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ACOUSTIC OR ELECTROMAGNETIC SCATTERING FROM THE PENETRABLE WEDGE

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ACOUSTIC OR ELECTROMAGNETIC SCATTERING FROM THE PENETRABLE WEDGE

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28 February 1993

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ABSTRACT

An integral transform analysis of the static and time-harmonic scattering of the scalar (two-dimensional) field radiated by a line source in the vicinity of a penetrable and impedance boundary wedge is presented. The Mellin transform is used to derive the exact static solution to Laplace's equation for the dielectric wedge, in the form of a modal series. The important dielectric edge condition behavior is explicitly contained in this analytic solution. Application of the Kontorovich-Lebedev transform to the acoustic scattering by the density contrast wedge also gives analytical solutions of the Helmholtz equation, in a form suitable for the asymptotic extraction of the high-frequency ray components. Similarly, the transform analysis of the impedance boundary wedge results in a difference equation to be solved in the transform variable. A special inhomogeneous surface impedance yields purely algebraic equations for the transforms, which can be solved in closed-form and inverted. At each stage, the mathematics and discussion are guided by the physics of both the acoustic and equivalent electromagnetic scattering problems.

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I. INTRODUCTION

The intentional "or" in the title of this report emphasizes the exact, mathematical equivalence between the acoustic and electromagnetic scattering problems for a two-dimensional, penetrable wedge. A z -directed, time-harmonic line source at the transverse (x, y) position $\vec{r}'$ is external and parallel to the edge of an infinite wedge of included angle 2α . Both the exterior (region 1) and interior (region 2) are composed of simple media having linear, isotropic, and homogeneous properties. See Fig. 1 of page 7, for example.

In the acoustic problem for the velocity/density contrast wedge, which is generally called the transmission problem in the mathematics literature, the media ($j = 1$ or 2) constitutive parameters are conveniently taken to be the ambient density ρ_{0j} and the wave speed c_j . In the electromagnetic problem, each medium is characterized by its electric permittivity ϵ_j and magnetic permeability μ_j , which naturally appear together as the wave speed $c_j = 1/\sqrt{\mu_j \epsilon_j}$ and intrinsic impedance $\eta_j = \sqrt{\mu_j / \epsilon_j}$. The electromagnetic scatterer is referred to as the dielectric wedge, even though μ as well as ϵ can independently differ in regions 1 and 2.

At the constant angular frequency ω , associated with each unbounded medium is the wavenumber $k_j = \omega/c_j$. The reduced wave or Helmholtz equation

$$(\nabla^2 + k_j^2) \psi(\vec{r}) = -\delta(\vec{r} - \vec{r}')$$

governs the total field in each of these three different physical situations:

1. *Acoustic line source*, with $\psi(\vec{r})$ the scalar pressure field. Boundary conditions at the material interfaces are continuity of the pressure and normal velocity,

$$\psi_1 = \psi_2 \quad \text{and} \quad \frac{1}{\rho_{01}} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_1 = \frac{1}{\rho_{02}} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_2.$$

2. *Magnetic line source*, where the scalar field of interest is the single z -directed component of magnetic field $H_z(\vec{r}) = \psi(\vec{r})$ in this often-called case of TE or transverse-electric polarization. A unique solution to Maxwell's equations requires continuity of tangential magnetic and electric fields,

$$\psi_1 = \psi_2 \quad \text{and} \quad \frac{1}{\epsilon_1} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_1 = \frac{1}{\epsilon_2} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_2.$$

3. *Electric line source*, where the scalar field of interest is the single z -directed component of electric field $E_z(\vec{r}) = \psi(\vec{r})$ in this often-called case of TM or transverse-magnetic polarization. Boundary conditions of continuity of tangential electric and magnetic fields are

$$\psi_1 = \psi_2 \quad \text{and} \quad \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_1 = \frac{1}{\mu_2} \frac{1}{r} \frac{\partial}{\partial \phi} \psi_2.$$

All three of these different physical problems are governed by the same differential equation, with different physical origins for the constants appearing in the mathematically identical boundary conditions. Therefore, this acoustic and electromagnetic problem is properly treated as one boundary value problem, and liberal use is made of the archival contributions of acousticians, electricians, and wave mechanics.

The perfectly soft (Dirichlet or TM boundary condition) or perfectly hard (Neumann or TE boundary condition) impenetrable wedges are historically solved by separation of variables. Familiarity with these classical solutions is important for guidance and sanity checks of any approximate methods applied to the penetrable wedge. The Kontorovich-Lebedev transform (a spectral superposition of cylindrical waves) of D. S. Jones [49] is extended in Sections III and IV to handle the density contrast and impedance boundary wedges, respectively. The analysis of the impedance boundary wedge results in a difference equation in the transform variable, which is interesting in view of the apparently unrelated difference equation that appears in a 1959 paper by W. E. Williams [7].

The analytic solution of the Leontovich impedance boundary conditions by Maliuzhinets [6] in 1958 is still being asymptotically evaluated [9-12,25-26] to derive high-frequency diffraction coefficients in the spirit of the geometric theory of diffraction (GTD) or related wedge assemblage [41]. Heuristic diffraction coefficients [27] consisting of the product of the accepted forms for perfect conductors and plane-wave reflection coefficients at a dielectric interface are absolutely invalid. A seemingly ad-hoc correction to the physical optics or Kirchhoff approximation [24] gives results that agree with numerical experiments in some instances, but offer little hope of getting closer to the important wave physics. Obviously then, the prospect of rigorously deriving the high-frequency ray behavior is a prime motivation for the present research effort.

Although it may initially appear strange to the uninitiated, the static solution is a critical component of any dynamic scattering problem, especially with regard to edge conditions and behavior in source regions. Section II is a complete and exact solution to the static dielectric wedge problem. Based on literature searches and discussion with E. Marx (author of [20-22, 28]), it is presently believed that this static solution is new. The anticipated peer review of a refereed journal submission based on Section II should confirm or refute this. It should be noted that the special case with the line source lying on the x -axis (the wedge bisector) does appear in the Russian text [53]. Furthermore, Smythe [50] presents a formal treatment of the static dielectric wedge, which is unfortunately flawed with divergent integral representations. In any event, the exact static solution of Section II is now available to confidently answer any questions about the near-in edge behavior. The static problem is solved using the Mellin transform, which is itself the static limit of the Kontorovich-Lebedev transform employed in Sections III and IV.

It is important to address the literature on the penetrable wedge, and to put this research in historical perspective. Rawlins [1] constructs a perturbation solution of a volume Fredholm integral equation of the first kind (actually a surface equation in $\mathbb{R}^2$), which converges for low values of the relative permittivity or dielectric constant $\epsilon_r = \epsilon_2/\epsilon_1$. Several authors, including Kleinman and Martin [4], prove convergence for $1 < \sqrt{\epsilon_r} < 2$, and divergence of the Neumann iteration series for $\sqrt{\epsilon_r} > 2$. Except for largely numerical schemes, the 1979 observation "... little progress has been made with solving the canonical

problem ... " by D. S. Jones [48] has remained true. Fortunately, the progress and solid foundation of the present research is encouraging, especially with regard to the novel solution of the impedance boundary wedge.

In competition with the integral transforms that have become the method of choice in this first phase of the project, surface integral equations are applicable to the transmission problem. Colton and Kress [59] give one of the most complete, and therefore theoretical, accounts of coupled Fredholm integral equations for equivalent surface distributions. The infinite extent of the surface may appear a little discouraging, but perhaps no more so than the infinite domains in the integral transforms. Davey [5] gets some mileage out of a physically intuitive subtraction of the expected far ($r \rightarrow \infty$) surface behavior, which is to first order the behavior at an infinite, planar boundary. Glisson [19], Marx [20], and Kleinman and Martin [4] combine the pair of coupled integral equations into a double integral operator of a single unknown surface function, which they all claim is preferable to two unknown functions. Two coupled, single integral operators could in fact be analytically and computationally preferable, especially in view of the "hypersingular" kernel in the double operator.

The substantial contribution by Chu [45], in 1989, for the impulse response of the density contrast wedge can serve as a future check on the time-harmonic results of Section III, after a suitable Fourier transformation between time and frequency. A three-dimensional plane wave reflection coefficient using a finite number of images is assembled by Deane and Tindle [46], in one of many JASA papers that underscore the applicability of propagation in wedge-shaped regions to shallow ocean acoustics. Westwood [47] employs an approximate summation (integral) of complex ray contributions in the spirit of Felsen, and then computes some wideband transient responses via the Fourier transform.

The research summarized in this report is a first-principles effort toward a fundamental understanding of the static and time-harmonic excitation of the dielectric (penetrable) wedge. Arbitrary transient waveforms are not addressed in this phase of the research, as a Fourier transform of the time-harmonic response is the traditional and logical approach to time domain scattering. The case of plane wave excitation is obtained by letting the source recede to infinity, i.e. $r' \rightarrow \infty$. Each section is mostly self-contained, although frequent reference is made to the details of the completed static solution of Section II. A concise summary and suggestions for the most fruitful avenues to pursue for the coming year appear in Section V.

