AFIT/DS/AA/93-2

SIMULTANEOUS DESIGN OF ACTIVE VIBRATION CONTROL AND PASSIVE DAMPING

DISSERTATION

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Presented to the Faculty of the Graduate School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

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December 1993

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Preface

This research was undertaken in an attempt to bring the worlds of active controls and structures a little closer together. I hoped that by viewing damping from an active controls point of view and from a structural point of view at the same time, I could make a little headway in the area of simultaneous design of controls and structures. I am encouraged by my results, and hope that you find this dissertation enlightening. I especially hope that my approach to viscoelastic damping will be of use to researchers currently struggling with structures containing viscoelastic materials.

Although I'd love to take full credit for this work, the truth is it would never have become a reality without the support of my professional associates, friends, and family. My original sponsor from the Vibrations Group, Structural Dynamics Branch, Stuctures Division, Flight Dynamics Directorate, Wright Laboratories, Major Al Janiszewski (now Lieutenant Colonel), deserves the credit for planting the seed which grew into my research topic. To his successor, Major Steve Whitehouse, I am grateful not only for his sponsorship, but also for his patience and understanding when things took a little longer than originally planned.

My committee members have also been very patient, and I thank them for sticking it out. I am especially grateful to Dr Liebst and Major Warhola for the many hours I spent in their offices. There is not enough room to express my appreciation and gratitude to my advisor, Colonel Bagley. From the moment he first suggested I consider pursuing a doctorate, he has been my staunchest supporter. He had faith in me and my abilities at times when I was about to give up hope. My husband Pierre has also been a wonderful supporter. He has done more than his share in taking care of the house and girls.

This dissertation is dedicated to my daughters, Marie and Liana, both of whom were born during its development. They have given my life a whole new meaning and purpose.

Michele Lynn Devereaux Gaudreault

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List of Symbols

A	area
$\mathbf{A}$	plant matrix
$\hat{\mathbf{A}}$	approximate plant matrix
b, b_m	parameters of viscoelastic model
$\mathbf{b_k}$	matrix coefficient of additional stiffness forces
$\mathbf{b_u}$	matrix coefficient of the control vector
$\mathbf{b_v}$	matrix coefficient of the passive forces
$\mathbf{B}$	active control input matrix
$\mathbf{B_v}$	passive forces input matrix
$\mathbf{c}$	vector of damping coefficients
$\mathbf{C}$	matrix of damping coefficients
$\mathbf{C_k}$	matrix of additional stiffness coefficients
$\mathbf{D}$	damping matrix
$\mathbf{e}_k$	unit vector in the direction of the k_{th} coordinate
E	Young's modulus
E_n	parameters of viscoelastic model
$\mathbf{E}$	error matrix
G'	real part of complex modulus
G''	imaginary part of complex modulus
$G(s)$	complex modulus
$G_{rel}(t)$	shear relaxation modulus
$\mathbf{G_r}$	matrix of active control gains
$\mathbf{H}$	matrix product equal to $\mathbf{S}^{-1/2}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P})$
$\mathbf{I}$	identity matrix
$\mathbf{I}_A$	connectivity matrix equal to $\sum_{i=0}^{k} b_i(\mathbf{A}^T)^i$
$\mathbf{I_r}$	connectivity matrix made up of identity matrices
$\widehat{\mathbf{I}}_1$	connectivity matrix made up of identity matrices
$\mathbf{I}_{12}$	connectivity matrix equal to $\sum_{i=0}^{k} b_i \left(\widehat{\mathbf{I}}_1\right)^i$

J	performance index
$\overline{J}$	average value of performance index
J_{max}	maximum value of the performance index
$\mathbf{K}$	stiffness matrix
$\mathbf{K_v}$	visco-stiffness matrix when multiplied by the complex modulus
$\mathbf{M}$	mass matrix
m	number of damping coefficients, denominator of α
m_j	modal coefficient
n	number of elements in state vector
$\mathbf{n}$	unit vector normal to the surface
$\mathbf{P}$	solution matrix of Riccati equation
$\hat{\mathbf{P}}$	constrained solution matrix of Riccati equation
$\mathbf{Q}$	state weighting matrix
$\hat{\mathbf{Q}}$	weighting matrix equal to $\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi}$
$\mathbf{Q}_x$	weighting matrix on position and velocity
$\mathbf{R}$	control weighting matrix
s	Laplace parameter
$\mathbf{S}$	passive forces weighting matrix
$\mathbf{W}$	matrix product equal to $-\mathbf{S}^{-1}\mathbf{B_v}^T\mathbf{P}$
t	time
$\mathbf{u}$	active control forces
$\mathbf{u}$	continuous vector function in Section 2.2.2
$\mathbf{v}$	passive forces
V	volume
$\mathbf{w}$	disturbance
$\mathbf{x}$	column vector of structural displacements
$\mathbf{y}$	state vector
α	order of the fractional derivative

$\Gamma(\alpha)$	gamma function of α
$\epsilon(t)$	strain history
ζ	damping ratio
η	modal vector
λ	eigenvalue
$\boldsymbol{\lambda}$	Lagrange multiplier
$\boldsymbol{\Lambda}$	eigenvalue matrix
$\overline{\boldsymbol{\Lambda}}$	conjugate of $\boldsymbol{\Lambda}$
$\tilde{\boldsymbol{\Lambda}}$	eigenvalue matrix
ρ	density
$\sigma(t)$	stress history
[illegible]	connectivity matrix
ψ	eigenvector
$\boldsymbol{\Psi}$	eigenvector matrix
$\overline{\boldsymbol{\Psi}}$	conjugate of $\boldsymbol{\Psi}$
$\tilde{\boldsymbol{\Psi}}$	modal matrix
ω_n	natural frequency

Subscripts

0	initial value
c0	active control only solution
f	final value
F	Frobenius norm solution
i	i^{th} element of vector
ij	element of matrix in the i^{th} row, j^{th} column
k	stiffness
tr	minimum trace solution

Superscripts

T	transpose of matrix
-1	inverse of matrix
$1/2$	square root

Operators

$\|\cdot\|$	vector two norm = $\sqrt{\mathbf{y}^T\mathbf{y}}$
$\|\cdot\|_2$	matrix two-norm = magnitude of the greatest singular value
$\|\cdot\|_F$	Frobenius norm
$\dot{(\ )}, \ddot{(\ )}$	first and second time derivatives
$\frac{d^n}{dt^n}$	n^{th} time derivative
div	divergence
$\mathbf{A}(t, \mathbf{y})$	time dependent vector function of the state
$D^{\alpha}[\ ]$	generalized derivative of order α
$F[\ ]$	Fourier transform operator
$Im[\ ]$	Imaginary part
$L[\ ]$	Laplace transform operator
δ	variation
Δ	change
$\sum$	summation

Abstract

A modified form of the standard linear quadratic regulator (LQR) cost functional is used to optimally blend active vibration control and passive structural damping, whether viscous or viscoelastic. Viscoelastic damping is first modelled by a standard linear model, and then by a fractional derivative model. For the viscous damping case and the classically modelled viscoelastic damping case, a sub-optimal closed form solution is derived that is independent of the initial conditions. An iterative technique that minimizes the average value of the cost functional and one that minimizes the maximum value are developed. Both techniques are applicable to viscous and viscoelastic damping and are independent of the initial state. The advantages and disadvantages of the different solution techniques are given with respect to computation requirements and performance. Several numerical examples illustrate the similarities of and differences between the various techniques.

SIMULTANEOUS DESIGN OF ACTIVE VIBRATION CONTROL AND PASSIVE DAMPING

I. Introduction

The need for simultaneous optimization of damping and active vibration control is driven primarily by the potential use of large flexible structures in space. Simultaneous design of active vibration control and passive damping increases the effectiveness and efficiency of vibration suppression systems for these structures. In many methods of vibration suppression, the structure is made as rigid as possible, the residual damping is approximated as viscous, and then an optimal controller is designed with respect to a predetermined cost function. Although the controller effectively damps vibrations at specific frequencies, the controller may excite one mode while trying to damp out another. To stabilize the associated modes, passive dampers are added. But this changes the whole system – how "optimum" is the controller now? Is control effort still optimized? What is the "best" size for the passive dampers? The efficiency as well as the effectiveness of the resulting vibration system is suspect. By designing passive and active control elements simultaneously, the efficiency and effectiveness of the vibration suppression system is increased.

Thus a need exists for a procedure to simultaneously design active control and structural damping. The procedure should be applicable to structures with viscous or viscoelastic damping. Mar alluded to this need at the Vibration Damping 1984 Workshop when he encouraged viewing damping as a creative force in design [22]. Others have placed a greater emphasis on the interaction between viscoelastic damping and active vibration control.

One of the efforts in this area was the development of numerical schemes for the modeling and control of longitudinal vibrations in a rod with Boltzmann-type viscoelastic damping by Burns and Fabiano [7]. Another effort was Hannsgen and Wheeler's research on the dynamic behavior of viscoelastic structures which emphasized the interaction between passive viscoelastic damping and active feedback damping. They determined the interacting effects of viscoelastic dissipation and feedback dissipation in the damping of oscillations in certain viscoelastic rods and beams [15].

Simonian, Major, and Gluck pointed out that incorporating passive damping in an active control design must be done so that the deficiencies of one technology are compensated by the strengths of the other [29]. They proposed an iterative scheme that uses modal strain energy analysis to determine the modal damping. A hybrid cost function was used to determine the tradeoffs. Fowler et al. implemented a variation of this method in a computer program [14].

Smith, et al. looked at redesigning a structure to make it easier to control. They assumed an ideal controller had been designed, and adjusted the mass, damping stiffness, and control to give the same performance with less effort [30].

Onoda and Watanbe used a direct numerical optimization approach, which accounted for the uncontrolled residual modes, for the design of an optimal controller composed of a regulator and an observer. The approach was incorporated into a structure/controller simultaneous optimization scheme [28]. Onoda and Watanbe demonstrated their technique for a beam, and optimized the structure by changing its cross-sectional area.

This dissertation takes a different approach, attempting a closed form solution to the problem of how to achieve an optimal blending of active control feedback gains and passive structural damping parameters (e.g., dashpot constants, thickness of viscoelastic pad, or area of contact of viscoelastic pad). A variety of techniques are derived, some of which are closed form approximate methods, and some that require iteration to achieve the optimum.

The techniques are developed for structures in which the equations of motion can be expressed in terms of finite elements. In addition, the damping matrix in the equations of motion must be symmetric and be able to be expressed as a product of three matrices, the middle one being a diagonal matrix of the damping parameters.

Chapter II contains the fundamental development of the dissertation. Viscous damping is assumed; Chapters III and IV will extend the development to structures with viscoelastic damping. Two performance indices, each a modification of the standard linear quadratic regulator (LQR) cost functional, are used. In the first one, the damping force enters the cost functional in the same form as the control force. This cost functional leads to a closed form, approximate, solution for the damping parameters and active control feedback gains. Next, an iterative technique for determining the optimal blend of damping parameters and active control feedback gains that minimize the the average value of the cost functional over a representative sample of initial conditions is derived. An iterative technique that minimizes the maximum value of the cost functional is also derived. Examples are included to illustrate and compare the techniques.

In the second performance index, damping is treated as a one-time cost. Thus the damping parameters appear explicitly in the cost functional. Although a closed form solution is not obtained, two iterative techniques similar to those derived for the first modification of the cost functional are presented. An example is given to illustrate these techniques.

In Chapter III, the development of Chapter II for the first performance index is extended to the case of viscoelastic damping. Viscoelasticity is modelled using a classical approach in which the relationship between stress and stain is expressed in terms of integer order derivatives on stress and strain. As in Chapter II, a closed form, approximate, solution for the damping parameters and active control feedback gains is derived. The derivation holds for any (finite) number of derivatives on stress and strain. For the case in which only first derivatives on stress and strain are used in the model, techniques that minimize the maximum value of the cost functional and the average value of the cost functional over a representative sample of the initial conditions are derived. Examples similar to the ones in Chapter II are presented.

In Chapter IV, the development of Chapter II for the first performance index is also extended to the case of viscoelastic damping, but in this chapter viscoelasticity is modelled using fractional order derivatives on stress and strain. Although a closed form solution is not obtained in this case, two iterative solutions analogous to those in Chapter II are derived. One minimizes the maximum value of the cost functional, and the other minimizes the average value of the cost functional over a representative sample of the initital conditions. Examples similar to the ones in Chapters II and III are presented.

A variety of solution techniques for determining the optimum blending of active vibration control and passive structural damping is offered to the design engineer in this dissertation. The closed form solutions give him quick, adequate results, and provide good starting points for the iterative techniques. Minimizing the average value of the performance index tends to give better performance of the two iterative techniques.

II. Fundamental Development

2.1 Preliminaries

In addition to developing techniques for simultaneously designing active vibration control and passive viscous damping, this chapter lays the groundwork for the following two chapters. These three chapters rely heavily on a modified version of the linear quadratic regulator (LQR) performance index, which is commonly used in control theory [1]. In this section, a short review of the linear quadratic regulator is given. The next section applies the development of the linear quadratic regulator to the problem of simultaneous optimization of passive viscous damping and active control.

In this thesis, all problems considered are in finite element formulation. The equations of motion of a structure modelled using finite elements is

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{b_u}\mathbf{u}(t) = \mathbf{0} \tag{1}$$

where $\mathbf{x}$ is the position vector, $\mathbf{M}$ is the mass matrix, $\mathbf{D}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, and $\mathbf{b_u}$ is the matrix coefficient of the vector of control forces, $\mathbf{u}$.

It will be convenient to have the equations of motion in first order form:

$$\left\{ \begin{array}{c} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{array} \right\} = \left[\begin{array}{cc} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{array} \right] \left\{ \begin{array}{c} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{array} \right\} + \left[\begin{array}{c} 0 \\ -\mathbf{M}^{-1}\mathbf{b_u} \end{array} \right] \mathbf{u}(t) \tag{2}$$

By dropping the notation for dependence on time t for brevity, and defining a state vector $\mathbf{y}$ composed of the position and velocity vectors, $\mathbf{x}$ and $\dot{\mathbf{x}}$

$$\mathbf{y} = \left\{ \begin{array}{c} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{array} \right\} \tag{3}$$

Eq (2) can be written more compactly

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} \tag{4}$$

where $\mathbf{A}$ is the plant matrix and $\mathbf{B}$ is the input matrix. The initial conditions at time t_0 are assumed known:

$$\mathbf{y}(t_0) = \mathbf{y}_0 \tag{5}$$

Since the goal of this thesis is vibration suppression, the final condition at a specified time t_f is zero:

$$\mathbf{y}(t_f) = \mathbf{0} \tag{6}$$

The LQR performance index depends on the state $\mathbf{y}$ and on the control $\mathbf{u}$:

$$J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{7}$$

The state weighting matrix $\mathbf{Q}$ is symmetric and positive semi-definite, while the control weighting matrix $\mathbf{R}$ is symmetric and positive definite. The relative magnitude of the diagonal elements of the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ reflect the relative importance of the state $\mathbf{y}$ and the control $\mathbf{u}$. *The optimal control is that which minimizes the performance index, Eq (7), subject to the state equations, Eq (4).*

Using the method of Lagrange multipliers, the state equations, Eq (4), can be appended to the performance index [12:263]:

$$J = \int_{t_0}^{t_f} \left[\frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u}) - \boldsymbol{\lambda}^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{u})\right] dt$$

To minimize the performance index, it is necessary to compute the variation of the performance index due to variations in the state and the control.

$$\delta J = \int_{t_0}^{t_f} \left[(\mathbf{y}^T\mathbf{Q}\delta\mathbf{y} + \mathbf{u}^T\mathbf{R}\delta\mathbf{u}) - \delta\boldsymbol{\lambda}^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{u}) - \boldsymbol{\lambda}^T(\delta\dot{\mathbf{y}} - \mathbf{A}\delta\mathbf{y} - \mathbf{B}\delta\mathbf{u})\right] dt$$

The independence of the three unknowns $\mathbf{y}$, $\mathbf{u}$, and $\boldsymbol{\lambda}$ leads to three equations:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} \tag{8}$$

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda} \tag{9}$$

$$\mathbf{0} = \dot{\boldsymbol{\lambda}} + \mathbf{A}^T\boldsymbol{\lambda} + \mathbf{Q}\mathbf{y} \tag{10}$$

Assuming the Lagrange multiplier $\boldsymbol{\lambda}$ is linearly dependent upon the state $\mathbf{y}$

$$\boldsymbol{\lambda} = \mathbf{P}\mathbf{y}$$

Eqs (9) and (10) can be written as

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} \tag{11}$$

$$\mathbf{0} = \dot{\mathbf{P}}\mathbf{y} + \mathbf{P}\dot{\mathbf{y}} + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} \tag{12}$$

Substituting Eq (11) into Eq (8), and then Eq (8) into Eq (12), one obtains

$$[\dot{\mathbf{P}} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{A}^T\mathbf{P} + \mathbf{Q}]\mathbf{y} = \mathbf{0}$$

This equation holds true for general $\mathbf{y}$ only if the expression within the brackets is zero, which in turn implies that $\mathbf{P}$ satisfies the matrix equation

$$\dot{\mathbf{P}} = -\mathbf{PA} + \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} - \mathbf{A}^T\mathbf{P} - \mathbf{Q} \tag{13}$$

One of the basic characteristics of the regulator is that the final time t_f is very far in the future. So the steady state solution of Eq (13), i.e., $\dot{\mathbf{P}} = \mathbf{0}$, is of interest — which implies $\mathbf{P}$ satisfies the algebraic Riccati equation

$$\mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{A}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{14}$$

Since the algebraic Riccati equation is symmetric (i.e., taking the transpose of Eq (14) results in the same equation), its solution $\mathbf{P}$ is symmetric ($\mathbf{P} = \mathbf{P}^T$). Another characteristic of the steady state solution $\mathbf{P}$ is its independence of initial conditions. The control $\mathbf{u}$ that minimizes the performance index J is given by Eq (11) where $\mathbf{P}$ satisfies Eq (14).

In the next section, viscous damping is considered analogous to active control, and the problem of optimizing damping and control simultaneously is addressed.

2.2 Simultaneous Optimization of Viscous Damping and Active Control

To optimize damping and control simultaneously, consider a modification of the LQR cost functional to include passive damping forces. The passive damping forces are treated as co-equals to the active control forces, and are weighted in a similar manner. Let $\mathbf{v}$ be a vector of passive damping forces in the system, and $\mathbf{S}$ its positive definite weighting matrix. The modified performance index is taken to be

$$J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v})dt \tag{15}$$

There are specific reasons for weighting the passive damping forces like the active control forces. When implementing a damping design using viscous fluid dampers or viscoelastic solids, one must take into account the temperature sensitivity of the damping medium. Modest changes in temperature due to absorbed mechanical energy can dramatically alter the damping properties of both fluids and solids. Hence one is motivated to limit in some fashion the mechanical energy absorbed by any given damper as well as limit the peak value of its damping force. The quadratic damping term appearing in the performance index, Eq (15), serves this end rather nicely. One should note that in the case where the passive forces $\mathbf{v}$ are proportional to the velocity vector, this quadratic damping term is reminiscent of the Rayleigh dissipation function used in conjunction with energy methods in classical dynamics [25:89].

To determine the form of $\mathbf{v}$ and how it fits into the equation of motion, a closer look at the equations of motion is needed. In the problems addressed in this chapter, the damping matrix is symmetric and is of the form

$$\mathbf{D} = \mathbf{b_v}\mathbf{C}\mathbf{b_v}^T \tag{16}$$

where $\mathbf{C}$ is a diagonal matrix of the desired viscous damping coefficients. By expressing the damping matrix $\mathbf{D}$ in this form, the damping coefficients have been separated out of the damping matrix. Since viscous damping coefficients are (usually) constant and positive, $\mathbf{C}$ is positive semi-definite. The matrix $\mathbf{b_v}$ is a matrix of constants chosen to reflect the attachment pattern of the viscous dampers. Then the state equations can be written as

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b_v} \end{bmatrix} \mathbf{C}\mathbf{b_v}^T\dot{\mathbf{x}}(t) \tag{17}$$

The quantity $\mathbf{C}\mathbf{b_v}^T\dot{\mathbf{x}}$ represents the passive damping forces.

If stiffness is allowed to vary as a design parameter, and the change in stiffness can be expressed as

$$\Delta \mathbf{K} = \mathbf{b_k C_k b_k}^T$$

then the third term on the right hand side of Eq (17) becomes

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b_k} & -\mathbf{M}^{-1}\mathbf{b_v} \end{bmatrix} \begin{bmatrix} \mathbf{C_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{b_k}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{b_v}^T \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{Bmatrix}$$

Hence, the passive structural forces $\mathbf{v}$ depend linearly upon the state $\mathbf{y}$ according to

$$\mathbf{v} = \mathbf{C\Phi y} \tag{18}$$

where $\mathbf{C}$ contains the desired passive damping coefficients (and/or changes in stiffness) along its diagonal. The matrix $\mathbf{\Phi}$ is defined as either

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{0} & \mathbf{b_v}^T \end{bmatrix} \quad \text{or} \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{b_k}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{b_v}^T \end{bmatrix} \tag{19}$$

depending on whether stiffness is allowed to vary or not. If the number of stiffnesses that are allowed to vary plus the number of damping coefficients is equal to the length of the state vector, then $\mathbf{\Phi}$ can usually be chosen such that it is invertible.

By defining the matrix $\mathbf{B_v}$ in a similar manner,

$$\mathbf{B_v} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b_v} \end{bmatrix} \quad \text{or} \quad \mathbf{B_v} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b_k} & -\mathbf{M}^{-1}\mathbf{b_v} \end{bmatrix} \tag{20}$$

the state equations can be written in a modified, compact form:

$$\dot{\mathbf{y}} = \mathbf{Ay} + \mathbf{Bu} + \mathbf{B_v v} \tag{21}$$

The optimization problem can now be expressed as the minimization of the performance index, Eq (15), subject to the state equations, Eq (21). Two additional constraints are that the passive forces $\mathbf{v}$ are of the form given in Eq (18) and the matrix of damping coefficients $\mathbf{C}$ is a constant, positive semi-definite, diagonal matrix since damping coefficients are non-negative. The method of Lagrange multipliers can be used to append the first two constraints to the performance index:

$$J = \int_{t_0}^{t_f} \left[\frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v}) - \boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{u} - \mathbf{B}_\mathbf{v}\mathbf{v}) - \boldsymbol{\lambda}_2^T(\mathbf{v} - \mathbf{C}\boldsymbol{\Phi}\mathbf{y})\right] dt$$

Now the vectors $\mathbf{y}, \mathbf{u}$, and $\mathbf{v}$ are taken to be independent. So Eq (18) implies that the matrix $\mathbf{C}$ is independent of these vectors also. By definition, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are independent of each other and of $\mathbf{y}, \mathbf{u}, \mathbf{v}$, and $\mathbf{C}$. Hence, to minimize the performance index J, one takes its variation with respect to $\mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{C}$, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ and sets the variation equal to zero. The variation of the performance index J is

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} \Big[&(\mathbf{y}^T\mathbf{Q}\delta\mathbf{y} + \mathbf{u}^T\mathbf{R}\delta\mathbf{u} + \mathbf{v}^T\mathbf{S}\delta\mathbf{v}) - \delta\boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{u} - \mathbf{B}_\mathbf{v}\mathbf{v}) \\ &-\boldsymbol{\lambda}_1^T(\delta\dot{\mathbf{y}} - \mathbf{A}\delta\mathbf{y} - \mathbf{B}\delta\mathbf{u} - \mathbf{B}_\mathbf{v}\delta\mathbf{v}) - \delta\boldsymbol{\lambda}_2^T(\mathbf{v} - \mathbf{C}\boldsymbol{\Phi}\mathbf{y}) - \boldsymbol{\lambda}_2^T(\delta\mathbf{v} - \delta\mathbf{C}\boldsymbol{\Phi}\mathbf{y} - \mathbf{C}\boldsymbol{\Phi}\delta\mathbf{y})\Big] dt\end{aligned}$$

Since initial and final conditions are specified (Eqs (5) and (6)), the term $\boldsymbol{\lambda}_1^T\delta\dot{\mathbf{y}}$ can be integrated by parts:

$$\int_{t_0}^{t_f} \boldsymbol{\lambda}_1^T\delta\dot{\mathbf{y}}dt = \boldsymbol{\lambda}_1^T\delta\mathbf{y}\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}_1^T\delta\mathbf{y}dt = -\int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}_1^T\delta\mathbf{y}dt$$

Substituting this result into the variation of the performance index and combining like terms yields

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} \Big[&(\mathbf{y}^T\mathbf{Q} + \dot{\boldsymbol{\lambda}}_1^T + \boldsymbol{\lambda}_1^T\mathbf{A} + \boldsymbol{\lambda}_2^T\mathbf{C}\boldsymbol{\Phi})\delta\mathbf{y} + (\mathbf{u}^T\mathbf{R} + \boldsymbol{\lambda}_1^T\mathbf{B})\delta\mathbf{u} \\ &+(\mathbf{v}^T\mathbf{S} + \boldsymbol{\lambda}_1^T\mathbf{B}_\mathbf{v} - \boldsymbol{\lambda}_2^T)\delta\mathbf{v} - \delta\boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{u} - \mathbf{B}_\mathbf{v}\mathbf{v}) \\ &-\delta\boldsymbol{\lambda}_2^T(\mathbf{v} - \mathbf{C}\boldsymbol{\Phi}\mathbf{y}) + \boldsymbol{\lambda}_2^T(\delta\mathbf{C})\boldsymbol{\Phi}\mathbf{y}\Big] dt\end{aligned}$$

The independence of the six unknowns leads to six simultaneous equations

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + \mathbf{B}_\mathbf{v}\mathbf{v} \tag{22}$$

$$\mathbf{v} = \mathbf{C}\boldsymbol{\Phi}\mathbf{y} \tag{23}$$

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}_1 \tag{24}$$

$$\mathbf{0} = \dot{\boldsymbol{\lambda}}_1 + \mathbf{A}^T\boldsymbol{\lambda}_1 + \mathbf{Q}\mathbf{y} + \boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\lambda}_2 \tag{25}$$

$$\boldsymbol{\lambda}_2 = \mathbf{B}_\mathbf{v}{}^T\boldsymbol{\lambda}_1 + \mathbf{S}\mathbf{v} \tag{26}$$

$$\boldsymbol{\lambda}_2 = \mathbf{0} \tag{27}$$

From Eqs (26) and (27) the passive force vector $\mathbf{v}$ is proportional to the Lagrange multiplier $\boldsymbol{\lambda}_1$:

$$\mathbf{v} = -\mathbf{S}^{-1}\mathbf{B_v}^T\boldsymbol{\lambda}_1 \tag{28}$$

Substituting Eqs (24), (27), and (28) into Eqs (22) and (25) yields expressions for the time derivatives of the state vector $\mathbf{y}$ and the Lagrange multiplier $\boldsymbol{\lambda}_1$:

$$\begin{aligned}\dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}_1 - \mathbf{B_v}\mathbf{S}^{-1}\mathbf{B_v}^T\boldsymbol{\lambda}_1 \\ \dot{\boldsymbol{\lambda}}_1 &= -\mathbf{Q}\mathbf{y} - \mathbf{A}^T\boldsymbol{\lambda}_1\end{aligned}$$

From Eqs (23), (27), and (28) the relation between the parameter matrix $\mathbf{C}$, the state vector $\mathbf{y}$, and the Lagrange multiplier $\boldsymbol{\lambda}_1$ is

$$\mathbf{C}\boldsymbol{\Phi}\mathbf{y} = -\mathbf{S}^{-1}\mathbf{B_v}^T\boldsymbol{\lambda}_1 \tag{29}$$

Since the matrices $\mathbf{C}$, $\boldsymbol{\Phi}$, $\mathbf{S}$, and $\mathbf{B_v}$ are constant, Eq (29) implies that the Lagrange multiplier $\boldsymbol{\lambda}_1$ is equal to a constant matrix times the state:

$$\boldsymbol{\lambda}_1 = \mathbf{P}\mathbf{y} \tag{30}$$

where the square matrix $\mathbf{P}$ is constant. This is the form of the standard LQR solution. In fact, Eqs (24) and (30) can be combined to give an expression for the active control $\mathbf{u}$, which is the same as that derived in the last section:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} \tag{11}$$

From Eqs (29) and (30) the relation between the damping coefficients, the state, and the constant matrix $\mathbf{P}$ is

$$\mathbf{C}\boldsymbol{\Phi}\mathbf{y} = -\mathbf{S}^{-1}\mathbf{B_v}^T\mathbf{P}\mathbf{y} \tag{31}$$

while Eqs (25) and (30) yield the following relationship between the state $\mathbf{y}$, its time derivative $\dot{\mathbf{y}}$, and the matrix $\mathbf{P}$:

$$\mathbf{0} = \mathbf{P}\dot{\mathbf{y}} + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} \tag{32}$$

Substituting for the time derivative of the state $\dot{\mathbf{y}}$ in Eq (32) yields the expression

$$\mathbf{P}(\mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + \mathbf{B}_v\mathbf{v}) + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} = \mathbf{0}$$

Using Eqs (11) and (18) to substitute for the active control vector $\mathbf{u}$ and the passive force vector $\mathbf{v}$, one obtains an equation in terms of the matrix $\mathbf{P}$ and the state vector $\mathbf{y}$:

$$\mathbf{P}\mathbf{A}\mathbf{y} + \mathbf{P}[\mathbf{B}(-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y}) + \mathbf{B}_v(-\mathbf{S}^{-1}{\mathbf{B}_v}^T\mathbf{P}\mathbf{y})] + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} = \mathbf{0}$$

This equation can be written in terms of a matrix times the state vector:

$$[\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T + \mathbf{B}_v\mathbf{S}^{-1}{\mathbf{B}_v}^T)\mathbf{P} + \mathbf{Q}]\mathbf{y} = \mathbf{0}$$

For general $\mathbf{y}$, the above equation leads to the algebraic Riccati equation

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T + \mathbf{B}_v\mathbf{S}^{-1}{\mathbf{B}_v}^T)\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{33}$$

After determining the solution $\mathbf{P}$ to the above equation, the active control $\mathbf{u}$ is determined from Eq (11). To determine the viscous damping coefficients, one notes from Eq (31) that for general $\mathbf{y}$,

$$\mathbf{C}\mathbf{\Phi} = -\mathbf{S}^{-1}{\mathbf{B}_v}^T\mathbf{P} \tag{34}$$

If $\mathbf{\Phi}$ is invertible, it would seem that $\mathbf{C}$ could be determined from Eq (34)

$$\mathbf{C} = -\mathbf{S}^{-1}{\mathbf{B}_v}^T\mathbf{P}\mathbf{\Phi}^{-1} \tag{35}$$

where $\mathbf{P}$ satisfies the above algebraic Riccati equation, Eq (33). However, in general, $\mathbf{C}$, as computed from Eq (35), is fully populated, while $\mathbf{C}$ must be a positive semi-definite diagonal matrix as the problem was originally posed. This requirement was not specifically addressed in the above derivation, and is normally not encountered in an LQR problem. The assumption that the solution has the form of the standard regulator (i.e. $\boldsymbol{\lambda}_1 = \mathbf{P}\mathbf{y}$) is what led to a non-diagonal $\mathbf{C}$. So the optimal solution derived in this section (Eqs (11), (33)), and (34)) is not attainable since it led to a fully populated $\mathbf{C}$. In the next section, approximate solutions to the optimization problem of Eq (15) are proposed.

Unlike the standard LQR problem, the solution to the optimization problem posed by Eq (15) and $\mathbf{C}$ constrained positive semi-definite diagonal is not given by a Riccati equation, even when $t_f \to \infty$. The solution to the present problem does, however, require an iteration involving a Riccati equation similar to Eq (33) and is dependent upon initial conditions. It is presented briefly in the following section.

2.2.1 Solution Technique for Minimizing the Cost Functional Error. The purpose of this section is to find good approximate solutions to the problem of optimizing active control and passive viscous damping simultaneously. Two approaches that are independent of initial conditions are given.

One approach is to minimize the distance between the optimal solution derived in the previous section and the set of allowable solutions. The optimal solution is the matrix product $\mathbf{C\Phi}$ that satisfies

$$\mathbf{C\Phi} = -\mathbf{S}^{-1}\mathbf{B_v}^T\mathbf{P} \tag{34}$$

where $\mathbf{P}$ is the solution of Eq (33). The set of allowable solutions is the set of matrices of the form $\mathbf{C\Phi}$ where $\mathbf{C}$ is a positive semi-definite diagonal matrix and $\mathbf{\Phi}$ is defined by Eq (19). This set is a convex subset of the space of all matrices that have the same dimensions as $\mathbf{C\Phi}$. The distance will be defined as the Frobenius norm of the difference:

$$\|\mathbf{C\Phi} + \mathbf{S}^{-1}\mathbf{B_v^T}\mathbf{P}\|_F \tag{36}$$

Minimizing this norm is equivalent to a least squares fit. This norm will be minimized by computing the appropriate viscous damping coefficients, which are the elements of the diagonal matrix $\mathbf{C}$.

The Frobenius norm of an $m \times n$ matrix $\mathbf{A}$ is defined as the square root of the sum of the squares of the elements of the matrix:

$$\|\mathbf{A}\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2\right]^{1/2}$$

Minimizing Eq (36) over the set of positive semi-definite diagonal matrices is a minimization problem that can be solved in closed form (see Appendix A). If

$$\mathbf{W} = -\mathbf{S}^{-1}\mathbf{B_v}^T\mathbf{P}$$

the diagonal elements of the diagonal matrix **C** are

$$C_{ii} = \max\left(\frac{\sum_{j=1}^{n} \Phi_{ij} W_{ij}}{\sum_{j=1}^{n} \Phi_{ij}^2}, 0\right) \tag{37}$$

If m is the number of damper coefficients, and n is the length of **y**, **Φ** and **W** are $m \times n$ matrices.

The second approach involves examining the minimum performance index (Eq (15)) when the damping coefficients **C** are set before computing the active control gains rather than computing them simultaneously. The performance index becomes

$$J = \int_{t_0}^{t_f} \frac{1}{2}\left[\mathbf{y}^T(\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi})\mathbf{y} + \mathbf{u}^T\mathbf{Ru}\right] dt \tag{38}$$

subject to the equations of motion:

$$\dot{\mathbf{y}} = (\mathbf{A} + \mathbf{B_v C\Phi})\mathbf{y} + \mathbf{Bu} \tag{39}$$

With the damping coefficients **C** fixed in advance, the minimizaton is a standard LQR problem with modified state weighting matrix **Q** and modified plant matrix **A**. The minimum performance index J is achieved by Eq (11):

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{Py} \tag{11}$$

where **P** (in the limit as $t_f \to \infty$) is a solution to the algebraic Riccati equation

$$\mathbf{P}(\mathbf{A} + \mathbf{B_v C\Phi}) + (\mathbf{A} + \mathbf{B_v C\Phi})^T\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi} = \mathbf{0} \tag{40}$$

This algebraic Riccati equation expresses the matrix **P** in terms of the matrix **C**. In effect, the active control **u** is now in terms of the damping coefficients **C**. The next step is to determine how to select the damping coefficients **C** such that the cost functional (Eq (38)) is minimized.

One approach is to express the value of the performance index as $t_f \to \infty$ in terms of **P** and the initial conditions (see Appendix B):

$$J = \lim_{t_f \to \infty} J(t_f) = \frac{1}{2}\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0 \tag{41}$$

Thus, if the initial conditions $\mathbf{y}_0$ are known, one can iterate with respect to the damping coefficients (i.e., positive semi-definite, diagonal $\mathbf{C}$), using Eq (40) to determine $\mathbf{P}$ during each iteration, to minimize the performance index J given by Eq (41). However, optimizing for fixed $\mathbf{y}_0$ may result in poor performance if initial conditions change (due to an unexpected disturbance, for instance). In contrast, the standard LQR problem does not require this iteration because the solution to the Riccati equation (Eq (14) is independent of the initial conditions $\mathbf{y}_0$, and hence results in inherently robust performance with respect to the initial conditions. To achieve robustness with respect to initial conditions, solutions independent of $\mathbf{y}_0$ will be sought.

