1}^{n} \left(\sum_{l=1}^{j-1} d^i G_{\cdot l} \right)^2 \right)^{\frac{1}{2}}
\end{aligned}$$

and with the previous inequalities we obtain

$$\|H_x\| \leq h^{\frac{3}{2}} \left(\sum_{j=1}^{n} \frac{D_1 D_2^{-2}}{h^6 k'^2} \right)^{\frac{1}{2}} \leq \frac{D_3 h^{\frac{3}{2}}}{h^{\frac{7}{2}} k'} \leq \frac{D_3}{h^2 k'}.$$

By similar computation we can show that for any $l, 1 \leq l \leq n$

$$|d^l H_{\cdot n}| = |h \sum_{j=1}^{n} d^i G_{\cdot j}| \leq \frac{D_4}{h^2 k'}$$

hence

$$\|H_x\| + \max_l |d^l H_{\cdot n}| \leq \frac{D}{h^2 k'}$$

where $D = D_3 + D_4$.
Returning now to eqs (3.31) and (3.33),

$$\begin{aligned} |z|_1 &= \left(h \sum_{j=1}^{n} |d^l z|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\max_{1<l<n} |d^l H_{.n}| + \|H_x\| \right) |r_1|_1 \\ &\leq \frac{D}{h^2 k'} C' h^2 k^2 |u|_1 \\ &\leq kCD \left(\frac{k}{k'} \right) |u|_1. \end{aligned}$$

From the Taylor series expansion (3.21) we see that

$$\frac{k'}{k} = 1 + \frac{k^2 h^2}{6} - \frac{3k^4 h^4}{640} \pm \dots$$

is bounded for sufficiently small kh. Hence there exists a constant E not depending on h and k s.t.

$$|z|_1 \leq Ek|u|_1. \tag{3.38}$$

We then have

$$|v|_1 = |u + z|_1 \leq (1 + Ek)|u|_1,$$

hence there exists, for sufficiently large k, a constant F s.t.

$$|u|_1 \geq \frac{F}{k}|v|_1$$

and left inequality of the statement follows from the definition of z and the Gårdings-type inequality (2.14).

To prove the right inequality we construct, in analogy to section 2, a function z_o for which continuity holds with Ck^{-1}.
Consider the function

$$Z(x) = \varphi(x)\, w(x)$$

where $\varphi(x) \in C^\infty(0,1)$ and

$$w(x) = \frac{\sin k'x}{k}$$

is a fundamental solution of the discrete system eq (3.13). Let $z_o(x) \in V_h$ be the piecewise linear interpolant of $Z(x)$ on X_h. Again we assume that φ does not depend on the paramter k and is selected such that

$$\varphi(0) = \varphi(1) = \varphi'(1) = 0$$

and there exists $\alpha > 0$ s.t.

$$|z_o|_1 \geq \alpha$$

independently on k. Then

$$\forall v \in V_h : \quad \frac{|B(z_o, v)|}{|z_o|_1} \leq \frac{1}{\alpha} |B(z_o, v)|.$$

Turn to the estimation of $|\mathcal{B}(z_o, v)|$ (we omit the subscript o from now on):

$$\begin{aligned}\mathcal{B}(z,v) &= \int_0^1 z'v' - k^2 \int_0^1 zv \\ &= h\sum_{j=1}^{n} d^j z d^j v - \frac{k^2}{6} h \sum_{j=1}^{n} (z_{j-1} + 4z_j + z_{j+1})\, v_j\end{aligned}$$

(let formally $z_{n+1} := z_{n-1}$).
Summation by parts then yields

$$\mathcal{B}(z,v) = -h\sum_{j=1}^{n}\left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1})\right)\bar{v}_j + \frac{1}{h}(z_{n-1}\bar{v}_n - z_o\bar{v}_o)$$

The term outside the sum is $O(h)$. Indeed, $z_o = 0$ and

$$\varphi_{n-1} = \varphi(1) - h\varphi'(1) + \frac{h^2}{2}\varphi''(1) + O(h^3).$$

Consequently, since $\varphi(1) = \varphi'(1) = 0$, we have $h^{-1}z_{n-1} = h^{-1}\varphi_{n-1}w_{n-1} = O(h)$. Hence, omitting the terms $O(h)$,

$$\mathcal{B}(z,v) = -h\sum_{j=1}^{n}\left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1})\right)\bar{v}_j.$$

For arbitrarily fixed j we write the second differences as

$$\begin{aligned}D^j(d^j z) &= D^j(d^j(\varphi w)) = D^j\left((d^j\varphi)w_{j-1} + \varphi_j d^j w\right) \\ &= D^j(d^j\varphi)w_{j-1} + 2D^j\varphi d^j w + \varphi_j D^j(d^j w)\end{aligned}$$

and the weighted sum as

$$\begin{aligned}z_{j-1} + 4z_j + z_{j+1} &= (\varphi w)_{j-1} + 4(\varphi w)_j + (\varphi w)_{j-1} \\ &= w_{j-1}(\varphi_j - h\varphi_j' + O(h^2)) + 4w_j\varphi_j + w_{j+1}(\varphi_j + h\varphi_j' + O(h^2)) \\ &= \varphi_j(w_{j-1} + 4w_j + w_{j+1}) + 2h^2\varphi_j' w_j' + O(h^2)).\end{aligned}$$

Then, neglecting all terms that are $O(h)$ we can write

$$D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1}) =$$
$$\varphi_j\left[D^j(d^j w) + \frac{k^2}{6}(w_{j-1} + 4w_j + w_{j+1})\right] + D^j(d^j\varphi)w_{j-1} + 2D^j\varphi d^j w.$$

Since w has been selected as a fundamental solution of the discrete system, the expression in square brackets vanishes.
We now define the piecewise linear function u as the linear interpolant of the meshfunction u_h defined by

$$u_h(x_i) := h\sum_{j=1}^{i}\left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1})\right).$$

Then, on the one hand,

$$|\mathcal{B}(z,v)| = \left| u(1)\bar{v}(1) - \int_0^1 u(x)\bar{v}'(x)\,dx \right| \leq (|u(1)| + \|u\|)|v|_1$$

and on the other hand

$$\begin{aligned} \|u\| &= \left(h \sum_{i=1}^{n} \left(h \sum_{j=1}^{i-1} \left(D^j(d^j\varphi) w_{j-1} + 2 D^j \varphi d^j w \right) \right)^2 \right)^{\frac{1}{2}} \\ &= \left(h \sum_{i=1}^{n} \left(h \sum_{j=1}^{i-1} \left(-D^j(d^j\varphi) w_{j-1} + 2 \left(D^{i-1}\varphi w_{i-1} - w_1 D^0 \varphi \right) \right) \right)^2 \right)^{\frac{1}{2}} \end{aligned}$$

Making use of the smoothness of the function φ we have for all j

$$\begin{aligned} D^j(d^j\varphi) &= \varphi''(jh) + O(h^2) \\ D^{j-1}\varphi &= \varphi'((j-1)h) + O(h) \end{aligned}$$

and we obtain

$$\|u\| \leq h \sum_{i=1}^{n} \left(hi|w| \left(\|\varphi''\|_\infty + (\|\varphi'\|_\infty + 2\|\varphi'\|_\infty + O(h)) \right)^2 \right)^{\frac{1}{2}}$$

where the function $w = k^{-1} \sin k'x$ can be estimated by

$$|w| \leq \frac{1}{k}.$$

and the term $O(h)$ does not depend on k.

By similar estimates for $|u(1)|$ we conclude that for sufficiently small h there exists a constant C with

$$(\|u\| + |u(1)|) \leq \frac{C}{k}.$$

It then follows that

$$\forall v \in V_h : \quad |\mathcal{B}(z,v)| \leq \frac{C}{k} |v|_1$$

and the proof is completed.

Remark 11: We recapitulate that, for $f \in L^2(0,1)$, both approximability (theorem 2) and the discrete stability condition hold under the assumption the hk is sufficiently small. It then follows from a fundamental theorem [BA, p.187] that the FE-solution exists and is uniquely determined. We emphasize that the latter stability result is thus obtained by restricting the magnitude of hk only (compare to the more severe restriction of hk^2 in Theorem 3!).

3.5 A preasymptotic error estimate

In this subsection, an error estimate will be given that is suited to *bound the error at finite range* also for high wavenumbers k. First we have to prove:

Lemma 3 *Let $u_{fe} \in V_h$ be the finite element solution to the variational problem (2.10) for given data $f \in L^2(0,1)$.*
Then, if h is small s.t. $hk \leq 1$, there exists a constant C not depending on h and k s.t.

$$\|u'_{fe}\| \leq C\|f\|.$$

Proof: Since u_{fe} is piecewise linear, we have

$$\|u'_{fe}\| = \left(h \sum_{i=1}^{n} \left(d^i u_{fe} \right)^2 \right)^{\frac{1}{2}}.$$

Write $u_h := u_{fe}|_{X_h}$ in terms of the discrete Green's function as

$$u_i = h \sum_{j=1}^{n} G_{ij} r_j$$

then

$$d^i u = h \sum_{j=1}^{n} d^i G_{\cdot j} r_j$$

and

$$|d^i u| \leq \|d^i G\| \|r\|. \tag{3.39}$$

with

$$\|d^i G\| = \left(h \sum_{j=1}^{n} \left(d^i G_{\cdot j} \right)^2 \right)^{\frac{1}{2}}$$

$$\|r\| = \left(h \sum_{j=1}^{n} r_j^2 \right)^{\frac{1}{2}}.$$

The mesh function r_h is related to the function $f \in L^2(0,1)$ by eq (3.15) from which it is easy to see that there exists a constant C_1 s.t.

$$\|r\| \leq C_1 h^2 \|f\|.$$

The derivatives of the Green's function are - cf. eqs (3.36, 3.37) -

$$d^i G_{\cdot l} = \frac{1}{h^2 \cos \frac{k'h}{2}} \begin{cases} \cos\left(\frac{k'h}{2}(2i-1)\right)\left(A \sin k's_l + \cos k's_l\right) & i \leq l \\ \sin k's_l \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right) & i \geq l \end{cases}.$$

Obiously $h^2 |d^i G_{\cdot l}|$ is bounded provided that $hk' \leq \alpha < \pi/2$. From the Taylor series expansion of hk', eq (3.21), we conclude that such α exists for sufficiently (say, $hk < 1$) small hk.

