PARAMETER ESTIMATION FOR ARMA MODELS WITH INFINITE VARIANCE INNOVATIONS

by

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Abstract

We consider a standard ARMA process of the form $\Phi(B)X_t = \Theta(B)Z_t$, where the innovations Z_t belong to the domain of attraction of a stable law, so that neither the Z_t nor the X_t have a finite variance. Our aim is to estimate the coefficients of Φ and Θ. Since maximum likelihood estimation is not a viable possibility (due to the unknown form of the marginal density of the innovation sequence) we adopt the so-called "Whittle estimator", based on the sample periodogram of the X sequence. Despite the fact that the periodogram does not, *a priori*, seem like a logical object to study in this non-$\mathcal{L}^2$ situation, we show that our estimators are consistent, obtain their asymptotic distributions, and show that they converge to the true values faster than in the usual $\mathcal{L}^2$ case.

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1. Introduction

This paper considers two closely related, but distinct, subjects. We commence with the discrete moving average process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $(Z_t)_{t\in \mathbb{Z}}$ is a noise sequence of iid random variables (r.v.'s) having not necessarily a finite variance. Two preceding papers by Klüppelberg and Mikosch (1991), (1992), studied the asymptotic behaviour of periodogram-type estimators for the process $(X_t)_{t\in \mathbb{Z}}$ under the condition that Z_1 is in the domain of normal attraction of an α-stable law for some $\alpha \in (0,2]$. In Klüppelberg and Mikosch (1992) it was shown that the normalized periodogram

$$\widetilde{I}_{n,X}(\lambda) = \left|\sum_{t=1}^{n} X_t e^{-i\lambda t}\right|^2 \Big/ \sum_{t=1}^{n} X_t^2, \quad -\pi < \lambda \leq \pi,$$

converges in distribution to

$$\frac{|\psi(\lambda)|^2}{\psi^2} \, \frac{\alpha^2(\lambda)+\beta^2(\lambda)}{\gamma^2}$$

where $\psi(\lambda) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j}$ is the transfer function, $\psi^2 = \sum_{j=-\infty}^{\infty} \psi_j^2$, and the vector $(\alpha(\lambda), \beta(\lambda), \gamma^2)$ has a mixed stable distribution such that $(\alpha(\lambda), \beta(\lambda))$ are jointly α-stable and γ^2 is positive $\alpha/2$-stable. Furthermore, the vector of different periodogram ordinates $\left(\widetilde{I}_{n,X}(\lambda_i)\right)_{i=1,\ldots,m}$ converges weakly, and the components of the limit vector have exponentially fast decreasing tails and are uncorrelated. Smoothed versions of the normalized periodogram were also studied, and their weak convergence to the normalized power transfer function $|\psi(\lambda)|^2/\psi^2$ established.

In this paper we weaken the above assumptions on Z_1 considerably: we only require that $E|Z_1|^d < \infty$ for some $d > 0$ and that $\left(\sum_{t=1}^{n} Z_t^2\right)_{n\geq 1}$ satisfies some tightness condition. Under such general conditions one cannot expect to derive distributional limits for $\widetilde{I}_{n,X}(\lambda)$; but we prove weak convergence for the smoothed normalized periodogram to $|\psi(\lambda)|^2/\psi^2$ for a large class of smoothing filters (Section 3). In Section 4 we obtain under the same mild conditions on the noise variables weak convergence of the sample autocorrelations $\sum_{t=1}^{n} X_{t+h} X_t \Big/ \sum_{t=1}^{n} X_t^2$ to $\sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j \Big/ \psi^2$ for $h \in \mathcal{N}$.

In Section 5 we turn to the second and main subject of this paper: parameter estimation for an ARMA(p, q)-process of the form

$$X_t - \varphi_1 X_{t-1} - \ldots - \varphi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$$

for fixed, known values of p and q. Adapting an idea of Whittle (1953) we construct a weakly consistent estimator β_n of the parameter vector $\beta = (\varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q)^T$. Moreover, if Z_1 belongs to the domain of normal attraction of an α-stable law, $\alpha \in (0, 2)$, the rate of convergence is $(n/\log n)^{1/\alpha}$. The proofs of the results of Sections 3-5 are given in Sections 6-9. In the concluding Section 10 we discuss how to use our estimator in practice, and give the results of a small simulation study which indicate that the estimator seems to perform, in practice, as good as the well known MLE estimators in the corresponding model with Gaussian innovations.

2. Assumptions and notation

We consider the moving average process $(X_t)_{t \in Z}$ defined by (1.1). To formulate the conditions on the noise $(Z_t)_{t \in Z}$ we introduce the following functions for $x > 0$

$$\begin{aligned} G(x) &= P\left(Z_1^2 > x\right) \\ K(x) &= x^{-2} E Z_1^4 I\left(Z_1^2 \le x\right) \\ Q(x) &= G(x) + K(x) = E\left[1 \wedge \left(x^{-1} Z_1^2\right)^2\right] . \end{aligned}$$

Since Q is strictly decreasing and continuous on $(0, \infty)$ the identity

$$Q\left(a_n^2\right) = \frac{1}{n}, \quad n \in \mathcal{N}, \tag{2.1}$$

defines a sequence of positive numbers a_n such that $a_n \uparrow \infty$ as $n \to \infty$. Furthermore, define

$$\gamma_{n,Z}^2 = a_n^{-2} \sum_{t=1}^{n} Z_t^2, \quad n \in \mathcal{N}. \tag{2.2}$$

For the moving average process as in (1.1) we introduce the following assumptions: There exists some $d > 0$ such that

$$\text{(A1)} \quad E\,|Z_1|^d < \infty\,;$$

$$\text{(A2)} \quad \sum_{j=-\infty}^{\infty} |j|\;|\psi_j|^\delta < \infty \qquad \text{for} \quad \delta = 1 \wedge d\,;$$

$$\text{(A3)} \quad n/a_n^{2\delta} \to 0\,, \quad n \to \infty\,, \qquad \text{for} \quad \delta = 1 \wedge d\,;$$

$$\text{(A4)} \quad \lim_{x \to 0} \limsup_{n \to \infty} P\left(\gamma_{n,Z}^2 \le x\right) = 0\,.$$

Remarks. 1) (A1) and (A2) imply absolute a.s. convergence of the series (1.1) for every $t \in \mathcal{Z}$. This is a consequence of the three-series theorem.
2) (A2) is obviously satisfied for every ARMA(p,q)-process. In this case the ψ_j decrease exponentially.
3) The conditions $E\,Z_1^2 < \infty$, (A3) and (A4) cannot hold together, since (A3) and the SLLN imply that $\gamma_{n,Z}^2 \xrightarrow{\text{a.s.}} 0$ contradicting (A4).
4) (A4) is a stochastic compactness condition on $\gamma_{n,Z}^2$. A necessary and sufficient condition for $\gamma_{n,Z}^2$ to be stochastically compact is

$$\liminf_{x \to \infty} K(x)/G(x) > 0\,.$$

[e.g. Maller (1981)]. Furthermore, if $\gamma_{n,Z}^2$ is stochastically compact, then there exists some constant $c > 0$ such that for all $n \in \mathcal{N}$

$$P\left(\gamma_{n,Z}^2 \le x\right) \le c\,x\,, \quad x \ge 0\,,$$

[Griffin (1983)] which implies (A4).

A natural class of noise variables to satisfy conditions (A1), (A3) and (A4) is the domain of attraction of an α-stable random variable, which we denote by $DA(\alpha)$. For the definition and properties of α-stable r.v.'s, their domain of attraction and regularly and slowly varying functions see e.g. Feller (1971) or Bingham, Goldie and Teugels (1987).

Now if $Z_1 \in DA(\alpha)$ for some $\alpha \in (0,2)$, then $Z_1^2 \in DA(\alpha/2)$ and

$$\lim_{x \to \infty} G(x)/K(x) = (4-\alpha)/\alpha\,.$$

Then G is a regularly varying function and the norming constants in (2.2) can be chosen as

$$a_n^2 = G^{\leftarrow}\left(n^{-1}\right) = \inf\left\{x; G(x) < n^{-1}\right\}\,;$$

i.e. $G^{\leftarrow}$ is the generalized inverse of G. This implies that $a_n^2 = n^{2/\alpha}L(n)$ where L is a slowly varying function and $\gamma_{n,Z}^2 \xrightarrow{d} \gamma^2$ for some positive $\alpha/2$-stable r.v. γ^2. Furthermore, $E|Z_1|^d < \infty$ for $d < \alpha$. In the following lemma we summarize these relations.

Lemma 2.1. *Suppose $Z_1 \in DA(\alpha)$ for some $\alpha \in (0,2)$, then (A1), (A3) and (A4) hold for some $d > 0$ and $a_n^2 = n^{2/\alpha}L(n)$ where L is a slowly varying function.* □

The following notation will be used throughout the paper: For any sequence of r.v.'s $(A_t)_{t\in Z}$ and a sequence of positive constants $(a_n)_{n\in\mathcal{N}}$ we introduce

$$
\begin{aligned}
\gamma_{n,A}^2 &= a_n^{-2}\sum_{t=1}^{n} A_t^2\,,\\
I_{n,A}(\lambda) &= a_n^{-2}\left|\sum_{t=1}^{n} A_t\, e^{-i\lambda t}\right|^2, \quad \lambda \in (-\pi,\pi]\,,\\
\widetilde{A}_t &= a_n^{-1}\; A_t/\gamma_{n,A} = A_t \Bigg/ \left(\sum_{s=1}^{n} A_s^2\right)^{1/2},\\
\widetilde{I}_{n,A}(\lambda) &= I_{n,A}(\lambda)/\gamma_{n,A}^2 = \left|\sum_{t=1}^{n} \widetilde{A}_t\, e^{-i\lambda t}\right|^2, \quad \lambda \in (-\pi,\pi]\,.
\end{aligned}
$$

3. Consistency of the smoothed normalized periodogram

Klüppelberg and Mikosch (1991) considered Z_1 in the domain of normal attraction of an α-stable r.v. $(Z_1 \in DNA(\alpha))$, $\alpha \in (0,2)$; this means that $Z_1 \in DA(\alpha)$ with norming constants $a_n^2 = n^{2/\alpha}$. In that case both the periodogram $I_{n,X}(\lambda)$ and the normalized periodogram $\widetilde{I}_{n,X}(\lambda)$ converge in distribution for every $\lambda \in (-\pi,\pi]$. A common technique to obtain consistency is to apply some smoothing operation, and for $\widetilde{I}_{n,X}(\lambda)$ this provides a consistent estimator for the normalized power transfer function $|\psi(\lambda)|^2/\psi^2$.

Now we shall show consistency of the smoothed normalized periodogram

$$\widetilde{T}_{n,X}(\lambda) = \sum_{|k|\le m} W_n(k)\, \widetilde{I}_{n,X}(\lambda_k)$$

under less restrictive conditions. Here $W_n(k)$ are nonnegative weights at points $\lambda_k = \lambda + k/n$, $|k| \le m$, $n \in \mathcal{N}$, satisfying

$$m = m_n \to \infty, \quad m_n/n \to 0, \quad n \to \infty, \tag{3.1a}$$

$$W_n(k) = W_n(-k), \quad |k| \le m, \tag{3.1b}$$

$$\sum_{|k|\le m} W_n(k) = 1, \tag{3.1c}$$

$$\sum_{|k|\le m} W_n^2(k) = o(1), \quad n \to \infty. \tag{3.1d}$$

If $\lambda_k = \lambda + k/n \notin (-\pi, \pi]$ the term $\widetilde{I}_{n,X}(\lambda_k)$ in $\widetilde{T}_{n,X}(\lambda)$ will be evaluated by defining $\widetilde{I}_{n,X}$ to have period 2π. The same convention will be used to define $\psi(\lambda)$, $\lambda \notin (-\pi, \pi]$.

