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The required inner products in the Galerkin solution of the integral equation, using a basis of exponential-weighted Laguerre polynomials, are expressed as series of algebraic forms. We are presently evaluating these series to get a feel for the invertibility of the integral equation (transformed to a system of linear equations), prior to any inclusion of the anticipated asymptotic behavior from the Sommerfeld half-space problem. If this unmodified system of the first kind inverts nicely, then its inverse can be used to express the boundary value problem as a system of the second kind; resulting in a whole new set of opportunities for iterative or perturbation series approaches.

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PENETRABLE WEDGE ANALYSIS
GRANT NO. N00014-93-1-6014

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3 February 1994

I. INTEGRAL EQUATION STATUS

The required inner products in the Galerkin solution of the integral equation, using a basis of exponential-weighted Laguerre polynomials, are expressed as series of algebraic forms. We are presently evaluating these series to get a feel for the invertibility of the integral equation (transformed to a system of linear equations), prior to any inclusion of the anticipated asymptotic behavior from the Sommerfeld half-space problem. If this unmodified system of the first kind inverts nicely, then its inverse can be used to express the boundary value problem as a system of the second kind; resulting in a whole new set of opportunities for iterative or perturbation series approaches.

There are several important issues to be addressed in connection with the far ($r \to \infty$) behavior:

(1) The far behavior

$$\frac{\partial}{\partial y}\psi_1(x \to \infty, y \to 0) \sim \frac{k_1 a}{\sqrt{8\pi i k_1}} \frac{e^{ik_1 x}}{x^{3/2}}$$

is not the proper form close to the apex. An abrupt jump from the Laguerre series to this oscillatory function is likely to lead to numerical instabilities, which suggests that a scaling funtion that filters out its effect for small r could be useful.

(2) Inclusion of this far behavior is complicated in context of the even/odd symmetry decomposition. One simplification, based on the same line of physical intuition used to insure uniqueness in these types of scattering problems, is to invoke a radiation condition inside the wedge material. That is, assume the wedge to be slightly lossy so that the effect of the image source (in the even or odd case) is negligible (in comparison with the original line source), which is the domain of the surface integral equation. Admittedly, this is only a conjecture at the present, and its utility remains to be tested via experiments (both analytical and numerical).

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II. Conferences, Meetings, Proposals, and Papers

(1) We submitted an abstract titled "Single Surface Integral Equation for Penetrable Wedge Scattering" to the Acoustical Society of America Meeting to be held in Cambridge, MA during June 1994 (copy attached).

(2) We will present a seminar on this research at the Electrical Engineering Department of the University of Mississippi on 3 March 1994. While in Oxford, we will also meet with acoustics researchers from the National Center for Physical Acoustics.

(3) We submitted a proposal "Analytical and Asymptotic Diffraction Theory for Penetrable Rough Surfaces" to ONR through the DEPSCoR program to continue and extend this acoustic scattering research.

(4) The wedge paper [62], which is a result of last year's grant and has been the basis for our present grant, has appeared in print (copy attached).

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Single Surface Integral Equation for Penetrable Wedge Scattering.

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The transmission problem of time-harmonic acoustic scattering by the velocity/density contrast wedge is formulated, through symmetry arguments and with the construction of suitable Green's functions, as a pair of uncoupled surface integral equations, each with one unknown function defined on a single wedge face. The convergent solution to the Fredholm integral equation of the first kind is expressed as a Galerkin series of Laguerre polynomials, and the development takes account of the distant behavior anticipated from the asymptotic solution to the related Sommerfeld half-space problem. The required inner products for the Galerkin projection scheme, which are integrals of products of the weighted Laguerre polynomials and the pair of Green's functions for the separate homogeneous regions, are written as Taylor series coefficients of an auxiliary function. Efficient numerical implementation of the physically-based analysis produces an accurate and intuitive picture of the wave interactions with this canonical scatterer. [Work supported by the Naval Research Laboratory - Stennis Space Center.]