Hence there is a constant C_2 s.t.

$$\forall i, j: \quad \left| d^i G_{\cdot j} \right| \leq \frac{C_2}{h^2}.$$

Then also

$$\forall i: \quad \|d^t G\| \leq \frac{C_2}{h^2}$$

and the statement follows from eq (3.39) with $C = C_1 C_2$. The proof is completed.

We proceed to the proof of the error estimate:

Theorem 5 *Let $u \in H^2(0,1)$ be the exact solution of the variational problem (2.10) with data $f \in L^2(0,1)$ and let $u_{fe} \in S_h[0,1[$ be the finite element solution of (2.10). Then if the stepsize h is such that $hk \leq 1$ the error estimate*

$$|u - u_{fe}|_1 \leq \left(\frac{hk}{\pi} + C \left(\frac{hk}{\pi} \right)^2 (1+k) \right) \|f\| \tag{3.40}$$

holds with constant C not depending on h and k.

Proof: Let $u_I \in V_h = S_h[0,1[$ be the interpolant of u and define $z \in V_h$ by

$$z := u_{fe} - u_I.$$

From

$$\forall v \in V_h: \quad \mathcal{B}(u, v) = \mathcal{B}(u_{fe}, v)$$

and since $\mathcal{B}$ is sesquilinear we have

$$\mathcal{B}(u - u_I, v) = \mathcal{B}(z, v).$$

On the other hand, for $v \in V_h$:

$$\forall i: \quad \int_{x_{i-1}}^{x_i} (u - u_I)' v' = \left[(u - u_I) v' \right]_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (u - u_I) v'' = 0$$

since $(u - u_I)|_{X_h} = 0$ and $v''|_{(x_{i-1}, x_i)} = 0$ and therefore

$$\mathcal{B}(u - u_I, v) = k^2 \int_0^1 (u - u_I) v.$$

Hence z is a solution of

$$\forall v \in V_h: \quad \mathcal{B}(z, v) = k^2 (u - u_I, v)$$

and from Lemma 3 we have the estimate

$$\|z'\| \leq C k^2 \|u - u_I\|.$$

Then

$$\begin{aligned} |e|_1 &= |u - u_{fe}|_1 = |u - u_I + u_I - u_{fe}|_1 \\ &\leq |u - u_I|_1 + |z|_1 \\ &\leq |u - u_I|_1 + C k^2 \|u - u_I\| \end{aligned}$$

We now invoke the approximation properties of the space V_h from Lemma 2 to obtain

$$|e|_1 \leq \left(\frac{h}{\pi} + C \frac{k^2 h^2}{\pi^2} \right) \|u''\|$$

The statement of the theorem now follows from Lemma 1.

3.6 Comments

In this section we have given different proofs of existence-uniqueness for the FE-solution. The main results are that

- the discrete problem is stable provided that proper restrictions are made for the magnitudes of hk only (theorem 4) and
- the error of the finite element solution can be controlled by restricting the magnitudes of hk and $hk^{\frac{3}{2}}$ (theorem 5).

It had been shown that the error bounds of subsection 1 tend towards 0 as k is increased. This is not the case for the estimates of theorem 5 since only boundedness of hk is assumed. There is, however, a close relation between both estimates. Namely, the corollary 1 from theorem 3:

$$|u - u_{fe}|_1 \leq C|u - u_I|_1 \leq C\,h\,(1+k)\,\|f\|$$

follows also from theorem 5 if *the magnitude of k^2h is bounded.* In other words, both error estimates lead to the same conclusion that the stability constant C_s does not depend on k if k^2h is bounded.

We will show by numerical experiment that this condition is also necessary, i.e. the constant C_s grows with k if k^2h is not restricted.

The assumption of uniform mesh is due to technical necessities of the proofs for theorems 4 and 5. All statements of this section should hold for nonuniform mesh as well.

4 Numerical Evaluation

The first and obvious purpose of the numerical evaluation lies in the illustration and application of the theoretical results by computational experiment. Beyond this, we will draw a qualitative conclusion concerning the assumptions of some propositions of the previous section.

Throughout this section, we will present FE-solutions to the variational problem (2.10) with constant right hand side $f(x) \equiv -1$ on uniform mesh.

4.1 Error of the best approximation

Consider in Fig. 1 the errors e_a of the best approximation (interpolant) computed for different k and h such that $0.2 \leq hk \leq 2$, plotted in log-log-scale.

As predicted by theorem 2, all error curves decrease with constant slope of -1 in the log-log-plot (the theoretical rate of convergence being O(h)).

The inequality of the theorem gives, however, a crude upper bound for the relative error

$$\tilde{e}_a := \frac{e_a}{\|u'\|}$$

in the case that k is large and $\|u'\|$ is not bounded from below independently on k. For $u \in H^2(0,1)$, an estimate is obtained from Lemma 2 as

$$\tilde{e}_a \leq \frac{h}{\pi}\,\frac{\|u''\|}{\|u'\|}. \tag{4.1}$$

In our example the relation

$$\frac{\|u''\|}{\|u'\|} = k \left(\frac{3 - 2\cos k - \frac{\sin k(2-\cos k)}{k}}{3 - 2\cos k + \frac{\sin k(2-\cos k)}{k}} \right)^{\frac{1}{2}}$$

is easily computed, hence for sufficiently large k we can by

$$n_\varepsilon = \frac{1}{h} \geq \frac{k}{\pi \tilde{e}_a}.$$

predict the meshsize needed for approximation with a given tolerance $\varepsilon = \tilde{e}_a$.

Let, for example, $\varepsilon = 0.1$ be a maximal tolerance, then

$$n \geq \frac{10\,k}{\pi} \tag{4.2}$$

is the "rule of the thumb" for the number of elements. As table 1 shows, this rule works well for large k.

Table 1: Number of elements needed for a relative error of interpolation less than 0.1: number obtained from numerical experiment compared to bound computed from eq (4.2).

k	2	10	40	100
n_ε computed from eq (4.2)	6	31	127	310
n_ε measured from Fig. 1	8	30	120	300

Consider now the results plotted in Fig. 2. Clearly the relative error of interpolation cannot exceed 100%. From the plots we observe that for each wavenumber k the error stays at 100% on coarse mesh and starts to decrease at a certain meshsize. We are interested in the point where the descend starts. More precisely, we seek the critical number of degrees of freedom according to the following definition:

Define - for any fixed k and f - the *critical number of degrees of freedom (DOF)* $N_o(k)$ as the minimal number $N(k, f)$ of DOF for which

1. $\tilde{e}(n, k) < 1$ and

2. $\tilde{e}(n, k)$ is monotone decreasing w.r. to n

for all $n > N(k, f)$.

For the best approximation, the critical number of DOF is determined by the rule that the stepwidth of interpolation by piecewise linears should be smaller than one half of the wavelength of the exact solution:

$$hk < \pi.$$

In Fig. 3, the critical point n_o computed from

$$n_o = \left[\frac{k}{\pi} \right] \tag{4.3}$$

is plotted for different k. The predicted critical number of DOF is close to the actual one in all cases.

Finally we whish to see experimentally that the error of the best approximation is controlled by bounding the magnitude hk. To this end, consider the fat line plotted in Fig. 3. In the error curves for the different k, this line connects the points that are computed from $hk \equiv$ const. $= 0.2$. As we see, this line does neither increase nor decrease significantly with the change of k. For more detailed observation, the relative error of the best approximation, computed for all integer k from 1 to 500 and for $hk \equiv 0.1$, is plotted in Fig. 4. The error oscillates with decaying amplitude around the horizontal line

$$|\tilde{e}_a|_1 = 0.02887.$$

The upper estimate from eq (4.1) is

$$|\tilde{e}_a|_1 \leq \frac{0.1}{\pi} = 0.03183.$$

The figure can be further analyzed as follows: we find for the relative error the expression ($t = hk$):

$$|\tilde{e}_a|_1 = \frac{t}{\sqrt{12}} \left(1 - \frac{2}{k} \frac{\sin 2k - 4 \sin k}{6 - 4 \cos k - \frac{\sin 2k - 4 \sin k}{k}} \right)^{\frac{1}{2}}$$

under the assumption that t^2 and higher terms of t can be neglected. For the case $t = 0.1$, plotted in Fig. 3, this expansion predicts for high k the value

$$|\tilde{e}_a|_1 = \frac{0.1}{\sqrt{12}} = 0.02886751.$$

Remark 12: In the one-dimensional case one can by means of a Galerkin least squares method ([HH1]) obtain a modified finite element solution that is identical with the interpolant of the exact solution. Therefore the conclusions drawn above for the minimal error in H^1-seminorm hold for this solution as well.