Theorem 3.1. *Suppose $(X_t)_{t\in Z}$ satisfies (A1)-(A4) and (3.1) holds. Then*

$$\widetilde{T}_{n,X}(\lambda) \xrightarrow{P} |\psi(\lambda)|^2/\psi^2, \quad n \to \infty. \quad \square$$

4. Consistency of the sample autocorrelation function

For $h \in Z$ define

$$\begin{aligned} \widetilde{\gamma}_{n,X}(h) &= \gamma_{n,X}(h)/\gamma_{n,X}^2 \\ \widetilde{\gamma}(h) &= \gamma(h)/\psi^2 \end{aligned}$$

where

$$\gamma_{n,X}(h) = a_n^{-2} \sum_{t=1}^{n-|h|} X_t X_{t+|h|}$$

$$\gamma(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|h|}\,.$$

Obviously, if $E\,Z_1^2 < \infty$, $\tilde{\gamma}_{n,X}(h)$ is a consistent estimator of the autocorrelation function $\tilde{\gamma}(h)$ of $(X_t)_{t\in Z}$. One of the results of Davis and Resnick (1986) is the following: For $Z_1 \in DNA(\alpha)$, $\alpha \in (0,2)$, Z_1 symmetric

$$\Big((n/\log n)^{1/\alpha}\,(\tilde{\gamma}_{n,X}(h) - \tilde{\gamma}(h))\Big)_{h=1,\dots,m} \xrightarrow{d} \left(\sum_{j=1}^{\infty}(\tilde{\gamma}(j+h) - \tilde{\gamma}(j-h) - 2\tilde{\gamma}(j)\,\tilde{\gamma}(h))\,\frac{Y_j}{Y_0}\right)_{h=1,\dots,m}, \tag{4.1}$$

where $Y_0, Y_1, Y_2, \dots$ are independent r.v.'s, Y_0 is positive $\alpha/2$-stable and $(Y_j)_{j\in\mathcal{N}}$ are iid standard α-stable. (4.1) implies that $\tilde{\gamma}_{n,X}(h)$ is weakly consistent with limit $\tilde{\gamma}(h)$ and the rate of convergence is faster than in the finite variance case. Under our more general conditions (A1)-(A4) a precise result as (4.1) cannot be expected but we prove weak consistency.

Proposition 4.1. *Suppose $(X_t)_{t\in Z}$ satisfies (A1)-(A4), then*

$$\tilde{\gamma}_{n,X}(h) \xrightarrow{P} \tilde{\gamma}(h), \quad h \in \mathcal{N}, \quad n \to \infty. \quad \square$$

As shown in the appendix by replacing conditions (A3) and (A4) by a slightly more restrictive condition it is possible to obtain a.s. convergence of $\tilde{\gamma}_{n,X}(h)$ to $\tilde{\gamma}(h)$ along some known subsequence. In particular, this condition is satisfied for $Z_1 \in DA(\alpha)$, $\alpha \in (0,2)$. Moreover, we give an example to show that a.s. convergence need not hold in general under (A1)-(A4).

5. Parameter estimation for ARMA(p,q) processes

We consider a causal invertible ARMA(p,q) process $(X_t)_{t\in Z}$ satisfying for every t the ARMA equations

$$X_t - \varphi_1 X_{t-1} - \dots - \varphi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$$

for iid $(Z_t)_{t\in Z}$. Denote

$$\begin{aligned}\varphi(z) &= 1 - \varphi_1 z - \dots - \varphi_p z^p\\ \theta(z) &= 1 + \theta_1 z + \dots + \theta_q z^q\end{aligned}$$

and

$$\beta = (\varphi_1, \ldots, \varphi_p\,,\, \theta_1, \ldots, \theta_q)^T\,.$$

Then in the infinite moving average representaion of our process we have that $\psi(\lambda) \equiv \varphi(e^{-i\lambda})/\theta(e^{-i\lambda})$.

We introduce the parameter set

$$\begin{aligned} C \;&=\; \Big\{\beta \in \mathcal{R}^{p+q}\,; \varphi_p \neq 0\,, \theta_q \neq 0\,, \varphi(z) \text{ and } \theta(z) \text{ have no common zeros}\,, \\ &\qquad \varphi(z)\,\theta(z) \neq 0 \quad \text{for } |z| \leq 1\Big\}\,. \end{aligned}$$

Denote by $g(\lambda, \beta)$ the power transfer function corresponding to $\beta \in C$; i.e.

$$g(\lambda, \beta) = \left|\frac{\theta\,(e^{-i\lambda})}{\varphi\,(e^{-i\lambda})}\right|^2 = |\psi\,(\lambda)|^2\,,$$

and define

$$\sigma_n^2(\beta) = \int_{-\pi}^{\pi} \frac{\tilde{I}_{n,X}(\lambda)}{g(\lambda, \beta)}\,d\lambda, \qquad \bar{\sigma}_n^2(\beta) = \frac{2\pi}{n}\sum_j \frac{\tilde{I}_{n,X}(\lambda_j)}{g(\lambda_j; \beta)},$$

where the sum is taken over all Fourier frequencies

$$\lambda_j = \frac{2\pi j}{n} \in (-\pi, \pi].$$

Clearly, as $n \to \infty$, the sum and the integral should converge to the same limit.

Suppose $\beta_0 \in C$ is the true, but unknown parameter vector. Then two natural estimators of β_0 are given by

$$\beta_n = \operatorname*{argmin}_{\beta \in C}\; \sigma_n^2(\beta), \qquad \bar{\beta}_n = \operatorname*{argmin}_{\beta \in C}\; \bar{\sigma}_n^2(\beta)\,.$$

Given the assumption that $\sigma_n^2(\beta) \sim \bar{\sigma}_n^2(\beta)$, it seems reasonable to assume, as is in fact the case, that $\beta_n \sim \bar{\beta}_n$, and that therefore the two estimators are asymptotically equivalent. It is clear that, in practice, $\bar{\beta}_n$ is the only applicable estimator, since the integral defining $\sigma_n^2(\beta)$ will always have to be evaluated by an approximating sum. Nevertheless, throughout this paper we shall give proofs of convergence for the estimator based on $\sigma_n^2(\beta)$, since here the notation is much lighter.

The choice of these estimators is motivated by the fact that the function

$$\int_{-\pi}^{\pi} \frac{g\,(\lambda, \beta_0)}{g(\lambda, \beta)}\,d\lambda$$

has its absolute minimum at $\beta = \beta_0$ in C. [cf. Brockwell and Davis (1991), Proposition 10.8.1.]. Moreover, by Theorem 3.1., $\widetilde{I}_{n,X}(\lambda)$ can be applied to estimate $g(\lambda, \beta_0)/\psi^2(\beta_0)$, where $\psi^2(\beta_0)$ is the quantity ψ^2 corresponding to β_0.

For Gaussian $(X_t)_{t\in Z}$ the estimator β_n is closely related to least squares and maximum likelihood estimators and it is a standard estimator for ARMA processes with finite variance. The idea goes back to Whittle (1953), see also Dzhaparidze (1986), Fox and Taqqu (1986) and Dahlhaus (1989). It is well-known that in the classical case β_n is consistent and asymptotically normal [cf. Brockwell and Davis (1991)]. We show that β_n is also for ARMA processes with infinite variance a weakly consistent estimator for the true parameter vector β_0.

Theorem 5.1. *Suppose $(X_t)_{t\in Z}$ is a causal invertible ARMA(p,q) process and conditions (A1)-(A4) hold. Then*

$$\beta_n \xrightarrow{P} \beta_0 \quad \text{and} \quad \sigma_n^2(\beta_n) \xrightarrow{P} 2\pi\psi^{-2}(\beta_0)\,, \quad n \to \infty\,.$$

Furthermore, the same limit relationships hold also for $\bar{\beta}_n$ and $\bar{\sigma}_n^2$. □

As shown in the Appendix, it is possible to obtain a.s. convergence along some specified subsequence under more restrictive conditions which hold e.g. for $Z_1 \in DA(\alpha)$, $\alpha \in (0,2)$.

For ARMA(p,q) processes with finite variance β_n is asymptotically normal with rate of convergence of order $n^{-1/2}$. An analogous result gives in our case a rate of convergence of order $(n/\log n)^{-1/\alpha}$:i.e. the convergence is considerably faster (since $\alpha < 2$).

To obtain a representation of the limit vector we restrict ourselves to symmetric $Z_1 \in DNA(\alpha)$ for $\alpha \in (0,2)$ such that

$$n^{-1/\alpha}\sum_{t=1}^{n} Z_t \quad \xrightarrow{d} \quad Y \tag{5.1}$$

where Y is α-stable.

Recall that a random variable Y is said to have a stable distribution ($Y \overset{d}{=} S_\alpha(\sigma,\beta,\mu)$) if there are parameters $0 < \alpha \le 2$, $\sigma \ge 0$, $-1 \le \beta \le 1$, and μ real such that its characteristic function has the form:

$$E\left(e^{i\theta Y}\right) = \begin{cases} \exp\left\{-\sigma^\alpha|\theta|^\alpha\left(1 - i\beta(\operatorname{sign}\theta)\tan\frac{\pi\alpha}{2}\right) + i\mu\theta\right\} & \text{if } \alpha \ne 1, \\ \exp\left\{-\sigma|\theta|\left(1 + \frac{2i\beta}{\pi}(\operatorname{sign}\theta)\ln|\theta|\right) + i\mu\theta\right\} & \text{if } \alpha = 1. \end{cases}$$

If $\beta = \mu = 0$ then Y is symmetric and we say that Y has a "symmetric α stable" distribution, denoted by $Y \stackrel{d}{=} S\alpha S$.

For later use, let C_α be the constant defined by

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} & \text{if } \alpha = 1. \end{cases} \tag{5.2}$$

Theorem 5.2. *Suppose $(X_t)_{t \in Z}$ is an ARMA(p,q) process and $(Z_t)_{t \in Z}$ are iid symmetric such that (5.1) holds. Then*

$$\left(\frac{n}{\log n}\right)^{1/\alpha} (\beta_n - \beta_0) \quad \xrightarrow{d} \quad 4\pi\, W^{-1}(\beta_0)\, \frac{1}{Y_0} \sum_{k=1}^{\infty} Y_k\, b_k\,, \tag{5.3}$$

where $Y_0, Y_1, Y_2, \ldots$ are independent r.v.'s, $Y_0 \stackrel{d}{=} S_{\alpha/2}(C_{\alpha/2}^{-2/\alpha}, 1, 0)$ is positive $\alpha/2$-stable, $(Y_t)_{t \in \mathcal{N}}$, are iid $S\alpha S$ with scale parameter $\sigma = C_\alpha^{1/\alpha}$, $W^{-1}(\beta_0)$ is the inverse of the matrix

$$W(\beta_0) = \int_{-\pi}^{\pi} \left[\frac{\partial \ln g(\lambda, \beta_0)}{\partial \beta}\right] \left[\frac{\partial \ln g(\lambda, \beta_0)}{\partial \beta}\right]^T d\lambda\,,$$

and, for $k \in \mathcal{N}$, b_k is the vector

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\lambda}\, g(\lambda, \beta_0)\, \frac{\partial g^{-1}(\lambda, \beta_0)}{\partial \beta}\, d\lambda\,.$$

Furthermore, (5.3) holds also with β_n replaced by $\bar{\beta}_n$ □

The limit vector in (5.3) is the ratio of an α-stable $(p+q)$-dimensional vector over a positive $\alpha/2$-stable r.v. It is not difficult to see that for $AR(p)$ processes β_n is just the formal analogue of the Yule-Walker estimates. Their weak limit behaviour was derived by Davis and Resnick (1986) using time domain methods.