II. EXACT STATIC SOLUTION FOR THE DIELECTRIC WEDGE

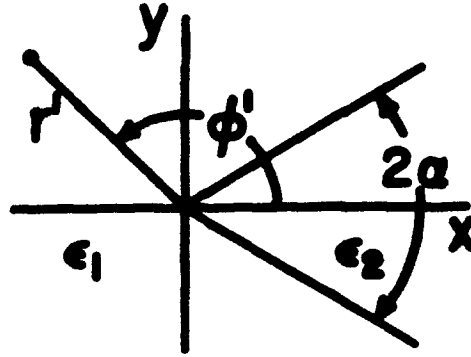


FIG. 1. Dielectric Wedge and Static Line Charge

The static ($k = 0$) excitation of the penetrable wedge is treated in the electrical parlance, where the line charge of unit lineal density is located at the source coordinates (r', ϕ') of Fig. 1. The permittivity of the wedge of angle 2α is ϵ_2 , which is surrounded by a medium with permittivity ϵ_1 . Consistent with the other chapters of this report, all geometry is of infinite extent and invariant in the z -dimension, which restricts the physical domain to $\mathbb{R}^2$.

The irrotational electrostatic field is uniquely characterized by the scalar potential $\psi(\vec{r})$, which is a solution of Poisson's equation subject to appropriate boundary conditions. Denote by $\psi_1(\vec{r})$ the potential field in the external region where the source is

$$\nabla^2 \psi_1(r, \phi) = -\frac{1}{\epsilon_1 r} \delta(r - r') \delta(\phi - \phi') \quad (\alpha \leq \phi \leq 2\pi - \alpha) \quad (1)$$

and let $\psi_2(\vec{r})$ be the source-free field inside the wedge

$$\nabla^2 \psi_2(r, \phi) = 0 \quad (-\alpha \leq \phi \leq \alpha). \quad (2)$$

Boundary conditions at the material interfaces $\phi = \pm\alpha$ for this scalar potential are continuity of ψ (from continuity of the tangential electric field) and continuity of the normal electric flux density $\epsilon_j \partial \psi / \partial n$ (absence of any free surface charge).

In order to simplify the ensuing analysis, it is expedient to decompose the desired solution $\psi(x, y)$ for the boundary value problem of Fig. 1 into its odd $\psi^o(x, y)$ and even $\psi^e(x, y)$ components

$$\psi(x, \pm y) = \frac{1}{2} [\psi^e(x, y) \pm \psi^o(x, y)] \quad (y \geq 0) \quad (3)$$

with respect to the x -axis which bisects the wedge. Since $\psi^o(x, y)$ is an odd function of y , it vanishes on the $y = 0$ plane and is therefore the solution in the upper half-space $y \geq 0$ for the problem having a soft bisecting plane (Dirichlet boundary condition). This is equivalent to an out-of-phase image source. Similarly, $\psi^e(x, y)$ is the solution for the hard

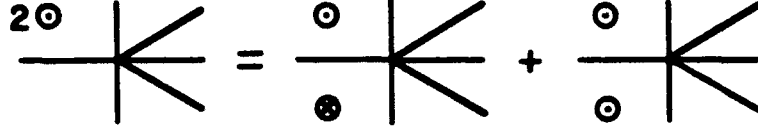


FIG. 2. Odd and Even Symmetry Components

bisecting plane (Neumann boundary condition), which sustains an in-phase image source. This symmetry is depicted in Fig. 2.

A. Mellin Transform for the Case of Odd Symmetry.

The source coordinate $\phi = \phi'$ divides region (1) into two source-free regions, resulting in a total of three subregions to consider for the half-space above the perfectly soft plane:

$$\psi^o(x, y) = \begin{cases} \psi_2(r, \phi), & 0 \leq \phi \leq \alpha \\ \psi_1^-(r, \phi), & \alpha < \phi \leq \phi' \\ \psi_1^+(r, \phi), & \phi' < \phi \leq \pi. \end{cases} \quad (4)$$

The Dirichlet boundary condition on the soft plane is

$$\psi_2(r, 0) = 0 \quad (5)$$

$$\psi_1^+(r, \pi) = 0 \quad (6)$$

and at the material interface

$$\psi_2(r, \alpha) = \psi_1^-(r, \alpha) \quad (7)$$

$$\epsilon_2 \frac{\partial}{\partial \phi} \psi_2(r, \alpha) = \epsilon_1 \frac{\partial}{\partial \phi} \psi_1^-(r, \alpha). \quad (8)$$

Consistent with the conventional Green's function ansatz, the potential is continuous everywhere across the source plane $\phi = \phi'$

$$\psi_1^-(r, \phi') = \psi_1^+(r, \phi'), \quad (9)$$

while the discontinuity in normal derivative

$$\frac{\partial}{\partial \phi} \psi_1^-(r, \phi') - \frac{\partial}{\partial \phi} \psi_1^+(r, \phi') = \frac{r}{\epsilon_1} \delta(r - r') \quad (10)$$

results from integrating (1) from $\phi' -$ to $\phi' +$.

The Mellin transform [52] is used to transform the radial variable r to a complex variable s , whereupon the remaining differential equation with boundary conditions in ϕ is solved in

closed-form. The physical solution $\psi(r, \phi)$ is recovered by careful evaluation of the inverse Mellin transform. The Mellin transform of $f(r)$ is

$$F(s) = \mathcal{M}\{f(r); s\} = \int_0^{\infty} r^{s-1} f(r) dr \quad (11)$$

and has

$$f(r) = \mathcal{M}^{-1}\{F(s); r\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s} F(s) ds \quad (12)$$

as its inversion formula. If $r^{a-1} f(r)$ is absolutely integrable on the positive real axis for some $a > 0$, then the inversion is valid for $c > a$.

The Mellin transform of the product of r^2 and Poisson's equation (1)

$$\left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \phi^2} \right) \psi_1(r, \phi) = -\frac{r}{\epsilon_1} \delta(r - r') \delta(\phi - \phi') \quad (13)$$

is the simple form

$$\left(s^2 + \frac{\partial^2}{\partial \phi^2} \right) \Psi_1(s, \phi) = -\frac{r'^2}{\epsilon_1} \delta(\phi - \phi'). \quad (14)$$

This fortunate property of the Mellin transform of the r dependence in the two-dimensional Laplacian is responsible for its successful application [52]–[54] to potential problems in wedge-shaped regions. In region (2) where there is no forcing term, this procedure gives

$$\left(s^2 + \frac{\partial^2}{\partial \phi^2} \right) \Psi_2(s, \phi) = 0. \quad (15)$$

The complex variable s is a parameter in the above pair of ordinary differential equations in ϕ , with solutions

$$\Psi_2(s, \phi) = A(s) \sin(s\phi) \quad (0 \leq \phi \leq \alpha) \quad (16)$$

$$\Psi_1^-(s, \phi) = B(s) \sin(s\phi) + C(s) \cos(s\phi) \quad (\alpha < \phi \leq \phi') \quad (17)$$

$$\Psi_1^+(s, \phi) = D(s) \sin[s(\phi - \pi)] \quad (\phi' < \phi \leq \pi) \quad (18)$$

in view of the soft boundary conditions (5) and (6). Transformation of the four remaining conditions (7)–(10) gives the set of simultaneous equations

$$\begin{bmatrix} -\sin(s\alpha) & \sin(s\alpha) & \cos(s\alpha) & 0 \\ \frac{\epsilon_2}{\epsilon_1} \cos(s\alpha) & -\cos(s\alpha) & \sin(s\alpha) & 0 \\ 0 & \sin(s\phi') & \cos(s\phi') & \sin[s(\pi - \phi')] \\ 0 & \cos(s\phi') & -\sin(s\phi') & -\cos[s(\pi - \phi')] \end{bmatrix} \begin{bmatrix} A(s) \\ B(s) \\ C(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r'^2}{\epsilon_1 s} \end{bmatrix} \quad (19)$$

to be solved for the coefficient functions in (16)-(18). Solution of these yields the (soft) Mellin transforms

$$\Psi_2(s, \phi) = \frac{1 - \Gamma}{\epsilon_1} \frac{r'^s}{s} \sin[s(\pi - \phi')] \sin(s\phi) / \Delta(s) \quad (0 \leq \phi \leq \alpha) \quad (20)$$

$$\Psi_1^-(s, \phi) = \frac{r'^s}{\epsilon_1 s} \sin[s(\pi - \phi')] \{ \sin(s\phi) + \Gamma \sin[s(\phi - 2\alpha)] \} / \Delta(s) \quad (\alpha < \phi \leq \phi') \quad (21)$$

$$\Psi_1^+(s, \phi) = \frac{r'^s}{\epsilon_1 s} \sin[s(\pi - \phi)] \{ \sin(s\phi') + \Gamma \sin[s(\phi' - 2\alpha)] \} / \Delta(s) \quad (\phi' < \phi \leq \pi) \quad (22)$$

with denominator function

$$\Delta(s) = \sin(s\pi) + \Gamma \sin[s(\pi - 2\alpha)] \quad (23)$$

and dielectric contrast parameter

$$\Gamma = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}. \quad (24)$$

B. Modifications for the Case of Even Symmetry.

In the case of a hard ground plane, the Neumann boundary conditions

$$\frac{\partial}{\partial \phi} \psi_2(r, 0) = 0 \quad (25)$$

$$\frac{\partial}{\partial \phi} \psi_1^+(r, \pi) = 0 \quad (26)$$

replace the Dirichlet boundary conditions (5) and (6) of the previous section. A similar application of the Mellin transform and the other unchanged boundary conditions yields the (hard) Mellin transforms

$$\Psi_2(s, \phi) = \frac{\Gamma - 1}{\epsilon_1} \frac{r'^s}{s} \cos[s(\pi - \phi')] \cos(s\phi) / \Delta(s) \quad (0 \leq \phi \leq \alpha) \quad (27)$$

$$\Psi_1^-(s, \phi) = \frac{r'^s}{\epsilon_1 s} \cos[s(\pi - \phi')] \{ \Gamma \cos[s(\phi - 2\alpha)] - \cos(s\phi) \} / \Delta(s) \quad (\alpha < \phi \leq \phi') \quad (28)$$

$$\Psi_1^+(s, \phi) = \frac{r'^s}{\epsilon_1 s} \cos[s(\pi - \phi)] \{ \Gamma \cos[s(\phi' - 2\alpha)] - \cos(s\phi') \} / \Delta(s) \quad (\phi' < \phi \leq \pi) \quad (29)$$

where in this case the denominator function is

$$\Delta(s) = \sin(s\pi) - \Gamma \sin[s(\pi - 2\alpha)]. \quad (30)$$

Note that, except for the simple scaling by ϵ_1 which persists from the original source strength chosen in (1), the presence of two different dielectrics is entirely accounted for by Γ in all of the above transforms.