4.2 Error of the finite element solution

Discrete wavenumber: Unlike the best approximation, the FE-solution is, in general, not in phase with the exact solution. On uniform mesh this numerical effect is highlighted by the notation of a "discrete wave number" k' that governs the periodicity of the finite solution. In other words, we observe a phase lag ([HH1, p.71], cf. Fig. 4) between the exact solution and it's best approximation on the one and the FE-solution on the other hand.

The determining equation for the discrete solution on uniform mesh had been found in subsection 3.2. as

$$\cos k'h = -\frac{S(t)}{R(t)}$$

where $t = hk$ and the r.h.s. is a rational function of t.

In Fig. 5 the functions $y_1 = -S(t)/R(t)$, $y_2 = \cos t$ and $|y_3| = 1$ are plotted. We observe that:

- at $t_o = \sqrt{12}$ the function y_1 reaches absolute value 1; the numerical solution switches from the propagating case to the decaying case;

- for fixed k, the convergence $k' \to k$ is vizualized by $\cos k'h \to \cos t = \cos kh$ as $h \to 0$. The curves begin to deviate significantly at about $hk = 1$.

Rate of convergence: In Fig. 6 the relative errors of the FE-solutions for different k are plotted. The meshes are such that the magnitude of kh is in the same range as in the error plot for the best approximation in Fig. 1. We observe the following:

1. The relative error of the FE-solution exceeds (for higher k on relatively coarse mesh) 100%.
2. For low k (represented by $k = 3$ in the figure) the rate of convergence is nearly constant throughout the region considered, i.e. the fe-solution behaves essentially like the best approximation.
3. For high k, the relative error oscillates above 100% before it starts to descend after a critical value N_0 of meshpoints has been reached. The decrease first occurs with a rate greater than -1 in the log-log-scale but becomes -1 for small h.
4. Unlike the error of the best approximation, the error of the FE-solution cannot be controlled by bounding the magnitude of hk. The relative error clearly grows with k on all lines $hk \equiv$ const.

The last observation is further emphasized by the results plotted in Fig. 7, together with table 2. The "rule of the thumb" to place a certain number of elements per wavelength does obviously not hold for high k.

Asymptotic stability and quasioptimality: Consider now in Fig. 8 the plots of the relative errors of the FE-solution together with the relative errors of the best approximation. This figure is well suited to enhance the quasioptimal stability estimate in corollary 2, section 3.2. To this end, lines are plotted connecting h and k s.t.

$$hk^2 = \alpha \equiv \text{const} \tag{4.4}$$

for $\alpha = 2, \alpha = 1$ and $\alpha = 0.5$. The corollary states that on these lines, if α is sufficiently small, the ratio of the errors of the best approximation and the FE-solution does not depend on k, i.e. the distances between both curves in the log-log-plot do not grow along the lines (4.4).

The statement is vizualized in the plot; even more: we see that for the example considered the stability constant is close to 1 for sufficiently small α.

In Fig. 9 the stability constant C_s from corollary 2, computed with the restriction $hk^2 = 1$, is plotted for all integer k from 1 to 200. Obviously, the constant computed with constrained hk^2 does neither decrease nor grow with increasing k (except for small k, then hk is the leading member in the estimate of theorem 5 - cf. comments to Fig. 13).

On the other hand, it is easy to verify from Fig. 8 that the error ratio *does depend* on k on all lines $hk^\beta = \alpha$ with $\beta < 2$. In particular, C_s is increasing with k on the line defined by $hk = 1$ (Fig. 10) and $hk^{\frac{3}{2}} = 1$ (Fig. 11).

Preasymptotic stability and error estimate: We have thus shown experimentally that the assumptions of theorem 3 and corollary 2 are indeed inherent to the problem: for the ratio of the FE-solution error and the best approximation to be bounded it is necessary

to restrict the magnitude of hk^2. However, it is not necessary (though sufficient) to bound this ratio for the practical purpose of limiting the error of the FE-solution at finite range. Indeed C_s grows with k on the line of constant (relative) error of the FE-solution (Fig. 12).

According to theorem 5, the relative error is bounded at any range by the magnitudes of h^2k^3 and hk. This statement is vizualized by the results plotted in Fig. 13. Here, the relative error has been computed for all integer k from 1 to 1000 on meshes with $h = \left(k^{\frac{3}{2}}\right)^{-1}$. We observe the following:

1. For low k ($1 \leq k \leq 30$) the relative error decreases rapidly with k. In this range, the FE-solution is still close to the best approximation ($hk^2 = 5.48$ for $k = 30$) and hence the term hk is the significant member in the estimate (3.40).

2. For large k ($k \geq 100$) the error is bounded by $\tilde{c} = 0.05$. The term h^2k^3 is leading in estimate (3.40).

Let us consider how these effects might influence the results of applied computations. To this end, we write the estimate of theorem 5 in the form

$$|e|_1 \leq \left(\alpha + C(1+k)\alpha^2\right) \|f\| \tag{4.5}$$

with the "rule of the thumb"

$$\frac{hk}{\pi} = \alpha.$$

In most practical computations with low ($k \leq 10$) wavenumbers, intuitively a good resolution (like $\alpha = 0.1$, i.e. 20 elements per wavelength) is choosen. In this case, $\alpha^2 = 0.01$ and $k\alpha^2 = 0.1$: both terms in the estimate (4.5) are of the same magnitude and hence the phase lag does not affect the error significantly. In other words, no negative effects will be observed in benchmark tests. However, for high wavenumber (say, $k = 100$) the second member equals 1 for the same resolution $\alpha = 0.1$ and hence is prevalent in the estimate.

These effects become much more visible if, for cost reduction of the computations, there are choosen lower resolutions like $\alpha = 0.2$ or $\alpha = 0.5$ (cited as "acceptable resolution" or "limit of resolution", respectively, in [HI1]). For $k = 10$, the magnitudes $\alpha = 0.2$ and $k\alpha^2 = 0.4$ are still of the same order for acceptable resolution but differ considerably for the limit of resolution ($\alpha = 0.5$ and $k\alpha^2 = 2.5$). For the latter resolution, both magnitudes are roughly of the same order up to $k = 4$.

For high wavenumber ($k = 100$) the second member of the estimate is clearly dominating for both resolutions: we have $\alpha = 0.2$ vs. $k\alpha^2 = 4$ and, for the limit of resolution, $\alpha = 0.5$ and $k\alpha^2 = 25$.

Finally, we demonstrate that also the critical number of DOF for the FE-solution error is governed by the magnitude of h^2k^3. Consider in Fig. 14 the curves of the relative error computed for different k from $k = 10$ to $k = 1000$ and the predicted critical number of DOF where the latter has been computed from the formula

$$N_c = \sqrt{\frac{k^3}{24}} \tag{4.6}$$

(a physical argument for this formula will be given below). Again, the predicted critical number of DOF is close to the actual one.

4.3 Summary

We are now in a position to comment on the behavior of both approximation and FE-solution error throughout the whole range of existence of a propagating numerical solution (i.e. for $hk \leq \sqrt{12}$ in the case of piecewise linear approximation).

Consider the case $k = 100$ in Fig. 15. We have marked on the abscissa three significant points for the meshsize $n = h^{-1}$. By these points the range of degrees of freedom is divided into four regions, namely:

1. $1 \leq n \leq n_o$:
   The mesh is too coarse to allow for neither approximation nor FE-solution of the Helmholtz equation with given wavenumber k. The number
   $$n_o = \left[\frac{k}{\pi}\right]$$
   is the critical number of DOF ("limit of resolution" [HH1]) for approximability.

2. $n_o \leq n \leq N_o$:
   The relative error of approximation is smaller than 100 % but the relative error of the FE-solution is still above this range. Though we have approximability and stability, the stability constant is too large to bound the error.

3. $N_o \leq n \leq N_s$:
   Both the FE-solution error and the approximation error are in the range of convergence. In the error estimate (3.40) the magnitude h^2k^3 is the leading member. A considerable phase lag is present between the exact and the FE-solution. The stability constant
   $$C_s = \frac{|u - u_{fe}|_1}{\inf |u - v|_1}$$
   depends still on k but is "under control" since the magnitude of $hk^{3/2}$ is bounded. With the leading member of the estimate being $O(h^2)$ (for any fixed k), the rate of convergence of the FE-solution is higher than the rate of convergence of the best approximation. The FE-error curve descends towards the line of the optimal error.

4. $n \geq N_s$:
   The critical number N_s has been computed from the relation $hk^2 = 1$ (cf. eq (4.4) and related comments). The stability constant C_s does not depend on h and k, the magnitude hk is leading in estimate (3.40). Both the FE-solution error and the error of best approximation have the same rate of convergence $O(h)$, i.e. the statement of quasioptimality (corollary 2) holds.

