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James F. Geer



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# RATIONAL TRIGONOMETRIC APPROXIMATIONS USING FOURIER SERIES PARTIAL SUMS

*James F. Geer*[1]
Department of Systems Science
Watson School of Engineering and Applied Science
State University of New York
Binghamton, NY 13902

## ABSTRACT

A class of approximations $\{S_{N,M}\}$ to a periodic function $f$ which uses the ideas of Padé, or rational function, approximations based on the Fourier series representation of $f$, rather than on the Taylor series representation of $f$, is introduced and studied. Each approximation $S_{N,M}$ is the quotient of a trigonometric polynomial of degree $N$ and a trigonometric polynomial of degree $M$. The coefficients in these polynomials are determined by requiring that an appropriate number of the Fourier coefficients of $S_{N,M}$ agree with those of $f$. Explicit expressions are derived for these coefficients in terms of the Fourier coefficients of $f$. It is proven that these "Fourier-Padé" approximations converge point-wise to $(f(x^+) + f(x^-))/2$ more rapidly (in some cases by a factor of $1/k^{2M}$) than the Fourier series partial sums on which they are based. The approximations are illustrated by several examples and an application to the solution of an initial, boundary value problem for the simple heat equation is presented.



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**1. Introduction.** Fourier series are used widely in many branches of applied mathematics. For example, they are often used together with separation of variables to construct analytical solutions to boundary value problems for differential equations and with a variety of spectral methods to find approximate solutions to these problems numerically. For practical purposes, approximate solutions to these problems are often obtained using only a finite number of the terms in a Fourier series. This truncation procedure often leads to nonuniformly valid approximations. For example, when the function being approximated has a point of discontinuity, the Gibbs phenomena is present. The "oscillations" caused by this phenomena typically propagate into regions away from the singularity, and, hence, degrade the quality of the partial sum approximation in these regions. Even if the function being modeled is analytic, but has a region of large slope or curvature, there can be significant oscillations in the partial sums outside these regions, which again degrade the quality of the approximation.

Recently, Gottlieb and Shu [6] and Gottlieb, et.al. [7] have proposed a way of overcoming the Gibbs phenomena. Their technique involves the construction of a new series using the Gegenbauer polynomials $C_n^{\lambda}(x)$. For a function $f$ that is analytic on the interval $[-1,1]$, but is not periodic, they prove that their technique leads to a series which converges exponentially to $f$ in the maximum norm. To do this, they require that the parameter $\lambda$, which appears in the weight factor $(1-x^2)^{\lambda-1/2}$, grows with the number of Fourier modes considered. As we shall demonstrate below, the family of approximations we shall introduce can be defined explicitly in terms of the known Fourier coefficients. This will prove to be particularly useful for certain applications (as we shall demonstrate) when the Fourier coefficients are themselves functions of one or more other variables. Although the approximations we shall define do not "eliminate" the Gibbs phenomena, they do mitigate its effect, as we shall show. This is especially true outside a "small" neighborhood of a point of discontinuity of $f$, where, for practical purposes, the "unwanted" oscillations *can* essentially be eliminated.

To fix notation, we let $f(x)$ be a piece-wise smooth, $2\pi$-periodic function. Then we can associate with $f(x)$ its Fourier series $S(x)$ defined by

$$S(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), \tag{1}$$

$$a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \tag{2}$$

$n = 0, 1, 2, \ldots$. (For some of our formulas below, it is convenient to regard the coefficients $\{a_n, b_n\}$ as being defined by (2) for negative as well as positive values of n. Thus, $a_{-n} = a_n$ and $b_{-n} = -b_n$ for all integers $n$.) It is well known (see [2], for example) that $S(x)$ converges to $f(x)$ at each point $x$ where $f$ is continuous and to $(f(x^+)+f(x^-))/2$ at each point $x$ where $f$ is not continuous. For practical purposes, $f(x)$ is often approximated by its Fourier partial sum $S_{N,0}(x)$ defined by

$$S_{N,0}(x) = a_0/2 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)). \tag{3}$$

We now define a class of approximations to $f$ which uses the ideas of Padé approximants (see, e.g., [1]), except that the approximations are based on the Fourier series representation of $f$, rather than on the Taylor series representation of $f$. Thus, for any two non-negative integers $N$ and $M$, we define a family of "Fourier-Padé" approximations $S_{N,M}(x)$ by

$$S_{N,M}(x) = \frac{A_0/2 + \sum_{n=1}^{N}(A_n \cos(nx) + B_n \sin(nx))}{1 + \sum_{m=1}^{M}(C_m \cos(mx) + D_m \sin(mx))}. \tag{4}$$

The $2M + 2N + 1$ constants $\{A_n, B_n, C_m, D_m\}$ which appear in the definition of $S_{N,M}$ are determined by the condition that $2M + 2N + 1$ of the Fourier coefficients of $S_{N,M}$ agree with those of $f$, i.e.

$$(1/\pi)\int_{-\pi}^{\pi} S_{N,M}(x)\cos(jx)dx = a_j,\ 0 \le j \le J \tag{5}$$

$$(1/\pi)\int_{-\pi}^{\pi} S_{N,M}(x)\sin(kx)dx = b_k,\ 1 \le k \le K \tag{6}$$

where $J + K = 2M + 2N$.

The idea of constructing Padé approximations based on series representations of functions other than the classical power series representation has been suggested and studied by several other investigators. For example, Maehly [9] has suggested an approach to determine the coefficients in rational approximations based on Chebyshev series, and this approach has since been described in numerical terms, with examples, by Ralston[10] and by Fike[5]. An account of the basic theory of such approximations has been given by Cheney [3], who considered the more general case of expansions in terms of a basis $\{\varphi_j\}$, whose elements satisfy relations of the form $\varphi_i\varphi_j = \sum a_{ijk}\varphi_k$. Clenshaw and Lord [4] have reviewed rational approximations based on Chebyshev series and present a well-conditioned method for constructing the elements of a "Chebyshev-Padé" table. The general form of a Fourier-Padé representation, such as equation (4), has also been suggested by Cheney. He has proven the existence of a rational trigonometric function which "best" approximates a continuous function $f$, but does not discuss any detailed algorithms for the computation of the coefficients, rates of convergence of the approximations, etc.

In the sections 2-4 below we consider the special case of equation (4) when $M = 1$. In particular, in sections 2-3 we develop explicit formulas for the coefficients which appear in the definition of $S_{N,1}$ in terms of the Fourier coefficients of $f$. Several results concerning the behavior and convergence of the approximations $\{S_{N,1}\}$ are presented in section 4. In sections 5-6, the results of sections 2-4 are generalized to values of $M \ge 2$. In section 7 the approximations are applied to a simple initial, boundary-value problem for the heat equation. Some observations and insights about this class of approximations are discussed in section 8.

**2. Odd functions of $x$.** Since $f$ can be expressed as the sum of an even function of $x$ and an odd function of $x$, it is sufficient to consider separately the cases when $f$ is either even or odd. Thus, in this section, we consider the case when $f$ is an odd function of $x$, while the case of an even function will be discussed in the next section. For this case, the Fourier series of $f$ can be expressed as (1) with each $a_n = 0,\ n \geq 0$, and we define a class of Fourier-Padé approximations $S_{N,M}(x)$ of the form

$$S_{N,M}(x) = \frac{\sum_{n=1}^{N} B_n \sin(nx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)}. \tag{7}$$

In this section we examine in detail the case $M = 1$. The case $M \geq 2$ will be discussed in section 5. Thus we consider first approximations of the form

$$S_{N,1}(x) = \frac{\sum_{n=1}^{N} B_n \sin(nx)}{1 + C_1 \cos(x)}. \tag{8}$$

The constants $\{B_1, B_2, \ldots, B_N, C_1\}$ are determined by conditions (6), which for this case become

$$\sum_{j=1}^{N} d_{k,j} B_j = b_k, \quad k = 1, \ldots, N+1, \tag{9}$$

where

$$d_{k,j} = (1/\pi) \int_{-\pi}^{\pi} \frac{\sin(jx)\sin(kx)}{1 + C_1 \cos(x)} dx = d_{j,k}.$$

To solve (9), we first use the identity

$$\sin(jx)\sin(kx) = (1/2)(\cos((k-j)x) - \cos((k+j)x)$$

to express the coefficients $\{d_{k,j}\}$ as

$$d_{k,j} = I_{|k-j|,1} - I_{k+j,1}, \text{ where } I_{n,1} = (1/\pi) \int_{0}^{\pi} \frac{\cos(nx)}{1 + C_1 \cos(x)} dx,\ n \geq 0. \tag{10}$$

Assuming that $0 < C_1^2 < 1$, we find (see, e.g., [8], p. 113)

$$I_{n,1} = \rho^n \frac{1 + \rho^2}{1 - \rho^2}, \text{ for } n \geq 0, \text{ where } \rho = \frac{\sqrt{1 - C_1^2} - 1}{C_1} \text{ or } C_1 = -\frac{2\rho}{1 + \rho^2}. \tag{11}$$

Using this result in (10) we can write

$$d_{k,j} = \rho^{k-j}(1 + \rho^2)(1 + \rho^2 + \ldots + \rho^{2j-2}), \text{ for } k \geq j.$$

Using these expressions, (9) can be expressed in matrix-vector form as

$$\Re \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{pmatrix} = \frac{1}{1 + \rho^2} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N+1} \end{pmatrix} \tag{12}$$

where the $(N+1) \times N$ matrix $\Re$ is defined by

$$\Re = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\ \rho & 1+\rho^2 & \rho(1+\rho^2) & \cdots & \rho^{N-2}(1+\rho^2) \\ \rho^2 & \rho(1+\rho^2) & 1+\rho^2+\rho^4 & \cdots & \rho^{N-3}(1+\rho^2+\rho^4) \\ & \cdot & \cdot & \cdots & \cdot \\ & \cdot & \cdot & \cdots & \cdot \\ \rho^{N-1} & \rho^{N-2}(1+\rho^2) & \cdot & \cdots & 1+\rho^2+\cdots+\rho^{2N-2} \\ \rho^N & \rho^{N-1}(1+\rho^2) & \cdot & \cdots & \rho(1+\rho^2+\cdots+\rho^{2N-2}) \end{pmatrix}.$$

To solve (12), we first note that if we subtract $\rho$ times the next to last equation from the last equation we obtain the relation $0 = (b_{N+1} - \rho b_N)/(1+\rho^2)$, from which we find (assuming $b_N \neq 0$)

$$\rho = \frac{b_{N+1}}{b_N}, \text{ or } C_1 = -\frac{2\rho}{1+\rho^2} = -\frac{2b_N b_{N+1}}{b_N^2 + b_{N+1}^2}. \tag{13}$$

In a similar manner, subtracting $\rho$ times the $(k-1)^{th}$ equation from the $k^{th}$ equation we find

$$B_k = \frac{b_k - \rho b_{k-1}}{1+\rho^2} - \sum_{j=k+1}^{N} \rho^{k-j} B_j, \quad 1 \le k \le N.$$

Using this recursive definition of the $\{B_k\}$, we can use induction to show that the $\{B_k\}$ are given explicitly by

$$B_k = b_k - \frac{\rho(b_{k-1} + b_{k+1})}{1+\rho^2} = b_k + \frac{C_1}{2}(b_{k-1} + b_{k+1}) \quad 1 \le k \le N. \tag{14}$$

(We note that equations (14) also follow from equating (8) to $\sum_{j=1}^{N+1} b_j \sin(jx)$, multiplying by $1 + C_1 \cos(x)$, and equating coefficients of $\sin(nx)$ on each side of the resulting expression, for $1 \le n \le N$.) Thus, the coefficients $\{B_k\}$ and $C_1$, which appear in the definition (8) of $S_{N,1}$, are completely defined in terms of the coefficients $\{b_k\}$ by equations (13) and (14).

**Example 1**: As an example of these results, we let $f(x) = x/2$, for $-\pi \le x < \pi$, and $f(x+2\pi) = f(x)$. Then $b_k = (-1)^{k+1}/k$, for $k = 1, 2, \ldots$. Using the formulas above, we find $\rho = -N/(N+1)$ and hence

$$C_1 = \frac{2N(N+1)}{2N(N+1)+1}, \; B_1 = \frac{3N(N+1)+2}{4N(N+1)+2},$$

$$B_k = (-1)^k \left( \frac{2N(N+1)}{2N(N+1)+1} \frac{k}{k^2-1} - \frac{1}{k} \right), \; k = 2, 3, \ldots, N.$$

Explicit values for $C_1$ and $B_1, \ldots, B_N$ for a few small values of $N$ are shown in Table I.

*TABLE I (Example 1: $M = 1$)*

| $N$ | $C_1$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ |
|---|---|---|---|---|---|---|---|
| 1 | 4/5 | 4/5 | | | | | |
| 2 | 12/13 | 10/13 | 3/26 | | | | |
| 3 | 24/25 | 19/25 | 7/50 | −2/75 | | | |
| 4 | 40/41 | 31/41 | 37/246 | −4/123 | 5/492 | | |
| 5 | 60/61 | 46/61 | 19/122 | −13/366 | 3/244 | −3/610 | |
| 6 | 84/85 | 64/85 | 27/170 | −19/510 | 23/1700 | −1/170 | 7/2550 |

To illustrate these results, in Figure 1 we have plotted the function $f(x)$ (solid line), along with the approximations $S_{4,0}(x)$ (dashed line) and $S_{3,1}(x)$ (dotted line). As the figure illustrates, except possibly in a neighborhood of the discontinuities of $f$, $S_{3,1}$ is a better approximation to $f$ than is $S_{4,0}$. In particular, the oscillations present in $S_{4,0}$ away from the singularities of $f$ have been virtually eliminated in $S_{3,1}$. However, near the points of discontinuity of $f$, the Gibbs phenomenon is still present in $S_{3,1}$, although its effect appears to be somewhat less than the corresponding phenomena present in $S_{4,0}$. We shall discuss this example further in sections 4 and 8.