C. Inverse Mellin Transforms - Preliminaries.

The zeros of the denominator functions (23) and (30)

$$\Delta(s) = \sin(s\pi) \pm \Gamma \sin[s(\pi - 2\alpha)] \quad \left\{ \begin{array}{l} \text{soft} \\ \text{hard} \end{array} \right\} \quad (31)$$

are central to the Mellin inversion (12). These real, simple zeros can be computed via Muller's algorithm [56] for arbitrary half-angle α , or solved analytically as the roots of a trigonometric polynomial when α is a rational multiple of π . This procedure is demonstrated for the particular case $\alpha = \pi/3$, whereupon the variable change $u = s\pi/3$ in (31) gives

$$\sin(3u) \pm \Gamma \sin(u) = 0 \quad (32)$$

which factors into

$$[3 \pm \Gamma - 4 \sin^2(u)] \sin(u) = 0. \quad (33)$$

The required roots are now explicitly given by

$$s_n = \left\{ \begin{array}{l} \frac{3}{\pi} \sin^{-1} \sqrt{\frac{3 \pm \Gamma}{4}} \\ -\frac{3}{\pi} \sin^{-1} \sqrt{\frac{3 \pm \Gamma}{4}} \\ 0 \end{array} \right\} + 3m \quad (m = 0, \pm 1, \pm 2, \dots), \quad (34)$$

where it is recalled that the $+\Gamma$ ($-\Gamma$) denotes the case of soft (hard) symmetry. The effect of the dielectric material ($\epsilon_2 \neq \epsilon_1 \Rightarrow \Gamma \neq 0$) on the potential above both symmetry planes is a regular displacement of the integer poles for the homogeneous case ($\epsilon_2 = \epsilon_1 \Rightarrow \Gamma = 0$). The index n in (34) is a denumerable ordering of these poles.

D. Inverse Mellin Transform for the Case of Odd Symmetry.

As $r \rightarrow 0$ the odd potential $\psi^o \rightarrow 0$ and the dipole behavior $\psi^o \sim 1/r$ prevails as $r \rightarrow \infty$ in the far field. A sufficient Bromwich contour for the complex integration (12) is therefore guaranteed for the choice of real constant $0 < c < 1$. Complete details of the Mellin inversion for the potential $\psi_2(r, \phi)$ of (4) inside the dielectric sector are provided, whereupon the final forms for $\psi_1^-(r, \phi)$ and $\psi_1^+(r, \phi)$ are immediately written by comparison.

The $\sin(s\phi)$ factor in $\Psi_2(s, \phi)$ of (20) together with the transform property [52]

$$\mathcal{M}^{-1} \{ \sin(s\phi) F(s); s \} = -\Im [f(re^{i\phi})] \quad (35)$$

renders

$$f(r) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\sin[s(\pi - \phi')](r'/r)^s}{\underbrace{s \{ \sin(s\pi) + \Gamma \sin[s(\pi - 2\alpha)] \}}_{G(s)}} ds \quad (36)$$

the desired function. The integration path in (36) has been pushed flush against the imaginary axis, and the principal value notation invoked to properly account for the pole at the origin. For $r > r'$, closure at infinity in the right-half plane gives a convenient

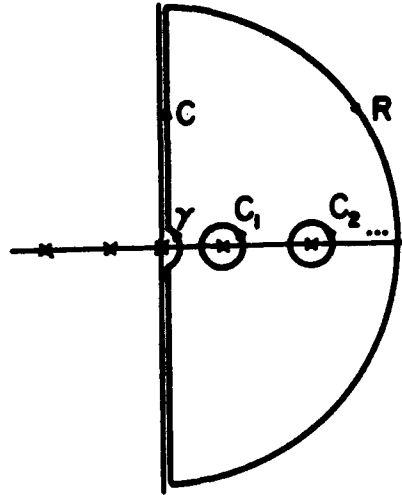


FIG. 3. Bromwich Contour in the Complex s -Plane

contour on which to apply Cauchy's integral theorem, and is shown in Fig. 3. Let C_n be a circle of vanishing radius $\rho \rightarrow 0$, centered on the pole s_n with residue

$$\lim_{\rho \rightarrow 0} \frac{1}{2\pi i} \oint_{C_n} G(s) ds = \frac{\sin[s_n(\pi - \phi')](r'/r)^{s_n}}{s_n \Delta'(s_n)} \quad (37)$$

where the derivative of the soft denominator function is

$$\Delta'(s_n) = \pi \cos(s_n \pi) + (\pi - 2\alpha) \Gamma \cos[s_n(\pi - 2\alpha)]. \quad (38)$$

The integral around the origin

$$\lim_{\rho \rightarrow 0} \frac{1}{2\pi i} \oint_{C_0} G(s) ds = \frac{\pi - \phi'}{\Delta'(0)} \quad (39)$$

follows from the limit $s_n \rightarrow 0$. Let positive $n = 1, 2, \dots$ be the indices of the poles in the right-half plane, and let negative $n = -1, -2, \dots$ identify the poles in the left-half plane. The odd symmetry of the function $\Delta(s)$ provides

$$s_{-n} = -s_n \quad \text{and} \quad \Delta'(-s_n) = \Delta'(s_n). \quad (40)$$

The contribution from the integral around the infinite semicircle R of Fig. 3 is zero for $r > r'$, while closure in the left-half plane is appropriate when $r < r'$. Cauchy's integral theorem now yields

$$f(r) = \pm \left\{ \frac{\pi - \phi'}{2\Delta'(0)} + \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')]}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} \right\} \quad (r \leq r'), \quad (41)$$

which together with (35) gives

$$\psi_2(r, \phi) = \frac{\Gamma - 1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')]}{s_n \Delta'(s_n)} \sin(s_n \phi) (r/r')^{\pm s_n} \quad (r \leq r') \quad (42)$$

for the sector $0 \leq \phi \leq \alpha$

as the inverse Mellin transform of (20). Since all three of the transforms (20)-(22) are of the same general form, the remaining two field expressions are apparently

$$\psi_1^-(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi')] \{ \sin(s_n \phi) + \Gamma \sin[s_n(\phi - 2\alpha)] \}}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} \quad (r \leq r') \quad \text{for the sector } \alpha < \phi \leq \phi' \quad (43)$$

and

$$\psi_1^+(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\sin[s_n(\pi - \phi)] \{ \sin(s_n \phi') + \Gamma \sin[s_n(\phi' - 2\alpha)] \}}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} \quad (r \leq r') \quad \text{for the sector } \phi' < \phi \leq \pi. \quad (44)$$

Note the reciprocity in the last two expressions for $\phi \leftrightarrow \phi'$. The above reduce to the correct potential due to a unit line charge located at $\phi' = \pi/2$ above a soft ground plane in the special case of no dielectric wedge ($\epsilon_2 = \epsilon_1$).

E. Inverse Mellin Transform for the Case of Even Symmetry.

The monopole potential $\ln r$ due to the original line source and the in-phase image is the dominant feature of ψ^e in the far field as $r \rightarrow \infty$. Therefore, the condition of integrability following the transform pair of (11) and (12) is not satisfied, and the Mellin inversion formula cannot be directly applied to the functions in (27)-(29). Mathematically, operating with the ϕ -derivative of the potential temporarily removes this troublesome logarithmic variation, which is then restored following the convergent Mellin inversion.

The ϕ -derivative of (27)

$$\frac{\partial}{\partial \phi} \Psi_2(s, \phi) = \frac{1 - \Gamma \cos[s(\pi - \phi')]}{\epsilon_1} \frac{\sin(s\phi)}{\Delta(s)} r'^s \quad (45)$$

is transformed in the manner of the previous section to

$$\frac{\partial}{\partial \phi} \psi_2(r, \phi) = \frac{\Gamma - 1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\cos[s_n(\pi - \phi')] \sin(s_n \phi)}{\Delta'(s_n)} (r/r')^{\pm s_n} \quad (r \leq r'), \quad (46)$$

where the s_n are now the hard poles in (34) and $-\Gamma$ replaces Γ in (38) for this hard denominator function (31). The anti-derivative with respect to ϕ yields

$$\psi_2(r, \phi) = \frac{1 - \Gamma}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\cos[s_n(\pi - \phi')] \cos(s_n \phi)}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} + w_{\mp}(r) \quad (r \leq r') \quad \text{for the sector } 0 \leq \phi \leq \alpha, \quad (47)$$

where the boundedness at $r = 0$ and the known behavior at $r \rightarrow \infty$ specify

$$w_{\mp}(r) = \begin{cases} 0, & \text{for } r < r' \\ \frac{-1}{\pi \epsilon_1} \ln(r/r'), & \text{for } r > r' \end{cases} \quad (48)$$

as the ϕ -independent term in this solution of Laplace's equation. Similarly, the spatial potentials from the related transforms (28) and (29) are thus

$$\psi_1^-(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\cos[s_n(\pi - \phi')] \{ \Gamma \cos[s_n(\phi - 2\alpha)] - \cos(s_n \phi) \}}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} + w_{\mp}(r) \quad (r \leq r') \quad \text{for the sector } \alpha < \phi \leq \phi' \quad (49)$$

and

$$\psi_1^+(r, \phi) = \frac{-1}{\epsilon_1} \sum_{n=1}^{\infty} \frac{\cos[s_n(\pi - \phi)] \{ \Gamma \cos[s_n(\phi' - 2\alpha)] - \cos(s_n \phi') \}}{s_n \Delta'(s_n)} (r/r')^{\pm s_n} + w_{\mp}(r) \quad (r \leq r') \quad \text{for the sector } \phi' < \phi \leq \pi. \quad (50)$$

These results are also verified for the special case of a unit line charge located at $\phi' = \pi/2$ in a homogeneous half-space ($\epsilon_2 = \epsilon_1$) above the hard symmetry plane.