It will be helpful to consider the difference between the value of the performance index at the (unattainable) solution of Section 2.2 and that achieved by enforcing the constraint that the damping coefficient matrix $\mathbf{C}$ must be diagonal. Letting $\mathbf{P}$ represent the unconstrained optimal solution and $\hat{\mathbf{P}}$ a constrained solution, the change in the performance index is

$$\Delta J = \mathbf{y}_0^T\hat{\mathbf{P}}\mathbf{y}_0 - \mathbf{y}_0^T\mathbf{P}\mathbf{y}_0 = \mathbf{y}_0^T(\hat{\mathbf{P}} - \mathbf{P})\mathbf{y}_0 = \mathbf{y}_0^T\Delta\mathbf{P}\mathbf{y}_0 \tag{42}$$

In solving for $\Delta\mathbf{P}$, one first recalls that $\mathbf{P}$ satisfies Eq (33)

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{BR}^{-1}\mathbf{B}^T + \mathbf{B_v S}^{-1}\mathbf{B_v}^T)\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{33}$$

while $\hat{\mathbf{P}}$ satisfies Eq (40):

$$\hat{\mathbf{P}}(\mathbf{A} + \mathbf{B_v C\Phi}) + (\mathbf{A} + \mathbf{B_v C\Phi})^T\hat{\mathbf{P}} - \hat{\mathbf{P}}\mathbf{BR}^{-1}\mathbf{B}^T\hat{\mathbf{P}} + \mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi} = \mathbf{0} \tag{43}$$

Subtracting Eq (33) from Eq (43) leads to an expression for $\Delta\mathbf{P}$

$$\begin{aligned}\Delta\mathbf{PA} + \mathbf{A}^T\Delta\mathbf{P} + \Delta\mathbf{PB_v C\Phi} + \mathbf{PB_v C\Phi} + (\mathbf{B_v C\Phi} + \mathbf{PB_v C\Phi})^T\Delta\mathbf{P}& \\ + (\mathbf{B_v C\Phi} + \mathbf{PB_v C\Phi})^T\mathbf{P} - \Delta\mathbf{P}(\mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P}) - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}& \\ - \Delta\mathbf{P}(\mathbf{BR}^{-1}\mathbf{B}^T)\Delta\mathbf{P} + \mathbf{\Phi}^T\mathbf{CSC\Phi} + \mathbf{PB_v S}^{-1}\mathbf{B_v}^T\mathbf{P} &= \mathbf{0}\end{aligned}$$

which can also be written in the form of an algebraic Riccati equation

$$\begin{aligned}\Delta\mathbf{P}(\mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{B_v C\Phi}) + (\mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{B_v C\Phi})^T\Delta\mathbf{P}& \\ - \Delta\mathbf{P}(\mathbf{BR}^{-1}\mathbf{B}^T)\Delta\mathbf{P} + \mathbf{E} = \mathbf{0}&\end{aligned} \tag{44}$$

where

$$\mathbf{E} = \mathbf{\Phi}^T\mathbf{CSC\Phi} + \mathbf{\Phi}^T\mathbf{CB_v}^T\mathbf{P} + \mathbf{PB_vC\Phi} + \mathbf{PB_vS}^{-1}\mathbf{B_v}^T\mathbf{P} \tag{45}$$

This Riccati equation determines $\Delta\mathbf{P}$, which in turn determines ΔJ. Standard Riccati solvers find the real, symmetric solution to the algebraic Riccati equation that stabilizes the plant matrix. Now the plant matrix $(\mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{B_vC\Phi})$ is stable. Therefore, if $\mathbf{E} = \mathbf{0}$, then $\Delta\mathbf{P} = \mathbf{0}$ is the only real symmetric solution to Eq (44) [36]. So, driving $\mathbf{E}$ as small as possible yields a small $\Delta\mathbf{P}$ and thus drives $\hat{\mathbf{P}}$ to $\mathbf{P}$. This is highly desirable because the unconstrained standard LQR solution $\mathbf{P}$ is highly robust with respect to initial conditons.

So this approach to an approximate solution is based on finding those damping coefficients that make the matrix $\mathbf{E}$ as close to the zero matrix as possible. Based on Eq (45), E can be written as

$$\begin{aligned}
\mathbf{E} &= \mathbf{\Phi}^T\mathbf{C}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P}) + \mathbf{PB_vS}^{-1}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P}) \\
&= (\mathbf{\Phi}^T\mathbf{C} + \mathbf{PB_vS}^{-1})(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P}) \\
&= (\mathbf{\Phi}^T\mathbf{CS} + \mathbf{PB_v})\mathbf{S}^{-1}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P}) \\
&= [\mathbf{S}^{-1/2}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P})]^T[\mathbf{S}^{-1/2}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P})] \\
&= \mathbf{H}^T\mathbf{H}
\end{aligned} \tag{46}$$

where

$$\mathbf{H} = \mathbf{S}^{-1/2}(\mathbf{SC\Phi} + \mathbf{B_v}^T\mathbf{P})$$

So choosing the damping coefficients such that the diagonal matrix $\mathbf{C}$ makes the matrix product $\mathbf{SC\Phi}$ as close to the matrix product $(-\mathbf{B_v}^T\mathbf{P})$ as one can will make $\mathbf{E}$ small. Since $\|\mathbf{H}^T\mathbf{H}\|_2 = \|\mathbf{H}\|_2^2$, choosing the damping coefficients that minimizes the two-norm of $\mathbf{H}$ minimizes the two-norm of $\mathbf{E}$ and drives ΔJ small. This minimization can be accomplished using the MATLAB routine FMINS, which uses the Nelder-Mead simplex search algorithm [23].

It is interesting to note how similar the matrix $\mathbf{H}$ is to the error term in Eq (36). In fact, minimizing the Frobenius norm of Eq (36) gives the same result as minimizing $\|\mathbf{H}\|_F$. Because $\|\mathbf{H}^T\mathbf{H}\|_F \leq \|\mathbf{H}\|_F^2$, choosing the damping coefficients that minimize the Frobenius norm of $\mathbf{H}$ will minimize an upper bound on the Frobenius norm of $\mathbf{E}$ and thus drive $\Delta J \to 0$. The advantages of the Frobenius minimization over minimizing $\|\mathbf{H}\|_2$ are that it is guaranteed to return positive damping coefficients and it can be solved in closed form (see Appendix A). As will be shown later, the latter advantage makes the Frobenius minimization computationally more efficient.

In this section, two methods of obtaining an approximate solution to the optimization problem posed in Section 2.2 have been derived. One is a minimum singular value method (minimize $\|\mathbf{H}\|_2$), and the other is a Frobenius norm minimization (minimize $\|\mathbf{H}\|_F$). However, these are ***approximate*** solutions. It is desirable to achieve an actual optimization. In the next two subsections, the performance index is redefined in two different ways, and the minimums of the resulting cost functionals sought.

2.2.2 Solution Technique for Minimizing the Mean Performance lex. In each of the next two subsections, a cost functional is defined based on the form of the performance index given in Eq (41). The cost functional of the next section is the maximum value of the performance index. In this subsection, a cost functional is the average of the performance index over the initial conditions. Since the magnitude of the initial conditions is irrelevant to the value of viscous damping coefficients **C** and the corresponding solution to the modified Riccati equation, Eq (40), initial conditions used in the next two subsections are those whose magnitude is unity, in other words, the unit ball $\|\mathbf{y}_0\| = 1$.

Minimizing the average value of the performance index, denoted by $\overline{J}$, over the unit ball $\|\mathbf{y}_0\| = 1$ does have the potential to lead to large values of the performance index for some initial conditions, but the large values will be offset by low values of the performance index at other initial conditions. Since the damping coefficients and active control gains that are calculated will give the lowest average value of the performance index, one would expect that this solution will give better performance over more than half of the initial condition compared to any other combination of damping coefficients and active control gains.

The average value of the performance index will be calculated by integrating the performance index over the surface of the unit ball, then dividing by the surface area of the unit ball. First the performance index will be expressed in terms of the elements of the initial state vector and the solution to the Riccati equation. Let $\mathbf{y}_0 = [y_1 \; y_2 \; \cdots \; y_n]^T$. Since **P** is symmetric, the performance index is

$$J = \frac{1}{2}\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0 = \frac{1}{2}\sum_{i}^{n}\sum_{j=1}^{n} P_{ij}y_iy_j = \frac{1}{2}\sum_{i=1}^{n} P_{ii}y_i^2 + \sum_{i}^{n}\sum_{j=i+1}^{n} P_{ij}y_iy_j \tag{47}$$

The expression for the average value of the performance index is

$$\overline{J} = \frac{\int_{\|\mathbf{y}_0\|=1} \left(\frac{1}{2} \sum_{i=1}^{n} P_{ii} y_i^2 + \sum_{i}^{n} \sum_{j=i+1}^{n} P_{ij} y_i y_j \right) dA}{\int_{\|\mathbf{y}_0\|=1} dA}$$

Since the sums are finite, the integrals and summations can be interchanged:

$$\overline{J} = \frac{\left[\frac{1}{2} \sum_{i=1}^{n} P_{ii} \left(\int_{\|\mathbf{y}_0\|=1} y_i^2 \, dA \right) + \sum_{i}^{n} \sum_{j=i+1}^{n} P_{ij} \left(\int_{\|\mathbf{y}_0\|=1} y_i y_j \, dA \right) \right]}{\int_{\|\mathbf{y}_0\|=1} dA} \tag{48}$$

To simplify the evaluation of the integrals in Eq (48), Gauss' divergence theorem is used to express the surface integals as volume integrals [19:355]:

$$\int \mathbf{n} \cdot \mathbf{u} \, dA = \int \operatorname{div} \mathbf{u} \, dV \tag{49}$$

Let $\mathbf{e}_k$ represent the unit vector in the direction of the k_{th} coordinate. Then the unit vector normal to the surface of the unit ball is $\mathbf{n} = \sum_{k=1}^{n} y_k \mathbf{e}_k$. The vector $\mathbf{u}$ must be determined for each of the three integrals in Eq (48). (Please note that in this discussion, $\mathbf{u}$ is some continuous vector function, *not* the active control vector.) If $\mathbf{n} \cdot \mathbf{u}$ is taken to be y_i^2, one can infer that $\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{e}_k = y_i \mathbf{e}_i$. Therefore the divergence of the vector $\mathbf{u}$ is unity:

$$\operatorname{div} \mathbf{u} = \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_k} = \frac{\partial y_i}{\partial y_i} = 1 \tag{50}$$

So evaluation of the first integral in the numerator in Eq (48) is the volume of the unit ball:

$$\int_{\|\mathbf{y}_0\|=1} y_i^2 \, dA = \int_{\|\mathbf{y}_0\|=1} dV = V \tag{51}$$

In the second integral in the numerator of Eq (48), $\mathbf{n} \cdot \mathbf{u}$ is taken to be $y_i y_j$. It follows that $\mathbf{u} = \frac{1}{2}(y_i \mathbf{e}_j + y_j \mathbf{e}_i)$. Then the divergence of $\mathbf{u}$ is zero:

$$\operatorname{div} \mathbf{u} = \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_k} = \frac{\partial(\frac{1}{2} y_j)}{\partial y_i} + \frac{\partial(\frac{1}{2} y_i)}{\partial y_j} = 0 \tag{52}$$

Hence the second integral in the numerator of Eq (48) integrates to zero:

$$\int_{\|\mathbf{y}_0\|=1} y_i y_j \, dA = 0 \tag{53}$$

The integral in the denominator of Eq (48) is just the surface area of the unit ball, but it can be expressed in terms of the volume. In the integral, the quantity $\mathbf{n} \cdot \mathbf{u}$ is unity:

$$\mathbf{n} \cdot \mathbf{u} = 1$$

It is sufficient that $\mathbf{u} = \mathbf{n}$. Thus the divergence of $\mathbf{u}$ is equal to the number of terms in the state vector:

$$\operatorname{div} \mathbf{u} = \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_k} = \sum_{k=1}^{n} \frac{\partial y_k}{\partial y_k} = n \tag{54}$$

Thus the surface area of the unit ball is n times the volume:

$$\int_{\|\mathbf{y}_0\|=1} dA = \int_{\|\mathbf{y}_0\|=1} n \, dV = nV \tag{55}$$

Combining Eqs (48), (51), (53), and (55),

$$\overline{J} = \frac{\left[\frac{1}{2}\sum_{i=1}^{n} P_{ii} \cdot V + \sum_{i}^{n} \sum_{j=i+1}^{n} P_{ij} \cdot 0\right]}{nV} = \frac{\frac{1}{2}V}{nV} \sum_{i=1}^{n} P_{ii} \tag{56}$$

$$= \frac{1}{2n} \operatorname{trace} \mathbf{P} \tag{57}$$

Hence, choosing the viscous parameters $\mathbf{C}$ that minimize the trace of the matrix $\mathbf{P}$ minimizes the average value of the performance index J over the unit ball $\|\mathbf{y}_0\| = 1$, regardless of system order.

This subsection has derived a technique for minimizing the average value of the cost functional over a representative sample of initial conditions. The next subsection will present another technique for optimizing viscous damping and active control simultaneously.

2.2.3 Solution for Minimizing the Maximum Performance Index. Another approach to optimizing viscous damping and active control simultaneously is to minimize the maximum value of the performance index J over the unit ball $\|\mathbf{y}_0\| = 1$. One way to accomplish this is to minimize the two-norm of $\mathbf{P}$, since the performance index is less than or equal to the two-norm of $\mathbf{P}$:

$$J = \frac{1}{2}|\mathbf{y}_0^T \mathbf{P} \mathbf{y}_0| \leq \frac{1}{2}||\mathbf{y}_0||^2 ||\mathbf{P}||_2 = \frac{1}{2}||\mathbf{P}||_2 \tag{58}$$

The quantity on the right hand side of the above equation will be referred to as J_{max}:

$$J_{max} = \frac{1}{2}||\mathbf{P}||_2$$

In this method, iteration with respect to the viscous parameters $\mathbf{C}$ is carried out until $||\mathbf{P}||_2$ is minimized where $\mathbf{P}$ satisfies Eq (40). This approach might be very conservative in general, since the initial conditions $\mathbf{y}_0$ that maximize the performance index may not be encountered very often.

The last three subsections have produced four solution techniques that are independent of initial conditions. In summary, the four solution methods are

1. min $||\mathbf{H}||_F$: Closed form solution based on finding a diagonal $\mathbf{C}$ that minimizes the error in the performance index with respect to the Frobenius norm. See Eqs (40)-(46).
2. min $||\mathbf{H}||_2$: Iterative solution technique based on finding a diagonal $\mathbf{C}$ that minimizes the error in the performance index with respect to the two-norm. See Eqs (40)-(46).
3. min $||\mathbf{P}||_2$: Iterative solution technique based on finding a diagonal $\mathbf{C}$ that minimizes the maximum value of the performance index. See Eqs (40) and (58).
4. min (trace $\mathbf{P}$): Iterative solution technique based on finding a diagonal $\mathbf{C}$ that minimizes the average value of the performance index. See Eqs (44) and (47)-(57).

The minimization of $||\mathbf{H}||_F$ and the minimization of $||\mathbf{H}||_2$ do not guarantee a positive solution for the damping parameters. The same applies to minimizing trace($\mathbf{P}$) and $||\mathbf{P}||_2$. However, as will be seen later, positive damping parameters are obtained in all the example problems. Following are example problems which illustrate the above techniques. The example problems are also used to compare the different solution techniques.

2.2.4 Example Problems.

2.2.4.1 Example #1 - Single DOF System. In this example problem, single degree of freedom spring-mass-damper system, illustrated in Figure 1, is used to

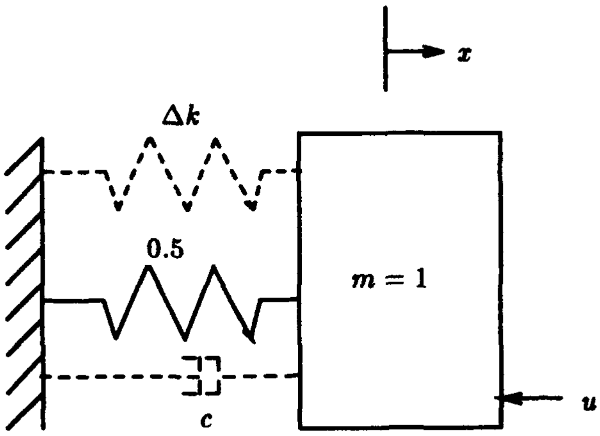

Figure 1. Simple Spring-Mass-Damper System

demonstrate and compare the four solution techniques derived in this section. The equation of motion is given by the scalar equation

$$\ddot{x} + c\dot{x} + (0.5 + \Delta k)x = -u \tag{59}$$

where Δk and c are the passive design parameters, representing change in stiffness and viscous damping respectively. Letting $y_1 = x, y_2 = \dot{x}$, then

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix} \mathbf{y} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} u + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta k & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}$$

$$= \mathbf{A}\mathbf{y} + \mathbf{B}u + \mathbf{B}_v \mathbf{C} \mathbf{\Phi} \mathbf{y}$$

The weighting matrices are chosen to give equal weighting on position, active control, and viscous damping:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1, \quad \mathbf{S} = \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix} \tag{60}$$

The weight on the change in stiffness, Δk, is chosen large to assure negligible change in stiffness ($\Delta k \approx 0$). This isolates damping as the design parameter of interest.

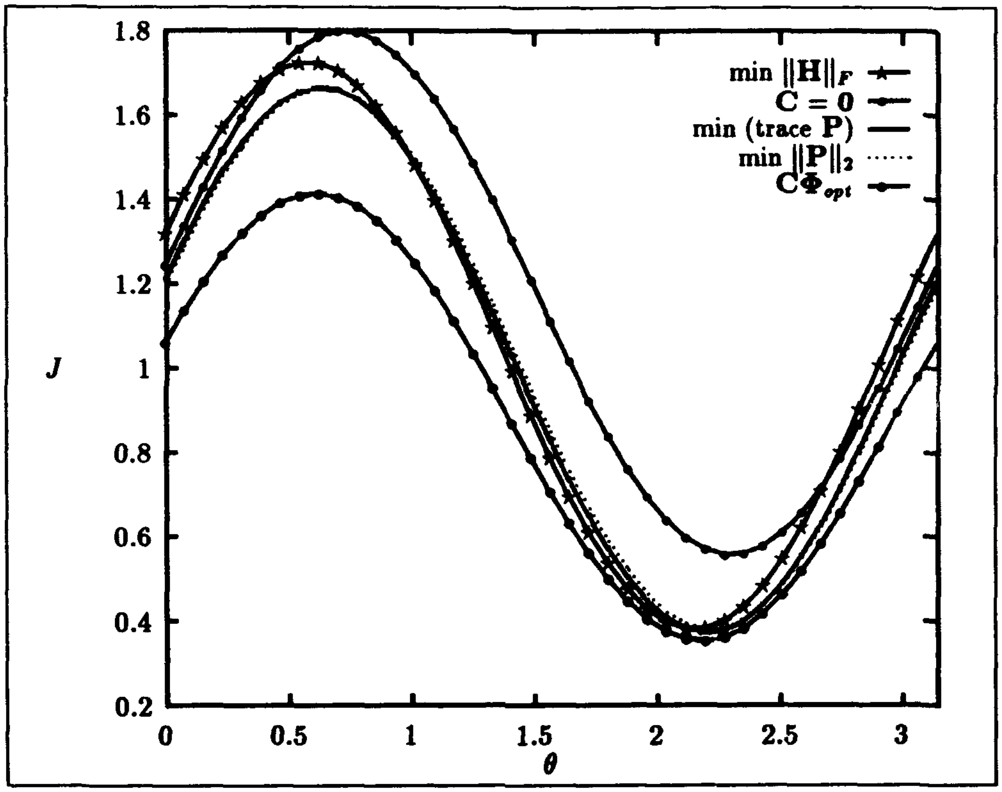

Figure 2. Performance Index vs. Initial Conditions

The optimal unconstrained solution of Section 2.2 (i.e., the solution of Eq (35)) for this problem yields a non-diagonal parameter matrix:

$$\mathbf{C} = \begin{bmatrix} 0.000 & 0.000 \\ .500 & .707 \end{bmatrix} \tag{61}$$

For the single degree of freedom case, $\|\mathbf{y}_0\| = 1$ can be represented by a single parameter θ where $\mathbf{y}_0 = [\cos\theta \;\; \sin\theta]^T$. The value of the performance index for the optimal unconstrained parameter matrix is shown as a solid line with circles in Figure 2. Also included for comparison is the result for purely active vibration control (i.e., C=0).

Table 1 summarizes various results for the single degree of freedom system. For this example, the results for the alternative suboptimal solutions, the min $\|\mathbf{H}\|_2$ and min $\|\mathbf{H}\|_F$ cases, are the same to two decimal places, but in general the results will be different. Although the damping coefficients for the min $\|\mathbf{H}\|_2$ and min $\|\mathbf{H}\|_F$ cases are nearly twice those for for the min $\|\mathbf{P}\|_2$ and min (trace P) cases, they nonetheless serve as reasonable

Table 1. Damping Parameters and Feedback Matrices for Example #1

	c	feedback matrix
min $\|\|\mathbf{P}\|\|_2$	.395	[11.6679 4.4679]
min trace **P**	.455	[13.4997 4.7807]
min $\|\|\mathbf{H}\|\|_F$	.707	[21.2052 5.8816]
min $\|\|\mathbf{H}\|\|_2$	.707	[21.2052 5.8816]

first guesses for these solution techniques. The importance of this in higher order systems will become evident in Example #3.

For the single degree of freedom problem, minimizing $||\mathbf{P}||_2$ resulted in a diagonal **C**:

$$\mathbf{C} = \begin{bmatrix} 0.000 & 0 \\ 0 & .395 \end{bmatrix} \tag{62}$$

The result is plotted in Figure 2. The graph shows this solution to be more robust than the purely active solution. For at least one initial condition, any other value of damping coefficient (or change in stiffness) will produce a larger value of the performance index than the maximum of this solution.

Minimizing the trace of **P** for Example #1 gave the following diagonal parameter matrix:

$$\mathbf{C} = \begin{bmatrix} 0.000 & 0 \\ 0 & .455 \end{bmatrix} \tag{63}$$

These results are also shown in Figure 2. Although this graph does have points above the one for min $||\mathbf{P}||_2$, the average value of the performance index is lower for this solution than for any other. The min $||\mathbf{H}||_2$ and the min $||\mathbf{H}||_F$ solutions are equal to two decimal places, so only the min $||\mathbf{H}||_F$ solution is shown on Figure 2.

In this example, the different solution techniques were compared using the original performance index (Eq (15)). In the following example, three solution techniques, min $||\mathbf{H}||_F$, min $||\mathbf{P}||_2$, and min (trace **P**) are compared using the modified performance indices introduced in Section 2.2.2.

2.2.4.2 Example #2 – Single DOF System with No Additional Stiffness. In the example just presented, the change in stiffness (Δk) was made small by choosing the corresponding weight large. The following example constrains the change in stiffness to be zero from the outset ($\Delta k \equiv 0$). The scalar equation of motion is

$$\ddot{x} + c\dot{x} + 0.5x = -u$$

which leads to a first order system in which the parameter matrix is a scalar and the matrix $\mathbf{\Phi}$ is nonsquare:

$$\begin{aligned} \dot{\mathbf{y}} &= \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix} \mathbf{y} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} u + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} c \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{y} \\ &= \mathbf{A}\mathbf{y} + \mathbf{B}u + \mathbf{B_v} c \mathbf{\Phi} \mathbf{y} \end{aligned} \tag{64}$$

Before giving the weightings on the state, control, and passive damping, a short discussion on how to choose the weightings is appropriate. The weightings determine the relative importance of driving the state to zero, the expenditure of control energy, and the absorption of damping energy [26:782]. In the the previous example, no weighting was placed on velocity, implying that driving velocity to zero was not important. Choosing the weightings on the state, control, and passive damping is "more of an art than a science" [18:219]. In this example, the weighting matrices are chosen to minimize energy. Hence, velocity is weighted more heavily than position since the magnitude of the mass is greater than the magnitude of the stiffness. Active control and passive damping are weighted equally, and less than position:

$$\mathbf{Q} = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = .1, \quad S = .1 \tag{65}$$

In practice, the design engineer may have to experiment with different weightings to get a design that is appropriate to his needs. For example, if the weightings on active control and passive damping are too large, the damping ratios may be unacceptably low. If the weightings on active control and passive damping are too small, the amount of control and damping material required may be unacceptably high. For a more detailed discussion of the choice of weighting matrices, see Reference [2].

Table 2. Active Control Feedback Gains for SDOF System

c	feedback matrices
2.52	[1.72 2.61]
0.89	[1.72 3.00]
0.47	[1.72 3.27]

Table 3. Comparison of Solutions for SDOF System

c	2.52	0.89	0.47
trace **P**	1.31	1.15	1.16
$\|\mathbf{P}\|_2$	1.09	0.90	0.89

In this case, the optimal unconstrained solution of Section 2.2 (i.e., the solution of Eq (34)) for this problem yields a 1×2 matrix:

$$c\mathbf{\Phi} = \begin{bmatrix} 1.35 & 2.52 \end{bmatrix}$$

The Frobenius norm solution yields $c = 2.52$, while minimizing trace **P** gives $c = 0.89$, and minimizing $\|\mathbf{P}\|_2$ gives $c = 0.47$. The active control feedback gains are given in Table 2.

The disparity of the values for the damping parameter c indicate that there is some insensitivity to the magnitude of the damping parameter c. This is supported by the fact that there is little difference in trace **P** and $\|\mathbf{P}\|_2$ for the three solution techniques (Table 3). Figure 3 shows that there is little change in $\|\mathbf{P}\|_2$ due to the value of c between $c = 0$ and $c = 1$. The graph of the trace of **P** is varies by less than 0.05 between $c = 0.5$ and $c = 1.5$.

2.2.4.3 Example #3 - Fourteen DOF System. In this example, a more realistic and more complex structure is considered, the plane aluminum truss in Figure 4. The finite element model equations of motion can be written in the state space form:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + \mathbf{B}_{\mathbf{v}}\mathbf{C}\mathbf{\Phi}\mathbf{y} \tag{66}$$

Since the fourteen physical degrees of freedom lead to twenty-eight mathematical degrees of freedom in a dynamical system, the dimension of state vector **y** is 28×1. There are four control actuators and five dashpots on the truss, so the dimension of the active control vector **u** is 4×1 and the dimension of the damping parameter matrix **C** is 5×5. The dimensions of the remaining matrices are given as follows: plant matrix **A**, 28×28; active

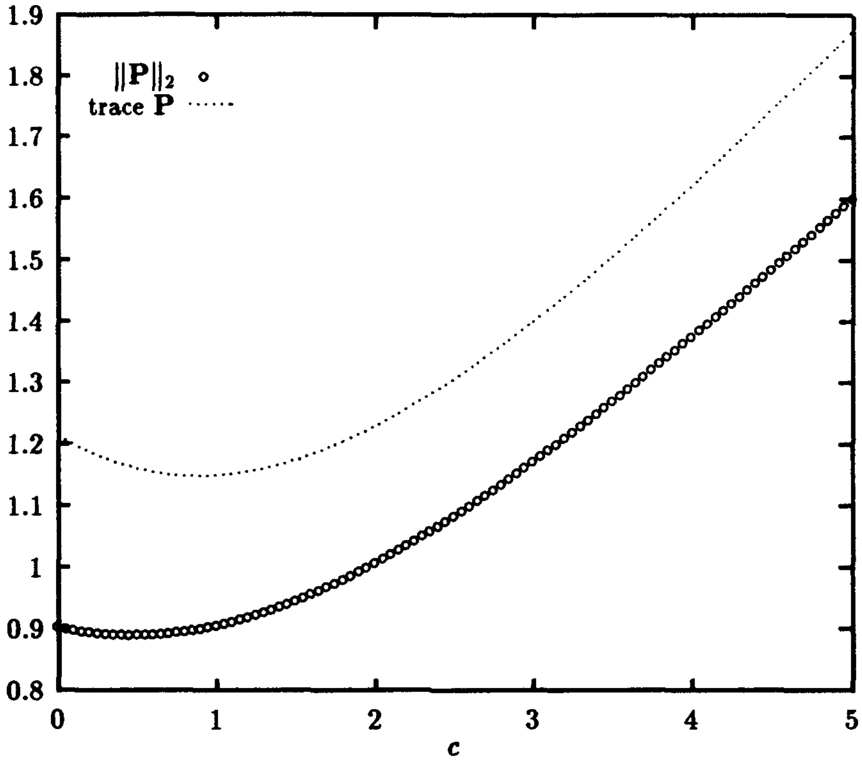

Figure 3. Example Problem #2 — Trace **P** and $\|\mathbf{P}\|_2$ vs. c

control input matrix **B**, 28×4; passive force input matrix $\mathbf{B_v}$, 28×5; and $\mathbf{\Phi}$, 5×28. With E representing the elastic modulus of aluminum, A the cross sectional area of the aluminum elements, and ρ the density of the aluminum, the parameters of the system are

$$EA = \quad 1.5179 \times 10^6 \text{ N} \quad (3.4125 \times 10^5 \text{ lb})$$
$$\rho A = \quad 8.3564 \text{ N-s}^2/\text{m}^2 \quad (1.212 \times 10^{-3} \text{ lb-s}^2/\text{in}^2)$$

The length of the horizontal elements is 4.572 m (15 ft) while the vertical elements are 3.048 m (10 ft). The length of the two elements with dampers c_4 and c_5 is 4.819 m (15.81 ft). Thus, the diagonal elements with dampers c_1, c_2, and c_3 are 5.495 m (18.03 ft) in length. In addition to the consistent mass matrix due to consideration of the mass of the aluminum elements, there are 0.454 kg (1 lbm) point masses at every node, with an 0.907 kg (2 lbm) point mass at the tip. The undamped natural frequencies range from 7.7 rad/sec to 319.3 rad/sec (1.2 hz to 50.8 hz).

The state weighting matrix **Q** is chosen such that $\mathbf{y}^T\mathbf{Q}\mathbf{y}$ equals the total mechanical energy in the system. Thus **Q** is formed using the mass and stiffness matrices, **M** and **K**.

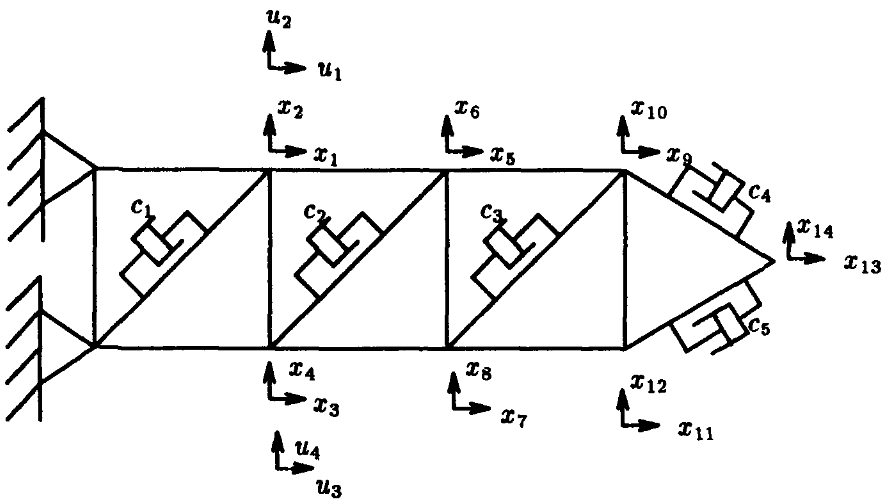

Figure 4. Example Problem #3 Truss Structure

The active control weighting matrix is chosen such that its diagonal elements are smaller than the smallest diagonal value of **Q**.

In the previous two examples, the weighting on active control and passive damping were equal. In Example #1, it was shown that using simultaneously designed active feedback control and passive damping gave better performance than using active control alone. In general, one would expect that passive damping is cheaper than active control – it doesn't need an actuator or power for the actuator. Hence for this example, it is assumed that active control is more costly than passive damping, so the passive damping weighting matrix S is chosen to be 10^{-3} times identity, and the active control weighting matrix **R** is chosen to be 10^{-2} times identity:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad \mathbf{R} = 10^{-2}\mathbf{I}, \quad \mathbf{S} = 10^{-3}\mathbf{I} \tag{67}$$

Using the weightings given above gives the resulting damping parameters for the different solution techniques in Table 4. At the bottom of Table 4 are the approximate computation times on a VAX 6420. The solution techniques can be run with simple MATLAB routines. The Frobenius norm minimization uses the algorithm given in Appendix A, while the other minimizations utilized the MATLAB routine FMINS, using default tolerances in the minimization routines. The minimization algorithm used in FMINS is the Nelder-Mead simplex search algorithm [23].

Table 4. Damping Parameters and CPU Times for 14 DOF Truss

		min $\|\|\mathbf{H}\|\|_F$	min $\|\|\mathbf{H}\|\|_2$	$\min_0 \|\|\mathbf{H}\|\|_2$	
dashpot constants	c_1	60.0169	59.6515	59.6514	
	c_2	41.7915	46.2314	46.2314	
	c_3	41.9507	14.7370	14.7372	
	c_4	57.8418	59.2816	59.2816	
	c_5	55.2145	45.9636	45.9636	
computation time		17 s	34 s	61 s	
		min trace $\mathbf{P}$	$\min_0$ trace $\mathbf{P}$	min $\|\|\mathbf{P}\|\|_2$	$\min_0 \|\|\mathbf{P}\|\|_2$
dashpot constants	c_1	54.3826	54.3826	76.1842	76.1839
	c_2	35.4502	35.4503	17.5549	17.5542
	c_3	38.4443	38.4443	28.5870	28.5873
	c_4	42.1846	42.1846	34.2785	34.2786
	c_5	46.8419	46.8420	45.5261	45.5255
computation time		22 min	52 min	38 min	78 min

The zero subscript on the minimization operator (e.g., $\min_0$) indicates that an initial guess of $\mathbf{C} = \mathbf{0}$ was used. The other cases used the damping coefficients of the min $\|\mathbf{H}\|_F$ solution as the initial guess. Using the min $\|\mathbf{H}\|_F$ solution as an initial guess significantly decreased the computation time, by more than one half in some cases. The active gain feedback matrices are given in Appendix F. The results in Table 4 clearly demonstrate that if one chooses not to spend the additional computer time to optimize the dampers that the inexpensive min $\|\mathbf{H}\|_F$ solution is a reasonably close estimate. For very large systems this solution may be the only affordable alternative.

In fact, as shown in Table 5, the min $\|\mathbf{H}\|_F$ solution gave almost identical closed loop damping ratios and natural frequencies as the min (trace $\mathbf{P}$) and min $\|\mathbf{P}\|_2$ solutions for this example. Also shown in Table 5 are the open loop poles of the system (no active or passive control). The average change in frequency due to adding viscous damping and active feedback damping was about 3 rad/sec.