Concluding this subsection we give the argument for the computation of the critical number of DOF for the FE-solution eq (4.6). Assume that the solutions are given by $u = \sin kx$ and $u_h = u_{fe}|_{X_h} = \sin k'x_h$ and consider the error in the L_∞-norm. Then, if the phase lag $k - k'$ is smaller than $\frac{\pi}{2}$, the maximal difference of amplitudes $|\sin kx_h - \sin k'x_h|$ occurs at the end of the interval $[0, 1]$. Since $\|\sin kx\|_\infty = 1$ we require for $\|\tilde{e}\|_\infty \leq \|\sin kx\|_\infty^{-1} \|u - u_{fe}\|_\infty$:

$$|\sin k - \sin k'| = 2\left|\cos\frac{k + k'}{2}\right|\left|\sin\frac{k - k'}{2}\right| \leq 1.$$

This inequality certainly holds if

$$\left|\sin \frac{k-k'}{2}\right| \leq \frac{1}{2}$$

or, equivalently,

$$k - k' \leq \frac{\pi}{3} \approx 1.$$

With this, eq. (4.6) follows from the Taylor expansion eq (3.21).

5 Conclusions

The numerical solution of the Helmholtz equation with the h-version of the FEM is studied on a one-dimensional model problem. Following the proof of new analytical statements, the investigation is completed with the results of computational experiments.

While it is evident from the oscillatory character of the exact solution that the meshsize h has to be adapted to the magnitude of the wavenumber k it is not obvious how exactly this adaption should be properly designed. This question is the starting point and the practical motivation of the present investigation.

On the one hand, "rules of the thumb" restricting the product hk had reportedly failed for high k. On the other hand, the restriction of k^2h assumed in existing proofs of asymptotic stability and convergence in the analytical literature are practically inapplicable in the very case of high wavenumbers.

The results of the present study - confined to the case of uniform mesh - reveal that:

- the finite element solution is stable given only restrictions on the magnitude of hk;
- in the preasymptotic range, the error of the finite element solution is governed by the term h^2k^3 and hence can be controlled restricting this magnitude;
- the Babuška-Brezzi stability constant is of order k^{-1} both in the infinite-dimensional and the finite-dimensional level;
- the restriction of hk^2 is indeed necessary for quasioptimality of the finite element solution w. r. to k.

In physical terms, if hk^2 is small, then the FE-solution is in the *asymptotic* range of convergence where it is close to the interpolant of the exact solution and hence is quasioptimal, i.e. the FE-error is proportinal (independently of k) to the interpolation error.

In the *preasymptotic* range, the difference between the FE-solution and the interpolant (the *phase lag* of the FE-solution) is the prevalent part of the FE-error. To bound this error it is both necessary and sufficient to restrict the magnitude h^2k^3.

Referring to the originally posed question we see that the answer for the proper choice of the meshwidth lies "in the middle" (between hk and hk^2). Consequently, for large k there still has to be chosen a quite fine mesh (egs., $h = 10^{-3}$ for $k = 100$) if the h-version of the FEM is applied.

In part II of this paper, results will be presented for the h-p-version. Following these conclusions we have to investigate what can be gained on the global level (in terms of the number of DOF) by investing locally (in terms of the order of elemental approximation).

Further research will be directed to the generalization of the results presented herein to higher-dimensional cases and to applied problems of fluid-structure interaction.

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References

[A] N.I. Achieser, Vorlesungen über Approximationstheorie, Akademieverlag Berlin (1953)

[AKS] A.K. Aziz, R.B. Kellogg and A.B. Stephens, A two point boundary value problem with a rapidly oscillating solution, Numer. Math. 53, 107-121 (1988)

[BA] I. Babuška and A.K. Aziz, The mathematical foundations of the finite element method, in: A.K. Aziz (ed.), The mathematical foundations of the finite element method with applications to partial differential equations, Academic Press, New York 1972, pp. 5-359

[Bu] D. Burnett, A 3-D acoustic infinite element based on a generalized multipole expansion, Journ Acoust. Soc. (submitted)

[D] Demkowicz, L.,Asymptotic convergence in finite and boundary element methods: part I: theoretical results, preprint 1993

[DL] Dautray, R. and Lions, L., Mathematical Analysis and Numerical Methods for Science and Technology, v. 1, Springer-Verlag New York - Berlin 1990

[DSSS] J. Douglas Jr., J.E. Santos, D. Sheen, L. Schreiyer, Frequency domain treatment of one-dimensional scalar waves, Mathematical Models and Methods in Applied Sciences, Vol. 3, No. 2 (1993) 171-194

[HH1] I. Harari and T.J.R. Hughes, Finite element method for the Helmholtz equation in an exterior domain: model problems, Comp. Meth. Appl. Mech. Eng. 87 (1991), 59-96

[HH2] I. Harari and T.J.R. Hughes, A cost comparison of boundary element and finite element methods for problems of time-harmonic acoustics, Comp. Meth. Appl. Mech. Eng. 97 (1992), 77-102

[HH3] I. Harari and T.J.R. Hughes, Galerkin/least squares finite element methods for the reduced wave equation with non-reflecting boundary conditions in unbounded domains, Comp. Meth. Appl. Mech. Eng. 98 (1992) 411-454

[J] F. John, Partial Differential Equations, Fourth ed., Springer New York etc., 1982

[JF] M.C. Junger and D. Feit, Sound, structures and their interaction, 2nd ed., MIT Press, Cambridge MA, 1986

[Sa] A.A. Samarskii, Introduction to the theory of difference schemes, Moscow, Nauka edition, 1971

[SF] G. Strang and G.J. Fix, An Analysis of the Finite Element Method, Prentice Hall, Englewood Cliffs, NJ, 1973

[S] A. Schatz, An observation concerning Ritz-Galerkin methods with indefinite bilinear forms, Math. Comp. 28 (1974), 959-962

[SB] B. Szabó and I. Babuška, Finite Element Analysis, J. Wiley, New York etc., 1991

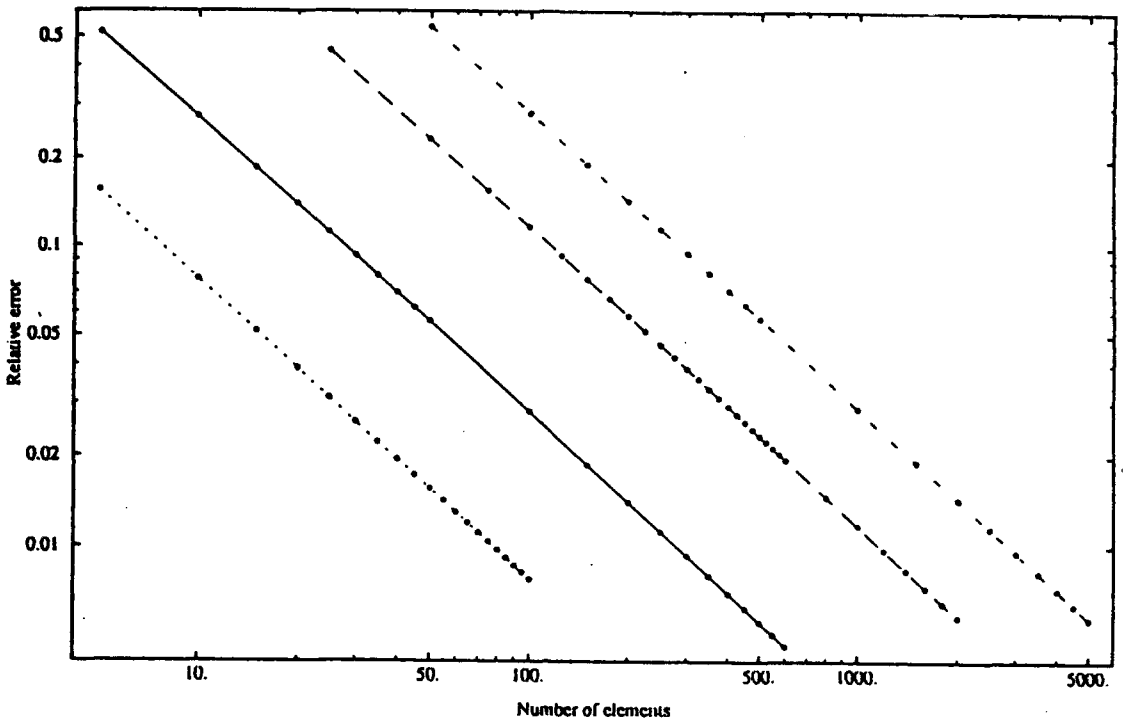

Figure 1: Relative error in H^1-seminorm for $k = 2, 10, 40$ and $k = 100$.

