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Perfect 0, ± Matrices

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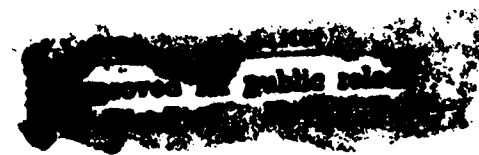Università di Padova
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Abstract

Perfect graphs and perfect $0, 1$ matrices are well studied in the literature. Here we introduce perfect 0 ± 1 matrices. Our main result is a characterization of these matrices in terms of a family of perfect $0, 1$ matrices.

1   Introduction

Given a $0, \pm 1$ row vector a, let $\nu(a)$ denote the number of negative entries in a. The inequality $ax \leq 1 - \nu(a)$ is called a *generalized set packing inequality*. Given a $0, \pm 1$ matrix A, let $\nu(A)$ denote the column vector whose i^{th} component is the number of -1's in the ith row of A. The *generalized set packing polytope* is $Q(A) = \{x \in R^n : Ax \leq 1 - \nu(A), 0 \leq x \leq 1\}$. Note that the inequalities $x_i \leq 1$ and $-x_i \leq 0$ are the generalized set packing inequalities with exactly one nonzero element. Since these bounds appear explicitly in the description of $Q(A)$, we assume w.l.o.g. that every row of A contains at least two nonzero entries. The *generalized set packing problem* consists of finding a $0, 1$ vector $x \in Q(A)$ which maximizes some linear objective function cx. The generalized set packing problem is equivalent to the following logic problem: given a set of clauses (here, a clause is a set of literals and a literal is an atomic proposition or its negation) and weights associated with the atomic propositions, find an assignment of "true" or "false" to the atomic propositions such that each clause contains at most one false literal and the sum of the weights of the false atomic propositions is minimized.

A $0, \pm 1$ matrix A is *perfect* if $Q(A)$ has only $0, 1$ vertices. When A is perfect, the generalized set packing problem can be solved as a linear program. For $0, 1$ matrices, the concept of perfection is well studied. It is well-known that a $0, 1$ matrix is perfect if and only if it is the clique-node matrix of a perfect graph, a concept introduced by Berge [1]. Two books, several conferences and well over a hundred papers have already been devoted to the subject. Therefore it seems natural to relate the notion of perfection for $0, \pm 1$ matrices to that for $0, 1$ matrices.

Given a $0, \pm 1$ matrix A, the matrix A' obtained from A by multiplying by -1 all entries in a subset S of the columns is said to be obtained from A by *switching signs* in the columns of S. Note that the transformation

2

$y_i = x_i$, $i \notin S$ and $y_i = 1 - x_i$, $i \in S$ maps $Q(A)$ into $Q(A')$. In particular, A is perfect if and only if A' is perfect.

We say that a polytope Q contained in the unit hypercube $[0,1]^n$ is *irreducible* if, for each j, both polytopes $Q \cap \{x_j = 0\}$ and $Q \cap \{x_j = 1\}$ are nonempty. Irreducibility of a polytope Q defined by a system of linear inequalities can be checked using linear programming and it is a natural assumption to make for the generalized set packing problem since, when $Q(A)$ is reducible, some variables can be fixed to 0 or 1, and the resulting problem is still a generalized set packing problem, in a lower dimensional space.

For $0, \pm 1$ row vectors $a = (a_1, \ldots, a_n)$ and $d = (d_1, \ldots, d_n)$, the inequality $ax \leq 1 - \nu(a)$ *dominates* $dx \leq 1 - \nu(d)$ if $d_j \neq 0$ implies $a_j = d_j$ or, equivalently, if $\{0 \leq x \leq 1 : ax \leq 1 - \nu(a)\} \subseteq \{0 \leq x \leq 1 : dx \leq 1 - \nu(d)\}$. Given a $0, \pm 1$ matrix A, the *completion* of A is the matrix A^* obtained by adding to A all row vectors a, with at least two nonzero entries, that induce a generalized set packing inequality $ax \leq 1 - \nu(a)$ which is valid for $Q(A)$ and not dominated by any other inequality in A^*. Obviously, $Q(A^*) = Q(A)$. A $0, 1$ matrix B obtained from A^* by switching signs in some columns and replacing all negative entries of the resulting matrix by 0 is called a *monotone completion* of A.