In closing we note that "more rapid than Gaussian" rates of convergence for estimators in heavy tailed problems seems to be the norm rather than the exception. For example, Feigin and Resnick (1992, 1993) study parameter estimation for autoregressive processes with *positive*, heavy tailed innovations, and obtain rates of convergence for their estimator of the same order as ours, but without the logarithmic term. Their estimators, however, are different to ours both in spirit and detail, and involve the numerical solution of a non-trivial linear programming problem.

6. Auxiliary results

We shall frequently make use of the following decomposition of the periodogram. Its proof is given in Proposition 2.1 of Klüppelberg and Mikosch (1991).

Proposition 6.1. *Suppose $(X_t)_{t\in Z}$ is a moving average process as in (1.1) and (A.1), (A.2) are satisfied. Then*

$$I_{n,X}(\lambda) = |\psi(\lambda)|^2\, I_{n,Z}(\lambda) + R_n(\lambda)\,, \quad -\pi < \lambda \le \pi$$

where

$$\begin{aligned}
R_n(\lambda) &= \psi(\lambda)\, J_n(\lambda)\, Y_n(-\lambda) + \psi(-\lambda)\, J_n(-\lambda)\, Y_n(\lambda) + |Y_n(\lambda)|^2 \\
J_n(\lambda) &= a_n^{-1} \sum_{t=1}^{n} Z_t\, e^{-i\lambda t} \\
Y_n(\lambda) &= a_n^{-1} \sum_{j=-\infty}^{\infty} \psi_j\, e^{-i\lambda j}\, U_{nj}(\lambda) \\
U_{nj}(\lambda) &= \sum_{t=1-j}^{n-j} Z_t\, e^{-i\lambda t} - \sum_{t=1}^{n} Z_t\, e^{-i\lambda t}\,. \qquad \square
\end{aligned}$$

The following Lemma is similar to Davis and Resnick (1986), p. 549, see also Lemma 5.1 of Klüppelberg and Mikosch (1992).

Lemma 6.2. *Suppose $(X_t)_{t\in Z}$ satisfies (A1)-(A4), then*

$$\gamma_{n,X}^2 = \psi^2\, \gamma_{n,Z}^2 (1 + o_P(1))\,, \quad n \to \infty\,.$$

Proof.

$$\begin{aligned}
\gamma_{n,X}^2 &= a_n^{-2} \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} \psi_j^2\, Z_{t-j}^2 + a_n^{-2} \sum_{t=1}^{n} \sum_{i\neq j} \psi_i\, \psi_j\, Z_{t-i}\, Z_{t-j} \\
&=: V_1 + V_2\,.
\end{aligned}$$

Then the triangle inequality gives

$$E\,|V_2|^\delta \le a_n^{-2\delta}\, n \sum_{i\neq j} (|\psi_i|\,|\psi_j|)^\delta\; E\,|Z_1 Z_2|^\delta \quad \to \quad 0$$

and

$$E\left|V_1 - r^2\,\gamma_{n,Z}^2\right|^{\delta/2} \;\le\; E\left|a_n^{-2}\sum_{j=-\infty}^{\infty}\psi_j^2\sum_{t=1}^{n}\left(Z_{t-j}^2 - Z_t^2\right)\right|^{\delta/2}$$

$$\le\; 2a_n^{-\delta}\sum_{j=-\infty}^{\infty}|\psi_j|^{\delta}\;|j|\;E\,|Z_1|^{\delta} \to 0\,. \qquad \square$$

Lemma 6.3. *Let $(Z_t)_{t\in Z}$ be a sequence of iid r.v.'s. Then the following relations hold for $n \to \infty$:*

(a)

$$E\,\widetilde{Z}_1^2\,\widetilde{Z}_2^2 = O\left(n^{-2}\right)\,. \tag{6.1a}$$

(b) If (A1), (A3) and (A4) hold, then

$$E\,\widetilde{Z}_1\,\widetilde{Z}_2 = o\left(n^{-1}\right)\,, \tag{6.1b}$$

$$E\,\widetilde{Z}_1^2\,\widetilde{Z}_2\,\widetilde{Z}_3 = o\left(n^{-2}\right)\,, \tag{6.1c}$$

$$E\,\widetilde{Z}_1\,\widetilde{Z}_2\,\widetilde{Z}_3\,\widetilde{Z}_4 = o\left(n^{-2}\right)\,. \tag{6.1d}$$

(c) If (A1) and (A3) hold, then

$$a_n^{-2}\sum_{t=1}^{n-h} Z_t\,Z_{t+h} \;\xrightarrow{P}\; 0\,, \quad h \in \mathcal{N}_0\,.$$

Proof.

(a)

$$n(n-1)E\,\widetilde{Z}_1^2\,\widetilde{Z}_2^2 = E\sum_{\substack{t,s=1\\ t\neq s}}^{n}\widetilde{Z}_t^2\,\widetilde{Z}_s^2 \le E\left(\sum_{t=1}^{n}\widetilde{Z}_t^2\right)^2 = 1\,.$$

(c)

$$E\left|a_n^{-2}\sum_{t=1}^{n-h} Z_t\,Z_{t+h}\right|^{\delta} \le a_n^{-2\delta}(n-h)\,E\,|Z_1\,Z_2|^{\delta} \xrightarrow{P} 0\,.$$

(b) By Hölder's inequality, $\left|\sum_{t=1}^{n-1} \widetilde{Z}_t \widetilde{Z}_{t+1}\right|$ $le1$, $n \in \mathcal{N}$, thus the sequence $\left(\sum_{t=1}^{n-1} \widetilde{Z}_t \widetilde{Z}_{t+1}\right)_{n\in\mathcal{N}}$ is uniformly integrable.

Part (c) and (A4) imply that

$$\sum_{t=1}^{n-1} \widetilde{Z}_t \widetilde{Z}_{t+1} = \gamma_{n,Z}^{-2} a_n^{-2} \sum_{t=1}^{n-1} Z_t Z_{t+1} \xrightarrow{P} 0 .$$

Hence

$$E \sum_{t=1}^{n-1} \widetilde{Z}_t \widetilde{Z}_{t+1} = (n-1) E \widetilde{Z}_1 \widetilde{Z}_2 \to 0 .$$

This proves (6.1b). The proofs of (6.1c) and (6.1d) are similar; we only prove (6.1d):

$$([n/2]-1)(n-[n/2]-1) \left|E \widetilde{Z}_1 \widetilde{Z}_2 \widetilde{Z}_3 \widetilde{Z}_4\right|$$

$$= \left| E\left[\left\{\sum_{t=1}^{[n/2]-1} Z_t Z_{t+1} \sum_{s=[n/2]+1}^{n-1} Z_s Z_{s+1}\right\} \Big/ \left\{\sum_{t=1}^{[n/2]} Z_t^2 + \sum_{t=[n/2]+1}^{n} Z_t^2\right\}^2\right]\right|$$

$$\leq E\left|\frac{\sum_{t=1}^{[n/2]-1} Z_t Z_{t+1}}{\sum_{t=1}^{[n/2]} Z_t^2}\right| E\left|\frac{\sum_{t=[n/2]+1}^{n-1} Z_t Z_{t+1}}{\sum_{t=[n/2]+1}^{n} Z_t^2}\right| \xrightarrow{P} 0$$

by (6.1c). □

7. Proof of Theorem 3.1.

As a structural part of $\widetilde{T}_{n,X}(\lambda)$ we define

$$c_{ts} := \sum_{|k|\leq m} W_n(k) \cos \lambda_k (t-s)$$

and prove some asymptotic relations for $n \to \infty$.

Lemma 7.1.

(a)

$$\sum_{t,s=1}^{n} c_{ts} = O(1)\,,$$

(b)

$$\sum_{t,s=1}^{n} c_{ts}^2 = o\left(n^2\right) = \sum_{\substack{t,s=1\\ t\neq s}}^{n} c_{ts}^2\,,$$

(c)

$$\sum_{t,s,r=1}^{n} c_{ts}\,c_{tr} = O(n)\,,\quad \sum_{\substack{t,s,r=1\\ t\neq s\neq r}}^{n} c_{ts}\,c_{tr} = o\left(n^2\right)\,,\quad \sum_{\substack{t,s,r,v=1\\ t\neq s\neq r\neq v}}^{n} c_{ts}\,c_{rv} = O\left(n^2\right)\,.$$

Proof. (a)

$$\begin{aligned}\sum_{t,s} c_{ts} &= \sum_{|k|\leq m} W_n(k) \sum_{t,s} \cos\lambda_k(t-s)\\ &= \sum_{|k|\leq m} W_n(k) \left\{\left(\sum_t \cos\lambda_k t\right)^2 + \left(\sum_k \sin\lambda_k t\right)^2\right\} = O(1)\,.\end{aligned}$$

(b) $\sum_{t,s} c_{ts}^2 = \sum_{t\neq s} c_{ts}^2 + n$, and the result follows from equation (6.2) of Klüppelberg and Mikosch (1992).

(c) Using trigonometric sum formulas we obtain

$$\begin{aligned}\left|\sum_{t,s,r} c_{ts}\,c_{tr}\right| &\leq \sum_{|k_1|,|k_2|\leq m} W_n(k_1)\; W_n(k_2) \left|\sum_{t,s,r} \cos\lambda_{k_1}(t-s)\cos\lambda_{k_2}(t-r)\right|\\ &\leq \sum_{|k_1|,|k_2|\leq m} W_n(k_1)\; W_n(k_2) \left\{\left|\sum_t \cos\lambda_{k_1}t\cos\lambda_{k_2}t \sum_s \cos\lambda_{k_1}s \sum_r \cos\lambda_{k_2}r\right|\right.\\ &\quad + \left|\sum_t \cos\lambda_{k_1}t\sin\lambda_{k_2}t \sum_s \cos\lambda_{k_1}s \sum_r \sin\lambda_{k_2}r\right|\\ &\quad + \left|\sum_t \sin\lambda_{k_1}t\sin\lambda_{k_2}t \sum_s \sin\lambda_{k_1}s \sum_r \sin\lambda_{k_2}r\right|\end{aligned}$$

$$+\left|\sum_t \sin\lambda_{k_1}t\cos\lambda_{k_2}t\sum_s \sin\lambda_{k_1}s\sum_r\cos\lambda_{k_2}r\right|\Bigg\}$$

$$\leq \left(\sum_{|k_1|,|k_2|\leq m} W_n(k_1)\, W_n(k_2)\right) d_0\, n = O(n)$$

for some constant $d_0 > 0$. Furthermore, there exist constants d_1, d_2, d_3 such that by (a) and (b)

$$\begin{aligned}\sum_{t\neq s\neq r} c_{ts}\, c_{tr} &= \sum_{t,s,r} c_{ts}\, c_{tr} + d_1\sum_t c_{tt}^2 + d_2\sum_{t,r} c_{tt}\, c_{tr} + d_3\sum_{t,s} c_{ts}^2 \\ &= \sum_{t,s,r} c_{ts}\, c_{tr} + d_1\, n + d_2\sum_{t,r} c_{tr} + d_3\sum_{t,s} c_{ts}^2 \\ &= o\left(n^2\right).\end{aligned}$$

That the sum $\sum_{t\neq s\neq r\neq v} c_{ts}\, c_{rv} = O\left(n^2\right)$ follows similarly. □

