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POLYNOMIAL INTERPOLATION AND APPROXIMATION OF REAL FUNCTIONS
II: SYMMETRICAL INTERPOLATION FOR THE TRIANGLE

by

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and

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20. ABSTRACT

This is the second paper in a series to discuss the approximation efficacy of polynomial interpolation of functions. In particular, the approximation accuracy depends sensitively on the locations of the interpolation nodes. We address the problem of finding the "optimal" symmetrical polynomial interpolation schemes for the triangle. The table for the symmetrical mean minimal interpolation sets for the triangle is given in this paper. An adaptive scheme for determining the interpolation order is also presented.

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Polynomial Interpolation and Approximation of Real Functions II: Symmetrical Interpolation for the Triangle

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Abstract

This is the second paper in a series to discuss the approximation efficacy of polynomial interpolation of functions. In particular, the approximation accuracy depends sensitively on the locations of the interpolation nodes. We address the problem of finding the "optimal" symmetrical polynomial interpolation schemes for the triangle. The table for the symmetrical mean minimal interpolation sets for the triangle is given in this paper. An adaptive scheme for determining the interpolation order is also presented.

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I. Introduction

In this paper, we continue our discussion of the approximation power of polynomial interpolation of real functions. We shall discuss the approximation accuracy of polynomial interpolation in a triangular domain and find the "optimal" polynomial interpolation schemes for the triangle.

For reason of efficiency and aesthetics, among all possible interpolation schemes, symmetrical polynomial interpolation schemes are of most interest. The distribution of the interpolation nodes of a symmetrical interpolation obeys the symmetries of the triangle symmetry group D_3, *i.e.*, for the standard triangle shown in Fig. 1, the nodal distribution is invariant under the reflections of the 3 triangle symmetry axes.

The approach we have used in finding the optimal interpolation sets in the interval [1] can be generalized to finding the optimal interpolation sets in the triangle. We shall estimate the interpolation error of a function in terms of its least deviation from the polynomial interpolation space and the norm of the interpolation operator. The latter can be minimized by redistributing the interpolation nodes. The resulting sets are called the minimal sets and they have good interpolation properties.

We shall restrict our attention to finding the symmetrical minimal sets. Unlike polynomial interpolation in the interval where the minimal sets are symmetrical under reflection, we don't know if the actual minimal sets of the interpolation operator are indeed symmetrical under the triangle group D_3.

The main difficulty in finding the minimal sets is the complication due to the exponential explosion in the number of local minima in the search space. We shall devise procedures to circumvent this.

In section II, we introduce the notation and review some known results for polynomial interpolation in the triangle. In section III, we introduce the concept of the minimal sets and find the mean minimal sets and discuss their properties. We show some numerical interpolation examples in section IV. In section V, we give an adaptive procedure for determining the interpolation order and discuss the C^0 continuity for functions approximated in a triangular mesh.

II. Formulation and Notations

Let D be the standard triangular domain shown in Fig. 1, I_1, I_2, I_3 be the 3 edges (3 closed intervals) of the standard triangle. We shall denote $\mathbf{x} = (x_1, x_2)$ the Cartesian coordinates in the triangle. Let $C(D)$ be the space of real continuous functions on D, $\mathcal{P}_n(D)$ be the space of algebraic polynomials on D of degree $\leq n$, *i.e.*, the space $\mathcal{P}_n(D)$ consists of all linear combinations of monomials: $\mathcal{P}_n(D) = Span\{x_1^i x_2^j, i \geq 0, j \geq 0, i + j \leq n\}$. The dimension of the space $\mathcal{P}_n(D)$ is $N_n = (n+1)(n+2)/2$.

Let I be a closed 1-dimensional interval, we shall denote $C(I)$ the space of real continuous functions on I, $\mathcal{P}_n(I)$ the space of algebraic polynomials on I of degree $\leq n$.

For $f \in C(D)$, we can find a unique n-th order polynomial $p_n \in \mathcal{P}_n(I)$ which takes the value of f at given N_n points in the triangle, provided some nondegenerate conditions on these interpolation points are satisfied. These N_n points will be called interpolation nodes, and denoted by $T = (\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_{N_n})$, where $\mathbf{t}_i \in D, i = 1, ..., N_n$. The

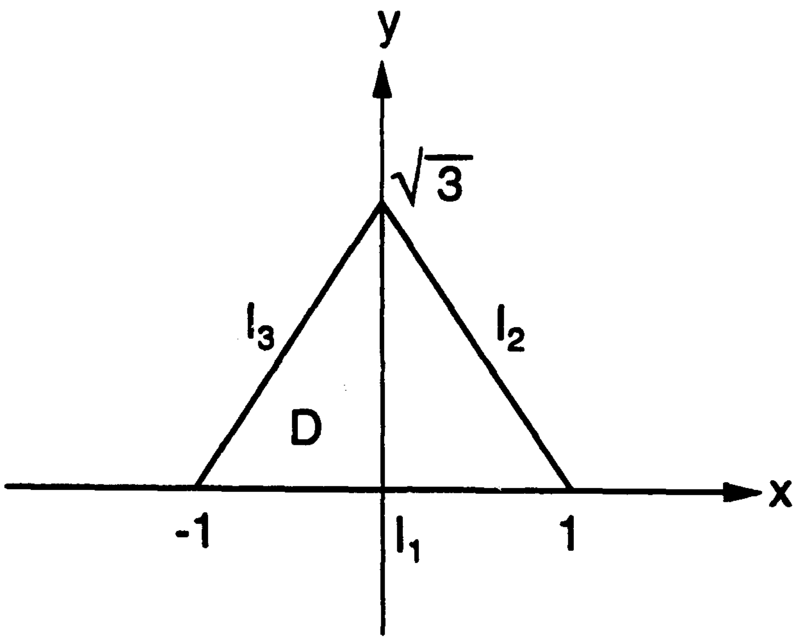

Figure 1: The standard triangle.

n-th order polynomial interpolant is given by $p_n(\mathbf{x}) = \mathcal{L}_T f(\mathbf{x}) := \sum_{k=1}^{N_n} f(\mathbf{t}_k) L_k(\mathbf{x})$, where $L_k(\mathbf{x}) \in \mathcal{P}_n(D), k = 1, ..., N_n$ are, under the nondegenerate condition, uniquely determined by: $L_i(\mathbf{t}_j) = \delta_{ij}$.

