AD-A276 078

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NRL/FR/7140--94-9700

Detection of a Chi-square Fluctuating Target in Gaussian Noise

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February 7, 1994

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave Blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	February 7, 1994	Final

4. TITLE AND SUBTITLE

Detection of a Chi-square Fluctuating Target in Gaussian Noise

5. FUNDING NUMBERS

PR-R035B03-04

6. AUTHOR(S)

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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Research Laboratory
Washington, DC 20375-5320

8. PERFORMING ORGANIZATION REPORT NUMBER

NRL/FR/7140--94-9700

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Naval Research
Arlington, VA 22217

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

Detection of a target is often performed by applying a binary hypothesis test on the output of a matched-filter receiver. This report presents a derivation of the probability density function and cumulative probability function of the matched-filter output for a chi-square fluctuating target in Gaussian noise. The cumulative probability function is used to determine the probability of target detection for a given false alarm probability.

14. SUBJECT TERMS

Target detection   Chi-square fluctuation
Gaussian noise   Probability of detection

15. NUMBER OF PAGES

13

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std 239-18
298-102

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DETECTION OF A CHI-SQUARE FLUCTUATING TARGET IN GAUSSIAN NOISE

INTRODUCTION

Detection of a target in an echo-location system is often accomplished through the use of a matched-filter receiver. Shown in Fig. 1, it is composed of a multiplier to form the product of the return (echo) $x(t)$ and the processing signal $g(t)$, which is followed by an integrator, and an envelope detector. The output r is then compared to a threshold η to make one of two decisions:

$$H_0 \,(\text{no target present})\colon x(t) = n(t), \tag{1}$$

where $n(t)$ is white Gaussian noise, or

$$H_1 \,(\text{target present})\colon x(t) = \rho e^{j\theta} f(t-\tau) + n(t), \tag{2}$$

where $f(t)$ is the transmit signal, τ is the range-delay of the target, and $\rho \geq 0$ and θ are the range and phase of the reflection coefficient associated with the point target. The target strength is $w = \rho^2$. This represents a fairly simple model of the echo-iocation problem; a more complicated model of the return would include the effects of target Doppler or range-spreading. The output of the matched-filter is given by

$$r(\hat{\tau}) = \left| \int_{-\infty}^{\infty} x(t) g^*(t)\, dt \right| = \left| \int_{-\infty}^{\infty} x(t) \frac{f^*(t-\hat{\tau})}{||f||}\, dt \right|, \tag{3}$$

where $\hat{\tau}$ is the hypothesized range-delay of the target. Note that $g(t)$ is taken to be a delayed and normalized replica of the transmit signal $f(t)$. For our purpose, we assume that we have correctly hypothesized the range delay of the target, that is, $\hat{\tau} = \tau$. This implies that $r \doteq r(\tau) = |\rho e^{j\theta} + n|$, where n is the matched-filter response to the noise.

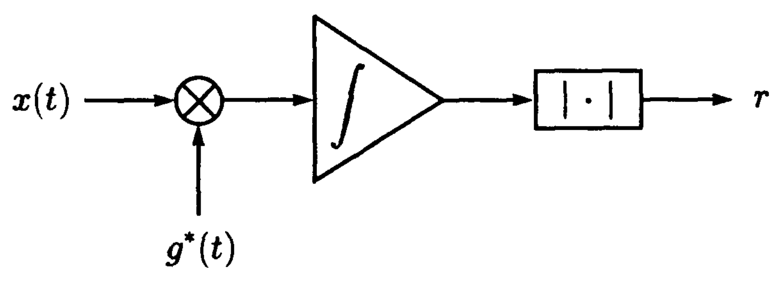

Fig. 1 – The matched-filter receiver structure

The task is to find the probability distribution and cumulative probability function $p(r)_{T+C}$ given that the noise is a white Gaussian noise process, and that ρ^2 is a chi-square random variable[1]. This will allow us to quantify the performance of the matched-filter for target detection.

Manuscript approved November 16, 1993.

[1] T, C, and $T+C$ are used to denote the density function and cumulative probability function of the amplitude of the matched-filter response to the target, clutter, and target plus clutter, respectively.