We apply Proposition 6.1. and Lemma 6.2. to $\widetilde{T}_{n,X}$ and obtain

$$\begin{aligned}\widetilde{T}_{n,X}(\lambda) &= \sum_{|k|\leq m} W_n(k)\,\widetilde{I}_{n,X}(\lambda_k) \\ &= \Big\{\psi^{-2}\sum_{|k|\leq m} W_n(k)\,|\psi(\lambda_k)|^2\,\widetilde{I}_{n,Z}(\lambda_k) \\ &\quad + \psi^{-2}\,\gamma_{n,Z}^{-2}\sum_{|k|\leq m} W_n(k)\, R_n(\lambda_k)\Big\}\,(1+o_P(1)).\end{aligned} \tag{7.1}$$

Since $\max_{|k|\leq m}|\lambda_k-\lambda|\to 0$ as $n\to\infty$ and $|\psi(\lambda)|^2$ is uniformly continuous,

$$\max_{|k|\leq m}\left||\psi(\lambda_k)|^2-|\psi(\lambda)|^2\right|\to 0,\quad n\to\infty.$$

Thus we conclude that

$$\begin{aligned}\sum_{|k|\leq m} W_n(k)\,|\psi(\lambda_k)|^2\,\widetilde{I}_{n,Z}(\lambda_k) &= (1+o(1))\,|\psi(\lambda)|^2\sum_{|k|\leq m} W_n(k)\,\widetilde{I}_{n,Z}(\lambda_k) \\ &= (1+o(1))\,|\psi(\lambda)|^2\left(1+\widetilde{Q}_n(\lambda)\right),\end{aligned} \tag{7.2}$$

where

$$\widetilde{Q}_n(\lambda) = \sum_{|k|\leq m} W_n(k)\sum_{\substack{t,s=1\\ t\neq s}}^{n}\cos\lambda_k(t-s)\,\widetilde{Z}_t\,\widetilde{Z}_s.$$

Lemma 7.2. $\widetilde{Q}_n(\lambda) \xrightarrow{P} 0$, $n \to \infty$.

Proof. We prove $E\left(\widetilde{Q}_n(\lambda)\right)^2 \to 0$ as $n \to \infty$: There exist constants d_1, d_2, d_3 such that

$$\begin{aligned} \widetilde{Q}_n^2(\lambda) &= d_1 \sum_{t \neq s} c_{ts}^2 \, \widetilde{Z}_t^2 \, \widetilde{Z}_s^2 + d_2 \sum_{t \neq s \neq r} c_{ts} \, c_{tr} \, \widetilde{Z}_t^2 \, \widetilde{Z}_s \, \widetilde{Z}_r \\ &\quad + d_3 \sum_{t \neq s \neq r \neq v} c_{ts} \, c_{rv} \, \widetilde{Z}_t \, \widetilde{Z}_s \, \widetilde{Z}_r \, \widetilde{Z}_v \,, \end{aligned}$$

and the result follows from Lemmas 6.3. and 7.1. $\square$

In view of (7.1), (7.2) and Lemma 7.2. it remains to prove that

$$\gamma_{n,Z}^{-2} \sum_{|k| \leq m} W_n(k) \, R_n(\lambda_k) \xrightarrow{P} 0 \,.$$

By the decomposition of Proposition 6.1. and by Hölder's inequality, we have for some constant $c > 0$

$$\left| \sum_{|k| \leq m} W_n(k) \, R_n(\lambda_k) \right|$$

$$\leq c \left\{ \left(\sum_{|k| \leq m} W_n(k) \, |J_n(\lambda_k)|^2 \right)^{1/2} \left(\sum_{|k| \leq m} W_n(k) \, |Y_n(\lambda_k)|^2 \right)^{1/2} + \sum_{|k| \leq m} W_n(k) \, |Y_n(\lambda_k)|^2 \right\}$$

By (7.2) and Lemma 7.2.,

$$\gamma_{n,Z}^{-2} \sum_{|k| \leq m} W_n(k) \, |J_n(\lambda_k)|^2 = \sum_{|k| \leq m} W_n(k) \, \widetilde{I}_{n,Z}(\lambda_k) = O_P(1) \,,$$

hence it suffices to show that

$$\gamma_{n,Z}^{-2} \sum_{|k| \leq m} W_n(k) \, |Y_n(\lambda_k)|^2 \xrightarrow{P} 0 \,. \tag{7.3}$$

By the decomposition of Proposition 6.1. we have

$$|Y_n(\lambda_k)|^2 \leq 2 \, (A_{1k} + A_{2k})$$

where

$$A_{1k} = a_n^{-2} \left| \sum_{|t| > n} \psi_t \, e^{-i\lambda_k t} \, U_{nt}(\lambda_k) \right|^2$$

$$A_{2k} = a_n^{-2} \left| \sum_{|t| \le n} \psi_t \, e^{-i\lambda_k t} \, U_{nt}(\lambda_k) \right|^2 .$$

Lemma 7.3. $\gamma_{n,Z}^{-2} \sum_{|k| \le m} W_n(k) \, A_{1k} \xrightarrow{P} 0.$

Proof. We have for some $c < \infty$

$$\sum_{|k| \le m} W_n(k) \, A_{1k} \le c \, (V_1 + V_2)$$

where

$$V_1 = a_n^{-2} \sum_{|k| \le m} W_n(k) \left| \sum_{|t| > n} \psi_t \, e^{-i\lambda_k t} \sum_{r=1-t}^{n-t} Z_r \, e^{-i\lambda_k r} \right|^2$$

$$V_2 = a_n^{-2} \sum_{|k| \le m} W_n(k) \left| \sum_{|t| > n} \psi_t \, e^{-i\lambda_k t} \right|^2 I_{n,Z}(\lambda_k) .$$

Note that

$$\gamma_{n,Z}^{-2} \, V_2 \le \left(\sum_{|t| > n} |\psi_t| \right)^2 \sum_{|k| \le m} W_n(k) \, \tilde{I}_{n,Z}(\lambda_k) = o_P(1) .$$

By (A4), it remains to show that $V_1 \xrightarrow{P} 0$. We restrict ourselves to prove that

$$\begin{aligned} V_{11} &= a_n^{-2} \sum_{|k| \le m} W_n(k) \left| \sum_{t > n} \psi_t \, e^{-i\lambda_k t} \sum_{r=1-t}^{n-t} Z_r \, e^{-i\lambda_k r} \right|^2 \\ &= a_n^{-2} \sum_{|k| \le m} W_n(k) \left| \sum_{t=-\infty}^{-1} Z_t \, e^{-i\lambda_k t} \sum_{r=(n+1) \wedge (1-t)}^{n-t} \psi_r \, e^{-i\lambda_k r} \right|^2 \xrightarrow{P} 0 . \end{aligned}$$

We have for some positive c

$$E \, |V_{11}|^{\delta/2} \le a_n^{-\delta} \, E \left| \sum_{|k| \le m} W_n(k) \left(\sum_{t=-\infty}^{-1} |Z_t| \sum_{r=(n+1) \wedge (1-t)}^{n-t} |\psi_r| \right)^2 \right|^{\delta/2}$$

$$= \; a_n^{-\delta}\, E\left|\sum_{t=-\infty}^{-1} |Z_t| \sum_{r=(n+1)\wedge(1-t)}^{n-t} |\psi_r|\right|^{\delta}$$

$$\leq \; c\, a_n^{-\delta} \sum_{t=-\infty}^{-1} \; \sum_{r=(n+1)\wedge(1-t)}^{n-t} |\psi_r|^{\delta}$$

$$\leq \; c\, a_n^{-\delta} \sum_{t=-\infty}^{\infty} |\psi_t|^{\delta}\; |t| \to 0$$

and an application of Markov's inequality proves $V_{11} \xrightarrow{P} 0$. □

Lemma 7.4. $\gamma_{n,Z}^{-2} \sum_{|k|\leq m} W_n(k)\, A_{2k} \xrightarrow{P} 0.$

Proof. In view of (A4) it suffices to show that $\sum_{|k|\leq m} W_n(k)\, A_{2k} \xrightarrow{P} 0$. We restrict ourselves to show that

$$V_3 \;=\; a_n^{-2} \sum_{|k|\leq m} W_n(k) \left|\sum_{t=1}^{n} \psi_t\, e^{-i\lambda_k t}\, U_{nt}(\lambda_k)\right|^2$$

$$=\; a_n^{-2} \sum_{|k|\leq m} W_n(k) \left|\sum_{t=1}^{n} \psi_t\, e^{-i\lambda_k t} \left(\sum_{\ell=1-t}^{0} Z_\ell\, e^{-i\lambda_k \ell} - \sum_{\ell=n-t+1}^{n} Z_\ell\, e^{-i\lambda_k \ell}\right)\right|^2 \;\xrightarrow{P}\; 0.$$

We have for some positive c

$$E\,|V_3|^{\delta/2} \;\leq\; a_n^{-\delta}\, E\left|\sum_{t=1}^{n} |\psi_t| \left(\sum_{\ell=1-t}^{0} |Z_\ell| + \sum_{\ell=n-t+1}^{n} |Z_\ell|\right)\right|^{\delta}$$

$$\leq\; c\, a_n^{-\delta} \sum_{t=-\infty}^{\infty} |\psi_t|^{\delta}\; |t| \to 0.$$

Markov's inequality proves $V_3 \xrightarrow{P} 0$. □

Finally we combine Lemmas 7.3. and 7.4. which gives (7.3); this proves Theorem 3.1. □

8. Proof of Proposition 4.1.

We mimic the proof of Davis and Resnick (1986), pp. 548-550. For any $h \in \mathcal{N}$ we have

$$\begin{aligned}\sum_{t=1}^{n} X_t\, X_{t+h} - \tilde{\gamma}(h) \sum_{t=1}^{n} X_t^2 &= \sum_{t=1}^{n} \sum_{i \neq j} \psi_i\,(\psi_{j+h} - \tilde{\gamma}(h)\,\psi_j)\, Z_{t-i}\, Z_{t-j} \\ &+ \sum_{t=1}^{n} \sum_{i} \psi_i\,(\psi_{i+h} - \tilde{\gamma}(h)\,\psi_i)\,(Z_{t-i}^2 - Z_t^2) =: V_1 + V_2\end{aligned} \tag{8.1}$$

where we used the fact that $\sum_i \psi_i\,(\psi_{i+h} - \tilde{\gamma}(h)\,\psi_i) = 0$. By (A1)-(A3) we obtain for some $c_i > 0$, $i = 1, 2, 3, 4$.

$$\begin{aligned} E\left|a_n^{-2}\, V_1\right|^{\delta} &\leq c_1\, n\, a_n^{-2\delta} \sum_{i \neq j} |\psi_i\,(\psi_{j+h} - \tilde{\gamma}(h)\,\psi_j)|^{\delta} \\ &\leq c_2\, n\, a_n^{-2\delta} \quad \to \quad 0\,, \end{aligned}$$

and

$$\begin{aligned} E\left|a_n^{-2}\, V_2\right|^{\delta/2} &\leq c_3\, a_n^{-\delta} \sum_{i} |\psi_i\,(\psi_{i+h} - \tilde{\gamma}(h)\,\psi_i)|^{\delta/2}\, |i| \\ &\leq c_4\, a_n^{-\delta} \quad \to \quad 0\,. \end{aligned}$$

By Markov's inequality this implies

$$a_n^{-2}\,(V_1 + V_2) \quad \xrightarrow{P} \quad 0\,. \tag{8.2}$$

Furthermore, by Lemma 6.2.

$$\gamma_{n,X}^2 = \psi^2\, \gamma_{n,Z}^2\,(1 + o_P(1))\,.$$

This, (8.1), (8.2) and (A4) imply that

$$\tilde{\gamma}_{n,X}(h) - \tilde{\gamma}(h) = \frac{\sum_{t=1}^{n} X_t\, X_{t+h} - \tilde{\gamma}(h) \sum_{t=1}^{n} X_t^2}{\sum_{t=1}^{n} X_t^2} - \frac{\sum_{t=n-h+1}^{n} X_t\, X_{t+h}}{\sum_{t=1}^{n} X_t^2}$$

$$= \frac{V_1 + V_2}{\sum_{t=1}^{n} X_t^2} - \frac{\sum_{t=n-h+1}^{n} X_t X_{t+h}}{\sum_{t=1}^{n} X_t^2} = o_P(1). \tag{8.3}$$

In the latter relation we also used (A4) together with the fact that $a_n^{-2} \sum_{t=n-h+1}^{n} X_t X_{t+h} \xrightarrow{P} 0$ for every h. □

9. Proofs of the results in Section 5

The proofs in this section are modelled on those in the finite variance case, due initially to Hannan (1973). [cf. the treatment in Brockwell and Davis (1991), Section 10.8, which we follow closely]. The technical differences in the infinite variance case are, however, substantial.