F. Results for the Complete Static Solution.

The important edge-behavior of the *dynamic* fields in the immediate vicinity of the apex $r = 0$ is obtained from the above *static* analysis. A quasi-static philosophy of extracting the dominant behavior of a solution to the Helmholtz (wave) equation

$$(\nabla^2 + k^2) \psi(\vec{r}) = 0 \quad (51)$$

from the static ($k = 0$) solution to Laplace's equation

$$\nabla^2 \psi(\vec{r}) = 0 \quad (52)$$

in the neighborhood of boundary discontinuities and sources is well known and successful ([55], [57], and [58]). This analytic solution for the static wedge problem is a valuable resource for continuing wave studies.

Contours of constant potential are illustrated in Figs. 4-8, all for the case of wedge half-angle $\alpha = \pi/3$. Even and odd potentials are computed for the separate boundary value problems and combined according to (3). The line source is located a unit distance ($r' = 1$) from the wedge apex at the (x, y) coordinate origin. The domain of Figs. 4-7 is all internal to the unit circle, while Fig. 8 encompasses a larger area. Unfortunately, the poor numerical convergence of the modal series at $r = r'$ results in slight kinks in the contours near this unit circle. In fact, the data of Fig. 8 is already smoothed by a numerical averaging based on the mean value property of potential solutions. This slight blemish in Fig. 8 has no effect on either the physical interpretation or mathematical integrity of this analysis. Figs. 4-7 show the effect of varying both the source location ($\phi' = \pi/2$ or π) and the dielectric constant ($\epsilon_2 = 10$ or 100) of the wedge relative to the external medium of permittivity $\epsilon_1 = 1$. When $\epsilon_2 = 100$ (Figs. 6 and 7), the dielectric wedge is essentially an equipotential region, which is expected since an increasingly dense dielectric ($\epsilon_2 \rightarrow \infty$) behaves electrically like a perfect conductor.

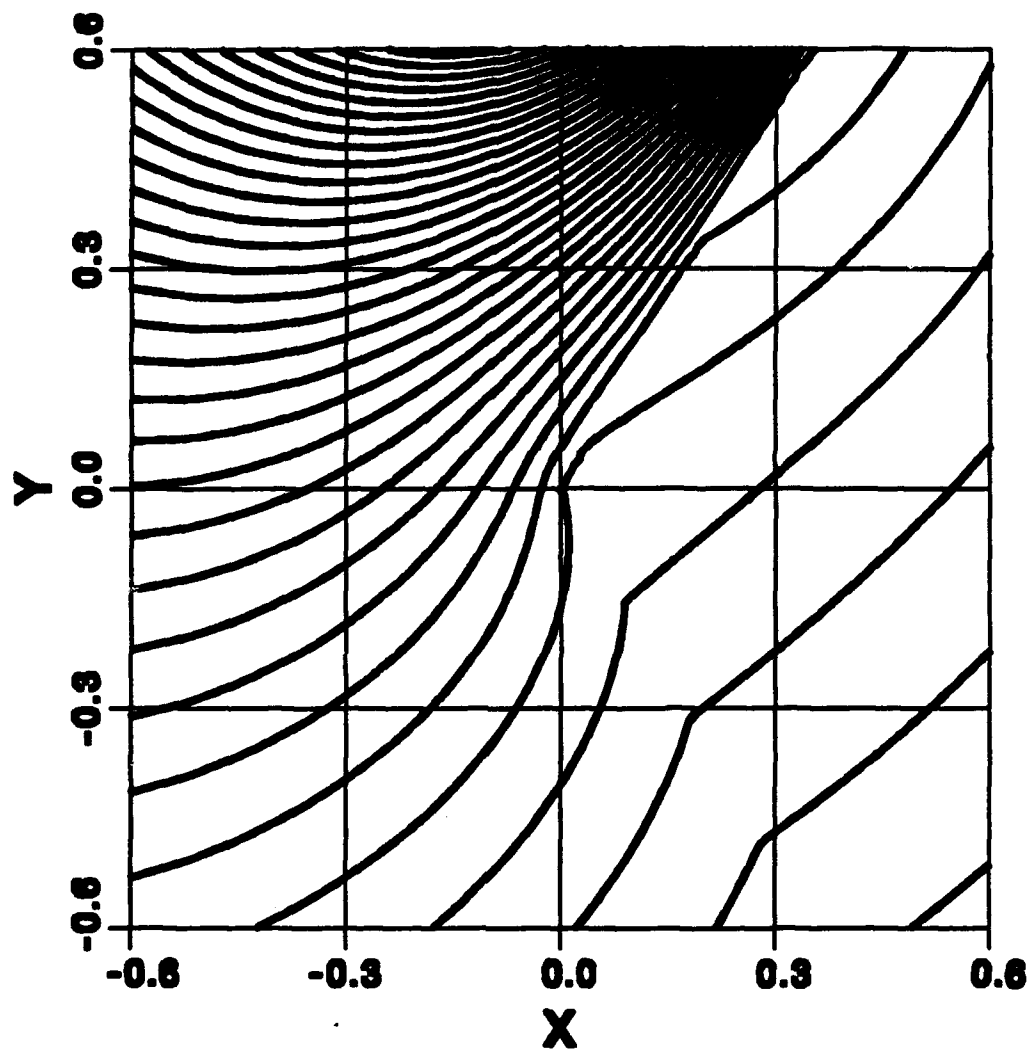


FIG. 4. Contours of Constant Potential for the Dielectric Wedge. Case: $r' = 1$, $\phi' = \pi/2$, $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 10$.

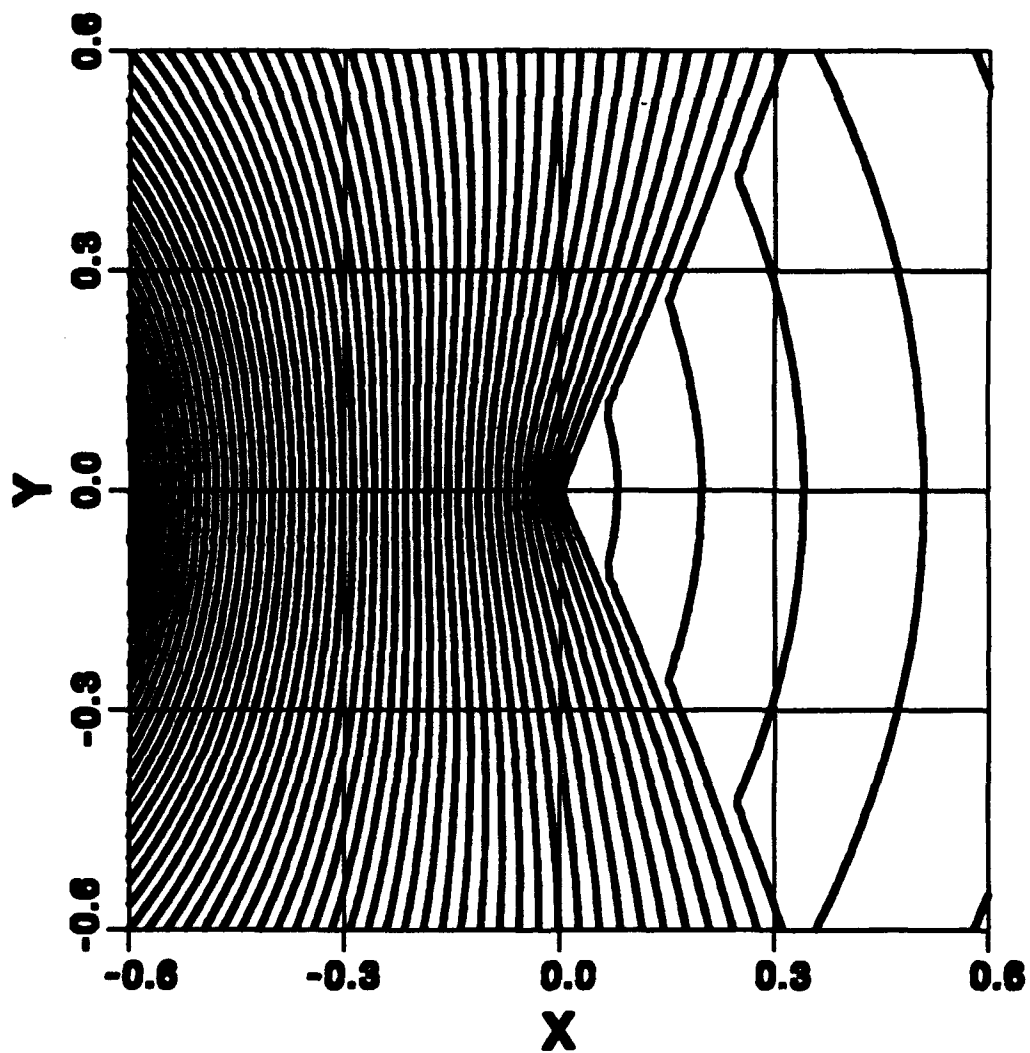


FIG. 5. Contours of Constant Potential for the Dielectric Wedge. Case: $r' = 1$, $\phi' = \pi$, $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 10$.