It should be noted that an expression for the density function for the matched-filter output r under the assumptions given here has been reported by others [1, 2]. A short derivation of the same result is given here with the intent of producing an available reference. Furthermore, we derive expressions for the cumulative probability function and moments of the matched-filter output. To the best of the author's knowledge, these results have not been reported.

DERIVATION OF THE DENSITY FUNCTION

Consider the case of using a matched-filter for detecting a nonfluctuating target (ρ fixed) in white Gaussian noise. The probability density function for r is the well known Rician density function [3-5] given by

$$p(r|\rho)_{T+C} = \frac{2r}{b^2} e^{-(r^2+\rho^2)/b^2} I_0\left(\frac{2r\rho}{b^2}\right), \tag{4}$$

where I_0 denotes the modified Bessel function of order zero and b^2 is the expected value of r^2 when only noise is present in the return[2]. This density function can be derived assuming that the target phase θ is either fixed, or is a uniform random variable that assumes its values in the interval $[0, 2\pi)$. Thus, explicit use of the target phase in the derivation that follows is unnecessary, and will be eliminated.

The target strength w is often modelled as a chi-square random variable, for which the distribution is

$$p(w)_T = \frac{N^N}{\Gamma(N)\sigma^{2N}} w^{N-1} e^{-Nw/\sigma^2}, \tag{5}$$

where N is any positive real number. However, it is more convenient to use the density function of the target reflection amplitude r_T. Thus, by the transformation of random variable $r_T = \sqrt{w}$ [6], the density function for r_T is

$$f(r_T) = \frac{2N^N}{\Gamma(N)\sigma^{2N}} r_T^{2N-1} e^{-Nr_T^2/\sigma^2}, \tag{6}$$

for which it can be shown that

$$E\{\rho\} = \frac{\Gamma(N+1/2)}{\Gamma(N)} \sqrt{\frac{\sigma}{N}}, \tag{7}$$

$$E\{\rho^2\} = \sigma. \tag{8}$$

Note that Eq. (8) shows the (expected) target strength. This distribution defines what is known as the Nakagami target fluctuation model [2].

Figure 2 shows the curves of the density function in Eq. (6) for several values of N. Note that as N increases, the variance tends to zero, and the density function approaches a delta function. Thus, as N increases, the target amplitude fluctuations decrease, and in the limit as $N \to \infty$ the target is nonfluctuating. Moreover, $E\{\rho\} \to \sqrt{\sigma}$ as $N \to \infty$.

[2]Since $r \geq 0$, $\rho \geq 0$, and $w \geq 0$, it should be understood that the density functions and cumulative probability functions given in this report are only defined for positive semi-definite values of their arguments.

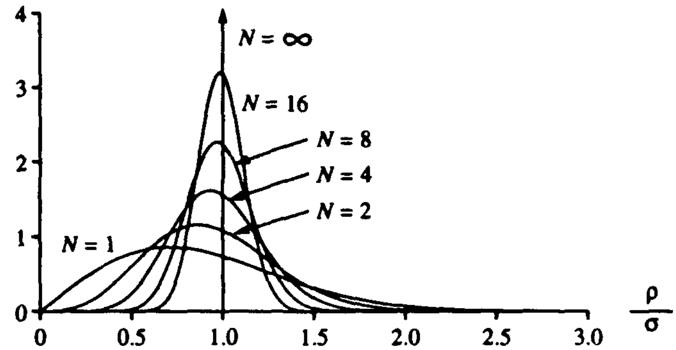

Fig. 2 – Probability distributions of the target reflection coefficient amplitude

For certain values of N, the density function in Eq. (6) reduces to well-known target fluctuation models: $N = 1$ corresponds to a Rayleigh fluctuating target, and $N = 2$ corresponds to a one-dominant-plus-Rayleigh fluctuating target.