Throughout this section we shall treat only the estimator β_n, based on minimising an integral of the integrated periodogram. The estimator $\bar{\beta}_n$, based on the summed periodogram, can be treated similarly. Unfortunately, as in the Gaussian case, there seems to be no easy way to exploit the "obvious" asymptotic equivalence of β_n and $\bar{\beta}_n$, so as to obtain the consistency and asymptotic distribution of one directly from the other. Since a full proof of this equivalence involves treating the differences between the higher order terms in the Taylor expansions of $\sigma_n^2(\beta_n - \beta_0)$ and $\bar{\sigma}_n^2(\bar{\beta}_n - \beta_0)$, and this is no easier than deriving the results for $\bar{\beta}_n$ directly, we refer the reader to the thesis Gadrich (1993), where essentially the same arguments used below for β_n are applied to $\bar{\beta}_n$.

We start with some auxiliary results.

Lemma 9.1. *Suppose $(X_t)_{t\in\mathbb{Z}}$ is a causal invertible ARMA(p,q) process and conditions (A1)-(A4) hold. Then for every $\beta \in C$*

$$\sigma_n^2(\beta) \quad \xrightarrow{P} \quad \psi^{-2}(\beta_0) \int_{-\pi}^{\pi} \frac{g(\lambda, \beta_0)}{g(\lambda, \beta)}\, d\lambda \tag{9.1}$$

and for every $\delta > 0$

$$\sup_{\beta\in\overline{C}}\left|\sigma^2_{n,\delta}(\beta) - \psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda\right| \xrightarrow{P} 0 \tag{9.2}$$

where

$$g_\delta(\lambda,\beta) = \frac{\left|\theta\left(e^{-i\lambda}\right)\right|^2 + \delta}{\left|\varphi\left(e^{-i\lambda}\right)\right|^2}$$

and

$$\sigma^2_{n,\delta}(\beta) = \int_{-\pi}^{\pi}\frac{\widetilde{I}_{n,X}(\lambda)}{g_\delta(\lambda,\beta)}\,d\lambda\,.$$

Proof. We restrict ourselves to prove that (9.2) is satisfied. The proof of (9.1) is analogous. We adopt the proof of Proposition 10.8.2 in Brockwell, Davis (1991). We define

$$q_m(\lambda,\beta) = \frac{1}{m}\sum_{j=0}^{m-1}\sum_{|k|\le j} b_k\, e^{-i\lambda k} = \sum_{|k|<m}\left(1-\frac{|k|}{m}\right)b_k\,e^{-i\lambda k}\,,$$

where

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda k}\, g_\delta^{-1}(\lambda,\beta)\,d\lambda\,.$$

Fix $\epsilon > 0$. Then there exists some $m \in \mathcal{N}$ such that

$$\left|q_m(\lambda,\beta) - g_\delta^{-1}(\lambda,\beta)\right| < \epsilon/(4\pi)$$

for all $(\lambda,\beta) \in [-\pi,\pi]\times\overline{C}$. Hence

$$\left|\sigma^2_{n,\delta}(\beta) - \int_{-\pi}^{\pi}\widetilde{I}_{n,X}(\lambda)\,q_m(\lambda,\beta)\,d\lambda\right| \le \frac{\epsilon}{4\pi}\int_{-\pi}^{\pi}\widetilde{I}_{n,X}(\lambda)\,d\lambda = \epsilon/2\,,\quad \forall\beta\in\overline{C}\,.$$

Hence for fixed ϵ

$$\begin{aligned}
& P\left(\sup_{\beta\in\overline{C}}\left|\sigma^2_{n,\delta}(\beta) - \psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda\right| \ge \epsilon\right) \\
\le\ & P\left(\sup_{\beta\in\overline{C}}\left|\int_{-\pi}^{\pi}\widetilde{I}_{n,X}(\lambda)q_m(\lambda,\beta)d\lambda - \psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda\right| \ge \frac{\epsilon}{2}\right) \\
=\ & P\left(\sup_{\beta\in\overline{C}}\left|2\pi\sum_{|h|\le m}\widetilde{\gamma}_{n,X}(h)\left(1-\frac{|h|}{m}\right)b_h - \psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda\right| \ge \frac{\epsilon}{2}\right)
\end{aligned}$$

$$\leq P\left(\sup_{\beta\in\overline{C}}\left|2\pi\sum_{|h|\leq m}(\tilde{\gamma}_{n,X}(h)-\tilde{\gamma}(h))\left(1-\frac{|h|}{m}\right)b_h\right|\geq\frac{\epsilon}{4}\right)$$
$$+P\left(\sup_{\beta\in\overline{C}}\left|2\pi\sum_{|h|\leq m}\tilde{\gamma}(h)\left(1-\frac{|h|}{m}\right)b_h-\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda\right|\geq\frac{\epsilon}{4}\right)$$

The first summand on the rhs converges to zero in view of Proposition 4.1. and because the b_h are uniformly bounded for $\beta\in\overline{C}$ and m fixed. The second summand is zero provided m is chosen sufficiently large (see p.379 in Brockwell, Davis (1991)). □

Proof of Theorem 5.1. We adapt the proof of Theorem 10.8.1 in Brockwell and Davis (1991). We suppose that β_n does not converge in probability to β_0. We have by Lemma 9.1. that

$$P(\sigma_n^2(\beta_n)\leq t)\geq P(\sigma_n^2(\beta_0)\leq t)\rightarrow P(2\pi\psi^{-2}(\beta_0)\leq t). \tag{9.3}$$

for every t. By the Helly-Bray theorem and the compactness of $\overline{C}$ there exists a non-random subsequence n_k such that β_{n_k} converges in distribution to a random variable β which is different from β_0 on a set of positive probability. The functional $F(f,x)=f(x)$ mapping $C(\overline{C})\times\overline{C}$ to $\mathcal{R}$ is continuous where $C(\overline{C})$ is the space of continuous functions on $\overline{C}$ equipped with the supnorm. According to Lemma 9.1.,$\sigma_{n,\delta}^2(\cdot)$ converges in probability to $\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\cdot)}d\lambda$. Hence $\sigma_{n,\delta}^2$ is tight. Since $\beta_{n_k}\xrightarrow{d}\beta$ the sequence β_{n_k} is tight as well. Thus $(\sigma_{n_k,\delta}^2,\beta_{n_k})$ is tight in $C(\overline{C})\times\overline{C}$ and there exists a further subsequence (we use n_k for the ease of notation) such that $(\sigma_{n_k,\delta}^2,\beta_{n_k})$ converges in distribution. By the continuous mapping theorem we conclude that

$$F(\sigma_{n_k,\delta}^2,\beta_{n_k})=\sigma_{n_k,\delta}^2(\beta_{n_k})\xrightarrow{d}\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda$$

Thus we have

$$P(\sigma_{n_k}^2(\beta_{n_k})\leq t)\leq P(\sigma_{n_k,\delta}^2(\beta_{n_k})\leq t)$$

$$
\begin{aligned}
&\to P(\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda \le t)\\
= \quad & P(\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)} \le t, \beta=\beta_0) + \\
&+P(\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda \le t, \int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda > 2\pi, \beta\neq\beta_0)\\
&+P(\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda \le t, \int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g_\delta(\lambda,\beta)}d\lambda \le 2\pi, \beta\neq\beta_0)
\end{aligned}
$$

Choosing δ close to zero the last summand can be made arbitrarily small. Thus we conclude that

$$
\begin{aligned}
&\limsup_{k\to\infty} P(\sigma^2_{n_k}(\beta_{n_k}) \le t)\\
\le \quad & P(\psi^{-2}(\beta_0)2\pi \le t, \beta=\beta_0) + P(\psi^{-2}(\beta_0)\int_{-\pi}^{\pi}\frac{g(\lambda,\beta_0)}{g(\lambda,\beta)}d\lambda \le t, \beta\neq\beta_0)\\
\le \quad & P(\psi^{-2}(\beta_0)2\pi \le t, \beta=\beta_0) + P(\psi^{-2}(\beta_0)2\pi < t, \beta\neq\beta_0) \qquad (9.4)
\end{aligned}
$$

Now choose $t = 2\pi\psi^{-2}(\beta_0)$. We obtain from (9.3) and (9.4) that

$$1 = P(\psi^{-2}(\beta_0)2\pi \le t) \le P(\psi^{-2}(\beta_0)2\pi \le t, \beta=\beta_0)$$

which yields a contradiction since the event $\{\beta\neq\beta_0\}$ has positive probability.
□

Lemma 9.2. *Suppose the assumptions of Theorem 5.2. hold. Furthermore, let η be a continuous function on $[-\pi,\pi]$ such that*

$$\left(\frac{n}{\log n}\right)^{1/\alpha}\int_{-\pi}^{\pi}\eta(\lambda)\, g(\lambda,\beta_0)\, \widetilde{I}_{n,Z}(\lambda)\, d\lambda = O_p(1)\,, \quad n\to\infty\,.$$

Then

$$\left(\frac{n}{\log n}\right)^{1/\alpha}\int_{-\pi}^{\pi}\left(\widetilde{I}_{n,X}(\lambda) - \psi^{-2}(\beta_0)\, g(\lambda,\beta_0)\, \widetilde{I}_{n,Z}(\lambda)\right)\eta(\lambda)\, d\lambda \xrightarrow{P} 0\,.$$

Proof. Set $x_n = (n/\log n)^{1/\alpha}$ and note that $a_n = cn^{1/\alpha}$ for some constant $c>0$. By the decomposition of Proposition 6.1. and Lemma 6.2. we get

$$\int_{-\pi}^{\pi}\widetilde{I}_{n,X}(\lambda)\,\eta(\lambda)\,d\lambda \quad = \quad \psi^{-2}(\beta_0)\,(1+o_P(1))\int_{-\pi}^{\pi}\gamma_{n,Z}^{-2}\, I_{n,X}(\lambda)\,\eta(\lambda)\,d\lambda$$

$$= \psi^{-2}(\beta_0)(1+o_P(1))\int_{-\pi}^{\pi} \tilde{I}_{n,Z}(\lambda)\, g(\lambda,\beta_0)\; \eta(\lambda)\, d\lambda$$

$$+\, \psi^{-2}(\beta_0)(1+o_P(1))\; \gamma_{n,Z}^{-2} \int_{-\pi}^{\pi} R_n(\lambda)\; \eta(\lambda)\, d\lambda\, .$$