$\mathcal{L}_T$ is a linear projection operator which maps a real continuous function to its corresponding polynomial interpolant. n is called the order of the interpolation T.

We list some general results on polynomial approximation of functions.

Theorem 1 [3] (Weierstrass) Every continuous function in $C(D)$ can be uniformly approximated by polynomials on D to any degree of accuracy.

The least deviation $d_n(f)$ of a function f in $C(D)$ is the distance between f and $\mathcal{P}_n(D)$: $d_n(f) := inf\{sup_{\mathbf{x}\in D}|q(\mathbf{x}) - f(\mathbf{x})|, q(\cdot) \in \mathcal{P}_n(D)\}$. This least deviation is always attained.

Theorem 2 [2] Given $f(\cdot) \in C(D)$, there exists $w_n(\cdot) \in \mathcal{P}_n(D)$, such that for all

$q_n(\cdot) \in \mathcal{P}_n(D)$, $\|f - w_n\|_\infty \leq \|f - q_n\|_\infty$, where $\|\cdot\|$ is the L^∞ norm in $C(D)$ $\|f\|_\infty = sup_{\mathbf{x}\in D}|f(\mathbf{x})|$.

Jackson type results concerning the behaviors of $d_n(f)$ and its relation to the analyticity of the function f are lacking for functions defined in the triangular domain.

Since $\mathcal{L}_T$ is a linear projection operator from $C(D)$ to $\mathcal{P}_n(D)$, we define the L^p norm of $\mathcal{L}_T$ as $\|\mathcal{L}_T\|_p := sup_{f\neq 0, f\in C(D)} \frac{\|\mathcal{L}_T f\|_p}{\|f\|_\infty}$, where $1 \leq p \leq \infty$ and $\|f\|_p = \int_D(|f(\mathbf{x})|^p d\mathbf{x})^{1/p}$.

The L^∞ norm of $\mathcal{L}_T$ is called the Lebesgue constant of $\mathcal{L}_T$ and is denoted by $\lambda(T)$. It is given by: $\lambda_n(T) = \|\mathcal{L}_T\|_\infty = sup_{\mathbf{x}\in D} \sum_{k=1}^{N_n} |L_k(\mathbf{x})|$. Let $_IT$ be the interpolation nodes restricted to the edge I_1, $\mathcal{L}_{IT}$ be the interpolation operator in I_1 with interpolation nodes $_IT$. Obviously, $\|\mathcal{L}_T\|_\infty \geq \|\mathcal{L}_{IT}\|_\infty$.

We also have $\|\mathcal{L}_T\|_p \leq |\mathcal{L}_T|_p := (\int_D(\sum_{k=1}^{N_n} |L_k(\mathbf{x})|)^p d\mathbf{x})^{1/p}$, where $\int_D d\mathbf{x}$ denotes integration in the standard triangle. We shall call $|\mathcal{L}_T|_p$ the L^p pseudonorm or the $|L|^p$ norm of the interpolation operator $\mathcal{L}_T$. We define the "mean norm" of $\mathcal{L}_T$ as $\|\langle\mathcal{L}_T\rangle\| := (\int_D \sum_{k=1}^{N_n} |L_k(\mathbf{x})|^2 d\mathbf{x})^{1/2}$.

Theorem 3 For all $f(\cdot) \in C(D)$, $\|f - \mathcal{L}_T f\|_p \leq (3^{\frac{1}{2p}} + \|\mathcal{L}_T\|_p) d_n(f)$.

Note that the n-th order interpolant of the best approximant w_n in Theorem 2 is itself, hence

$\|f - \mathcal{L}_T f\|_p = \|f - w_n + \mathcal{L}_T(w_n - f)\|_p \leq \|f - w_n\|_p + \|\mathcal{L}_T(w_n - f)\|_p \leq \|f - w_n\|_\infty(3^{\frac{1}{2p}} + \|\mathcal{L}_T\|_p) = (3^{\frac{1}{2p}} + \|\mathcal{L}_T\|_p) d_n(f)$.

Note for a given interpolation T, the bound $\|\mathcal{L}_T\|_p d_n(f)$ for $\|\mathcal{L}_T(w_n - f)\|_p$ is sharp.

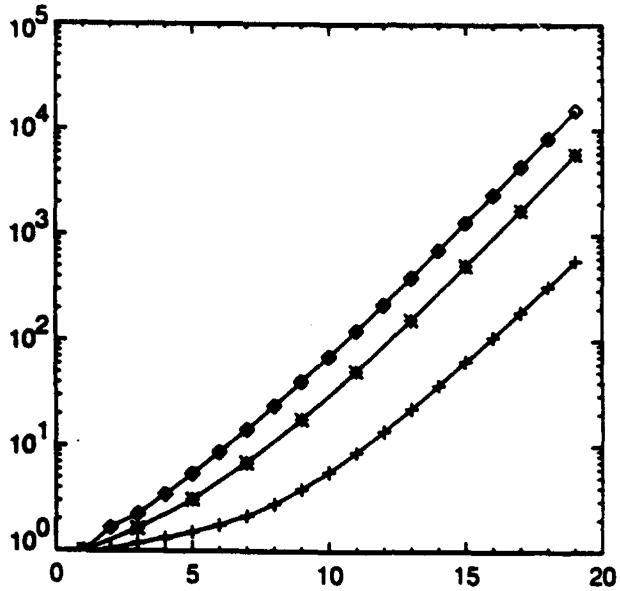

Figure 2: The mean norm (cross) and the L^∞ norm (diamond) for the n-th order uniform interpolation operator as a function of n. The asterisks are the L^∞ norms for the uniform interpolation operator on the standard interval I: $\|\mathcal{L}_{I T}\|_\infty$

i.e., there exists a nonpolynomial continuous function f such that $\|\mathcal{L}_T\|_p d_n(f) = \|\mathcal{L}_T(w_n - f)\|_p$. In this sense, we say the inequality in Theorem 3 is optimal.