Since it is not possible to show a two-dimensional plot of J over the unit ball as in Example 1, another method of comparing the solutions is needed. It would be useful to have a way to compare the solutions with respect to the original performance index, Eq (15), regardless of system order. To compare the solution given in Table 4, recall from Eq (42) that the difference in J between two different solutions can be given by

$$\Delta J = \mathbf{y}_0^T(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{y}_0 \tag{68}$$

Table 5. Damping Ratios and Natural Frequencies for 14 DOF Truss

undamped frequency (rad/s)	min $\|\mathbf{H}\|_F$		min trace $\mathbf{P}$		min $\|\mathbf{P}\|_2$	
	ζ	ω_n (rad/s)	ζ	ω_n (rad/s)	ζ	ω_n (rad/s)
319.3471	0.1249	315.7222	0.1077	316.8960	0.0712	316.9671
290.2645	0.0790	281.1909	0.0763	283.5066	0.0534	288.0130
281.1347	0.0728	279.7743	0.0620	279.4916	0.0729	275.2599
271.9703	0.0417	265.4633	0.0406	267.0294	0.0377	267.9009
264.1012	0.0282	258.9797	0.0297	260.8798	0.0327	262.1410
236.3806	0.1376	245.6173	0.0965	243.8296	0.0857	243.0665
223.1620	0.0853	224.8248	0.0728	224.4934	0.0693	224.1446
165.0056	0.1353	167.7850	0.1166	167.0317	0.1008	166.6382
150.4895	0.1090	153.7449	0.0862	152.4666	0.0811	152.1203
86.1426	0.0884	86.4941	0.0792	86.4090	0.0682	86.2700
64.8809	0.1016	65.2589	0.0952	65.1966	0.0962	65.1802
54.9372	0.0341	55.0353	0.0331	55.0221	0.0337	55.0725
31.9937	0.1096	32.2309	0.1087	32.2221	0.1103	32.2677
7.7196	0.0986	7.7581	0.0986	7.7580	0.0986	7.7582

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are two different solutions. If the matrix $(\mathbf{P}_1 - \mathbf{P}_2)$ is positive definite, then $\Delta J > 0$ for all $\mathbf{y}_0$. Hence, $\mathbf{P}_2$ would give a lower value of J than $\mathbf{P}_1$ for all $\mathbf{y}_0$. As shown in Table 6, the min $\|\mathbf{H}\|_F$ solution ($\mathbf{P}_F$), the min (trace $\mathbf{P}$) solution ($\mathbf{P}_{tr}$), and the min $\|\mathbf{P}\|_2$ solution ($\mathbf{P}_2$) give lower values of J for all $\mathbf{y}_0$ than the solution using only active damping ($\mathbf{P}_{c0}$); hence, for this example, they are all better solutions than using only active control. Table 6 also shows that $\mathbf{P}_{tr}$ gives a lower value of J than $\mathbf{P}_2$ in fourteen directions, and a higher value of J in six directions. Since most of the positive eigenvalues of $(\mathbf{P}_2 - \mathbf{P}_{tr})$ are greater in magnitude than the negative eigenvalues, it is argued that $\mathbf{P}_{tr}$ is a better overall solution than $\mathbf{P}_2$. This is consistent with the derivation of the min (trace $\mathbf{P}$) and min $\|\mathbf{P}\|_2$ solutions since min (trace $\mathbf{P}$) minimizes the average of the performance index whereas min $\|\mathbf{P}\|_2$ only minimizes the performance index for the worst case. So one would expect that the min trace $\mathbf{P}$ solution would be a better solution for the majority of initial state vector directions.

Another method of comparing the solutions is comparing the relative differences between the solutions and the optimal, unconstrained solution (Eqs (11), (33)), and (34)). Table 7 gives the relative differences between the performance index attained with the optimal, unconstrained solution and the performance indices attained with the three solution techniques. All three solution techniques resulted in increases in the performance index

Table 6. Eigenvalues of Delta P for 14 DOF Truss

$\mathbf{P}_{c0} - \mathbf{P}_F$	$\mathbf{P}_{c0} - \mathbf{P}_{tr}$	$\mathbf{P}_{c0} - \mathbf{P}_2$	$\mathbf{P}_2 - \mathbf{P}_{tr}$	$\mathbf{P}_F - \mathbf{P}_{tr}$	$\mathbf{P}_F - \mathbf{P}_2$
50062.43	50235.92	50178.14	592.16	778.51	1092.02
31627.88	31654.71	31349.79	468.76	171.34	-501.75
24620.13	24776.08	24465.41	-324.75	111.50	-608.94
11273.39	11227.77	10975.97	215.36	-157.09	-268.53
9964.37	10089.19	9818.28	185.94	-116.79	-237.21
9061.48	9043.97	8936.96	138.59	-69.31	-177.37
7178.83	7266.41	7214.76	122.97	31.17	-168.83
6473.13	6536.19	6495.66	-39.70	-48.81	-108.45
6114.26	6326.30	6265.98	87.88	-43.49	65.78
1879.97	1833.14	1707.37	79.01	-31.22	-79.37
751.17	661.39	683.74	52.84	-16.95	23.78
449.52	416.79	371.62	25.30	-5.84	-24.24
60.82	54.98	60.69	-5.85	2.45	-0.30
1.81	1.60	1.53	0.07	-0.21	0.17
0.61	0.61	0.61	-0.03	-0.06	-0.07
0.40	0.39	0.35	-0.03	0.01	-0.03
0.35	0.35	0.39	0.03	-0.01	-0.02
0.03	0.03	0.02	0.01	0.01	0.01
0.06	0.28	0.28	-0.01	-0.01	0.01
0.07	0.05	0.23	0.01	-0.01	-0.01
0.28	0.24	0.22	0.00	0.00	-0.01
0.10	0.23	0.20	0.00	0.00	0.00
0.16	0.08	0.06	0.00	0.00	0.00
0.21	0.09	0.11	0.00	0.00	-0.01
0.14	0.20	0.13	0.00	0.00	0.00
0.24	0.12	0.08	0.00	0.00	0.00
0.24	0.16	0.16	0.00	0.00	0.00
0.12	0.14	0.08	0.00	0.00	0.00

Table 7. Example #3 Relative Increase in Performance Index

	$\|\|\mathbf{H}\|\|_F$	trace(P)	$\|\|\mathbf{P}\|\|_2$
$\frac{J-J_{opt}}{J_{opt}}$	0.1276	0.1120	0.1534

Table 8. Relative Changes in Eigenvalues

ζ	ω_n	$\|\|\mathbf{H}\|\|_F$	trace(P)	$\|\|\mathbf{P}\|\|_2$
0.1169	7.7733	0.0185	0.0185	0.0185
0.1490	32.3715	0.0398	0.0408	0.0391
0.0714	55.0899	0.0373	0.0383	0.0377
0.1486	65.5923	0.0476	0.0539	0.0530
0.1415	87.0082	0.0537	0.0629	0.0739
0.1097	151.5400	0.0146	0.0244	0.0290
0.1237	166.3686	0.0145	0.0082	0.0232
0.0791	224.1759	0.0069	0.0065	0.0098
0.0841	238.6824	0.0619	0.0250	0.0184
0.0396	263.7692	0.0214	0.0147	0.0092
0.0595	271.9458	0.0296	0.0260	0.0262
0.0640	282.0354	0.0118	0.0093	0.0256
0.0826	290.0382	0.0307	0.0234	0.0300
0.0936	320.6543	0.0348	0.0183	0.0251

of less than 20%; minimizing the trace of the solution to the algebraic Riccati equation resulted in the lowest increase, 11%.

Table 8 gives a measure of how close the attainable system is to the optimum system produced by the optimal, unconstrained solution. It does this by computing the eigenvalues according to the formula

$$\frac{|\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A}_{opt})|}{|\lambda_i(\mathbf{A}_{opt})|}$$

The damping ratio and natural frequency listed in the table are those associated with the optimum system. Minimizing the trace of **P** is closer to the optimum solution than the other two solution techniques at higher frequencies. At lower frequencies, it is not clear whether the Frobenius norm solution or the minimum two-norm solution is closer to the optimum solution; both are closer than the minimum trace solution. Table 9 lists the changes in eigenvalues in increasing order. Looking at this table, it is clear that the largest eigenvalue change is 7% and occurs in the minimum two-norm solution. Tables 7-9 indicate that the solution techniques are good approximations to the the optimal, unconstrained solution.

Table 9. Example #3 Relative Changes in Eigenvalues in Increasing Order

$\|\mathbf{H}\|_F$	trace(**P**)	$\|\mathbf{P}\|_2$
0.0069	0.0065	0.0092
0.0118	0.0082	0.0098
0.0145	0.0093	0.0184
0.0146	0.0147	0.0185
0.0185	0.0183	0.0232
0.0214	0.0185	0.0251
0.0296	0.0234	0.0256
0.0307	0.0244	0.0262
0.0348	0.0250	0.0290
0.0373	0.0260	0.0300
0.0398	0.0383	0.0377
0.0476	0.0408	0.0391
0.0537	0.0539	0.0530
0.0619	0.0629	0.0739

In this example, the different solution techniques were demonstrated for a fourteen degree of freedom truss. A method of comparing the different solutions was developed that is independent of system order and initial conditions. Before proceeding to the case in which the damping is provided by viscoelastic materials, an alternate form of the cost functional will be touched on in the next section.

2.3 Alternate Form of The Performance Index

The cost functional of Eq (15) treats the passive damping forces as if they were similar to the active damping forces, as do the following chapters on viscoelastic damping. This section takes a short side trip to address the argument that passive damping is an initial one-time cost item and ought to be weighted as such. Hence, the weighting on damping is accomplished by weighting the damping constants directly:

$$J = \frac{1}{2}\mathbf{c}^T\mathbf{S}\mathbf{c} + \int_{t_0}^{\infty} \frac{1}{2}\left(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{69}$$

The equations of motion are of the same form as Eq (39):

$$\dot{\mathbf{y}} = (\mathbf{A} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi})\mathbf{y} + \mathbf{B}\mathbf{u} \tag{39}$$

The vector **c** in Eq (69) is a vector of the passive damping parameters and the matrix **C** again is a diagonal matrix of the damping parameters (i.e., the elements of **c**). Regardless of the values of the damping parameters, the minimum of the integral term is given by $\frac{1}{2}\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0$ where the matrix $\mathbf{P}$ satisfies

$$\mathbf{P}(\mathbf{A}+\mathbf{B}_v\mathbf{C}\boldsymbol{\Phi})+(\mathbf{A}+\mathbf{B}_v\mathbf{C}\boldsymbol{\Phi})^T\mathbf{P}-\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{Q}=\mathbf{0} \tag{70}$$

Hence the performance index can be written as the sum of two terms:

$$J=\frac{1}{2}(\mathbf{y}_0\mathbf{P}\mathbf{y}_0+\mathbf{c}^T\mathbf{S}\mathbf{c}) \tag{71}$$

To obtain solutions independent of initial conditions, the same approach used in Section 2.2.2 will be used here. Since

$$J=\frac{1}{2}|\mathbf{y}_0\mathbf{P}\mathbf{y}_0+\mathbf{c}^T\mathbf{S}\mathbf{c}|\leq\frac{1}{2}(\|\mathbf{y}_0\|^2\|\mathbf{P}\|_2+\mathbf{c}^T\mathbf{S}\mathbf{c})=\frac{1}{2}(\|\mathbf{P}\|_2+\mathbf{c}^T\mathbf{S}\mathbf{c})$$

the maximum value of the performance index is minimized for $\|\mathbf{y}_0\|=1$ by minimizing

$$J_{max}=\frac{1}{2}(\|\mathbf{P}\|_2+\mathbf{c}^T\mathbf{S}\mathbf{c}) \tag{72}$$

where $\mathbf{P}$ is found by solving Eq (70). This is done by iterating with respect to the viscous damping parameters $\mathbf{c}$ in the same manner as in Section 2.2.2.

To minimize the average value of the performance index minimize

$$\overline{J}=\frac{\frac{1}{2}\int_{\|\mathbf{y}_0\|=1}\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0\,dA}{\int_{\|\mathbf{y}_0\|=1}dA}+\frac{\frac{1}{2}\int_{\|\mathbf{y}_0\|=1}\mathbf{c}^T\mathbf{S}\mathbf{c}\,dA}{\int_{\|\mathbf{y}_0\|=1}dA} \tag{73}$$

$$=\frac{1}{2n}\operatorname{trace}\mathbf{P}+\frac{1}{2}\mathbf{c}^T\mathbf{S}\mathbf{c} \tag{74}$$

where $\mathbf{P}$ is again given by Eq (70). Here also, one needs to iterate with respect to the viscous damping parameters $\mathbf{c}$ in the same manner as in Section 2.2.2.

For both of the above solution techniques, the form of the active control is the same as with the original performance index Eq (15):

$$\mathbf{u}=-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} \tag{11}$$

An example demonstrating both of the above solution techniques follows.

Table 10. Dashpot Constants and Performance Index Values for Example #4

Solution Technique		min $\bar{J}$	min J_{max}
	c_1	130.1174	132.6323
dashpot	c_2	76.3213	290.2810
constants	c_3	84.9125	7.1011
	c_4	63.8805	33.1698
	c_5	77.5260	60.7109
$\bar{J}$		484	648
J_{max}		4,587	4,180

2.3.1 Example #4 - Fourteen DOF with Alternate Version of the Performance Index. This example uses the same structure as Example #3, the aluminum truss shown in Figure 4. The same weighting matrix for control was used. The magnitude of the weighting matrix **S** for the damping coefficients is the same as in Example #3, but the dimensions are different since the damping coefficients are being weighted directly. The two solutions cannot be compared in the same manner as the solutions in Example #3 due to the form of the performance index (refer to Eq (71)), so the values of $\bar{J}$ and J_{max} will be compared, as was done in Example #2.

The solution techniques were run with modified versions of the MATLAB routines used in Example #3. An initial guess of $\mathbf{c} = \mathbf{0}$ was used since a closed form solution was not obtained for this form of the performance index (Eq (69)).

The resulting damping parameters for the min $\bar{J}$ and min J_{max} solutions are given in Table 10. The dashpot constants are generally much greater in magnitude than those found in Example #3, even though the magnitudes of the weighting matrices were the same. This is due to the difference in the way damping was weighted and the different dimensions on the damping weighting matrix. Increasing the weighting on the damping parameters would decrease the magnitude of the dashpot constants in Table 10.

The active control gains were determined using Eq (11), where the matrix **P** is the solution to the algebraic Riccati equation given by Eq (70). The active control gains for the two solution techniques are listed in Appendix F.

Choosing the dashpot constants and active control gains such that the average value of the performance index was minimized (min $\bar{J}$) gave a 23% reduction in the value of the performance index over choosing the dashpot constants and active control gains such that the maximum value of the performance index was minimized (min J_{max}). The solution for the minimization of the maximum value of the performance index gave a 12% reduction in

Table 11. Damping Ratios and Natural Frequencies for Example #4

undamped frequency (rad/s)	min $\overline{J}$		min J_{max}	
	ζ	ω_n (rad/s)	ζ	ω_n (rad/s)
319.3471	0.2747	305.9835	1.0000	372.6796
			1.0000	144.3131
290.2645	0.0421	264.6086	0.0342	297.7766
281.1347	0.0911	261.0290	0.0322	266.2804
271.9703	0.3560	258.1865	0.0790	265.7667
264.1012	0.0236	256.3547	0.1430	256.2697
236.3806	0.2604	245.0388	0.0500	254.9518
223.1620	0.1130	239.0424	0.0683	236.8992
165.0056	0.2803	178.0419	0.1579	181.8822
150.4895	0.1637	157.9386	0.0934	153.5848
86.1426	0.1927	88.3404	0.1719	92.5748
64.8809	0.1588	66.3746	0.1125	69.4808
54.9372	0.0420	55.3605	0.0418	55.2103
31.9937	0.1202	32.5571	0.1187	32.4441
7.7196	0.0986	7.7597	0.0988	7.7609

J_{max} over the solution for the minimization of the average value of the performance index. Therefore, using the min $\overline{J}$ solution produces only a 12% increase in the maximum value of the performance index, but using the min J_{max} solution increases the average value of the performance index 23%.

Shown in Table 11 are the open loop frequencies (no active feedback control or passive damping) and the closed loop damping ratios and natural frequencies for the minimization of the average value of the performance index (min $\overline{J}$) and the minimization of the maximum value of the performance index (min J_{max}). The (min J_{max}) solution has two poles on the negative real axis ($\zeta = 1$). However, the other thirteen modes all have damping ratios less than 0.2 while the (min $\overline{J}$) solution has four modes with damping ratios greater than 0.2.

In this section two methods of obtaining solutions to the case in which the dashpot constants are weighted as a one-time cost have been derived. These solution techniques are similar to those developed in Section 2.2.2 in that one minimizes the maximum value of the performance index (Eq (72)), and the other minimizes its average value (Eq (74)). An example has been given to illustrate the methods and how they differ from the methods presented in the earlier portion of the chapter.

2.4 Summary

This chapter presented several new design techniques to determine optimal blending of passive viscous damping and active control. The techniques are based on modified versions of the standard linear quadratic regulator performance index of optimal control theory. It was shown that it is not possible to find the optimum by solving a single algebraic Riccati equation as in the standard linear quadratic regulator. However, several iterative techniques involving a Riccati equation were developed.

The first version of the proposed performance index treated passive damping as a separate control force, which resulted in an additional control energy term in the performance index. This formulation led to two solutions (one of which is closed form) that attempt to minimize the error in the performance index due to using viscous dampers rather than a truly active control force. The closed form solution gave a reasonable "first-cut" design, especially for larger systems. It is useful when a "quick and dirty" design of active control and passive damping is needed. The closed form solution can also be used as a starting estimate for the damping coefficients in two additional iterative techniques.

One of the additional iterative techniques minimized the maximum value of the proposed performance index with respect to all initial conditions represented by the unit ball $||\mathbf{y}_0|| = 1$. If it is pertinent that the performance index remain below a certain level, this is the method the design engineer is likely to prefer. Starting the iteration at the closed form solution saved approximately 40 CPU minutes in computation time over starting at zero for the 28th order example problem given in Section 2.2.4.3.

The other iterative technique minimized the average value of the proposed performance index with respect to all initial conditions represented by the unit ball $||\mathbf{y}_0|| = 1$. This technique gives good overall response, and is more likely to give better performance over the lifetime of the system since the average of the performance index is minimized for all initial conditions represented by the unit ball $||\mathbf{y}_0|| = 1$. Using the closed form solution as an estimate to start the iteration saved about 30 CPU minutes over starting at zero. This method ran about 16 CPU minutes faster than the method that minimized the maximum value of the proposed performance index. Both techniques yielded similar performance relative to each other, and much improved performance relative to a system with no passive damping.

The second version of the performance index treats passive damping as a one-time initial cost. Although a closed form solution was not developed for this case, iterative techniques for minimizing the average value and the maximum value of this performance index were developed. An example problem illustrated the two solution techniques. Minimizing the average value of the performance index gave higher damping ratios for the majority of the oscillatory modes than minimizing the maximum value of the performance index. Hence, minimizing the average value of the performance index led to shorter settling times for those modes. So the overall structure would tend to stop vibrating sooner.

Although the development of the second version of the performance index was only carried out for the case of viscous damping, the extension to structures with viscoelastic damping is straightforward, if the development for the first version of the performance index (where damping is weighted in the same manner as active control) has been carried out. Hence, the following two chapters on viscoelastic damping only present the development for the first version of the performance index.

III. *Viscoelastic Damping Using a Classical Model*

3.1 *Viscoelastic Materials*

Before proceeding to the problem of simultaneous optimization of active vibration control and passive viscoelastic damping, it is appropriate to discuss the characteristics of viscoelastic materials and how the behavior of viscoelastic materials is modeled in this chapter.

The basic characteristic of viscoelastic materials at constant uniform temperature is that the modulus varies with frequency, which leads to three behavioral regions for the material [3]. In the rubbery region, which occurs at low frequencies, the viscoelastic material dissipates energy well, which is why viscoelastic material is good for damping out vibrations at low frequencies. In this region, the real part of the modulus stays fairly constant while the imaginary part increases with increasing frequency. The glassy region occurs at high frequencies. The real part of the modulus is again fairly constant, but the imaginary part decreases with increasing frequency. In this region it is necessary to rely mainly on active control for vibration suppression. The region between the rubbery region and the glassy region is called the transition region. At these intermediate frequencies, both the real and imaginary parts of the modulus increase with frequency, with the rate of increase of the real part slowly overtaking that of the imaginary part. The delineation between the regions is sometimes unclear, and is highly dependent on the type of viscoelastic material being used.

The problems under consideration in this dissertation are those in which the viscoelastic dampers have been constructed so that one component of strain dissipates the energy. Viscoelastic damping pads and constrained layer damping are examples of this type of damping treatment. Damping treatments that typically don't fall in this category are viscoelastic tuned-mass dampers and compression-type dampers.

Since one component of strain dissipates the energy, the mechanical properties of the damper can be derived from a scalar relationship between stress and strain in the material. A standard linear viscoelastic model relating stress $\sigma(t)$ and strain $\epsilon(t)$ is [8:14]

$$\sigma(t) + \sum_{i=1}^{k} b_i \frac{d^i\sigma(t)}{dt^i} = E_0\epsilon(t) + \sum_{i=1}^{j} E_i \frac{d^i\epsilon(t)}{dt^i} \tag{75}$$

where the values of the constants b_i, E_0, and E_i depend on what viscoelastic material is used. Usually the number of derivatives on strain is equal to the number of derivatives on stress ($j = k$). The next section assumes the more general case of the number of derivatives on strain being less than or equal to the number of derivatives on stress ($j \leq k$).

3.2 Simultaneous Optimization of Viscoelastic Damping and Active Control

This section is analogous to Section 2.2 of Chapter II, in which the equations that minimize a chosen performance index were derived and an approximate solution was proposed, since the solution that satisfies the derived equations was not obtainable. The first task in this section is to derive the first order form of the equations of motion. Next a quadratic performance index similar to the one used in Section 2.2 is proposed. Along the way, expressions for the active control and a representative damping term are given in terms of the state. Finally, the equations which minimize the performance index will be derived, and an approximate solution presented.

In deriving the first order form of the equations of motion the viscoelastic material and the structure are assumed undisturbed for $t < 0$. At $t = 0$, it is assumed that a disturbance $\mathbf{w}_0$ occurs. Using the classical model Eq (75) for viscoelastic damping, the equations of motion of the structure in the Laplace domain are

$$s^2\mathbf{M}\mathbf{x}(s) + G(s)\mathbf{K_v}\mathbf{x}(s) + \mathbf{K}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) + \mathbf{w}_0(s) = \mathbf{0} \tag{76}$$

where

$$G(s) = \frac{G_0 + \sum_{i=1}^{j} G_i s^i}{1 + \sum_{i=1}^{k} b_i s^i} \tag{77}$$

The use of G_i instead of E_i indicates that the stresses are shear stresses only.

The matrix $\mathbf{K_v}$ is analogous to the damping matrix $\mathbf{D}$ of Section 2.2. Like the matrix $\mathbf{D}$, the matrix $\mathbf{K_v}$ is symmetric and can be written as $\mathbf{b_v}\mathbf{C}\mathbf{b_v}^T$ where $\mathbf{C}$ is a diagonal matrix of the unknown damping coefficients.

Defining $b_0 = 1$ and multiplying Eq (76) by the summation $\sum_{i=0}^{k} b_i s^i$ clears the denominator in the viscoelastic modulus $G(s)$:

$$\mathbf{M}\sum_{i=0}^{k} b_i s^{i+2}\mathbf{x}(s) + \mathbf{b_v}\left[\mathbf{C}\sum_{i=0}^{j} G_i s^i \mathbf{b_v}^T\mathbf{x}(s)\right] + \mathbf{K}\sum_{i=0}^{k} b_i s^i\mathbf{x}(s) + \mathbf{B}\sum_{i=0}^{k} b_i s^i\mathbf{u}(s) + \sum_{i=0}^{k} b_i s^i\mathbf{w}_0(s) = \mathbf{0} \tag{78}$$

With some manipulation, Eq (78) can be written in the time domain as a first order system. This will be helpful in applying the approach used in Chapter II. But first, the time domain representation of the viscoelastic forces and the disturbance will be addressed.

The Laplace transform of the damping forces is given by the second term in Eq (76), $G(s)\mathbf{K}_v\mathbf{x}$. For simplicity, the damping will be represented by the inverse Laplace transform of the term in brackets in Eq (78):

$$\mathbf{v}(t) = \mathbf{C}\mathbf{b}_v{}^T \sum_{i=0}^{j} G_i \frac{d^i}{dt^i}\mathbf{x}(t) \tag{79}$$

A new disturbance is defined as the inverse Laplace transform of the disturbance term in Eq (78):

$$\mathbf{w}(t) = L^{-1}\left\{\sum_{i=0}^{k} b_i s^i \mathbf{w}_0(s)\right\} \tag{80}$$

Making use of Eqs (79) and (80), Eq (78) is written in the time domain with the highest derivative on the position written separately from the summation:

$$b_k\mathbf{M}\frac{d^{k+2}}{dt^{k+2}}\mathbf{x}(t) + \mathbf{M}\sum_{i=2}^{k+1} b_{i-2}\frac{d^i}{dt^i}\mathbf{x}(t) + \mathbf{b}_v\mathbf{v}(t) + \mathbf{K}\sum_{i=0}^{k} b_i\frac{d^i}{dt^i}\mathbf{x}(t) + \mathbf{B}\sum_{i=0}^{k}\frac{d^i}{dt^i}\mathbf{u}(t) + \mathbf{w}(t) = \mathbf{0} \tag{81}$$

The next step is to express the highest derivative on the position as a function of the other terms in Eq (81) (the dependence on time t has been dropped for convenience):

$$\begin{aligned}\frac{d^{k+2}}{dt^{k+2}}\mathbf{x} = &-\frac{1}{b_k}\mathbf{M}^{-1}\left[\mathbf{M}\sum_{i=2}^{k+1} b_{i-2}\frac{d^i}{dt^i}\mathbf{x} + \mathbf{K}\sum_{i=0}^{k} b_i\frac{d^i}{dt^i}\mathbf{x}\right]\\ &-\frac{1}{b_k}\mathbf{M}^{-1}\mathbf{B}\sum_{i=0}^{k} b_i\frac{d^i}{dt^i}\mathbf{u} - \frac{1}{b_k}\mathbf{M}^{-1}\mathbf{b}_v\mathbf{v} - \frac{1}{b_k}\mathbf{M}^{-1}\mathbf{w}\end{aligned}$$

This relation can also be written as

$$\begin{aligned}\frac{d^{k+2}}{dt^{k+2}}\mathbf{x} = &-\frac{1}{b_k}\left[\mathbf{M}^{-1}\mathbf{K}\mathbf{x} + \mathbf{M}^{-1}\mathbf{K}b_1\dot{\mathbf{x}} + \sum_{i=2}^{k}\left(b_{i-2} + \mathbf{M}^{-1}\mathbf{K}b_i\right)\frac{d^i}{dt^i}\mathbf{x} + b_{k-1}\frac{d^{k+1}}{dt^{k+1}}\mathbf{x}\right]\\ &-\frac{1}{b_k}\mathbf{M}^{-1}\mathbf{B}\sum_{i=0}^{k} b_i\frac{d^i}{dt^i}\mathbf{u} - \frac{1}{b_k}\mathbf{M}^{-1}\mathbf{b}_v\mathbf{v} - \frac{1}{b_k}\mathbf{M}^{-1}\mathbf{w}\end{aligned} \tag{82}$$

To write this equation as a first order system will require the construction of a higher order state vector $\mathbf{y}$. While the state vector in the previous chapter contained only position and velocity, this state vector will include derivatives on position up to and including order $k+1$. By letting

$$
\begin{aligned}
\mathbf{y} &= [\ \mathbf{x}^T \quad \dot{\mathbf{x}}^T \quad \ddot{\mathbf{x}}^T \quad \cdots \quad d^{k+1}\mathbf{x}^T/dt^{k+1}\ ]^T \\
\mathbf{a}_i &= -\tfrac{b_i}{b_k}\mathbf{M}^{-1}\mathbf{K} && i = 0, 1 \\
\mathbf{a}_i &= -\tfrac{1}{b_k}[b_{i-2}\mathbf{I} + \mathbf{M}^{-1}\mathbf{K}b_i] && i = 2, \ldots, k \\
\mathbf{a}_{k+1} &= -\tfrac{b_{k-1}}{b_k}\mathbf{I}
\end{aligned}
$$

Eq (82) can be written as the following first order system:

$$
\begin{aligned}
\dot{\mathbf{y}} &= \begin{bmatrix} 0 & \mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathbf{I} \\ \mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_{k+1} \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{b_k}\mathbf{M}^{-1}\mathbf{B} \end{bmatrix} \sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} \\
&\quad + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{b_k}\mathbf{M}^{-1}\mathbf{b_v} \end{bmatrix} \mathbf{v} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{b_k}\mathbf{M}^{-1} \end{bmatrix} \mathbf{w} \\
&= \mathbf{A}\mathbf{y} + \mathbf{B}_1 \sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} + \mathbf{B_v}\mathbf{v} + \mathbf{B_w}\mathbf{w}
\end{aligned}
$$

Now the disturbance $\mathbf{w}_0(t)$ is assumed to be relatively short-lived compared to the response of the structure. Assume the disturbance stops at $t_0 > 0$. The result of the disturbance is

$$\mathbf{y}(t_0) = \mathbf{y}_0 \tag{5}$$

If the disturbance is known, $\mathbf{y}_0$ can be calculated, but since solutions independent of the initial conditions $\mathbf{y}_0$ are sought, the actual value of $\mathbf{y}_0$ is of little or no importance.

Hence the equations of motion will be expressed as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1 \sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} + \mathbf{B}_v\mathbf{v} \tag{83}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0 \tag{5}$$

Although the equations of motion are now in first order form, the derivatives on the active control will cause a problem if full state feedback is used. If the active control is allowed to have full state feedback, then the active control term in Eq (82) will have derivatives on position that are higher than that on the left hand side of Eq (82). Since interest lies primarily in active feedback control that consists of the constant gain feedback of position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$, the active control $\mathbf{u}$ will be so constrained:

$$\mathbf{u} = \mathbf{G}_r[\ \mathbf{x}^T \quad \dot{\mathbf{x}}^T\ ]^T = \mathbf{G}_r{\mathbf{I}_r}^T\mathbf{y} \tag{84}$$

$$\text{where} \qquad {\mathbf{I}_r}^T = \begin{bmatrix} \mathbf{I} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{I} & 0 & \cdots & 0 \end{bmatrix}$$

The matrix $\mathbf{G}_r$ is the matrix of feedback gains that will be determined by simultaneous design. (Note that $\mathbf{G}_r$ is a matrix and is not related to the modulus coefficients G_i.)

It will be convenient to express the damping term $\mathbf{v}$ in terms of the state also. The resulting form of $\mathbf{v}$ is identical to the form of the damping forces in Chapter II:

$$\mathbf{v} = \mathbf{C}\left[\ G_0{\mathbf{b}_v}^T \quad G_1{\mathbf{b}_v}^T \quad \cdots \quad G_j{\mathbf{b}_v}^T \quad 0 \quad \cdots \quad 0\ \right]\mathbf{y}$$

$$\mathbf{v} = \mathbf{C}\boldsymbol{\Phi}\mathbf{y} \tag{29}$$

To simultaneously design the active feedback control gains and the damping coefficients, consider the performance index used in the previous chapter in which the passive damping term is weighted as well as displacement, velocity, and active control:

$$J = \int_{t_0}^{t_f} \frac{1}{2}\left([\mathbf{x}^T \ \ \dot{\mathbf{x}}^T]\mathbf{Q}_x \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v} \right) dt \tag{85}$$

To express the performance index in terms of the state vector $\mathbf{y}$, a new weighting matrix $\mathbf{Q}$ is defined such that

$$\mathbf{y}^T\mathbf{Q}\mathbf{y} = \begin{bmatrix} \mathbf{x}^T & \dot{\mathbf{x}}^T \end{bmatrix} \mathbf{Q}_x \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} \tag{86}$$

Since the state vector $\mathbf{y}$ contains derivatives on position $\mathbf{x}$ up through the $k+1$ derivative, the matrix $\mathbf{Q}$ is semi-definite; it will have zeros on the diagonal corresponding to all derivatives higher than the first derivative on $\mathbf{x}$. The performance index can now be expressed in terms of the state $\mathbf{y}$:

$$J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v})dt \tag{87}$$

Now that the performance index, the equations of motion, the active control, and the passive damping term are all expressed in terms of the state $\mathbf{y}$, the next step is solving the simultaneous design problem of active vibration control and passive viscoelastic damping. The goal is to determine the damping coefficients (positive semi-definite, diagonal $\mathbf{C}$) and the active control feedback gains ($\mathbf{G_r}$) that

$$\text{minimize: } J = \int_{t_0}^{\infty} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v})dt \tag{87}$$

$$\text{subject to: } \dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1 \sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} + \mathbf{B_v}\mathbf{v} \tag{83}$$

$$\mathbf{u} = \mathbf{G_r}\mathbf{I_r}^T\mathbf{y} \tag{84}$$

$$\mathbf{v} = \mathbf{C}\mathbf{\Phi}\mathbf{y} \tag{29}$$

The approach used in Section 2.2 will be followed as closely as possible in this section. Using the method of Lagrange multipliers to append the constraints to the performance index gives

$$J = \int_{t_0}^{t_f} \Bigg[\frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v}) - \boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}_1 \sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} - \mathbf{B_v}\mathbf{v})$$
$$- \boldsymbol{\lambda}_2^T(\mathbf{u} - \mathbf{G_r}\mathbf{I_r}^T\mathbf{y}) - \boldsymbol{\lambda}_3^T(\mathbf{v} - \mathbf{C}\mathbf{\Phi}\mathbf{y}) \Bigg] dt$$

Now the vectors $\mathbf{y}, \mathbf{u}$, and $\mathbf{v}$ are taken to be independent. So Eqs (29) and (84) imply that the matrices $\mathbf{G_r}$ and $\mathbf{C}$ are independent of these quantities also. By definition, the

Lagrange multipliers $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3$ are independent of each other and of $\mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{G}_r$ and $\mathbf{C}$. From the calculus of variations, the necessary condition for minimizing the performance index is that its change due to variations in the independent variables vanish for arbitrary values of the variations [12:589]. Hence, to minimize the performance index J, compute its change due to the variation of the independent variables:

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} \Bigg[&(\mathbf{y}^T\mathbf{Q}\delta\mathbf{y} + \mathbf{u}^T\mathbf{R}\delta\mathbf{u} + \mathbf{v}^T\mathbf{S}\delta\mathbf{v}) - \delta\boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}_1\sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} - \mathbf{B}_v\mathbf{v}) \\ &- \boldsymbol{\lambda}_1^T(\delta\dot{\mathbf{y}} - \mathbf{A}\delta\mathbf{y} - \mathbf{B}_1\sum_{i=0}^{k} b_i\delta\left(\frac{d^i}{dt^i}\mathbf{u}\right) - \mathbf{B}_v\delta\mathbf{v}) - \delta\boldsymbol{\lambda}_2^T(\mathbf{u} - \mathbf{G}_r\mathbf{I}_r{}^T\mathbf{y}) \\ &- \boldsymbol{\lambda}_2^T(\delta\mathbf{u} - (\delta\mathbf{G}_r)\mathbf{I}_r{}^T\mathbf{y} - \mathbf{G}_r\mathbf{I}_r{}^T\delta\mathbf{y}) - \delta\boldsymbol{\lambda}_3^T(\mathbf{v} - \mathbf{C}\boldsymbol{\Phi}\mathbf{y}) - \boldsymbol{\lambda}_3^T(\delta\mathbf{v} - \delta\mathbf{C}\boldsymbol{\Phi}\mathbf{y} - \mathbf{C}\boldsymbol{\Phi}\delta\mathbf{y})\Bigg] dt\end{aligned}$$

As in Chapter II, the final value of the state is assumed zero (Eq (6)) and the initial condition is specified (Eq (5)), so integration by parts yields

$$\int_{t_0}^{t_f} \boldsymbol{\lambda}_1^T\delta\dot{\mathbf{y}}dt = \boldsymbol{\lambda}_1^T\delta\mathbf{y}\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}_1^T\delta\mathbf{y}dt$$

Integration by parts on the derivatives of the active control yields

$$\begin{aligned}\int_{t_0}^{t_f} \boldsymbol{\lambda}_1^T\mathbf{B}_1 b_i\delta\left(\frac{d^i}{dt^i}\mathbf{u}\right) = &\sum_{l=0}^{i-1}(-1)^l\left(\frac{d^l}{dt^l}\boldsymbol{\lambda}_1^T\right)\mathbf{B}_1 b_i\delta\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\mathbf{u}\right)\Bigg|_{t_0}^{t_f} \\ &+ \int_{t_0}^{t_f}(-1)^i\left(\frac{d^i}{dt^i}\boldsymbol{\lambda}_1^T\right)\mathbf{B}_1 b_i\delta\mathbf{u}dt\end{aligned}$$

Then the variation of the performance index δJ becomes

$$\begin{aligned}\delta J = \int_{t_0}^{t_f} \Bigg[&(\mathbf{y}^T\mathbf{Q} + \dot{\boldsymbol{\lambda}}_1^T + \boldsymbol{\lambda}_1^T\mathbf{A} + \boldsymbol{\lambda}_2^T\mathbf{G}_r\mathbf{I}_r{}^T + \boldsymbol{\lambda}_3^T\mathbf{C}\boldsymbol{\Phi})\delta\mathbf{y} + (\mathbf{u}^T\mathbf{R} + \sum_{i=0}^{k}\frac{d^i}{dt^i}\boldsymbol{\lambda}_1^T\mathbf{B}_1 b_i - \boldsymbol{\lambda}_2^T)\delta\mathbf{u} \\ &+ (\mathbf{v}^T\mathbf{S} + \boldsymbol{\lambda}_1^T\mathbf{B}_v - \boldsymbol{\lambda}_3^T)\delta\mathbf{v} + \boldsymbol{\lambda}_2^T\delta\mathbf{G}_r\mathbf{I}_r{}^T\mathbf{y} + \boldsymbol{\lambda}_2^T(\delta\mathbf{C})\boldsymbol{\Phi}\mathbf{y} \\ &- \delta\boldsymbol{\lambda}_1^T(\dot{\mathbf{y}} - \mathbf{A}\mathbf{y} - \mathbf{B}_1\sum_{i=0}^{k} b_i\frac{d^i}{dt^i}\mathbf{u} - \mathbf{B}_v\mathbf{v}) - \delta\boldsymbol{\lambda}_2^T(\mathbf{u} - \mathbf{G}_r\mathbf{I}_r{}^T\mathbf{y}) \\ &- \delta\boldsymbol{\lambda}_3^T(\mathbf{v} - \mathbf{C}\boldsymbol{\Phi}\mathbf{y})\Bigg] dt + \sum_{i=0}^{k}\sum_{l=0}^{i-1}(-1)^l\left(\frac{d^l}{dt^l}\boldsymbol{\lambda}_1^T\right)\mathbf{B}_1 b_i\delta\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\mathbf{u}\right)\Bigg|_{t_0}^{t_f}\end{aligned}$$

Since the variations of the dependent variables are independent of each other, the following equations need to be satisfied in addition to the three constraint equations:

$$\mathbf{0} = \dot{\boldsymbol{\lambda}}_1 + \mathbf{A}^T\boldsymbol{\lambda}_1 + \mathbf{Q}\mathbf{y} \tag{10}$$

$$\mathbf{u} = -\mathbf{R}^{-1}\sum_{i=0}^{k}(-1)^i b_i \mathbf{B}_1^T \frac{d^i}{dt^i}\boldsymbol{\lambda}_1 \tag{88}$$

$$\mathbf{v} = -\mathbf{S}^{-1}\mathbf{B_v}^T\boldsymbol{\lambda}_1 \tag{28}$$

$$\boldsymbol{\lambda}_2^T = \boldsymbol{\lambda}_3^T = \mathbf{0} \tag{89}$$

The minimization of the performance index is now expressed as a system of six equations. The task is to find the active control gains and passive damping parameters that satisfy these equations. The first step is to eliminate the multiple derivatives on $\boldsymbol{\lambda}_1$ in Eq (88). Due to the special form of the problem, this can be accomplished rather nicely.