The following theorem is inspired by a similar result due to Hooker [5] for the generalized set covering polytope.

Theorem 1 *Let A be a $0, \pm 1$ matrix such that the generalized set packing polytope $Q(A)$ is irreducible. Then A is perfect if and only if all the monotone completions of A are perfect $0, 1$ matrices.*

For a monotone completion B of A, obtained by switching signs in the column set S of A^* (and then setting the -1's to 0's), let B^* be the matrix obtained from B by switching back the signs in the column set S. Let $\mathcal{B}^*$ be the family of all such matrices B^*. Since $Q(A) = Q(A^*) = \bigcap_{B^* \in \mathcal{B}^*} Q(B^*)$, the above theorem provides an interesting example of a polytope Q obtained as the intersection of a family of polytopes Q_k such that Q has $0, 1$ vertices if and only if each Q_k has $0, 1$ vertices.

3

2 Proof of Theorem 1

The proof of the theorem uses some lemmas. A set $S \in R^r$ is the *projection* of the set $Q \in R^n$ into the subspace of variables $x_1, \ldots, x_r$ if S contains all vectors $(x_1^*, \ldots, x_r^*)$ such that there exists a vector $(x_1^*, \ldots, x_r^*, x_{r+1}^*, \ldots, x_n^*) \in Q$. A well-known procedure for computing the projection of a polyhedron Q into the space of variables $x_1, \ldots, x_r$ is the Fourier-Motzkin elimination procedure, see [8].

Lemma 2 *Let A be a $0, \pm 1$ matrix such that the generalized set packing polytope $Q(A)$ is irreducible, and let $ax \leq 1 - \nu(a)$ and $dx \leq 1 - \nu(d)$ be two generalized set packing inequalities which are valid for $Q(A)$. If $a_j = -d_j \neq 0$ for some j, then either $a_k d_k = 0$ for every $k \neq j$, or a and d each have exactly two nonzero entries and $a = -d$.*

Proof: W.l.o.g. we assume that $a_n = -d_n = 1$, so the inequalities $ax \leq 1 - \nu(a)$ and $dx \leq 1 - \nu(d)$ can be written as

$$\sum_{j \in P_1} x_j + \sum_{j \in N_1} (1 - x_j) + x_n \leq 1$$

$$\sum_{j \in P_2} x_j + \sum_{j \in N_2} (1 - x_j) + (1 - x_n) \leq 1.$$

Adding up these two inequalities, we obtain a valid inequality for $Q(A)$, namely

$$\sum_{j \in P_1} x_j + \sum_{j \in P_2} x_j + \sum_{j \in N_1} (1 - x_j) + \sum_{j \in N_2} (1 - x_j) \; \leq \; 1. \tag{1}$$

If $j \in P_1 \cap P_2$, then $x_j < 1$ for every $x \in Q(A)$, contradicting the assumption. So $P_1 \cap P_2 = \emptyset$. Similarly, $N_1 \cap N_2 = \emptyset$.

Now consider $(P_1 \cap N_2) \cup (P_2 \cap N_1)$. If this set has cardinality greater than one, then inequality (1) is inconsistent, implying that $Q(A)$ is empty, a contradiction to the assumption that $Q(A)$ is irreducible. If $(P_1 \cap N_2) \cup (P_2 \cap N_1) = \emptyset$, then n is the only index where $a_j = -d_j \neq 0$ and we are done. Finally, assume $(P_1 \cap N_2) \cup (P_2 \cap N_1)$ has cardinality one. Then inequality (1) implies that $x_j = 0$ for $j \in P_1 \cup P_2 \setminus N_1 \cup N_2$ and that $x_j = 1$ for $j \in N_1 \cup N_2 \setminus P_1 \cup P_2$. Since $Q(A)$ is irreducible, these two sets must be empty. This implies that a and d each have exactly two nonzero entries and $a = -d$. This proves the lemma. $\square$