Before continuing, we briefly note two special cases. First, if $b_{N+1} = 0$, then $\rho = 0$ and hence $C_1 = 0$. Thus $B_k = b_k$, for $1 \le k \le N$, and $S_{N,1}$ reduces to the Fourier partial sum $S_{N,0}$, for this case. Second, suppose $b_k = 0$ when $k$ is *even*. In this case, we modify the form of $S_{N,1}$ and define

$$S_{N,1}(x) = \frac{\sum_{n=1}^{N} B_{2n-1}\sin((2n-1)x)}{1 + C_2\cos(2x)}. \tag{15}$$

Then equations (6) for this case have the same form as (9) with $b_k$ replaced by $b_{2k-1}$ and $d_{k,j} = d_{j,k} = I_{|k-j|} - I_{k+j-1}$, from which we find

$$d_{k,j} = \rho^{k-j}\frac{(1+\rho^2)(1+\rho+\rho^2+\ldots+\rho^{2j-2})}{1+\rho}, \text{ for } k \ge j.$$

Using these expressions and the same type of arguments as presented above, we find for this case (assuming $b_{2N-1} \neq 0$)

$$C_2 = -\frac{2\rho}{1+\rho^2},\ \rho = \frac{b_{2N+1}}{b_{2N-1}},\ B_{2k-1} = b_{2k-1} + \frac{C_2}{2}(b_{2k-3} + b_{2k+1}),\ 1 \le k \le N. \tag{16}$$

(Recall that we have defined $b_{-1} = -b_1$.)

**3. Even functions of $x$.** When $f$ is an even function of $x$, its Fourier series has the form (1) with each $b_n = 0$. Then we define the family of Fourier-Padé approximations

$$(17) \qquad S_{N,M}(x) = \frac{A_0/2 + \sum_{n=1}^{N} A_n \cos(nx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)}.$$

In this section we consider the case $M = 1$. The case $M \geq 2$ will be discussed in section 5. Thus we consider approximations of the form

$$(18) \qquad S_{N,1}(x) = \frac{A_0/2 + \sum_{n=1}^{N} A_n \cos(nx)}{1 + C_1 \cos(x)}.$$

The requirement that the first $N+2$ Fourier coefficients of $S_{N,1}$ agree with those of $f$ leads to a system of equations for the $N+2$ unknowns $\{C_1, A_0, ..., A_N\}$, which can be solved using the techniques of section 2. In particular, we find (assuming $a_N \neq 0$)

$$(19) \quad C_1 = -\frac{2\rho}{1+\rho^2}, \text{ where } \rho = \frac{a_{N+1}}{a_N}, \ A_k = a_k + \frac{C_1}{2}(a_{k-1} + a_{k+1}), \ 0 \leq k \leq N.$$

In the special case when $a_{N+1} = 0$, we see that $\rho = 0$ and hence $C_1 = 0$. Then each $A_k = a_k$ and $S_{N,1}(x) = S_{N,0}(x)$ for this case. Also, in the special case that $a_k = 0$ when $k$ is *odd*, we redefine

$$(20) \qquad S_{N,1}(x) = \frac{A_0/2 + \sum_{n=1}^{N} A_{2n} \cos(2nx)}{1 + C_2 \cos(2x)}$$

and find (assuming $a_{2N} \neq 0$)

$$(21) \quad C_2 = -\frac{2\rho}{1+\rho^2}, \text{ where } \rho = \frac{a_{2N+2}}{a_{2N}}, \ A_{2k} = a_{2k} + \frac{C_2}{2}(a_{2k-2} + a_{2k+2}), \ 0 \leq k \leq N.$$

**Example 2**: As an application of these results, we let $f(x) = 1/\sqrt{1 + \alpha \cos(x)}$, where $\alpha^2 < 1$. Here $f$ is analytic for $-\pi \leq x \leq \pi$, but develops a sharp, narrow peak near $x = 0$ as $\alpha \to -1$. In Table II we have recorded the Fourier coefficients $\{a_n\}$ of $f$, as well as the coefficients $\{C_1, A_0, ..., A_N\}$ for a few small values of N when $\alpha = -0.95$.

*Table II* (*Example* 2: $\alpha = -0.95$, $M = 1$)

| $N$ | $a_N$ | $C_1$ | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|
| 0 | 2.94734 | −0.68380 | 2.15059 | | | | |
| 1 | 1.16518 | −0.85289 | 1.95357 | −0.37012 | | | |
| 2 | 0.65289 | −0.89142 | 1.90867 | −0.43948 | −0.04495 | | |
| 3 | 0.40050 | −0.90812 | 1.88922 | −0.46953 | −0.05802 | −0.01235 | |
| 4 | 0.25636 | −0.91741 | 1.87839 | −0.48626 | −0.06529 | −0.01657 | −0.00452 |

To illustrate these results, in Figure 2 we have plotted the function $f(x)$ (solid line), along with the approximations $S_{4,0}(x)$ (dashed line) and $S_{3,1}(x)$ (dotted line). As the figure illustrates, $S_{3,1}$ is a noticeably better approximation to $f$ than is $S_{4,0}$. In particular, the oscillations present in $S_{4,0}$ have been virtually eliminated in $S_{3,1}$. We shall discuss this example further in sections 4 and 8.

**4. Analysis of the Case $M = 1$.** We now prove certain results which express the asymptotic behavior of the coefficients $C_1$ and $\{A_k\}$ or $\{B_k\}$ as $N \to \infty$, and which also allow us to make some statements concerning the convergence of the family of approximations $\{S_{N,1}\}$, as $N \to \infty$.

It is well known (see [2], for example) that if $f$ has $q$ continuous derivatives on $-\pi \leq x \leq \pi$ and the derivative of $f$ of order $q+1$ is piece-wise continuous, then the Fourier coefficients of $f$ are $O(1/k^{q+2})$ as $k \to \infty$. For such a function, we now show that the coefficients in $S_{N,1}$ decay as $k \to \infty$ at a faster rate than the Fourier coefficients on which they are based.

THEOREM 1. *Let $f$ be an odd, $2\pi$-periodic, piece-wise smooth function and let its Fourier coefficients $\{b_k\}$ satisfy the condition that $b_k = O(1/k^p)$, as $k \to \infty$, where $p$ is a real, positive number. Then $|C_1| = 1 + O(1/N^2)$ and $B_k = O(1/k^{p+2})$ as $k, N \to \infty$. More precisely, let $b$, $\beta$, and $\gamma$ be constants (independent of $k$). Then, if $b_k = (b/k^p)(1 + \beta/k + \gamma/k^2 + O(1/k^3))$, as $k \to \infty$, it follows that*

$$C_1 = -1 + \frac{p^2}{2N^2} + O(1/N^3), \quad B_k = -\frac{bp}{2}\left(\frac{1}{k^{p+2}}\right)\left(1 + p(1 - (k/N)^2)\right) + O(\frac{1}{k^{p+3}}),$$

*as $k, N \to \infty$. If $f$ is an even, $2\pi$-periodic, piece-wise smooth function, then these results hold with $b_k$ replaced by $a_k$ and $B_k$ replaced by $A_k$.*

*Proof:* We shall outline the proof only for the case when $f$ is an odd, $2\pi$-periodic, piece-wise smooth function, since the proof for the case when $f$ is an even function follows the same line of reasoning. Using the assumed form of the asymptotic behavior of the coefficients $\{b_k\}$, we note first that we can write

$$\begin{aligned} b_{N+1} &= (b/(N+1)^p)(1 + \beta/(N+1) + \gamma/(N+1)^2 + O(1/(N+1)^3)) \\ &= (b/N^p)(1+\epsilon)^{-p}(1 + \beta\epsilon/(1+\epsilon) + \gamma\epsilon^2/(1+\epsilon)^2 + O(1/N^3)) \\ &= (b/N^p)(1 + (\beta - p)\epsilon + (\gamma - \beta(p+1) + p(p+1)/2)\epsilon^2 + O(1/N^3)), \end{aligned} \tag{22}$$

where we have defined $\epsilon = 1/N$. Then, using this expression and the definition of $\rho$ in (13), we can write

$$\rho = \frac{b_{N+1}}{b_N} = 1 - \frac{p}{N} + \frac{p + p^2 - 2\beta}{2N^2} + O(1/N^3), \text{ as } N \to \infty, \tag{23}$$

and hence

$$C_1 = -\frac{2\rho}{1+\rho^2} = -1 + \frac{p^2}{2N^2} + O(1/N^3), \text{ as } N \to \infty,$$

as stated in the Theorem.

In a similar manner, for large values of $k$ we can use the assumed form of the coefficients $\{b_k\}$ to write

$$\begin{aligned} b_{k\pm 1} &= (b/(k\pm 1)^p)(1+\beta/(k\pm 1)+\gamma/(k\pm 1)^2+O(1/k^3)) \\ &= (b/k^p)(1\pm\epsilon)^{-p}(1+\beta\epsilon/(1\pm\epsilon)+\gamma\epsilon^2/(1\pm\epsilon)^2+O(1/k^3)) \\ &= (b/k^p)(1+(\beta\mp p)\epsilon+(\gamma\mp\beta(p+1)+p(p+1)/2)\epsilon^2+O(1/k^3)), \end{aligned}$$

where we have now defined $\epsilon = 1/k$. Using these expressions, along with an analogous expression for $b_k$ and the asymptotic expression above for $C_1$, in the definition of $B_k$ in (14), we find for large values of $k$ and $N$ (with $k \le N$)

$$\begin{aligned} B_k &= b_k + (C_1/2)(b_{k-1}+b_{k+1}) \\ &= (b/k^p)\{1+\beta\epsilon+\gamma\epsilon^2+O(\epsilon^3) \\ &\quad +(1/2)(-1+p^2/(2N^2)+O(1/N^3))(2+2\beta\epsilon+(2\gamma+p+p^2)\epsilon^2+O(\epsilon^3))\} \\ &= -(bp/2)(1/k^{p+2})(1+p(1-(k/N)^2))+O(1/k^{p+3}), \end{aligned}$$

which completes our proof.

Thus, Theorem 1 establishes that the coefficients $\{B_n\}$ in the approximations $S_{N,1}$ decay to zero more rapidly (by a factor of $k^{-2}$) than the coefficients in the partial sum $S_{N,0}$ on which they are based.

Before continuing, we remark that results very similar to those stated in Theorem 1 hold if the coefficients $\{b_k\}$ of $f$ have the same asymptotic form as that stated in the theorem, except for a multiplicative factor of $(-1)^k$, i.e., if $b_k = (b/k^p)(-1)^k(1+\beta/k+\gamma/k^2+O(1/k^3))$, as $k\to\infty$. In this case, it is easy to show that the Fourier coefficients $\{\tilde{b}_k\}$ of $\tilde{f} \equiv f(x+\pi)$ are related to those of $f$ by $\tilde{b}_k = (-1)^k b_k$. It then follows that the coefficients $\{\tilde{b}_k\}$ satisfy the conditions of the theorem and hence the corresponding coefficients $\tilde{C}_1$ and $\{\tilde{B}_k\}$ have the asymptotic form indicated in the theorem. In particular, it follows from equations (13) and (14) that $C_1 = -\tilde{C}_1$ and $B_k = (-1)^k\tilde{B}_k$. Thus, the asymptotic behavior of $C_1$ is the negative of that indicated for $C_1$ in the theorem, while the asymptotic form of $B_k$ is $(-1)^k$ times the form indicated for $B_k$.

To illustrate these results, in Figure 3 we have used the coefficients $\{B_k\}$ from Example 1 and have plotted $k^3|B_k|$ as a function of $1/k$, for $3 \le N \le 20$. The figure clearly illustrates the $O(1/k^3)$ decay of the coefficients $\{B_k\}$, and also indicates an interesting asymptotic behavior of $B_{N-q}$, for a fixed value of $q$, as $N\to\infty$.

If the function $f$ is analytic for $-\pi \le x \le \pi$, then (see, e.g., [2]) the Fourier coefficients of $f$ decay exponentially as $k\to\infty$. That is, there exists a constant $\theta$, with $0<|\theta|<1$, such that $\{a_k, b_k\}$ are $O(\theta^k)$ as $k\to\infty$. The next theorem shows that, again for this case, the coefficients $\{B_k\}$ decay to zero more rapidly than the Fourier coefficients on which they are based.