By the assumptions it suffices to show that

$$x_n \int_{-\pi}^{\pi} R_n(\lambda)\; \eta(\lambda)\, d\lambda = o_P(1)\, .$$

We apply Hölder's inequality and obtain for some $c>0$

$$\left|\int_{-\pi}^{\pi} R_n(\lambda)\; \eta(\lambda)\, d\lambda\right| \le c\int_{-\pi}^{\pi} |R_n(\lambda)|\; d\lambda$$

$$\le\; c\left\{\left(\int_{-\pi}^{\pi} I_{n,Z}(\lambda)\, d\lambda\right)^{1/2}\left(\int_{-\pi}^{\pi} |Y_n(\lambda)|^2\; d\lambda\right)^{1/2} + \int_{-\pi}^{\pi} |Y_n(\lambda)|^2\; d\lambda\right\}.$$

Thus it remains to show that

$$x_n^2 \int_{-\pi}^{\pi} |Y_n(\lambda)|^2\; d\lambda \;\xrightarrow{P}\; 0\, .$$

We have

$$\int_{-\pi}^{\pi} |Y_n(\lambda)|^2\; d\lambda \le cn^{-2/\alpha}\left\{\int_{-\pi}^{\pi}\left|\sum_{j>n}\psi_j\, e^{-i\lambda j}\sum_{t=1-j}^{n-j} Z_t\, e^{-i\lambda t}\right|^2 d\lambda\right.$$

$$+\int_{-\pi}^{\pi}\left|\sum_{j>n}\psi_j\, e^{-i\lambda j}\sum_{t=1}^{n} Z_t\, e^{-i\lambda t}\right|^2 d\lambda + \int_{-\pi}^{\pi}\left|\sum_{j=1}^{n}\psi_j\, e^{-i\lambda j}\sum_{t=1-j}^{0} Z_t\, e^{-i\lambda t}\right|^2 d\lambda$$

$$\left.+\int_{-\pi}^{\pi}\left|\sum_{j=1}^{n}\psi_j\, e^{-i\lambda j}\sum_{t=n-j+1}^{n} Z_t\, e^{-i\lambda t}\right|^2 d\lambda\right\}$$

$$=:\; cn^{-2/\alpha}\,(V_1+V_2+V_3+V_4)\, .$$

It suffices to show that the V_i are stochastically bounded. We will show this for V_1. The other estimates are similar. Note that $V_1 = \int_{-\pi}^{\pi} |Q(\lambda)|^2\, d\lambda$ where

$$Q(\lambda) \;=\; \sum_{j=-\infty}^{-1} Z_j \sum_{t=(n+1)\wedge(1-j)}^{n-j} \psi_t e^{-i\lambda(t+j)}.$$

Let B_1 and B_2 be two independent Brownian motions on $[-\pi,\pi]$ and suppose that they are independent of (Z_t). Then

$$\begin{aligned}
& Ee^{ir\left(\int_{-\pi}^{\pi}\mathrm{Re}(Q(\lambda))dB_1(\lambda)+\int_{-\pi}^{\pi}\mathrm{Im}(Q(\lambda))dB_2(\lambda)\right)} \\
= \; & E\left(E\left(e^{ir\left(\int_{-\pi}^{\pi}\mathrm{Re}(Q(\lambda))dB_1(\lambda)+\int_{-\pi}^{\pi}\mathrm{Im}(Q(\lambda))dB_2(\lambda)\right)}\middle|(Z_t)\right)\right) \\
= \; & E\left(E\left(e^{ir\left(\left(\int_{-\pi}^{\pi}(\mathrm{Re}(Q(\lambda)))^2d\lambda\right)^{1/2}N_1+\left(\int_{-\pi}^{\pi}(\mathrm{Im}(Q(\lambda)))^2d\lambda\right)^{1/2}N_2\right)}\middle|(Z_t)\right)\right) \\
= \; & E\left(E\left(e^{ir\left(\int_{-\pi}^{\pi}((\mathrm{Re}(Q(\lambda)))^2+(\mathrm{Im}(Q(\lambda)))^2)d\lambda\right)^{1/2}N_1}\middle|(Z_t)\right)\right) \\
= \; & Ee^{-\frac{r^2}{2}\int_{-\pi}^{\pi}|Q(\lambda)|^2d\lambda} \\
= \; & Ee^{-\frac{r^2}{2}V_1}.
\end{aligned}$$

Here N_1, N_2 are i.i.d. standard Gaussian r.v.'s independent of (Z_t).

In order to show stochastic boundedness of V_1 it thus suffices to prove that the real and the imaginary parts of $\int_{-\pi}^{\pi} Q(\lambda)dB_1(\lambda)$ are stochastically bounded. We restrict ourselves to the real part.

We introduce the gauge function Λ_α for any r.v. A by

$$\Lambda_\alpha(A) = \left(\sup_{t>0} t^\alpha\, P(|A|>t)\right)^{1/\alpha}.$$

Then for any sequence $(a_i)_{i\in\mathcal{N}}$ of real numbers we have for some constant $c_\alpha > 0$

$$\Lambda_\alpha^\alpha\left(\sum_{i=1}^n a_i\, Z_i\right) \le c_\alpha \sum_{i=1}^n |a_i|^\alpha\, \Lambda_\alpha^\alpha\,(Z_0)$$

[see e.g. Klüppelberg and Mikosch (1991), Lemma 3.4]. Then for fixed $\epsilon > 0$

$$\begin{aligned}
& P\left(\left|\int_{-\pi}^{\pi}\mathrm{Re}\,(Q(\lambda))\,dB_1(\lambda)\right| \ge \epsilon\right) \\
\le \; & \epsilon^{-\alpha} E\left(E\left(\sup_{s>0} s^\alpha P\left(\left|\sum_{j=-\infty}^{-1} Z_j \sum_{t=(n+1)\wedge(1-j)}^{n-j} \psi_t \int_{-\pi}^{\pi}\mathrm{Re}\left(e^{-i\lambda(t+j)}\right) dB_1(\lambda)\right| > s\right)\middle| B_1\right)\right)
\end{aligned}$$

$$\leq c_\alpha \Lambda_\alpha^\alpha(Z_0) \sum_{j=-\infty}^{-1} E \left| \sum_{t=(n+1)\wedge(1-j)}^{n-j} \psi_t \int_{-\pi}^{\pi} \mathrm{Re}\left(e^{-i\lambda(t+j)}\right) dB_1(\lambda) \right|^\alpha$$

$$= c_\alpha \Lambda_\alpha^\alpha(Z_0) \sum_{j=-\infty}^{-1} E|N_j|^\alpha \left(\sum_{t=(n+1)\wedge(1-j)}^{n-j} |\psi_t|^2 \right)^{\alpha/2}$$

$$\leq c \sum_{j=-\infty}^{-1} |\psi_j|^{\alpha\wedge 1} j < \infty$$

Here (N_j) is a sequence of identically distributed (but dependent) Gaussian r.v.'s and c is a positive constant. In the last step we made use of condition (A2). This proves the stochastic boundedness of V_1. □

Lemma 9.3. *Suppose the assumptions of Theorem 5.2. hold. Furthermore, let η be an odd continuous function on $[-\pi, \pi]$ and the Fourier coefficients f_k, $k \in \mathbb{Z}$, of $\eta(\lambda)\ g(\lambda, \beta_0)$ satisfy $\sum_{k=-\infty}^{\infty} |f_k|^\mu < \infty$ for some $\mu \in (0, 1 \wedge \alpha)$. Then*

$$\left(\frac{n}{\log n}\right)^{1/\alpha} \int_{-\pi}^{\pi} \widetilde{I}_{n,X}(\lambda)\ \eta(\lambda)\, d\lambda \quad \xrightarrow{d} \quad 4\pi\ \psi^{-2}(\beta_0) \sum_{k=1}^{\infty} \frac{Y_k}{Y_0} f_k$$

where $Y_0, Y_1, Y_2, \ldots$ are independent r.v.'s, Y_0 is positive $\alpha/2$-stable and $(Y_t)_{t\in\mathcal{N}}$ are iid symmetric α-stable with ch.f. $E\, e^{itY_1} = e^{-C_\alpha |t|^\alpha}$, $t \in \mathcal{R}$.

Proof. We adapt the proof of Proposition 10.8.6 of Brockwell and Davis (1991). In view of Lemma 9.2. it suffices to show that

$$x_n \int_{-\pi}^{\pi} \widetilde{I}_{n,Z}(\lambda)\ \eta(\lambda)\ g(\lambda, \beta_0)\ d\lambda \quad \to \quad 4\pi \sum_{k=1}^{\infty} \frac{Y_k}{Y_0} f_k$$

where $x_n = (n/\log n)^{1/\alpha}$. Set

$$\chi(\lambda) = \eta(\lambda)\ g(\lambda, \beta_0)$$

and

$$\chi_m(\lambda) = \sum_{|k|\leq m} f_k\, e^{i\lambda k} \quad \text{with} \quad f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda k}\, \chi(\lambda)\, d\lambda\,.$$

The assumptions on (f_k) imply that uniformly in λ

$$\chi_m(\lambda) \quad \to \quad \chi(\lambda) = \sum_{k=-\infty}^{\infty} f_k\, e^{i\lambda k}\,, \quad m \to \infty\,.$$

Moreover, $f_0 = 0$. We show that for all $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(x_n \left|\int_{-\pi}^{\pi} \widetilde{I}_{n,Z}(\lambda)\,(\chi(\lambda) - \chi_m(\lambda))\, d\lambda\right| > \varepsilon\right) = 0\,. \tag{9.5}$$

For $n \in \mathcal{N}$ and $h \in \mathcal{Z}$ we set

$$\widetilde{\gamma}_{n,Z}(h) = \widetilde{\gamma}_{n,Z}(|h|) = \sum_{t=1}^{n-|h|} Z_t\, Z_{t+|h|} \Big/ \sum_{t=1}^{n} Z_t^2$$

and $y_n = (n \log n)^{1/\alpha}$. Then for $n > m$ there exists some $c_1 > 0$

$$\begin{aligned}
V_1 &= x_n \int_{-\pi}^{\pi} \widetilde{I}_{n,Z}(\lambda)\,(\chi(\lambda) - \chi_m(\lambda))\, d\lambda \\
&= x_n \int_{-\pi}^{\pi} \left(\sum_{|h|<n} \widetilde{\gamma}_{n,Z}(h)\, e^{-i\lambda h} \sum_{|k|>m} f_k\, e^{i\lambda k}\right) d\lambda \\
&= x_n\, 2\pi \sum_{m<|h|<n} \widetilde{\gamma}_{n,Z}(h)\, f_h \\
&= c_1\, \gamma_{n,Z}^{-2}\, y_n^{-1} \left\{\sum_{h=m+1}^{n-1} f_h \sum_{t=1}^{n-h} Z_t\, Z_{t+h}\right\} \\
&= c_1\, \gamma_{n,Z}^{-2}\, y_n^{-1} \sum_{t=1}^{n-m-1} Z_t \sum_{h=m+1}^{n-t} f_h\, Z_{t+h} \\
&= c_1\, \gamma_{n,Z}^{-2}\, y_n^{-1} \sum_{t=1}^{n-m-1} Z_t \sum_{h=m+t+1}^{n} f_{h-t}\, Z_h \\
&= c_1\, \gamma_{n,Z}^{-2}\, V_2\,.
\end{aligned}$$

Since $\gamma_{n,Z}^2 \xrightarrow{d} \gamma^2$ for some positive $\alpha/2$-stable r.v. γ^2 for (9.5) it suffices to show that for every $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(|V_2| > \varepsilon\right) = 0\,.$$