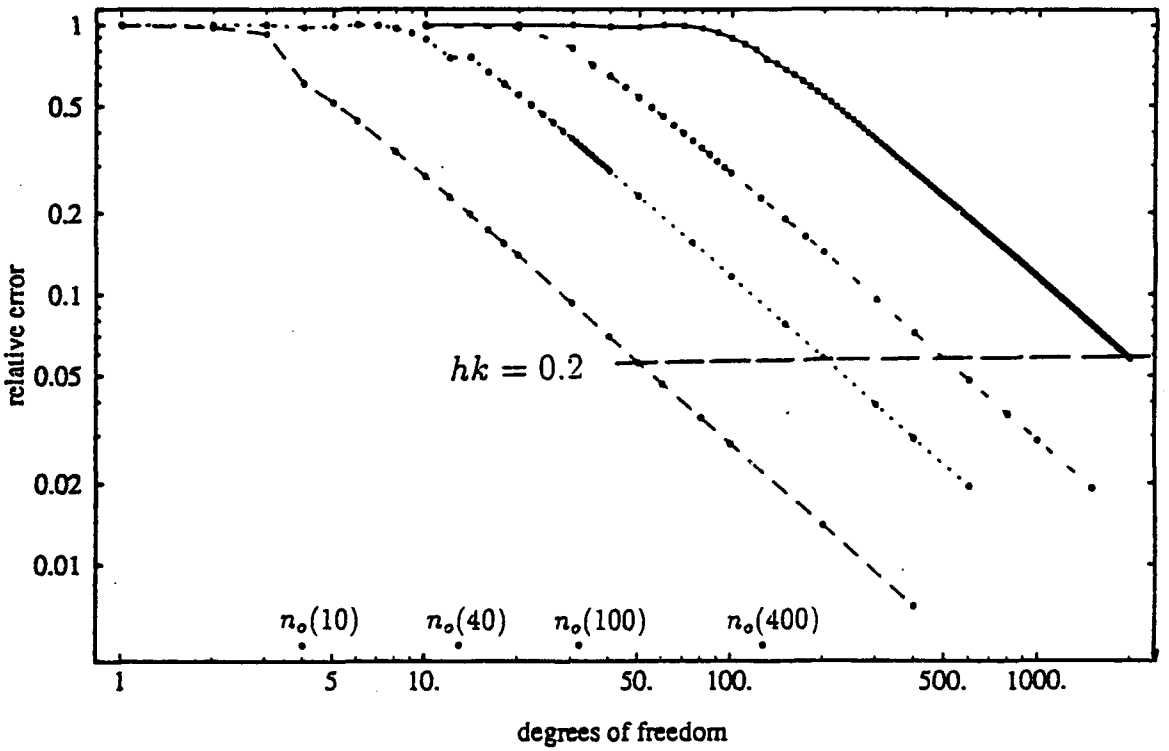

Figure 2: Relative error of the best approximation in H^1-seminorm and predicted critical numbers of DOF for $k = 10, k = 40, k = 100$ and $k = 400$

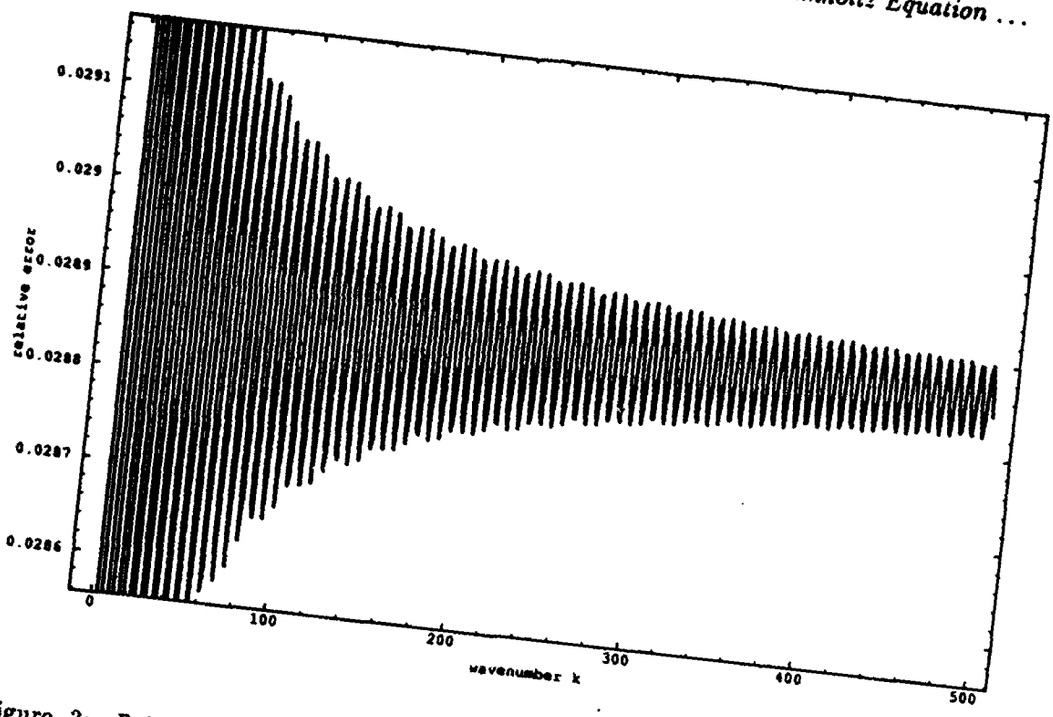

Figure 3: Relative error of the best approximation in H^1-seminorm computed for $k = 1 \ldots 500$ with stepwidth h determined by $hk = 0.1$

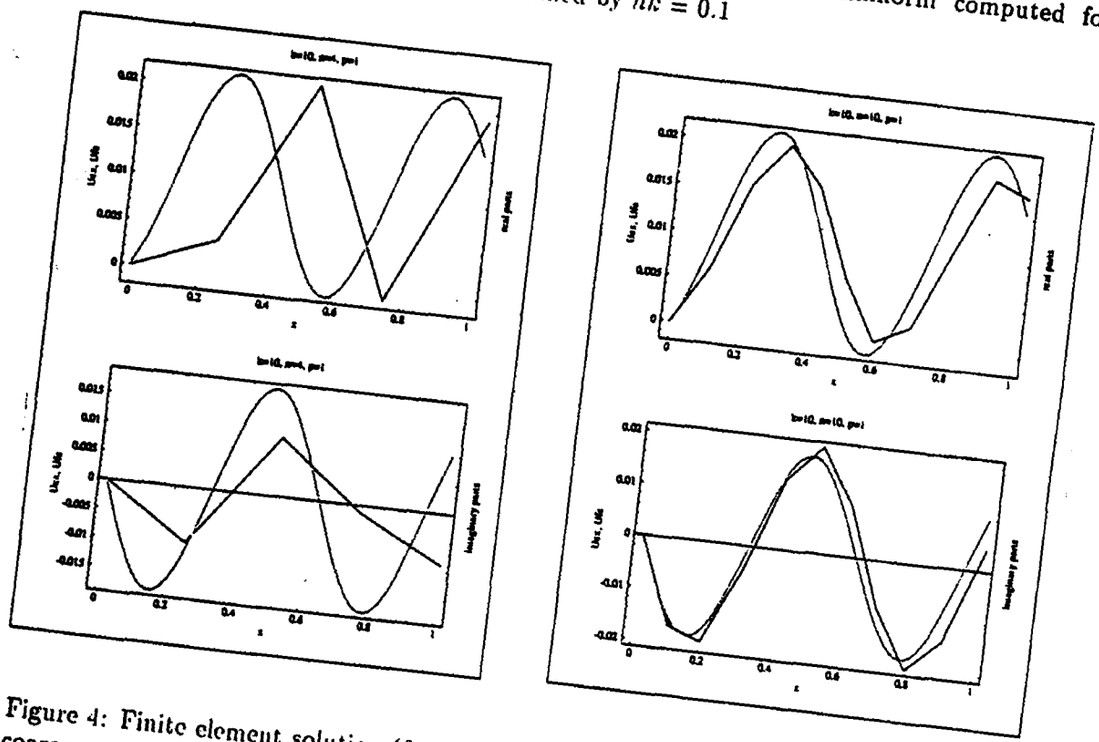

Figure 4: Finite element solution (fat line) versus exact solution for $k = 10$: phase lag on coarse mesh ($n = 4$) compared to phase lag on refined mesh ($n = 10$).