In order to insure C^0 continuity on the interelement boundaries, we shall consider the following type of interpolation schemes. For n-th order interpolation, the interpolation set includes exactly $n+1$ points on each edge I_i of the triangle. Therefore the polynomial space $\mathcal{P}_n(D)$ restricted to the edge I_i reduces to $\mathcal{P}_n(I_i)$, and the polynomial interpolant in the triangle restricted to I_i is the same as the polynomial interpolant in the interval I_i. For such an interpolation scheme, one can design proper procedure to guarantee the C^0 continuity of the interpolating function on the edges of the triangle when the function is approximated using piecewise polynomial interpolation in a triangle mesh. (See section V for more details).

The simplest and widely used interpolation scheme in the triangle is the uniform interpolation scheme. The n-th order uniform interpolation set is given by $T^n_{unif} = \{\frac{2i+j-n}{n}, \frac{\sqrt{3}j}{n}, 0 \leq i, 0 \leq j, 0 \leq i+j \leq n\}$. Indeed, the isoparametric formulation of Finite Element Method uses the uniform interpolation scheme to approximate functions in the triangle. Since $\|\mathcal{L}_T\|_\infty \geq \|\mathcal{L}_{_I T}\|_\infty \geq Cexp(n/2)$, the norm for the n-th order uniform interpolation operator grows at least exponentially with the interpolation order n. This behavior is illustrated in Fig. 2 for the L^∞ norm as well as the mean norm. Therefore as interpolation order increases, the isoparametric formulation may lead to spurious results in high order calculations if the least deviation of the function to be approximated does not decrease fast enough.

III. Symmetrical Minimal Interpolation Sets

Theorem 3 gives the upper bound for the interpolation error of a function f in L^∞ norm as $(1 + \|\mathcal{L}_T\|_\infty)d_n(f)$. We can minimize this optimal error upper bound by minimizing the Lebesgue constant of the interpolation operator $\|\mathcal{L}_T\|_\infty$ through the redistribution of the interpolation nodes (the least deviation $d_n(f)$ by definition is already the smallest possible error). Similarly, since $\|\mathcal{L}_T\|_p \leq |\mathcal{L}_T|_p$, $(3^{\frac{1}{2p}} + |\mathcal{L}_T|_p)d_n(f)$ also gives an upper bound for the interpolation error of a function in L^p norm. We can also minimize $|\mathcal{L}_T|_p$ to minimize the interpolation error bound in L^p norm. We shall say that an interpolation set is minimal in $|L|^p$ or L^∞ if it minimizes $|\mathcal{L}_T|_p$ or $\|\mathcal{L}_T\|_\infty$. We shall also say an interpolation set is minimal in the mean if it minimizes $\|\langle\mathcal{L}_T\rangle\|$.

We are not aware of any computationally feasible way to determine an arbitrary order L^∞ symmetrical minimal set. Therefore, we shall concentrate on finding the mean minimal sets. Notice the integrand for the mean norm of the interpolation operator is a polynomial, hence, can be integrated exactly using numerical quadrature. The symmetrical n-th order L^∞ and mean minimal sets will be denoted by $T^n_{L^\infty}$ and $T^n_{\langle L \rangle}$ respectively.

One key result from our study of polynomial interpolation in the interval is that n-th order symmetrical mimimal sets are close to each other and their interpolation properties are quite similar. All minimal sets are good for interpolation purpose. Similarly, for symmetrical interpolation in the triangle, numerical evidence indicates that the norm of the interpolation operator for the n-th order mean minimal set is close to that of the n-th order $|L|^1$ minimal set. Therefore, we believe this norm is also close to the norm of the n-th order L^∞ minimal sets.

To find the interpolation nodes on the edges of the standard triangle, we note that for a good interpolation set, the interpolation error of a function on the 3 edges of the triangle must be small. For n-th order interpolation, by mapping linearly the standard interval $I = [-1, 1]$ to the 3 edges of the standard triangle, we shall place the n-th order mean minimal set in the interval (which is also symmetrical) as the interpolation nodes on the edges. These nodes automatically satisfy the triangle symmetries. The n-th order minimal set under the above constraint is in fact close to the actual n-th order mean minimal set. We have verified that the difference in the mean norm as well as the L^∞ norm between these two sets are quite small.

Now we consider the distribution of interpolation nodes inside the triangle. There