Given $p(r|\rho)_{T+C}$ and $p(\rho)_T$, it follows that

$$p(r)_{T+C} = \int_0^\infty p(r|\rho)_{T+C}\, p(\rho)_T \, d\rho. \tag{9}$$

Thus, substituting Eq. (4) and Eq. (6) in Eq. (9), using the formula ([7], page 716, Eq. 6.631.1),

$$\int_0^\infty x^\mu e^{-\alpha x^2} J_0(\beta x)\, dx = \frac{\Gamma\left(\frac{\mu}{2} + \frac{1}{2}\right)}{2\alpha^{(\mu+1)/2}}\, {}_1F_1\left(\frac{\mu}{2} + \frac{1}{2}\,;\, 1\,;\, -\frac{\beta^2}{4\alpha}\right), \tag{10}$$

where J_0 denotes the Bessel function of order zero, and ${}_1F_1$ denotes the confluent hypergeometric function, and realizing that $J_0(\jmath x) = I_0(x)$, one finds that

$$p(r)_{T+C} = \frac{2re^{-r^2/b^2}}{b^2\left(1 + \frac{\sigma^2}{Nb^2}\right)^N}\, {}_1F_1\left(N\,;\, 1\,;\, \frac{r^2/b^2}{1 + \frac{Nb^2}{\sigma^2}}\right). \tag{11}$$

Two other forms of the density function in Eq. (11) can be found. If we define the signal-to-noise ratio (SNR) as

$$R \doteq \frac{E\{r^2\}}{E\{r^2\}_{\sigma=0}} = \frac{\sigma^2}{b^2}, \tag{12}$$

then the density function in Eq. (11) can be rewritten as

$$p(r)_{T+C} = \frac{2re^{-r^2/b^2}}{b^2\left(1 + \frac{R}{N}\right)^N}\, {}_1F_1\left(N\,;\, 1\,;\, \frac{r^2/b^2}{1 + \frac{N}{R}}\right). \tag{13}$$

Furthermore, if we define y to be the matched-filter output normalized by the rms value of the (un-normalized) matched-filter response to noise, i.e.,

$$y \doteq \frac{r}{\sqrt{E\{r^2\}_{\sigma=0}}} = \frac{r}{b}, \tag{14}$$

then, by the linear transformation of a random variable [6],

$$p(y)_{T+C} = \frac{2ye^{-y^2}}{\left(1+\frac{R}{N}\right)^N} \, {}_1F_1\left(N\,;\,1\,;\,\frac{y^2}{1+\frac{N}{R}}\right). \tag{15}$$

Calculation of the density function is easily accomplished using *Mathematica*, a software package for performing numerical and symbolic mathematics [8], since the hypergeometric functions are available as function calls. A user-defined *Mathematica* function for the density function in Eq. (15) is

```
Density[n_,R_,y_] := 2 y Exp[-y^2]
                        Hypergeometric1F1[n,1,(y^2)/(1+n/R)]/(1+R/n)^n;
```

Figures 3 and 4 show plots of the probability density functions for R equal to 1 and 10, and N equal to 0.5, 1, 3, and 7. Note that for $R = 1$ the density functions do not differ significantly. This is reasonable since it can be shown that in the limit as $R \to 0$, the density function in Eq. (15) becomes equal to $2ye^{-y^2}$.

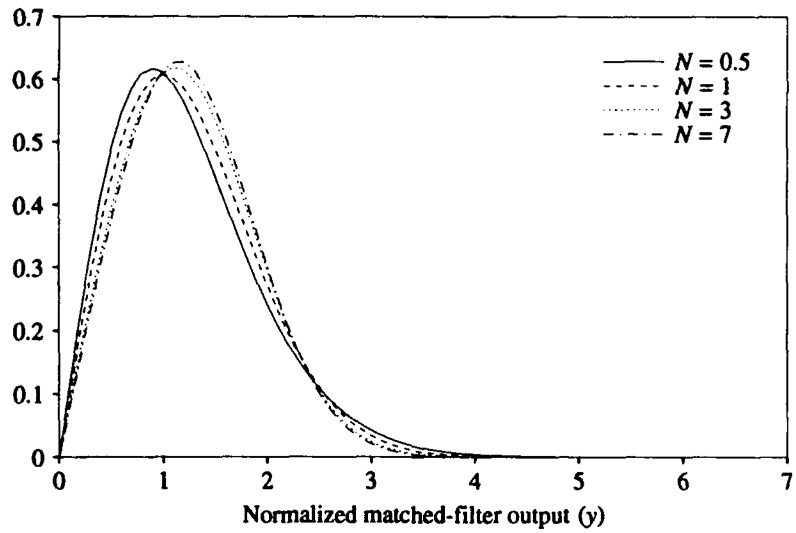

Fig. 3 – Probability density functions for $R = 1$

DERIVATION OF THE CUMULATIVE PROBABILITY FUNCTION

We note that the confluent hypergeometric function has the series representation

$${}_1F_1(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{16}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$, and $(a)_0 = 1$, and is known as Pochhammer's symbol. Substituting Eq. (16) into Eq. (15), integrating from zero to y, using the definition of the incomplete gamma function defined as