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Mellin Transform Solution for the Static Line-Source Excitation of a Dielectric Wedge

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***Abstract*—An integral transform analysis of the static scattering of the two-dimensional potential radiated by a line source in the vicinity of a penetrable wedge is presented. The Mellin transform is used to derive the exact static solution to Laplace's equation for the dielectric wedge, in the form of a modal series. The important dielectric edge condition behavior is explicitly contained in this analytic solution.**

I. Introduction

THE static solution is a critical component of any dynamic scattering problem, especially with regard to edge conditions and behavior in source regions. A quasistatic philosophy of extracting the dominant behavior of a solution to the Helmholtz equation $(\nabla^2 + k^2)\psi(r) = 0$ from the static ($k = 0$) solution to Laplace's equation $\nabla^2\psi(r) = 0$ in the neighborhood of boundary discontinuities and sources is well known and successful [1]–[4], and is naturally advocated by Anderson and Solodukhov [5] for the dielectric wedge. The important edge behavior of the *dynamic* fields in the immediate vicinity of the apex $r = 0$ is obtained from the *static* analysis, since all physical dimensions are scaled by the wavelength, and $kr \to 0$ is the limit of interest. The numerical results of Marx [6] indicate that the dynamic-field behavior at the apex of the dielectric wedge can differ from the static solution. However, continued mathematical analysis of the wave problem is required before initiating a comprehensive physical interpretation.

This paper presents the complete and exact solution to the static dielectric wedge problem. It should be noted that the special case with the line source lying on the x axis (the wedge bisector) does appear in the Russian text [7]. Furthermore, Smythe [8] presents a formal treatment of the static dielectric wedge, which is unfortunately flawed with divergent integral representations. In any event, the exact static solution is now available for this canonical geometry. The potential problem is solved using the Mellin transform, which is itself the static limit of the Kontorovich–Lebedev transform employed by Jones [9] for the scattering of time-harmonic waves from impenetrable wedges having Dirichlet and Neumann boundary conditions. The static modes of [5] appear as the residues in the Mellin inversion integral.

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II. Statement of the Boundary-Value Problem

The static excitation of the penetrable wedge is due to the line charge of unit lineal density, located at the source coordinates (r', ϕ') of Fig. 1. The permittivity of the wedge of angle 2α is ϵ_2, which is surrounded by a medium with permittivity ϵ_1. All geometry and therefore all resultant fields are of infinite extent and invariant in the z dimension, which restricts the physical domain to $\mathbb{R}^2$.

The irrotational electrostatic field is uniquely characterized by the scalar potential $\psi(r)$, which is a solution of Poisson's equation subject to appropriate boundary conditions. Denote by $\psi_1(r)$ the potential field in the external region where the source is

$$\nabla^2\psi_1(r,\phi) = -\frac{1}{\epsilon_1 r}\delta(r-r')\delta(\phi-\phi') \quad (\alpha \le \phi \le 2\pi - \alpha), \quad (1)$$

and let $\psi_2(r)$ be the source-free field inside the wedge

$$\nabla^2\psi_2(r,\phi) = 0 \quad (-\alpha \le \phi \le \alpha). \quad (2)$$

Boundary conditions at the material interfaces $\phi = \pm\alpha$ for this scalar potential are continuity of ψ (from continuity of the tangential electric field) and continuity of the normal electric flux density $\epsilon_j \partial\psi_j/\partial n$ (absence of any free surface charge).

In order to simplify the ensuing analysis, it is expedient to decompose the desired solution $\psi(x, y)$ for the boundary-value problem of Fig. 1 into its odd $\psi^o(x, y)$ and even $\psi^e(x, y)$ components

$$\psi(x, \pm y) = \frac{1}{2}[\psi^e(x,y) \pm \psi^o(x,y)] \quad (y \ge 0) \quad (3)$$

with respect to the x axis which bisects the wedge. Since $\psi^o(x, y)$ is an odd function of y, it vanishes on the $y = 0$ plane and is therefore the solution in the upper half-space $y \ge 0$ for the problem having a soft bisecting plane (Dirichlet boundary condition). This is equivalent to an out-of-phase image source. Similarly, $\psi^e(x, y)$ is the solu-

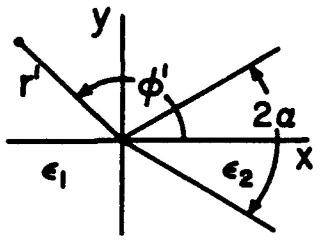

Fig. 1. Dielectric wedge and static line charge.