The higher derivatives on $\boldsymbol{\lambda}_1$ occur in matrix products $(\mathbf{B}_1^T\frac{d^i}{dt^i}\boldsymbol{\lambda}_1)$, so concentration will be on these matrix products, with the goal being to express the summation of matrix products in Eq (88) as a single matrix product. The form of the input matrix $\mathbf{B}_1$, the state weighting matrix $\mathbf{Q}$, and the plant matrix $\mathbf{A}$ will be important in this development. One should remember that the sizes of these matrices is determined by the value of k — the order of the highest derivative on stress (see Eq (75)) — and on the Lagrange multiplier $\boldsymbol{\lambda}_1$ in Eq (88). From Eq (10), the first derivative of the Lagrange multiplier is

$$\dot{\boldsymbol{\lambda}}_1 = -\mathbf{Q}\mathbf{y} - \mathbf{A}^T\boldsymbol{\lambda}_1$$

so the product of the input matrix and Lagrange multiplier is

$$\mathbf{B}_1^T\dot{\boldsymbol{\lambda}}_1 = \mathbf{B}_1^T(-\mathbf{Q}\mathbf{y} - \mathbf{A}^T\boldsymbol{\lambda}_1) = -\mathbf{B}_1^T\mathbf{A}^T\boldsymbol{\lambda}_1 \tag{90}$$

since, for systems in which there is at least one derivative on stress ($k \geq 1$),

$$\mathbf{B}_1^T\mathbf{Q} = 0$$

This occurs because the one non-zero matrix element in the composite matrix $\mathbf{B}_1$ is multiplied by a zero matrix element in the composite matrix $\mathbf{Q}$ (see Eqs (83) and (86)). If the highest order derivative on stress is second order or greater ($k \geq 2$), the second term in the summation in Eq (88) is given by

$$\mathbf{B}_1^T \ddot{\boldsymbol{\lambda}}_1 = -\mathbf{B}_1^T \mathbf{A}^T \dot{\boldsymbol{\lambda}}_1 = \mathbf{B}_1^T \mathbf{A}^T (\mathbf{Q}\mathbf{y} + \mathbf{A}^T \boldsymbol{\lambda}_1) = (-1)^2 \mathbf{B}_1^T (\mathbf{A}^T)^2 \boldsymbol{\lambda}_1 \tag{91}$$

In this case $\mathbf{B}_1^T \mathbf{A}^T \mathbf{Q} = 0$. It follows easily that the general form for the matrix product in the i^{th} term of the summation in Eq (88) is

$$\mathbf{B}_1^T \frac{d^i}{dt^i} \boldsymbol{\lambda}_1 = (-1)^i \mathbf{B}_1^T (\mathbf{A}^T)^i \boldsymbol{\lambda}_1 \qquad \text{for } i \leq k \tag{92}$$

Substituting Eq (92) into Eq (88) produces an expression for the active control that does not contain derivatives on $\boldsymbol{\lambda}_1$:

$$\mathbf{u} = -\mathbf{R}^{-1} \mathbf{B}_1^T \left(\sum_{i=0}^{k} b_i (\mathbf{A}^T)^i \right) \boldsymbol{\lambda}_1 \tag{93}$$

This can be written more compactly by letting $\mathbf{I}_A = \sum_{i=0}^{k} b_i (\mathbf{A}^T)^i$:

$$\mathbf{u} = -\mathbf{R}^{-1} \mathbf{B}_1^T \mathbf{I}_A \boldsymbol{\lambda}_1 \tag{94}$$

Substituting Eq (94) into Eq (84), and Eq (28) into Eq (29) gives the following relationships between the active control gains, the damping parameters, and the Lagrange multiplier $\boldsymbol{\lambda}_1$:

$$\mathbf{G}_r \mathbf{I}_r{}^T \mathbf{y} = -\mathbf{R}^{-1} \mathbf{B}_1^T \mathbf{I}_A \boldsymbol{\lambda}_1 \tag{95}$$

$$\mathbf{C} \boldsymbol{\Phi} \mathbf{y} = -\mathbf{S}^{-1} \mathbf{B}_v{}^T \boldsymbol{\lambda}_1 \tag{96}$$

Since $\mathbf{G}_r$ and $\mathbf{C}$ are constrained to be constant, solutions of the form $\boldsymbol{\lambda}_1 = \mathbf{P}\mathbf{y}$ where $\dot{\mathbf{P}} = \mathbf{0}$ will be sought. With this constraint and $\mathbf{y}$ not specified, Eqs (95) and (96) lead to

$$\mathbf{G}_r \mathbf{I}_r{}^T = -\mathbf{R}^{-1} \mathbf{B}_1^T \mathbf{I}_A \mathbf{P} \tag{97}$$

$$\mathbf{C} \boldsymbol{\Phi} = -\mathbf{S}^{-1} \mathbf{B}_v{}^T \mathbf{P} \tag{98}$$

Now $\mathbf{I}_r{}^T \mathbf{I}_r = \mathbf{I}$, so the active control gains are given by

$$\mathbf{G}_r = -\mathbf{R}^{-1} \mathbf{B}_1^T \mathbf{I}_A \mathbf{P} \mathbf{I}_r \tag{99}$$

The Lagrange multiplier has been effectively eliminated from the derivation and the active control gains and damping parameters expressed in terms of the matrix **P**. Before solving for **P**, one needs to address the summation term involving the higher derivatives on active control **u** that appear in Eq (83). It will be shown that this summation can be expressed as a single term that varies linearly with the state **y**.

From Eq (84), the first derivative on active control can be written in terms of the state:

$$\dot{\mathbf{u}} = \mathbf{G}_r \left\{ \begin{array}{c} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{array} \right\} = \mathbf{G}_r \mathbf{I}_r^T \hat{\mathbf{I}}_1 \mathbf{y} \tag{100}$$

where

$$\hat{\mathbf{I}}_1 = \begin{bmatrix} 0 & \mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \mathbf{I} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{101}$$

Similarly, the second derivative on active control can also be written in terms of the state:

$$\ddot{\mathbf{u}} = \mathbf{G}_r \left\{ \begin{array}{c} \ddot{\mathbf{x}} \\ \frac{d^3}{dt^3}\mathbf{x} \end{array} \right\} = \mathbf{G}_r \mathbf{I}_r^T \hat{\mathbf{I}}_1^{\,2} \mathbf{y} \tag{102}$$

Extrapolating from Eqs (100) and (102), the i^{th} derivative on active control can be expressed in terms of the state:

$$\frac{d^i}{dt^i}\mathbf{u} = \mathbf{G}_r \mathbf{I}_r^T \left(\hat{\mathbf{I}}_1\right)^i \mathbf{y}$$

Hence, the sum of the derivatives on the control **u** can be expressed in terms of the full state vector, **y**:

$$\sum_{i=0}^{k} b_i \frac{d^i}{dt^i}\mathbf{u} = \sum_{i=0}^{k} b_i \mathbf{G}_r \mathbf{I}_r^T \left(\hat{\mathbf{I}}_1\right)^i \mathbf{y} = \mathbf{G}_r \mathbf{I}_r^T \left[\sum_{i=0}^{k} b_i \left(\hat{\mathbf{I}}_1\right)^i\right] \mathbf{y} = \mathbf{G}_r \mathbf{I}_r^T \mathbf{I}_{12} \mathbf{y} \tag{103}$$

where the matrix $\mathbf{I}_{12}$ is defined as

$$\mathbf{I}_{12} = \left[\sum_{i=0}^{k} b_i \left(\hat{\mathbf{I}}_1\right)^i\right]$$

Substituting Eq (99) into Eq (103), and Eqs (28) and (103) into Eq (83) gives an expression for the first derivative of the state that is in terms of the state and the matrix **P**

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} - \mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\hat{\mathbf{I}}_r\mathbf{I}_{12}\mathbf{y} - \mathbf{B}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P}\mathbf{y} \tag{104}$$

where

$$\hat{\mathbf{I}}_r = \mathbf{I}_r\mathbf{I}_r{}^T \tag{105}$$

Recalling that the Lagrange multiplier can be expressed as $\boldsymbol{\lambda}_1 = \mathbf{P}\mathbf{y}$, Eq (10) can be written in terms of the state vector:

$$\mathbf{0} = \mathbf{P}\dot{\mathbf{y}} + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} \tag{106}$$

Substituting for the time derivative of the state vector $\dot{\mathbf{y}}$ (Eq (104)) in Eq (106) produces an equation solely in terms of the matrix **P** and the state **y**:

$$\mathbf{P}(\mathbf{A}\mathbf{y} - \mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\hat{\mathbf{I}}_r\mathbf{I}_{12}\mathbf{y} - \mathbf{B}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P}\mathbf{y}) + \mathbf{A}^T\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{y} = \mathbf{0} \tag{107}$$

For general **y**, the above equation leads to a non-symmetric quadratic matrix equation:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\hat{\mathbf{I}}_r\mathbf{I}_{12} - \mathbf{P}\mathbf{B}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{108}$$

This equation serves the same role as the algebraic Riccati equation Eq (40) derived in Chapter II. However, due to the asymmetry of the third term, it is not a Riccati equation. Because the equation is non-symmetric, its solution is also non-symmetric. The solution matrix **P** of Eq (108) can be found using the technique outlined in Appendix C.

Once the solution matrix **P** is determined, the damping coefficients **C** can be determined in the same manner as in Section 2.2.1 — minimize the Frobenius norm

$$\|\mathbf{C}\boldsymbol{\Phi} + \mathbf{S}^{-1}\mathbf{B}_v^T\mathbf{P}\|_F \tag{36}$$

where **P** is the solution of Eq (108). With

$$\mathbf{W} = -\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P}$$

the elements of the damping coefficient matrix **C** are

$$C_{ii} = \max\left(\frac{\sum_{j=1}^{n} \Phi_{ij} W_{ij}}{\sum_{j=1}^{n} \Phi_{ij}^2}, 0\right); \qquad C_{ij} = 0 \quad \text{for} \quad i \neq j \tag{37}$$

Thus, a closed form solution for the damping coefficients has been found. The active control gains are given by Eq (99).

The damping coefficients and active control gains just determined are an approximate solution to the optimization problem posed at the beginning of this section. The multiple derivatives on the control in the first order state equation Eq (83) make the mathematics intractable when trying to refine the choice of active feedback gains. Specifically, the ability to express the higher derivatives of Lagrange multiplier $\boldsymbol{\lambda}_1$ in terms of the state was due largely in part to the state weighting matrix **Q** having (at most) only two non-zero matrix elements, one weighting position **x** and one weighting velocity $\dot{\mathbf{x}}$. If the damping parameter matrix **C** is chosen a priori, then weighting on the state in the performance index Eq (87) becomes $\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi}$ (refer back to Eqs (15) and (38) of Chapter II). In general $\mathbf{\Phi}^T\mathbf{CSC\Phi}$ does not have the nice form of **Q**; it may contain non-zero weightings for the derivatives of the position vector contained in the state vector up to the k^{th} derivative. So Eqs (99) and (37) represent the general (approximate) solution to the minimization problem posed in this section.

In an alternative approach to optimizing viscoelastic damping and active control simultaneously, the following two subsections take advantage of the fact that the matrix $\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi}$ has no weighting on the $k+1$ derivative of position $(\frac{d^{k+1}}{dt^{k+1}}\mathbf{x})$.

3.2.1 Solution Technique for Minimizing the Mean Performance Index. If the viscoelastic material is modelled using only one derivative on stress and only one on strain, two alternative methods of determining damping coefficients and active control gains can be developed that are independent of initial conditions. One method consists of minimizing the average value of the cost functional over the unit ball $\|\mathbf{y}_0\| = 1$. The second method minimizes the maximum value of the cost functional over the unit ball $\|\mathbf{y}_0\| = 1$.

Given the active control gains $\mathbf{G}_r$ and the damping coefficients **C**, the first order state equations are:

$$\dot{\mathbf{y}} = (\mathbf{A} + \mathbf{B}_1\mathbf{G}_r\mathbf{I}_r{}^T\mathbf{I}_{12} + \mathbf{B}_v\mathbf{C\Phi})\mathbf{y} \tag{109}$$

Then, using the same procedure outlined in Appendix B, the value of the performance index, Eq (87) can be expressed in terms of the initial state:

$$J = \frac{1}{2}\mathbf{y}_0^T \mathbf{P}_1 \mathbf{y}_0 \tag{110}$$

The matrix $\mathbf{P}_1$ satisfies the following Lyapunov equation (refer to Appendix B for its derivation):

$$\begin{aligned}\mathbf{P}_1(\mathbf{A} + \mathbf{B}_1\mathbf{G}_r\mathbf{I}_r{}^T\mathbf{I}_{12} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi}) + (\mathbf{A} + \mathbf{B}_1\mathbf{G}_r\mathbf{I}_r{}^T\mathbf{I}_{12} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi})^T\mathbf{P}_1 \\ + (\mathbf{Q} + \mathbf{I}_r\mathbf{G}_r{}^T\mathbf{R}\mathbf{G}_r\mathbf{I}_r{}^T + \mathbf{\Phi}^T\mathbf{C}\mathbf{S}\mathbf{C}\mathbf{\Phi}) = \mathbf{0}\end{aligned} \tag{111}$$

The subscript 1 on $\mathbf{P}_1$ is to distinguish it from the solution of Eq (108). From Section 2.2.2, The average value of the cost functional over the unit ball is given by

$$\overline{J} = \frac{1}{2n}\mathrm{trace}\,\mathbf{P}_1 \tag{112}$$

Hence, minimizing the trace of $\mathbf{P}_1$ minimizes the average value of J over the unit ball $\|\mathbf{y}_0\| = 1$, regardless of system order. In this method, given the damping coefficients $\mathbf{C}$, one solves first for the active gain matrix $\mathbf{G}_r$ using the same procedure as in Section 3.2. The active gain matrix is of the same form as that in Section 3.2,

$$\mathbf{G}_r = -\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\mathbf{I}_r \tag{99}$$

but the matrix $\mathbf{I}_A$ has a slightly different definition:

$$\mathbf{I}_A = \mathbf{I} + b_1(\mathbf{A} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi})^T$$

The matrix $\mathbf{P}$ satisfies a different non-symmetric quadratic matrix equation than the one given by Eq (108):

$$\mathbf{P}(\mathbf{A} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi}) + (\mathbf{A} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi})^T\mathbf{P} - \mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\widehat{\mathbf{I}}_r\mathbf{I}_{12} + (\mathbf{Q} + \mathbf{\Phi}^T\mathbf{C}\mathbf{S}\mathbf{C}\mathbf{\Phi}) = \mathbf{0} \tag{113}$$

Like the matrix $\mathbf{I}_A$, the matrix $\mathbf{I}_{12}$ also has a slightly different definition in Eq (113) than in Eq (108):

$$\mathbf{I}_{12} = \mathbf{I} + b_1\widehat{\mathbf{I}}_1$$

where $\widehat{\mathbf{I}_1}$ is defined by Eq (101).

In summary, once an initial guess for the damping coefficients $\mathbf{C}$ is made, the matrix $\mathbf{P}$ is found by solving Eq (113), the active control gains are calculated via Eq (99), the matrix $\mathbf{P}_1$ is found by solving Eq (111), and finally the average value of the performance index is calculated using Eq (112). Iteration with respect to $\mathbf{C}$ is carried out until the trace of $\mathbf{P}_1$ is minimized.

This subsection has derived a technique for minimizing the average value of the cost functional over a representative sample of initial conditions. The next subsection will present another technique for optimizing viscoelastic damping and active control simultaneously when there is only a first derivative on stress and strain.

3.2.2 Solution for Minimizing the Maximum Performance Index. A similar approach to the one just presented for minimizing the average value of the performance index is used for minimizing the maximum value of the performance index for all $\|\mathbf{y}_0\| = 1$. As in Section 2.2.3, this approach requires determining the two norm of $\mathbf{P}_1$, since

$$J \leq \frac{1}{2}\|\mathbf{P}_1\|_2 \tag{114}$$

In this method, iteration with respect to $\mathbf{C}$ is carried out until $\|\mathbf{P}_1\|_2$ is minimized. In each iteration step, the matrix $\mathbf{P}$ is found by solving Eq (113) for the current value $\mathbf{C}$, the active control gains are calculated via Eq (99), the matrix $\mathbf{P}_1$ is found by solving Eq (111), and finally the maximum value of the performance index is calculated using Eq (114). Based on this value, new damping parameters are selected for the next iteration. This approach might be very conservative in general, since the initial condition $\mathbf{y}_0$ that maximizes the performance index may not be encountered very often.

Examples follow which illustrate the three different solution techniques developed in this section.

3.3 Example Problems

3.3.0.1 Example Problem #1 – Single DOF System. In this example, the solution techniques developed in this section will be applied to the single degree of freedom spring-mass-damper system of Chapter II. A hypothetical viscoelastic damper will be modelled with one derivative on stress and one on strain. The parameters for the model are

$$G_0 = 6.8948 \times 10^3 \ \text{N/m}^2 \quad (1.0 \ \text{lb/in}^2)$$
$$G_1 = 4.8264 \times 10^4 \ \text{N-s/m}^2 \quad (7.0 \ \text{lb-sec/in}^2)$$
$$b_1 = 0.001 \ \text{sec}$$

The the equation of motion in the Laplace domain is

$$s^2\mathbf{x} + \frac{1+7s}{1+0.001s}c\mathbf{x} + 0.5\mathbf{x} = -u \tag{115}$$

Converting to the time domain,

$$\frac{d^3}{dt^3}\mathbf{x} = -500\mathbf{x} - 0.5\dot{\mathbf{x}} - 1000\ddot{\mathbf{x}} - 1000(u + 0.001\dot{u}) - 1000c(\mathbf{x} + 7\dot{\mathbf{x}}) \tag{116}$$

The state space equation is

$$\dot{\mathbf{y}} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \\ \frac{d^3}{dt^3}\mathbf{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -500 & -0.5 & -1000 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix}(u + 0.001\dot{u})$$
$$+ \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix} c \begin{bmatrix} 1 & 7 & 0 \end{bmatrix} \mathbf{y}$$

The weightings on the state, active control, and passive viscoelastic damping are

$$\mathbf{Q} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = 0.1, \quad S = 0.1 \tag{117}$$

The solutions for the three different solution techniques developed in this chapter are given in Table 12. The active gain matrices $\mathbf{G_r}$ are included as well as the viscoelastic coefficients c. The Frobenius norm solution (min $\|\mathbf{H}\|_F$) gave a higher value of the viscoelastic coefficients and lower feedback gains than either of the other two solutions. Its performance, as measured by trace $\mathbf{P}$ and $\|\mathbf{P}\|_2$, was not quite as good as the other solutions, but it was within 10%. It is not surprising that the value of the Frobenius norm for the solution minimizing trace $\mathbf{P}$ (and therefore the average value of the performance index) is

Table 12. Example Problem #1 Results

Solution Technique	min $\|\mathbf{H}\|_F$	min trace $\mathbf{P}$	min $\|\mathbf{P}\|_2$
c	0.3795	0.1813	0.1122
$\mathbf{G}_r$	[1.3568 2.5293]	[1.6676 2.8054]	[1.7132 3.0486]
trace $\mathbf{P}$	1.1924	1.1256	1.1374
$\|\mathbf{P}\|_2$	0.9692	0.8902	0.8830
$\|\mathbf{H}\|_F$	0.3096	0.5406	0.6731

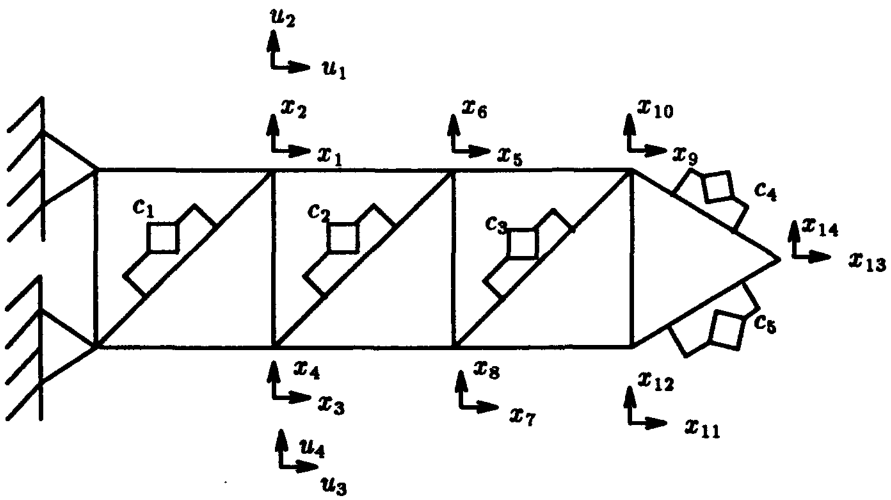

Figure 5. Example Problem – Truss Structure

smaller than that for the solution minimizing $\|\mathbf{P}\|_2$ since the former solution is closer to the Frobenius norm solution.

3.3.0.2 Example Problem #2 – Fourteen DOF System. In this example, the solution techniques developed in this chapter will be applied to a viscoelastically damped version of the planar aluminum truss used in Example #3 in Chapter II. In Figure 5, the u_i's represent active control forces, and the c_i's represent the unknown damping parameters to be determined. The hypothetical viscoelastic material was modelled using one derivative on stress and one on strain. The material is similar to the one used in the previous example problem, but the parameters G_1 and b_1 are chosen to be ten times their counterparts in the previous example. They are chosen in this manner so that the natural frequencies lie in the transition region. The parameters for the model are

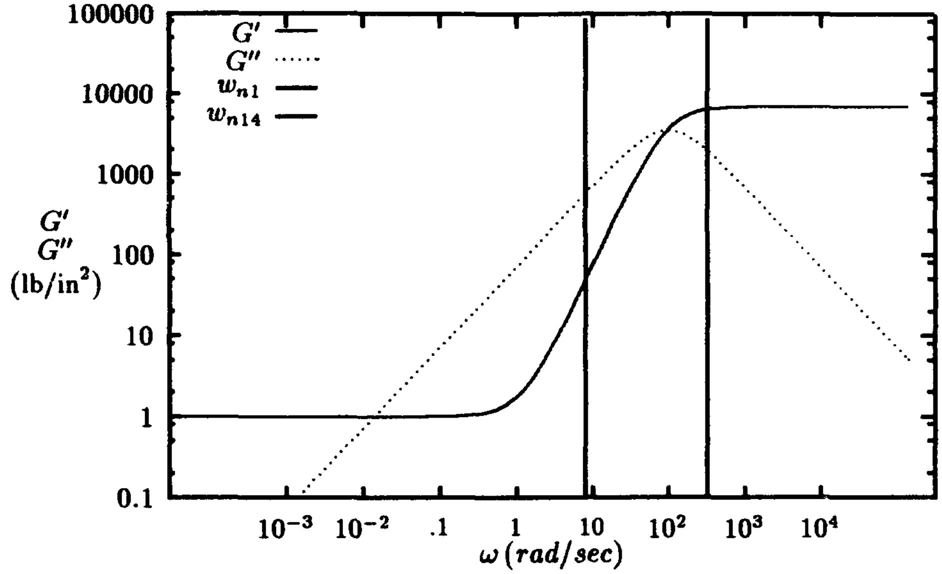

Figure 6. Viscoelastic Modulus Plot

$$\begin{aligned} G_0 &= 6.8948 \times 10^3 \text{ N/m}^2 \quad (1.0 \text{ lb/in}^2) \\ G_1 &= 4.8264 \times 10^5 \text{ N-s/m}^2 \quad (70 \text{ lb-sec/in}^2) \\ b_1 &= 0.01 \text{ sec} \end{aligned}$$

The real part (G') and imaginary part (G'') of the modulus $G(s)$ is shown in Figure 6. The natural undamped frequencies of the truss lie in the transition region, with most of the natural frequencies near the glassy region. The two vertical lines indicate the range of the natural frequencies.

Using the development presented in Section 3.2, the equations of motion can be written in the state space form:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1 \sum_{i=0}^{1} b_i \frac{d^i}{dt^i}\mathbf{u} + \mathbf{B}_v \mathbf{v} \tag{118}$$

Since the state vector contains position, velocity, and acceleration, the dimensions of the state vector $\mathbf{y}$ are 42×1. There are four control inputs, so the control vector $\mathbf{u}$ is 4×1. The five viscoelastic dampers lead to a damping parameter matrix $\mathbf{C}$ that is 5×5 and diagonal. Therefore, $\mathbf{A}$ is 42×42, $\mathbf{B}$ is 42×4, $\mathbf{B}_v$ is 42×5, and $\mathbf{\Phi}$ is 5×42.

The state weighting matrix $\mathbf{Q}_x$ was chosen such that $\mathbf{y}^T\mathbf{Q}\mathbf{y}$ equalled the total mechanical energy in the system. Thus $\mathbf{Q}_x$ was formed using the mass and stiffness matrices,

Table 13. Example Problem #2 Results

	min $\|\|\mathbf{H}\|\|_F$	min trace $\mathbf{P}$	min $\|\|\mathbf{P}\|\|_2$
c_1	3.335	2.019	0.372
c_2	2.443	1.554	1.092
c_3	2.441	1.605	0.896
c_4	3.330	2.012	0.902
c_5	2.973	1.857	0.715
computation time	3 min	10 hr 54 min	23 hr 32 min
ζ – Passive only	.0066 to .1137	.0042 to .0875	.0018 to .0511
ζ – Complete Solution	.0217 to .2491	.0193 to .2591	.0097 to .2591

M and **K**. The weighting matrices are

$$\mathbf{Q}_x = \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad \mathbf{R} = 0.1\mathbf{I}, \quad \mathbf{S} = 0.01\mathbf{I} \tag{119}$$

The solution techniques were run on a VAX 6420 using MATLAB routines. Special routines were written that calculate the solutions to the non-symmetric quadratic matrix equations derived in this chapter, using the algorithm given in Appendix C. The minimization algorithm utilized is the Nelder-Mead simplex method [23].

The values of the damping parameters for the three solution techniques, along with their approximate computation times, are given in Table 13. The active control gains for each set of damping parameters are computed using Eq (99) and the corresponding matrix **P**. For the min $||\mathbf{H}||_F$ solution, this is the matrix that solves Eq (108). For the other two solutions, it is the matrix that satisfies Eq (113) for the specific damping parameter matrix **C**. The active control gains are listed in Appendix F. Because the active control gains for these two solutions are calculated for a certain diagonal damping parameter matrices (in fact, the active gains and damping coefficients are essentially calculated simultaneously), the damping ratios for the complete solutions are better than those for the min $||\mathbf{H}||_F$ solution, whose active control gains are based on a non-diagonal parameter matrix (the damping coefficients are those that give a diagonal damping parameter matrix that best approximates this optimal solution). The range of damping ratios with just the viscoelastic damping portion of the solution implemented (i.e., setting the active gains to zero) is shown in Table 13, as well as the range of damping ratios for the complete solution.

How well the complete solutions do in approximating the optimal solution, Eqs (98) and (99), is indicated in Table 14. It gives a measure of how close the attainable systems

Table 14. Example #2 Relative Changes in Eigenvalues

ζ	ω_n	$\|\mathbf{H}\|_F$	trace(**P**)	$\|\mathbf{P}\|_2$
0.0863	322.7332	0.1612	0.1130	0.0890
0.0843	291.9234	0.1578	0.1045	0.0831
0.0637	283.4323	0.1275	0.0859	0.0589
0.0642	272.6237	0.1159	0.0735	0.0594
0.0383	264.3360	0.0830	0.0584	0.0401
0.0943	241.8979	0.1034	0.0952	0.0857
0.0932	226.7390	0.1245	0.1039	0.0850
0.1814	172.5763	0.2119	0.1725	0.1548
0.1689	156.5123	0.1940	0.1585	0.1408
0.2862	93.6310	0.2286	0.1989	0.1986
0.3002	70.6018	0.2496	0.2131	0.1985
0.1585	55.9026	0.0985	0.0966	0.0943
0.2859	34.2871	0.1373	0.1181	0.1031
0.1930	7.8979	0.0783	0.0781	0.0780
1.0000	94.9204	0.1546	0.4417	0.3909
1.0000	109.3135	0.5831	0.2399	0.2751
1.0000	107.3431	0.4012	0.1400	0.1261
1.0000	105.7911	0.4360	0.2859	0.1649
1.0000	102.6897	0.0999	0.3035	0.1397
1.0000	101.6959	0.4445	0.3626	0.1822
1.0000	99.5807	0.0343	0.3158	0.1842
1.0000	99.9061	0.4822	0.0338	0.0368
1.0000	100.0000	0.0000	0.0000	0.0000
1.0000	100.0000	0.0000	0.0000	0.0000
1.0000	100.0000	0.0000	0.0000	0.0000
1.0000	100.0000	0.0000	0.0000	0.0000
1.0000	100.0000	0.0000	0.0000	0.0000
1.0000	100.0000	0.0000	0.0000	0.0000

are to the optimum system produced by the optimal solution. The relative changes in the eigenvalues are computed according to the formula

$$\frac{|\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A}_{\mathbf{opt}})|}{|\lambda_i(\mathbf{A}_{\mathbf{opt}})|}$$

The damping ratio and natural frequency listed in the table are those associated with the optimum system.

Table 15 lists the changes in the optimum system's oscillatory eigenvalues in increasing order. The relative changes range from 4% (minimum two-norm solution) to 25% (Frobenius norm solution). Tables 14 and 15 indicate that the solution techniques are

Table 15. Example #2 Relative Changes in Eigenvalues in Increasing Order

$\|\mathbf{H}\|_F$	trace($\mathbf{P}$)	$\|\mathbf{P}\|_2$
0.0783	0.0584	0.0401
0.0830	0.0735	0.0589
0.0985	0.0781	0.0594
0.1034	0.0859	0.0780
0.1159	0.0952	0.0831
0.1245	0.0966	0.0850
0.1275	0.1039	0.0857
0.1373	0.1045	0.0890
0.1578	0.1130	0.0943
0.1612	0.1181	0.1031
0.1940	0.1585	0.1408
0.2119	0.1725	0.1548
0.2286	0.1989	0.1985
0.2496	0.2131	0.1986

good approximations to the the optimal solution. The relative changes are higher than in Chapter II, Example #3, but this is to be expected since the state vector is larger (42 states versus 28 in Example #3) and there is a constraint applied to active control as well as to passive damping.

This section has developed three solution techniques for a classically modelled viscoelastic system analogous to the techniques developed for a viscous system in Chapter II. A closed form solution was derived, as well as two iterative solutions, one of which minimized the average value of the performance index over the unit ball, and one that minimized its maximum over the unit ball.

3.4 Multiple Damping Materials

Up to now, it has been assumed that only one type of viscoelastic material is used for damping. This section will expand the results of Section 3.2 to the case where more than one type of viscoelastic material is present in the structure. As stated in Section 3.1, the viscoelastic dampers are assumed to have been constructed so that one component of strain dissipates the energy, so a scalar relationship exists between stress and strain in the material.

Only two different damping materials will be considered, but the derivation can be adapted for more materials. Both materials will be assumed to have a frequency dependent modulus of the form

$$\frac{G_0 + G_1 s}{1 + bs}$$

Although the assumption that there is only one derivative on stress has been made here, the derivation can be expanded for higher derivatives on stress. Two hypothetical materials with simple moduli are used so that it is easier to follow the derivation.