THEOREM 2. *Let $f$ be an odd, analytic, $2\pi$-periodic function and let its Fourier coefficients $\{b_k\}$ satisfy the condition that $b_k = O(|\theta|^k/k^p)$, as $k\to\infty$, where $0<|\theta|<1$, and $p$ is a real number. Then $C_1 = 2\theta/(1+\theta^2)+O(1/N)$ and $B_k = O(|\theta|^k/k^{p+1})$ as $k, N\to\infty$. More precisely, let $b$, $\beta$, and $\gamma$ be constants (independent of $k$). Then, if $b_k = (b\theta^k/k^p)(1+\beta/k+\gamma/k^2+O(1/k^3))$, as $k\to\infty$, it follows that*

$$C_1 = -\frac{2\theta}{1+\theta^2}\left\{1-\frac{p}{N}\left(\frac{1-\theta^2}{1+\theta^2}\right)+\frac{(p-2\beta)(1-\theta^4)+p^2(1-6\theta^2+\theta^4)}{2(1+\theta^2)^2N^2}+O(1/N^3)\right\},$$

$$B_k = -\frac{b\theta^k}{k^{p+1}}\left\{p\frac{1-\theta^2}{1+\theta^2}(1-k/N) - \frac{1}{k}\left[\frac{1}{2}\frac{1-\theta^2}{1+\theta^2}(2\beta p + (2\beta + p)(1+k/N))(1-k/N)\right.\right.$$

$$\left.\left. -\frac{p^2}{2(1+\theta^2)^2}\left((1+\theta^2)^2 - (k/N)(1-6\theta^2+\theta^4)\right)(1-k/N) - \frac{p}{1+\theta^2}\right]\right\} + O(\frac{\theta^k}{k^{p+3}}),$$

*as $k, N \to \infty$. If $f$ is an even, analytic, $2\pi$-periodic function, then these results hold with $b_k$ replaced by $a_k$ and $B_k$ replaced by $A_k$.*

*Proof:* As in the proof of Theorem 1, we shall outline the proof only for the case when $f$ is an odd function, since the proof for the case when $f$ is an even function follows the same line of reasoning. Using the assumed form of the asymptotic behavior of the coefficients $\{b_k\}$, we note first that $b_{N+1}$ has the asymptotic form indicated on the right side of equation (22), multiplied by $\theta^{N+1}$. Then, using this expression in the definition of $\rho$ in (13), we find that $\rho$ has the asymptotic form indicated by the right side of (23) multiplied by $\theta$. Using these expressions in the definition of $C_1$ in (13) we find

$$C_1 = -\frac{2\rho}{1+\rho^2} = -2\theta\frac{1-p/N+(p+p^2-2\beta)/(2N^2)+O(1/N^3)}{1+\theta^2\left(1-p/N+(p+p^2-2\beta)/(2N^2)+O(1/N^3)\right)^2}$$

$$= -\frac{2\theta}{1+\theta^2}\left\{1-\frac{p}{N}\left(\frac{1-\theta^2}{1+\theta^2}\right)+\frac{(p-2\beta)(1-\theta^4)+p^2(1-6\theta^2+\theta^4)}{2(1+\theta^2)^2N^2}+O(1/N^3)\right\},$$

as $N \to \infty$.

In a similar manner, using the assumed asymptotic form of the coefficients $\{b_k\}$ we can write

$$b_{k\pm1} = (b\theta^{k\pm1}/k^p)(1+(\beta\mp p)\epsilon + (\gamma \mp \beta(p+1) + p(p+1)/2)\epsilon^2 + O(1/k^3)),$$

as $k \to \infty$, where $\epsilon = 1/k$. Thus, for large values of both $k$ and $N$ we have

$$B_k = b_k + (C_1/2)(b_{k-1} + b_{k+1})$$

$$= \frac{\theta^k b}{k^p}\left\{1+\beta\epsilon+\gamma\epsilon^2 - \frac{1}{1+\theta^2}\left[1-\frac{p}{N}\left(\frac{1-\theta^2}{1+\theta^2}\right)+\frac{(p-2\beta)(1-\theta^4)+p^2(1-6\theta^2+\theta^4)}{2(1+\theta^2)^2N^2}\right]\right.$$

$$\cdot\Big[1+(\beta+p)\epsilon+(\gamma+\beta(p+1)+p(p+1)/2)\epsilon^2$$

$$\left. +\theta^2\left(1+(\beta-p)\epsilon+(\gamma-\beta(p+1)+p(p+1)/2)\epsilon^2\right)\Big]\right\} + O(\theta^k/k^{p+3})$$

$$= -\frac{b\theta^k}{k^{p+1}}\left\{p\frac{1-\theta^2}{1+\theta^2}(1-k/N) - \frac{1}{k}\left[\frac{1}{2}\frac{1-\theta^2}{1+\theta^2}(2\beta p + (2\beta+p)(1+k/N))(1-k/N)\right.\right.$$

$$-\frac{p^2}{2(1+\theta^2)^2}\left((1+\theta^2)^2-(k/N)(1-6\theta^2+\theta^4)\right)(1-k/N)-\frac{p}{1+\theta^2}\Big]\Big]\Big\}+O(\frac{\theta^k}{k^{p+3}}),$$

as $k \to \infty$. This completes our proof.

We note that the rate at which the coefficients $\{B_k\}$ decay to zero changes slightly as $k \to N$. In particular, for $k < N$, $B_k = O(\theta^k(1-\theta^2)(1-k/N)/k^{p+1}$, but $B_N = O(\theta^N/N^{p+2})$, as $k, N \to \infty$. Thus, while each $B_k$ decays to zero faster than the corresponding Fourier coefficient $b_k$, we see that $B_{k_1}$ decays somewhat faster than $B_{k_2}$, when $k_2 < k_1 \leq N$.

To illustrate these results, in Figure 4 we have used the coefficients $\{A_k\}$ from Example 2 and have plotted $k^2|A_k|/|a_k|$ as a function of $1/k$, for $3 \leq N \leq 20$. The figure clearly illustrates that the coefficients $\{A_k\}$ decay at a faster rate (asymptotically by a factor of $k^{-2}$) than the coefficients $\{a_k\}$, and also indicates, as in Example 1, an interesting asymptotic behavior of $A_{N-q}$, for a fixed value of $q$, as $N \to \infty$.

Finally, we use the results of the previous two theorems to show the manner in which the approximations $\{S_{N,1}(x)\}$ converge to the original function $f(x)$.

THEOREM 3. *Let $f$ be an odd, $2\pi$-periodic, piece-wise smooth function and let its Fourier coefficients $\{b_k\}$ satisfy the conditions of Theorem 1. Then the sequence of approximations $\{S_{N,1}(x)\}$ converges to $(f(x^+)+f(x^-))/2$, as $N \to \infty$, for all $-\pi \leq x \leq \pi$. Moreover, the sequence $\{S_{N,1}(x)\}$ converges like a Fourier series whose terms are $O(1/N^{p+2})$. These results also hold if $f$ is an even, $2\pi$-periodic, piece-wise smooth function, if the Fourier coefficients $\{a_k\}$ of $f$ satisfy the conditions stated above with $b_k$ replaced by $a_k$.*

*If $f$ is an odd, analytic, $2\pi$-periodic function and its Fourier coefficients satisfy the conditions of Theorem 2, then the sequence of approximations $\{S_{N,1}(x)\}$ converges to $f(x)$, as $N \to \infty$, for all $-\pi \leq x \leq \pi$. Moreover, the sequence $\{S_{N,1}(x)\}$ converges like a Fourier series whose terms are $O(\theta^N/N^{p+1})$. These results also hold if $f$ is an even, analytic, $2\pi$-periodic function, if the Fourier coefficients $\{a_k\}$ of $f$ satisfy the conditions stated above with $b_k$ replaced by $a_k$.*

*Proof.* As in the proofs of the previous theorems, we shall outline only the proof for the case when $f$ is an odd, $2\pi$-periodic, piece-wise smooth function.

To begin, we note that, since $f$ is an odd function, we can restrict our attention to values of $x$ in the *open* interval $0 < x < \pi$ (since $S_{N,1}(0) = (f(0^+)+f(0^-))/2 = 0 = S_{N,1}(\pi) = (f(\pi^+)+f(\pi^-))/2$, and hence convergence is assured at $x = 0$ and $x = \pi$ in a trivial way). Consequently, we let $x$ be any fixed number in the interval $0 < x < \pi$ and let $\epsilon > 0$ be any fixed positive number. We then define

$$E_{N,1}(x) \equiv (f(x^+)+f(x^-))/2 - S_{N,1}(x) = E_N^{(1)}(x) + E_N^{(2)}(x), \tag{24}$$

$$E_N^{(1)}(x) = (f(x^+)+f(x^-))/2 - S_{N,0}(x),\ E_N^{(2)}(x) = S_{N,0}(x) - S_{N,1}(x). \tag{25}$$

By the hypotheses of the theorem concerning $f$, the Fourier partial sums $S_{N,0}(x)$ converge to $(f(x^+)+f(x^-))/2$, as $N \to \infty$. Hence, there exists a positive integer

$N_1 = N_1(x,\epsilon)$ such that

$$(26) \qquad \left|E_N^{(1)}(x)\right| = \left|(f(x^+)+f(x^-))/2 - S_{N,0}(x)\right| < \frac{\epsilon}{2}, \text{ for all } N > N_1.$$

Also, using the definitions of $S_{N,0}$ and $S_{N,1}$, we can write

$$E_N^{(2)}(x) = S_{N,0}(x) - S_{N,1}(x) = \sum_{n=1}^{N} b_n \sin(nx) - \frac{\sum_{n=1}^{N} B_n \sin(nx)}{1 + C_1\cos(x)}$$

$$(27) \qquad = \frac{C_1}{2}\frac{b_N \sin((N+1)x) - b_{N+1}\sin(Nx)}{1 + C_1\cos(x)}.$$

Using the assumed asymptotic form of the coefficients $\{b_n\}$, there exists a positive integer $N_2 = N_2(x,\epsilon) > (4\,|b|\,/(\epsilon(1-\cos(x)))^{1/p}$ such that $|b_n| < 2\,|b|\,/N^p$ and $C_1 > -1$ for all $N > N_2$. Then, using equation (27), we can write

$$\left|E_N^{(2)}(x)\right| = |S_{N,0}(x) - S_{N,1}(x)| \leq \frac{|C_1|}{2}\frac{|b_N| + |b_{N+1}|}{1-\cos(x)}$$

$$(28) \qquad < \frac{4\,|b|}{2(1-\cos(x))N_2^p} < \frac{\epsilon}{2}, \text{ for all } N > N_2.$$

Using the bounds (26) and (28), from equation (25) we find

$$(29) \qquad \left|(f(x^+)+f(x^-))/2 - S_{N,1}(x)\right| \leq \left|E_N^{(1)}(x)\right| + \left|E_N^{(2)}(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $N > N_3 = Max(N_1, N_2)$. Thus, the sequence of approximations $\{S_{N,1}\}$ converges to $(f(x^+)+f(x^-))/2$ as $N \to \infty$.

To demonstrate the manner in which these approximations converge, we write $S_{N,1}$ as

$$(30) \quad S_{N,1} = \sum_{k=1}^{N} r_k, \text{ where } r_1 = S_{1,1} \text{ and } r_k = S_{k,1}(x) - S_{k-1,1}(x), \quad 2 \leq k \leq N,$$

and then examine the quantities $r_k$ for large values of $k$. Using the definition (8), we can write

$$(31) \qquad r_k(x) = \frac{\sum_{n=1}^{k} B_n^{(k)} \sin(nx)}{1 + C_1^{(k)}\cos(x)} - \frac{\sum_{n=1}^{k-1} B_n^{(k-1)} \sin(nx)}{1 + C_1^{(k-1)}\cos(x)},$$

where we have placed a superscript on the coefficients $B_n$ and $C_1$, to remind us of the dependence of these quantities on the parameter $k$. Combining the terms on the right side of equation (31) and using various trigonometric identities, we can write

$$(32) \qquad r_k(x) = \frac{Num}{Den},$$

where

$$Num = \frac{\sin((k-1)x)}{2} C_1^{(k)} \left\{ b_k + \frac{C_1^{(k-1)}}{2}(b_{k-1} + b_{k+1}) \right\}$$

$$+ \frac{\sin(kx)}{2} \left\{ b_k + \frac{C_1^{(k-1)}}{2} b_{k-1} + \frac{C_1^{(k)}}{2} b_{k+1}) \right\}$$

$$+ \frac{\sin((k+1)x)}{2} C_1^{(k-1)} \left\{ b_k + \frac{C_1^{(k)}}{2}(b_{k-1} + b_{k+1}) \right\},$$

and

$$Den = (1 + C_1^{(k)} \cos(x))(1 + C_1^{(k-1)} \cos(x)). \tag{33}$$

Using the asymptotic form of the coefficients $B_n^{(k)}$ and $C_1^{(k)}$ from Theorem 1, we find

$$Num = -\frac{bp}{2} \frac{1}{k^{p+2}}(1 - \cos(x)) \sin(kx) + O(1/k^{p+3}),$$

$$Den = (1 - \cos(x))^2 + \frac{p^2}{k^2}(1 - \cos(x)) \cos(x) + O(1/k^4), \text{ as } k \to \infty. \tag{34}$$

Using the estimates (34) in (32), we find

$$r_k = -\frac{bp}{2} \frac{1}{1 - \cos(x)} \frac{\sin(kx)}{k^{p+2}} + O(1/k^{p+3}), \text{ as } k \to \infty. \tag{35}$$

This completes the proof of our theorem.