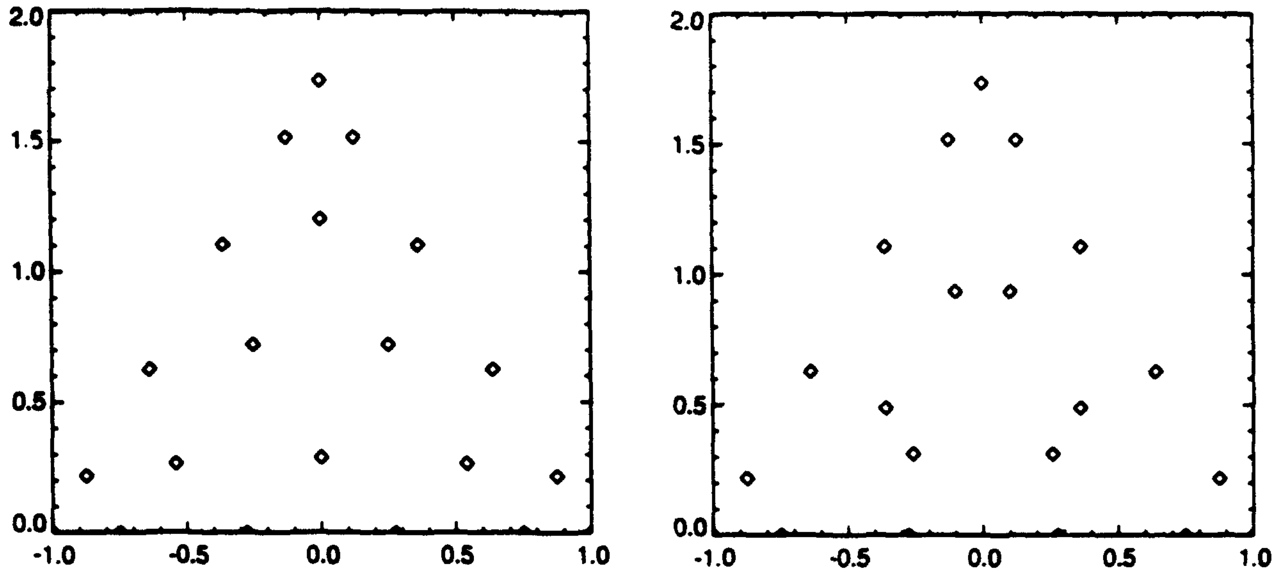

Figure 3: The positions of the interpolation nodes of two local minima for the mean norm. Case 1 (the left graph) corresponds to the global minimum of the mean norm; while for case 2 (the right graph), the mean norm is of order 4.9×10^{14}.

Appendix for interpolation order $n = 1$ to $n = 12$.

The minimum of the mean norm is found by using a minimization procedure. All minimization procedures only find local minimum, hence good initial guess of the actual locations of the minimal sets is essential for the success of the search algorithm. The landscape for the mean norm is extremely complicated. A rule for constructing good initial guess from the one dimensional minimal sets has also been successfully devised and is used to find the symmetrical minimal sets up to order 20.

In the appendix, we also list the approximate minimal sets in the triangle barycentric coordinates: $b_1 = (1 - x_1 - x_2/\sqrt{3})/2$, $b_2 = (1 + x_1 - x_2/\sqrt{3})/2$, $b_3 = x_2/\sqrt{3}$. We note that these sets have the smallest mean norms from our computation, and may not correspond to the actual global minimal sets. However, we believe that these are many possible symmetrical patterns for such a distribution. A node can be at the center of the standard triangle, or on one of the symmetrical axes (the meridians) of the triangle or neither (this node is located inside one of the 6 subtriangles bounded by the symmetry axes). By symmetry, this node corresponds a point of a singlet, a triplet, or a sextuplet respectively. We shall denote the number of singlets, triplets, sextuplets by $n_1 (n_1 = 0 \text{ or } 1)$, n_3, n_6. Since the total number of singlets, triplets, sextuplets equals the number of interpolating nodes inside the triangle, we have $n_1 + 3n_3 + 6n_6 = (n-1)(n-2)/2$.

The integer solution for the above equation is nonunique when $n \geq 5$. Different integer pairs of (n_1, n_3, n_6) correspond to different symmetry patterns of the nodal set. Each symmetry pattern has a mean norm minimum, we want to find the smallest mean norm (the global minimum) among these minima. Not all minima are good interpolation sets, and minima of distinct symmetry patterns may exhibit drastically different interpolation properties. For example, Fig. 3 shows in the case of interpolation order 5, the symmetry patterns of the two local minima for the mean norm. For pattern 1 $(n_1 = 0, n_3 = 2, n_6 = 0)$, which in fact is the global minimum, the mean norm is 1.24: while for pattern 2 $(n_1 = 0, n_3 = 0, n_6 = 1)$, the approximate minimum of the mean norm is so huge ($\sim 4.9 \times 10^{14}$, this number is computed from a double precision algorithm) that interpolation using this set becomes meaningless.

From all symmetry patterns of the interpolation nodes, we have extracted a unique symmetry which appears to be the correct symmetry for the global minimum. The symmetry patterns (n_1, n_3, n_6) for our predicted global minimal sets are shown in the

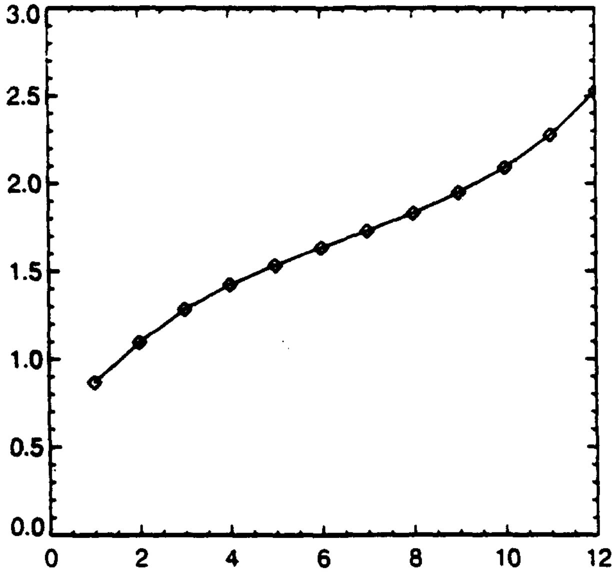

Figure 4: Square of the mean norm for the n-th order mean minimal set as a function of n.

sets are close to the actual minimal sets. Interpolation properties for these sets are illustrated in the following.

The mean minimal sets are close to the actual L^∞ minimal sets. For example, for order 3 minimal sets, the minimum for the L^∞ norm is 2.10845, while the L^∞ norm for the order 3 mean minimal set is 2.11162.

In Fig. 4, we show the mean norm for the mean minimal sets. In Fig. 5, we show the approximate $|L|^1$ and the L^∞ norms for the mean minimal sets. The approximate $|L|^1$ norm is found by overintegration in the numerical quadrature. The L^∞ norm is found by navigating the landscape of the function $\sum_{k=1}^{N_n} |L_k(\mathbf{x})|$.

Since $\|\mathcal{L}_T\|_\infty \geq \|\mathcal{L}_{IT}\|_\infty$, the norm of the interpolation operator increases at least logarithmically. It appears that the norm of interpolation operator for the minimal sets in the triangle increases much faster than that.