Lemma 3 *Let A be a $0, \pm 1$ matrix. If $Q(A)$ is irreducible, then the projection of $Q(A)$ into the subspace of variables $x_1, \ldots, x_r$ is an irreducible generalized set packing polytope $Q(A^r) = \{x \in R^r : A^r x \leq 1 - \nu(A^r), \ 0 \leq x \leq 1\}$.*

Proof: The fact that the projection of $Q(A)$ is irreducible follows immediately from the definition. Hence, to prove the lemma, it suffices to establish that any nontrivial inequality obtained by the Fourier-Motzkin elimination of one variable from two inequalities of $Q(A)$ is a generalized set packing inequality. Then the result follows by induction. Consider any inequality obtained from two inequalities $ax \leq 1 - \nu(a)$ and $dx \leq 1 - \nu(d)$ of $Q(A)$ by the Fourier-Motzkin elimination of x_n. It follows from Lemma 2 that the resulting inequality is either the trivial inequality $0 \leq 0$ or it is of the form $bx \leq 1 - \nu(b)$ where b is a $0, \pm 1$ vector, proving the result. $\square$

Theorem 4 *Let A be a $0, \pm 1$ matrix such that the generalized set packing polytope $Q(A)$ is irreducible. Every row of A^* is either a row of A or it is generated from the inequalities of $Q(A)$ by the Fourier-Motzkin elimination procedure.*

Proof: Let a be a row of A^* but not A and suppose that $ax \leq 1 - \nu(a)$ is not generated from $Q(A)$ by the Fourier-Motzkin elimination procedure. By switching signs in some columns of A^* if necessary, we can assume that, for some $2 \leq r \leq n$, $a_1 = \ldots = a_r = 1$ and $a_{r+1} = \ldots = a_n = 0$. Since the inequality $ax \leq 1$ is valid for $Q(A)$, it is also valid for $Q(A^r)$, so it must be a positive combination of the inequalities defining $Q(A^r)$. By Lemma 3, the polytope $Q(A^r)$ is an irreducible generalized set packing polytope. By Lemma 2, any inequality $dx \leq 1 - \nu(d)$ defining $Q(A^r)$ which has a negative coefficient is either a bound inequality $-x_i \leq 0$ or, in the case where $r = 2$, the inequality $-x_1 - x_2 \leq -1$. All other inequalities defining $Q(A^r)$ are strictly dominated by $ax \leq 1$, i.e. they are of the form $dx \leq 1$ where $d_j = 0$ or 1 for all $j = 1, \ldots, r$ and $d_j = 0$ for at least one $j = 1, \ldots, r$.

When $r = 2$, $Q(A^r)$ is defined by the bound constraints $0 \leq x_1, x_2 \leq 1$ and possibly $-x_1 - x_2 \leq -1$. It is easy to check that $ax \leq 1$ is not a positive combination of these inequalities, a contradiction.

When $r \geq 3$, $Q(A^r)$ is defined by the bound constraints and inequalities $dx \leq 1$ which are strictly dominated by $ax \leq 1$. Again, it follows that

5

$ax \leq 1$ cannot be obtained as a positive combination of these inequalities, a contradiction. $\square$

Lemma 5 *Let $x^* = (x_1^*, \ldots, x_n^*)$ be a vector in $Q(A)$ such that $x_n^* = 0$ or 1. Then x^* is a vertex of $Q(A)$ if and only if $(x_1^*, \ldots, x_{n-1}^*)$ is a vertex of the projection of $Q(A)$ into the subspace of variables $x_1, \ldots, x_{n-1}$.*

Proof: Let P be this projection.