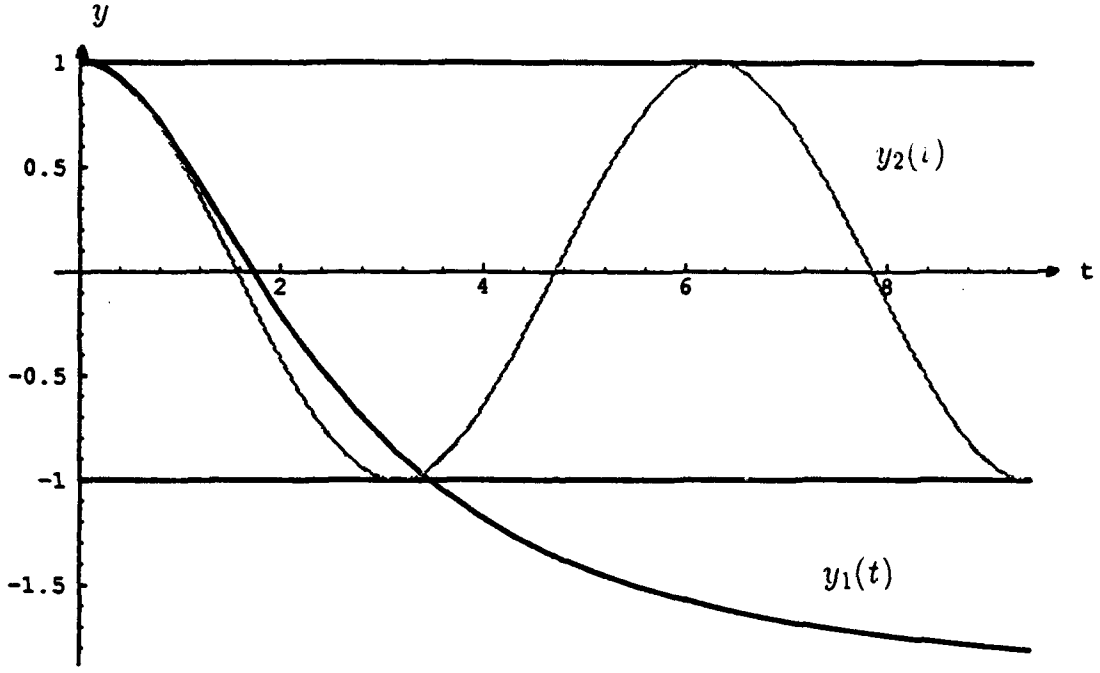

Figure 5: Functions $y_1(t) = -\frac{S(t)}{R(t)}$ and $y_2(t) = \cos t$

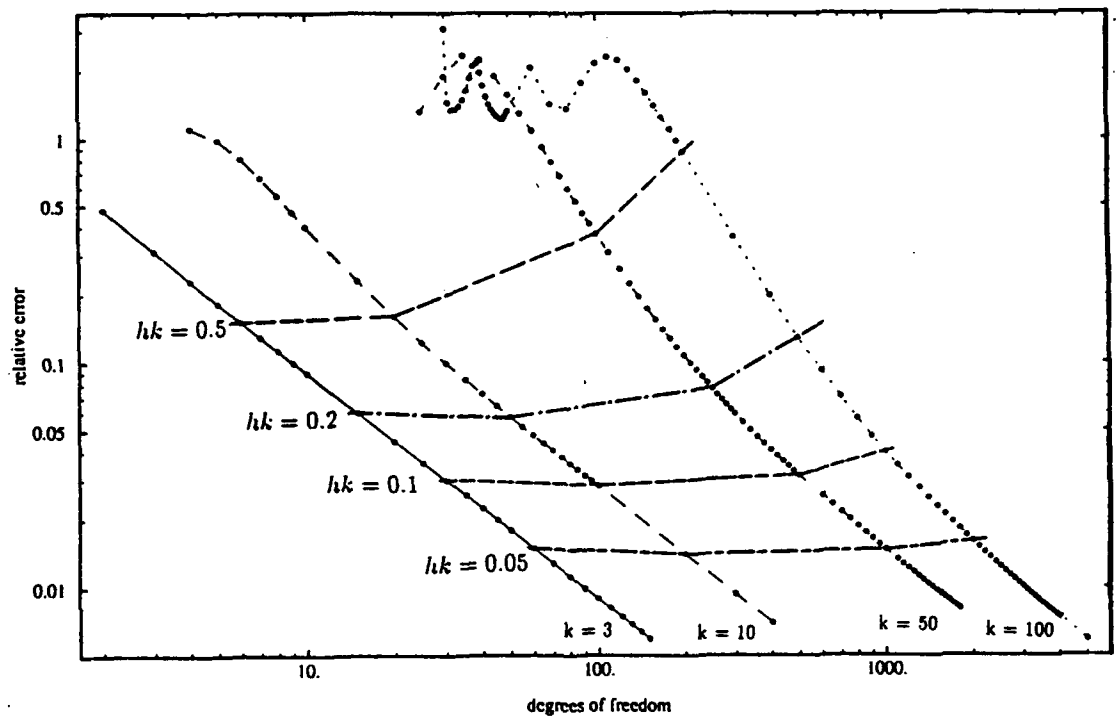

Figure 6: Relative error in H1-seminorm: Finite element solutions for $k = 3, k = 10, k = 50$ and $k = 100$

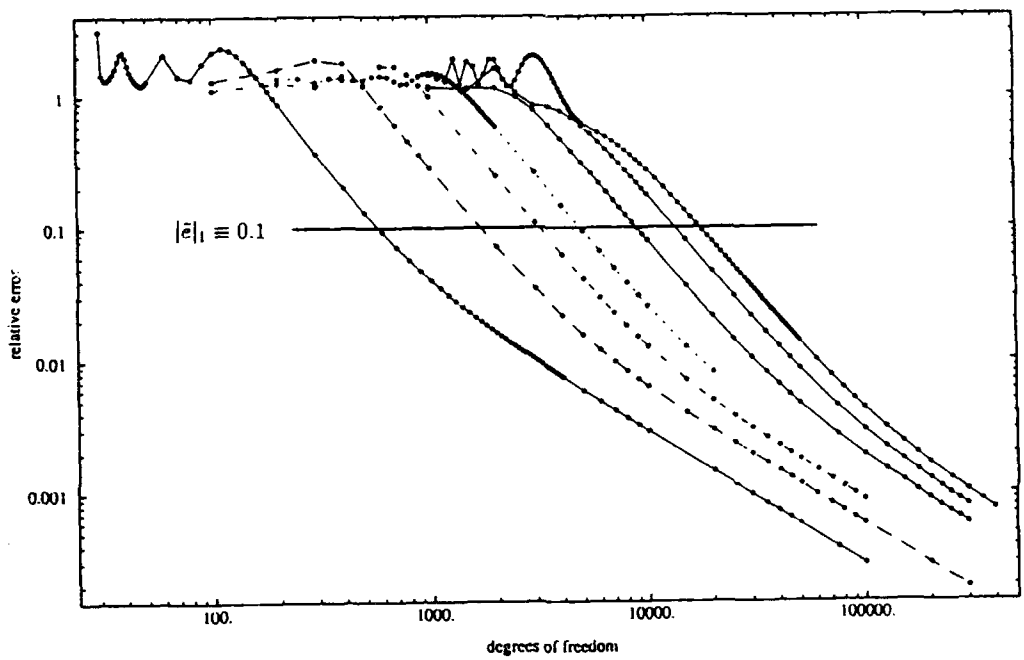

Figure 7: Relative error in H^1-seminorm: Finite element solutions for $k = 100, 200, 300, 400, 600, 800$ and $k = 1000$

Table 2: Number of elements per wavelength necessary for accuracy of 10% in H^1-seminorm

k	100	200	300	400	600	800	1000
# of elements	38	57	63	82	94	107	120

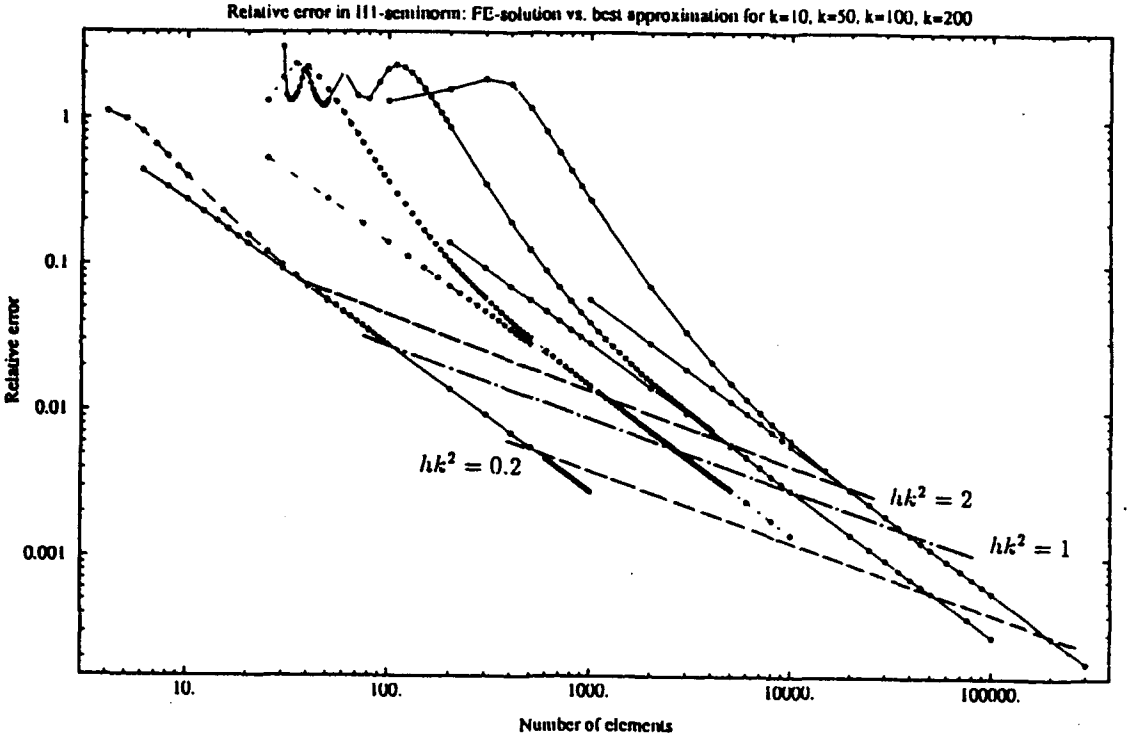

Figure 8: Relative error in H1-seminorm: Finite element solutions versus best approximations for $k = 10, k = 50, k = 100$ and $k = 200$ with lines of constraint $hk^2 = \alpha$