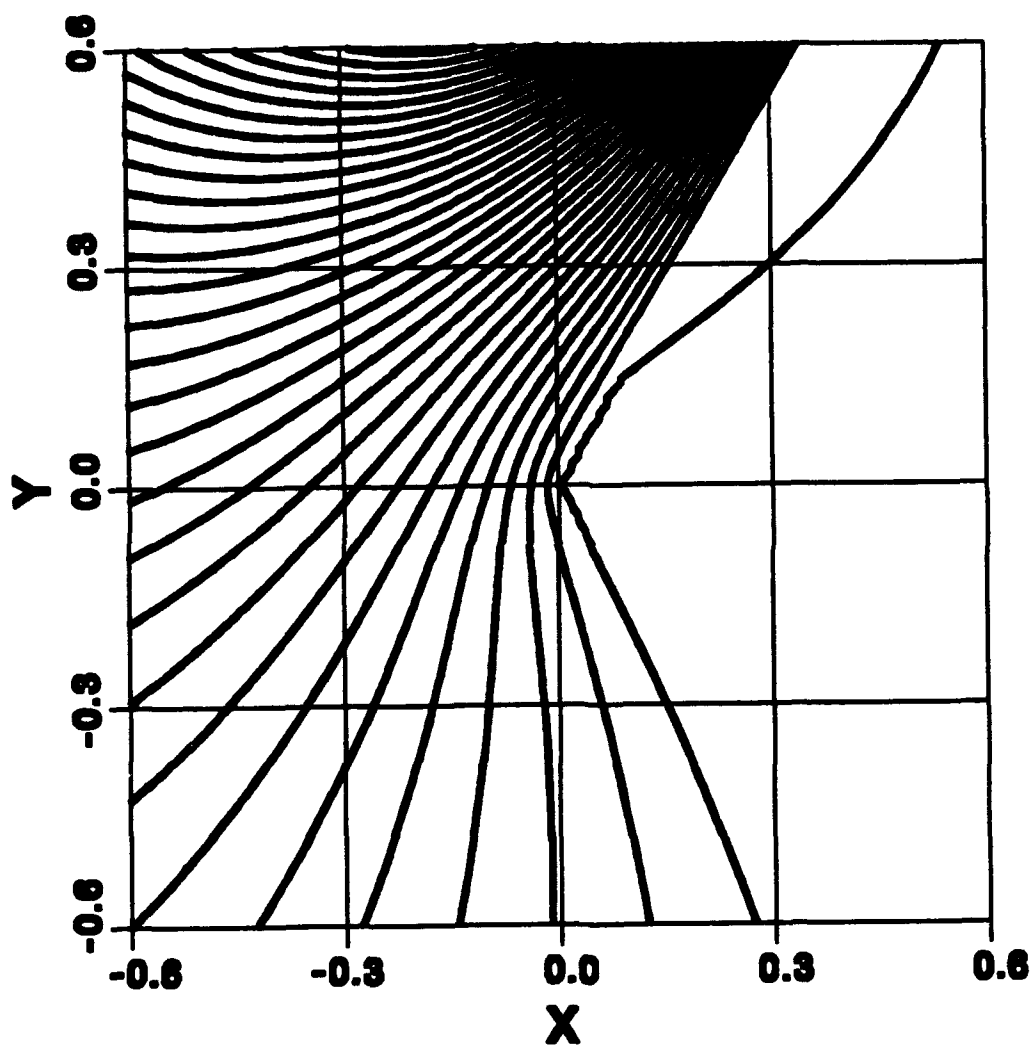


FIG. 6. Contours of Constant Potential for the Dielectric Wedge. Case: $r' = 1$, $\phi' = \pi/2$, $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 100$.

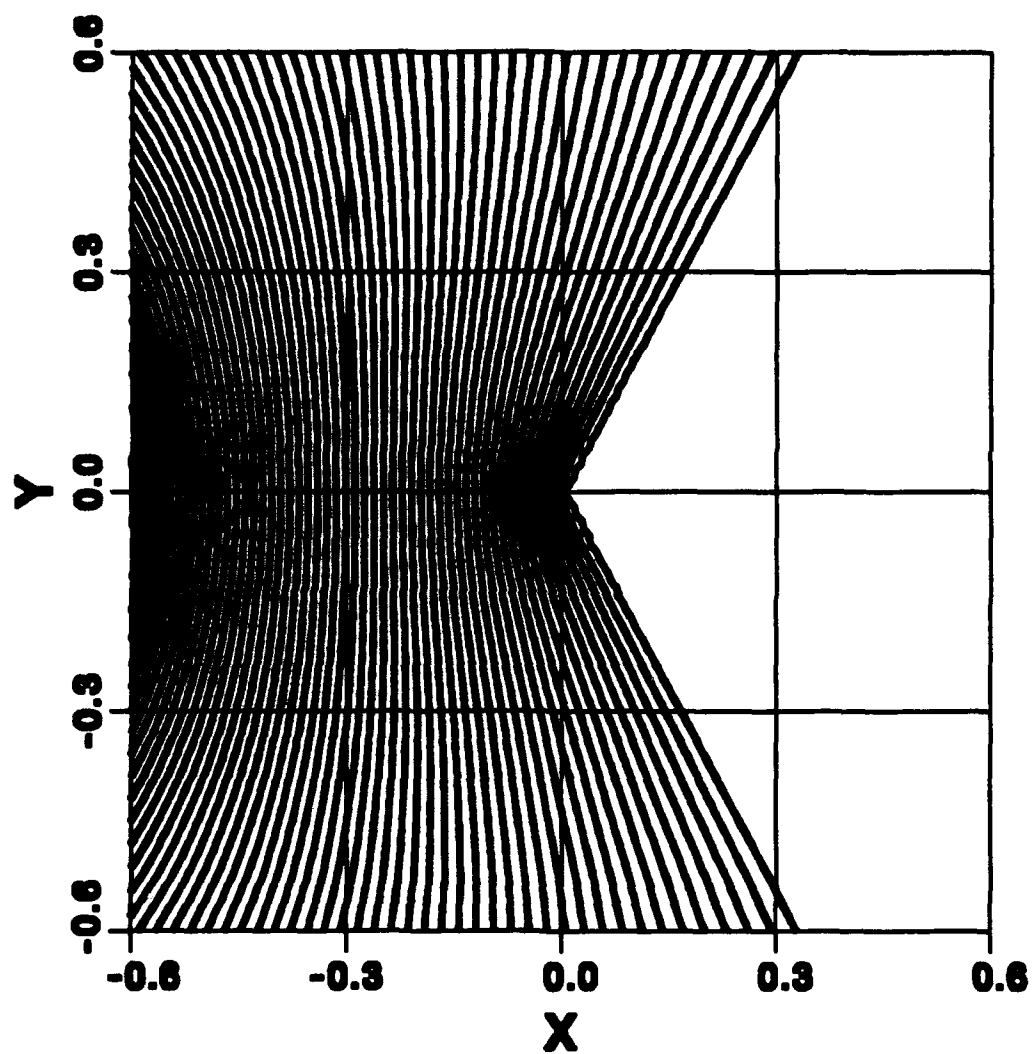


FIG. 7. Contours of Constant Potential for the Dielectric Wedge. Case: $r' = 1$, $\phi' = \pi$, $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 100$.