$\Rightarrow$ Let $A'x = 1 - \nu(A')$, $x_j = 0$ for $j \in K$, $x_j = 1$ for $j \in L$ be a subsystem of $Ax = 1 - \nu(A)$, $x = 0$, $x = 1$ which has x^* as its unique solution. If $x_j^* = 1$, we assume w.l.o.g. that L contains equation $x_j = 1$ and if $x_j^* = 0$, we assume w.l.o.g. that K contains equation $x_j = 0$. Let $\bar{A}'$ be the matrix obtained from A' by removing the last column. Then $\bar{A}'x \leq 1 - \nu(\bar{A}')$ is a system of valid inequalities for P. Furthermore, since $A'x^* = 1 - \nu(A')$, then if $x_n^* = 0$, the nth column of A' is a $0, 1$ vector, and if $x_n^* = 1$, the nth column of A' is a $0, -1$ vector. This shows that $(x_1^*, \ldots, x_{n-1}^*)$ is the unique solution of the system $\bar{A}'x = 1 - \nu(\bar{A}')$, $x_j = 0$ for $j \in K \setminus \{n\}$, $x_j = 1$ for $j \in L \setminus \{n\}$. Hence, $(x_1^*, \ldots, x_{n-1}^*)$ is a vertex of P.

$\Leftarrow$ Assume not. Then x^* is the convex combination of vectors $x^1, \ldots, x^k \in Q(A) \setminus \{x^*\}$. Since $x_n^1 = \ldots = x_n^k = x_n^*$, then the vectors $(x_1^j, \ldots, x_{n-1}^j)$, $j = 1, \ldots, k$, belong to P and are distinct from $(x_1^*, \ldots, x_{n-1}^*)$, contradicting the assumption that $(x_1^*, \ldots, x_{n-1}^*)$ is a vertex of P. $\square$

Proof of Theorem 1: $\Rightarrow$ Assume not and let A be a $0, \pm 1$ matrix with the smallest number of columns such that the generalized set packing polytope $Q(A)$ has $0, 1$ vertices but, for some monotone completion B, the polytope $Q(B)$ has a fractional vertex x^*.

First note that every component of x^* is fractional. For if not, say $x_n^* = 0$ or 1, then Lemma 3 shows that the projection of $Q(A)$ into the space of variables $x_1, \ldots, x_{n-1}$ is an irreducible generalized set packing polytope $Q(\bar{A})$. Since B is a $0, 1$ matrix, the projection of $Q(B)$ is $Q(\bar{B})$, where $\bar{B}$ is the submatrix of B obtained by removing the last column. It follows from the Fourier-Motzkin elimination procedure that $\bar{B}$ is a monotone completion of $\bar{A}$. Furthermore, Lemma 5 shows that $(x_1^*, \ldots, x_{n-1}^*)$ is a vertex of $Q(\bar{B})$. This contradicts our choice of the matrix A with smallest number of columns.

By changing variables $y_j = 1 - x_j$ if necessary in $Q(A)$, we assume w.l.o.g. that the $0, 1$ matrix B is obtained from A^* without any switching of signs in

6

the columns. Let $ax \leq 1 - \nu(a)$ be a row of $Ax \leq 1 - \nu(A)$ which is violated by x^*. Since B is a monotone completion of A, there exists at least one index t such that $a_t = -1$.

Case 1: There exist $\beta x = 1$ (among the equations from $Bx = 1$ which define x^*), and columns t and t' such that $a_t = a_{t'} = -1$ and $\beta_t = \beta_{t'} = 1$.

By Lemma 2, $\beta x \leq 1$ is the inequality $x_t + x_{t'} \leq 1$ and $ax \leq 1 - \nu(a)$ is the inequality $(1 - x_t) + (1 - x_{t'}) \leq 1$. Using the fact that $\beta x^* = 1$, it now follows that $ax^* = 1 - \nu(a)$, contradicting the assumption.

Case 2: For every equality $\beta x = 1$ from the equations $Bx = 1$ which define x^*, there exists at most one t such that $\beta_t = 1$ and $a_t = -1$.