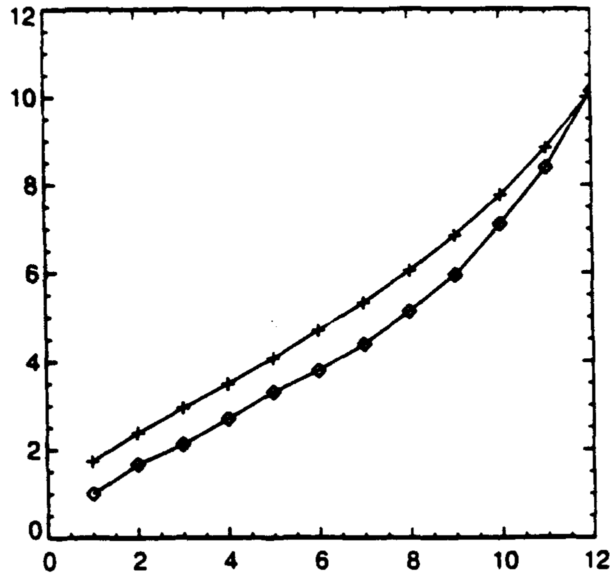

Figure 5: The approximate $|L|^1$ norm (cross) and the L^∞ (diamond) norm for the n-th order mean minimal set as a function of n. The asterisks are the bound for the L^{infty} norm for the rectangular minimal sets

V. Examples

In Fig. 6, we interpolate the following rational function $g = \frac{4}{(1+(2b_1-1)^2)(1+(2b_2-1)^2)(1+(2b_3-1)^2)}$, where b_1, b_2, b_3 are the barycentric coordinates $b_1 = (1 - x_1 - x_2/\sqrt{3})/2$, $b_2 = (1 + x_1 - x_2/\sqrt{3})/2$, $b_3 = x_2/\sqrt{3}$. The error $g - \mathcal{L}_T g$ is illustrated for the order 9 approximation. We also show in Fig. 7 the L^2 norm for the interpolation error as a function of interpolation order. The error decreases exponentially in accordance with the fact that g is infinitely differentiable.

To see the efficiency of the triangle interpolation. In Fig. 8, we plot the L^2 norm of the interpolation error for the function $f(x, y) = \frac{1}{(1+x^2)(1+y^2)}$ in the rectangular domain $[-1, 1] \times [-1, 1]$ using the the direct product of 2 one-dimensional minimal sets (henceforth referred as product interpolation). We also plot the L^2 norm of interpo-

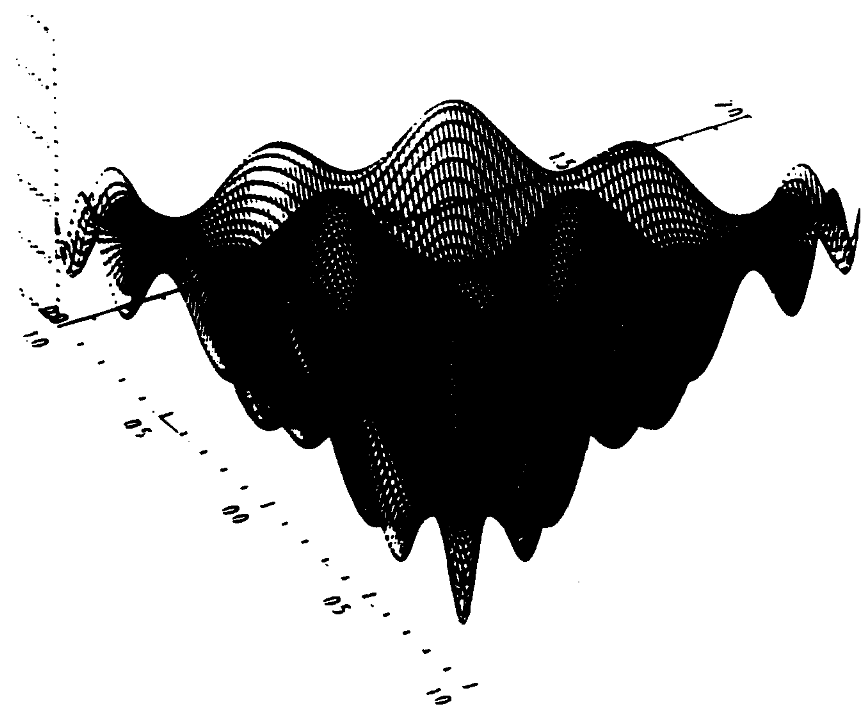

Figure 6: The error $g - \mathcal{L}_T g$ for order 9 interpolation, g is defined in the text.

lation error in the same graph using interpolations in 2 diagonally meshed triangles. We see that the error is of similar order. The rectangular product interpolation appears to be more efficient. This is expected since the dimension of the n-th order polynomial product interpolation space is almost twice that of $P_n(\mathcal{D})$.

V. Adaptive Determination of Approximation Order

Suppose we want to approximate a function in a general 2 dimensional domain. we can mesh this domain with triangles and approximate the function in each triangle by polynomial interpolation. However, it is usually inefficient to use the same order polynomial interpolation for the function in the whole triangle meshed domain. For example, in Fig. 9, we plot the function $h(x, y) = \frac{1}{(1+(x+1)^2)(1+(y+1)^2)}$ in the rectangular domain $[0, 4] \times [0, 4]$. The triangle mesh for the rectangular domain is shown in Fig. 10. If we use order 5 interpolation to approximate the function in all triangles.