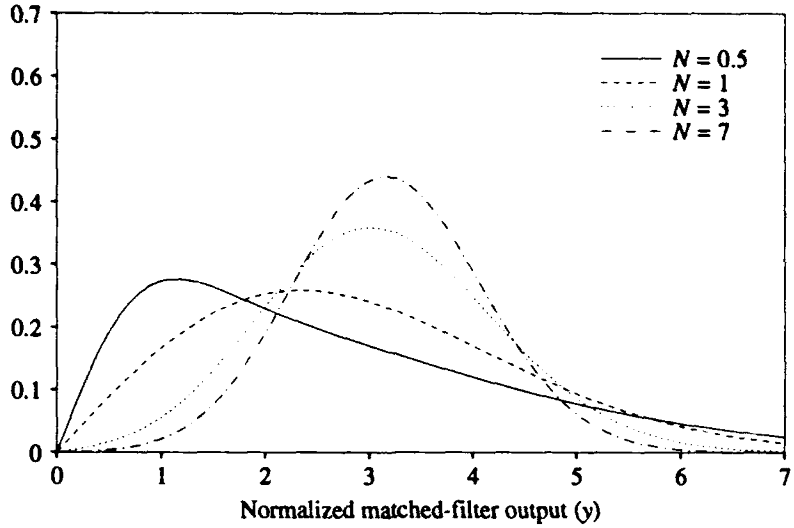

Fig. 4 – Probability density functions for $R = 10$

$$\gamma(a,y) \doteq \int_0^y x^{a-1} e^{-x}\, dx, \tag{17}$$

and realizing that $(1)_n = n!$, one finds that the cumulative probability function is given by

$$P(y)_{T+C} = \int_0^y p(x)_{T+C} dx = \frac{1}{\left(1+\frac{R}{N}\right)^N} \sum_{n=0}^{\infty} \frac{(N)_n}{(n!)^2} \frac{\gamma(n+1, y^2)}{\left(1+\frac{N}{R}\right)^n}. \tag{18}$$

As with the probability density function, calculation of the series in Eq. (18) is easy since the incomplete gamma function is available as a function call in *Mathematica.* Of course a truncated form of the series must be used, and it was found that the terms in the series often increased before they decayed. Thus, practical experience revealed that the first 10 terms should always be used when summing the series. Once beyond the tenth term, the summation can be terminated if the current term differs from the previous term by less than 0.000001. A user-defined *Mathematica* function for the calculation of the cumulative probability function in Eq. (18) using the method described here is

```
CumProb[n_,R_,y_] := Module[{error, p, plast, k, temp},
                            error = 1; p = 0; k = -1;
                            While[(error > 0.000001) || (k < 10),
                                  plast = p;
                                  k = k+1;
                                  temp = Pochhammer[n,k] Gamma[k+1,0,y^2]
                                         / (Factorial[k]^2 (1+n/R)^k);
                                  p = p + N[temp/(1+R/n)^n];
                                  error = Abs[p-plast]];
                            p];
```

Figures 5 and 6 show the cumulative probability functions for R equal to 1 and 10, and N equal to 0.5, 1, 3, and 7. An expanded horizontal axis was used in Fig. 5 since the curves are nearly indistinguishable.

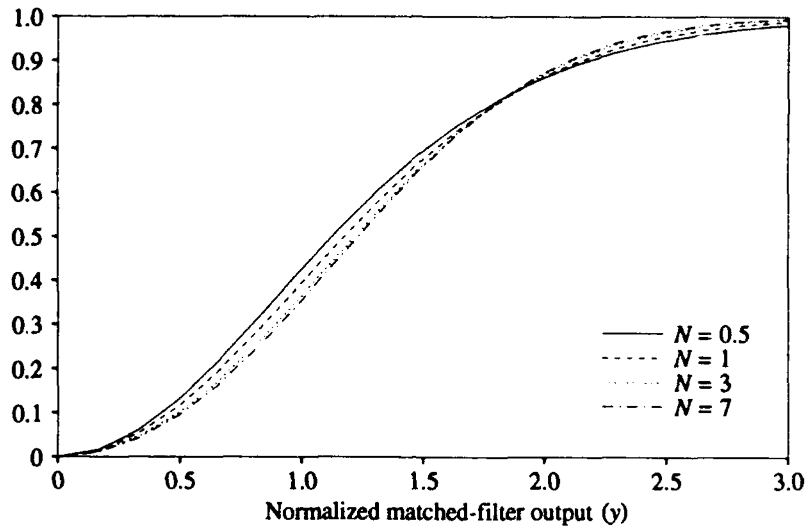

Fig. 5 – Cumulative probability function for $R = 1$, and for N equal to 0.5, 1, 3, and 7

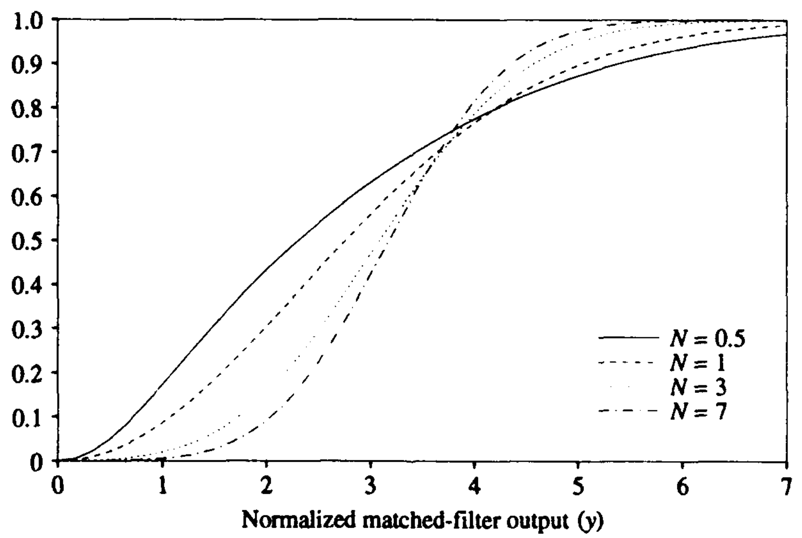

Fig. 6 – Cumulative probability functions for $R = 10$, and for N equal to 0.5, 1, 3, and 7

The convergence of the series in Eq. (18) can be proved by using the binomial theorem. First, we note that from Eq. (17), and the definition of the gamma function, it follows that

$$\gamma(n+1, y^2) \leq \int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n!\ . \tag{19}$$

This inequality implies that

$$P(y)_{T+C} \leq \frac{1}{\left(1+\frac{R}{N}\right)^N} \sum_{n=0}^{\infty} \frac{(N)_n}{n!} \left(\frac{1}{1+\frac{N}{R}}\right)^n . \tag{20}$$

It can be shown that $(N)_n/n!$ is equal to the binomial coefficient. Furthermore, $1/(1 + N/R) < 1$ since $R > 0$. Thus, by the Binomial Theorem, the right side of Eq. (20) converges, yielding

$$P(y)_{T+C} \leq \frac{1}{\left(1+\frac{R}{N}\right)^N} \frac{1}{\left(1-\frac{1}{1+N/R}\right)^N} = 1 . \tag{21}$$

This bound implies that the series in Eq. (18) converges.

MOMENTS ABOUT ZERO

The m-th moment about zero is defined as

$$E\{y^m\} \doteq \int_0^{\infty} y^m p(y)_{T+C}\, dy. \tag{22}$$

Substituting Eq. (15) into Eq. (22), and using the formula ([7], page 860, Eq. 7.621.4)

$$\int_0^{\infty} e^{-sx} x^{b-1} {}_1F_1(a;c;\alpha x)\, dx = \frac{\Gamma(b)}{s^b} {}_2F_1\left(a\,,\, b\,;\, c\,;\, \frac{\alpha}{s}\right), \tag{23}$$

where ${}_2F_1$ is the hypergeometric function, one finds that

$$E\{y^m\} = \frac{1}{(1+\frac{R}{N})^N} \Gamma\left(\frac{m}{2}+1\right) {}_2F_1\left(N\,,\, \frac{m}{2}+1\,;\, 1\,;\, \frac{1}{1+\frac{N}{R}}\right). \tag{24}$$

However, the hypergeometric function obeys the property

$$\frac{1}{(1+x)^a} {}_2F_1\left(a\,,\, b\,;\, c\,;\, \frac{1}{1+\frac{1}{x}}\right) = {}_2F_1(a\,,\, c-b\,;\, c\,;\, -x), \tag{25}$$

and can be used to derive an alternate form of the m-th moment about zero given by

$$E\{y^m\} = \Gamma\left(\frac{m}{2}+1\right) {}_2F_1\left(N\,,\, -\frac{m}{2}\,;\, 1\,;\, -\frac{R}{N}\right). \tag{26}$$

The hypergeometric function has the series representation

$${}_2F_1(a\,,\, b\,;\, c\,;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}, \tag{27}$$

which is convergent for $|x| < 1$. However, if the series terminates, it is convergent for all x. One can exploit this property by noting that for $m = 2k$ $(k = 0, 1, 2, \ldots)$, the series for the hypergeometric function in Eq. (26) terminates at the k-th term, thus

$$E\{y^{2k}\} = k!\left[1 + \sum_{n=1}^{k} k(k-1)\ldots(k-n+1)\frac{(N)_n}{(n!)^2}\left(\frac{R}{N}\right)^n\right]. \tag{28}$$

In particular,

$$E\{y^2\} = 1 + R, \tag{29}$$

$$E\{y^4\} = 2 + 4R + \frac{N+1}{N}R^2. \tag{30}$$

Of course, Eq. (28) can be used to prove the trivial result $E\{y^0\} = E\{1\} = 1$.

A user-defined *Mathematica* function that calculates the moments about zero is

```
Moment[n_,R_,m_] :=
    Gamma[m/2+1] Hypergeometric2F1[n,m/2+1,1,1/(1+n/R)]/(1+R/n)^n;
```

CALCULATION OF DETECTION PROBABILITIES

If no target is present, then $R = 0$, and, as indicated in Section 2, Eq. (15) reduces to

$$p(y)_C = 2ye^{-y^2}. \tag{31}$$

If the detection threshold is η, then the probability of false alarm is

$$P_{fa} = \int_{\eta}^{\infty} 2ye^{-y^2}\,dy = e^{-\eta^2}. \tag{32}$$

Solving for the threshold in Eq. (32) yields

$$\eta = \sqrt{-\ln(P_{fa})}. \tag{33}$$

The associated probability of detection follows from Eq. (18):

$$\begin{aligned} P_d &= \int_{\eta}^{\infty} p(y)_{T+C}\,dy \\ &= 1 - \int_0^{\eta} p(y)_{T+C}\,dy \\ &= 1 - \frac{1}{\left(1+\frac{R}{N}\right)^N}\sum_{n=0}^{\infty}\frac{(N)_n}{(n!)^2}\frac{\gamma(n+1,\eta^2)}{\left(1+\frac{N}{R}\right)^n}. \end{aligned} \tag{34}$$

Figures 7 through 9 show the probability of detection for N equal to 0.5, 1, 3, and 7, and a probability of false alarm of 0.01, 0.001, and 0.00001.

CONCLUSIONS

We have presented a concise derivation of the probability density function, cumulative probability function, and moments about zero of the normalized matched-filter output when detecting a target buried in white Gaussian noise, and whose target strength follows a chi-square distribution. It was shown that the cumulative probability function has a series representation that lends itself to calculation using a mathematical software package such as *Mathematica*.

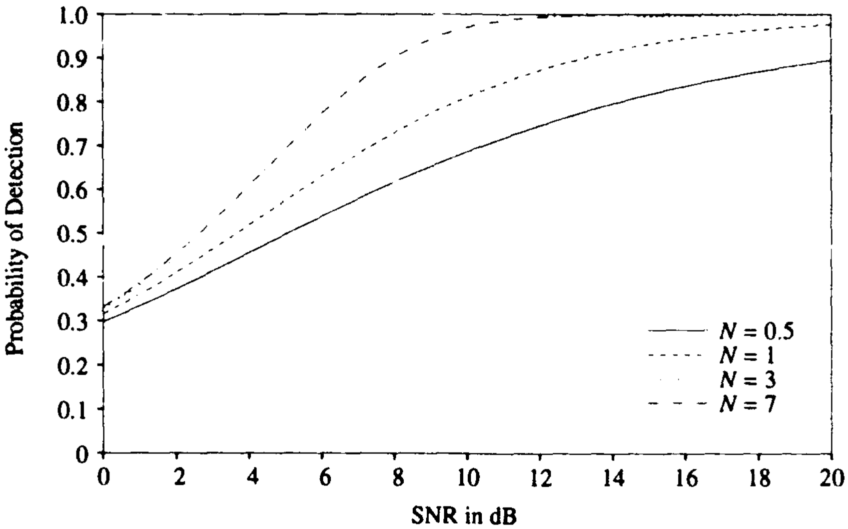

Fig. 7 – Probability of detection as a function of SNR for $P_{fa} = 0.01$

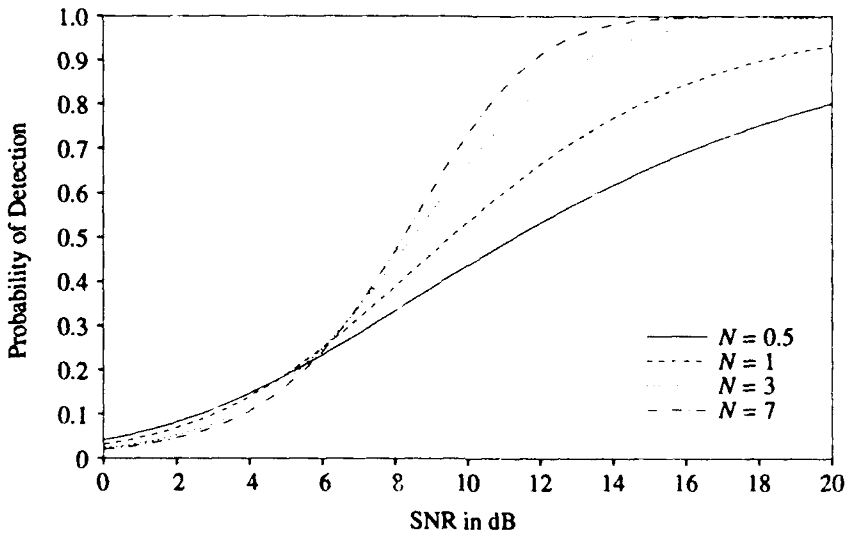

Fig. 8 – Probability of detection as a function of SNR for $P_{fa} = 0.001$

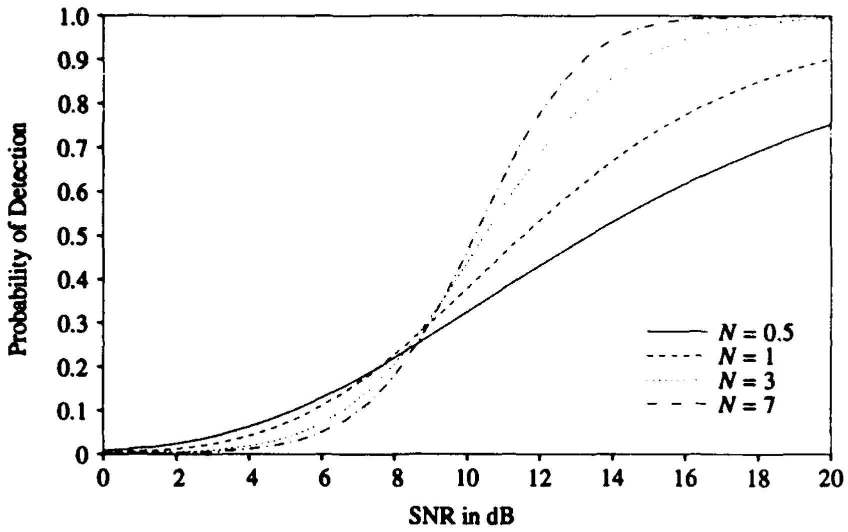

Fig. 9 – Probability of detection as a function of SNR for P_{fa} = 0.00001

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