We write $ax \leq 1 - \nu(a)$ as

$$\sum_{j=1}^{k}(1 - x_j) + \sum_{j=k+1}^{n} a_j x_j \leq 1 \tag{2}$$

with $a_j = 0$ or 1. For each $t = 1, \ldots, k$, there is an inequality defining x^* in $Bx \leq 1$ such that

$$x_t^* + \sum_{j=k+1}^{n} b_{tj} x_j^* = 1 \tag{3}$$

with $b_{tj} = 0$ or 1. Adding up, we get that $\sum_{j=k+1}^{n}(a_j + \sum_{t=1}^{k} b_{tj})x_j \leq 1$ is valid for $Q(A)$. Since $Q(A)$ is irreducible, the coefficients $a_j + \sum_{t=1}^{k} b_{tj}$ are equal to 0 or 1 for all $j = k+1, \ldots, n$. Since B is a monotone completion of A, the above inequality is dominated by an inequality in $Bx \leq 1$, $0 \leq x \leq 1$. Therefore, $\sum_{j=k+1}^{n}(a_j + \sum_{t=1}^{k} b_{tj})x_j^* \leq 1$ holds. Now using (3) it follows that $ax^* \leq 1 - \nu(a)$ holds, a contradiction.

$\Leftarrow$ Assume not and let A be $0, \pm 1$ matrix with the smallest number of columns such that the generalized set packing polytope $Q(A)$ has a fractional vertex x^* but, for every monotone completion B, the polytope $Q(B)$ has $0, 1$ vertices.

First note that every component of x^* is fractional. For if not, say $x_n^* = 0$ or 1, then Lemma 3 shows that the projection of $Q(A)$ into the space of variables $x_1, \ldots, x_{n-1}$ is an irreducible generalized set packing polytope $Q(\bar{A})$. Also the projection of $Q(B)$ into the space of variables $x_1, \ldots, x_{n-1}$ is a monotone completion of $\bar{A}$. Furthermore, Lemma 5 shows that $(x_1^*, \ldots, x_{n-1}^*)$

is a vertex of $Q(\bar{A})$. This contradicts our choice of the matrix A with smallest number of columns.

Since x^* is a vertex of $Q(A)$, there is a subset $A'x = 1 - \nu(A')$ of n equations from $Ax = 1 - \nu(A)$ which has x^* as its unique solution. We will construct such a subset of n equations with the property that, in each column of A', all the nonzero entries have the same sign (when this is the case, we say that $A'x = 1 - \nu(A')$ is *monotone*). Note that the existence of a monotone system immediately implies the existence of a monotone completion with a nonintegral vertex, namely let B be the monotone completion of A obtained by switching signs in the columns of A^* for which A' has nonpositive entries and let $y_j^* = 1 - x_j^*$ for all such columns, whereas $y_j^* = x_j^*$ for the columns that have not changed sign. Then y^* is a vertex of $Q(B)$ since $y^* \in Q(B)$ and y^* is the unique solution of n equations from the inequalities defining $Q(B)$.

Now we prove the existence of a monotone system. If $A'x = 1 - \nu(A')$ is not monotone, there is some t such that for two rows, say k_1 and k_2, we have $a_{k_1 t} = 1$ and $a_{k_2 t} = -1$.

Note that t is the only column where $a_{k_1 j} = -a_{k_2 j} \neq 0$ since, otherwise, by Lemma 2, the rows k_1 and k_2 are linearly dependent, a contradiction.

In fact, it follows from Lemma 2 that the rows k_1 and k_2 can be written as

$$
\begin{aligned}
x_t^* + \sum_{j \in P_1} x_j^* + \sum_{j \in N_1}(1 - x_j^*) = 1 \\
(1 - x_t^*) + \sum_{j \in P_2} x_j^* + \sum_{j \in N_2}(1 - x_j^*) = 1
\end{aligned}
\tag{4}
$$

where the sets P_1, P_2, N_1 and N_2 are pairwise disjoint. By adding the two inequalities of $Ax \leq 1 - \nu(A)$ which correspond to rows k_1 and k_2, we obtain that

$$
\sum_{j \in P_1} x_j + \sum_{j \in P_2} x_j + \sum_{j \in N_1}(1 - x_j) + \sum_{j \in N_2}(1 - x_j) \leq 1
\tag{5}
$$

is valid for $Q(A)$. Therefore, the inequality (5) is dominated by an inequality of $A^*x \leq 1 - \nu(A^*)$. In fact, since x^* has only fractional components and satisfies (5) at equality, it follows that (5) is one of the inequalities in $A^*x \leq 1 - \nu(A^*)$. We claim that the equation obtained from (5) can be used to replace either of the two equations (4) in the system $A'x = 1 - \nu(A')$ whose unique solution is x^*. This is because either of the equations in (4) is a linear combination of the other equation in (4) and the equation resulting from (5). Consider the new linear system resulting from this interchange. In column