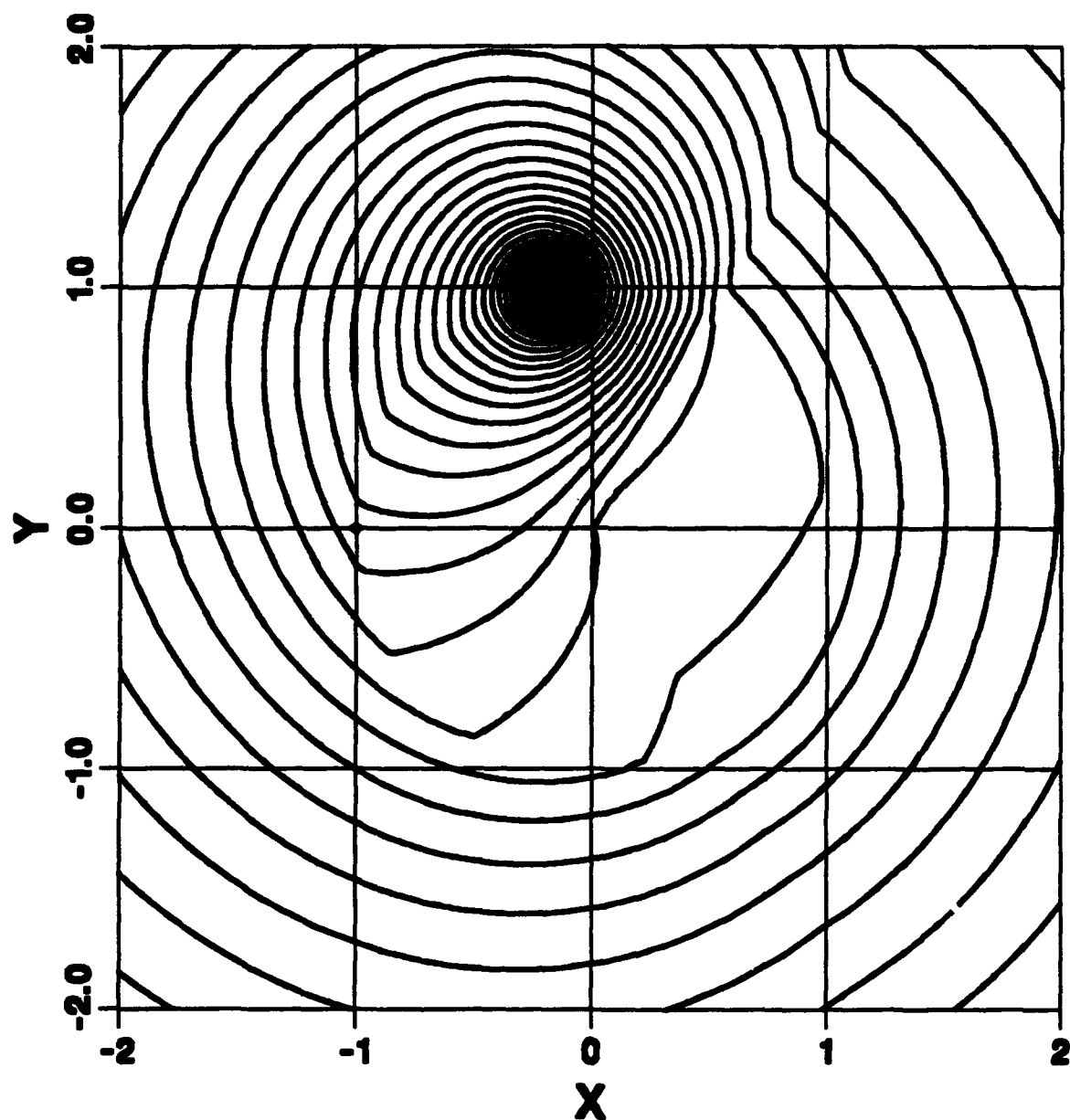


FIG. 8. Contours of Constant Potential for the Dielectric Wedge. Case: $r' = 1$, $\phi' = 100$ (deg), $\alpha = \pi/3$, $\epsilon_1 = 1$, $\epsilon_2 = 10$.

III. DENSITY CONTRAST WEDGE VIA THE KONTOROVICH-LEBEDEV TRANSFORM

Following D. S. Jones [49], the Kontorovich-Lebedev transform pair

$$g(\nu) = \int_0^{\infty} f(y) H_{\nu}^{(2)}(y) dy \quad (\Re \nu = 0) \quad (1)$$

$$xf(x) = \lim_{\epsilon \rightarrow +0} -\frac{1}{2} \int_{-i\infty}^{i\infty} e^{\epsilon \nu^2} \nu J_{\nu}(x) g(\nu) d\nu \quad (2)$$

is adopted for the time-harmonic analysis ($\exp[+i\omega t]$) of an acoustic plane wave incident upon a wedge having the same intrinsic wave-speed c as the surrounding medium, but with a different ambient density ρ_0 . Convergence of the transform is assured if

$$\int_0^1 |f(y) \ln y| dy < \infty \quad (3)$$

and if

$$\int_a^{\infty} f(y) y^{-\frac{1}{2}} e^{iy} dy < \infty \quad \text{for any } a > 0. \quad (4)$$

The geometry is still that of Fig. 1, but the acoustic notation is adopted in this section. Specifically, a line source of unit strength is located at the cylindrical coordinates (r', ϕ') in the external medium of ambient density ρ_{01} , and radiates time-harmonic waves at the radian frequency ω . A wedge-shaped scatterer of ambient density ρ_{02} occupies the sector $-\alpha \leq \phi \leq \alpha$. The key to applying the Kontorovich-Lebedev transform to this idealized transmission problem is precisely this restriction of a homogeneous wave speed c . The simplification in the mathematics is obvious in view of the ray physics at planar interfaces between media with differing density but with the same wave speed and therefore wavenumber $k = \omega/c$. This observation is certainly not new, as evidenced by Chu's [45] elegant derivation of the impulse response of this density-contrast/isovelocity wedge-scatterer.

As in the static case, the mathematics is less cumbersome if the solution is split into its odd and even components with respect to the symmetry plane $y = 0$. In particular, the total scalar field is

$$\psi(x, \pm y) = \frac{1}{2} [\psi^e(x, y) \pm \psi^o(x, y)] \quad (y \geq 0) \quad (5)$$

where the odd field ψ^o exists above the soft or Dirichlet ground plane and the even field ψ^e is the field in the half-space above the hard or Neumann ground plane (see also Fig. 2).

A. Odd Symmetry - the Soft or Dirichlet Ground Plane.

A statement of the boundary value problem consists of the Helmholtz equations

$$(\nabla^2 + k^2) \psi_1(r, \phi) = -\frac{1}{r'} \delta(r - r') \delta(\phi - \phi') \quad (\alpha < \phi \leq \pi) \quad (6)$$

and

$$(\nabla^2 + k^2) \psi_2(r, \phi) = 0 \quad (0 \leq \phi \leq \alpha) \quad (7)$$

together with the Dirichlet boundary conditions on the soft plane

$$\psi_2(r, 0) = 0 \quad \text{and} \quad \psi_1(r, \pi) = 0, \quad (8)$$

and continuity of both the pressure

$$\psi_2(r, \alpha) = \psi_1(r, \alpha) \quad (9)$$

and the normal velocity

$$\frac{1}{\rho_{01}} \frac{\partial}{\partial \phi} \psi_1(r, \alpha) = \frac{1}{\rho_{02}} \frac{\partial}{\partial \phi} \psi_2(r, \alpha) \quad (10)$$

at the material interface. Additionally, sufficient decay as $r \rightarrow \infty$ (the Sommerfeld radiation condition)

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial \psi}{\partial r} + ik\psi \right) = 0 \quad (11)$$

is automatically incorporated into the chosen integral representation by the Hankel function.

Application of the integral operator

$$\int_0^\infty dr r H_\nu^{(2)}(kr) \quad (12)$$

to the partial differential equation (6)

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \psi_1(r, \phi) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \psi_1(r, \phi) + k^2 \psi_1(r, \phi) = -\frac{1}{r'} \delta(r - r') \delta(\phi - \phi') \quad (13)$$

is followed by two consecutive integrations by parts to remove the r -derivatives. Explicitly, the first term above is transformed to

$$\begin{aligned} & \int_0^\infty \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \psi_1(r, \phi) \right] H_\nu^{(2)}(kr) dr \\ &= \underbrace{r \frac{\partial}{\partial r} \psi_1(r, \phi) H_\nu^{(2)}(kr) \Big|_{r=0}^\infty}_{=0 \text{ by (8) and (11)}} - k \int_0^\infty r \frac{\partial}{\partial r} \psi_1(r, \phi) H_\nu^{(2)'}(kr) dr \\ &= \underbrace{-kr \psi_1(r, \phi) H_\nu^{(2)'}(kr) \Big|_{r=0}^\infty}_{=0 \text{ by (8) and (11)}} + k \int_0^\infty \psi_1(r, \phi) \left[H_\nu^{(2)'}(kr) + kr H_\nu^{(2)''}(kr) \right] dr. \end{aligned} \quad (14)$$

Thus, the operation (12) on (13) gives

$$\int_0^\infty \frac{1}{r} \psi_1(2, \phi) \underbrace{\left[(kr)^2 H_\nu^{(2)''}(kr) + kr H_\nu^{(2)'}(kr) + (kr)^2 H_\nu^{(2)}(kr) \right]}_{= \nu^2 H_\nu^{(2)}(kr) \text{ by Bessel's equation}} + \frac{\partial^2}{\partial \phi^2} \int_0^\infty \frac{1}{r} H_\nu^{(2)}(kr) \psi_1(r, \phi) dr = -H_\nu^{(2)}(kr') \delta(\phi - \phi'), \quad (15)$$

which is the simple differential equation

$$\left(\frac{\partial^2}{\partial \phi^2} + \nu^2 \right) f_1(\nu, \phi) = -\delta(\phi - \phi') \quad (16)$$

in terms of the Kontorovich-Lebedev transform

$$f_1(\nu, \phi) = \frac{1}{H_\nu^{(2)}(kr')} \int_0^\infty \frac{1}{r} H_\nu^{(2)}(kr) \psi_1(r, \phi) dr. \quad (17)$$

By inspection, the source-free equation (7) now transforms to

$$\left(\frac{\partial^2}{\partial \phi^2} + \nu^2 \right) f_2(\nu, \phi) = 0 \quad (18)$$

with

$$f_2(\nu, \phi) = \frac{1}{H_\nu^{(2)}(kr')} \int_0^\infty \frac{1}{r} H_\nu^{(2)}(kr) \psi_2(r, \phi) dr. \quad (19)$$

As in the static analysis of Section II, the source coordinate $\phi = \phi'$ divides the external medium (1) into two source-free subregions, with fields (transforms) distinguished by a $\mp$ superscript:

$$f_2(\nu, \phi) = A(\nu) \sin(\nu\phi) \quad (0 \leq \phi \leq \alpha) \quad (20)$$

$$f_1^-(\nu, \phi) = B(\nu) \sin(\nu\phi) + C(\nu) \cos(\nu\phi) \quad (\alpha < \phi \leq \phi') \quad (21)$$

$$f_1^+(\nu, \phi) = D(\nu) \sin[\nu(\phi - \pi)] \quad (\phi' < \phi \leq \pi). \quad (22)$$