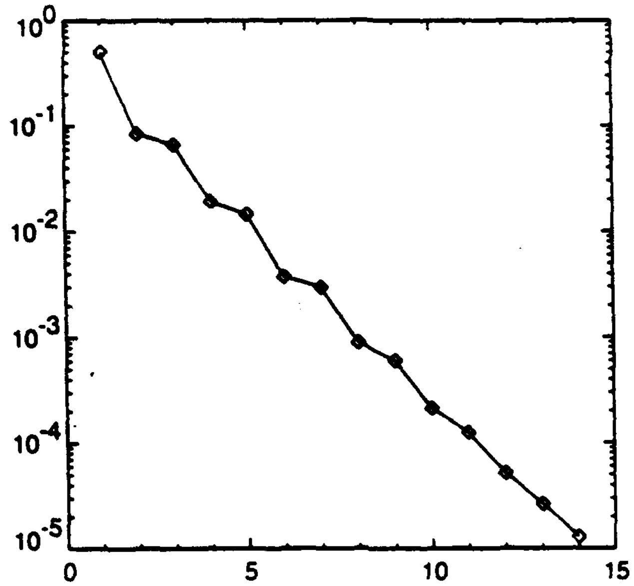

Figure 7: The L^2 norm of the error $g - \mathcal{L}_T g$

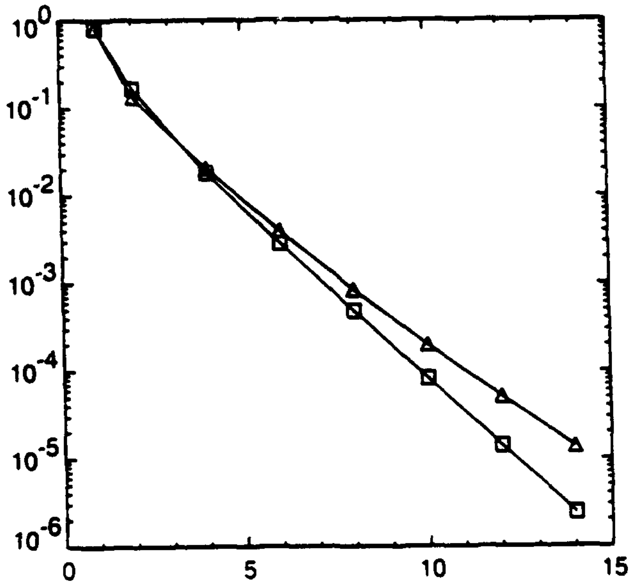

Figure 8: The L^2 norm of the error $f - \mathcal{L}_T f$ for rectangle (squares) and triangle interpolation (triangles)

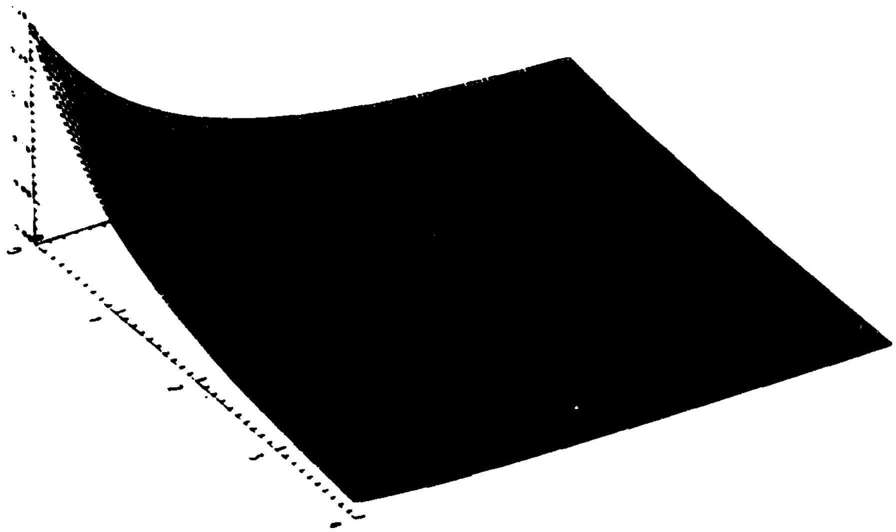

Figure 9: The function h in $[0,4] \times [0,4]$

the interpolation error has quite different order of magnitude (Fig. 11). Except in triangles 1 2, 3, 5, errors in other triangles are almost negligible. In fact, in these triangles, we can use much lower order approximation to achieve the same accuracy. In Fig. 12, we show the error using approximation order 5,4,4,3,4,3,3,3 for triangles 1 through 8. The error has the similar order of magnitude in all triangles(10^{-4}).

The adaptive procedure for the determination of interpolation order is described as follows. Let $\{ T^i = (\mathbf{t}_1^{(i)}, \ldots, \mathbf{t}_{N_i}^{(i)}), i = 1, \ldots, n, \ldots\}$ be the interpolation scheme in the triangle D, f be the function to be approximated. The approximation error at level i can be estimated from lower level interpolation. At order n, define the estimated interpolation error at order n by $e_n(f) = max_{j=1,\ldots,N_i;i=1,\ldots,n-1}|f(\mathbf{t}_j^{(i)}) - (\mathcal{L}_{T^n} f)(\mathbf{t}_j^{(i)})|$. For a prescribed error $\rho(f)$, calculate the estimated interpolation error $e_n(f)$. If $e_n(f) < \rho(f)$, stop; otherwise, continue to go to higher order polynomial