An application of Theorem 3.1 of Rosinski and Woyczynski (1987) yields for some $c_2 > 0$

$$P(|V_2| > \varepsilon) \le c_2 \frac{1}{n} \sum_{t=1}^{n-m-1} \sum_{h=m+t+1}^{n} |f_{h-t}|^{\alpha} \left(1 + \log^{+} \frac{1}{|f_{h-t}|}\right).$$

Note that for $x \in (0,1)$,

$$x^{\alpha}\left(1 + \log^{+} \frac{1}{x}\right) \le x^{\mu}$$

where $\mu \in (0, 1 \wedge \alpha)$. Hence for constants c_2, $c_3 > 0$

$$\begin{aligned} P(|V_2| > \varepsilon) &\le c_2 \frac{1}{n} \sum_{t=1}^{n-m-1} \sum_{h=m+t+1}^{n} |f_{h-t}|^{\mu} \\ &\le c_3 \frac{1}{n} \sum_{\ell=m+1}^{n} (n-\ell)\, |f_{\ell}|^{\mu} \le c_3 \sum_{\ell=m+1}^{\infty} |f_{\ell}|^{\mu} \end{aligned}$$

and, by the assumptions, the rhs converges to 0 as $m \to \infty$. This proves (9.5). Now it remains to show [cf. Proposition 6.3.9 in Brockwell and Davis (1991)] that

$$V_3 = x_n \int_{-\pi}^{\pi} \widetilde{I}_{n,Z}(\lambda)\, \chi_m(\lambda)\, d\lambda \ \xrightarrow{d}\ 2\pi \sum_{|k| \le m} f_k \frac{Y_k}{Y_0}. \tag{9.6}$$

For $n > m$ we have

$$\begin{aligned} V_3 &= x_n \int_{-\pi}^{\pi} \left(\sum_{|h|<n} \widetilde{\gamma}_{n,Z}(h)\, e^{-i\lambda h} \sum_{|k| \le m} f_k\, e^{i\lambda k} \right) d\lambda \\ &= x_n\, 2\pi \sum_{|h| \le m} \widetilde{\gamma}_{n,Z}(h)\, f_h \\ &= 2\pi\, \gamma_{n,Z}^{-2} \sum_{|h| \le m} f_h \left(y_n^{-1} \sum_{t=1}^{n-|h|} Z_t\, Z_{t+|h|} \right). \end{aligned}$$

Theorem 3.3 of Davis and Resnick (1986) gives for $h > 0$

$$\left(\gamma_{n,Z}^{2},\ y_n^{-1} \sum_{t=1}^{n-1} Z_t\, Z_{t+1}, \ldots, y_n^{-1} \sum_{t=1}^{n-h} Z_t\, Z_{t+h} \right) \ \xrightarrow{d}\ (Y_0, Y_1, \ldots, Y_h).$$

The specific scaling constants in the statement of the lemma, and in Theorem 5.2, then follow from the representation of the Y_i given in Davis and Resnick (1986) and the results of Le Page (1980).

This together with the continuous mapping theorem proves (9.6). □

Proof of Theorem 5.2. We adapt the proof of Theorem 10.8.2 of Brockwell and Davis (1991). A Taylor expansion gives

$$\begin{aligned}\frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} &= \frac{\partial \sigma_n^2(\beta_n)}{\partial \beta} - (\beta_n - \beta_0)\,\frac{\partial^2 \sigma_n^2(\beta_n^*)}{\partial \beta^2} \\ &= -(\beta_n - \beta_0)\,\frac{\partial^2 \sigma_n^2(\beta_n^*)}{\partial \beta^2}\end{aligned}$$

for some β_n^* with $\|\beta_n^* - \beta_n\| \le \|\beta_n - \beta_0\|$ where $\|\cdot\|$ denotes the Euclidean norm. Now

$$\frac{\partial^2 \sigma_n^2(\beta_n^*)}{\partial \beta^2} = \int_{-\pi}^{\pi} \tilde{I}_{n,X}(\lambda)\,\frac{\partial^2 g^{-1}(\lambda, \beta_n^*)}{\partial \beta^2}\,d\lambda$$

and since $\beta_n^* \xrightarrow{P} \beta_0$ similar arguments as in the proof of Lemmas 9.1. yield that

$$\frac{\partial^2 \sigma_n^2(\beta_n^*)}{\partial \beta^2} \xrightarrow{P} \psi^{-2}(\beta_0)\int_{-\pi}^{\pi} g(\lambda, \beta_0)\,\frac{\partial^2 g^{-1}(\lambda, \beta_0)}{\partial \beta^2}\,d\lambda\,.$$

Following the lines of the proof in Brockwell and Davis (1991) after (10.8.39) the same arguments lead to

$$\frac{\partial^2 \sigma_n^2(\beta_n^*)}{\partial \beta^2} \xrightarrow{P} \psi^{-2}(\beta_0)\,W(\beta_0)\ ,.$$

Hence it suffices to show that

$$x_n\,\frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} \xrightarrow{d} 4\pi\psi^{-2}(\beta_0)\sum_{k=1}^{\infty}\frac{Y_k}{Y_0}\,b_k$$

where b_k is defined in Theorem 5.2., or, equivalently, by the Cramér-Wold device that for all vectors $c \in \mathcal{R}^{p+q}$

$$x_n\,c^T\,\frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} \xrightarrow{d} 4\pi\psi^{-2}(\beta_0)\sum_{k=1}^{\infty}\frac{Y_k}{Y_0}\,c^T\,b_k\,.$$

We have

$$c^T \frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} = c^T \int_{-\pi}^{\pi} \widetilde{I}_{n,X}(\lambda) \frac{\partial g^{-1}(\lambda,\beta_0)}{\partial \beta} d\lambda$$

$$=: \int_{-\pi}^{\pi} \widetilde{I}_{n,X}(\lambda)\, \eta(\lambda)\, d\lambda$$

where

$$\eta(\lambda) = c^T \frac{\partial g^{-1}(\lambda,\beta_0)}{\partial \beta}$$

is an odd continuous function. Furthermore, it is not difficult to see that the Fourier coefficients of $\eta(\lambda)\, g(\lambda,\beta_0)$ satisfy the conditions of Lemma 9.3. An application of this lemma implies that

$$x_n \int_{-\pi}^{\pi} \widetilde{I}_{n,X}(\lambda)\, \eta(\lambda)\, d\lambda \xrightarrow{d} 4\pi\, \psi^{-2}(\beta_0) \sum_{k=1}^{\infty} \frac{Y_k}{Y_0} f_k$$

where f_k are the Fourier coefficients of $\eta(\lambda)\, g(\lambda,\beta_0)$; i.e.

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\lambda}\, c^T \frac{\partial g^{-1}(\lambda,\beta_0)}{\partial \beta}\, g(\lambda,\beta_0)\, d\lambda\,.$$

Thus

$$x_n\, c^T \frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} \xrightarrow{d} 4\pi\, \psi^{-2}(\beta_0) \sum_{k=1}^{\infty} \frac{Y_k}{Y_0} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda k}\, c^T \frac{\partial g^{-1}(\lambda,\beta_0)}{\partial \beta}\, g(\lambda,\beta_0)\, d\lambda$$

for all $c \in \mathcal{R}^{p+q}$. This implies that

$$x_n \frac{\partial \sigma_n^2(\beta_0)}{\partial \beta} \xrightarrow{d} 4\pi\, \psi^{-2}(\beta_0) \sum_{k=1}^{\infty} \frac{Y_k}{Y_0} b_k$$

where b_k is the vector

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda k} \frac{\partial g^{-1}(\lambda,\beta_0)}{\partial \beta}\, g(\lambda,\beta_0)\, d\lambda\,. \qquad \square$$

10. An application to simulated data

To get some idea of how the Whittle estimator behaves in the heavy tailed situation, we ran a small simulation study. Before describing the results, we make some comments about the application of the estimator.

As noted earlier, in application it is the estimator $\bar{\beta}_n$, based on the summed periodogram, that is used. In fact, whereas until now we worked with the self-normalized sample periodogram $\tilde{I}_{n,X}$, in practice it makes more sense to work with the regular periodogram $\hat{I}_{n,X}(\lambda)$ defined by

$$\hat{I}_{n,X}(\lambda) = \left| n^{-1} \sum_{t=1}^{n} X_t e^{-i\lambda t} \right|^2, \qquad -\pi < \lambda \leq \pi.$$

In this case, it is immediate from the definition of $\bar{\beta}_n$ that it could have also been defined as the minimiser of

$$\hat{\sigma}_n^2(\beta) = \sum_j \frac{\hat{I}_{n,X}(\lambda_j)}{g(\lambda_j;\beta)}, \tag{10.1}$$

where the sum, as before, is taken over all Fourier frequencies. (The difference between $\hat{\sigma}_n^2$ and $\bar{\sigma}_n^2$ lies in factors of n and the normalisation $\sum X_j^2$, neither of which affect the minimisation.)

It should be emphasised that minimisation of (10.1) requires knowledge of neither the stability parameter α nor the scale parameter σ of the data. (This, of course, is not true if one wants to determine the convergence rate of the estimator.) This fact has two important consequences. The first is that although there exist methods for estimating stable exponents [e.g. Dzhaparidze (1986), Hahn and Weiner (1991), Hsing (1991) and Koutrouvelis (1980)] none of these have very good small sample behaviour, and so it extremely comforting to have an estimator that is α independent.

The second consequence is that since it is well known that the Whittle estimator is asymptotically equivalent to the MLE in the Gaussian case, the fact that in the stable case the Whittle estimator is an identical function of the data implies a robustness property for both the Whittle and maximum likelihood estimators in the Gaussian case as well.

The following table includes the result of a small scale simulation study. We generated 100 observations from each of the models

1. $X_t - 0.4\, X_{t-1} = Z_t$

2. $X_t = Z_t + 0.8\ Z_{t-1}$

3. $X_t - 0.4\ X_{t-1} = Z_t + 0.8\ Z_{t-1}$

where the innovations sequence $\{Z_t\}$ was either iid α-stable with $\alpha = 1.5$ and scale parameter equal to 2.0, or, for comparison purposes, $N(0,2)$. (In the stable case we relied on the algorithm given by Chambers, Mallows and Stuck (1976) for generation of the innovation process.) We ran 1,000 such simulations for each model. In the stable example we estimated the ARMA parameters via the estimator $\bar{\beta}_n$, and in the Gaussian case via the usual MLE estimator. The results were as follows:

Model No.	True values	Whittle estimate mean	Whittle estimate st. dev.	Maximum-likelihood mean	Maximum-likelihood st. dev.
1	$\varphi = 0.4$	0.384	0.093	0.394	0.102
2	$\theta = 0.8$	0.782	0.097	0.831	0.099
3	$\varphi = 0.4$	0.397	0.100	0.385	0.106
	$\theta = 0.8$	0.736	0.124	0.815	0.082

Table 10.1: Estimating the parameters of stable and normal ARMA processes via Whittle and MLE estimates.

We shall not attempt to interpret these results for the reader, but merely point out that the accuracy of the Whittle estimator in the stable case seems indistinguishable from that of the MLE in the Gaussian case.

Finally, a comment about estimating p, q, α and the scale parameter of the stable innovations. We have assumed throughout, including in the simulation above, that p and q are known. When this is not that case, Bhansali (1984, 1988) has proposed a technique for estimating p and q that seems to work well in practice. Estimation of α can be done either from the raw data, or from the residuals calculated after parameter estimation. Limited experience with simulations indicates that it is best done on the (supposedly iid) residuals.