To illustrate some of these results, in Figure 5 we have plotted $\log|E_{N,1}|/\log|N|$ for Example 1, as a function of $1/\log|N|$, for $3 \leq N \leq 50$, with $x = \pi/2$. We note that, if $E_{N,1} \sim aN^{-p}$, as $N \to \infty$, then $\log|E_{N,1}|/\log|N| \sim -p + \log|a|/\log|N|$, as $N \to \infty$. Thus, the plot should be approximately linear for small values of $1/\log|N|$, with an intercept of $-p$. The value of $p = 3$, as predicted by Theorem 3, is clearly suggested by the figure. In this figure we have also plotted the corresponding error for the Fourier partial sums, corresponding to $M = 0$. For $M = 0$, the figure suggests a value of $p = 1$. Thus, the improvement in the rate of convergence suggested by the figure is consistent with the increase predicted by Theorem 3, i.e., an increase in $p$ of 2.

**5. Case $M \geq 2$.** We now generalize the results of the previous sections. In particular, when $f$ is an odd function of $x$, we consider the family of approximations

$$S_{N,M}(x) = \frac{\sum_{n=1}^{N} B_n \sin(nx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)}, \tag{36}$$

where $M$ is any positive integer. The constants $\{B_n, C_m\}$ are determined by the conditions (6), which for this case become

$$\sum_{j=1}^{N} d_{k,j} B_j = b_k, \quad k = 1, ..., N + M, \tag{37}$$

where

$$d_{k,j} = (1/\pi) \int_{-\pi}^{\pi} \frac{\sin(jx)\sin(kx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)} dx = d_{j,k} = I_{|k-j|,M} - I_{k+j,M}, \tag{38}$$

$$I_{n,M} = (1/\pi) \int_{0}^{\pi} \frac{\cos(nx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)} dx, \; n \geq 0. \tag{39}$$

To evaluate $I_{n,M}$, we denote the denominator of the integrand by $P$ and express $\cos(mx)$ as a polynomial of degree $m$ in $\cos(x)$. Then $P$ becomes a polynomial of degree $M$ in $\cos(x)$ and, assuming $C_M \neq 0$, we can write

$$P = 1 + \sum_{m=1}^{M} C_m \cos(mx) = 2^{M-1} C_M \prod_{i=1}^{M} (\cos(x) + z_i), \tag{40}$$

and hence

$$\frac{1}{1 + \sum_{m=1}^{M} C_m \cos(mx)} = \frac{1}{2^{M-1} C_M} \sum_{i=1}^{M} \frac{\alpha_i}{\cos(x) + z_i}, \; \alpha_i = \prod_{k=1, k\neq i}^{M} (z_k - z_i)^{-1}. \tag{41}$$

Here $\{-z_i\}$ are the roots of the polynomial $P$, when $P$ is regarded as a polynomial in the variable $z = \cos(x)$, and we have assumed that the $\{z_i\}$ are all distinct. Substituting (41) into (39) and using (11), we can write

$$I_{n,M} = -\frac{2}{C_M} \sum_{i=1}^{M} \beta_i \frac{\rho_i^{n+1}}{1 - \rho_i^2}, \tag{42}$$

$$\beta_i = \prod_{k=1, k\neq i}^{M} \frac{\rho_i \rho_k}{(\rho_k - \rho_i)(1 - \rho_i \rho_k)}, \; \rho_i = \sqrt{z_i^2 - 1} - z_i.$$

Using this result in (38) we find

$$d_{k,j} = -\frac{2}{C_M} \sum_{i=1}^{M} \beta_i \rho_i^{k-j+1} \sum_{l=0}^{j-1} \rho_i^{2l}, \; k \geq j. \tag{43}$$

To solve equations (37), we first define the quantities

$$s_0 = 1, \; s_m = (-1)^m \sum_{1 \leq i_1 < i_2 < ... < i_m \leq M} \rho_{i_1} \cdots \rho_{i_m}, \; 1 \leq m \leq M. \tag{44}$$

(We note that $s_m$ is just the coefficient of $\rho^{M-m}$ in the polynomial $\prod_{i=1}^{M}(\rho-\rho_i)$.) We then denote the $k^{th}$ equation in (37) by $eq(k)$ and form the linear combinations

$$lc(k) \equiv \sum_{m=0}^{M} s_m eq(k-m),\ k = N+M, N+M-1, ..., 1. \tag{45}$$

Here we define $eq(0)$ to be the trivial equation $0 = 0$, and $eq(n) = -eq(-n)$, for $n < 0$. For $k = N+M, N+M-1, ..., N+1$, we find that the left side of each $lc(k)$ vanishes. This leads to the relations

$$\sum_{m=1}^{M} s_m b_{N+M+1-p-m} = -b_{N+M+1-p},\ p = 1, ..., M, \tag{46}$$

which are a system of $M$ linear equations for the $M$ quantities $s_1, ..., s_M$. We shall assume that the determinant of the matrix of coefficients in (46) is nonzero so that these equations can be solved uniquely for $s_1, ..., s_M$. Once these quantities have been determined, we can use equations (44) and (42) to express each $s_m$ in terms of the quantities $z_i$, and then use equation (41) to express each of the coefficients $C_m$ in terms of the $z_i$. We write out these equations for a few small values of M and then use induction to show that the coefficients $\{C_m\}$ can be expressed explicitly in terms of the known quantities $\{s_i\}$ as

$$C_m = \frac{2}{D}\sum_{k=0}^{M-m} s_k s_{m+k},\ 1 \le m \le M,\ \ D = \sum_{j=0}^{M} s_j^2. \tag{47}$$

To determine the coefficients $\{B_k\}$, for $k = N, N-1, ..., 1$, we solve $lc(k)$ for $B_k$ and then use induction to show that the coefficients $\{B_k\}$ are given explicitly by

$$B_k = b_k + \frac{1}{2}\sum_{m=1}^{M} C_m(b_{k+m} + b_{k-m}),\ 1 \le k \le N. \tag{48}$$

(Equations (48) also follow from equating (36) to $\sum_{j=1}^{N+M} b_j \sin(jx)$, multiplying by $1 + \sum_{m=1}^{M} C_m \cos(mx)$, and equating coefficients of $\sin(nx)$ on each side of the resulting expression, for $1 \le n \le N$.) Thus, the coefficients $\{C_m\}$ and $\{B_n\}$ are completely determined by equations (48), (47), and (46).

**Example 3:** As an application of these results, we consider the odd, $2\pi$-periodic function $f$ defined by $f(x) = \epsilon x(\pi - x)/(\epsilon^2 + (x-\delta)^2(x-\pi+\delta)^2)^{1/2}$ for $0 \le x \le \pi$, where $\epsilon$ and $\delta$ are real, positive parameters. For this function, we find $b_k = 0$ when $k$ is *even* and hence $S_{N,0} = \sum_{k=1}^{N} b_{2k-1}\sin((2k-1)x)$. Then, for $M = 2$, we define

$$S_{N,2} = \frac{\sum_{k=1}^{N} B_{2k-1}\sin((2k-1)x)}{1 + C_1\cos(2x) + C_2\cos(4x)},$$

where the coefficients $\{B_{2k-1}\}$ and $\{C_m\}$ are defined by equations (46)-(48), with $b_j$ replaced by $b_{2j-1}$. Explicit values for $\{C_1, C_2, B_1, ..., B_{2N-1}\}$ are shown for a few small values of $N$ in Table III when $\epsilon = 0.1$ and $\delta = 0.5$.

*Table III* $(M = 2)$

| $N$ | $b_{2N-1}$ | $C_1$ | $C_2$ | $B_1$ | $B_3$ | $B_5$ | $B_7$ | $B_9$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.41814 | −.99477 | 0.29067 | 0.43574 | | | | |
| 2 | 0.31809 | −1.3981 | 0.46568 | 0.43655 | −.16521 | | | |
| 3 | 0.09684 | −1.3559 | 0.54610 | 0.42554 | −.17564 | 0.02412 | | |
| 4 | −.11150 | −1.3462 | 0.58487 | 0.42077 | −.18343 | 0.02989 | 0.00716 | |
| 5 | −.17153 | −1.3539 | 0.59107 | 0.42047 | −.18703 | 0.02987 | 0.00817 | 0.00234 |

In Figure 6, we have plotted $S_{6,0}$, $S_{5,1}$ (using equation (15)), and $S_{4,2}$, as well as $f$, for $0 \leq x \leq \pi$. From the figure it is clear that $S_{4,2}$ is a better approximation to $f$ than either $S_{5,1}$ or $S_{6,0}$. In fact, for this example, $S_{5,1}$ and $S_{6,0}$ lie close to each other and are each a "poor" approximation to $f$. We shall comment further on this example in section 8.

If $f$ is an *even* function of $x$, then results very similar to those above hold with each $b_k$ replaced by $a_k$. In particular, for approximations of the form

$$S_{N,M}(x) = \frac{A_0/2 + \sum_{n=1}^{N} A_n \cos(nx)}{1 + \sum_{m=1}^{M} C_m \cos(mx)}$$

we find

$$A_k = a_k + \frac{1}{2}\sum_{m=1}^{M} C_m(a_{k+m} + a_{k-m}),\ 0 \leq k \leq N,$$

$$C_m = \frac{2}{D}\sum_{k=0}^{M-m} s_k s_{m+k},\ 1 \leq m \leq M,\quad D = \sum_{j=0}^{M} s_j^2,$$

where the $\{s_m\}$ are now determined from the equations

$$\sum_{m-1}^{M} s_m a_{N+M+1-p-m} = -a_{N+M+1-p},\ p = 1, \ldots, M. \tag{49}$$

**6. Analysis of the case $M \geq 2$.** In this section, we prove some results analogous to those proved in section 4 for the case $M = 1$.

THEOREM 4. *Let $f$ be an odd, $2\pi$-periodic, piece-wise smooth function and let its Fourier coefficients $\{b_k\}$ satisfy the condition that $b_k = O(1/k^p)$, as $k \to \infty$, where $p$ is a real, positive number. Then, for any positive integer $M \geq 1$, $C_m = c_{m,M} + O(1/N^2)$, for $1 \leq m \leq M$, where $c_{m,M}$ are certain constants independent of $N$, and $B_k = O(1/k^{p+2M})$, as $k, N \to \infty$. More precisely, let $b$, $\beta$, and $\gamma$ be constants (independent of $k$). Then if $b_k = (b/k^p)(1 + \beta/k + \gamma/k^2 + O(1/k^3))$, as $k \to \infty$,*

$$C_m = \frac{(-1)^m\binom{2M}{M-m}}{\binom{2M-1}{M-1}}\left\{1 - \frac{m^2(p+M-1)(p+2(M-1))}{2(2M-1)N^2}\right\} + O(1/N^3),$$

$$B_k = \frac{bp(p+1)\cdots(p+M-1)(-1)^M}{2\binom{2M-1}{M-1}k^{p+2M}}\left(\sum_{i=0}^{M}(-1)^i\binom{M}{i}\prod_{j=1}^{M}(p+2M-j-i)(k/N)^{2i}\right)$$

$$+O(1/k^{p+2M+1}),$$

*as $k, N \to \infty$. Here $\binom{n}{j} \equiv \frac{n!}{j!(n-j)!}$ is the usual binomial coefficient. If $f$ is an even, $2\pi$-periodic, piece-wise smooth function, then these results hold with $b_k$ replaced by $a_k$ and $B_k$ replaced by $A_k$.*

*Proof:* As in the proof of Theorems 1-3, we shall only outline the proof for the case when $f$ is an odd, $2\pi$-periodic, piece-wise smooth function.

To see that the coefficients $\{C_m\}$ have the stated form as $N \to \infty$, we first use the assumed form of the asymptotic form of the coefficients $\{b_k\}$ to write

$$\begin{aligned} b_{N+j} &= (b/(N+j)^p)\,(1+\beta/(N+j)+\gamma/(N+j)^2+O(1/N^3)) \\ &= b\epsilon^p\,(1+(\beta-jp)\epsilon+(\gamma-j\beta(p+1)+j^2p(p+1)/2)\epsilon^2+O(\epsilon^3))\,, \end{aligned} \tag{50}$$

where we have defined $\epsilon = 1/N$. Next, we examine the behavior of the quantities $\{s_m\}$ as $N \to \infty$. Using the *linear* equations (46), we solve for $\{s_m\}$ explicitly in terms of the coefficients $\{b_k\}$ for a few small values of M (see the Appendix). We then insert the expansions (50) into these explicit expressions for the $\{s_m\}$ and expand the resulting expressions for small values of $\epsilon$. In this way we obtain an expansion for each $s_m$ which is valid as $\epsilon \to 0$, i.e. as $N \to \infty$. We then use induction to show that these expressions, for any positive integer $M$, can be written as

$$s_m = (-1)^m\left\{\binom{M}{m} - M\binom{M-1}{m-1}\frac{p+M-1}{N} + \right.$$

$$\left. M\binom{M-1}{m-1}\frac{(p+M-1)(-2\beta+(M+1+m(M-2))p+mp^2)}{2pN^2}\right\} + O(1/N^3), \tag{51}$$

for $m = 1, 2, ..., M$, as $N \to \infty$. Inserting equations (51) into (47) and expanding the resulting expressions for large values of $N$, we find that each coefficient $C_m$ has the asymptotic form indicated in the statement of the theorem.