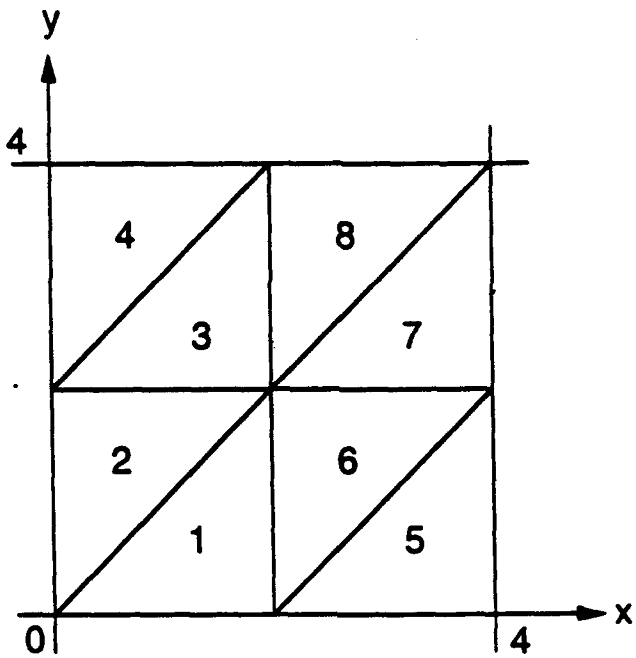

Figure 10: The triangle mesh

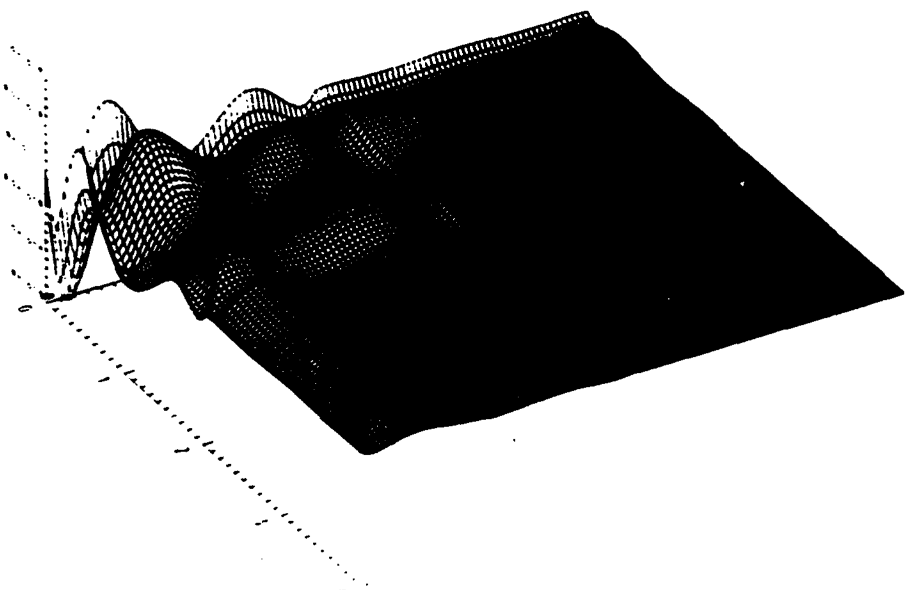

Figure 11: The error in the triangle mesh using uniform order 5 interpolation in all the triangles.

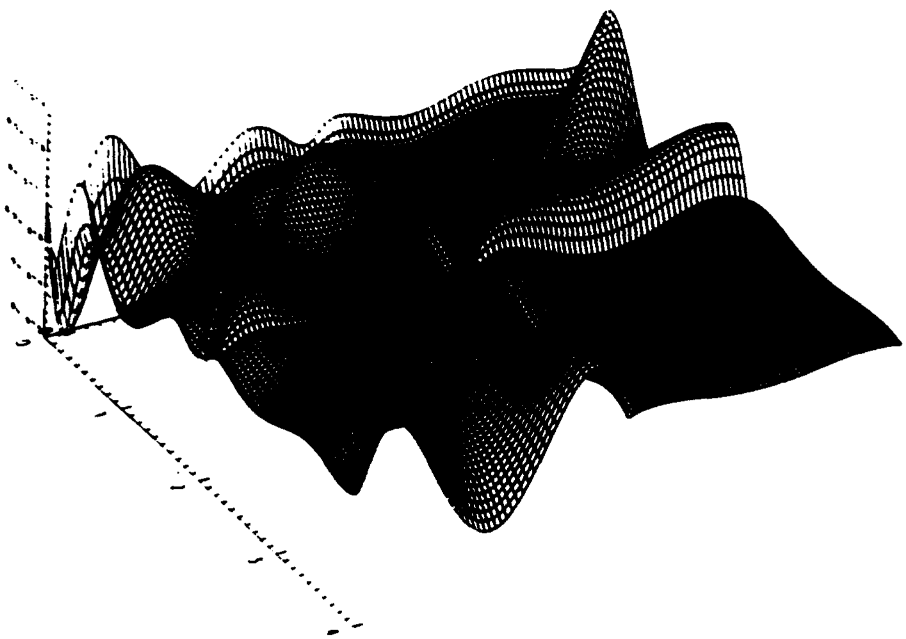

Figure 12: The error in the traingle mesh using adaptive approximation

interpolations till $e_n(f) < \rho(f)$. Fig. 9 is generated using the above procedure.

Note that if we interpolate a function with different order in different triangles in a simple minded way, the C^0 continuity of the interpolating function can be violated on the edges shared by triangles with different interpolation order. This is shown in Fig. 12 on those triangle edges shared by different approximation order triangles. most visible on the edges between triangles 3,4 and 5,6. The reason is that the interpolation nodes on a shared edge are different for different triangle interpolations. hence the resulting polynomial interpolant restricted to the edge are also different. To achieve the continuity of the polynomial interpolant on a shared edge. since we are only interested in approximating the function. we can interpolate first in the lower interpolation order triangle, then use the lower order interpolant values on the shared edge as the interpolation values for the higher order interpolation. Notice the higher order interpolant on the edge reduces to the same polynomial as the lower order interpolant. therefore guaranteeing the C^0 continuity of the interpolating function on the shared edge.

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Appendix

The barycentric coordinates of the Symmetrical Mean Minimal Sets.

$n = 1, N_n = 3, n_6 = 0, n_3 = 0, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

$n = 2, N_n = 6, n_6 = 0, n_3 = 0, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

$n = 3, N_n = 10, n_6 = 0, n_3 = 0, n_1 = 1$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.7153323836874790 0.2846676163125210 0.0000000000000000E+00

0.3333333333333333 0.3333333333333333 0.3333333333333333

$n = 4, N_n = 15, n_6 = 0, n_3 = 1, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.8181632611352384 0.1818367388647614 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

0.2211896848879315 0.2211896848879315 0.5576206302241369

$n = 5, N_n = 21, n_6 = 0, n_3 = 2, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.8742875052784925 0.1257124947215076 0.0000000000000000E+00

0.6382594790542118 0.3617405209457882 0.0000000000000000E+00

0.1528511668833901 0.1528511668833901 0.6942976662332199

0.4168253271522609 0.4168253271522609 0.1663493456954783

$n = 6, N_n = 28, n_6 = 1, n_3 = 1, n_1 = 1$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9080633118611157 9.1936688138884221E-02 0.0000000000000000E+00