Appendix

In Theorem 5.1. we proved weak convergence of the estimated coefficient vector β_n to its true value β_0. In the finite variance case this holds even almost surely. In the infinite variance case (which is e.g. satisfied for $Z_1 \in DA(\alpha)$, $\alpha \in (0,2)$) we obtain a.s. convergence under a more restrictive condition if we take the limit along a well-specified sequence in $\mathcal{N}$. We introduce the following condition:

(A5) There exists a sequence of positive numbers e_n such that

$$\liminf_{n\to\infty} e_n^{-2} \sum_{t=1}^{n} Z_t^2 = 1 \quad \text{a.s.} \tag{AP.1}$$

where the norming constants e_n satisfy the following conditions: There exist some $d > 0$ and $\nu \in \mathcal{N}$ such that for $n_k = k^\nu$, $k \in \mathcal{N}$, $\sum_{k=1}^{\infty} \left(n_k\, e_{n_k}^{-2\delta} + e_{n_k}^{-\delta}\right) < \infty$, for $\delta = 1 \wedge d$, and (e_n/e_{n_k}) is bounded away from 0 and ∞ uniformly for $n \in [n_k, n_{k+1}]$ for all $k \in \mathcal{N}$.

A survey of results of type (AP.1) can be found in Pruitt (1990, p. 1149). Fristedt and Pruitt (1971) proved under the restriction $E\,|Z_1|^d < \infty$ for some $d > 0$ that (AP.1) holds with

$$e_n^2 = \frac{\log\log n}{\eta(\xi \log\log n/n)} \tag{AP.2}$$

for some constant $\xi > 1$ where $\eta(\cdot) = \left(-\log E\, e^{-\cdot Z_1^2}\right)^{\leftarrow}$.

If $Z_1 \in DA(\alpha)$ for $\alpha \in (0,2)$ we deduce from (AP.2) the following Lemma.

Lemma 1 *Suppose $F \in DA(\alpha)$, $\alpha \in (0,2)$. Then (A5) is satisfied for $d < \alpha$ and $e_n^2 = n^{2/\alpha}\,\tilde{L}(n)$ for some slowly varying function $\tilde{L}$. The number ν in (A5) can be chosen to satisfy $\nu > \alpha/(2\delta - \alpha) \vee (\alpha/\delta)$ provided $\delta > \alpha/2$.* $\square$

The following result complements Proposition 4.1.

Proposition 2 *Suppose $(X_t)_{t\in Z}$ satisfies (A1), (A2) and (A5). Then*

$$\tilde{\gamma}_{n_k,X}(h) \overset{\text{a.s.}}{\to} \tilde{\gamma}(h), \quad h \in \mathcal{N}, \quad n \to \infty.$$

Proof. We use the decomposition of (8.1) and obtain

$$\max_{n\in[n_k,n_{k+1}]} |V_1| \leq \sum_{t=1}^{n_{k+1}} \sum_{i\neq j} |\psi_i (\psi_{j+h} - \tilde{\gamma}(h)\psi_j))|\ |Z_{t-i} Z_{t-j}| .$$

By (A1), (A2) and (A5) we obtain for all $\varepsilon > 0$

$$\begin{aligned}\sum_{k=1}^{\infty} P\left(\max_{n\in[n_k,n_{k+1}]} |V_1| > \varepsilon\, e_{n_k}^2\right) &\leq c_1 \sum_{k=1}^{\infty} e_{n_k}^{-2\delta}\, E \max_{n\in[n_k,n_{k+1}]} |V_1|^\delta \\ &\leq c_2 \sum_{k=1}^{\infty} e_{n_k}^{-2\delta}\, n_{k+1} < \infty\end{aligned}$$

for some c_1, $c_2 > 0$. A Borel-Cantelli argument yields

$$\lim_{k\to\infty} \max_{n\in[n_k,n_{k+1}]} |V_1|\, e_n^{-2} = 0 \quad \text{a.s.}$$

Now to estimate V_2 set

$$f_i = \psi_i (\psi_{i+h} - \tilde{\gamma}(h)\psi_i) .$$

Then

$$\begin{aligned}V_2 &= \sum_{i>0} f_i \sum_{t=1}^{n} (Z_{t-i}^2 - Z_t^2) + \sum_{i<0} f_i \sum_{t=1}^{n} (Z_{t-i}^2 - Z_t^2) \\ &= V_3 + V_4 .\end{aligned}$$

We restrict ourselves to show that $\lim_{k\to\infty} e_{n_k}^{-2} V_3 = 0$ a.s., the proof for $e_{n_k}^{-2} V_4$ is similar. We have

$$\begin{aligned}V_3 &= \sum_{i>n} f_i \sum_{t=1-i}^{n-i} Z_{t-i}^2 - \sum_{i>n} f_i \sum_{t=1}^{n} Z_t^2 + \sum_{1\leq i\leq n} f_i \sum_{t=1-i}^{0} Z_t^2 - \sum_{1\leq i\leq n} f_i \sum_{t=n-i+1}^{n} Z_t^2 \\ &= V_5 - V_6 + V_7 - V_8 .\end{aligned}$$

We restrict ourselves to show that $\lim_{k\to\infty} e_{n_k}^{-2} V_7 = 0$ a.s., the proof for V_5, V_6 and V_8 is analogous. Again by (A1), (A2) and (A5) we have

$$\sum_{k=1}^{\infty} E\left|e_{n_k}^{-2} V_7\right|^{\delta/2} \le c \sum_{k=1}^{\infty} e_{n_k}^{-\delta} \sum_{i>0} |f_i|^{\delta/2}\ |i| < \infty$$

and a Borel-Cantelli argument yields the desired result. Similar arguments show that

$$e_{n_k}^{-2} \sum_{t=1}^{n_k} X_t^2 = e_{n_k}^{-2} \psi^2 \sum_{t=1}^{n_k} Z_t^2 + o(1) \quad \text{a.s.}$$

So we obtain as in (8.3) that

$$\widetilde{\gamma}_{n_k,X}(h) - \widetilde{\gamma}(h) \quad \to \quad 0 \quad \text{a.s.} \qquad \square$$

A result similar to Proposition 2 for $Z_1 \in DA(p)$ was obtained by Bhansali (1988). The following example shows that Proposition 2 is in general not valid if the subsequence (n_k) is replaced by (n).

Example. Consider the $MA(1)$ process

$$X_t = Z_t + \theta Z_{t-1}, \quad t \in \mathcal{N}, \quad |\theta| < 1,$$

for a symmetric $Z_1 \in DNA(\alpha)$ for some $\alpha \in (0,2)$. Then as mentioned in Section 2 (A1)-(A4) and (A5) are satisfied where (e_n) can be chosen as

$$e_n = n^{1/\alpha} (\log\log n)^{(1-\alpha/2)/\alpha}.$$

Now consider

$$\begin{aligned}\widetilde{\gamma}_{n,X}(1) &= \frac{\sum_{t=1}^{n-1} (Z_t + \theta Z_{t-1})(Z_{t+1} + \theta Z_t)}{\sum_{t=1}^{n} (Z_t + \theta Z_{t-1})^2} \\ &= \frac{\sum_{t=1}^{n-1} Z_t Z_{t+1} + \theta \sum_{t=1}^{n-1} Z_{t-1} Z_{t+1} + \theta \sum_{t=1}^{n-1} Z_t^2 + \theta^2 \sum_{t=1}^{n-1} Z_{t-1} Z_t}{\sum_{t=1}^{n} Z_t^2 + \theta^2 \sum_{t=1}^{n} Z_{t-1}^2 + 2\theta \sum_{t=1}^{n} Z_{t-1} Z_t}.\end{aligned}$$

Rosinski and Woyczynski (1987) have shown that for some $c > 0$

$$P(Z_1 Z_2 > x) \le c\, x^{-\alpha} \left(1 + \log^+ x^{-1}\right) .$$

Similar arguments as in the proof of Heyde's SLLN [see Stout (1974)] and the fact that

$$\sum_{h=1}^{\infty} P\left(Z_1 Z_2 > e_n^2\right) < \infty$$

imply that

$$\lim_{n\to\infty} \frac{e_n^{-2} \sum_{t=1}^{n-1} \left(Z_t Z_{t+1} + \theta^2 Z_t Z_{t-1} + \theta Z_{t-1} Z_{t+1}\right)}{e_n^{-2} \sum_{t=1}^{n-1} Z_t^2} = 0 \quad \text{a.s.}$$

Thus

$$\tilde{\gamma}_{n,X}(1) = \frac{\theta + o(1)}{\theta^2 + 1 + Z_n^2 \Big/ \sum_{t=1}^{n-1} Z_t^2 + o(1)} \quad \text{a.s.} \tag{AP.3}$$

We shall show that

$$\limsup_{n\to\infty} Z_n^2 \Big/ \sum_{t=1}^{n-1} Z_t^2 = \infty \quad \text{a.s.} \tag{AP.4}$$

Define

$$A_n := \left\{ Z_n^2 > \varepsilon\, n^{2/\alpha} \right\}, \quad B_n := \left\{ \sum_{t=1}^{n-1} Z_t^2 < n^{2/\alpha} \right\},$$

then for every $\varepsilon > 0$

$$P\left(Z_n^2 \Big/ \sum_{t=1}^{n-1} Z_t^2 > \varepsilon \text{ i.o.} \right) \ge P\left(A_n \cap B_n \text{ i.o.}\right) .$$

Note that $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $\liminf_{n\to\infty} P(B_n) > 0$. Since A_n and $\{B_1, B_2, \ldots, B_n\}$ are independent for each $n \ge 1$, an application of a standard Borel-Cantelli lemma [e.g. Petrov (1975), Lemma 5, Section IX.2] yields $P(A_n \cap B_n \text{ i.o.}) > 0$, hence

$P(A_n \cap B_n \text{ i.o.}) = 1$ which implies (AP.4). From (AP.3) and (AP.4) we conclude that for almost every ω there exists a subsequence $n' = n'(\omega)$ such that

$$\lim_{n' \to \infty} Z_{n'}^2 \Big/ \sum_{t=1}^{n'-1} Z_t^2 = \infty .$$

Hence 0 is an a.s. limit point of $\tilde{\gamma}_{n,X}(1)$. □

We apply the above Proposition 2 to give an analogous result for $\sigma_n^2(\beta)$ and finally for β_n and $\sigma_n^2(\beta_n)$.

Lemma 3 *Suppose $(X_t)_{t \in Z}$ satisfies (A1), (A2) and (A5). Then for every fixed $\delta > 0$, uniformly for $\beta \in \overline{C}$ and for $\delta = 0$ and $\beta \in C$*

$$\sigma_{n_k,\delta}^2(\beta) \quad \xrightarrow{\text{a.s.}} \quad \psi^{-2}(\beta_0) \int_{-\pi}^{\pi} \frac{g(\lambda, \beta_0)}{g_\delta(\lambda, \beta)} \, d\lambda$$

where g_δ and $\sigma_{n_k,\delta}^2$ are defined as in Lemma 9.1.

Proof. It follows by an adaption of the proof of Proposition 10.8.2 in Brockwell and Davis (1991) and in view of Proposition 2. □

This lemma and an adaption of the proof of Theorem 10.8.1 of Brockwell and Davis (1991) imply the following result.

Theorem 4 *Suppose $(X_t)_{t \in Z}$ is a causal invertible ARMA(p, q) process and conditions (A1), (A2) and (A5) hold. Then*

$$\beta_{n_k} \quad \xrightarrow{\text{a.s.}} \quad \beta_0 \qquad \textit{and} \qquad \sigma_n^2(\beta_{n_k}) \quad \xrightarrow{\text{a.s.}} \quad \psi^{-2}(\beta_0) . \qquad \square$$

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