8

t, the number of pairs i, k where $a_{it} = -a_{kt} \neq 0$ is strictly smaller than in the original linear system. By repeating this procedure, we can remove all such pairs in column t, thus making column t monotone. Then, applying the procedure to another column with pairs i, k such that $a_{ij} = -a_{kj} \neq 0$, we note that the monotonicity in column t is not destroyed. So, by induction, we can construct a monotone linear system whose unique solution is x^*. $\square$

3   Extensions

Note that the "if" part of Theorem 1 does not hold if the irreducibility assumption is dropped, as shown by the example $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. In this case $(\frac{1}{2}, \frac{1}{2})$ is a vertex of $Q(A)$ but every monotone completion of A is perfect since all two-column $0, 1$ matrices are. However, the irreducibility assumption can be dropped for the "only if" part of the theorem. This is so because when A is a perfect $0, \pm 1$ matrix and, say $Q(A) \cap \{x_n = 1\} = \emptyset$, then $Q(A) \subset \{x_n = 0\}$ is identical to the projection of $Q(A)$ into the space of variables $x_1, \ldots, x_{n-1}$. Using an argument similar to that used in the proof of Lemma 3, one shows that the projection can be described only by generalized set packing inequalities. By repeating this projection argument if necessary, the polytope $Q(A)$ can be assumed to be irreducible.

A submatrix B of A^* is called a *row monotone completion* of A if B is a row submatrix of A^* such that, in each column, all the nonzero entries have the same sign. The matrix B is a *maximal* row monotone completion if it is not properly contained in any other row monotone completion of A. Note that there are at most 2^n maximal row monotone completions of A. When A is irreducible, it would be interesting to know whether a sharper bound is possible.

Theorem 6 *Let A be a $0, \pm 1$ matrix such that the generalized set packing polytope $Q(A)$ is irreducible. Then A is perfect if and only if all the maximal row monotone completions of A are perfect.*

Proof: The proof is the same as for Theorem 1 except for the following three statements.

9

1) $\Rightarrow$ "Every component of x^* is fractional."

2) $\Rightarrow$ **Case 2** "Since B is a monotone completion of A, the inequality $\sum_{j=k+1}^{n}(a_j + \sum_{t=1}^{k} b_{tj})x_j \leq 1$ is dominated by an inequality in $Bx \leq 1$, $0 \leq x \leq 1$."

3) $\Leftarrow$ "Every component of x^* is fractional."

1) and 3) "Every component of x^* is fractional."
We assume w.l.o.g. that B is a $0,1$ matrix. We need to show that the projection of $Q(B)$ in the space of variables $x_1, \ldots, x_{n-1}$ is a maximal row monotone completion of $\bar{A}$. Assume not. Then there exists a row $ax \leq 1 - \nu(a)$ of $\bar{A}$ such that $(a_1, \ldots, a_{n-1})$ is a $0,1$ vector, $a_n = -1$, and no row of B dominates $(a_1, \ldots, a_{n-1})$. Since B is maximal, there exists a $0,1$ row b of B where $b_n = 1$ and b is not a unit row. Eliminating variable x_n from $ax \leq 1 - \nu(a)$ and $bx \leq 1$, we get either a contradiction to the irreducibility assumption or a $0,1$ row dominating $(a_1, \ldots, a_{n-1})$.