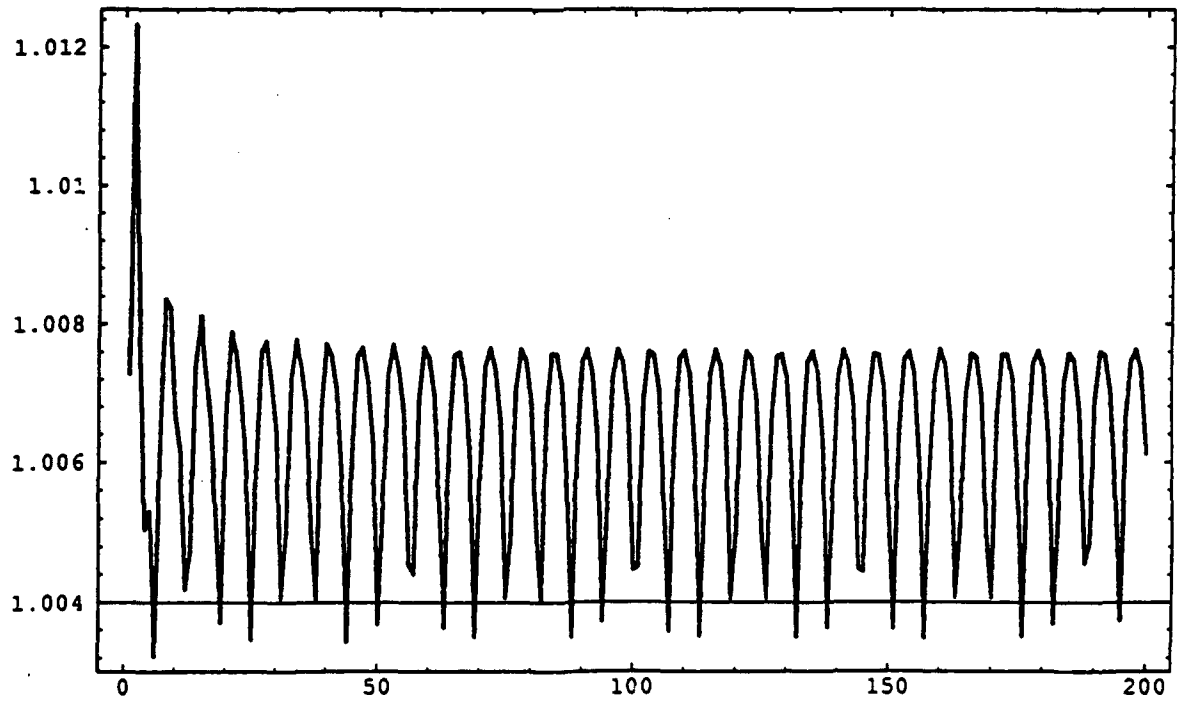

Figure 9: Stability constant C_s in H^1-seminorm computed with constraint $hk^2 = 1$ for $k = 1, 200, 1$.

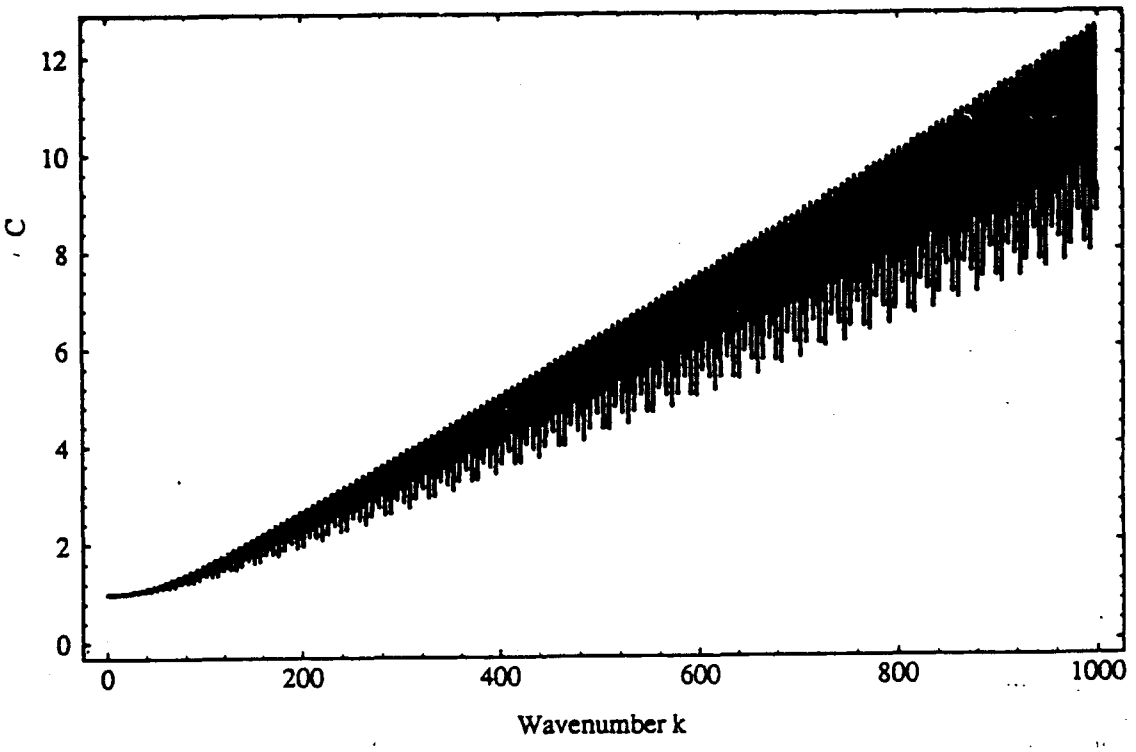

Figure 10: Stability constant C_s in H^1-seminorm computed with constraint $hk = 0.1$ for $k = 1, 1000, 1$.

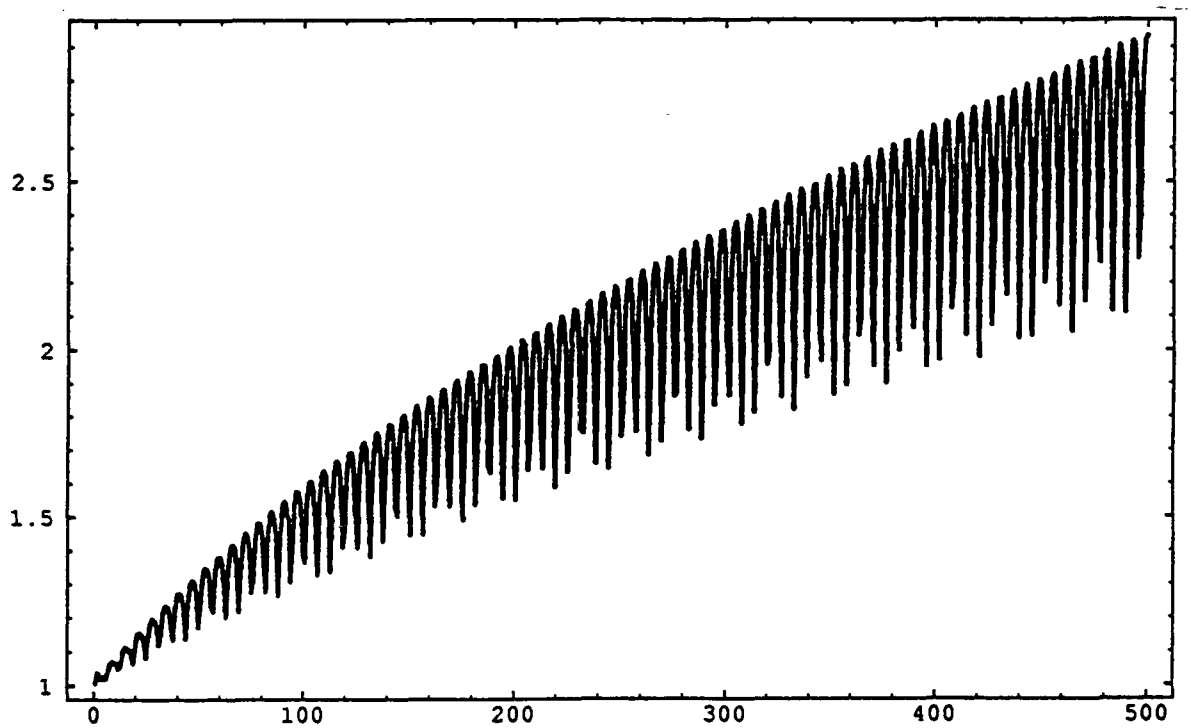

Figure 11: Stability constant C_s in H^1-seminorm computed with constraint $hk^{3/2} = 1$ for $k = 1, 500, 1$.

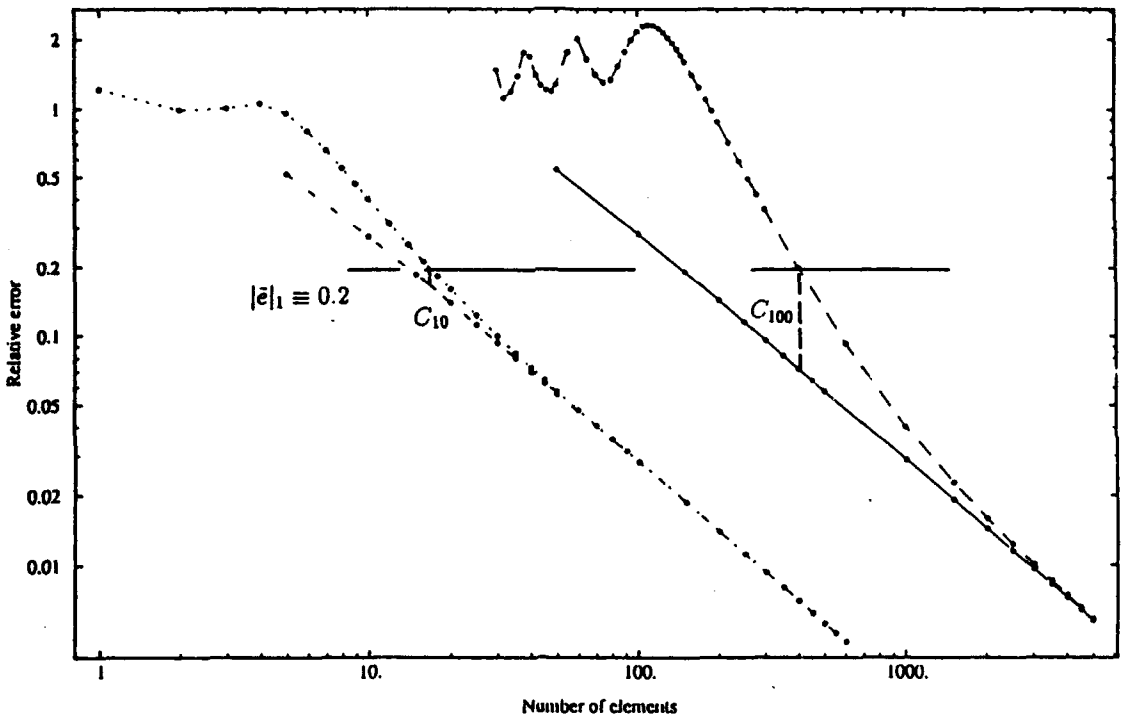

Figure 12: Relative error in H1-seminorm: Finite element solutions for $k = 10$ and $k = 100$