In a similar manner, using the expressions (50) with $N$ replaced by $k$, along with the asymptotic form of the $\{C_m\}$, in equations (48), we find that the coefficients $\{B_k\}$ have the asymptotic form indicated in the statement of the theorem, as both $k, N \to \infty$. This completes the proof of our theorem.

To illustrate these results, in Figure 7 we have plotted $\log|B_k|/\log k$ vs. $1/\log k$ for Example 1 with $M = 1, 2, 3$, and 4, for $N = 20$. The figure clearly illustrates how much more rapidly the coefficients $B_k$ decay to zero (especially with increasing values of $M$) than the coefficients $b_k$, which decay only like $1/k$.

The results of Theorems 2 and 3 generalize to the case when $M \geq 2$ in a straightforward manner. Using the results of Theorem 4, the proofs of the following two theorems follow closely the proofs of Theorems 2 and 3, respectively, and, hence, we state them without proof.

THEOREM 5. *Let $f$ be an odd, analytic, $2\pi$-periodic function and let its Fourier coefficients $\{b_k\}$ satisfy the condition that $b_k = (b\theta^k/k^p)(1+\beta/k+\gamma/k^2+O(1/k^3))$, as $k \to \infty$, where $b$, $\beta$, and $\gamma$ are constants (independent of $k$), $p$ is a real positive number, and $0 < |\theta| < 1$. Then*

$$C_m = 2(-1)^m\theta^m \frac{\sum_{k=0}^{M-m} \theta^{2k}\binom{M}{k}\binom{M}{m+k}}{\sum_{k=0}^{M}\binom{M}{k}^2\theta^{2k}} + O(1/N),\ m = 1, ..., M,$$

*and $B_k = O(|\theta|^k(1-\theta^2)(1-k/N)/k^{p+M}) + O(|\theta|^k/N^{p+M+1})$, as $k, N \to \infty$. If $f$ is an even, analytic, $2\pi$-periodic function, then these results hold with $b_k$ replaced by $a_k$ and $B_k$ replaced by $A_k$.*

THEOREM 6. *Let $f$ be an odd, $2\pi$-periodic, piece-wise smooth function and let its Fourier coefficients $\{b_k\}$ satisfy the conditions of Theorem 1. Then the sequence of approximations $\{S_{N,M}(x)\}$ converges to $(f(x^+)+f(x^-))/2$, as $N \to \infty$, for all $-\pi \le x \le \pi$. Moreover, the sequence $\{S_{N,M}(x)\}$ converges like a Fourier series whose terms are $O(1/N^{p+2M})$. These results also hold if $f$ is an even, $2\pi$-periodic, piece-wise smooth function, if the Fourier coefficients $\{a_k\}$ of $f$ satisfy the conditions stated above with $b_k$ replaced by $a_k$.*

*If $f$ is an odd, analytic, $2\pi$-periodic function and its Fourier coefficients satisfy the conditions of Theorem 2, then the sequence of approximations $\{S_{N,M}(x)\}$ converges to $f(x)$, as $N \to \infty$, for all $-\pi \le x \le \pi$. Moreover, the sequence $\{S_{N,M}(x)\}$ converges like a Fourier series whose terms are $O(\theta^N/N^{p+M})$. These results also hold if $f$ is an even, analytic, $2\pi$-periodic function, if the Fourier coefficients $\{a_k\}$ of $f$ satisfy the conditions stated above with $b_k$ replaced by $a_k$.*

To illustrate Theorem 6, in Figure 5 we have also plotted $\log|E_{N,M}|/\log|N|$ for Example 1, with $M = 2$ and 3, as a function of $1/\log|N|$, for $3 \le N \le 50$, with $x = \pi/2$. Here $E_{N,M}(x) \equiv (f(x^+)+f(x^-))/2 - S_{N,M}(x)$. Using the ideas discussed after Theorem 3, the results presented in Figure 5 are consistent with the convergence rates of $p = 5$ (for $M = 2$) and $p = 7$ ($M = 3$), as predicted by Theorem 6 for the function considered in Example 1.

**7. Application.** Applications of the results presented above to several classes of problems usually solved by Fourier series alone will be presented and discussed elsewhere. In this section, we present an application to a simple heat conduction problem to illustrate some of the potential of the method to improve the accuracy of approximate solutions obtained by partial sums of Fourier series.

We consider the problem of determining the transient behavior of the temperature $u(x,t)$ which satisfies the conditions

$$u_t = u_{xx},\ \text{for } 0 < x < \pi \text{ and } t > 0, \tag{52}$$

with

$$u(0,t) = 0 = u(\pi,t),\ \text{for all } t \ge 0, \text{ and } u(x,0) = x/2,\ \text{for } 0 < x < \pi.$$

(In equation (52), the subscripts denote partial differentiation.) Using separation of variables, the solution is found in a straightforward manner to be

$$(53)\qquad u(x,t) = \lim_{N\to\infty} u^{(N,0)}(x,t), \text{ where } u^{(N,0)}(x,t) \equiv \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} e^{-n^2 t}\sin(nx).$$

This solution obviously has a discontinuity at $x=\pi$ when $t=0$.

Using the formulas of section 2, we define a new class of approximate solutions $\{u^{(N,1)}\}$ by

$$(54)\qquad u^{(N,1)}(x,t) \equiv \frac{\sum_{n=1}^{N} B_n(t)\sin(nx)}{1+C_1(t)\cos(x)},$$

where the coefficients $B_n(t)$ and $C_1(t)$ are defined by equations (13) and (14) with $b_k$ replaced by $(-1)^{k+1}e^{-k^2 t}/k$. Thus, we find

$$C_1(t) = \frac{2N(N+1)e^{-(2N+1)t}}{(N+1)^2 + N^2 e^{-(4N+2)t}}, \; B_1 = e^{-t} - \frac{C_1}{4}e^{-4t},$$

$$(55)\qquad B_k(t) = (-1)^{k+1}\left\{\frac{e^{-k^2 t}}{k} - \frac{C_1}{2}\left(\frac{e^{-(k-1)^2 t}}{k-1} + \frac{e^{-(k+1)^2 t}}{k+1}\right)\right\}, \; 2 \le k \le N.$$

In Figure 8 we have plotted $u^{(4,0)}$ (dashed line) and $u^{(3,1)}$ (dotted line), along with the exact solution $u^{(\infty,0)}$ (solid line) for $t=0.01$ and $t=0.05$. As the figure illustrates, $u^{(3,1)}$ is consistently a better approximation than $u^{(4,0)}$ to the exact solution, and the quality of this approximation improves as $t$ becomes larger. The corresponding comparison at $t=0$ is the same as shown in Figure 1 for $0 \le x \le \pi$.

**8. Conclusions and Discussion.** We now make a few observations about the family of Fourier-Padé approximations discussed above, and also comment on several of their properties that need further investigation.

First, we note that, using only the "information" contained in the first few Fourier coefficients of a function $f$, the functions $S_{N,M}$ appear to provide new approximations to $f$ which are consistently "better" than the Fourier partial sums $S_{N,0}$ on which they are based. The sense in which these approximations are "better" can be interpreted in at least three different ways. First, Theorems 3 and 6 show that these approximations converge point-wise to $(f(x^+)+f(x^-))/2$ at a faster rate than the original sequence $\{S_{N,0}\}$, as the parameter $N$ increases. Secondly, although the Gibbs phenomena is still present in the family $\{S_{N,M}\}$, the amplitude of the oscillations near a point of discontinuity of $f$ appears to be mitigated, when compared with the oscillations present in $S_{N,0}$. Finally, oscillations in $S_{N,0}$, which lie *outside* a neighborhood of a point of discontinuity of $f$ (or outside a region of large curvature of $f$), are noticeably damped in $S_{N,M}$, for $M \ge 1$, especially as $N$ increases. These last two interpretations are illustrated in Examples 1 and 2.

The fact that explicit expressions were derived for the coefficients $A_n$, $B_n$ and $C_m$, which appear in the definition of $S_{N,M}$, should be emphasized. (From a practical point of view, it should be noted that it might be possible to develop an efficient, recursive algorithm, similar to algorithms presented by Baker [1], to compute these coefficients, since the equations which ultimately determine these quantities are recursive in nature. See, especially, equations (46)-(48), as well as the Appendix.) These explicit expressions not only facilitate the proofs of the stated theorems, but also illustrate some of the potential use of these approximations to construct approximate solutions to problems involving differential equations, especially partial differential equations. In particular, the Fourier coefficients $\{a_n, b_n\}$ may be functions of one or more "other" variables, such as $t$, as in the example of section 7. The application of the basic ideas presented here to problems involving several different classes of partial differential equations is currently under investigation and will be reported elsewhere.

As far as the mathematical properties of the family $\{S_{N,M}\}$ are concerned, only the most elementary properties have been investigated here. Many other questions, which are of both theoretical and practical importance, should be addressed. For example, for a fixed value of $N+M$, which approximation $S_{N,M}$ is "best"? In Figure 9 we have plotted $S_{6-q,q}$, for $q = 0, 1, ..., 5$ for Example 1. In this case, the figure seems to suggest that perhaps $S_{3,3}$ is "best" in some appropriate integral norm, although the improvement of the quality of the approximation of $S_{3,3}$, say, over $S_{5,1}$ is not dramatic. For this example, the function has essentially only *one* point of discontinuity (at $x = \pi$) and hence $S_{N,1}$ appears to provide a "good" approximation. By contrast, the function considered in Example 3 has essentially *two* points of large curvature (near $x = \delta$ and $x = \pi - \delta$). For this example, $S_{N,1}$ provided a poor approximation to $f$, while $S_{N,2}$ yields a much improved approximation. More generally, the "best" $S_{N,M}$ will undoubtedly depend on both the general "shape" and smoothness properties of the function $f$, as well as on the particular norm used.

From a practical point of view, a related issue concerns how best to represent a function with a discontinuity interior to $[-\pi, \pi]$. For example, consider the odd, piecewise continuous, $2\pi-$periodic function $f$ defined by $f(x) = \pi x/2$, for $-\pi/2 \leq x \leq \pi/2$, and $f(x) = 0$, otherwise, in $[-\pi, \pi]$. The Fourier coefficients $\{b_n\}$ of $f$ are given by $b_n = (-1)^{(n-1)/2}/n^2$, if $n$ is odd, and $b_n = (\pi/(2n))(-1)^{(n+2)/2}$, if $n$ is even. The Fourier partial sum $S_{10,0}$ associated with $f$ is plotted in Figure 10. To approximate $f$ by one of the functions $S_{N,M}$, one possibility is to use the formulas above "blindly" and construct an approximation, say $S_{9,1}$, in a straightforward manner. This approximation is also plotted in Figure 10. As the figure clearly illustrates, *neither* $S_{10,1}$ *nor* $S_{9,1}$ is a "good" approximation to $f$. However, we now observe that we can decompose $S_{2N+2,0}$ into the sum of two other partial sums, i.e.

$$S_{2N+2,0} = S^{(1)}_{N+1,0} + S^{(2)}_{N+1,0}, \quad \text{where}$$

$$S^{(1)}_{N+1,0} = \sin(x) - \frac{1}{9}\sin(3x) + \frac{1}{25}\sin(5x) - ...,$$

$$S^{(2)}_{N+1,0} = \frac{\pi}{2}\left\{\frac{1}{2}\sin(2x) - \frac{1}{4}\sin(4x) + \frac{1}{6}\sin(6x) - ...\right\}.$$

We can now apply our formulas to each of these sums, separately, and construct two new approximations $S_{N,1}^{(1)}$ and $S_{N,1}^{(2)}$. We then use $S_{N,1}^{(1)} + S_{N,1}^{(2)}$ as a new approximation to $f$. The approximation $S_{4,1}^{(1)} + S_{4,1}^{(2)}$ is also shown in Figure 10 and is clearly an improvement over either $S_{10,1}$ or $S_{9,1}$ .

This example also serves to illustrate a possible advantage of the more general form of $S_{N,M}$ defined in equation (4). In particular, the only approximations that have been investigated here correspond to setting each $D_m = 0$ in equation (4). More generally, the inclusion of *both* sine *and* cosine terms in the denominator of $S_{N,M}$ allows the possibility of "shifting" the location of an approximate "pole" of $S_{N,M}$ from either $x = 0$ or $x = \pm\pi$ to an arbitrary point interior to the interval $(-\pi, \pi)$. For example, when $M = 1$, the denominator of $S_{N,1}$ can be expressed as $1 + C_1 \cos(x) + D_1 \sin(x) = 1 + \tilde{C}_1 \cos(x - \delta)$. Thus, if $\tilde{C}_1 \approx -1$, for example, then the denominator will be small when $x$ is near $\delta$ and, hence, $S_{N,1}$ could potentially better simulate a function which has a singularity (or a "near" singularity) at $x = \delta$.

We also note that, while Theorems 3 and 6 establish the convergence of the sequence of approximations $\{S_{N,M}\}$ for a fixed value of $M$ as $N \to \infty$, the Gibbs phenomena has not been eliminated, although its effects seem to be mitigated. Thus, the rate of convergence of this sequence in regions near a point of discontinuity of $f$ needs further study. For example, in the proof of Theorem 3, equation (35) establishes that, for any fixed value of $x$, with $-\pi < x < \pi$, the terms $r_k$ are $O(1/k^{p+2})$ as $k \to \infty$. However, the "practical" rate of convergence of the series is mitigated somewhat, especially near $x = 0$, by the presence of the factor $(1 - \cos(x))$ in the denominator of the expression for $r_k$. This factor, in combination with the terms $\sin(kx)$ in the numerator, also foreshadows the Gibbs phenomena that does remain. However, the fact that an explicit form for these terms is available should assist in the investigation of ways to improve convergence in these regions. This observation, as well as the other observations, questions, and several related issues raised here, are the subject of some current investigations and the results of these investigations will be reported elsewhere.