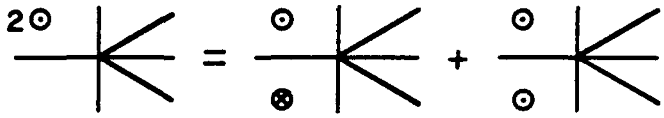

Fig. 2. Odd and even symmetry components.

tion for the hard bisecting plane (Neumann boundary condition), which sustains an in-phase image source. This symmetry is depicted in Fig. 2.

III. Mathematical Analysis

A. Mellin Transform for the Case of Odd Symmetry

The source coordinate $\phi = \phi'$ divides region (1) into two source-free regions, resulting in a total of three subregions to consider for the half-space above the perfectly soft plane:

$$\psi^o(x, y) = \begin{cases} \psi_2(r, \phi), & 0 \le \phi \le \alpha \\ \psi_1^-(r, \phi), & \alpha < \phi \le \phi' \\ \psi_1^+(r, \phi), & \phi' < \phi \le \pi. \end{cases} \quad (4)$$

The Dirichlet boundary condition on the soft plane is

$$\psi_2(r, 0) = 0 \quad (5)$$

$$\psi_1^+(r, \pi) = 0 \quad (6)$$

and at the material interface

$$\psi_2(r, \alpha) = \psi_1^-(r, \alpha) \quad (7)$$

$$\epsilon_2 \frac{\partial}{\partial \phi} \psi_2(r, \alpha) = \epsilon_1 \frac{\partial}{\partial \phi} \psi_1^-(r, \alpha). \quad (8)$$

Consistent with the conventional Green's function ansatz, the potential is continuous everywhere across the source plane $\phi = \phi'$

$$\psi_1^-(r, \phi') = \psi_1^+(r, \phi'), \quad (9)$$

while the discontinuity in normal derivative

$$\frac{\partial}{\partial \phi} \psi_1^-(r, \phi') - \frac{\partial}{\partial \phi} \psi_1^+(r, \phi') = \frac{r}{\epsilon_1} \delta(r - r') \quad (10)$$

results from integrating (1) from $\phi' -$ to $\phi' +$.

The Mellin transform [10] is used to transform the radial variable r to a complex variable s, whereupon the remaining differential equation with boundary conditions in ϕ is solved in closed form. The physical solution $\psi(r, \phi)$ is recovered by careful evaluation of the inverse Mellin transform. The Mellin transform of $f(r)$ is

$$F(s) = \mathcal{M}\{f(r); s\} = \int_0^\infty r^{s-1} f(r)\, dr, \quad (11)$$

and has

$$f(r) = \mathcal{M}^{-1}\{F(s); r\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s} F(s)\, ds \quad (12)$$

as its inversion formula. If $r^{a-1} f(r)$ is absolutely integrable on the positive real axis for some $a > 0$, then the inversion is valid for $c > a$.

The Mellin transform of the product of r^2 and Poisson's equation (1)

$$\left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \phi^2} \right) \psi_1(r, \phi) = -\frac{r}{\epsilon_1} \delta(r - r') \delta(\phi - \phi') \quad (13)$$

is the simple form

$$\left(s^2 + \frac{\partial^2}{\partial \phi^2} \right) \Psi_1(s, \phi) = -\frac{r'^s}{\epsilon_1} \delta(\phi - \phi'). \quad (14)$$

This fortunate property of the Mellin transform of the r dependence in the two-dimensional Laplacian is responsible for its successful application [7], [10], [11] to potential problems in wedge-shaped regions. In region (2) where there is no forcing term, this procedure gives

$$\left(s^2 + \frac{\partial^2}{\partial \phi^2} \right) \Psi_2(s, \phi) = 0. \quad (15)$$