The first step to expressing the equations of motion as a first order system is to write the equation of motion in the Laplace domain:

$$s^2\mathbf{M}\mathbf{x} + \mathbf{b}_{v1}\frac{G_{01} + G_{11}s}{1 + b_1 s}\mathbf{C}_1\mathbf{b}_{v1}^T\mathbf{x} + \mathbf{b}_{v2}\frac{G_{02} + G_{12}s}{1 + b_2 s}\mathbf{C}_2\mathbf{b}_{v2}^T\mathbf{x} + \mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u} = \mathbf{0} \tag{120}$$

It will be convenient to clear the denominators in Eq (120):

$$\begin{aligned}
&[1 + (b_1 + b_2)s + b_1 b_2 s^2]s^2\mathbf{M}\mathbf{x} + \mathbf{b}_{v1}(1 + b_2 s)(G_{01} + G_{11}s)\mathbf{C}_1\mathbf{b}_{v1}^T\mathbf{x} \\
&+ \mathbf{b}_{v2}(1 + b_1 s)(G_{02} + G_{12}s)\mathbf{C}_2\mathbf{b}_{v2}{}^T\mathbf{x} + \mathbf{K}[1 + (b_1 + b_2)s + b_1 b_2 s^2]\mathbf{x} \\
&+\mathbf{B}[1 + (b_1 + b_2)s + b_1 b_2 s^2]\mathbf{u} = \mathbf{0}
\end{aligned} \tag{121}$$

As in Section 3.2, the Laplace transform of the damping will be represented by the terms which contain the damping parameter matrices. For the first damping material, the Laplace transform of this term is

$$\mathbf{v}_1(s) = [G_{01} + (G_{01}b_2 + G_{11})s + G_{11}b_2 s^2]\mathbf{C}_1\mathbf{b}_{v1}^T\mathbf{x}(s) \tag{122}$$

In the time domain, this can be written as

$$\mathbf{v}_1(t) = \mathbf{C}_1\mathbf{b}_{v1}^T \begin{bmatrix} G_{01}\mathbf{I} & (G_{01}b_2 + G_{11})\mathbf{I} & G_{11}b_2\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} \tag{123}$$

Similarly, the expression for the representative damping due to the second material can be written in the time domain:

$$\mathbf{v}_2(t) = \mathbf{C}_2\mathbf{b}_{v2}{}^T \begin{bmatrix} G_{02}\mathbf{I} & (G_{02}b_1 + G_{12})\mathbf{I} & G_{12}b_1\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} \tag{124}$$

Rewriting the equation of motion Eq (121) in the time domain, and then substituting Eqs (123) and (124) into it leads to the following differential equation:

$$\begin{aligned} b_1 b_2 \mathbf{M}\frac{d^4}{dt^4}\mathbf{x} + (b_1+b_2)\mathbf{M}\frac{d^3}{dt^3} + \mathbf{M}\ddot{\mathbf{x}} + \mathbf{b}_{\mathbf{v}1}\mathbf{v}_1 + \mathbf{b}_{\mathbf{v}2}\mathbf{v}_2 + \mathbf{K}\mathbf{x} \\ + (b_1+b_2)\mathbf{K}\dot{\mathbf{x}} + b_1 b_2 \mathbf{K}\ddot{\mathbf{x}} + \mathbf{B}\mathbf{u} + (b_1+b_2)\mathbf{B}\dot{\mathbf{u}} + b_1 b_2 \mathbf{B}\ddot{\mathbf{u}} = \mathbf{0} \end{aligned} \tag{125}$$

Hence the highest derivative on $\mathbf{x}$, $\frac{d^4}{dt^4}\mathbf{x}$ can be written as

$$\begin{aligned} \frac{d^4}{dt^4}\mathbf{x} = & -\frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{K}\mathbf{x} - \frac{b_1+b_2}{b_1 b_2}\mathbf{M}^{-1}\mathbf{K}\dot{\mathbf{x}} + \left[\mathbf{M}^{-1}\mathbf{K} + \frac{\mathbf{I}}{b_1 b_2}\right]\ddot{\mathbf{x}} \\ & - \frac{b_1+b_2}{b_1 b_2}\frac{d^3}{dt^3}\mathbf{x} - \frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{B}\left[\mathbf{u} + (b_1+b_2)\dot{\mathbf{u}} + b_1 b_2 \ddot{\mathbf{u}}\right] \\ & - \frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{b}_{\mathbf{v}1}\mathbf{v}_1 - \frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{b}_{\mathbf{v}2}\mathbf{v}_2 \end{aligned} \tag{126}$$

Following the procedure of Section 3.2, Eqs (82) - (83), this equation can be written as a first order system:

$$\dot{\mathbf{y}} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \\ \frac{d^3}{dt^3}\mathbf{x} \\ \frac{d^4}{dt^4}\mathbf{x} \end{Bmatrix} = \begin{bmatrix} 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ -\frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{K} & -\frac{b_1+b_2}{b_1 b_2}\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{K} - \frac{\mathbf{I}}{b_1 b_2} & -\frac{b_1+b_2}{b_1 b_2}\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \\ \frac{d^3}{dt^3}\mathbf{x} \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{B} \end{bmatrix} [\mathbf{u} + (b_1 + b_2)\dot{\mathbf{u}} + b_1 b_2 \ddot{\mathbf{u}}]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{b}_{\mathbf{v}1} \end{bmatrix} \mathbf{v}_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{b_1 b_2}\mathbf{M}^{-1}\mathbf{b}_{\mathbf{v}2} \end{bmatrix} \mathbf{v}_2 \tag{127}$$

For compactness, the matrices can be represented by a single letter:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1 [\mathbf{u} + (b_1 + b_2)\dot{\mathbf{u}} + b_1 b_2 \ddot{\mathbf{u}}] + \mathbf{B}_{\mathbf{v}1}\mathbf{v}_1 + \mathbf{B}_{\mathbf{v}2}\mathbf{v}_2 \tag{128}$$

Eqs (123) and (124) can be written in terms of the state $\mathbf{y}$:

$$\mathbf{v}_1(t) = \mathbf{C}_1 \boldsymbol{\Phi}_1 \mathbf{y} \tag{129}$$

$$\mathbf{v}_2(t) = \mathbf{C}_2 \boldsymbol{\Phi}_2 \mathbf{y} \tag{130}$$

Now that the equations of motion have been expressed in first order form, a performance index is needed. Following the same approach as in the single damping material case, the performance index will be defined as

$$J = \int_{t_0}^{\infty} \frac{1}{2} \left([\mathbf{x}^T \ \ \dot{\mathbf{x}}^T] \mathbf{Q}_x \left\{ \begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right\} + \mathbf{u}^T \mathbf{R} \mathbf{u} + \mathbf{v}_1^T \mathbf{S}_1 \mathbf{v}_1 + \mathbf{v}_2^T \mathbf{S}_2 \mathbf{v}_2 \right) dt \tag{131}$$

As in Section 3.2, the state weighting matrix $\mathbf{Q}$ is constructed such that

$$\mathbf{y}^T \mathbf{Q} \mathbf{y} = \left[\begin{array}{cc} \mathbf{x}^T & \dot{\mathbf{x}}^T \end{array} \right] \mathbf{Q}_x \left\{ \begin{array}{c} \mathbf{x} \\ \dot{\mathbf{x}} \end{array} \right\} \tag{86}$$

and the matrix $\mathbf{I}_r{}^T$ is constructed such that

$$\mathbf{u} = \mathbf{G}_r [\ \mathbf{x}^T \ \ \dot{\mathbf{x}}^T \ ]^T = \mathbf{G}_r \mathbf{I}_r{}^T \mathbf{y} \tag{84}$$

Appending the constraints Eqs (128), (84), (129), and (130), to the cost functional Eq (131) and substituting in Eq (86),

$$\begin{aligned} J = \int_{t_0}^{\infty} \Big[& \frac{1}{2} (\mathbf{y}^T \mathbf{Q} \mathbf{y} + \mathbf{u}^T \mathbf{R} \mathbf{u} + \mathbf{v}_1^T \mathbf{S}_1 \mathbf{v}_1 + \mathbf{v}_2^T \mathbf{S}_2 \mathbf{v}_2) \\ & - \boldsymbol{\lambda}_1^T \{ \dot{\mathbf{y}} - \mathbf{A} \mathbf{y} - \mathbf{B}_1 [\mathbf{u} + (b_1 + b_2) \dot{\mathbf{u}} + b_1 b_2 \ddot{\mathbf{u}}] - \mathbf{B}_{v1} \mathbf{v}_1 - \mathbf{B}_{v2} \mathbf{v}_2 \} \\ & - \boldsymbol{\lambda}_2^T (\mathbf{u} - \mathbf{G}_r \mathbf{I}_r{}^T \mathbf{y}) - \boldsymbol{\lambda}_3^T (\mathbf{v}_1 - \mathbf{C}_1 \boldsymbol{\Phi}_1 \mathbf{y}) - \boldsymbol{\lambda}_4^T (\mathbf{v}_2 - \mathbf{C}_2 \boldsymbol{\Phi}_2 \mathbf{y}) \Big] \, dt \end{aligned}$$

Setting the variation of J equal to zero, and using the same procedures as those used in the single material case (Section 3.2), results in six simultaneous equations:

$$\dot{\mathbf{y}} = \mathbf{A} \mathbf{y} + \mathbf{B}_1 [\mathbf{u} + (b_1 + b_2) \dot{\mathbf{u}} + b_1 b_2 \ddot{\mathbf{u}}] + \mathbf{B}_{v1} \mathbf{v}_1 + \mathbf{B}_{v2} \mathbf{v}_2 \tag{132}$$

$$\mathbf{u} = \mathbf{G}_r \mathbf{I}_r{}^T \mathbf{y} \tag{133}$$

$$0 = \dot{\boldsymbol{\lambda}}_1 + \mathbf{A}^T \boldsymbol{\lambda}_1 + \mathbf{Q} \mathbf{y} \tag{134}$$

$$0 = \mathbf{R} \mathbf{u} + \mathbf{B}_1 \mathbf{I}_A \boldsymbol{\lambda}_1 \qquad [\mathbf{I}_A = \mathbf{I} + (b_1 + b_2) \hat{\mathbf{I}}_1 + b_1 b_2 \hat{\mathbf{I}}_1] \tag{135}$$

$$\mathbf{C}_1 \boldsymbol{\Phi}_1 \mathbf{y} = -\mathbf{S}_1{}^{-1} \mathbf{B}_{v1}{}^T \boldsymbol{\lambda}_1 \tag{136}$$

$$\mathbf{C}_2 \boldsymbol{\Phi}_2 \mathbf{y} = -\mathbf{S}_2{}^{-1} \mathbf{B}_{v2}{}^T \boldsymbol{\lambda}_1 \tag{137}$$

Since $\mathbf{C}_1$ and $\mathbf{C}_2$ are constant, Eqs (136) and (137) imply that $\boldsymbol{\lambda}_1$ is of the form $\boldsymbol{\lambda}_1 = \mathbf{P} \mathbf{y}$. As in Section 3.2, solving the above simultaneous equations leads to a non-

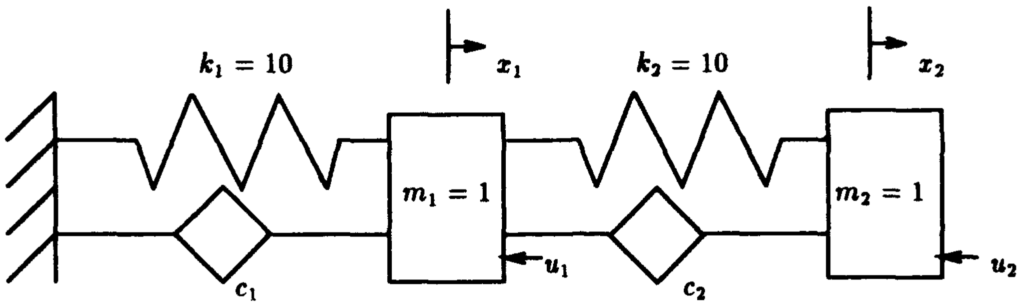

Figure 7. Two Degree of Freedom System

symmetric quadratic matrix equation:

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PB}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\hat{\mathbf{I}}_r\mathbf{I}_{12} - \mathbf{PB}_{v1}\mathbf{S}_1^{-1}\mathbf{B}_{v1}{}^T\mathbf{P} - \mathbf{PB}_{v2}\mathbf{S}_2^{-1}\mathbf{B}_{v2}{}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \quad (138)$$

This equation is solved in the same manner as the one for one material, Eq (108) (see Appendix C). The damping parameters are found by using the same closed form solution technique as in Section 3.2, which is given by Eqs (36) and (37). The two norms which need to be minimized to determine the damping coefficients are

$$\|\mathbf{C}_1\mathbf{\Phi}_1 + \mathbf{S}_1{}^{-1}\mathbf{B}_{v1}{}^T\mathbf{P}\|_F \quad (139)$$

and

$$\|\mathbf{C}_2\mathbf{\Phi}_2 + \mathbf{S}_2{}^{-1}\mathbf{B}_{v2}{}^T\mathbf{P}\|_F \quad (140)$$

This section derived simultaneous design of active vibration control and passive viscoelastic damping when more than one type of viscoelastic material is used in the structure. An example problem follows to demonstrate the techniques derived in this section.

3.4.1 Example Problem #3 - Two Viscoelastic Materials. To illustrate the case of using two different viscoelastic materials for damping in the same structure, the solution technique developed in this section will be applied to the two degree of freedom system in Figure 7. The viscoelastic behavior of the hypothetical dampers c_1 and c_2 will be given by

$$G_1(s) = \frac{1 + 7s}{1 + .001s} \qquad \text{and} \qquad G_2(s) = \frac{1 + 9s}{1 + .005s} \quad (141)$$

respectively.

Following the procedure outlined in Section 3.4, Eqs (120) through (140), and using the weightings

$$\mathbf{Q}_x = \begin{bmatrix} 20 & -10 & 0 & 0 \\ -10 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_1 = S_2 = 0.01, \quad \mathbf{R} = 0.1\mathbf{I}$$

the damping parameters are determined to be

$$c_1 = 1.75$$

$$c_2 = 1.16$$

The control gains were

$$\mathbf{G}_r = \begin{bmatrix} 2.88 & 0.02 & 1.56 & 0.44 \\ -0.45 & 2.03 & 0.44 & 1.06 \end{bmatrix}$$

This example has demonstated the feasiblility of designing passive viscoelastic damping and active feedback control when more than one viscoelastic material is used for damping.

3.5 Summary

In this chapter, several new design techniques to determine optimal blending of passive viscoelastic damping and active vibration control were presented. The techniques are based on a standard linear viscoelastic model and on a modified version of the standard linear quadratic regulator cost functional of optimal control theory. The proposed cost functional treats passive damping as a separate control force, which results in an additional energy term in the cost functional.

Two iterative techniques were developed in addition to a closed form solution. The closed form solution was a least squares solution. One iterative technique minimized the maximum value of the proposed cost functional. The other iterative technique minimized the average value of the proposed cost functional. All three techniques yielded similar performance relative to each other, although the closed form solution took considerably less computation time.

The three solution techniques were applied in Example #1 to a single degree of freedom spring-mass-damper system. These techniques were confirmed in Example #2 as being useful even when natural frequencies lie in the transition region of the structure. The applicability of the solution procedures were demonstrated in Example #3 for a structure in which damping is provided via two different viscoelastic materials.

One of the drawbacks of the techniques presented in this chapter is that the standard viscoelastic model may not adequately model the behavior of the viscoelastic material. Another drawback is that the formulation of the problem could yield very large state space matrices. During an attempt to solve a fourteen degree of freedom problem with five derivatives on stress strain, the large state space matrices caused computational difficulties in trying to solve the non-symmetric quadratic matrix equation, Eq (108). The approach presented in the following chapter overcomes these drawbacks by using a more accurate representation for the behavior of the viscoelastic material, and by keeping the system order down by using the more traditional state vector which consists of only position and velocity.

IV. Viscoelastic Damping Using a Four Parameter Fractional Derivative Model

4.1 Brief Overview of Generalized Derivatives as Applied to Viscoelastic Materials [11]

In this chapter, fractional order derivatives will be used to model viscoelasticity as opposed to the integer order derivatives used in the previous chapter. But before applying fractional order derivatives to structural problems, it is necessary to understand the properties of generalized derivatives and their use in the theory of viscoelasticity. As will be shown, generalized derivatives behave in much the same way as conventional derivatives. When used to model viscoelastic materials, generalized derivatives typically provide an excellent model over a broad range of frequencies [5]. To show how generalized derivatives can be used to model viscoelastic materials, it is appropriate to first present the properties of generalized derivatives, especially the Laplace and Fourier transforms. The generalized derivative is defined as [27:59]

$$D^{\alpha}[x(t)] \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad \text{for } 0 \leq \alpha < 1 \tag{142}$$

This definition is only valid for $\alpha < 1$. However, the definition requires only a slight modification for a generalized derivative of order greater than one. Let m be a nonnegative integer, and α defined as before. Then [27:59]

$$D^{m+\alpha}[x(t)] \equiv \frac{1}{\Gamma(1-\alpha)} \frac{d^{m+1}}{dt^{m+1}} \int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad \text{for } 0 \leq \alpha < 1 \tag{143}$$

Although imposing in the time domain, in the Laplace (or Fourier) domain the generalized derivative manifests itself as a fractional power of s (or ω). To calculate the Laplace transform, a change of variables will be useful:

$$\tau = t - \eta$$

This leads to

$$D^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(t-\eta)}{\eta^{\alpha}} d\eta \qquad \text{for } 0 \leq \alpha < 1 \tag{144}$$

Applying Leibnitz's rule,

$$D^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{\eta^{\alpha}} \frac{\partial}{\partial t} x(t-\eta)\, d\eta + \frac{x(0)}{\Gamma(1-\alpha)t^{\alpha}} \qquad \text{for } 0 \leq \alpha < 1 \tag{145}$$

Noting that the integral is a convolution integral, and that

$$L\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right] = \frac{1}{s^{1-\alpha}} \tag{146}$$

the Laplace transform is

$$L[D^{\alpha}[x(t)]] = \frac{1}{s^{1-\alpha}}\left(sL[x(t)] - x(0)\right) + \frac{x(0)}{s^{1-\alpha}} \tag{147}$$

or, more simply,

$$L[D^{\alpha}[x(t)]] = s^{\alpha}L[x(t)] \tag{148}$$

where the Laplace transform is defined as

$$L[x(t)] = \int_0^{\infty} x(t)\, e^{-st}\, dt \tag{149}$$

For initial conditions equal to zero, the Laplace transform of a generalized derivative of order α has the same property as the conventional derivative: the transform is s^{α} times the transform of the function. In fact, the generalized derivative satisfies many of the same properties as the conventional derivative, particularly linearity and the composition property [27:69-84]:

$$D^{\alpha}[y(t) + x(t)] = D^{\alpha}[y(t)] + D^{\alpha}[x(t)] \tag{150}$$

$$D^{\alpha}[D^{\beta}[x(t)] = D^{\alpha+\beta}[x(t)] \tag{151}$$

Linearity holds for α and β greater than or equal to zero. The compostion property holds for functions bounded at $t = 0$.

The Fourier transform is defined as

$$F[x(t)] \equiv \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt \tag{152}$$

If $x(t) = 0$ for $t < 0$, then the Fourier transform can be written as

$$F[x(t)] = \int_0^{\infty} x(t)\, e^{-i\omega t}\, dt \tag{153}$$

It is easily seen that the Fourier transform of a generalized derivative is

$$F[D^{\alpha}[x(t)]] = (i\omega)^{\alpha}F[x(t)] \tag{154}$$

In the preceding discussion, the only restriction placed on α was that it be a nonnegative real number less than one. However, for engineering applications, an irrational number can be approximated by a rational number. So α will now be restricted to be rational as well. Using the term "fractional derivative" will indicate this additional restriction.

The standard linear viscoelastic model relating stress and strain that was used in Chapter III will be useful in illustrating the use of fractional derivatives in viscoelastic theory:

$$\sigma(t) + \sum_{i=1}^{k} b_i \frac{d^i\sigma(t)}{dt^i} = E_0\epsilon(t) + \sum_{i=1}^{j} E_i \frac{d^i\epsilon(t)}{dt^i} \tag{75}$$

Recalling Scott-Blair's proposal that fractional derivatives could be used to relate time-dependent stress and strain in viscoelastic materials, replace the conventional derivatives in Eq (75) by derivatives of fractional order. The result is the general form of the fractional derivative viscoelastic model [4]:

$$\sigma(t) + \sum_{m=1}^{M} b_m D^{\beta_m}[\sigma(t)] = E_0\epsilon(t) + \sum_{n=1}^{N} E_n D^{\alpha_n}[\epsilon(t)] \tag{155}$$

A large number of materials can be modelled by replacing each sum in Eq (155) by a single term involving a fractional derivative:

$$\sigma(t) + b\, D^{\beta}[\sigma(t)] = E_0\epsilon(t) + E_1 D^{\alpha}[\epsilon(t)] \tag{156}$$

Invoking the Second Law of Thermodynamics requires that the parameters satisfy the following constraints [6]:

$$\begin{aligned} E_0 &\geq 0 & E_1 &\geq bE_0 \\ E_1 &\geq 0 & \alpha &= \beta \\ b &> 0 & & \end{aligned} \tag{157}$$

These constraints ensure nonnegative energy dissipation and nonnegative work. The stress-strain relation in the Laplace domain is

$$\frac{\sigma(s)}{\epsilon(s)} = \frac{E_0 + E_1 s^{\alpha}}{1 + bs^{\alpha}} \tag{158}$$

This is known as the four parameter model, and has been shown to be very accurate over several decades of frequency [5, 32, 33].

Unfortunately, using the same approach as in the classically modelled viscoelastic problem will lead to the appearance of a fractional order derivative on the control forces as well as on the position vector [34]. Integration by parts after taking the variation of J leads to an infinite sum. To find a way around these difficulties for a structure modeled using fractional derivatives, the fact that the position vector is a function comprised of exponentially decaying sinusoids and algebraically decaying relaxation modes will be important. The relaxation modes describe the non-oscillatory return of the structure to its equilibrium position. Hannsgen and Wheeler [16] point out the fallacy of trying to control relaxation modes: decay rates are essentially the same as those without feedback. They note that separating motion into creep modes and oscillatory modes appears to be of fundamental importance in stabilization and control problems involving viscoelastic damping. They also point out that systems in which viscoelasticity is modelled using fractional derivative models are among the systems that are the least susceptible to destabilization due to delays in boundary feedback [15].

Hence, the design techniques presented in this chapter only address the control of the exponentially decaying modes. The error introduced by this approximation is given in Appendix D, and is bounded. To control only exponentially decaying modes, one can approximate the state space equations by retaining only the exponential eigenvalues and eigenvectors. The next section derives this approximation.

4.2 Simultaneous Optimization of Viscoelastic Damping and Active Control

In this section, solution techniques for determining viscoelastic coefficients and active feedback gains that optimize performance indices like those in Section 2.2.2 are provided. A key point will be the derivation of the approximation of the first order state equations in the time domain.

Before deriving the time domain approximation of the first order state equations, the equations of motion will be expressed in the Laplace domain. The stress-strain relation in the Laplace domain is represented by $G(s)$:

$$G(s) = \frac{G_0 + G_1 s^\alpha}{1 + b s^\alpha} \tag{159}$$

The use of G_0 and G_1 instead of E_0 and E_1 indicates that the stresses are shear stresses. The equations of motion in the Laplace domain for a multiple degree of freedom system with viscoelastic damping modelled using the four parameter fractional derivative model are given by [4]

$$s^2\mathbf{M}\mathbf{x}(s) + G(s)\mathbf{K_v}\mathbf{x}(s) + \mathbf{K}\mathbf{x}(s) + \mathbf{b}\mathbf{u}(s) = \mathbf{0} \tag{160}$$

Before expressing the equations of motion in the time domain, it is appropriate to define the stress relaxation modulus. The stress relaxation modulus $G_{rel}(t)$ gives the time history of the stress in a viscoelastic material which is subjected to a unit step in strain at time $t = 0$ [35]. Thus the relaxation modulus is

$$G_{rel}(t) = L^{-1}\left\{\frac{G(s)}{s}\right\} \tag{161}$$

As in the previous two chapters, the matrix $\mathbf{K_v}$ is symmetric and an be expressed as

$$\mathbf{K_v} = \mathbf{b_v}\mathbf{C}\mathbf{b_v}^T$$

where $\mathbf{C}$ is a diagonal matrix of the damping coefficients to be selected by the design process.

To control the exponentially decaying modes of the structure, an approximation to the equations of motion represented by Eq (160) will be formulated in the time domain as a first order system that has identical eigenvectors amd eigenvalues as the harmonic exponential response of Eq (160). This will require an initial guess as to the values of the viscoelastic damping coefficients.

Taking the inverse Laplace transform of Eq (160), the equations of motion can be written in the time domain

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{b_v}\left[\mathbf{C}\mathbf{b_v}^T\frac{d}{dt}\int_0^t G_{rel}(t-\tau)\mathbf{x}(\tau)d\tau\right] + \mathbf{K}\mathbf{x}(t) + \mathbf{b}\mathbf{u}(t) = \mathbf{0} \tag{162}$$

In first order form, Eq (162) becomes

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_{\mathbf{v}} \frac{d}{dt} \int_0^t G_{rel}(t-\tau)\mathbf{x}(\tau)d\tau \end{Bmatrix} \right) + \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b} \end{bmatrix} \mathbf{u} \tag{163}$$

Defining a state vector in tems of position and velocity,

$$\mathbf{y} = \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix}$$

it is apparent that the term in parentheses in Eq (163) is a time dependent vector function of the state. It will be represented by an operator $\mathbf{A}$ acting on time and state, $\mathbf{A}(t,\mathbf{y})$. Hence, with the input matrix defined as

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{b} \end{bmatrix}$$

the equations of motion are expressed in compact, first order form:

$$\dot{\mathbf{y}} = \mathbf{A}(t,\mathbf{y}) + \mathbf{B}\mathbf{u} \tag{164}$$

The time dependent vector $\mathbf{A}(t,\mathbf{y})$ can be approximated by $\hat{\mathbf{A}}\mathbf{y}$ where $\hat{\mathbf{A}}$ is constructed using the eigenvectors and eigenvalues associated with the exponentially decaying modes of the open loop ($\mathbf{u} = \mathbf{0}$) equations of motion. Devereaux [11] presents an algorithm which calculates all the eigenvectors and the eigenvalues of systems in the form of Eq (160). The eigenvectors and the eigenvalues of interest here are those that lie in the upper half of the principal sheet of the s-plane (Im $(\lambda_i > 0)$). The matrix $\mathbf{\Psi}$ will represent the matrix of eigenvectors associated with exponential decay,

$$\mathbf{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \dots & \boldsymbol{\psi}_n \end{bmatrix}$$

where n is the length of the position vector $\mathbf{x}$. A diagonal matrix of the corresponding eigenvalues will be represented by $\mathbf{\Lambda}$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The approximation of $\mathbf{A}(t, \mathbf{y})$ will be constructed using the matrices of eigenvalues $\mathbf{\Lambda}$ and eigenvectors $\mathbf{\Psi}$, and their conjugates, $\overline{\mathbf{\Lambda}}$ and $\overline{\mathbf{\Psi}}$. Because the approximate plant matrix $\hat{\mathbf{A}}$ will be constructed using eigenvectors, eigenvalues, and their conjugates, only those systems that are underdamped will be considered.

The next step is to define a diagonal matrix $\tilde{\mathbf{\Lambda}}$ of the composite eigenvalues and their conjugates, and a modal matrix $\tilde{\mathbf{\Psi}}$:

$$\tilde{\mathbf{\Lambda}} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}} \end{bmatrix} \qquad \tilde{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} & \overline{\mathbf{\Psi}} \\ \mathbf{\Psi\Lambda} & \overline{\mathbf{\Psi\Lambda}} \end{bmatrix} \tag{165}$$

The approximate state matrix can now be written in terms of the eigenvalue matrix $\tilde{\mathbf{\Lambda}}$ and the modal matrix $\tilde{\mathbf{\Psi}}$:

$$\hat{\mathbf{A}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{\Psi}}^{-1} \tag{166}$$

An alternate but equivalent derivation of the approximation of $\mathbf{A}(t, \mathbf{y})$ is given in Appendix E. This alternate derivation shows that the approximation $\hat{\mathbf{A}}$ is a real matrix, even though the matrices $\tilde{\mathbf{\Psi}}$ and $\tilde{\mathbf{\Lambda}}$ are complex.

Although an approximation for the state equations has been found,

$$\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{B}\mathbf{u} \tag{167}$$

an approximation of the viscoelastic damping forces is still needed. A performance index similar to one used in Chapters II and III, Eq (15) will be used in this chapter also, so in order to weight the viscoelastic forces, an appropriate representation will be found.

As in the previous two chapters, again let $\mathbf{v}(t)$ represent the viscoelastic damping forces. Before attempting to approximate these forces, the position vector and its fractional derivative will be expressed in terms of a vector of modal coordinates $\boldsymbol{\eta}(t)$. The position can be expressed as a product of the eigenvalue matrix and the vector of modal coordinates:

$$\mathbf{x}(t) = \boldsymbol{\Psi}\boldsymbol{\eta}(t) \qquad (\boldsymbol{\eta} = \boldsymbol{\Psi}^{-1}\mathbf{x}) \tag{168}$$

Taking the time derivatives of both sides yields an expression for velocity:

$$\dot{\mathbf{x}}(t) = \boldsymbol{\Psi}\dot{\boldsymbol{\eta}}(t) \qquad (\dot{\boldsymbol{\eta}} = \boldsymbol{\Psi}^{-1}\dot{\mathbf{x}}) \tag{169}$$

The generalized derivative of the modal coordinate vector is just the eigenvalue matrix raised to the order of the derivative (neglecting the algebraic response):

$$D^{\alpha}\boldsymbol{\eta}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha}\boldsymbol{\eta}(t) \tag{170}$$

In particular, the first derivative of the modal vector is the eigenvalue matrix times the modal vector:

$$\dot{\boldsymbol{\eta}}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\eta}(t) \qquad (\boldsymbol{\eta} = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\eta}}) \tag{171}$$

From Eqs (170) and (171), it is apparent that

$$D^{\alpha}\dot{\boldsymbol{\eta}}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha}\dot{\boldsymbol{\eta}}(t) \tag{172}$$

Taking the α order derivative of both sides of Eqs (168) and (169) and applying Eqs (170) and (172) yields expressions for the α-order derivatives of position and velocity:

$$D^{\alpha}\mathbf{x}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha}\boldsymbol{\eta}(t) \tag{173}$$

$$D^{\alpha}\dot{\mathbf{x}}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha}\dot{\boldsymbol{\eta}}(t) \tag{174}$$

Recalling Eq (171), and also Eq (169), the α-order derivative of position can be written in terms of velocity:

$$D^{\alpha}\mathbf{x}(t) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha}\left(\boldsymbol{\Lambda}^{-1}\dot{\boldsymbol{\eta}}(t)\right) \tag{175}$$

$$= \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\alpha-1}\left(\boldsymbol{\Psi}^{-1}\dot{\mathbf{x}}(t)\right) \tag{176}$$

The relations given by Eqs (173), (174), and (176) will be used in the time domain formulation of viscoelastic forces.

In the Laplace domain, the viscoelastic forces are given by the second term in Eq (160):

$$\mathbf{v}(s) = \mathbf{C}\frac{G_0 + G_1 s^\alpha}{1 + bs^\alpha}\mathbf{b_v}^T\mathbf{x}(s)$$

Multiplying both sides by $(1 + bs^\alpha)$ and neglecting the algebraic portion of the fractional derivative once again, this equation can be written in the time domain:

$$\begin{aligned}(1 + bD^\alpha)\mathbf{v}(t) &= \mathbf{C}(G_0 + G_1 D^\alpha)\mathbf{b_v}^T\mathbf{x}(t) \\ &= G_0\mathbf{C}\mathbf{b_v}^T\mathbf{x}(t) + G_1\mathbf{C}\mathbf{b_v}^T D^\alpha\mathbf{x}(t)\end{aligned}$$

Substituting in Eq (176) yields the following expression:

$$(1 + bD^\alpha)\mathbf{v}(t) = G_0\mathbf{C}\mathbf{b_v}^T\mathbf{x}(t) + G_1\mathbf{C}\mathbf{b_v}^T\mathbf{\Psi}\mathbf{\Lambda}^{\alpha-1}\mathbf{\Psi}^{-1}\dot{\mathbf{x}} \tag{177}$$

It is desired that the time domain representation of the viscoelastic forces $\mathbf{v}(t)$ be linear in the state $\mathbf{y}$ and the matrix of viscoelastic parameters $\mathbf{C}$, as was the case in Chapters II and III. So the form of $\mathbf{v}(t)$ that will be sought is the same as that in the previous two chapters,

$$\mathbf{v}(t) = \mathbf{C}\mathbf{\Phi}\begin{Bmatrix}\mathbf{x} \\ \dot{\mathbf{x}}\end{Bmatrix} = \mathbf{C}\mathbf{\Phi}\mathbf{y} \tag{178}$$

where $\mathbf{\Phi}$ is a matrix to be determined. The α-derivative of Eq (178) will be shown to be linear in the state also. First the α-derivative of $\mathbf{v}(t)$ will be expressed in terms of the modal vector $\boldsymbol{\eta}(t)$ by taking the α-derivative of Eq (178) and using Eqs (173) and (174) to express the α-derivatives of position and velocity in tems of the modal vector:

$$D^\alpha\mathbf{v}(t) = \mathbf{C}\mathbf{\Phi}\begin{Bmatrix}D^\alpha\mathbf{x} \\ D^\alpha\dot{\mathbf{x}}\end{Bmatrix} = \mathbf{C}\mathbf{\Phi}\begin{Bmatrix}\mathbf{\Psi}\mathbf{\Lambda}^\alpha\boldsymbol{\eta} \\ \mathbf{\Psi}\mathbf{\Lambda}^\alpha\dot{\boldsymbol{\eta}}\end{Bmatrix}$$

The definition of the modal vector is used to bring position and velocity back into the expression for $D^\alpha\mathbf{v}(t)$:

$$D^\alpha\mathbf{v}(t) = \mathbf{C}\mathbf{\Phi}\begin{Bmatrix}\mathbf{\Psi}\mathbf{\Lambda}^\alpha\mathbf{\Psi}^{-1}\mathbf{x} \\ \mathbf{\Psi}\mathbf{\Lambda}^\alpha\mathbf{\Psi}^{-1}\dot{\mathbf{x}}\end{Bmatrix}$$

So the α-derivative of $\mathbf{v}(t)$ is linear with respect to the state $\mathbf{y}$:

$$D^{\alpha}\mathbf{v}(t) = \mathbf{C\Phi}\begin{bmatrix} \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} = \mathbf{C\Phi}\begin{bmatrix} \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\mathbf{y}$$

Multiplying this last equation by the model coefficient b and adding Eq (178) gives an alternate expression for the left hand side of Eq (177):

$$(1 + bD^{\alpha})\mathbf{v}(t) = \mathbf{C\Phi y} + b\mathbf{C\Phi}\begin{bmatrix} \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\mathbf{y} \tag{179}$$

Applying the distributive property,

$$(1 + bD^{\alpha})\mathbf{v}(t) = \mathbf{C\Phi}\left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + b\begin{bmatrix} \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\right)\mathbf{y} \tag{180}$$

The addition of the two composite matrices can be written as one composite matrix:

$$(1 + bD^{\alpha})\mathbf{v}(t) = \mathbf{C\Phi}\begin{bmatrix} \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\mathbf{y} \tag{181}$$

Setting the right hand sides of Eq (177) and (181) equal will help in finding an expression for the matrix $\mathbf{\Phi}$:

$$\begin{aligned} \mathbf{C\Phi}\begin{bmatrix} \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix}\mathbf{y} &= G_0\mathbf{Cb_v}^T\mathbf{x} + G_1\mathbf{Cb_v}^T\mathbf{\Psi\Lambda}^{\alpha-1}\mathbf{\Psi}^{-1}\dot{\mathbf{x}} \\ &= \begin{bmatrix} G_0\mathbf{Cb_v}^T & G_1\mathbf{Cb_v}^T\mathbf{\Psi\Lambda}^{\alpha-1}\mathbf{\Psi}^{-1} \end{bmatrix}\mathbf{y} \end{aligned}$$

Hence, for general $\mathbf{y}$,

$$\mathbf{C\Phi}\begin{bmatrix} \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + b\mathbf{\Psi\Lambda}^{\alpha}\mathbf{\Psi}^{-1} \end{bmatrix} = \mathbf{Cb_v}^T\begin{bmatrix} G_0\mathbf{I} & G_1\mathbf{\Psi\Lambda}^{\alpha-1}\mathbf{\Psi}^{-1} \end{bmatrix} \tag{182}$$

Post-multiplying this equation by the inverse of the last matrix on the left-hand side,

$$\begin{bmatrix} (\mathbf{I} + b\Psi\Lambda^{\alpha}\Psi^{-1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} + b\Psi\Lambda^{\alpha}\Psi^{-1})^{-1} \end{bmatrix} \tag{183}$$

yields an expression for the matrix product $\mathbf{C\Phi}$:

$$\mathbf{C\Phi} = \mathbf{Cb_v}^T \begin{bmatrix} G_0(\mathbf{I} + b\Psi\Lambda^{\alpha}\Psi^{-1})^{-1} & G_1\Psi\Lambda^{\alpha-1}\Psi^{-1}(\mathbf{I} + b\Psi\Lambda^{\alpha}\Psi^{-1})^{-1} \end{bmatrix} \tag{184}$$

If one pre-multiplies this expression by the inverse of $\mathbf{C}$, the matrix $\mathbf{\Phi}$ can be expressed explicitly. However, it always appears in the following derivation as part of the product $\mathbf{C\Phi}$. Also, since the eigenvalue and eigenvector matrices depend on the value of the viscoelastic coefficients, leaving $\mathbf{\Phi}$ expressed as part of the product $\mathbf{C\Phi}$ emphasizes that the two cannot be separated in this case.