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**A. Appendix.** In this appendix we present some explicit expressions for the quantities $\{s_m\}$ defined in section 5 in terms of the coefficients $\{b_k\}$ by equations (46) and (49) when $f$ is an odd function of $x$. Analogous expressions hold when $f$ is an even function of $x$, with each $b_k$ replaced by $a_k$. For $M = 1, 2$, and 3, these quantities are given by

$$M = 1: \qquad s_1 = -b_{N+1}/b_N;$$

$$M = 2: \qquad s_1 = \frac{b_{N+1}b_N - b_{N+2}b_{N-1}}{b_{N+1}b_{N-1} - b_N^2}, \; s_2 = \frac{b_N b_{N+2} - b_{N+1}^2}{b_{N+1}b_{N-1} - b_N^2};$$

$$M = 3:$$

$$s_1 = \frac{b_{N+1}(b_N^2 - b_{N-1}b_{N+1}) + b_{N+2}(b_{N+1}b_{N-2} - b_{N-1}b_N) + b_{N+3}(b_{N-1}^2 - b_N b_{N-2})}{\det},$$

$$s_2 = \frac{b_{N+1}(b_{N+2}b_{N-1} - b_N b_{N+1}) + b_{N+2}(b_N^2 - b_{N+2}b_{N-2}) + b_{N+3}(b_{N+1}b_{N-2} - b_N b_{N-1})}{\det},$$

$$s_3 = \frac{b_{N+1}(b_{N+1}^2 - b_N b_{N+2}) + b_{N+2}(b_{N+2}b_{N-1} - b_N b_{N+1}) + b_{N+3}(b_N^2 - b_{N+1}b_{N-1})}{\det},$$

$$\det = b_{N-2}(-b_{N+1}^2 + b_N b_{N+2}) + b_{N-1}(-b_{N-1}b_{N+2} + 2b_N b_{N+1}) + b_N(-b_N^2).$$

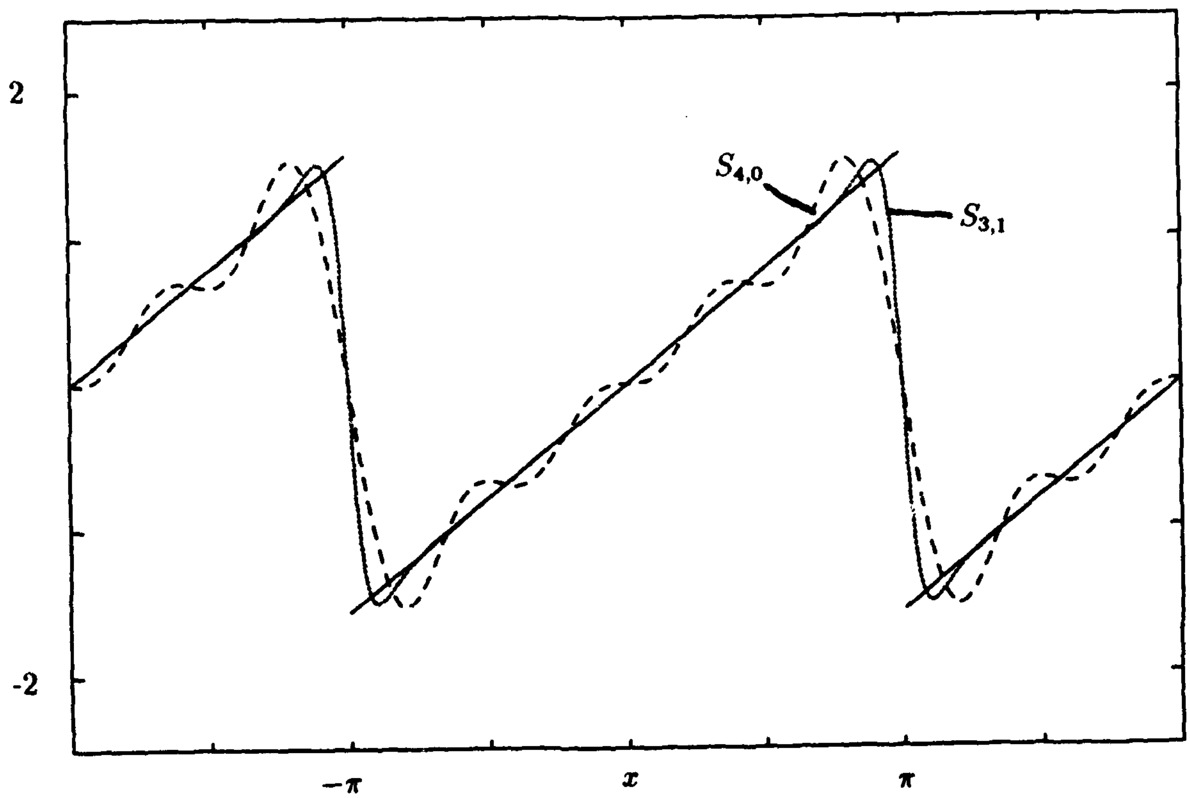


**Figure 1**: A plot of the function $f(x)$ (solid line), the approximations $S_{4,0}(x)$ (dashed line) and $S_{3,1}(x)$ (dotted line) for Example 1. Note that the oscillations present in $S_{4,0}$ away from the singularities of $f$ have been virtually eliminated in $S_{3,1}$.

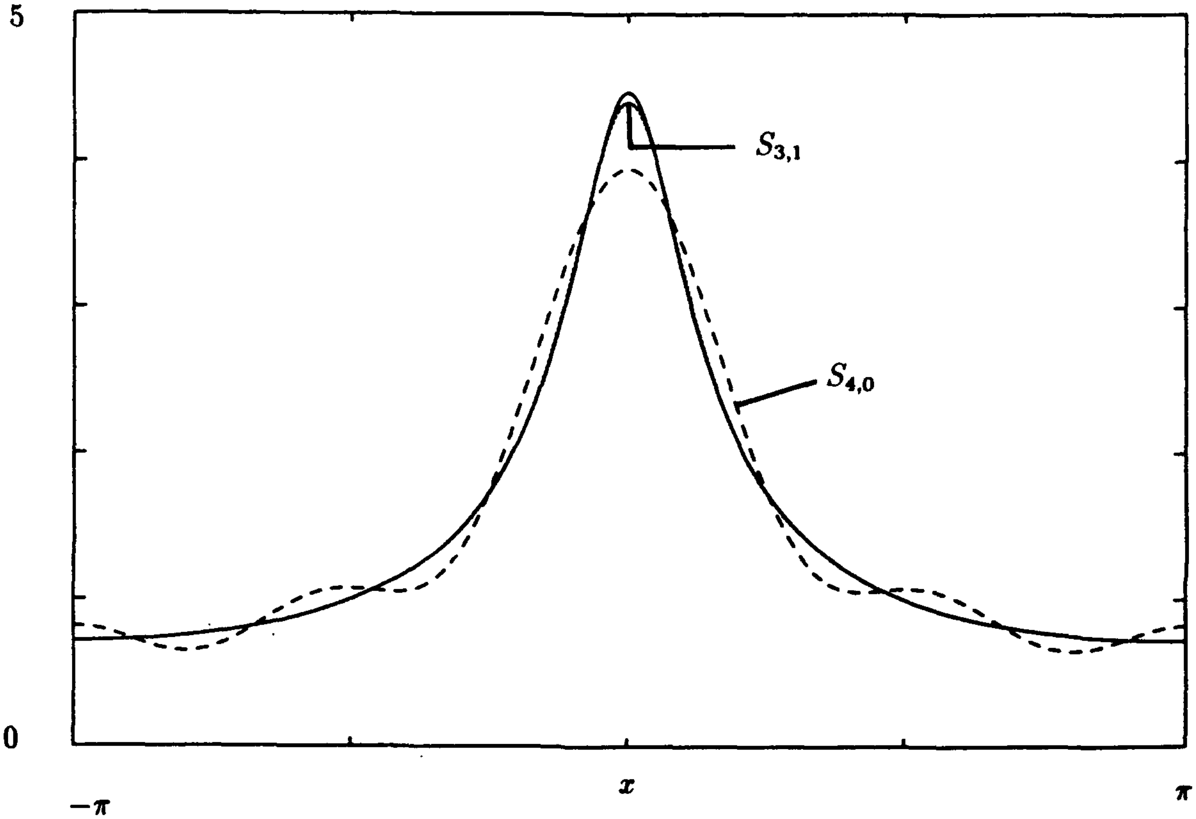


**Figure 2**: A plot of the function $f(x)$ (solid line), the approximations $S_{4,0}(x)$ (dashed line) and $S_{3,1}(x)$ (dotted line) for Example 2. Note that the oscillations present in $S_{4,0}$ away from the regions of large curvature of $f$ have essentially been eliminated in $S_{3,1}$.

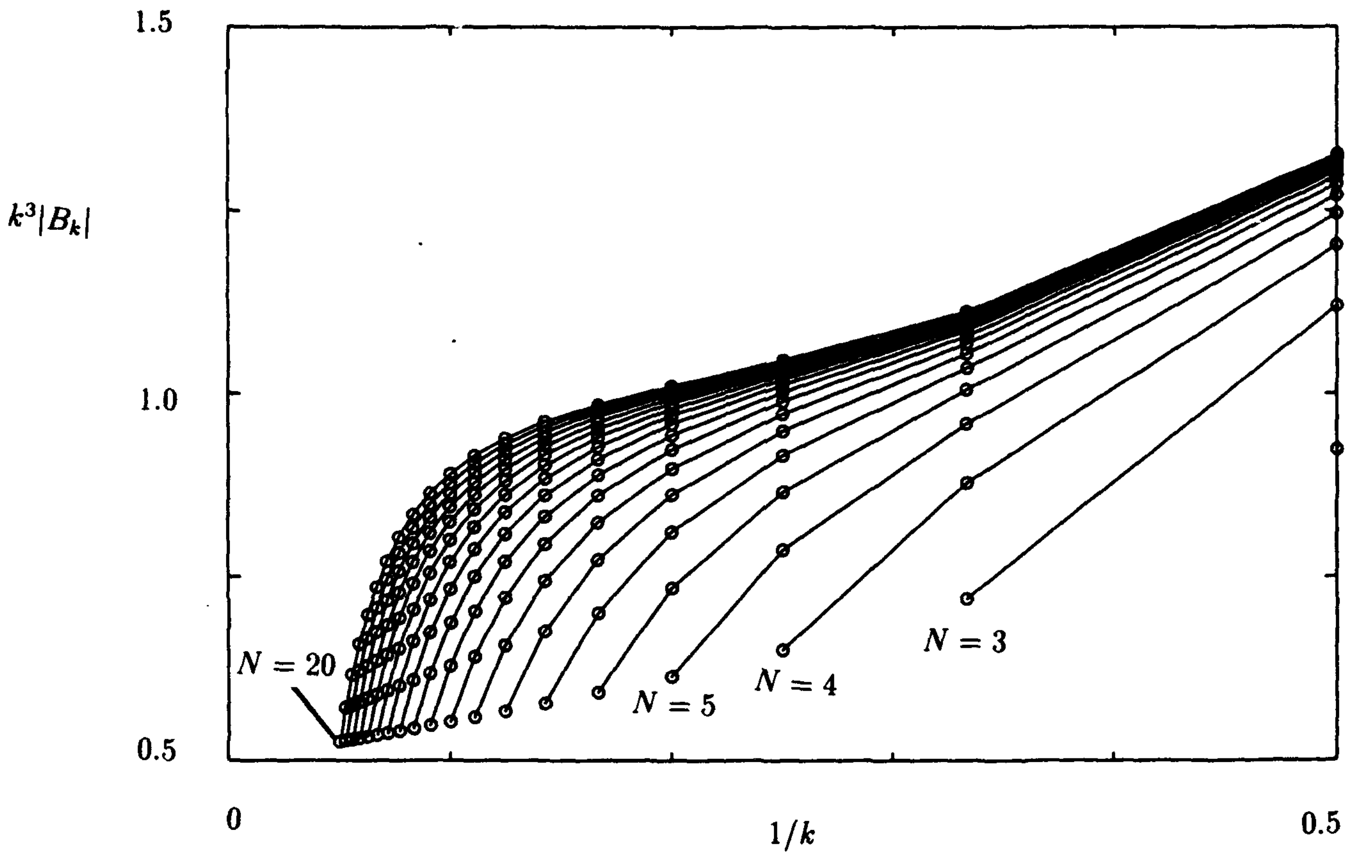


**Figure 3**: A plot of $k^3|B_k|$ (circles) for Example 1 as a function of $1/k$, for $2 \leq k \leq N$, with $3 \leq N \leq 20$. The $O(1/k^3)$ decay of the coefficients $\{B_k\}$ is clearly illustrated, as is an interesting asymptotic behavior of $B_{N-q}$, for a fixed value of $q$, as $N \to \infty$.