Note that the Dirichlet boundary conditions (8) have been used in writing (20) and (22). In the source neighborhood $\phi = \phi'$, the function f_1 is continuous

$$f_1^+(\nu, \phi') = f_1^-(\nu, \phi'), \quad (23)$$

leaving the derivative operator of (16) responsible for the singularity, which upon integration from $\phi' -$ to $\phi' +$ yields

$$f_1^{+'}(\nu, \phi') - f_1^{-'}(\nu, \phi') = -1. \quad (24)$$

These two conditions plus the transforms of (9) and (10) result in four simultaneous equations

$$\begin{bmatrix} -\sin(\nu\alpha) & \sin(\nu\alpha) & \cos(\nu\alpha) & 0 \\ \frac{\rho_{01}}{\rho_{02}} \cos(\nu\alpha) & -\cos(\nu\alpha) & \sin(\nu\alpha) & 0 \\ 0 & \sin(\nu\phi') & \cos(\nu\phi') & \sin[\nu(\pi - \phi')] \\ 0 & \cos(\nu\phi') & -\sin(\nu\phi') & -\cos[\nu(\pi - \phi')] \end{bmatrix} \begin{bmatrix} A(\nu) \\ B(\nu) \\ C(\nu) \\ D(\nu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\nu \end{bmatrix} \quad (25)$$

for the coefficient functions in (20)-(22).

The above is necessarily identical in form to the matrix equation (19) of Section II-A because of the action of both integral transforms on a single partial differential operator in both media, and the identical boundary conditions. Therefore, a trivial adjustment in the earlier static solution produces the desired transforms

$$f_2(\nu, \phi) = \frac{1 - \Gamma}{\nu} \sin[\nu(\pi - \phi')] \sin(\nu\phi) / \Delta(\nu) \quad (0 \leq \phi \leq \alpha) \quad (26)$$

$$f_1^-(\nu, \phi) = \frac{1}{\nu} \sin[\nu(\pi - \phi')] \{ \sin(\nu\phi) + \Gamma \sin[\nu(\phi - 2\alpha)] \} / \Delta(\nu) \quad (\alpha < \phi \leq \phi') \quad (27)$$

$$f_1^+(\nu, \phi) = \frac{1}{\nu} \sin[\nu(\pi - \phi)] \{ \sin(\nu\phi') + \Gamma \sin[\nu(\phi' - 2\alpha)] \} / \Delta(\nu) \quad (\phi' < \phi \leq \pi) \quad (28)$$

with denominator function

$$\Delta(\nu) = \sin(\nu\pi) + \Gamma \sin[\nu(\pi - 2\alpha)] \quad (29)$$

and density contrast parameter

$$\Gamma = \frac{\rho_{01} - \rho_{02}}{\rho_{01} + \rho_{02}}. \quad (30)$$

The Kontorovich-Lebedev transform pair (1) and (2), and the integral definitions of (17) and (19) express the physical fields in terms of complex integrals, such as

$$\psi_2(r, \phi) = \lim_{\epsilon \rightarrow +0} -\frac{1}{2} \int_{-i\infty}^{i\infty} e^{\epsilon\nu^2} \nu J_\nu(kr) H_\nu^{(2)}(kr') f_2(\nu, \phi) d\nu \quad (0 \leq \phi \leq \alpha) \quad (31)$$

and similarly for $\psi_1^-(r, \phi)$ and $\psi_1^+(r, \phi)$.

B. Even Symmetry - the Hard or Neumann Ground Plane.

The mathematical analysis of the hard ground plane with

$$\frac{\partial}{\partial \phi} \psi_2(r, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial \phi} \psi_1(r, \pi) = 0 \quad (32)$$

parallels the development above for the soft case. The convergence problem (due to the logarithmic *static* behavior) is avoided by working with the ϕ -derivative of the wave

functions, in the same manner as in Section II-E. In this case, it is desirable to start with the derivative of the field

$$\xi(r, \phi) = \frac{\partial}{\partial \phi} \psi(r, \phi) \quad (33)$$

and the governing partial differential equation (in region (1), for example)

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \xi_1(r, \phi) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \xi_1(r, \phi) + k^2 \xi_1(r, \phi) = -\frac{1}{r'} \delta(r - r') \delta'(\phi - \phi') \quad (34)$$

so that the boundary terms that result from the integrations by parts can be safely set to zero, as in (14) above. This procedure is carried through to give final expressions for the transform functions of (33) which must be evaluated by a clever combination of contour integration and possibly some asymptotics to extract the critical physical processes. The static analysis of Section II is good experience and background to continue to make real progress on this boundary value problem. Enough of the details and equations are presented here to enable a solid continuation, and to provide a basis for the discussion of the impedance boundary condition in Section IV.

The integral operator (12) applied to (34) yields

$$\left(\frac{\partial^2}{\partial \phi^2} + \nu^2 \right) g_1(\nu, \phi) = -\delta'(\phi - \phi') \quad (35)$$

in terms of the Kontorovich-Lebedev transform

$$g_1(\nu, \phi) = \frac{1}{H_\nu^{(2)}(k r')} \int_0^\infty \frac{1}{r} H_\nu^{(2)}(k r) \xi_1(r, \phi) dr. \quad (36)$$

The subscript and superscript notation is consistent with the soft case above. Physical boundary conditions on the ϕ -derivative of the fields can be directly applied to the transforms $g(\nu, \phi)$, but boundary conditions on ψ itself involve the anti-derivative of the $g(\nu, \phi)$ functions. That is, the Neumann conditions (32) are

$$g_2(\nu, 0) = 0 \quad \text{and} \quad g_1^+(\nu, \pi) = 0, \quad (37)$$

and continuity of pressure and normal velocity at the media interface become

$$\int_0^\alpha g_1^-(\nu, \phi) d\phi = \int_0^\alpha g_2(\nu, \phi) d\phi \quad (38)$$

and

$$\frac{1}{\rho_{01}} g_1^-(\nu, \alpha) = \frac{1}{\rho_{02}} g_2(\nu, \alpha), \quad (39)$$

respectively. Similarly, the boundary conditions due to the impulsive source at $\phi = \phi'$ are

$$\int_{\phi'}^{\phi} g_1^+(\nu, \phi) d\phi = \int_{\phi'}^{\phi} g_1^-(\nu, \phi) d\phi \quad (40)$$

and

$$g_1^+(\nu, \phi') - g_1^-(\nu, \phi') = -1. \quad (41)$$

Evidently, solutions of the form

$$g_2(\nu, \phi) = A(\nu) \sin(\nu\phi) \quad (0 \leq \phi \leq \alpha) \quad (42)$$

$$g_1^-(\nu, \phi) = B(\nu) \sin(\nu\phi) + C(\nu) \cos(\nu\phi) \quad (\alpha < \phi \leq \phi') \quad (43)$$

$$g_1^+(\nu, \phi) = D(\nu) \sin[\nu(\phi - \pi)] \quad (\phi' < \phi \leq \pi) \quad (44)$$

have coefficient functions that satisfy the linear equations

$$\begin{bmatrix} \frac{\rho_{01}}{\rho_{02}} \sin(\nu\alpha) & -\sin(\nu\alpha) & -\cos(\nu\alpha) & 0 \\ \cos(\nu\alpha) & -\cos(\nu\alpha) & \sin(\nu\alpha) & 0 \\ 0 & \sin(\nu\phi') & \cos(\nu\phi') & \sin[\nu(\pi - \phi')] \\ 0 & \cos(\nu\phi') & -\sin(\nu\phi') & -\cos[\nu(\pi - \phi')] \end{bmatrix} \begin{bmatrix} A(\nu) \\ B(\nu) \\ C(\nu) \\ D(\nu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (45)$$

Simple formulas for these transforms are also available through comparison with the appropriate static analysis, which in this hard case is Section II-E.

IV. IMPEDANCE BOUNDARY CONDITION

A. Homogeneous or Traditional Impedance Boundary.

The Leontovich impedance boundary condition avoids any consideration of the field interior to the wedge region, and therefore applies in the case of a highly lossy or effectively impenetrable wedge. As with all of the work reported here, the original boundary value problem is decomposed into odd (soft) and even (hard) symmetry components. Only the field in the external region $\alpha \leq \phi \leq \pi$ needs to be examined now, so the subscript notation is abandoned, but the superscript $\mp$ is kept to distinguish the fields in the sectors $\phi \leq \phi'$.

The impedance boundary condition in general coordinates

$$\psi(r, \alpha) + \eta \frac{\partial}{\partial n} \psi(r, \alpha) = 0 \quad (1)$$

specifies the ratio of the pressure to the normal velocity at the material boundary $\phi = \alpha$. The numerical value of this surface impedance η derives from the actual constitutive properties of the dissipative surface. For a highly lossy material, η is complex with a phase angle $\pi/4$. Note that the limiting values $\eta = 0$ and $\eta \rightarrow \infty$ are the classical Dirichlet and Neumann conditions, respectively, which can serve as partial validation cases. The mathematical treatment of the even symmetry component or hard ground plane problem (which is consistently the more *difficult* case) is illustrated here.

As before, a line source of unit strength is located at (r', ϕ') , in the space between the impedance boundary at $\phi = \alpha$ and the hard boundary at $\phi = \pi$. The forced Helmholtz equation is

$$(\nabla^2 + k^2) \psi(r, \phi) = -\frac{1}{r'} \delta(r - r') \delta(\phi - \phi') \quad (\alpha \leq \phi \leq \pi) \quad (2)$$

where the two physical boundary conditions in cylindrical coordinates are

$$\psi(r, \alpha) + \frac{\eta}{r} \frac{\partial}{\partial \phi} \psi(r, \alpha) = 0 \quad (3)$$

and

$$\frac{\partial}{\partial \phi} \psi(r, \pi) = 0, \quad (4)$$

plus the usual Sommerfeld radiation condition.

Let

$$\xi(r, \phi) = \frac{\partial}{\partial \phi} \psi(r, \phi) \quad (5)$$

be the ϕ -derivative of the field, and likewise for (2):

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \xi(r, \phi) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \xi(r, \phi) + k^2 \xi(r, \phi) = -\frac{1}{r'} \delta(r - r') \delta'(\phi - \phi'). \quad (6)$$

Duplicating the progression of Section III, application of the operator

$$\int_0^\infty dr r H_0^{(2)}(kr) \quad (7)$$

followed by integration by parts, with the aid of the boundary conditions and Bessel's equation gives

$$\left(\frac{\partial^2}{\partial \phi^2} + \nu^2\right) g(\nu, \phi) = -\delta'(\phi - \phi') \quad (8)$$

with transform

$$g(\nu, \phi) = \frac{1}{H_\nu^{(2)}(k r')} \int_0^\infty \frac{1}{r} H_\nu^{(2)}(k r) \xi(r, \phi) dr. \quad (9)$$

In terms of $\xi(r, \phi)$, the impedance condition (3) is

$$\int_\alpha^\pi \xi^-(r, \phi) d\phi + \frac{\eta}{r} \xi^-(r, \alpha) = 0. \quad (10)$$

The Kontorovich-Lebedev transform of $\xi(r, \phi)$ is denoted alternately by

$$g^-(\nu, \phi) = A(\nu) \sin(\nu \phi) + B(\nu) \cos(\nu \phi) \quad (\alpha \leq \phi \leq \phi') \quad (11)$$

and

$$g^+(\nu, \phi) = C(\nu) \sin[\nu(\phi - \pi)] \quad (\phi' < \phi \leq \pi), \quad (12)$$

which are the solutions of (8) in each source-free sector on either side of the source discontinuity. The Bessel recursion

$$\frac{H_\nu^{(2)}(k r)}{k r} = \frac{H_{\nu-1}^{(2)}(k r) + H_{\nu+1}^{(2)}(k r)}{2\nu} \quad (13)$$

and the anti-derivative of (11) give the transformed version of the impedance boundary condition

$$\begin{aligned} \frac{k\eta}{2} \{ & A(\nu-1) \sin[(\nu-1)\alpha] + B(\nu-1) \cos[(\nu-1)\alpha] \\ & + A(\nu+1) \sin[(\nu+1)\alpha] + B(\nu+1) \cos[(\nu+1)\alpha] \} \\ & - A(\nu) \cos(\nu\alpha) + B(\nu) \sin(\nu\alpha) = 0, \end{aligned} \quad (14)$$

which is a linear difference equation in the coefficient functions $A(\nu)$ and $B(\nu)$. The Dirac delta-function at $\phi = \phi'$ imparts the pair of boundary conditions

$$g^+(\nu, \phi') - g^-(\nu, \phi') = -1 \quad (15)$$

$$\int_{\phi'}^{\pi} g^+(\nu, \phi) d\phi = \int_{\alpha}^{\phi'} g^-(\nu, \phi) d\phi, \quad (16)$$

or equivalently

$$C(\nu) \sin[\nu(\phi' - \pi)] - A(\nu) \sin(\nu\phi') - B(\nu) \cos(\nu\phi') = -1 \quad (17)$$

$$-\frac{C(\nu)}{\nu} \cos[\nu(\phi' - \pi)] = -\frac{A(\nu)}{\nu} \cos(\nu\phi') + \frac{B(\nu)}{\nu} \sin(\nu\phi'). \quad (18)$$

Although apparently this is not the same approach or result of W. E. Williams [7], it is interesting and significant that he too solves a difference equation in modeling the impenetrable wedge.

B. Inhomogeneous or Pseudo-Impedance Boundary.

The difference equation above in the analysis of the true impedance boundary condition results from the $1/r$ factor in (3). In the perhaps improbable case of an inhomogeneous surface impedance that is proportional to radius, i.e. $\eta = r\eta'$, the surface condition is

$$\psi(r, \alpha) + \eta' \frac{\partial}{\partial \phi} \psi(r, \alpha) = 0 \quad (19)$$

and no shifting occurs in the argument ν of the transform coefficient functions. No difference equation need be considered, and the set of linear equations (for fixed ν)

$$\begin{bmatrix} \eta' \nu \sin(\nu \alpha) - \cos(\nu \alpha) & \eta' \nu \cos(\nu \alpha) + \sin(\nu \alpha) & 0 \\ \sin(\nu \phi') & \cos(\nu \phi') & -\sin[\nu(\phi' - \pi)] \\ \cos(\nu \phi') & -\sin(\nu \phi') & -\cos[\nu(\phi' - \pi)] \end{bmatrix} \begin{bmatrix} A(\nu) \\ B(\nu) \\ C(\nu) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (20)$$

is easily solved.

The inverse Kontorovich-Lebedev transform can be approached from an analytical advantage, and an inviting opportunity emerges for contour deformation and asymptotics. Of course the real impedance boundary will ultimately yield to these tactics, but in future continuation of this scattering research. For the present, consider these far-field ($kr \rightarrow \infty$) results

$$\psi^-(r, \phi) = \sqrt{\frac{i}{2\pi kr}} e^{-ikr} \int_{-i\infty}^{i\infty} i^\nu J_\nu(kr') \frac{\{\eta' \nu \cos[\nu(\phi - \alpha)] - \sin[\nu(\phi - \alpha)]\} \cos[\nu(\phi' - \pi)]}{\eta' \nu \sin[\nu(\pi - \alpha)] + \cos[\nu(\pi - \alpha)]} d\nu \quad (\alpha \leq \phi \leq \phi') \quad (21)$$

$$\psi^+(r, \phi) = \sqrt{\frac{i}{2\pi kr}} e^{-ikr} \int_{-i\infty}^{i\infty} i^\nu J_\nu(kr') \frac{\{\eta' \nu \cos[\nu(\phi' - \alpha)] - \sin[\nu(\phi' - \alpha)]\} \cos[\nu(\phi - \pi)]}{\eta' \nu \sin[\nu(\pi - \alpha)] + \cos[\nu(\pi - \alpha)]} d\nu \quad (\phi' \leq \phi \leq \pi) \quad (22)$$

with poles from the roots of the transcendental equation

$$\eta' \nu = -1 / \tan[\nu(\pi - \alpha)]. \quad (23)$$

V. PROGRESS SUMMARY AND RECOMMENDATIONS FOR CONTINUING RESEARCH

1. The complete, analytical solution for the static excitation (Poisson's equation) of the dielectric wedge is accomplished using the Mellin transform. This benchmark solution is believed to be a new result, and is a valuable tool in the study of static edge condition behavior at the apex of the material wedge.
2. The Kontorovich-Lebedev transform has been successfully applied to the density contrast wedge, resulting in *closed-form* expressions for the integral transform representation of the total fields in both the interior and exterior wedge regions. An asymptotic evaluation of these transforms at both the far field limit ($kr \rightarrow \infty$) and in the immediate vicinity of the apex ($kr \rightarrow 0$) is of great interest to both theoretical and numerical scattering researchers. Furthermore, this frequency domain result is a companion to Chu's [45] transient derivation.
3. The difference equation present in the transform variable of the Kontorovich-Lebedev integral needs to be addressed in order to rigorously solve for the *physical* case of the impedance boundary condition

$$\psi + \frac{\eta}{r} \frac{\partial \psi}{\partial \phi} = 0.$$

Such a rigorous solution provides for high frequency ray-launching coefficients, in the manner of the GTD [10] or Wedge Assemblage [41]. Any relationship between these promising results and the difference equation of Williams [7] and the Maliuzhinets function [6],[8-12] must be affirmed and explained.

4. In the case of an inhomogeneous surface impedance that is proportional to radius, i.e. $\eta = r\eta'$, the above Leontovich boundary condition is

$$\psi + \eta' \frac{\partial \psi}{\partial \phi} = 0,$$

resulting in *closed-form* integral transforms. Inversion to spatial coordinates (r, ϕ) will closely follow the completed Mellin inversion for the static wedge. If this inhomogeneous problem is deemed to be of physical interest, then this mathematical solution is of great value, offering great physical insight into the asymptotic interactions between the individual scattering mechanisms.

5. The coupled integral equations for equivalent surface distributions that were derived during the initial stages of the present research could form the basis for an alternate method of attack. However, in view of the success to date with the Kontorovich-Lebedev transform, surface integral equations will not likely be pursued for this proposed research.
6. Continuing consultation and interaction with active researchers engaged in related electromagnetics and acoustics work should be maintained, via paper submission, paper reviewing, telephone and written correspondence, the 1993 URSI National Radio Science Meeting, and the University library services.

CONCLUSIONS

The exact, modal series solution of Laplace's equation for the static field inside and outside of the dielectric wedge is obtained via the Mellin transform, and is readily evaluated. Similarly, the Kontorovich-Lebedev transform is effectively applied to the problem of time-harmonic acoustic interaction with the density contrast wedge. A high-frequency asymptotic analysis of this transform solution is a promising vehicle with which to understand and extract the basic physical mechanisms in this canonical scattering geometry. The integral transform solution of the wedge having impedance boundaries deserves substantially more mathematical analysis.

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