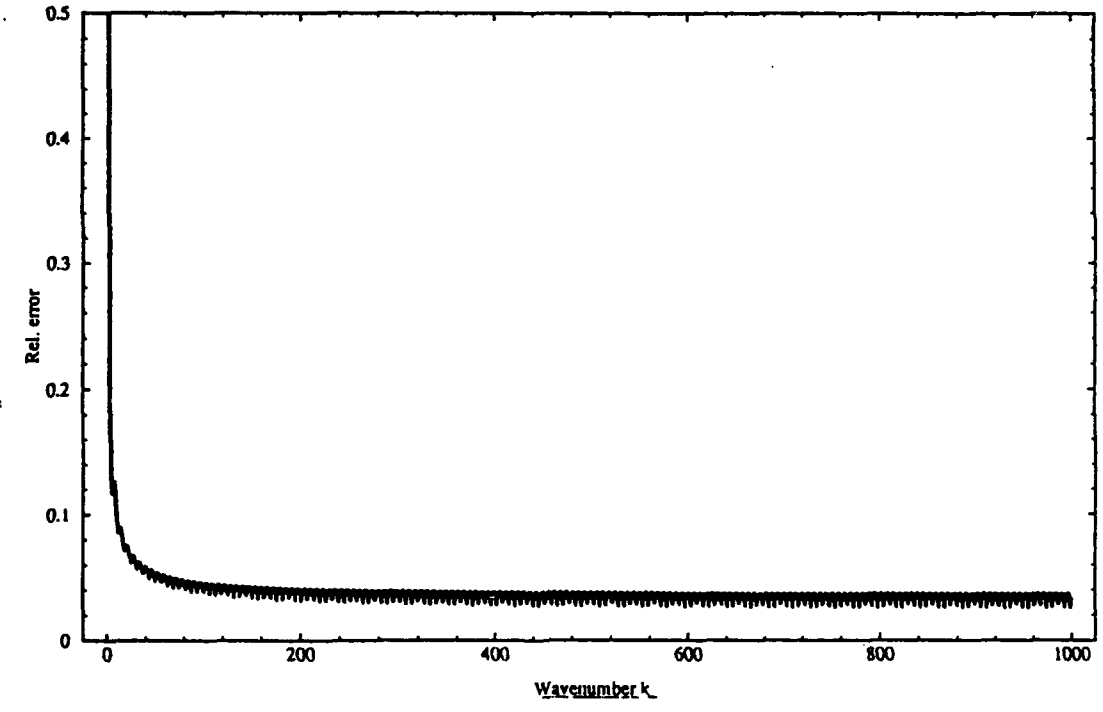

Figure 13: Relative error of FE-solution in H^1-seminorm computed with constraint $hk^{3/2} = 1$ for $k = 1, 1000.1$.

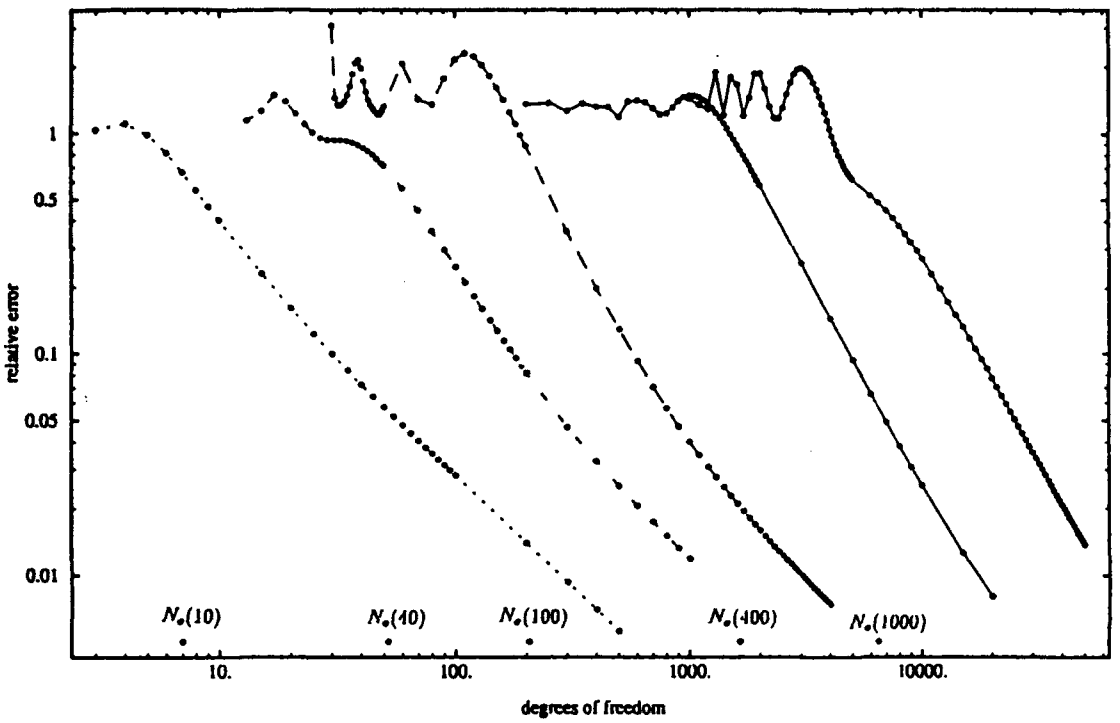

Figure 14: Relative error of the finite element solution in H^1-seminorm and predicted critical numbers of DOF for $k = 10, k = 100, k = 400$ and $k = 1000$

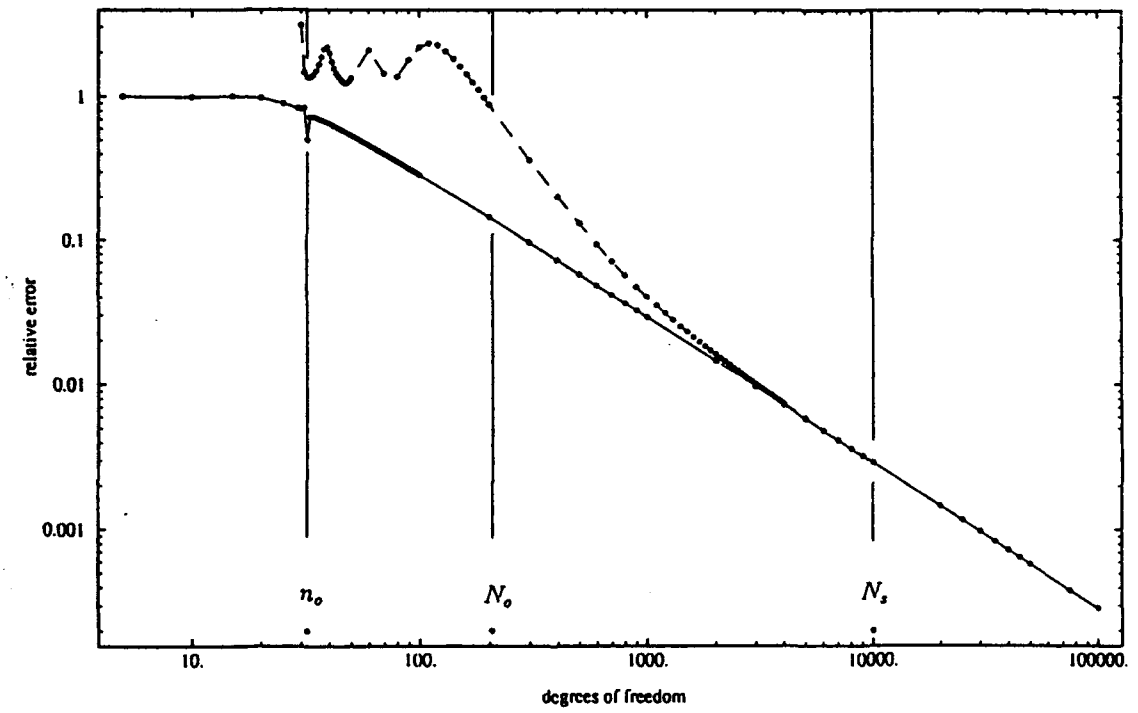

Figure 15: Relative errors of finite element solution and best approximation in H^1-seminorm for $k = 100$.

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- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

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