2) $\Rightarrow$ **Case 2** We show that "Since B is a *maximal row* monotone completion of A, the inequality $\sum_{j=k+1}^{n}(a_j + \sum_{t=1}^{k} b_{tj})x_j \leq 1$ is dominated by an inequality in $Bx \leq 1$, $0 \leq x \leq 1$."
Assume that $a_j + \sum_{t=1}^{k} b_{tj} = 1$ for $j = k+1, \ldots, l$ and $a_j + \sum_{t=1}^{k} b_{tj} = 0$ for $j = l+1, \ldots, n$. Let $cx \leq 1 - \nu(c)$ be an inequality of $A^*x \leq 1 - \nu(A^*)$ which is not in $Bx \leq 1$ that dominates $\sum_{j=k+1}^{n}(a_j + \sum_{t=1}^{k} b_{tj})x_j \leq 1$ and has the smallest number of -1's. Now $c_t = 0,1$ for all $t \leq l$, for if $c_t = -1$ for some $t \leq k$, by applying Fourier-Motzkin to $cx \leq 1 - \nu(c)$ and (3) to eliminate x_t we get a contradiction to the irreducibility assumption.

Since c does not belong to B, there exists a row d in B such that, for some $t \geq l+1$, we have $c_t = -1$ and $d_t = 1$. Since d is a $0,1$ vector with at least two 1's, by applying Fourier-Motzkin to $dx \leq 1$ and $cx \leq 1 - \nu(c)$ to eliminate x_t, we get either a contradiction to the irreducibility assumption or to the assumption that c has the smallest number of -1's. $\square$

Remark The following example shows that, even if A is irreducible, the number of rows of A^* may grow exponentially with the number of rows and of columns of A. Consider the $(n+1) \times 2n$ matrix $A = \begin{pmatrix} e & u \\ -I_n & I_n \end{pmatrix}$, where

e is the n-dimensional row vector whose components are all equal to $+1$, u is the n-dimensional row vector whose components are all equal to 0 and I_n is the $n \times n$ identity matrix; the generalized set packing polytope $Q(A)$ is irreducible and the matrix A^* is obtained by adding 2^n nonnegative rows to A. Note that A is totally unimodular. Moreover this example shows that the number of maximal row monotone completions of A may be exponential in the number of rows or columns of A.

4 A Conjecture

We know of two important classes of perfect $0, \pm 1$ matrices:

- the matrices obtained from perfect $0, 1$ matrices by switching signs in a subset of columns, and

- the *balanced* $0, \pm 1$ matrices, namely those for which, in every submatrix with two nonzeros per row and column, the sum of the entries is a multiple of four. Balanced $0, \pm 1$ matrices were introduced by Truemper [9]. They are shown to be perfect in [2] and their structure is well understood, see [4] for a survey.

Using the above matrices as building blocks, it is easy to construct perfect $0, \pm 1$ matrices that belong to neither class. But we do not know how to construct all perfect $0, \pm 1$ matrices. A $0, \pm 1$ matrix A is *minimally imperfect* if it is not perfect but, for every j, the polytopes $Q(A) \cap \{x_j = 0\}$ and $Q(A) \cap \{x_j = 1\}$ have only $0, 1$ vertices. The famous strong perfect graph conjecture of Berge [1] proposes a characterization of the minimally imperfect $0, 1$ matrices. See Lovász [6] and Padberg [7] for some properties that must be satisfied by minimally imperfect 0,1 matrices. We make the following conjecture.

Conjecture 7 *Let A be a $0, \pm 1$ matrix such that the generalized set packing polytope $Q(A)$ is irreducible. The matrix A is minimally imperfect if and only if it is either*

- *obtained from a minimally imperfect $0, 1$ matrix by switching signs in a subset of columns, or*

11

- *a matrix with two nonzeros per row and column where the sum of entries is equal to 2 mod 4.*

Partial evidence to support this conjecture can be found in the following result: if A is square and the pattern of nonzeros is circulant, i.e. for some positive integer k, $a_{ij} \neq 0$ for $j = i, \ldots, i + k$ (where indices are taken modulo n, the order of A) and $a_{ij} = 0$ otherwise, then the conjecture holds. The proof of this and other results on perfect and ideal $0, \pm 1$ matrices can be found in [3].

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