Now that an expression has been found for the matrix product $\mathbf{C\Phi}$, the damping forces can be weighted in the performance index:

$$J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v})dt \tag{15}$$

Using the relation in Eq (178) for the damping forces $\mathbf{v}(t)$ yields a performance index like the one in Section 2.2.1:

$$J = \int_{t_0}^{t_f} \frac{1}{2}[\mathbf{y}^T(\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi})\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u}]dt \tag{38}$$

Defining a new state weighting matrix,

$$\mathbf{Q} = (\mathbf{Q} + \mathbf{\Phi}^T\mathbf{CSC\Phi})$$

the performance index has a form identical to Eq (7) of Section 2.1:

$$J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \tag{185}$$

Now that expressions have been found for the performance index, the equations of motion, and the passive damping, the simultaneous design problem of active vibration control and passive viscoelastic damping can be solved. The problem is to find the damping coefficients and active feedback gains that

$$\text{minimize: } J = \int_{t_0}^{t_f} \frac{1}{2}(\mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt \qquad (185)$$

$$\text{subject to: } \dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{B}\mathbf{u} \qquad (167)$$

$$\mathbf{v} = \mathbf{C}\boldsymbol{\Phi}\mathbf{y} \qquad (178)$$

This resembles the standard LQR problem, which is solved by

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{y} \qquad (11)$$

and results in a value of the performance index given by

$$J = \frac{1}{2}\mathbf{y_0}^T\mathbf{P}\mathbf{y_0} \qquad (41)$$

where $\mathbf{P}$ satisfies

$$\mathbf{P}\hat{\mathbf{A}} + \hat{\mathbf{A}}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \hat{\mathbf{Q}} = \mathbf{0}$$

However, the performance index J has not yet been minimized with respect to the damping coefficients since the matrix $\mathbf{P}$ is a function of the unknown damping coefficient matrix $\mathbf{C}$. To minimize with respect to the damping coefficients, one can either specify the initial conditions $\mathbf{y}_0$ or minimize the average of the performance index J or the maximum value of the performance index J over the unit ball, $||\mathbf{y}_0|| = 1$. Since solutions independent of initial conditions are preferred, only the latter two techniques are considered in this chapter. The techniques follow the same algorithms as those in Sections 2.2.2 and 2.2.3. However, in computing $\hat{\mathbf{Q}}$, the matrix $\mathbf{C}\boldsymbol{\Phi}$ is computed using Eq (184). The matrix $\hat{\mathbf{A}}$ is computed using Eq (166).

The next section will demonstrate that the solution procedure is valid when the order of the fractional derivative is $\alpha = 1$. That is, when it reduces to the case of viscous damping.

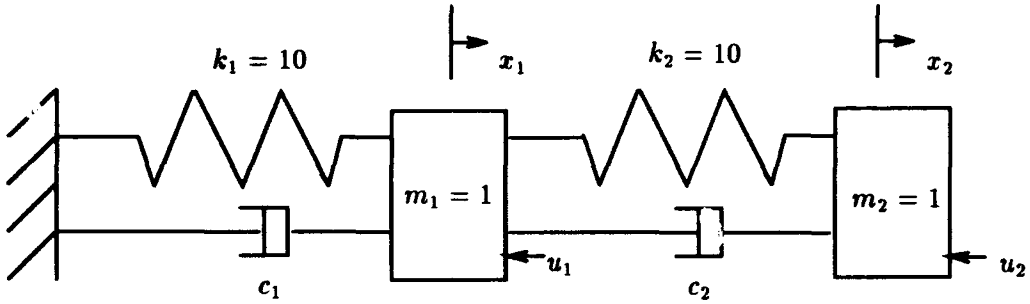

Figure 8. Two DOF Viscous System

4.3 Validation of Techniques for Viscous Case

This section is a verification of the algorithms and computer programs that implement the solution techniques presented in the previous section. A FORTRAN program that is a modification of the one written by Devereaux [11] computes the eigenvalues and eigenvectors and passes them to a MATLAB routine which makes the necessary calculations to minimize the performance index. If the programs return the same dashpot constants and feedback gains for viscous damping as those calculated using the techniques developed in Chapter II, there is a high degree of confidence that the results obtained for viscoelastic damping will be valid. The fractional derivative model reduces to that of a viscous damper if the order of the derivative on strain is one:

$$G_0 = 0 \quad \alpha = 1$$
$$G_1 = 1 \quad b = 0$$

Although G_1 is not required to be equal to one for the four parameter model to reduce to that of a viscous damper, it is set equal to one here so that the resulting equations of motion are identical to those in Chapter II. In this case, the technique described in the previous section ought to return the same parameters as the technique for viscous dampers given in Section 2.2.2. Damping parameters (or dashpot constants) were computed using both techniques for a single degree of freedom system, a two degree of freedom system, and a fourteen degree of freedom system. The single degree of freedom system used is the one described by Eqs (64) and (65). The damping parameters obtained using the min $\|\mathbf{P}\|_2$ solution procedures (c_2) are identical, as shown in Table 16. The damping parameters obtained using the min (trace $\mathbf{P}$) solution procedures (c_{tr}) are essentially equal also.

The two degree of freedom system is shown in Figure 8. The first order system of equations is

$$
\dot{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20 & 10 & 0 & 0 \\ 10 & -10 & 0 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{y}
$$

$$
= \mathbf{Ay} + \mathbf{Bu} + \mathbf{B_v C \Phi y}
$$

The weighting matrices were chosen to be

$$
\mathbf{Q} = \begin{bmatrix} 20 & -10 & 0 & 0 \\ -10 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \mathbf{I}, \quad \mathbf{S} = 0.5\mathbf{I} \tag{186}
$$

As seen in Table 16, the damping parameters for both techniques were identical for both the min $\|\mathbf{P}\|_2$ and min (trace $\mathbf{P}$) solution procedures.

The fourteen degree of freedom system is the same as in Section 2.2.4.3. Table 16 shows that the damping parameters obtained for this system for both techniques are in excellent agreement.

So by applying both the minimization of the average performance index (c_{tr}) and the minimization of the maximum value of the performance index (c_2) techniques from Sections 2.2.2 and 4.2, it has been shown that the techniques of Section 4.2 are equivalent to those of Section 2.2.2 for the case of viscous damping.

Table 16. Viscous vs Fractional Derviative

			c_1	c_2	c_3	c_4	c_5
SDOF	c_{tr}	vis	.8869				
		fv	.8872				
	c_2	vis	.4703				
		fv	.4703				
2DOF	$\mathbf{c}_{tr}$	vis	1.1330	1.0378			
		fv	1.1330	1.0378			
	$\mathbf{c}_2$	vis	1.0155	1.0155			
		fv	1.0155	1.0155			
14DOF	$\mathbf{c}_{tr}$	vis	54.3826	35.4502	38.4443	42.1846	46.8419
		fv	54.3825	35.4503	38.4458	42.1846	46.8414
	$\mathbf{c}_2$	vis	76.1842	17.5549	28.5870	34.2785	45.5261
		fv	76.1846	17.5585	28.5835	34.2784	45.5242

vis = viscous

fv = fractional derivative with $\alpha = 1$

4.4 Transition from Viscous to Viscoelastic

The last section verified that the solution techniques derived in this chapter for fractional order systems are valid for the case of viscous damping. This section will show that the viscoelastic damping parameters (the diagonal elements of $\mathbf{C}$) are smooth functions of the viscoelastic parameters G_0, α, and b. Hence, it can be inferred that the solution technique for fractional order systems is valid for all values of α from $\alpha = 0$ to $\alpha = 1$.

First, α will be varied incremently from $\alpha = 1$ to $\alpha = 0$, while keeping $G_0 = 0$ and $b = 0$. This is equivalent to varying the viscoelastic modulus $G(s)$ from pure viscous to pure elastic.

The two degree of freedom system of the previous section (Figure 8 and Eqs (186) and (186)) will be used in the calculations. Figure 9 starts with the damping parameters (c_i) for the pure viscous model ($\alpha = 1$, $G_0 = 0$, $b = 0$):

$$G(s) = 7.3s$$

The reasons for choosing $G_1 = 7.3$ will become apparent later. As α is decreased incrementally down to zero, Figure 9 shows the resulting damping coefficients for both solution techniques, min $\|\mathbf{P}\|_2$ and min trace($\mathbf{P}$). Figure 9 shows that the transition from pure viscous to pure elastic ($\alpha = 0$)

$$G(s) = 7.3$$

is smooth within the resolution of the data and monotonically decreasing.

Next, G_0 and b are varied. The end result will be the model which will be used in the example problem in the following section. Hence, the fractional order α will be set to $\alpha = 4/7$, and G_0 will be increased from 0 to 1. The parameter b will be increased to 0.0008. Thus, Figure 10 shows that the transition from the viscoelastic modulus

$$G(s) = 7.3s^{4/7}$$

to the viscoelastic modulus

$$G(s) = \frac{1 + 7.3s^{4/7}}{1 + .0008s^{4/7}} \tag{187}$$

is also smooth within the resolution of the data. Eq (187) is based on a model for 3M–467 adhesive at 75° F, a viscoelastic material used for damping [3:126]. The constants G_0, G_1, and b have the same value as the actual model, but 4/7 is an approximation for the order of the fractional derivatives on stress and strain. Reference [3] gives the order of the derivatives on stress and strain as .51 and .56 respectively. Since $4/7 \approx .57$, $\alpha = 4/7$ is a good approximation for order of the fractional derivative on strain, whose coefficient is four orders of magnitude greater than the fractional derivative on stress. Using a lower order fraction simplifies the calculation of the eigenvectors and eigenvalues.

4.5 *Example Problem - Fourteen DOF System*

In this section, the fractional order solution technique will be applied to the viscoelastically damped planar aluminum truss of the previous chapter (Figure 5). The viscoelastic material, 3M-467 adhesive, was modeled with the four parameter model given by Eq (187). The modulus plot of $G(s)$ is shown in Figure 11, where the real part is plotted as G' and the imaginary part as G''. The vertical lines in the graph indicate the lowest and highest frequencies, and show that the structure's natural frequencies lie well within the transition region. The state weighting matrix $\mathbf{Q}$ was chosen such that $\mathbf{y}^T\mathbf{Q}\mathbf{y}$ equals the total mechanical energy in the system, while the weightings on active control and passive damping were chosen to be

$$\mathbf{R} = 0.1\,\mathbf{I}, \qquad \mathbf{S} = 10^{-5}\mathbf{I}$$

The solutions were computed using the algorithms given in Section 4.2 and Section 2.2.2. The eigenvectors and eigenvalues of the system for a given set of damping

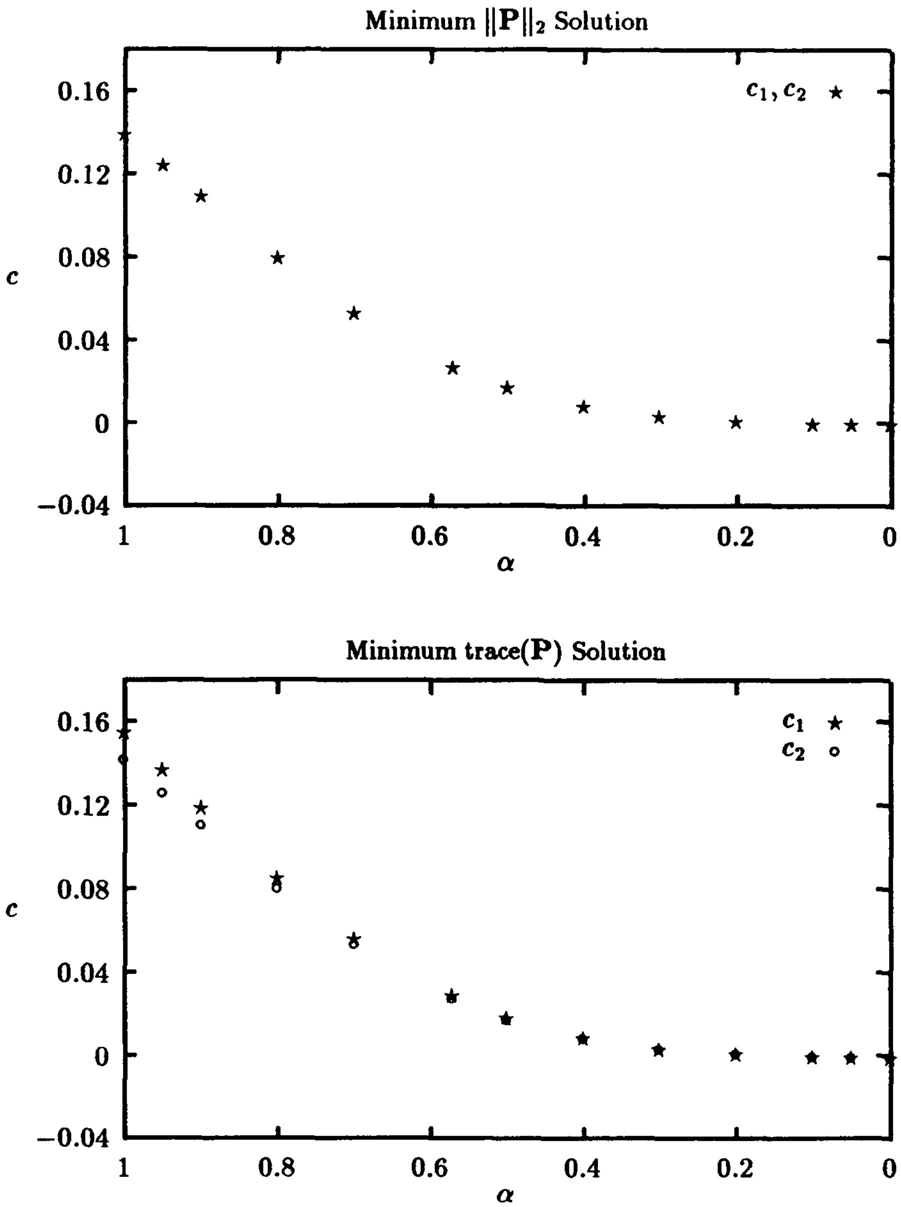

Figure 9. Two DOF Example — Transition From Pure Viscous to Pure Elastic

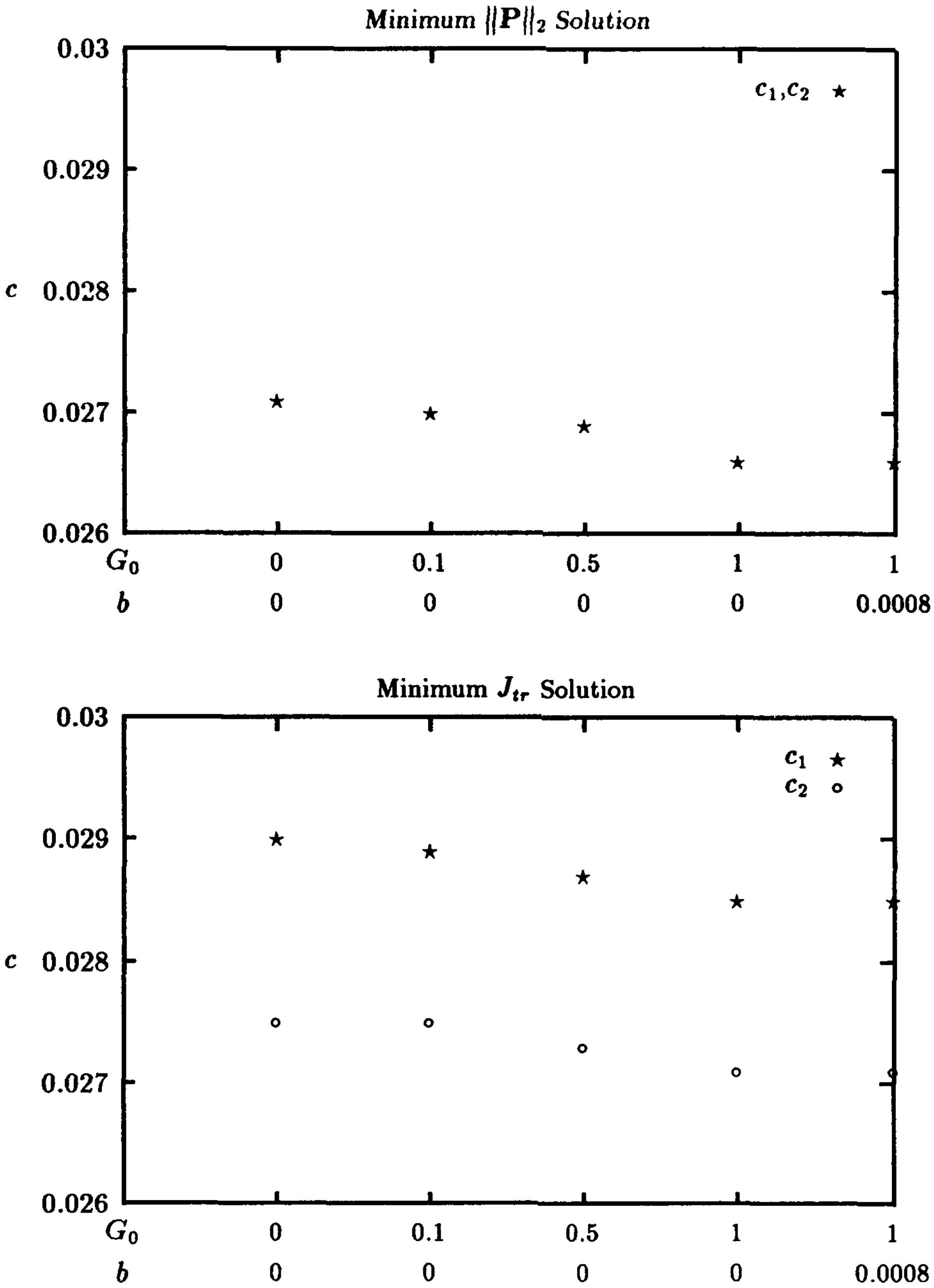

Figure 10. Example – Transition to Full Model

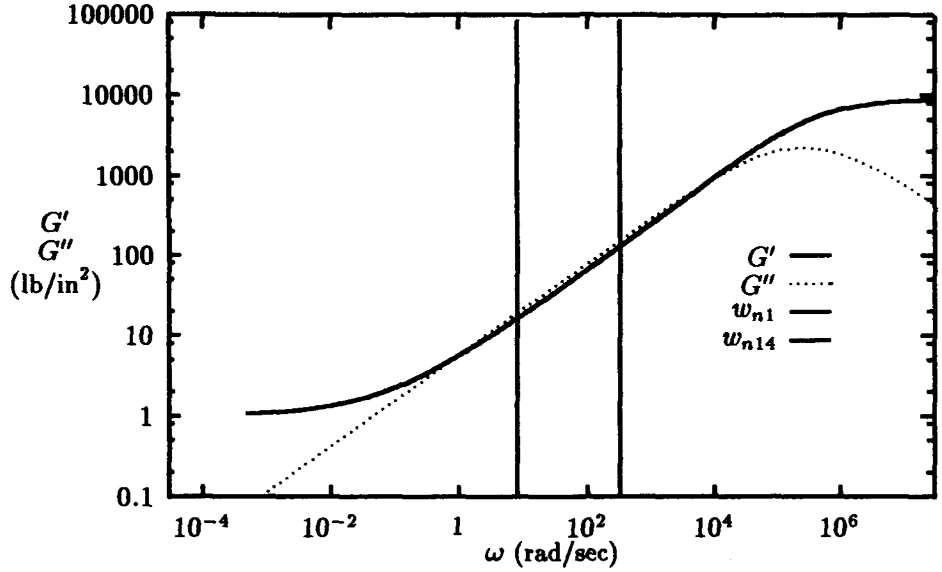

Figure 11. Viscoelastic Modulus – $G_1 = 7.3$, $b = .0008$

parameters **C** were computed by a FORTRAN program that used the algorithm developed by Devereaux (see Reference [11]). The eigenvectors and eigenvalues were passed to a MATLAB routine which computed the approximate state matrix, the appropriate gains for that state matrix, and the resulting value of $||\mathbf{P}||_2$ or trace(**P**), depending on which one was being minimized. A new set of damping coefficients was then selected using the MATLAB program FMINS, and the new eigenvalues and eigenvectors computed. The iteration continued until convergence occurred.

The values of the resulting damping parameters (c_i) for the two solution procedures, min $||\mathbf{P}||_2$ and min (trace **P**), are shown in Table 17. The system damping ratios with and without active control are included as well. The active control gains were computed using Eq (11) and are given in Appendix F.

Since the properties of viscoelastic materials are temperature dependent, it is conceivable that a drastic change in temperature could result in a rather drastic change in properties. Assume this has occurred, and the modulus $G(s)$ is now given by

$$G(s) = \frac{1 + 730s^{4/7}}{1 + .08s^{4/7}}$$

The real part of the modulus is plotted as G' in Figure 12, while the imaginary part is plotted as G''. Recomputing the viscoelastic parameters for this case produces the results shown in Table 18. Even though the natural frequencies are now very near the glassy

Table 17. Fractional Viscoelastic with $G_1 = 7.3$, $b = .0008$

	min trace $\mathbf{P}$	min $\|\mathbf{P}\|_2$
c_1	4.2782	3.6768
c_2	3.9100	4.4106
c_3	4.2057	5.8892
c_4	5.2116	6.0427
c_5	6.0598	6.5551
ζ – Passive only	.0003 to .0080	.0004 to .0092
ζ – Complete Solution	.0041 to .0340	.0044 to .0341

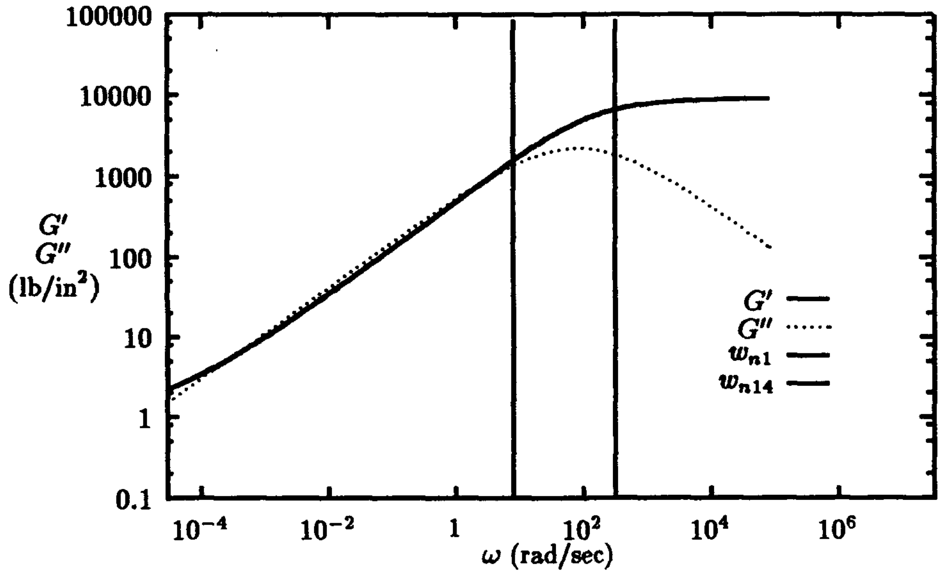

Figure 12. Viscoelastic Modulus Plot – $G_1 = 730$, $b = .08$

Table 18. Fractional Viscoelastic with $G_1 = 730$, $b = .08$

	min trace $\mathbf{P}$	min $\|\mathbf{P}\|_2$
c_1	0.0563	0.0085
c_2	0.0576	0.0001
c_3	0.0722	0.1592
c_4	0.0661	0.0497
c_5	0.1048	0.1194
ζ – Passive only	.0002 to .0029	.0001 to .0036
ζ – Complete Solution	.0023 to .0340	.0025 to .0340

region, the solution techniques can still be used. However, damping relies more heavily on active control. The active control gains are given in Appendix F.

The results above indicate that the viscoelastic material must be carefully chosen such that the natural frequencies of the problem structural modes lie in the transition region for significant viscoelastic damping to occur. This is not a concern with viscous dampers because they have no high and low frequency plateaus.

4.6 Summary

This chapter presents two new design techniques to determine optimal blending of passive viscoelastic damping and active vibration control. The techniques are iterative in nature, and are based on a fractional order derivative viscoelastic model and on modified versions of the standard linear quadratic regulator performance index of optimal control theory, given by Eq (7).

The performance index, Eq (15), treats passive damping as a separate control force, which results in an additional energy term in the performance index. By neglecting the algebraic modes due to the viscoelastic dampers, two of the solution techniques developed in Chapter II can be applied to the viscoelastically damped structure: the minimization of the average value of the performance index and the minimization of the maximum value of the performance index. These solution techniques are useful when the viscoelastic material is well modelled by a four parameter fractional order model.

V. Conclusions and Recommendations

The new design techniques presented in this dissertation provide design engineers with the ability to determine complementary feedback gains and passive damping parameters independent of initial conditions and the ability to handle viscoelastic dampers when determining the optimum blending of active vibration control and passive structural damping. In deriving new design techniques to determine optimal blending of passive structural damping and active vibration control, the treatment of passive structural damping forces in this dissertation is unique. Usually passive structural damping forces are just included as part of the state, but here they are weighted separately from the state in the linear quadratic regulator (LQR) performance index. The passive damping forces are treated as co-equals to the active control forces, and are weighted in a similar manner. The modification to the performance index is minor, but allows structural damping to be treated as a control parameter in LQR theory. Viewing structural damping as an LQR parameter increases the usefulness of LQR theory by enabling the design of complementary feedback gains and passive damping parameters. This reduces the active control effort needed to meet specifications, allowing the design engineer to take full advantage of each ounce of damping material. It also allows the design engineer to avoid implementing excess damping that inadvertently increases response times.

In addition, the techniques developed in this dissertation enable the design engineer to select complementary feedback gains and passive damping parameters that are independent of initial conditions. Usually in a problem dealing with active and passive control, initial conditions are set before working the problem. This restriction was not used in this dissertation, so the solution techniques presented here lead to designs that are more robust with respect to initial conditions.

Another restriction relaxed in this dissertation is related to the structural damping. In control problems, structural damping is often approximated by viscous damping, since other types of damping lead to complicated equations of motion. This dissertation develops the techniques for viscous damping, but extends the results to the more realistic case of viscoelastic damping.

Three techniques were developed for the viscous case — one closed form and two iterative. The closed form solution is a least squares solution. One iterative technique minimizes the maximum value of the proposed cost functional with respect to the initial state, while the other minimizes the average value of the proposed cost functional.

Both techniques yielded similar performance relative to each other, and much improved performance relative to a system with no passive damping.

Three similar techniques were also developed for viscoelasticity modelled with a classic model using integer order derivatives on stress and strain. The closed form solution proved to be much cheaper computationally than the iterative techniques, while giving similar performance.

The classic viscoelastic approach can be rather difficult to implement for large structures as the order of the state space equations increases dramatically with higher order derivatives on stress and strain. The order of the state space equations was kept low in the approach developed for the fractional derivative viscoelastic model. Another benefit of using the fractional derivative model is that it is often a more accurate description of the materials behavior. Two iterative techniques, similar to those for the viscous case, were developed for this case. A closed form solution was not possible due to the nature of the problem.

All of the techniques derived in this dissertation are applicable to structures that can be modelled with finite elements. The techniques are useful in cases where the baseline structure is well-defined and the design engineer wishes to use a combination of active and passive vibration suppression. If the damping is adequately represented by a viscous model, then the corresponding solution techniques should be used. The Frobenius norm solution gives a quick, rough-cut answer, and is adequate for many applications. If a better performance is required, the minimization of the average value of the performance index is recommended. It will tend to give better response times and takes less time to compute. The minimization of the maximum value of the performance index is recommended only if the design engineer is concerned that the minimization of the average value of the performance index may yield high values of the performance index for certain initial conditions.

If the damping is to be modelled with a classical viscoelastic model, the design engineer will use the Frobenius norm solution. If the model only has first derivatives on stress and strain, the design engineer may want to refine his design by minimizing the maximum value of the performance index. This method is recommended over minimizing the maximum value of the performance index for the same reasons as in the viscous case.

Also for the same reasons, minimizing the average value of the performance index is recommended for the case in which viscoelastic damping is modelled using a fractional derivative model. A four parameter fractional derivative model is recommended over a classical model for the reasons cited above: it keeps the order of the state space equations low and it is often a more accurate model.

For situations in which the design engineer wishes to weight passive damping as a one-time cost, two solution techniques were developed for structures with viscous damping. It would be a straightforward task to extend these solution techniques for viscoelastic damping. As in the case in which passive damping is weighted as an additional control force, minimizing the average value of the performance index is recommended over minimizing the maximum value since it will give better performance.

The weighting matrices in both types of performance indices are chosen by the design engineer based on the relative importance of passive damping, active vibration control, and required damping ratios. The weighting matrices may need to be adjusted to attain a solution that is acceptable to the design engineer. Two general guidelines in choosing the weighting matrices are that the diagonal elements of the weighting matrices for passive damping and active control be larger than the diagonal elements of the weighting matrix for the state, and that the diagonal elements of the weighting matrix for passive damping be less than or equal to those for active control. However, these guidelines do not have to be followed if the design engineer has a reason for choosing the weightings differently.

The solution techniques in this dissertation are limited to structures in which the structural damping is either viscous or viscoelastic. To use the solution techniques for viscoelastic damping, the viscoelastic dampers must be constructed such that only one component of strain dissipates the energy. Expanding the techniques to different types of dampers, such as viscoelastic dampers in which two components of strain dissipate energy, is an area for further research.

Another limitation is in the numerical implementation of the solution techniques. As noted in the summary of Chapter III, if the order of the finite element model is too large, computational difficulties may arise. This is especially critical in the case where viscoelasticity is modelled using integer order derivatives, since the order of the state equations are at least three times larger than the order of the finite element matrices defining the structure. The numerical stability for problems of more than fourteen degrees of freedom was not investigated.

Applying the solution techniques to larger scale problems would be an area for further research. Structures other than the spring-mass-damper systems and trusses in the example problems could also be considered. Other structures might be basic structures, such as plates and beams, and also more complex structures that are combinations of these simpler structures. Another area for further work would be to experimentally verify the solution techniques using some of these structures; the work presented in this dissertation is all theoretical.

Since this is an engineering dissertation, an engineering approach was taken in minimizing the cost functional subject to the constraints on the passive damping force (and on the active force when viscoelastic damping was modelled with a classical model). Following an exact analytic approach using vector space methods would be an opportunity for further research.

The two greatest contributions of this dissertation are the ability to determine complementary feedback gains and passive damping parameters independent of initial conditions and the ability to handle viscoelastic dampers. These two contributions combine to show that viscoelastic damping can be optimized in conjunction with active feedback control to provide better vibration reduction.

Appendix A. Calculation of Damping Coefficients

The derivation of the solution to the diagonal **C** that minimizes the Frobenius norm of Eq (36) is given in this appendix. Although the development is for vectors and matrices of order 4, it can be easily extended to vectors and matrices of order n.

Let **X** be the space of real valued vectors with four elements and let **Y** be the space of 4×4 real valued matrices. Define an operator $A : \mathbf{X} \rightarrow \mathbf{Y}$ by

$$\mathbf{y} = A(\mathbf{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}$$

where $\mathbf{x} \in \mathbf{X}$. It is easily verified that $A(\mathbf{x}_1 + \alpha\mathbf{x}_2) = A(\mathbf{x}_1) + \alpha A(\mathbf{x}_2)$ for any scalar α, so A is linear.

Given $\mathbf{y} \in \mathbf{Y}$, the vector $\mathbf{x}$ is sought that minimizes the Frobenius norm $\|\mathbf{y} - \mathbf{Ax}\|_F$. Hence, a solution to the following expression is sought:

$$\min_{\mathbf{x} \in \mathbf{X}} \|\mathbf{y} - \mathbf{Ax}\|_F \tag{188}$$

Define a functional on $\mathbf{Y} \times \mathbf{Y}$ by

$$(\mathbf{y}, \mathbf{z}) = \sum_{i,j=1}^{4} y_{ij} z_{ij} \;\; \forall \;\; \mathbf{y}, \mathbf{z} \in \mathbf{Y} \tag{189}$$

Since this functional satisfies the four requirements of a real inner product:

$$\begin{aligned} (\mathbf{y}, \mathbf{z}) &= (\mathbf{z}, \mathbf{y}) \\ (\mathbf{y}_1 + \mathbf{y}_2, \mathbf{z}) &= (\mathbf{y}_1, \mathbf{z}) + (\mathbf{y}_2, \mathbf{z}) \\ (\lambda\mathbf{y}, \mathbf{z}) &= \lambda(\mathbf{y}, \mathbf{z}) \\ (\mathbf{y}, \mathbf{y}) &\geq 0, \quad \text{with equality iff } \; \mathbf{y} = 0 \end{aligned}$$

define the functional in Eq (189) to be the inner product on **Y**. Then the norm on **Y** is given by

$$\|\mathbf{y}\| = (\mathbf{y},\mathbf{y})^{1/2} = \left[\sum_{i,j=1}^{4} y_{ij}^2\right]^{1/2}$$

Notice that this is the Frobenius norm for 4×4 matrices.

The conjugate operator $A^* : \mathbf{Y} \rightarrow \mathbf{X}$ is found from the relationship

$$(\mathbf{x}, A^*\mathbf{y})_{\mathbf{X}} = (A\mathbf{x}, \mathbf{y})_{\mathbf{Y}}$$

The subscripts **X** and **Y** denote the spaces in which the inner product is taken. Since

$$(A\mathbf{x},\mathbf{y})_{\mathbf{Y}} = \sum_{i,j=1}^{4}(A\mathbf{x})_{ij}y_{ij} = \sum_{i=1}^{4} x_i y_{ii} = (\mathbf{x}, A^*\mathbf{y})_{\mathbf{X}}$$

the conjugate operator is given by

$$A^*\mathbf{y} = \begin{Bmatrix} y_{11} \\ y_{22} \\ y_{33} \\ y_{44} \end{Bmatrix}$$

The solution which yields the smallest error in the Frobenius norm, Eq (188), is given by $(A^*A)^{-1}A^*\mathbf{y}$ if $(A^*A)^{-1}$ exists [21:160]. Find A^*A:

$$A^*A\mathbf{x} = A^*\left(\begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}\right) = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \mathbf{x}$$

So A^*A is the identity operator and its inverse is also the identity operator. Hence the solution that minimizes the Frobenius norm is

$$\mathbf{x} = A^*\mathbf{y}$$

In other words, **x** is the diagonal of **y**.

Using the same procedure outlined above, it can be shown that the diagonal **C** that minimizes the Frobenius norm $\|\mathbf{C\Phi} - \mathbf{W}\|_F$ is given by

$$\mathbf{c} = (A^*A)^{-1}A^*\mathbf{W} = \left\{ \begin{array}{c} \frac{\sum_{i=1}^{n} \Phi_{1i} W_{1i}}{\sum_{i=1}^{n} \Phi_{1i}^2} \\ \vdots \\ \frac{\sum_{i=1}^{n} \Phi_{mi} W_{mi}}{\sum_{i=1}^{n} \Phi_{mi}^2} \end{array} \right\}$$

where **c** is a vector containing the m elements of the diagonal of **C**, and $\mathbf{\Phi}$ and **W** are $m \times n$ matrices. In the case of Eq (35),

$$\mathbf{W} = -\mathbf{S}^{-1}\mathbf{B_v}^T\mathbf{P}$$

The diagonal elements of the positive semi-definite diagonal matrix **C** that minimizes the Frobenius norm $\|\mathbf{C\Phi} - \mathbf{W}\|_F$ are

$$C_{ii} = \max\left(\frac{\sum_{j=1}^{n} \Phi_{ij} W_{ij}}{\sum_{j=1}^{n} \Phi_{ij}^2}, 0\right) \tag{37}$$

The last sentence of the following theorem will be used to prove that **C** as defined above minimizes the norm $\|\mathbf{C\Phi} - \mathbf{W}\|_F$ over the set of positive semi-definite diagonal matrices:

Theorem [21:69]. *Let x be a vector in a Hilbert space H and let K be a closed convex subset of H. Then there is a unique vector $k_0 \in K$ such that*

$$\|x - k_0\| \leq \|x - k\|$$

for all $k \in K$. Furthermore, a necessary and sufficient condition that k_0 be the unique minimizing vector is that $(x - k_0, k - k_0) \leq 0$ for all $k \in K$.