The complex variable s is a parameter in the above pair of ordinary differential equations in ϕ, with solutions

$$\Psi_2(s, \phi) = A(s) \sin(s\phi) \quad (0 \le \phi \le \alpha) \quad (16)$$

$$\Psi_1^-(s, \phi) = B(s) \sin(s\phi) + C(s) \cos(s\phi) \quad (\alpha < \phi \le \phi') \quad (17)$$

$$\Psi_1^+(s, \phi) = D(s) \sin[s(\phi - \pi)] \quad (\phi' < \phi \le \pi) \quad (18)$$

in view of the soft boundary conditions (5) and (6). Transformation of the four remaining conditions (7)–(10) gives the set of simultaneous equations

$$\begin{bmatrix} -\sin(s\alpha) & \sin(s\alpha) & \cos(s\alpha) & 0 \\ \frac{\epsilon_2}{\epsilon_1} \cos(s\alpha) & -\cos(s\alpha) & \sin(s\alpha) & 0 \\ 0 & \sin(s\phi') & \cos(s\phi') & \sin[s(\pi - \phi')] \\ 0 & \cos(s\phi') & -\sin(s\phi') & -\cos[s(\pi - \phi')] \end{bmatrix} \begin{bmatrix} A(s) \\ B(s) \\ C(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r'^s}{\epsilon_1 s} \end{bmatrix} \quad (19)$$

to be solved for the coefficient functions in (16)–(18). Solution of these yields the (soft) Mellin transforms

$$\Psi_2(s,\phi) = \frac{1-\Gamma}{\epsilon_1}\frac{r'^s}{s}\sin[s(\pi-\phi')]\sin(s\phi)/\Delta_o(s) \quad (0 \le \phi \le \alpha) \quad (20)$$

$$\Psi_1^-(s,\phi) = \frac{r'^s}{\epsilon_1 s}\sin[s(\pi-\phi')]\{\sin(s\phi) + \Gamma\sin[s(\phi-2\alpha)]\}/\Delta_o(s) \quad (\alpha < \phi \le \phi') \quad (21)$$

$$\Psi_1^+(s,\phi) = \frac{r'^s}{\epsilon_1 s}\sin[s(\pi-\phi)]\{\sin(s\phi') + \Gamma\sin[s(\phi'-2\alpha)]\}/\Delta_o(s) \quad (\phi' < \phi \le \pi) \quad (22)$$

with denominator function

$$\Delta_o(s) = \sin(s\pi) + \Gamma\sin[s(\pi-2\alpha)], \quad (23)$$

and dielectric contrast parameter

$$\Gamma = \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}. \quad (24)$$

B. Modifications for the Case of Even Symmetry

In the case of a hard ground plane, the Neumann boundary conditions

$$\frac{\partial}{\partial\phi}\psi_2(r,0) = 0 \quad (25)$$

$$\frac{\partial}{\partial\phi}\psi_1^+(r,\pi) = 0 \quad (26)$$

replace the Dirichlet boundary conditions (5) and (6) of the previous section. A similar application of the Mellin transform and the other unchanged boundary conditions yields the (hard) Mellin transforms

$$\Psi_2(s,\phi) = \frac{\Gamma-1}{\epsilon_1}\frac{r'^s}{s}\cos[s(\pi-\phi')]\cos(s\phi)/\Delta_e(s) \quad (0 \le \phi \le \alpha) \quad (27)$$

$$\Psi_1^-(s,\phi) = \frac{r'^s}{\epsilon_1 s}\cos[s(\pi-\phi')]\{\Gamma\cos[s(\phi-2\alpha)] - \cos(s\phi)\}/\Delta_e(s), \quad (\alpha < \phi \le \phi') \quad (28)$$

$$\Psi_1^+(s,\phi) = \frac{r'^s}{\epsilon_1 s}\cos[s(\pi-\phi)]\{\Gamma\cos[s(\phi'-2\alpha)] - \cos(s\phi')\}/\Delta_e(s) \quad (\phi' < \phi \le \pi) \quad (29)$$

where in this case the denominator function is

$$\Delta_e(s) = \sin(s\pi) - \Gamma\sin[s(\pi-2\alpha)]. \quad (30)$$

Note that, except for the simple scaling by ϵ_1 which persists from the original source strength chosen in (1), the presence of two different dielectrics is entirely accounted for by Γ in all of the above transforms.

C. Inverse Mellin Transforms—Preliminaries

The zeros of the denominator functions (23) and (30)

$$\Delta(s) = \sin(s\pi) \pm \Gamma\sin[s(\pi-2\alpha)] \quad \begin{Bmatrix}\text{soft}\\ \text{hard}\end{Bmatrix} \quad (31)$$

are central to the Mellin inversion (12). These real, simple zeros can be computed via Muller's algorithm [12] for arbitrary half-angle α, or solved analytically as the roots of a trigonometric polynomial when α is a rational multiple of π. This procedure is demonstrated for the particular case $\alpha = \pi/3$, whereupon the variable change $u = s\pi/3$ in (31) gives

$$\sin(3u) \pm \Gamma\sin(u) = 0 \quad (32)$$

which factors into

$$[3 \pm \Gamma - 4\sin^2(u)]\sin(u) = 0. \quad (33)$$

The required roots are now explicitly given by

$$s_n = \begin{Bmatrix} \frac{3}{\pi}\sin^{-1}\sqrt{\frac{3\pm\Gamma}{4}} \\ -\frac{3}{\pi}\sin^{-1}\sqrt{\frac{3\pm\Gamma}{4}} \\ 0 \end{Bmatrix} + 3m \quad (m = 0, \pm 1, \pm 1, \cdots), \quad (34)$$

where it is recalled that the $+\Gamma$ $(-\Gamma)$ denotes the case of soft (hard) symmetry. The effect of the dielectric material ($\epsilon_2 \ne \epsilon_1 \Rightarrow \Gamma \ne 0$) on the potential above both symmetry planes is a regular displacement of the integer poles for the homogeneous case ($\epsilon_2 = \epsilon_1 \Rightarrow \Gamma = 0$). The index n in (34) is a denumerable ordering of these poles.

D. Inverse Mellin Transform for the Case of Odd Symmetry

As $r \to 0$ the odd potential $\psi^o \to 0$ and the dipole behavior $\psi^o \sim 1/r$ prevails as $r \to \infty$ in the far field. A sufficient Bromwich contour for the complex integration (12) is therefore guaranteed for the choice of real constant $0 < c < 1$. Complete details of the Mellin inversion for the potential $\psi_2(r,\phi)$ of (4) inside the dielectric sector are provided, whereupon the final forms for $\psi_1^-(r,\phi)$ and $\psi_1^+(r,\phi)$ are immediately written by comparison.

The $\sin(s\phi)$ factor in $\Psi_2(s,\phi)$ of (20) together with the transform property [10]

$$\mathcal{M}^{-1}\{\sin(s\phi)F(s); s\} = -\mathcal{I}[f(re^{i\phi})] \quad (35)$$

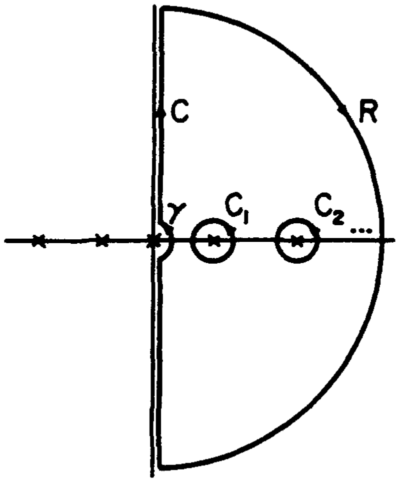

Fig. 3. Bromwich contour in the complex s plane.

renders

$$f(r) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \underbrace{\frac{\sin[s(\pi - \phi')](r'/r)^s}{s\{\sin(s\pi) + \Gamma \sin[s(\pi - 2\alpha)]\}}}_{G(s)} \, ds \tag{36}$$

the desired function. The integration path in (36) has been pushed flush against the imaginary axis, and the principal value notation invoked to properly account for the pole at the origin. For $r > r'$, closure at infinity in the right half-plane gives a convenient contour on which to apply Cauchy's integral theorem, and is shown in Fig. 3. Let C_n be a circle of vanishing radius $\rho \to 0$, centered on the pole s_n with residue

$$\lim_{\rho \to 0} \frac{1}{2\pi i} \oint_{C_n} G(s)\, ds = \frac{\sin[s_n(\pi - \phi')](r'/r)^{s_n}}{s_n \Delta_o'(s_n)}, \tag{37}$$

where the derivative of the soft denominator function is

$$\Delta_o'(s_n) = \pi \cos(s_n \pi) + (\pi - 2\alpha)\Gamma \cos[s_n(\pi - 2\alpha)]. \tag{38}$$

The integral around the origin

$$\lim_{\rho \to 0} \frac{1}{2\pi i} \oint_{C_0} G(s)\, ds = \frac{\pi - \phi'}{\Delta_o(0)} \tag{39}$$

follows from the limit $s_n \to 0$. Let positive $n = 1, 2, \cdots$ be the indices of the poles in the right half-plane, and let negative $n = -1, -2, \cdots$ identify the poles in the left half-plane. The odd symmetry of the function $\Delta_o(s)$ provides

$$s_{-n} = -s_n \quad \text{and} \quad \Delta_o'(-s_n) = \Delta_o'(s_n). \tag{40}$$

The contribution from the integral around the infinite semicircle R of Fig. 3 is zero for $r > r'$, while closure of the contour in the left half-plane is appropriate when $r < r'$. Cauchy's integral theorem now yields

$$f(r) = \pm \left\{ \frac{\pi - \phi'}{2\Delta'(0)} + \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')]}{s_n \Delta_o'(s_n)} (r/r')^{\pm s_n} \right\} \qquad (r \gtrless r'), \tag{41}$$

which together with (35) gives

$$\psi_2(r, \phi) = \frac{\Gamma - 1}{\epsilon_1} \cdot \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')] \sin(s_n \phi)}{s_n \Delta_o'(s_n)} (r/r')^{\mp s_n}$$
$$(r \gtrless r') \quad \text{for the sector } 0 \le \phi \le \alpha \tag{42}$$

as the inverse Mellin transform of (20). Since all three of the transforms (20)–(22) are of the same general form, the remaining two field expressions are evidently

$$\psi_1^-(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')]}{s_n \Delta_o'(s_n)} (r/r')^{\mp s_n} \cdot \{\sin(s_n \phi) + \Gamma \sin[s_n(\phi - 2\alpha)]\}$$
$$(r \gtrless r') \quad \text{for the sector } \alpha < \phi \le \phi' \tag{43}$$

and

$$\psi_1^+(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi)]}{s_n \Delta_o'(s_n)} (r/r')^{\pm s_n} \cdot \{\sin(s_n \phi') + \Gamma \sin[s_n(\phi' - 2\alpha)]\}$$
$$(r \gtrless r') \quad \text{for the sector } \phi' < \phi \le \pi. \tag{44}$$

Note the reciprocity in the last two expressions for $\phi \leftrightarrow \phi'$. The above reduce to the correct potential due to a unit line charge located at $\phi' = \pi/2$ above a soft ground plane in the special case of no dielectric wedge ($\epsilon_2 = \epsilon_1$).

E. Inverse Mellin Transform for the Case of Even Symmetry

The monopole potential $\ln r$ due to the original line source and the in-phase image is the dominant feature of ψ^e in the far field as $r \to \infty$. Therefore, the condition of integrability following the transform pair of (11) and (12) is not satisfied, and the Mellin inversion formula cannot be directly applied to the functions in (27)–(29). Mathematically, operating with the ϕ derivative of the potential temporarily removes this troublesome logarithmic variation, which is then restored following the convergent Mellin inversion.

The ϕ derivative of (27)

$$\frac{\partial}{\partial \phi} \Psi_2(s, \phi) = \frac{1 - \Gamma}{\epsilon_1} \frac{\cos[s(\pi - \phi')] \sin(s\phi)}{\Delta_e(s)} r'^s \tag{45}$$

is transformed in the manner of the previous section to

$$\frac{\partial}{\partial \phi} \psi_2(r, \phi) = \frac{\Gamma - 1}{\epsilon_1} \cdot \sum_{n=1}^{\infty} \frac{\cos[s_n(\pi - \phi')] \sin(s_n \phi)}{\Delta_e'(s_n)} (r/r')^{\mp s_n}$$
$$(r \gtrless r'), \tag{46}$$

where the s_n are now the hard poles in (34) and $-\Gamma$ replaces Γ in (38) for this hard denominator function (31).

The antiderivative with respect to ϕ yields

$$\psi_2(r,\phi) = \frac{1-\Gamma}{\epsilon_1}\sum_{n=1}^{\infty}\frac{\cos[s_n(\pi-\phi')]\cos(s_n\phi)}{s_n\Delta_e'(s_n)}(r/r')^{\pm s_n}$$
$$(r \lessgtr r') \quad \text{for the sector } 0 \le \phi \le \alpha, \tag{47}$$

where the boundedness at $r = 0$ and the known behavior at $r \to \infty$ specify

$$w_{\mp}(r) = \begin{cases} 0, & \text{for } r < r' \\ \dfrac{-1}{\pi\epsilon_1}\ln(r/r'), & \text{for } r > r' \end{cases} \tag{48}$$

as the ϕ-independent term in this solution of Laplace's equation. Similarly, the spatial potentials from the related transforms (28) and (29) are thus

$$\psi_1^-(r,\phi) = W_{\mp}(r) - \frac{-1}{\epsilon_1}\sum_{n=1}^{\infty}\frac{\cos[s_n(\pi-\phi')]}{s_n\Delta_e'(s_n)}(r/r')^{\pm s_n} \cdot\{\Gamma\cos[s_n(\phi-2\alpha)] - \cos(s_n\phi)\}$$
$$(r \lessgtr r') \quad \text{for the sector } \alpha < \phi \le \phi' \tag{49}$$

and

$$\psi_1^+(r,\phi) = w_{\mp}(r) - \frac{-1}{\epsilon_1}\sum_{n=1}^{\infty}\frac{\cos[s_n(\pi-\phi)]}{s_n\Delta_e'(s_n)}(r/r')^{\pm s_n} \cdot\{\Gamma\cos[s_n(\phi'-2\alpha)] - \cos(s_n\phi')\}$$
$$(r \lessgtr r') \quad \text{for the sector } \phi' < \phi \le \pi. \tag{50}$$

These results are also verified for the special case of a unit line charge located at $\phi' = \pi/2$ in a homogeneous half-space ($\epsilon_2 = \epsilon_1$) above the hard symmetry plane.

IV. Results for the Complete Static Solution

Contours of constant potential are illustrated in Fig. 4 for the case of wedge half-angle $\alpha = \pi/3$. Even and odd potentials are computed for the separate boundary-value problems and combined according to (3). The line source is located a unit distance ($r' = 1$) from the wedge apex at the (x, y) coordinate origin. The domain of Fig. 4 is all internal to the unit circle. Note that in this static problem with material boundaries at $\phi = \pm\alpha$, the source distance r' is the single length scale. As ϵ_2/ϵ_1 gets large, the dielectric wedge becomes essentially an equipotential region, which is expected since an increasingly dense dielectric ($\epsilon_2 \to \infty$) behaves electrically like a perfect conductor. As $r \to 0$, the dominant behavior of the scalar potential is given by the first terms of the modal series (42)–(44) and (47)–(50), as pointed out by Anderson and Solodukhov [5] and tabulated in the recent text by Van Bladel [13].

V. Conclusions

The exact, modal series solution of Laplace's equation for the static field inside and outside of the dielectric wedge is obtained via the Mellin transform, and is readily evaluated. This analytic solution is a useful benchmark with which to check any numerical and approximate methods that are applied to the static problem.

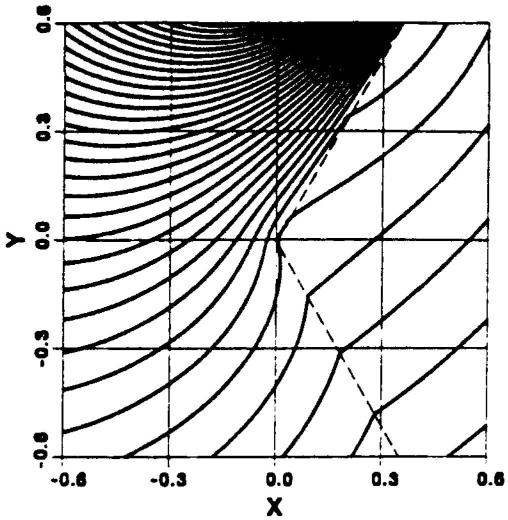

Fig. 4. Contours of constant potential for the dielectric wedge. Case: $r' = 1$, $\phi' = \pi/2$, $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 10$.

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