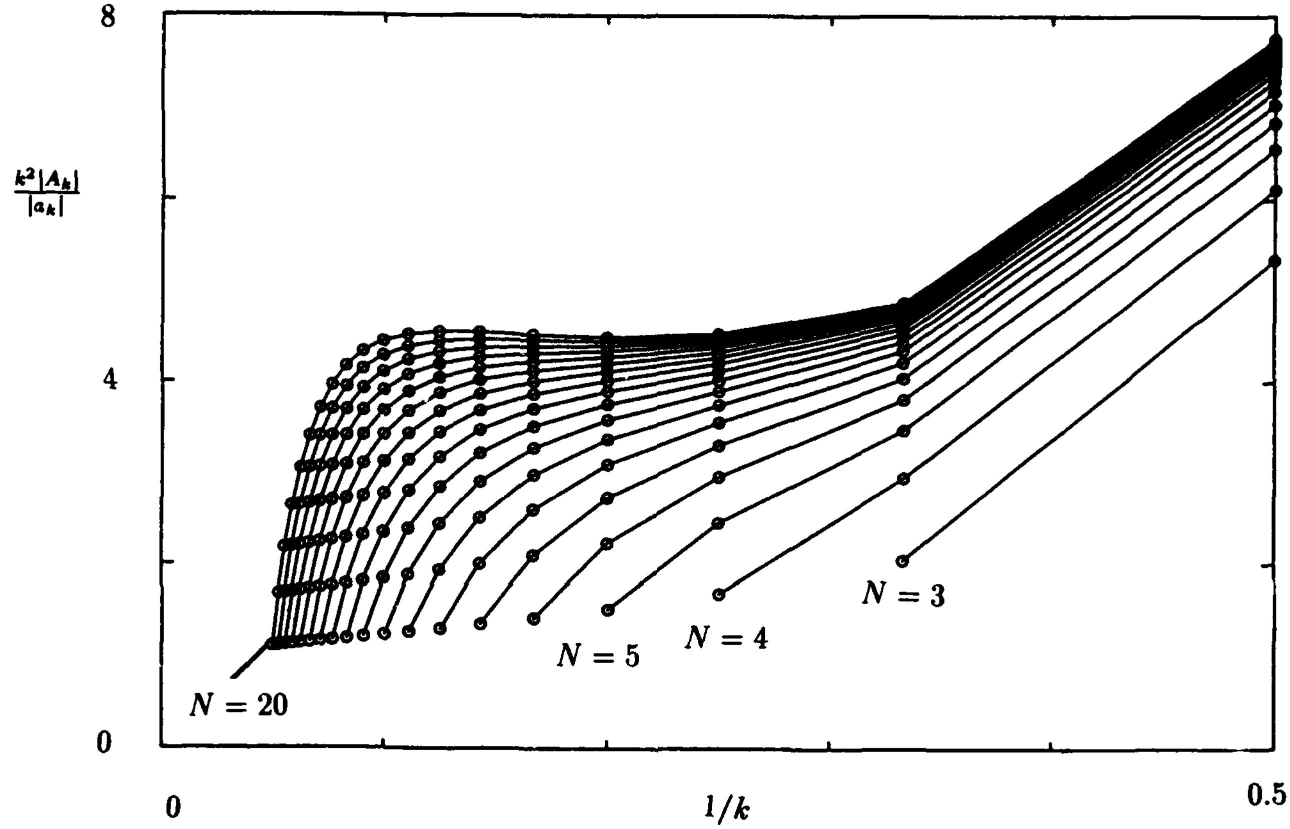


**Figure 4**: A plot of $k^2|A_k|/|a_k|$ (circles) for Example 2 as a function of $1/k$, for $2 \leq k \leq N$, with $3 \leq N \leq 20$. The figure illustrates that the coefficients $\{A_k\}$ decay faster than the coefficients $\{a_k\}$, asymptotically by a factor of $k^{-2}$, as predicted by Theorem 2. As in Example 1, an interesting asymptotic behavior of $A_{N-q}$, as $N \to \infty$ for a fixed value of $q$, is also predicted.

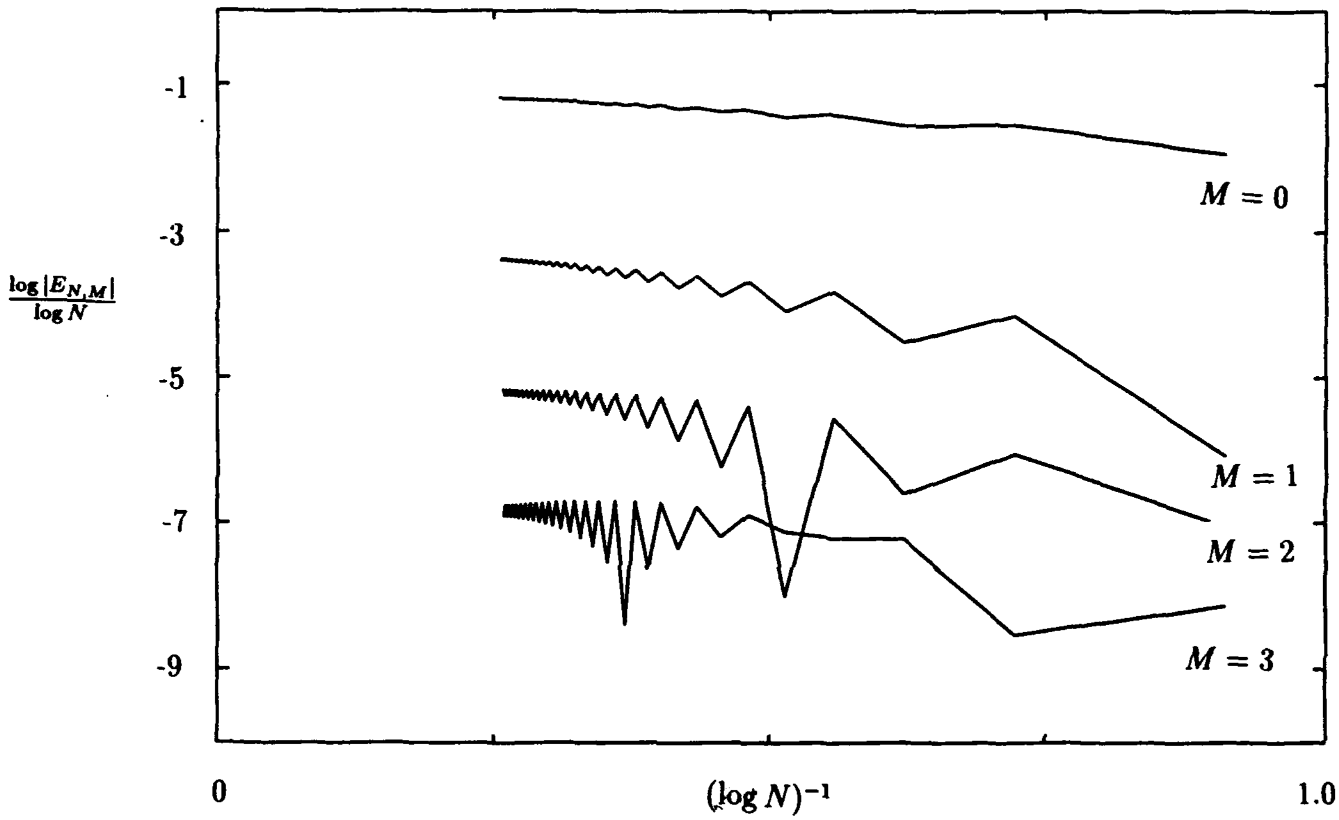


**Figure 5**: A plot of $\log |E_{N,M}| / \log |N|$ for Example 1 as a function of $1/\log |N|$ for $3 \leq N \leq 50$, at $x = \pi/2$, for $M = 0, 1, 2$, and 3. Here $E_{N,M}(x) \equiv (f(x^+) + f(x^-))/2 - S_{N,M}(x)$. We note that an intercept of $2M + 1$, as predicted by Theorems 3 and 6, is clearly consistent with these plots.

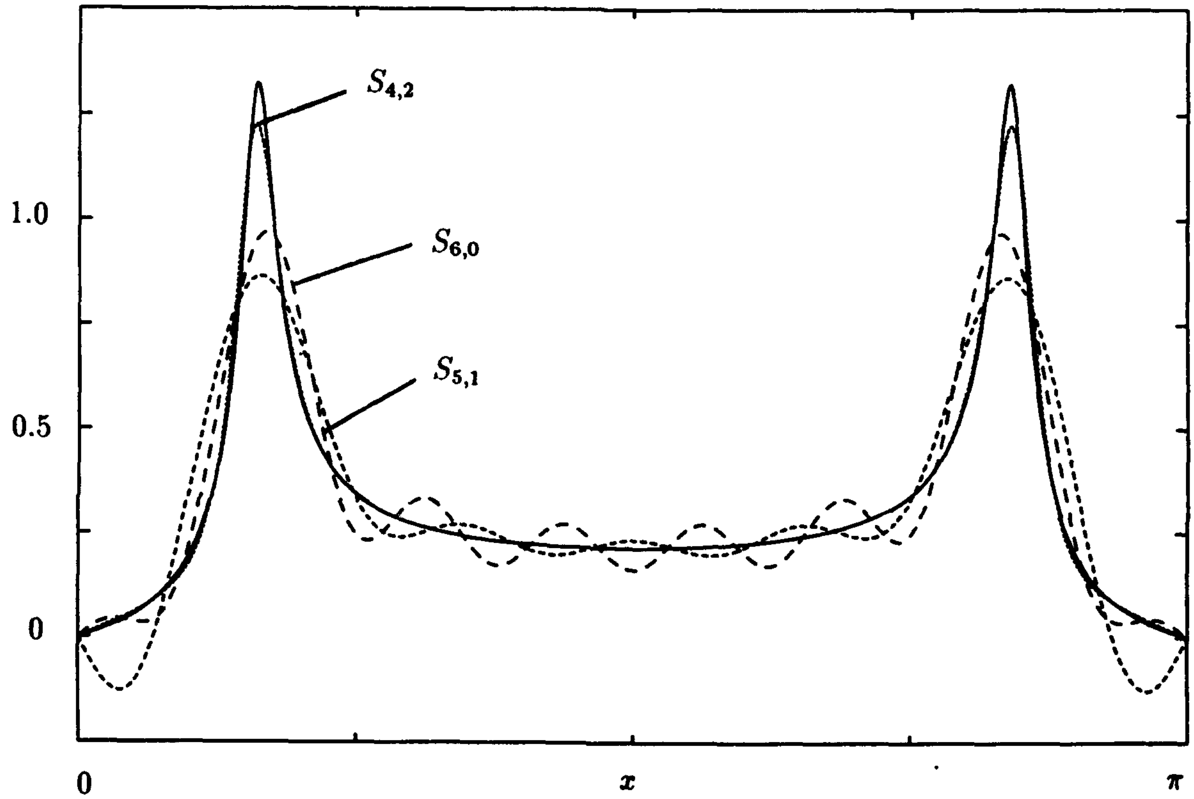


**Figure 6**: A plot of the function $f(x)$ (solid line), the approximations $S_{6,0}(x)$ (longer dashed line), $S_{5,1}(x)$ (shorter dashed line), and $S_{4,2}(x)$ (dotted line) for Example 3. Note that the oscillations present in $S_{6,0}$ away from the regions of large curvature of $f$ have essentially been eliminated in $S_{4,2}$, while $S_{5,1}$ is not much of an improvement over $S_{6,0}$.