Let **H** be the space of all $m \times n$ matrices. The following functional satisfies the four requirements of an inner product and will be defined as the inner product on **H**:

$$(\mathbf{y}, \mathbf{z}) = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij} z_{ij} \;\; \forall \;\; \mathbf{y}, \mathbf{z} \in \mathbf{H}$$

Let $\Omega = \{\mathbf{E} \in R^{m\times m} | \mathbf{E}$ is positive semi-definite diagonal$\}$ and let $K = \{k | k = \mathbf{E\Phi}$ where $\mathbf{E} \in \Omega\}$. Since Ω is convex, K is convex also $(0 < \alpha < 1)$:

$$\alpha\mathbf{E}_1 + (1-\alpha)\mathbf{E}_2 \in \Omega$$

therefore,

$$[\alpha\mathbf{E}_1 + (1-\alpha)\mathbf{E}_2]\mathbf{\Phi} = \alpha\mathbf{E}_1\mathbf{\Phi} + (1-\alpha)\mathbf{E}_2\mathbf{\Phi} \in K$$

Since K is equal to its closure, it is closed.

The inner product of $\mathbf{W} - \mathbf{C\Phi}$ with $\mathbf{E\Phi} - \mathbf{C\Phi}$ is

$$(\mathbf{W} - \mathbf{C\Phi}, \mathbf{E\Phi} - \mathbf{C\Phi}) = \sum_{i=1}^{m}\left[C_{ii}^2\sum_{j=1}^{n}\Phi_{ij}^2 - C_{ii}\sum_{j=1}^{n}\Phi_{ij}W_{ij} - C_{ii}E_{ii}\sum_{j=1}^{n}\Phi_{ij}^2 + E_{ii}\sum_{j=1}^{n}\Phi_{ij}W_{ij}\right]$$

For each i such that $\sum_{j=1}^{n}\Phi_{ij}W_{ij} > 0$, the corresponding term in the summation above is zero:

$$\left(\frac{\sum_{j=1}^{n}\Phi_{ij}W_{ij}}{\sum_{j=1}^{n}\Phi_{ij}^2}\right)^2\sum_{j=1}^{n}\Phi_{ij}^2 - \left(\frac{\sum_{j=1}^{n}\Phi_{ij}W_{ij}}{\sum_{j=1}^{n}\Phi_{ij}^2}\right)\left[\sum_{j=1}^{n}\Phi_{ij}W_{ij} - E_{ii}\sum_{j=1}^{n}\Phi_{ij}^2\right] + E_{ii}\sum_{j=1}^{n}\Phi_{ij}W_{ij} = 0$$

For each i such that $\sum_{j=1}^{n}\Phi_{ij}W_{ij} \leq 0$, the corresponding term in the summation over i in the inner product is negative for all $\mathbf{E}$:

$$E_{ii}\sum_{j=1}^{n}\Phi_{ij}W_{ij} \leq 0 \quad \text{for} \quad \sum_{j=1}^{n}\Phi_{ij}W_{ij} \leq 0 \text{ and } \mathbf{E} \in K$$

Hence, the summation over i in the inner product reduces to a sum of zeros and negative terms, and is therefore less than (or equal to) zero for all $\mathbf{E} \in K$. This proves that $\mathbf{C}$ as defined in Eq (37) minimizes the Frobenius norm $\|\mathbf{C\Phi} - \mathbf{W}\|_F$ over the set of positive semi-definite diagonal matrices.

The method of calculating the damping coefficients presented here is useful in both Section 2.2.1 and Section 3.2.

Appendix B. The Value of the Cost Functional in Terms of the Initial State

This appendix contains the derivation of a form of the cost functional in terms of the initial state, which is the form given in Eq (41). Define a time dependent function $J(t)$ as

$$J(t) = \int_{t_0}^{t} \frac{1}{2}(\mathbf{y}^T\mathbf{Q}\mathbf{y} + \mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{v}^T\mathbf{S}\mathbf{v})dt \tag{190}$$

Assume $J(t)$ is of the form

$$J(t) = c - \mathbf{y}^T\mathbf{G}\mathbf{y}$$

where c is an unknown constant and $\mathbf{G}$ is an unknown matrix. Then the derivative of $J(t)$ with respect to time is

$$\dot{J}(t) = -\dot{\mathbf{y}}^T\mathbf{G}\mathbf{y} - \mathbf{y}^T\dot{\mathbf{G}}\mathbf{y} - \mathbf{y}^T\mathbf{G}\dot{\mathbf{y}} \tag{191}$$

But from the definition of $J(t)$ and Eqs (23) and (11) in Chapter II,

$$J(t) = \int_{t_0}^{t} \mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y}\, d\tau \tag{192}$$

where $\hat{\mathbf{Q}} = \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{\Phi}^T\mathbf{C}^T\mathbf{S}\mathbf{C}\mathbf{\Phi}$. Taking the time derivative of Eq (192) gives

$$\dot{J}(t) = \mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y} \tag{193}$$

Now $\dot{\mathbf{y}} = \hat{\mathbf{A}}\mathbf{y}$ where $\hat{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{B}_v\mathbf{C}\mathbf{\Phi}$. The objective of this dissertation is vibration suppression, so $\mathbf{P}$ and $\mathbf{C}$ have been chosen such that $\hat{\mathbf{A}}$ is stable. Combining Eqs (191) and (193) gives

$$\mathbf{y}^T\hat{\mathbf{Q}}\mathbf{y} = -\mathbf{y}^T\hat{\mathbf{A}}\mathbf{G}\mathbf{y} - \mathbf{y}^T\dot{\mathbf{G}}\mathbf{y} - \mathbf{y}^T\mathbf{G}\hat{\mathbf{A}}\mathbf{y} \tag{194}$$

Now $\dot{\mathbf{G}} = \mathbf{0}$ since $\hat{\mathbf{Q}}$ is constant, so Eq (194) becomes a Lyapunov equation in $\mathbf{G}$:

$$\hat{\mathbf{A}}^T\mathbf{G} + \mathbf{G}\hat{\mathbf{A}} + \hat{\mathbf{Q}} = \mathbf{0} \tag{195}$$

In order to express $J(t)$ in terms of the initial state, $\mathbf{y}_0$, the state $\mathbf{y}$ is first expressed in terms of the initial state:

$$\mathbf{y} = e^{\hat{\mathbf{A}}(t-t_0)}\mathbf{y}_0$$

Then the cost functional can be written as

$$J(t) = c - \mathbf{y}_0^T e^{\hat{\mathbf{A}}^T(t-t_0)}\mathbf{G}e^{\hat{\mathbf{A}}(t-t_0)}\mathbf{y}_0$$

Since the initial value of the cost functional is zero, $J(t_0) = 0$, the constant c is $c = \mathbf{y}_0^T\mathbf{G}\mathbf{y}_0$.

The the cost functional J given in Eq (15) can be expressed as $J(t_f)$. As t_f approaches infinity, the cost functional can be expressed as

$$J = \lim_{t\to\infty} J(t)$$

Since $\hat{\mathbf{A}}$ is stable, the limit exists. Hence, the cost functional is

$$J = \lim_{t\to\infty} J(t) \; = \; \mathbf{y}_0^T\mathbf{G}\mathbf{y}_0 \tag{196}$$

where $\mathbf{G}$ is the solution of Eq (195). Comparing Eqs (40) and (195), it becomes clear that $G = \mathbf{P}$. Therefore Eq (196) is equivalent to Eq (41).

A similar development can also be found in Reference [1].

Appendix C. Solving the Non-Symmetric Quadratic Matrix Equation

In this appendix, the method for solving Eqs (108) and (113) is described. The method will be demonstrated using Eq (108):

$$\mathbf{Q} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PB}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\widehat{\mathbf{I}}_r\mathbf{I}_{12} - \mathbf{PB}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P} = \mathbf{0} \tag{108}$$

This equation is quadratic in **P**, but is not solvable by any of the standard MATLAB routines. Newton-Raphson is a standard technique and it is employed as follows: First the left hand side of Eq (108) is defined as a function of **P**:

$$\mathbf{Q(P)} = \mathbf{Q} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PB}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I}_A\mathbf{P}\widehat{\mathbf{I}}_r\mathbf{I}_{12} - \mathbf{PB}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P}$$

The goal is to find **P** such that **Q(P)=0**. The function **Q(P)** is approximated by

$$\mathbf{Q(P)} = \mathbf{Q(P_0)} + D\mathbf{Q}[\mathbf{P_0}](\mathbf{P} - \mathbf{P_0}) + o(\|\mathbf{P} - \mathbf{P_0}\|)$$

where the operator $D\mathbf{Q}[\mathbf{P_0}]$ represents the Fréchet derivative of **Q** at $\mathbf{P_0}$ and $o(\mathbf{P})$ represents terms that are at least quadratic in $(\|\mathbf{P} - \mathbf{P_0}\|)$. Let $\mathbf{H} = \mathbf{P} - \mathbf{P_0}$.
Since **Q** is a function of **P**,

$$D\mathbf{Q}[\mathbf{P_0}](\mathbf{H}) = \frac{d}{d\alpha}\mathbf{Q}(\mathbf{P}_0 + \alpha\mathbf{H})|_{\alpha=0} \tag{197}$$

Carrying out the calculation,

$$\begin{aligned} D\mathbf{Q}[\mathbf{P_0}](\mathbf{H}) = {} & \mathbf{HA} + \mathbf{A}^T\mathbf{H} - \mathbf{HB}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I_A}\mathbf{P}_0\widehat{\mathbf{I}}_r\mathbf{I}_{12} - \mathbf{P}_0\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1^T\mathbf{I_A}\mathbf{H}\widehat{\mathbf{I}}_r\mathbf{I}_{12} \\ & -\mathbf{HB}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{P}_0 - \mathbf{P}_0\mathbf{B}_v\mathbf{S}^{-1}\mathbf{B}_v{}^T\mathbf{H} \end{aligned}$$

This may not look much better than **Q(P)**, but at least it is linear in **H**. The equation to solve for **H** is

$$D\mathbf{Q}[\mathbf{P_0}](\mathbf{H}) + \mathbf{Q}(\mathbf{P_0}) = \mathbf{0} \tag{198}$$

Note that this equation is similar to a Lyapunov equation, although some of the terms do not have **H** on the left or right, but in the middle. To get around this difficulty, note that $\widehat{\mathbf{I}}_r$ is an identity matrix with the bottom third diagonal all zeros. Therefore approximate

$\mathbf{H\hat{I}_r}$ and $\mathbf{HI_{12}}$ by $\mathbf{H}$ where appropriate in $D\mathbf{Q}[\mathbf{P_0}](\mathbf{H})$. This results in the simplification,

$$D\mathbf{Q}[\mathbf{P_0}](\mathbf{H}) = \mathbf{HA} + \mathbf{A}^T\mathbf{H} - \mathbf{HB_1R^{-1}B_1^TI_AP_0\hat{I}_rI_{12}} - \mathbf{P_0B_1^TI_AH}$$
$$-\mathbf{HB_vS^{-1}B_v}^T\mathbf{P_0} - \mathbf{P_0B_vS^{-1}B_v}^T\mathbf{H}$$

With this approximation, Eq (198) becomes a Lyapunov equation, easily solved by the simple MATLAB routine LYAP. The initial guess, $\mathbf{P_0}$, is found by setting $\mathbf{I}_A$, $\mathbf{\hat{I}_r}$, and $\mathbf{I}_{12}$ equal to identity in Eq (108) and solving the resulting algebraic Riccati equation. The iteration scheme is

$$D\mathbf{Q}[\mathbf{P}_i](\mathbf{H}_i) + \mathbf{Q}(\mathbf{P})_i = \mathbf{0}$$
$$\mathbf{P}_{i+1} = \mathbf{P}_i + \mathbf{H}_i$$

Convergence occurs when $\mathbf{Q}(\mathbf{P})$ is sufficiently small. The sum of the absolute values of the elements of $\mathbf{Q}(\mathbf{P})$ was used as a measure of "smallness".

The method presented here for solving the non-symmetric quadratic matrix equation can be used to solve both Eq (108) of Section 3.2 and Eq (113) of Section 3.2.1.

Appendix D. Error Analysis

In this appendix, the error between the approximation $\hat{\mathbf{A}}\mathbf{y}$ and the operation $\mathbf{A}(t,\mathbf{y})$ is examined in detail. The operator $\mathbf{A}(t,\mathbf{y})$ is defined by Eqs (163) and (164) in Chapter IV. The approximate state matrix is defined by Eq (166):

$$\hat{\mathbf{A}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{\Psi}}^{-1} \tag{166}$$

The impulse response to Eq (160), represented by $\mathbf{x}(t)$, will be used to illustrate the error due to the approximation of $\mathbf{A}(t,\mathbf{y})$. From Reference [3], the impulse response can be expressed in terms of an integral that decays algebraically and a sum of decaying exponentials:

$$\mathbf{x}(t) = \frac{1}{\pi} Im \left[\int_0^\infty \hat{\mathbf{X}}(re^{-i\pi}) e^{-rt} dr \right] + \sum_{j=1}^{n} \frac{m\tilde{\lambda}_j^{m-1} \boldsymbol{\psi}_j^T \{1.\} (1 + b\tilde{\lambda}^{m\alpha})}{m_j} \boldsymbol{\psi}_j e^{\tilde{\lambda}_j^m t} \tag{199}$$

where

$$\hat{\mathbf{X}}(re^{-i\pi}) = \sum_{j=1}^{N} \frac{\boldsymbol{\psi}_j^T \{1.\}(1 + br^\alpha e^{-i\pi\alpha})}{m_j (r^{1/m} e^{-i\pi/m} - \tilde{\lambda}_j)} \boldsymbol{\psi}_j$$

$\alpha =$ order of fractional derivative ($\alpha = q/m$)

$m =$ denominator of α

$\tilde{\lambda}_j = j^{th}$ eigenvalue associated with the expanded equations of motion (see Reference [3])

$\boldsymbol{\psi}_j =$ eigenvector associated with the expanded equations of motion (see Reference [3])

$n =$ twice the length of the vector of displacements

$N =$ total number of eigenvalues associated with the expanded equations of motion (see Reference [3])

$b =$ viscoelastic parameter (see Eq (156))

$m_j =$ modal constant

$\{1.\} =$ column vector of ones

The approximate state matrix is calculated using the eigenvalues and exponential modes of the structure, so the error due to using the approximate state matrix is the integral term of the impulse response:

$$\mathbf{x}_E(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) = \frac{1}{\pi} Im \left[\int_0^\infty \hat{\mathbf{X}}(re^{-i\pi}) e^{-rt} dr \right] \quad (200)$$

The error will be shown to be real and continuous, and its asymptotic behavior for small and large time examined.

In a first glance at the behavior of the error, it appears to be singular at $t = 0$. But for real structures, the singular components cancel each other. Churchill's theorem on the existence of the inverse transform will be used to prove that the response is real, continuous, and causal [3:88]. Then it will follow that the error is real and continuous since it is the difference between two continuous functions.

Before applying the theorem, it will be shown that the Laplace transform does, indeed, satisfy the transformed equations as they appear in Eq (160):

$$s^2\mathbf{M}\mathbf{x}(s) + G(s)\mathbf{K}_v\mathbf{x}(s) + \mathbf{K}\mathbf{x}(s) + \mathbf{b}\mathbf{u}(s) = \mathbf{0} \quad (160)$$

The Laplace transform of the impulse response $\mathbf{x}(t)$ (Eq (199)) for which $-\mathbf{b}\mathbf{u}(t) = \delta(t)$ is $\hat{\mathbf{X}}(s)$ [3:92]:

$$\hat{\mathbf{X}}(s) = \sum_{j=1}^{N} \frac{\boldsymbol{\psi}_j^T \{1.\}(1 + bs^\alpha)}{m_j(s^{1/m} - \bar{\lambda}_j)} \boldsymbol{\psi}_j \quad (201)$$

To show that $\hat{\mathbf{X}}(s)$ does indeed satisfy the transformed equations of motion, the equations of motion will be posed in expanded form. The first step is to clear the denominator of $G(s)$:

$$G(s) = \frac{G_0 + G_1 s^\alpha}{1 + bs^\alpha}$$

Hence Eq (160) becomes

$$s^2(1 + bs^\alpha)\mathbf{M}\mathbf{x}(s) + (G_0 + G_1 s^\alpha)\mathbf{K}_v\mathbf{x}(s) + (1 + bs^\alpha)\mathbf{K}\mathbf{x}(s) = -(1 + bs^\alpha)\mathbf{b}\mathbf{u}(s) \quad (202)$$

Rearranging terms yields

$$[\mathbf{M}(s^2 + bs^{2+\alpha}) + (G_0\mathbf{K}_v + \mathbf{K}) + (G_1\mathbf{K}_v + b\mathbf{K})s^\alpha]\, \mathbf{x}(s) = -(1 + bs^\alpha)\mathbf{b}\mathbf{u}(s) \quad (203)$$

The left hand side of this equation can be written in terms of summations:

$$\left[M \sum_{j=0}^{J} c_j s^{j/m} + \sum_{j=0}^{J} \mathbf{K}_j s^{j/m} \right] \mathbf{x}(s) = -(1 + bs^{\alpha})\mathbf{b}\mathbf{u}(s) \tag{204}$$

where $J = 2q + m$ and c_j and $\mathbf{K}_j$ are zero where appropriate. Letting $\mathbf{A}_j = \mathbf{M}c_j + \mathbf{K}_j$ simplifies Eq (204) even further:

$$\sum_{j=0}^{J} \mathbf{A}_j s^{j/m} \mathbf{x}(s) = -(1 + bs^{\alpha})\mathbf{b}\mathbf{u}(s) \tag{205}$$

Hence, the equations of motion can now be posed in terms of two real, square, symmetric matrices:

$$s^{1/m} \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{A}_J \\ \vdots & & & \mathbf{A}_J & \mathbf{A}_{J-1} \\ \vdots & & & \vdots & \vdots \\ \mathbf{0} & \mathbf{A}_J & \cdots & \mathbf{A}_3 & \mathbf{A}_2 \\ \mathbf{A}_J & \mathbf{A}_{J-1} & \cdots & \mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix} \begin{Bmatrix} s^{\frac{J-1}{m}}\mathbf{x}(s) \\ s^{\frac{J-2}{m}}\mathbf{x}(s) \\ \vdots \\ s^{\frac{1}{m}}\mathbf{x}(s) \\ \mathbf{x}(s) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_J & \mathbf{0} \\ \vdots & & -\mathbf{A}_J & -\mathbf{A}_{J-1} & \mathbf{0} \\ \mathbf{0} & & & \vdots & \vdots \\ -\mathbf{A}_J & -\mathbf{A}_{J-1} & \cdots & -\mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{A}_0 \end{bmatrix} \begin{Bmatrix} s^{\frac{J-1}{m}}\mathbf{x}(s) \\ s^{\frac{J-2}{m}}\mathbf{x}(s) \\ \vdots \\ s^{\frac{1}{m}}\mathbf{x}(s) \\ \mathbf{x}(s) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \vdots \\ \vdots \\ \mathbf{0} \\ -(1 + bs^{\alpha})\mathbf{b}\mathbf{u}(s) \end{Bmatrix}$$

This equation can be expressed in more compact notation:

$$s^{1/m}\widetilde{\mathbf{M}}\tilde{\mathbf{x}}(s) + \tilde{\mathbf{K}}\tilde{\mathbf{x}}(s) = -\tilde{\mathbf{u}}(s) \tag{206}$$

The orthogonal transform matrix (or modal matrix) $\tilde{\mathbf{\Phi}}$ which simultaneously diagonalizes the expanded mass and stiffness matrices ($\widetilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$) is constructed from the eigenvectors associated with the eigenvalue problem for the expanded equations of motion:

$$\tilde{\lambda}_n \widetilde{\mathbf{M}} \tilde{\psi}_j = \mathbf{0} \tag{207}$$

The expanded displacement vector can be expressed in terms of the modal matrix and a set of modal coordinates $\mathbf{a}(s)$:

$$\tilde{\mathbf{x}}(s) = \tilde{\mathbf{\Phi}}\mathbf{a}(s) \tag{208}$$

Premultiplying Eq (206) by $\tilde{\mathbf{\Phi}}^T$ and substituting in Eq (208) produces the decoupled expanded equations of motion [24:159] [3:68]:

$$s^{1/m}\tilde{\mathbf{\Phi}}^T\widetilde{\mathbf{M}}\tilde{\mathbf{\Phi}}\mathbf{a}(s) + \tilde{\mathbf{\Phi}}^T\widetilde{\mathbf{K}}\tilde{\mathbf{\Phi}}\mathbf{a}(s) = -\tilde{\mathbf{\Phi}}^T\tilde{\mathbf{u}}(s) \tag{209}$$

To show the decoupled nature explicitly, the equations of motion are written in terms of the modal constants m_n and k_n where

$$m_n = \tilde{\boldsymbol{\psi}}_n^T\widetilde{\mathbf{M}}\tilde{\boldsymbol{\psi}}_n \tag{210}$$

$$k_n = \tilde{\boldsymbol{\psi}}_n^T\widetilde{\mathbf{K}}\tilde{\boldsymbol{\psi}}_n \tag{211}$$

Hence, Eq (209) becomes

$$s^{1/m}\begin{bmatrix} \ddots & & \\ & m_n & \\ & & \ddots \end{bmatrix}\mathbf{a}(s) + \begin{bmatrix} \ddots & & \\ & k_n & \\ & & \ddots \end{bmatrix}\mathbf{a}(s) = -\tilde{\mathbf{\Phi}}^T\tilde{\mathbf{u}}(s) \tag{212}$$

Premultiplying this equation by the matrix $\text{diag}(1/m_n)$ yields

$$s^{1/m}\mathbf{I}\mathbf{a}(s) + \begin{bmatrix} \ddots & & \\ & k_n/m_n & \\ & & \ddots \end{bmatrix}\mathbf{a}(s) = -\begin{bmatrix} \ddots & & \\ & 1/m_n & \\ & & \ddots \end{bmatrix}\tilde{\mathbf{\Phi}}^T\tilde{\mathbf{u}}(s) \tag{213}$$

Premultiplying Eq (207) by $\tilde{\boldsymbol{\psi}}_n$ and recalling Eqs (210) and (211) results in an equation relating the eigenvalues to their corresponding modal constants:

$$\tilde{\lambda}_n m_n + k_n = 0 \tag{214}$$

Hence, Eq (213) can be rewritten in terms of the eigenvalues:

$$s^{1/m}\mathbf{I}\mathbf{a}(s) - \begin{bmatrix} \ddots & & \\ & \tilde{\lambda}_n & \\ & & \ddots \end{bmatrix} \mathbf{a}(s) = - \begin{bmatrix} \ddots & & \\ & 1/m_n & \\ & & \ddots \end{bmatrix} \tilde{\mathbf{\Phi}}^T \tilde{\mathbf{u}}(s) \tag{215}$$

The individual elements of $\mathbf{a}(s)$ look like modal participation factors — in fact, they are. The individual elements of $\mathbf{a}(s)$ are of the form:

$$\mathbf{a}_n = \frac{\tilde{\boldsymbol{\psi}}_n^T \tilde{\mathbf{u}}(s)}{m_n(s^{1/m} - \tilde{\lambda}_n)}$$

From the form of $\tilde{\mathbf{x}}(s)$ and $\tilde{\boldsymbol{\psi}}_n$,

$$\tilde{\mathbf{x}}(s) = \begin{Bmatrix} s^{\frac{J-1}{m}}\mathbf{x}(s) \\ s^{\frac{J-2}{m}}\mathbf{x}(s) \\ \vdots \\ s^{\frac{1}{m}}\mathbf{x}(s) \\ \mathbf{x}(s) \end{Bmatrix} \qquad \tilde{\boldsymbol{\psi}}_n = \begin{Bmatrix} s^{\frac{J-1}{m}}\boldsymbol{\psi}_n \\ s^{\frac{J-2}{m}}\boldsymbol{\psi}_n \\ \vdots \\ s^{\frac{1}{m}}\boldsymbol{\psi}_n \\ \boldsymbol{\psi}_n \end{Bmatrix}$$

it follows that

$$\tilde{\boldsymbol{\psi}}_n^T \tilde{\mathbf{u}}(s) = (1 + bs^{\alpha})\boldsymbol{\psi}_n^T \mathbf{b}\mathbf{u}(s) \tag{216}$$

The Laplace transform of the structural displacements is of the form

$$\mathbf{x}(s) = \sum_{n=1}^{N} \mathbf{a}_n(s)\boldsymbol{\psi}_n$$

or

$$\mathbf{x}(s) = \sum_{n=1}^{N} \frac{\boldsymbol{\psi}_n^T \mathbf{b}\mathbf{u}(s)(1 + bs^{\alpha})}{m_n(s^{1/m} - \tilde{\lambda}_n)}\boldsymbol{\psi}_n$$

Therefore, $\hat{\mathbf{X}}(s)$ is the transform of the impulse response to Eq (160).

Churchill's theorem on the existence of the inverse transform can now be used to prove that the response is real, continuous, and causal.

Theorem [9:178]. *Let $f(s)$ be any function of the complex variable s that is analytic and of order $O(s^{-k})$ for all s ($s = x + iy$) in a half plane $x \geq x_0$, where $k > 1$; also let $f(x)$ be real ($x \geq x_0$). Then the inversion integral of $f(s)$ along any line $x = \gamma$, where $\gamma \geq x_0$, converges to a real-valued function $F(t)$ that is independent of γ,*

$$F(t) = L^{-1}\{f(s)\} \qquad (-\infty < t < \infty),$$

whose Laplace transform is the given function $f(s)$:

$$L\{F(t)\} = f(s) \qquad (x > x_0).$$

Furthermore $F(t)$ is $O(e^{x_0 t})$, it is continuous $(-\infty < t < \infty)$, and

$$F(t) = 0 \qquad \textit{when } t \leq 0.$$

The quantity $\hat{\mathbf{X}}(s)$ is analytic in the half plane $x \geq x_0$ when x_0 is positive and the branch cut of $s^{1/m}$ is chosen to lie along the negative, real axis in the s-plane and the poles of $\hat{\mathbf{X}}(s)$, which occur at

$$s = \tilde{\lambda}_n^m$$

do not appear in the right half s-plane. (If the poles appeared in the right half s-plane, then the viscoelastic material would be generating energy instead of dissipating energy.)

The quantity $\hat{\mathbf{X}}(x)$ is real for x positive. The only quantities appearing in Eq (201) that can be complex are m_j, ψ_j, and $\tilde{\lambda}_j$ because $(1 + bx^{\alpha})$ and $x^{1/m}$ are real for x positive. When m_j, ψ_j, and $\tilde{\lambda}_j$ are complex, they occur in conjugate pairs [3:89]. Hence, it follows directly that when the terms in Eq (201) are complex, they occur in complex conjugate pairs and $\hat{\mathbf{X}}(x)$ is real for x positive.

$\hat{\mathbf{X}}(s)$ also satisfies the condition that the transform be of order $O(s^{-k})$, $k > 1$. The direct proof that $\hat{\mathbf{X}}(s)$ is order s^{-2} for s large in the right half s-plane (i.e., putting all the terms over a common denominator and adding them up) is long and involved, so an indirect approach will be pursued. The transformed equations of motion as they appear in Eq (160) will be used:

$$s^2\mathbf{M}\mathbf{x}(s) + G(s)\mathbf{K}_v\mathbf{x}(s) + \mathbf{K}\mathbf{x}(s) + \mathbf{b}\mathbf{u}(s) = \mathbf{0} \qquad (160)$$

With $\mathbf{K}_t(s) = G(s)\mathbf{K}_v + \mathbf{K}$, the transformed equations of motion for simultaneous, unit impulsive loading ($-\mathbf{bu}(s) = \{1.\}$) are

$$[s^2\mathbf{M} + \mathbf{K}_t(s)]\,\hat{\mathbf{X}}(s) = \{1.\} \tag{217}$$

The only terms in $\mathbf{K}_t(s)$, other than the constant terms, are those terms proportional to $G(s)$, which behaves like a constant as $s \to \infty$. As a direct result, Eq (217) reduces to

$$s^2\mathbf{M}\hat{\mathbf{X}}(s) \sim \{1.\} \qquad\qquad \text{as } s \to \infty \tag{218}$$

Since the elements in the mass matrix are constant, $\hat{\mathbf{X}}(s)$ is order s^{-2} for s large in the right half s-plane.

Since $\hat{\mathbf{X}}(s)$ satisfies the conditions of the theorem, the response (Eq (199)) is real, continuous, and zero at initial time $t = 0$. It follows that the error (Eq (200)) is also continuous, since it equals the total response minus a sum of decaying exponentials. It also follows (from Eq (199)) that the initial value of the error is given by

$$\mathbf{x}_E(0) = -\sum_{j=1}^{n} \frac{m\tilde{\lambda}_j^{m-1}\boldsymbol{\psi}_j^T\{1.\}(1 + b\tilde{\lambda}^{m\alpha})}{m_j}\boldsymbol{\psi}_j$$

Before examining the short time behavior of the error, the short time behavior of the response will be derived. The response will be shown to monotone for short time; a Tauberian theorem will then be appled to determine the short time behavior of the response.

The monotone nature of the response for short time can be derived by examining the equations of motion (Eq (162) in the time domain wth the forcing term $-\mathbf{bu}(t)$ replaced by the impulse function $\delta(t)\{1.\}$:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}_v\frac{d}{dt}\int_0^t G_{rel}(t-\tau)\mathbf{x}(\tau)d\tau + \mathbf{K}\mathbf{x}(t) = \delta(t)\{1.\} \tag{219}$$

Integrating this equation from 0^- to t yields an expression in terms of the velocity:

$$\mathbf{M}\dot{\mathbf{x}}(t) + \mathbf{K}_v\int_0^t G_{rel}(t-\tau)\mathbf{x}(\tau)d\tau + \mathbf{K}\int_{0^-}^t \mathbf{x}(t) = \{1.\} \tag{220}$$

Now $G_{rel}(t)$ is a continuous function. This is evident when the Laplace transform of $G_{rel}(t)$ (see Eq (159) and (161)) is expressed as a sum of two terms:

$$L\{G_{rel}(t)\} = \frac{G_1/b}{s} + \frac{G_0 - G_1/b}{s(1+bs^{\alpha})}$$

The first term is clearly the Laplace transform of a step function, while the second term satisfies the conditions of Chuchill's theorem (where the x_0 of the theorem is equal to $b^{1/\alpha}$). Hence, as $t \downarrow 0$ the integrals in Eq (220) disappear since the integrands are continuous:

$$\lim_{t\downarrow 0} \mathbf{M}\dot{\mathbf{x}}(t) = \{1.\}$$

or

$$\lim_{t\downarrow 0} \dot{\mathbf{x}}(t) = \mathbf{M}^{-1}\{1.\}$$

Hence, there exists a $\delta > 0$ for each $\epsilon > 0$ such that the absolute value of each element of $\dot{\mathbf{x}}(t)$ minus each element of $\mathbf{M}^{-1}\{1.\}$ is less than ϵ whenever $0 \leq t < \delta$. Written in terms of vectors,

$$-\epsilon\{1.\} < \dot{\mathbf{x}}(t) - \mathbf{M}^{-1}\{1.\} < \epsilon\{1.\} \tag{221}$$

Solving for velocity,

$$\mathbf{M}^{-1}\{1.\} - \epsilon\{1.\} < \dot{\mathbf{x}}(t) < \mathbf{M}^{-1}\{1.\} + \epsilon\{1.\} \tag{222}$$

So for the smallest ϵ such that an element of $(\mathbf{M}^{-1}\{1.\} - \epsilon\{1.\})$ or $(\mathbf{M}^{-1}\{1.\} + \epsilon\{1.\})$ equals zero, there exists a $\delta > 0$ such that the elements of the velocity $\dot{\mathbf{x}}(t)$ do not change sign on $0 \leq t < \delta$. Therefore, each element of the response $\mathbf{x}(t)$ is a monotone function on the interval $0 \leq t < \delta$.

To examine the behavior of the response for short time in more detail, the following Tauberian theorem, slightly paraphrased from Feller, will be used. (Note that $\omega(s)$ is the Laplace transform of $u(t)$.)

Theorem [13:446]. *If U has an ultimately monotone derivative u then as $s \to \infty$ and $t \to 0$, respectively,*

$$\omega(s) \sim \frac{1}{s^2} \quad \text{iff} \quad u(t) \sim t. \tag{223}$$

From Eq (218), the Laplace transform of the response for large s is

$$\hat{\mathbf{X}}(s) \sim \frac{1}{s^2}\mathbf{M}^{-1}\{1.\}$$

It follows that the response is linear for short time:

$$\mathbf{x}(t) = L^{-1}\{\hat{\mathbf{X}}(s)\} \sim \mathbf{M}^{-1}\{1.\}\,t \qquad \text{when } t \to 0$$

Since the error is the response minus the exponential portion of the response, which is known exactly, the error can be expressed as

$$\mathbf{x}_E(t) \sim \mathbf{M}^{-1}\{1.\}\,t - \sum_{j=1}^{n} \frac{m\tilde{\lambda}_j^{m-1}\boldsymbol{\psi}_j^T\{1.\}(1+b\tilde{\lambda}^{m\alpha})}{m_j}\boldsymbol{\psi}_j e^{\tilde{\lambda}_j^m t} \qquad \text{when } t \to 0$$

It is appropriate to expand the exponential terms in terms of t. To make things easier, the vector coefficient of $e^{\tilde{\lambda}_j^m t}$ will be represented by $\mathbf{c}_j$. The vector coefficients appear in complex conjugate pairs and the complex conjugate of $\mathbf{c}_j$ will be assumed to be $\mathbf{c}_{n/2+j}$. The coefficient of t in the exponential will be expressed in terms of its real and imaginary parts:

$$\tilde{\lambda}_j^m = -\rho_j + i\gamma_j$$

Adding the complex conjugate pairs and carrying out the expansions yields an expression for the error in powers of t. Retaining only the first power of t yields

$$\mathbf{x}_E(t) \sim -2\sum_{j=1}^{n/2} Re(\mathbf{c}_j) + \left[\mathbf{M}^{-1}\{1.\} + 2\sum_{j=1}^{n/2}(\rho_j Re(\mathbf{c}_j) + \gamma_j Im(\mathbf{c}_j))\right] t$$

for small time. In long form, the constant term is

$$-2\sum_{j=1}^{n/2} Re(\mathbf{c}_j) = -2\sum_{j=1}^{n/2} m\left(1 + bRe(\tilde{\lambda}^{m\alpha})\right) Re\left(\frac{\tilde{\lambda}_j^{m-1}\boldsymbol{\psi}_j^T\{1.\}\boldsymbol{\psi}_j}{m_j}\right)$$

which is equal to the initial value of the error, $\mathbf{x}_E(0)$.

To examine the behavior of the error for large time, Watson's lemma will prove useful.

Watson's lemma [10:446]. *Let* $F(r) = f(r^a)r^b$, $a > 0$, $b > -1$. *Let* $f(x)$ *have a Maclaurin expansion for* $|x| < \delta$. *Let* $|F(r)| \leq Me^{cr}$ *for some constants* M *and* c *as*

$r \to \infty$. *Let* $f(x)$ *be continuous for all values of* x. *Then*

$$\int_0^\infty F(r)e^{-rt}dr \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(ka+b+1)}{t^{ka+b+1}}$$

as $t \to \infty$ *where* $a_k = \frac{f^{(k)}(0)}{k!}$.