0.7284330183910422 0.2715669816089578 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

0.3158942320998219 0.5585416574112235 0.1255641104889546

0.1101683475818847 0.1101683475818847 0.7796633048362307

0.3333333333333333 0.3333333333333333 0.3333333333333333

$n = 7, N_n = 36, n_6 = 1, n_3 = 3, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9299034955372345 7.0096504462765443E-02 0.0000000000000000E+00

0.7895072582937859 0.2104927417062141 0.0000000000000000E+00

0.6020311487196381 0.3979688512803619 0.0000000000000000E+00

0.2451281044713800 0.6579558759858357 9.6916019542784290E-02

8.2115803104864539E-02 8.2115803104864539E-02 0.8357683937902710

0.4493450658674361 0.4493450658674361 0.1013098682651278

0.2664181073381445 0.2664181073381445 0.4671637853237110

$n = 8, N_n = 45, n_6 = 2, n_3 = 3, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9448163437150148 5.5183656284985111E-02 0.0000000000000000E+00

0.8324512446943281 0.1675487553056718 0.0000000000000000E+00

0.6775748866685667 0.3224251133314333 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

0.3657785050915281 0.5523065120741060 8.1914982834365943E-02

0.1945715568399863 0.7289768787515726 7.6451564408441141E-02

6.2956298271008348E-02 6.2956298271008348E-02 0.8740874034579832

0.2153423770756083 0.2153423770756083 0.5693152458487836

0.3891308342783755 0.3891308342783755 0.2217383314432491

$n = 9, N_n = 55, n_6 = 3, n_3 = 3, n_1 = 1$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9554420839457645 4.4557916054235588E-02 0.0000000000000000E+00

0.8636610675074426 0.1363389324925574 0.0000000000000000E+00

0.7343658010099048 0.2656341989900952 0.0000000000000000E+00

0.5809026723382413 0.4190973276617587 0.0000000000000000E+00

0.3022979698394158 0.6305941125733549 6.7107917587229337E-02

0.1580235132291699 0.7802178829384695 6.1758603832360652E-02

0.3259838750983894 0.4887151048567661 0.1853010200448444

4.9667258949604278E-02 4.9667258949604278E-02 0.9006654821007913

0.4657791245700742 0.4657791245700742 6.8441750859851602E-02

0.1769224725585725 0.1769224725585725 0.6461550548828550

0.3333333333333333 0.3333333333333333 0.3333333333333333

$n = 10, N_n = 66, n_6 = 4, n_3 = 4, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9632760027314051 3.6723997268594987E-02 0.0000000000000000E+00

0.8869952049270331 0.1130047950729669 0.0000000000000000E+00

0.7778351605810001 0.2221648394189999 0.0000000000000000E+00

0.6450778849507119 0.3549221150492881 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

0.3965838101116214 0.5458726289486318 5.7543560939746889E-02

0.2541104747843801 0.6900362701651686 5.5853255050451355E-02

0.1308892070469940 0.8184976546269006 5.0613138326105477E-02

0.2759079443724909 0.5680265320840427 0.1560655235434665

3.9979823654525127E-02 3.9979823654525127E-02 0.9200403526909497

0.1477652510341296 0.1477652510341296 0.7044694979317407

0.4210302589291588 0.4210302589291588 0.1579394821416825

0.2860831530706470 0.2860831530706470 0.4278336938587060

$n = 11, N_n = 78, n_6 = 5, n_3 = 5, n_1 = 0$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9692151231856643 3.0784876814335779E-02 0.0000000000000000E+00

0.9048685364533428 9.5131463546657197E-02 0.0000000000000000E+00

0.8117034716938857 0.1882965283061142 0.0000000000000000E+00

0.6963587134442253 0.3036412865557747 0.0000000000000000E+00

0.5670428520428707 0.4329571479571293 0.0000000000000000E+00

0.3410061867540533 0.6100223949943824 4.8971418251564262E-02

0.2154463537311060 0.7371142274976481 4.7439418771245911E-02

0.1103911543297726 0.8474540426980305 4.2154802972196865E-02

0.3650161428097270 0.4995720679420922 0.1354117892481808

0.2363693386602579 0.6303683591358489 0.1332623022038932

3.2896313862711745E-02 3.2896313862711745E-02 0.9342073722745765

0.4754253185346134 0.4754253185346134 4.9149362930773268E-02

0.1254000862291636 0.1254000862291636 0.7491998275416729

0.2469463409895583 0.2469463409895583 0.5061073180208833

0.3752280933777087 0.3752280933777087 0.2495438132445825

$n = 12, N_n = 91, n_6 = 7, n_3 = 4, n_1 = 1$

1.000000000000000 0.0000000000000000E+00 0.0000000000000000E+00

0.9738238285434289 2.6176171456571193E-02 0.0000000000000000E+00

0.9188463030484114 8.1153696951588631E-02 0.0000000000000000E+00

0.8385306712170824 0.1614693287829176 0.0000000000000000E+00

0.7377419334160070 0.2622580665839930 0.0000000000000000E+00

0.6225769896251392 0.3774230103748608 0.0000000000000000E+00

0.5000000000000000 0.5000000000000000 0.0000000000000000E+00

0.4168281198422784 0.5408750717713540 4.2296808386367512E-02

0.2958645927885613 0.6620944951096916 4.2040912101747096E-02

0.1856877557735782 0.7736786953598021 4.0633548866619744E-02

9.4029154634774769E-02 0.8702379847289224 3.5732860636302765E-02

0.3187473818918873 0.5641272963702495 0.1171253217378632

0.2046743882806896 0.6799965980733892 0.1153290136459212

0.3305908816516884 0.4510620156781378 0.2183471026701738

2.7522667878192077E-02 2.7522667878192077E-02 0.9449546642436157

0.1076261576526970 0.1076261576526970 0.7847476846946060

0.4415231635034078 0.4415231635034078 0.1169536729931844

0.2153826974384240 0.2153826974384240 0.5692346051231521

0.3333333333333333 0.3333333333333333 0.3333333333333333

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

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