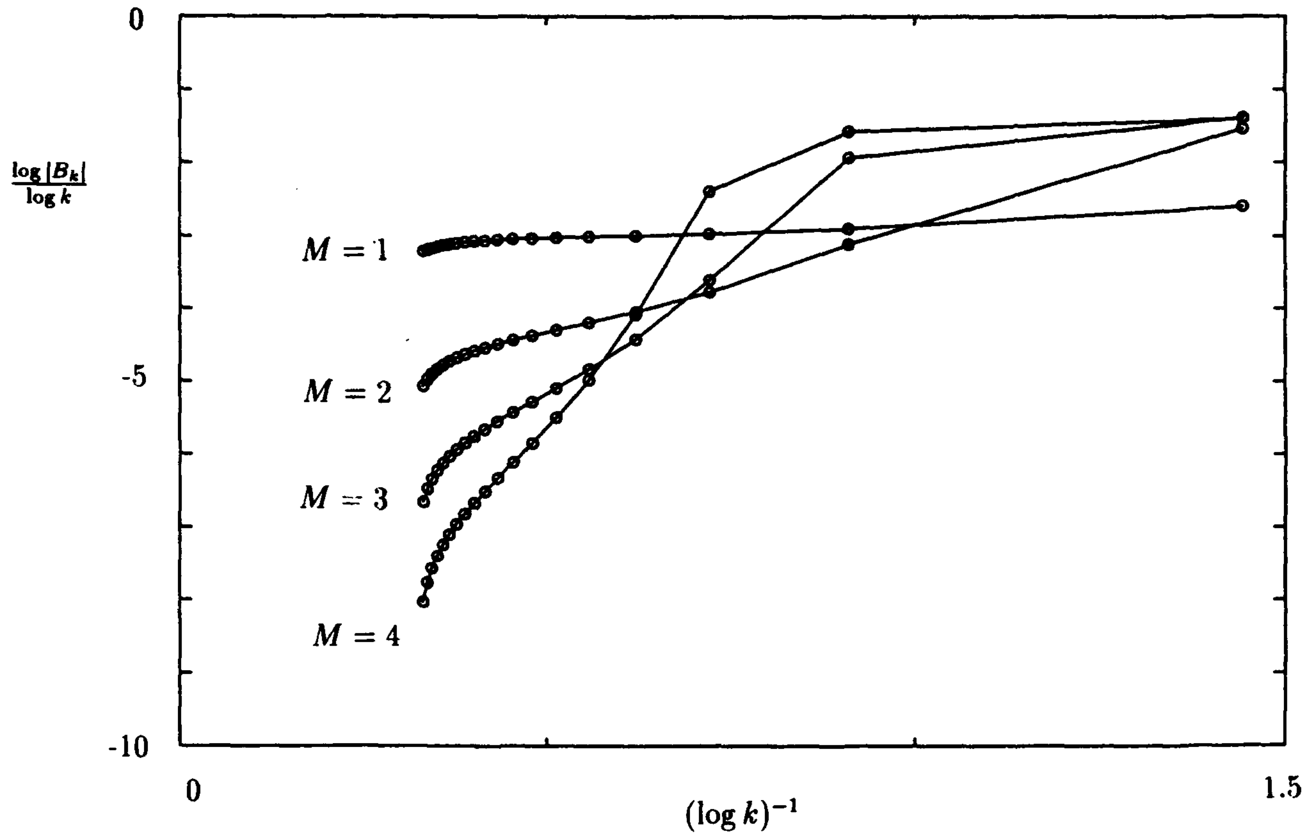


**Figure 7**: A plot of $\log|B_k|/\log k$ vs. $1/\log k$ for Example 1 with $M = 1, 2, 3$, and 4, for $N = 20$. The increased rate of decay to zero of the coefficients $B_k$ with increasing values of $M$ is clearly evident, especially when compared with original Fourier coefficients $b_k$, which decay only like $1/k$.

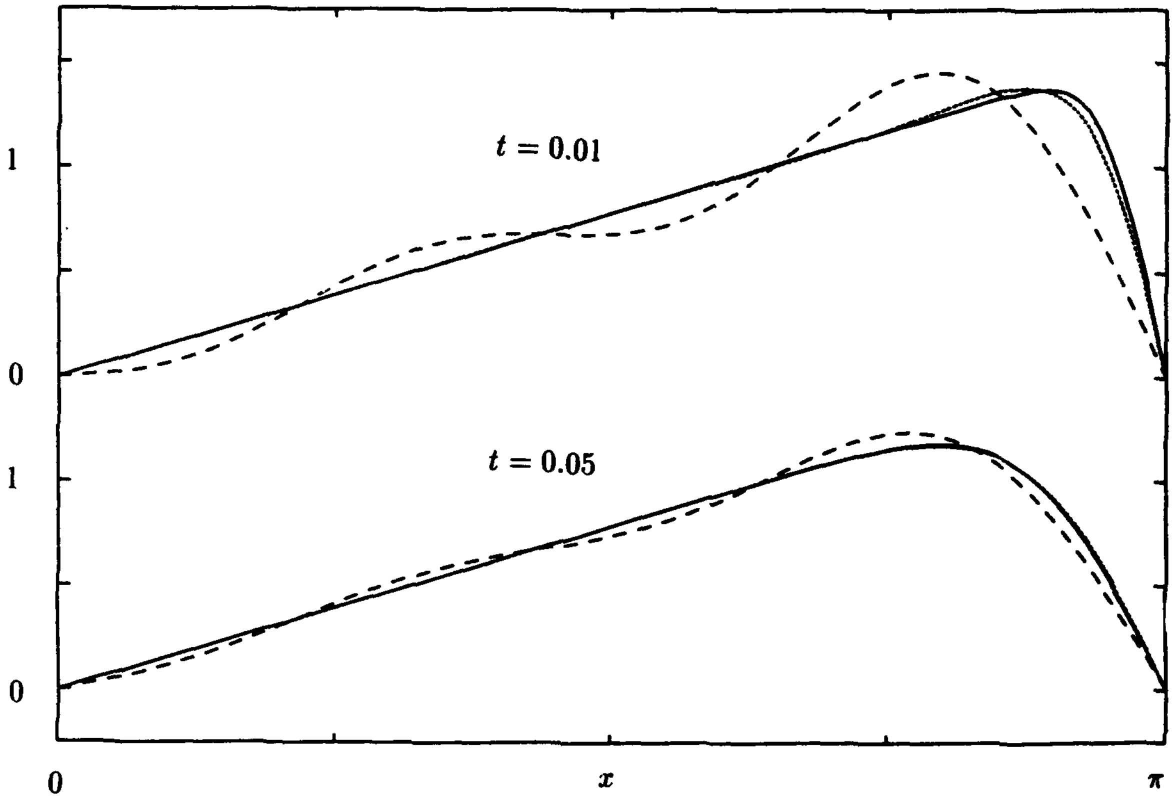


**Figure 8**: A plot of the exact solution $u^{(\infty,0)}$ (solid line), as well as the approximations $u^{(4,0)}$ (dashed line) and $u^{(3,1)}$ (dotted line), for the simple heat conduction problem of section 7, when $t = 0.01$ (upper set of curves) and $t = 0.05$ (lower set of curves). The corresponding comparison at $t = 0$ is the same as shown in Figure 1 for $0 \leq x \leq \pi$.

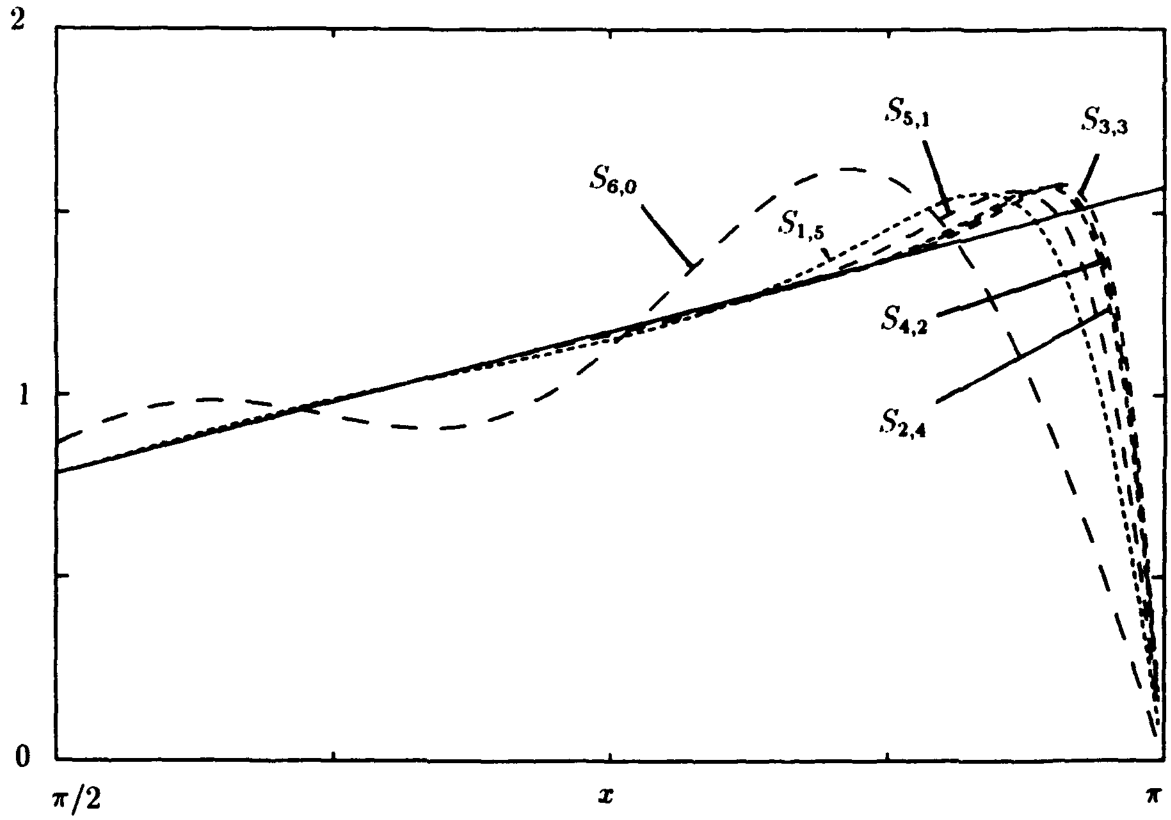


**Figure 9**: A comparison of the approximations $S_{6-q,q}$, for $q = 0, 1, ..., 5$ for Example 1. In this case, the figure seems to suggest that perhaps $S_{3,3}$ is "best" of the approximations $S_{N,M}$, with $N + M = 6$.

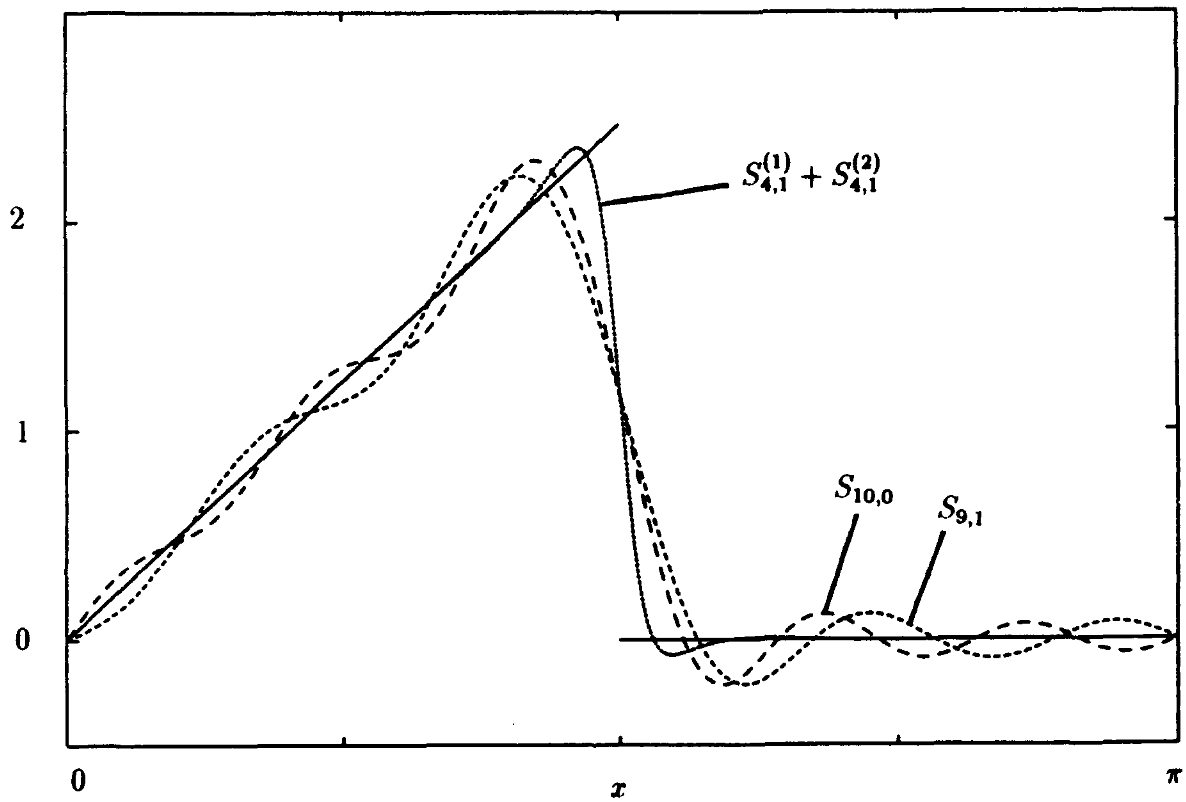


**Figure 10**: For the $2\pi$-periodic function $f$ defined by $f(x) = \pi x/2$, for $-\pi/2 \leq x \leq \pi/2$, and $f(x) = 0$, otherwise, in $[-\pi, \pi]$, a plot of $f(x)$ (solid lines) and the approximations $S_{10,0}$ (longer dashed line), $S_{9,1}$ (shorter dashed line), and $S_{4,1}^{(1)} + S_{4,1}^{(2)}$ (dotted line) for $0 \leq x \leq \pi$. The improved quality of the approximation $S_{4,1}^{(1)} + S_{4,1}^{(2)}$, over either $S_{10,1}$ or $S_{9,1}$, is apparent.

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A class of approximations $\{S_{N,M}\}$ to a periodic function f which uses the ideas of Padé, or rational function, approximations based on the Fourier series representation of f, rather than on the Taylor series representation of f, is introduced and studied. Each approximation $S_{N,M}$ is the quotient of a trigonometric polynomial of degree N and a trigonometric polynomial of degree M. The coefficients in these polynomials are determined by requiring that an appropriate number of the Fourier coefficients of $S_{N,M}$ agree with those of f. Explicit expressions are derived for these coefficients in terms of the Fourier coefficients of f. It is proven that these "Fourier-Padé" approximations converge point-wise to $(f(x^+)+f(x^-))/2$ more rapidly (in some cases by a factor of $1/k^{2M}$) than the Fourier series partial sums on which they are based. The approximations are illustrated by several examples and an application to the solution of an initial, boundary value problem for the simple heat equation is presented.

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