With $f(x)$ defined as

$$f(x) = \sum_{j=1}^{N} \frac{\psi_j^T \{1.\}(1 + bx^q e^{-i\pi\alpha})}{m_j(xe^{-i\pi/m} - \tilde{\lambda}_j)} \psi_j$$

the quantity $\hat{\mathbf{X}}(re^{-i\pi})$ in Eq (200) can be written as

$$\hat{\mathbf{X}}(re^{-i\pi}) = f(r^{1/m})$$

The Maclaurin expansion for $f(x)$ is

$$f(x) = \sum_{j=1}^{N} \frac{-\psi_j^T \{1.\}\psi_j}{m_j \tilde{\lambda}_j} \left[1 + \frac{xe^{-i\pi/m}}{\tilde{\lambda}_j} + \frac{x^2 e^{-2i\pi/m}}{\tilde{\lambda}_j^2} + \cdots + x^q \left(be^{-i\pi\alpha} + \frac{e^{-i\pi q/m}}{\tilde{\lambda}_j^q}\right) \right.$$
$$\left. + x^{q+1}\left(\frac{be^{-i\pi(q+1)/m}}{\tilde{\lambda}_j^1} + \frac{e^{-i\pi(q+1)/m}}{\tilde{\lambda}_j^{q+1}}\right) x^{q+2}\left(\frac{be^{-i\pi(q+2)/m}}{\tilde{\lambda}_j^2} + \frac{e^{-i\pi(q+2)/m}}{\tilde{\lambda}_j^{q+2}}\right) + \cdots \right]$$

for $|x| < |\lambda_1|$, where $|\lambda_1|$ is the eigenvalue with the smallest magnitude. Let the coefficient of x^k be represented by a_k:

$$a_k = \sum_{j=1}^{N} \frac{-\psi_j^T \{1.\}\psi_j}{m_j \tilde{\lambda}_j} \left(\frac{e^{-i\pi/m}}{\tilde{\lambda}_j}\right)^k \qquad \text{for } 0 \le k < q$$

$$a_k = \sum_{j=1}^{N} \frac{-\psi_j^T \{1.\}\psi_j}{m_j \tilde{\lambda}_j} \left(\frac{be^{-i\pi k/m}}{\tilde{\lambda}_j^{(k-q)}} + \frac{e^{-i\pi k/m}}{\tilde{\lambda}_j^k}\right) \qquad \text{for } k \ge q$$

Then the Maclaurin expansion can be written more compactly:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Hence, the asymptotic behavior of the integral in the error is given by a summation in terms of time t:

$$\int_0^\infty F(r)e^{-rt}dr \sim \sum_{k=0}^{\infty} \frac{a_k\Gamma(k/m+1)}{t^{k/m+1}} \qquad \text{for } t \to \infty \tag{224}$$

Expanding the first few terms of the summation in Eq (224) yields

$$\int_0^\infty F(r)e^{-rt}dr \sim \frac{a_0}{t} + \frac{a_1\Gamma(1+1/m)}{t^{1+1/m}} + \frac{a_2\Gamma(1+2/m)}{t^{1+2/m}} + \cdots \qquad \text{for } t \to \infty$$

Now a_0 can be expressed in terms of the eigenvectors ψ_j and modal constants m_j:

$$a_0 = \sum_{j=1}^{N} \frac{-\psi_j^T\{1.\}\psi_j}{m_j\tilde{\lambda}_j}$$

Since the eigenvectors, eigenvalues, and modal constants occur in conjugate pairs, a_0 is real. Hence, the error does not have a $1/t$ term:

$$\mathbf{x}_E(t) \sim \frac{1}{\pi} Im \left(\frac{a_1\Gamma(1+1/m)}{t^{1+1/m}} + \frac{a_2\Gamma(1+2/m)}{t^{1+2/m}} + \cdots \right)$$

To show that the imaginary part of a_1 is non-zero, a_1 will be computed. From the form of a_k,

$$a_1 = \sum_{j=1}^{N} \frac{-\psi_j^T\{1.\}\psi_j}{m_j\tilde{\lambda}_j} \left(\frac{e^{-i\pi/m}}{\tilde{\lambda}_j} \right)$$

Applying Euler's identity yields

$$a_1 = \sum_{j=1}^{N} \frac{-\psi_j^T\{1.\}\psi_j}{m_j(\tilde{\lambda}_j)^2} (\cos\pi/m - i\sin\pi/m)$$

Hence, by the same rationale that a_0 is real, a_1 has an imaginary part:

$$Im(a_1) = \sum_{j=1}^{N} \frac{\psi_j^T\{1.\}\psi_j}{m_j(\tilde{\lambda}_j)^2} \sin\pi/m$$

Since the error is asymptotic to $t^{-(1+1/m)}$ for large t, it does not decrease exponentially.

The rationale for ignoring the error due to the approximation is given at the end of Section 4.1. Although the solution procedure outlined in Section 4.2 does not take into account the relaxation modes (i. e., the error) due to the presence of viscoelastic dampers, the calculation of the oscillation modes does take into account the effect of the viscoelasticity of the damping material.

Appendix E. Alternate Method for Calculating the State Matrix

This appendix contains an alternate method of calculating the approximate state matrix to the one given in Section 4.2. The derivation of this method makes it clear that the approximate state matrix is real-valued.

In Section 4.2 $\hat{\mathbf{A}}$ is constructed using the principal value eigenvectors and eigenvalues

$$\tilde{\mathbf{\Lambda}} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}} \end{bmatrix} \qquad \tilde{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} & \overline{\mathbf{\Psi}} \\ \mathbf{\Psi\Lambda} & \overline{\mathbf{\Psi\Lambda}} \end{bmatrix} \qquad \hat{\mathbf{A}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{\Psi}}^{-1}$$

where $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues and $\mathbf{\Psi}$ is the matrix of corresponding eigenvectors. Then $\hat{\mathbf{A}}\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Lambda}}$. Writing $\hat{\mathbf{A}}$ in terms of four submatrices $\mathbf{A}_{ij}$, this becomes

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi} & \overline{\mathbf{\Psi}} \\ \mathbf{\Psi\Lambda} & \overline{\mathbf{\Psi\Lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi} & \overline{\mathbf{\Psi}} \\ \mathbf{\Psi\Lambda} & \overline{\mathbf{\Psi\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}} \end{bmatrix} \tag{225}$$

Carrying out the matrix multiplication

$$\begin{bmatrix} \mathbf{A}_{11}\mathbf{\Psi} + \mathbf{A}_{12}\mathbf{\Psi\Lambda} & \mathbf{A}_{11}\overline{\mathbf{\Psi}} + \mathbf{A}_{12}\overline{\mathbf{\Psi\Lambda}} \\ \mathbf{A}_{21}\mathbf{\Psi} + \mathbf{A}_{22}\mathbf{\Psi\Lambda} & \mathbf{A}_{21}\overline{\mathbf{\Psi}} + \mathbf{A}_{22}\overline{\mathbf{\Psi\Lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi\Lambda} & \overline{\mathbf{\Psi\Lambda}} \\ \mathbf{\Psi\Lambda}^2 & \overline{\mathbf{\Psi\Lambda}}^2 \end{bmatrix} \tag{226}$$

results in four equations. The top two will be considered first:

$$\mathbf{A}_{11}\mathbf{\Psi} + \mathbf{A}_{12}\mathbf{\Psi\Lambda} = \mathbf{\Psi\Lambda} \tag{227}$$

$$\mathbf{A}_{11}\overline{\mathbf{\Psi}} + \mathbf{A}_{12}\overline{\mathbf{\Psi\Lambda}} = \overline{\mathbf{\Psi\Lambda}} \tag{228}$$

Adding and subtracting these two equations gives

$$\mathbf{A}_{11}(\mathbf{\Psi} + \overline{\mathbf{\Psi}}) + \mathbf{A}_{12}(\mathbf{\Psi\Lambda} + \overline{\mathbf{\Psi\Lambda}}) = (\mathbf{\Psi\Lambda} + \overline{\mathbf{\Psi\Lambda}}) \tag{229}$$

$$\mathbf{A}_{11}(\mathbf{\Psi} - \overline{\mathbf{\Psi}}) + \mathbf{A}_{12}(\mathbf{\Psi\Lambda} - \overline{\mathbf{\Psi\Lambda}}) = (\mathbf{\Psi\Lambda} - \overline{\mathbf{\Psi\Lambda}}) \tag{230}$$

The quantities in brackets are either two times the real part of Ψ $(\Psi\Lambda)$ or $2i$ times the imaginary part of Ψ $(\Psi\Lambda)$. Hence, these two equations can be written as

$$\mathbf{A}_{11}Re(\Psi) + \mathbf{A}_{12}Re(\Psi\Lambda) = Re(\Psi\Lambda)$$
$$\mathbf{A}_{11}Im(\Psi) + \mathbf{A}_{12}Im(\Psi\Lambda) = Im(\Psi\Lambda)$$

Solving for $\mathbf{A}_{11}$ in each of the equations yields

$$\mathbf{A}_{11} = [Re(\Psi) - \mathbf{A}_{12}Re(\Psi\Lambda)]\,[Re(\Psi)]^{-1} = [Im(\Psi) - \mathbf{A}_{12}Im(\Psi\Lambda)]\,[Im(\Psi)]^{-1} \quad (231)$$

Combining the terms containing $\mathbf{A}_{12}$ gives

$$\mathbf{A}_{12}\left\{Re(\Psi\Lambda)\,[Re(\Psi)]^{-1} + Im(\Psi\Lambda)\;[Im(\Psi)]^{-1}\right\} = Re(\Psi\Lambda)\,[Re(\Psi)]^{-1} + Im(\Psi\Lambda)\,[Im(\Psi)]^{-1} \quad (232)$$

From this equation, it is apparent that

$$\mathbf{A}_{12} = \mathbf{I} \quad (233)$$

which, from Eq (227), implies

$$\mathbf{A}_{11} = \mathbf{0} \quad (234)$$

The second two equations,

$$\mathbf{A}_{21}\Psi + \mathbf{A}_{22}\Psi\Lambda = \Psi\Lambda^2 \quad (235)$$
$$\mathbf{A}_{21}\overline{\Psi} + \mathbf{A}_{22}\overline{\Psi\Lambda} = \overline{\Psi\Lambda}^2 \quad (236)$$

result in the two equations,

$$\mathbf{A}_{11}Re(\Psi) + \mathbf{A}_{12}Re(\Psi\Lambda) = Re(\Psi\Lambda^2) \quad (237)$$
$$\mathbf{A}_{11}Im(\Psi) + \mathbf{A}_{12}Im(\Psi\Lambda) = Im(\Psi\Lambda^2) \quad (238)$$

Solving for $\mathbf{A}_{21}$ in each of the equations yields

$$\mathbf{A}_{21} = [Re(\Psi^2) - \mathbf{A}_{22}Re(\Psi\Lambda)]\,[Re(\Psi)]^{-1} = [Im(\Psi^2) - \mathbf{A}_{22}Im(\Psi\Lambda)]\,[Im(\Psi)]^{-1} \quad (239)$$

Hence,

$$\mathbf{A}_{22}\left\{Re(\Psi\Lambda)\,[Re(\Psi)]^{-1}+Im(\Psi\Lambda)\;[Im(\Psi)]^{-1}\right\}$$
$$=Re(\Psi\Lambda^2)\,[Re(\Psi)]^{-1}+Im(\Psi\Lambda^2)\,[Im(\Psi)]^{-1} \tag{240}$$

Finally, solving for $\mathbf{A}_{22}$ yields an expression which is real.

$$\mathbf{A}_{22}=\left\{Re(\Psi\Lambda^2)\,[Re(\Psi)]^{-1}+Im(\Psi\Lambda^2)\,[Im(\Psi)]^{-1}\right\} \tag{241}$$
$$\times\left\{Re\Psi\Lambda)\,[Re(\Psi)]^{-1}+Im(\Psi\Lambda)\,[Im(\Psi)]^{-1}\right\}^{-1} \tag{242}$$

Substituting this expression into Eq (239) above yields $\mathbf{A}_{21}$. Therefore, $\mathbf{A}_{21}$ is real also. Since $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$ are real also, the state matrix $\hat{\mathbf{A}}$ is also real. Since this matrix is equivalent to the approximate state matrix derived in Section 4.2, the approximate state matrix is also real.

Appendix F. Feedback Matrices for the Truss Examples

This appendix contains the active gain feedback matrices for all the fourteen degree of freedom examples in this dissertation. Each feedback matrix has dimension 4×28. The gains for each chapter's examples are presented in separate sections.

F.1 Chapter II Gains

This section contains the feedback matrices as computed by Eq (11) for the four different solution techniques developed in Sections 2.2.1 and 2.2.2 for Example #3. It also includes the feedback matrices for the two solution techniques presented in Section 2.3 for Example #4. The matrices are on the following pages. Table 19 is the key to the solutions.

Table 19. Key to Matrix Names for Section F.1

Example	Matrix Name	Solution Technique
Example #3	$\mathbf{G}_f$	min $\lVert\mathbf{H}\rVert_F$
	$\mathbf{G}_{2h}$	min $\lVert\mathbf{H}\rVert_2$
	$\mathbf{G}_{tr}$	min trace $\mathbf{P}$
	$\mathbf{G}_2$	min $\lVert\mathbf{P}\rVert_2$
Example #4	$\mathbf{G}_{ave}$	min $\frac{1}{2n}$ trace $\mathbf{P} + \frac{1}{2}\mathbf{c}^T\mathbf{S}\mathbf{c}$
	$\mathbf{G}_{max}$	min $\frac{1}{2}(\lVert\mathbf{P}\rVert_2 + \mathbf{c}^T\mathbf{S}\mathbf{c})$

$G_f =$

Columns 1 through 7

60.8854	137.2134	-75.7822	-173.6511	26.2906	-34.7551	-8.8621
69.6113	135.6511	-52.4885	14.8513	-147.8120	-106.1664	168.0595
78.5921	104.6444	18.1517	-62.9460	-22.1549	-298.9220	103.0132
235.7793	136.3770	112.7833	50.3280	72.5824	-59.3887	-143.9000

Columns 8 through 14

48.0909	-8.4912	33.3340	-2.3179	-32.7392	5.9136	7.3087
19.5168	132.7994	30.9326	-134.7725	-65.6421	-21.4194	41.8575
312.1403	-23.4110	130.7269	-71.1841	-175.1829	42.5333	20.4618
-4.5445	-170.0058	-15.7503	108.4971	24.7631	24.4720	-10.5425

Columns 15 through 21

6.1961	-1.4179	0.1507	-0.1824	1.0709	-0.2874	1.4822
-1.4142	11.4926	0.7460	3.4615	0.3602	2.7212	-0.7177
0.1534	0.7517	7.6018	-0.8572	2.7187	1.2781	1.9533
-0.1758	3.4552	-0.8656	11.9417	-0.0934	6.7203	0.9148

Columns 22 through 28

-1.6310	1.5495	-1.7135	1.0639	-1.3217	2.6922	-3.8026
5.8867	0.1494	3.8181	0.2867	3.0427	0.8136	6.3550
1.4761	1.7575	0.6129	2.9212	1.4884	6.1589	2.5689
3.1828	0.7333	3.3875	0.3460	4.1678	1.4643	6.3543

G_{2h} =

Columns 1 through 7

62.9054	134.8581	-66.3055	-175.0679	18.3592	-26.0758	-16.5618
79.3484	144.3491	-37.5207	21.9577	-155.3707	-113.2804	163.3008
68.4578	95.8321	16.8779	-68.9060	-18.3349	-255.4717	67.5165
243.8168	145.9967	122.4900	58.4604	71.4145	-89.2929	-136.6324

Columns 8 through 14

38.2794	-4.4152	42.6332	-1.5232	-39.9349	3.4473	6.7362
-12.4108	144.4190	82.1159	-131.7938	-70.3460	-12.1800	23.5068
271.8900	0.4157	180.2037	-59.4202	-221.3717	38.4422	23.2290
-10.0895	-167.2828	-17.2681	118.7133	69.9598	23.8387	-30.7180

Columns 15 through 21

6.2410	-1.4277	0.1415	-0.1732	1.0468	-0.3986	1.5337
-1.4234	11.7814	0.7319	3.7110	0.4756	2.5659	-0.6713
0.1439	0.7369	7.6211	-0.9163	2.6754	1.2443	1.9206
-0.1663	3.7053	-0.9241	12.2309	-0.0261	6.5101	1.0204

Columns 22 through 28

-1.5967	1.5318	-1.7559	1.0256	-1.2416	2.7222	-3.8331
5.6853	0.1806	3.9373	0.3396	3.1580	0.9155	6.2907
1.5358	1.7642	0.6783	3.0228	1.3866	6.0855	2.5713
3.0823	0.7998	3.4006	0.3364	4.3195	1.5819	6.3279

$G_{tr} =$

Columns 1 through 7

62.5198	136.9156	-84.9311	-173.4920	19.8862	-36.8601	5.6607
70.7117	138.3706	-63.3180	17.3930	-158.8348	-102.9273	174.5300
88.3765	113.3171	19.2687	-63.9681	-29.4842	-302.0402	109.8547
230.0153	139.5345	114.5842	54.4877	79.6198	-75.2200	-152.8309

Columns 8 through 14

53.4789	5.0294	35.8599	-9.0953	-40.5677	-0.1541	10.6292
5.0429	144.8681	20.4038	-144.3306	-55.2011	-34.8869	46.1979
318.4706	-25.8206	128.6146	-63.7649	-179.7290	34.9756	21.2320
-0.1763	-180.1297	-4.7210	111.2053	15.1257	14.5805	-9.0842

Columns 15 through 21

6.1509	-1.3413	0.1804	-0.3025	1.1789	-0.4110	1.5747
-1.3369	11.7083	0.7295	3.7868	0.3732	2.7948	-0.7119
0.1831	0.7350	7.7083	-0.8737	2.8665	1.2882	2.0999
-0.2958	3.7803	-0.8823	12.1434	0.0067	6.6483	0.9537

Columns 22 through 28

-1.4605	1.6688	-1.6170	1.1401	-1.3985	2.8010	-3.8226
5.8102	0.1104	3.7480	0.3557	3.0772	0.9515	6.3343
1.6487	1.8831	0.7459	3.1165	1.4338	6.5630	2.4971
3.2901	0.8715	3.4161	0.3831	4.0682	1.5877	6.3704

G_2 =

Columns 1 through 7

59.4241	111.0129	-66.6250	-148.7496	11.8285	-44.6229	10.5544
129.2189	166.2954	-86.3966	9.1834	-162.3572	-72.3976	144.7425
85.9633	125.1300	20.7201	-50.8535	-34.4722	-309.3312	142.5468
244.6480	165.7833	95.4493	54.5290	101.1949	-136.9265	-144.0311

Columns 8 through 14

65.7956	0.8674	22.3552	-15.4089	-36.5680	2.0923	13.4014
-59.8052	148.0188	9.7843	-137.8634	-30.7585	-80.7885	51.7748
323.0306	-52.1648	133.3272	-62.2221	-194.8828	40.4834	20.4625
26.2492	-201.3158	13.9502	82.8149	-1.5140	-13.0974	1.0109

Columns 15 through 21

6.4417	-1.3751	0.1911	-0.3405	1.1791	-0.4520	1.5320
-1.3712	11.9444	0.5470	3.8516	0.2777	2.5384	-0.7196
0.1939	0.5529	8.1370	-0.8070	2.7553	1.2321	2.0935
-0.3339	3.8460	-0.8164	12.4175	-0.0345	6.1569	0.8482

Columns 22 through 28

-1.3896	1.6949	-1.5756	1.0828	-1.3600	2.7290	-3.8698
5.3474	-0.0081	3.8167	0.2607	3.1503	0.6429	6.4370
1.9055	1.7932	0.8369	3.0411	1.4902	6.6023	2.3926
3.0848	0.6363	3.4532	0.2701	4.1371	1.3288	6.5021

$G_{ave} =$

Columns 1 through 7

64.1695	144.5988	-50.0861	-183.6278	17.4595	-16.5797	-57.6956
177.3899	163.9929	-69.1734	-14.5471	-200.8170	-104.2373	213.3795
74.6711	131.9043	-7.9591	-94.1001	-13.6606	-307.3552	127.4211
328.3747	171.8565	134.3263	6.8416	145.0328	-40.5235	-181.5181

Columns 8 through 14

23.6747	-34.7961	22.5202	29.2920	-17.9947	10.8837	3.2705
41.7881	168.7589	0.6942	-159.4808	-55.8531	-57.0812	51.9287
323.8085	-3.0948	126.3508	-92.6014	-179.7254	45.7161	30.1199
13.7485	-230.9777	-6.5328	124.6472	6.2781	18.9491	-10.7131

Columns 15 through 21

2.6100	-2.0565	0.0589	0.3003	0.9235	-0.2244	1.2754
-2.0540	7.7074	0.8255	2.3542	0.2910	2.5540	-1.0319
0.0631	0.8334	4.5988	-0.9877	2.4842	1.3050	1.4064
0.3068	2.3489	-0.9968	8.3288	-0.6151	6.8957	0.6598

Columns 22 through 28

-1.8606	1.3283	-1.7513	0.9407	-1.3099	2.3204	-3.8801
6.0478	-0.0930	3.9044	-0.1169	3.0212	-0.0418	6.6334
1.0905	1.2959	0.5579	2.4848	1.6559	5.2348	2.6133
2.9706	0.1518	3.4567	0.0377	4.3167	0.5836	6.5274

G_{max} =

Columns 1 through 7

92.3936	154.8387	-58.1994	-199.3819	-57.3365	20.0614	-58.6918
195.5806	171.6741	-94.6911	-43.1889	-181.9405	-79.6736	212.1390
105.0575	141.6966	-45.9438	-166.4785	8.7368	-164.3084	74.2271
361.4868	183.6536	127.2080	-27.1159	225.2402	-36.0190	-144.7645

Columns 8 through 14

-25.4892	17.9443	5.2245	33.1036	-0.0822	-27.6275	8.3576
0.4522	272.0096	100.9069	-140.9888	-58.9851	-35.6951	10.2324
238.3149	25.3230	188.2920	-82.7019	-268.2625	41.4951	47.1680
22.7196	-239.5681	-9.4620	126.9080	88.0696	73.3432	-59.2934

Columns 15 through 21

2.8996	-2.0661	0.0321	0.4696	0.9380	-0.7617	1.2350
-2.0612	7.8104	1.0[illegible]40	2.1828	0.5056	3.0963	-0.8654
0.0378	1.0808	4.8366	-1.3009	2.9930	1.5393	0.8631
0.4781	2.1757	-1.3070	8.2418	-0.0999	6.5909	0.8618

Columns 22 through 28

-1.5725	1.2578	-1.5825	0.8400	-1.3211	1.6844	-3.8714
5.5949	-0.0739	3.4075	0.1453	2.8400	0.8144	6.8125
0.5598	0.9202	0.9424	2.3418	1.3532	4.3504	2.5067
3.5836	0.4557	3.0788	0.2729	3.7776	1.1053	6.8835

F.2 Chapter III Gains

This section contains the feedback matrices as computed by Eq (99) for the three different solution techniques developed in Sections 3.2 and 3.2.1 for Example #3. The matrices are on the following pages. Table 20 is the key to the solutions.

Table 20. Key to Matrix Names for Section F.2

Matrix Name	Solution Technique
$\mathbf{G}_f$	min $\|\mathbf{H}\|_F$
$\mathbf{G}_{tr}$	min trace $\mathbf{P}$
$\mathbf{G}_2$	min $\|\mathbf{P}\|_2$

$G_f =$

Columns 1 through 7

489.0862	28.6248	-71.3414	-1.5727	-72.1660	37.4579	37.6215
17.7314	515.4182	109.0362	-297.4294	31.6795	-146.2687	-20.8456
-44.9505	87.4180	504.7917	-50.9229	52.4029	-85.1045	-90.0218
105.2035	-271.3641	-18.1287	468.7974	-19.0522	97.2782	64.8550

Columns 8 through 14

-26.5444	7.8820	0.3209	-3.7119	0.8848	-2.5209	9.5871
130.7301	25.1807	21.3275	8.7155	-38.4993	33.8261	1.2374
59.3063	-56.4086	-14.8259	-10.3330	-0.5436	24.4887	1.9521
-120.3438	1.8507	-26.0600	24.9368	24.5943	32.8938	-12.1145

Columns 15 through 21

5.4058	-0.1816	-0.4012	0.2324	0.0592	0.2144	0.8973
-0.2802	8.3526	0.9596	-0.1919	0.1924	0.7148	-0.9564
-0.4365	0.9121	5.7926	-0.7942	1.2151	-0.3428	-0.0866
0.4123	-0.1276	-0.7620	7.8868	-0.7548	3.7372	0.5457

Columns 22 through 28

-0.7086	0.7830	-0.6640	0.3901	-0.5744	1.1016	-1.6770
3.7738	-0.3749	2.2375	-0.1275	1.6387	-0.3797	3.8566
0.4802	-0.1450	0.0443	0.8866	0.5386	1.9161	1.4046
0.8781	-0.1590	1.8312	-0.3545	2.3890	-0.4807	3.6853

$G_{tr} = 10^3 \times$

Columns 1 through 7

5.6539	0.6351	0.7863	-0.0965	-0.9973	-0.2201	-0.3016
0.4552	4.2154	-0.0477	-2.5036	0.0183	-0.7224	0.5553
0.3678	0.5871	6.7266	0.3822	-0.5905	-0.5118	-1.1454
1.4910	-1.8828	0.8736	3.7913	-0.0626	-0.2264	0.1935

Columns 8 through 14

0.1717	-0.2739	-0.3260	-0.7521	0.2527	0.2695	0.2142
0.2298	-0.0481	0.3316	0.1400	-0.3537	-0.1322	-0.0269
0.0057	-0.5344	0.2580	-0.3523	-0.4444	0.0257	-0.0366
-0.5208	-0.5672	-0.4576	-0.1986	0.3467	0.3019	0.0697

Columns 15 through 21

0.0861	0.0089	0.0119	-0.0025	0.0011	-0.0074	-0.0018
-0.0040	0.0758	0.0062	-0.0156	-0.0043	0.0052	0.0017
0.0031	0.0065	0.1020	0.0069	-0.0021	0.0018	0.0045
0.0135	-0.0135	0.0097	0.0765	-0.0042	0.0198	-0.0003

Columns 22 through 28

-0.0025	0.0018	-0.0101	-0.0018	-0.0040	0.0144	-0.0127
0.0223	0.0021	0.0156	0.0000	0.0081	-0.0037	0.0208
0.0004	0.0049	0.0027	0.0078	0.0037	0.0215	0.0113
0.0100	-0.0030	0.0093	-0.0014	0.0164	0.0056	0.0218

$G_2 = 10^3 \times$

Columns 1 through 7

6.0104	0.2557	0.9050	0.3489	-1.5184	-0.7742	-0.0029
0.6289	4.1351	0.2488	-2.2366	-0.2825	-1.0163	0.4448
0.3753	0.6546	6.8706	0.5112	-0.9782	-0.7147	-0.9912
1.5668	-1.8007	0.8274	3.8946	-0.2072	-0.6900	0.2627

Columns 8 through 14

0.7349	-0.1738	-0.5530	-0.9896	0.3881	0.3245	0.2369
0.0403	0.1154	0.5935	-0.0895	-0.3598	-0.1564	-0.0388
0.1271	-0.5036	0.3764	-0.3193	-0.6004	-0.1108	-0.0595
-0.4776	-0.4433	-0.5820	-0.2490	0.7063	0.0476	0.0356

Columns 15 through 21

0.0910	0.0048	0.0129	0.0016	-0.0032	-0.0128	0.0009
-0.0041	0.0765	0.0063	-0.0137	-0.0048	0.0037	0.0004
0.0053	0.0065	0.1017	0.0063	-0.0029	0.0024	0.0079
0.0146	-0.0123	0.0054	0.0773	-0.0032	0.0175	0.0017

Columns 22 through 28

0.0024	0.0037	-0.0121	-0.0049	-0.0021	0.0174	-0.0127
0.0202	0.0018	0.0175	-0.0014	0.0069	-0.0051	0.0203
0.0046	0.0065	0.0036	0.0093	0.0039	0.0217	0.0108
0.0117	-0.0036	0.0059	-0.0025	0.0195	0.0024	0.0213

F.3 Chapter IV Gains

This section contains the feedback matrices as computed by Eq (11) for the two different solution techniques developed in Section 4.2 for both cases of the fourteen degree of freedom example given in that chapter. The matrices are on the following pages. Table 21 is the key to the solutions.

Table 21. Key to Matrix Names for Section F.3

Example	Matrix Name	Solution Technique
$G_1 = 7.3$	$\mathbf{G}_{tr}$	min trace $\mathbf{P}$
	$\mathbf{G}_2$	min $\|\mathbf{P}\|_2$
$G_1 = 730$	$\mathbf{G}_{ave}$	min trace $\mathbf{P}$
	$\mathbf{G}_{max}$	min $\|\mathbf{P}\|_2$

G_{tr} =

Columns 1 through 7

-11.7488	3.2884	13.4986	-0.5558	17.9921	-13.8203	-16.7188
-9.9056	-16.9476	29.3157	0.7307	9.9056	-19.5694	-14.2028
-16.6610	-34.5369	-12.3350	26.5726	7.9554	26.8844	15.1763
-3.4407	-13.1607	-32.9885	-16.3686	0.2130	47.2543	12.5992

Columns 8 through 14

14.0544	-15.3505	-16.4734	15.3605	18.9762	-2.2705	-3.5532
39.0706	-24.1762	-4.5332	8.6402	-4.2683	10.3557	-2.2846
-29.0592	-3.8376	15.8942	-18.6049	-11.5161	10.3083	-0.9486
-23.9520	23.5508	-1.9967	-7.1669	-4.2558	-2.3863	3.4267

Columns 15 through 21

-5.1343	0.1716	-0.3069	-0.0404	-0.1499	0.3782	-0.4648
0.1692	-6.0402	-0.2692	-0.7329	-0.2864	-0.8601	0.0000
-0.3071	-0.2697	-5.4743	-0.0852	-0.5116	-0.3406	-0.5974
-0.0422	-0.7302	-0.0836	-6.2146	0.0723	-1.2644	-0.6249

Columns 22 through 28

0.0626	-0.3449	0.4453	-0.2715	0.5341	-0.9002	1.1157
-1.2409	-0.1138	-1.0375	-0.4907	-1.4566	-0.5478	-1.8195
-0.5316	-0.7306	-0.2372	-0.9703	-0.1914	-1.9094	-0.7640
-1.0689	-0.4573	-1.5007	-0.0909	-1.0637	-0.9986	-1.9607

$G_2 =$

Columns 1 through 7

-11.7935	3.6822	13.4141	-1.1972	19.0378	-14.1205	-15.4661
-9.7794	-16.1182	29.9985	0.9010	10.5674	-20.1413	-15.9227
-16.9675	-35.4135	-12.7730	27.7564	8.3364	25.3197	18.2846
-2.4169	-12.0715	-34.9031	-16.2340	1.2857	45.2595	15.8351

Columns 8 through 14

15.2768	-18.1244	-16.9983	14.6906	18.8738	-0.9257	-3.6643
36.[illegible]842	-23.8281	-3.3703	9.6093	-2.9354	8.9960	-3.0381
-26.6124	-5.6469	14.0735	-20.2280	-11.1045	11.0074	-0.6140
-23.8285	21.6735	-1.5667	-9.0406	-3.0817	-2.3342	2.8039

Columns 15 through 21

-5.1650	0.1681	-0.3420	-0.0606	-0.0630	0.3829	-0.4401
0.1657	-5.9862	-0.2674	-0.6652	-0.2846	-0.9341	0.0109
-0.3425	-0.2679	-5.5387	-0.1240	-0.4237	-0.3109	-0.4992
-0.0624	-0.6623	-0.1222	-6.1774	0.1294	-1.3122	-0.5896

Columns 22 through 28

0.0732	-0.3376	0.4428	-0.3043	0.5322	-0.9075	1.1218
-1.3018	-0.1148	-1.0139	-0.4733	-1.3978	-0.5568	-1.8391
-0.5020	-0.7731	-0.2456	-0.9706	-0.2047	-1.9324	-0.7673
-1.1380	-0.4672	-1.4513	-0.1094	-1.0390	-0.9881	-1.9717

$G_{ave} =$

Columns 1 through 7

-22.8709	22.8106	-6.5041	-28.3029	54.2380	0.8188	-20.0364
-16.5459	-20.1653	55.4339	6.3215	4.6520	-44.3262	-21.4800
-6.0725	-59.4756	-23.9072	54.6779	-5.3017	1.4459	69.3602
16.8109	-11.9735	-61.8080	-22.9949	4.4964	75.5195	28.6704

Columns 8 through 14

6.5656	-20.1957	-24.4718	21.3485	28.5355	-8.7677	-3.4032
62.5311	-31.8067	21.0302	3.1264	-31.6115	23.7250	-2.5897
-0.5412	-24.9753	48.2131	-56.3328	-52.2274	32.1660	0.1134
-41.2860	28.8172	-23.4676	-7.8490	14.2654	-8.1035	4.3680

Columns 15 through 21

-7.4396	0.5249	-0.8452	-0.7395	0.1190	0.9985	-0.4188
0.5212	-7.0872	-0.2929	-0.4252	-0.5485	-0.9977	-0.0066
-0.8442	-0.2941	-7.6642	-0.4416	0.2043	-0.4186	-0.4775
-0.7415	-0.4217	-0.4407	-7.5235	0.6443	-0.5406	-0.9764

Columns 22 through 28

-0.5363	-0.5846	0.7457	0.2731	0.3761	-0.9087	0.9068
-0.8074	-0.1145	-0.6507	-0.9885	-2.0818	-0.2452	-1.6534
-0.3056	-1.1602	-0.2901	-1.3372	-0.0700	-1.2076	-0.7120
-1.3012	-0.5369	-2.0333	0.3319	-0.6358	-1.3158	-1.9474

G_{max} =

Columns 1 through 7

-22.8709	22.8106	-6.5041	-28.3029	54.2380	0.8188	-20.0364
-16.5459	-20.1653	55.4339	6.3215	4.6520	-44.3262	-21.4800
-6.0725	-59.4756	-23.9072	54.6779	-5.3017	1.4459	69.3602
16.8109	-11.9735	-61.8080	-22.9949	4.4964	75.5195	28.6704

Columns 8 through 14

6.5656	-20.1957	-24.4718	21.3485	28.5355	-8.7677	-3.4032
62.5311	-31.8067	21.0302	3.1264	-31.6115	23.7250	-2.5897
-0.5412	-24.9753	48.2131	-56.3328	-52.2274	32.1660	0.1134
-41.2860	28.8172	-23.4676	-7.8490	14.2654	-8.1035	4.3680

Columns 15 through 21

-7.4396	0.5249	-0.8452	-0.7395	0.1190	0.9985	-0.4188
0.5212	-7.0872	-0.2929	-0.4252	-0.5485	-0.9977	-0.0066
-0.8442	-0.2941	-7.6642	-0.4416	0.2043	-0.4186	-0.4775
-0.7415	-0.4217	-0.4407	-7.5235	0.6443	-0.5406	-0.9764

Columns 22 through 28

-0.5363	-0.5846	0.7457	0.2731	0.3761	-0.9087	0.9068
-0.8074	-0.1145	-0.6507	-0.9885	-2.0818	-0.2452	-1.6534
-0.3056	-1.1602	-0.2901	-1.3372	-0.0700	-1.2076	-0.7120
-1.3012	-0.5369	-2.0333	0.3319	-0.6358	-1.3158	-1.9474

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Vita

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She was married in October of 1988, and has two beautiful daughters, two energetic dogs, and one tolerant cat.

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
December 1993

3. REPORT TYPE AND DATES COVERED
Dissertation

4. TITLE AND SUBTITLE
SIMULTANEOUS DESIGN OF ACTIVE VIBRATION CONTROL AND PASSIVE DAMPING

5. FUNDING NUMBERS

6. AUTHOR(S)
Michele Lynn Devereaux Gaudreault

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Institute of Technology, WPAFB OH 45433-6583

8. PERFORMING ORGANIZATION REPORT NUMBER
AFIT/DS/AA/93-2

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Col Robert L. Heklotz
WL/FI

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

Abstract

A modified form of the standard linear quadratic regulator (LQR) cost functional is used to optimally blend active vibration control and passive structural damping, whether viscous or viscoelastic. Viscoelastic damping is first modelled by a standard linear model, and then by a fractional derivative model. For the viscous damping case and the classically modelled viscoelastic damping case, a sub-optimal closed form solution is derived that is independent of the initial conditions. An iterative technique that minimizes the average value of the cost functional and one that minimizes the maximum value are developed. Both techniques are applicable to viscous and viscoelastic damping and are independent of the initial state. The advantages and disadvantages of the different solution techniques are given with respect to computation requirements and performance. Several numerical examples illustrate the similarities of and differences between the various techniques.

14. SUBJECT TERMS
control theory, damping, optimization, passive systems, structural response, viscoelasticity

15. NUMBER OF PAGES
147

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL