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Generalized Gaussian Quadrature Rules For Systems of Arbitrary Functions

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Research Report YALEU/DCS/RR-990
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In [6], a far-reaching generalization of the classical Gaussian quadrature rules is introduced, replacing the polynomials with a wide class of functions. While the rules of [6] possess most of the desirable properties of the classical Gaussian integration formulae (positivity of the weights, etc.), it is not clear from [6] how such quadrature rules can be obtained numerically. In this paper, we present a numerical scheme for the generation of such generalized Gaussian quadratures. The algorithm is applicable to a variety of functions, including smooth functions as well as functions with end-point singularities. The performance of the algorithm is demonstrated with several numerical examples.

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1 Introduction

Classical Gaussian quadrature rules are extremely efficient when the functions to be integrated are well approximated by polynomials. When the functions to be integrated are very different from polynomials, Gaussian quadratures do not perform well; a particularly difficult problem involves the integration of functions of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i \cdot \varphi_i, \tag{1}$$

where each of the functions φ_i has its own singularity at one of the ends of the interval, and the function f can only be evaluated *in toto*, the coefficients α_i being unavailable. This problem is encountered in the solution of integral equations with singular kernels, in the numerical complex analysis, in the numerical solution of elliptic partial differential equations on regions with corners, and in many other situations. While such problems are normally dealt with by means of various ad hoc procedures (see, for example, [1], [7]), these schemes lack the rapid convergence, stability, and elegance of the Gaussian rules.

In fact, in [6], a far-reaching generalization of the classical Gaussian quadratures is introduced, replacing the polynomials with functions from an extremely wide class. The quadrature rules of [6] possess most of the desirable properties of the classical Gaussian integration formulae, such as positivity of the weights, rapid convergence, mathematical elegance, etc. Unfortunately, it is not clear from [6] how such quadrature rules can be obtained numerically.

In this paper, we present a numerical scheme for the construction of such generalized Gaussian quadratures. The algorithm is applicable to a variety of functions, including smooth functions (not necessarily polynomials), as well as functions with end-point singularities.

The paper is organized as follows. In Section 2, we summarize the relevant results from [6], and in Section 3, we restate some numerical methods to be used in this paper. In Section 4, we develop analytical apparatus to be used in the numerical construction of the generalized Gaussian quadrature rules, then we extend those analytical tools to functions with end-point singularities in Section 5. The actual numerical algorithm is presented in Section 6, and the performance of the algorithm is demonstrated with numerical examples in Section 7.

2 Mathematical Preliminaries

In this section, we summarize several classical results from [6] to be used in Sections 4 and 5.

2.1 Chebychev Systems

Definition 2.1 *(Chebyshev System)*

A finite sequence of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$ *will be referred to as a* Chebyshev system *if and only if each of them is continuous on* $[a, b]$, *and the determinants*

$$\det \begin{pmatrix} \varphi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_m) \\ \varphi_2(x_1) & \varphi_2(x_2) & \cdots & \varphi_2(x_m) \\ \vdots & \vdots & & \vdots \\ \varphi_m(x_1) & \varphi_m(x_2) & \cdots & \varphi_m(x_m) \end{pmatrix} \tag{2}$$

are non-zero for any set of m *points* $x_1, x_2, \cdots, x_m \in [a, b]$ *such that* $x_i \neq x_j$ *for any* $i \neq j$.

Following are several important cases of Chebyshev systems (for more examples, see [6]).

Example 2.1 *For any natural* m, *the functions* $1, x, x^2, \cdots, x^m$ *constitute a Chebyshev system. Moreover, if* α_1, α_2, $\cdots$, α_m *is a sequence of distinct real numbers, then the system* $\{x^{\alpha_i}\}$ *is a Chebyshev system on any interval* $[a, b] \subset (0, \infty)$.

Example 2.2 *For any* n *distinct real numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$, *the functions* $e^{\alpha_1 x}$, $xe^{\alpha_1 x}$, $e^{\alpha_2 x}$, $xe^{\alpha_2 x}$, $\cdots$, $e^{\alpha_n x}$, $xe^{\alpha_n x}$ *constitute a Chebyshev system on any interval* $[a, b] \subset (-\infty, \infty)$.

Definition 2.2 *(Hermite System)*

A finite sequence of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ *will be referred to as an* Hermite system *on the interval* $[a, b]$ *if and only if* $\varphi_i \in C^1[a, b]$ *for all* $i = 1, 2, \cdots, 2n$, *and the determinants*

$$\det \begin{pmatrix} \varphi_1(x_1) & \varphi_1'(x_1) & \varphi_1(x_2) & \varphi_1'(x_2) & \cdots & \varphi_1(x_n) & \varphi_1'(x_n) \\ \varphi_2(x_1) & \varphi_2'(x_1) & \varphi_2(x_2) & \varphi_2'(x_2) & \cdots & \varphi_2(x_n) & \varphi_2'(x_n) \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varphi_{2n}(x_1) & \varphi_{2n}'(x_1) & \varphi_{2n}(x_2) & \varphi_{2n}'(x_2) & \cdots & \varphi_{2n}(x_n) & \varphi_{2n}'(x_n) \end{pmatrix} \tag{3}$$

are non-zero for any set of $2n$ *points* $x_1, x_2, \cdots, x_{2n} \in [a, b]$ *such that* $x_i \neq x_j$ *for any* $i \neq j$.

Definition 2.3 *(Extended Hermite System)*

A finite sequence of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ *will be referred to as an* extended Hermite system *if it is both Chebyshev and Hermite.*

Remark 2.1 *The Extended Hermite systems are a slight generalization of the extended Chebychev systems of [6].*

Following are several important cases of extended Hermite systems (for more examples, see [6]).

Example 2.3 *For any natural n, the functions* $1, x, x^2, \cdots, x^{2n}$ *constitute an extended Hermite system. Moreover, if* $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ *is a sequence of distinct real numbers, then the system*

$$x^{\alpha_1}, x^{\alpha_2}, \cdots, x^{\alpha_{2n}} \tag{4}$$

is an extended Hermite system on any interval $[a, b] \subset (0, \infty)$.

Example 2.4 *An important special case of the proceeding example is the finite sequence of functions*

$$1, x^{\alpha}, x, x^{1+\alpha}, x^2, x^{2+\alpha}, \cdots, x^{n-1}, x^{n-1+\alpha} \tag{5}$$

with α *an arbitrary non-integer real number.*

2.2 Gaussian Quadratures

We will be considering integrals of the form

$$\int_a^b \omega(x)\varphi(x)dx, \tag{6}$$

where $\omega : [a, b] \to R^1$ is a non-negative function to be referred to as the weight function, and $\varphi : [a, b] \to R^1$ is a function from a suitably chosen class. A quadrature rule is an expression of the form

$$T_n(\varphi) = \sum_{i=1}^{n} w_i \cdot \varphi(x_i), \tag{7}$$

with $x_i \in [a, b]$ and $w_i \in R^1$ for all $i = 1, 2, \cdots, n$. The points x_i and coefficients w_i are referred to as the nodes and weights of the quadrature formula (7), respectively, while the expression (7) itself is viewed as an approximation to the integral (6). Normally, quadrature formulae are chosen to be exact on certain chosen sets of functions, most frequently, polynomials up to some fixed order m. An $n-$point quadrature formula is referred to as a Gaussian quadrature if and only if it integrates exactly all polynomials of orders up to $2n - 1$.

We will generalize the notion of the classical Gaussian quadrature somewhat, by introducing the following definition.

Definition 2.4 *(Gaussian Quadrature)*

Suppose that

$$\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\} \tag{8}$$

is a set of integrable functions $[a, b] \to R^1$. *We will say that the* $n-$*point quadrature rule (7) is Gaussian with respect to the system (8) if and only if it integrates exactly all of the functions (8). In other words, a Gaussian rule is an* $n-$*point rule that is exact for* $2n$ *pre-chosen functions. We will refer to the nodes and weights of a Gaussian quadrature as the Gaussian nodes and weights, respectively.*

Remark 2.2 *Obviously, a classical Gaussian quadrature rule is a Gaussian quadrature rule for which*

$$\begin{aligned} \varphi_1(x) &= 1, \\ \varphi_2(x) &= x, \\ \cdots & \cdots \\ \varphi_{2n}(x) &= x^{2n-1}. \end{aligned} \tag{9}$$

The principal result we use from [6] is the following theorem.

Theorem 2.1 *(Karlin-Studden)*
Suppose that the functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ constitute a Chebyshev system on the interval $[a, b]$. Then there exists a unique n-point quadrature rule (7) that is Gaussian with respect to the functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$. Furthermore, all the weights $w_1, w_2, \cdots, w_n$ of the quadrature are positive.

As for smooth functions, Gaussian quadrature rules also exist for a variety of functions with end-point singularities. The following theorem is an immediate consequence of Theorem 2.1.

Theorem 2.2 *Suppose that functions $\varphi_i : (a, b] \to R^1$ are continuous, and integrable on $[a, b]$ for all $i = 1, 2, \cdots, 2n$. Suppose also that the function $w(x) > 0$ is continuous on $(a, b]$ and integrable on $[a, b]$. Suppose further that functions ψ_i are defined by the formula*

$$\psi_i(x) = \frac{\varphi_i(x)}{w(x)}, \tag{10}$$

and that

$$\lim_{x \to a} \psi_i(x) < \infty \tag{11}$$

for all $i = 1, 2, \cdots, 2n$. Suppose finally that the functions $\{\psi_1, \psi_2, \cdots, \psi_{2n}\}$ defined by (10) constitute a Chebyshev system on the closed interval $[a, b]$.

Then there exists a unique n-point quadrature rule (7) that is Gaussian with respect to the functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$. Furthermore, all the weights $w_1, w_2, \cdots, w_n$ of the quadrature are positive.

Proof: The theorem is proved by applying Theorem 2.1 to the new weight function

$$\tilde{\omega}(x) = \omega(x) \cdot w(x), \tag{12}$$

and the new set of functions $\{\psi_1, \psi_2, \cdots, \psi_{2n}\}$. ■

Example 2.5 *For any natural n, and real number $0 < \alpha < 1$, the unique n-point quadrature on the interval $[0, 1]$ with respect to the functions*

$$1, x^{-\alpha}, x, x^{1-\alpha}, x^2, x^{2-\alpha}, \cdots, x^{n-1}, x^{n-1-\alpha}. \tag{13}$$

can be obtained via the following Chebyshev system (see Example 2.3)

$$x^{\alpha}, 1, x^{1+\alpha}, x, x^{2+\alpha}, x^2, \cdots, x^{n-1+\alpha}, x^{n-1} \tag{14}$$

on the interval $[0, 1]$ with the weight function $\tilde{\omega}(x) = \omega(x) \cdot x^{\alpha}$.

3 Numerical Preliminaries

In this section, we collect the relevant numerical tools to be used in Sections 4 and 5. They can be found, for example, in [3], [4], [5].

3.1 Nested Chebyshev Approximation

For any non-negative integer n, the Chebyshev polynomial T_n of order n is defined by the formula

$$T_n(\cos\theta) = \cos(n\theta). \tag{15}$$

Clearly, $|T_n(x)| \leq 1$ for $x \in [-1,1]$.

The Chebyshev polynomials constitute an orthonormal basis for $L^2[-1,1]$ with respect to the inner product

$$(f,g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot f(x) \cdot g(x) dx. \tag{16}$$

Therefore, any function $f \in C^0[-1,1]$ can be represented by an expansion

$$f(x) = \sum_{i=0}^{\infty} a_i \cdot T_i(x), \tag{17}$$

with the coefficients a_i given by

$$a_i = (f, T_i). \tag{18}$$

Lemma 3.1 states that the Chebyshev series (17) converges rapidly for sufficiently smooth functions. Its proof can be found, for example, in [5].

Lemma 3.1 *Suppose that n and k are natural numbers, and that $f \in C^k[-1,1]$. Suppose further that the coefficients $a_0, a_1, \cdots, a_n$ are defined the formula (18). Then for any $x \in [-1,1]$,*

$$\left| f(x) - \sum_{i=0}^{n} a_i \cdot T_i(x) \right| = O(\frac{1}{n^{k-1}}). \tag{19}$$

In particular, if $f \in C^\infty$, then the expansion (17) converges to f superalgebraically.

Observation 3.1 *For functions with end-point singularities, such as $f(x) = \ln x$, we can build a structure on the given interval, consisting of subintervals clustering near the end points (see Figure 3.1), and then use the Chebyshev expansion (17) to approximate the functions on each subinterval. On each of the subintervals, the Chebyshev expansion converges superalgebraiclly.*

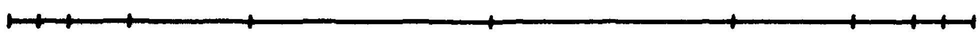

Figure 1: Subintervals clustering near the end-points

3.2 Nonlinear Equations

We will be considering systems of nonlinear equations of the form

$$F(\mathrm{x}) = 0, \tag{20}$$

where $\mathrm{x} = (x_1, \cdots, x_n)^T \in R^n$, and the mapping $F : R^n \to R^n$ is of the form

$$F(\mathrm{x}) = \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ \vdots \\ f_n(x_1, \cdots, x_n) \end{pmatrix}. \tag{21}$$

Definition 3.1 *The Jacobian matrix of mapping F in (21) is defined by the formula*

$$DF(\mathrm{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \tag{22}$$

The following two lemmas about the solution of systems of nonlinear equations (20) are well-known (see [3], [8], for example).

Lemma 3.2 *(Newton's Method)*
Suppose that $F : R^n \to R^n$ is continuously differentiable in an open convex set $D \subset R^n$, and the mapping $G : R^n \to R^n$ is defined by the formula

$$\mathrm{y} = \mathrm{x} - (DF(\mathrm{x}))^{-1} F(\mathrm{x}). \tag{23}$$

Suppose also that $\mathrm{x}^* \in R^n$ is the zero of F, and there exists $\beta > 0$ such that

$$||DF(\mathrm{x}^*)^{-1}|| \le \beta. \tag{24}$$

Suppose further that there exist two positive numbers r and γ such that $\mathrm{x} \in D$ for any x such that $||\mathrm{x} - \mathrm{x}^*|| < r$, and

$$||DF(\mathrm{x}) - DF(\mathrm{y})|| \le \gamma ||\mathrm{x} - \mathrm{y}|| \tag{25}$$

for any x, y such that $||\mathrm{x} - \mathrm{x}^*|| < r$, $||\mathrm{y} - \mathrm{x}^*|| < r$. Suppose finally that x_0 is an arbitrary point in R^n, and the sequence $\mathrm{x}_1, \mathrm{x}_2, \cdots$, of points in R^n is defined by the formula

$$\mathrm{x}_{k+1} = G(\mathrm{x}_k) \tag{26}$$

for all $k = 0, 1, 2, \cdots$.

Then there exist $\epsilon > 0$ *and* $\alpha > 0$ *such that the sequence generated by (26) converges to* x^*, *and*

$$||\mathrm{x}_{k+1} - \mathrm{x}^*|| \leq \alpha ||\mathrm{x}_k - \mathrm{x}^*||^2 \tag{27}$$

for any x_0 *such that* $||\mathrm{x}_0 - \mathrm{x}^*|| < \epsilon$.

Lemma 3.3 *(Modified Newton's Method)*

Suppose that under the assumptions of Lemma 3.2, x^* *is the zero of the mapping* $F : D \to R^n$. *Suppose also that*

$$A_0, A_1, A_2, \cdots \tag{28}$$

is a sequence of $n \times n$ *nonsingular matrices, and the mapping* $G : R^n \to R^n$ *is defined by the formula*

$$\mathrm{y} = \mathrm{x} - A_k^{-1} F(\mathrm{x}). \tag{29}$$

Suppose further that there exists a positive real number M *such that*

$$||A_k - DF(\mathrm{x}_k)|| \leq \mathrm{M} \cdot ||\mathrm{F}(\mathrm{x}_k)||. \tag{30}$$

Suppose finally that x_0 *is an arbitrary point in* R^n, *and the sequence* $\mathrm{x}_1, \mathrm{x}_2, \cdots$, *of points in* R^n *is defined by the formula*

$$\mathrm{x}_{k+1} = G(\mathrm{x}_k) \tag{31}$$

for all $k = 0, 1, 2, \cdots$.

Then there exist $\epsilon > 0$ *and* $\alpha > 0$ *such that the sequence generated by (31) converges to* x^*, *and*

$$||\mathrm{x}_{k+1} - \mathrm{x}^*|| \leq \alpha ||\mathrm{x}_k - \mathrm{x}^*||^2 \tag{32}$$

for any x_0 *such that* $||\mathrm{x}_0 - \mathrm{x}^*|| < \epsilon$,

3.3 Continuation Method

The Newton algorithm for the solution of systems of non-linear equations is an extremely powerful technique, provided that a satisfactory initial point is available. In many cases, a starting point is not available directly, but can be obtained by the process known as the continuation method (otherwise referred to as the homotopy method). Following is a brief description of the technique.

Suppose that we are trying to solve a system of non-linear equations

$$F(\mathrm{x}) = 0, \tag{33}$$

with $F : R^n \to R^n$ satisfying the conditions of Lemma 3.3, except for the initial point x_0, which is not available. Suppose further that we *do* have access to a mapping $G : [0, 1] \times R^n \to R^1$, satisfying the following conditions.

1. The mapping $F_0 : R^n \to R^n$ defined by the formula

$$F_0(\mathrm{x}) = G(0, \mathrm{x}) \tag{34}$$

has a simple structure, so that the solution of the equation

$$F_0(\mathrm{x}) = 0 \tag{35}$$

is unique and known.

2. For all $\mathrm{x} \in R^n$,

$$G(1, \mathrm{x}) = F(\mathrm{x}). \tag{36}$$

3. For all $t \in [0, 1]$, the equation

$$G(t, \mathrm{x}) = 0 \tag{37}$$

has a unique solution $\mathrm{x}(t) \in R^n$, and satisfies the conditions of Lemma 3.3 in the neighborhood of $\mathrm{x}(t)$.

4. x is a continuous (or better, Lipschitz) function of t.

Under the above conditions, the following procedure yields the solution of the equation (33).

1. For a sufficiently large m, construct the points $t_i = (i - 1)/(m - 1)$ on the interval $[0, 1]$, with $i = 1, 2, \cdots, m$, and consider the solutions of the equation (33) for $t = t_i$ with $i = 1, 2, \cdots, m$.

2. Clearly, we know the solution $\mathrm{x}(0)$ of the equation,

$$G(t_i, \mathrm{x}) = 0 \tag{38}$$

when $i = 1$, and for all $i = 2, 3, \cdots, m$, we solve the equation (37) by means of Lemma 3.3, using $\mathrm{x}(t_{i-1})$ as the initial approximation.

3. Since $t_n = 1$, the result of the final step of this process is the solution of the equation (33).

A detailed discussion of the continuation techniques can be found in [4], where the convergence of the above scheme is proven (in a much more general environment) for all sufficiently large m.

4 Analytical Apparatus

In this section, we develop analytical tools to be used in the numerical construction of the Gaussian quadratures whose existence follows from Theorem 2.1.

4.1 Construction of Gaussian Quadratures

The following lemma is an immediate consequence of Definition 2.3 of extended Hermite systems.

Lemma 4.1 *Suppose that the functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ constitute an Hermite system on the interval $[a, b]$, and $x_1, x_2, \cdots, x_n$ are n points on the interval $[a, b]$ such that $x_i \neq x_j$ for any $i \neq j$. Then there exist such unique coefficients $\alpha_{i,j}$, $\beta_{i,j}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 2n$ that*

$$\begin{cases} \sigma_i(x_k) &= 0, \\ \sigma_i'(x_k) &= \delta_{ik}, \end{cases} \tag{39}$$

$$\begin{cases} \eta_i(x_k) &= \delta_{ik}, \\ \eta_i'(x_k) &= 0, \end{cases} \tag{40}$$

for all $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots, n$, with $\delta_{i,k}$ denoting the Kronecker symbol, and the functions σ_i, η_i defined by the formulae

$$\sigma_i(x) = \sum_{j=1}^{2n} \alpha_{i,j} \cdot \varphi_j(x), \tag{41}$$

and

$$\eta_i(x) = \sum_{j=1}^{2n} \beta_{i,j} \cdot \varphi_j(x). \tag{42}$$

Furthermore, there exist unique coefficients a_{ij}, b_{ij} with $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, n$, such that

$$\varphi_i(x) = \sum_{j=1}^{n} a_{i,j} \cdot \sigma_j(x) + b_{i,j} \cdot \eta_j(x) \tag{43}$$

for all $i = 1, 2, \cdots, 2n$.

In other words, there exist unique linear combinations $\sigma_1, \sigma_2, \cdots, \sigma_n$, $\eta_1, \eta_2, \cdots, \eta_n$ of the functions $\varphi_1, \varphi_2, \cdots, \varphi_{2n}$ satisfying the conditions (39), (40), (43). Conversely, the functions $\varphi_1, \varphi_2, \cdots, \varphi_{2n}$ are linear combinations of the functions $\sigma_1, \sigma_2, \cdots, \sigma_n$, $\eta_1, \eta_2, \cdots, \eta_n$.

Theorem 4.1 below is the principal analytical tool of this paper. It establishes the necessary and sufficient conditions under which a quadrature is Gaussian with respect to a given Hermite system.

Theorem 4.1 *Suppose that functions*

$$\varphi_1, \varphi_2, \cdots, \varphi_{2n} \tag{44}$$

constitute an Hermite system on the interval $[a, b]$. Then the nodes $x_1, x_2, \cdots, x_n$ on $[a, b]$ are Gaussian with respect to the functions (44) if and only if

$$\int_a^b \omega(x)\sigma_i(x)dx = 0 \tag{45}$$

for all $i = 1, 2, \cdots, n$. In this case, the Gaussian weights $w_1, w_2, \cdots, w_n$ are given by the formula

$$w_i = \int_a^b \omega(x)\eta_i(x)dx, \tag{46}$$

with functions $\sigma_1, \sigma_2, \cdots, \sigma_n$, $\eta_1, \eta_2, \cdots, \eta_n$, in (45), (46) are defined by formulae (39) - (42).

Proof: First, we show that for any Gaussian quadrature rule (7), the conditions (45), (46) are satisfied. Indeed, since it integrates exactly all of the functions $\varphi_1, \varphi_2, \cdots, \varphi_{2n}$, it also integrates exactly all their linear combinations $\sigma_1, \sigma_2, \cdots, \sigma_n$, $\eta_1, \eta_2, \cdots, \eta_n$. Now, (45), (46) follow immediately from (39), (40).

Suppose now that the nodes $x_1, x_2, \cdots, x_n$ are such that the conditions (45) are satisfied, and the coefficients $w_1, w_2, \cdots, w_n$ are defined by the formula (46). We will show that the n-point quadrature (7) is Gaussian with respect to the system (44).

Due to Lemma 4.1, there exist coefficients $\alpha_{i,j}$, $\beta_{i,j}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 2n$ such that

$$\varphi_i = \sum_{j=1}^{n} (a_{ij}\sigma_j + b_{ij}\eta_j) \tag{47}$$

for any $i = 1, 2, \cdots, 2n$. Thus

$$\int_a^b \omega(x)\varphi_i(x)dx \;=\; \sum_{j=1}^{n} a_{ij} \int_a^b \omega(x)\sigma_j(x)dx + b_{ij} \int_a^b \omega(x)\eta_j(x)dx. \tag{48}$$

Combing (48) with (45), (46), we have

$$\int_a^b \omega(x)\varphi_i(x)dx = \sum_{j=1}^{n} b_{ij}w_j. \tag{49}$$

On the other hand, combining (39), (40), and (48), we obtain

$$\begin{aligned} \sum_{j=1}^{n} w_j\varphi_i(x_j) &= \sum_{j=1}^{n} w_j \sum_{k=1}^{n} (a_{ik}\sigma_k(x_j) + b_{ik}\eta_k(x_j)) \\ &= \sum_{j=1}^{n} b_{ij}w_j. \end{aligned} \tag{50}$$

Combining (49) and (50), we finally get

$$\int_a^b \omega(x)\varphi_i(x)dx = \sum_{j=1}^{n} w_j\varphi_i(x_j). \tag{51}$$

for all $i = 1, 2, \cdots, 2n$. ∎

Theorem 4.2 below follows immediately from Theorem 4.1. It describes the Gaussian nodes as the solution of a system of non-linear equations.

Theorem 4.2 *Suppose that the functions $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ constitute an Hermite system, and functions $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ and $\{\eta_1, \eta_2, \cdots, \eta_n\}$ are defined by the formulae (39), (40), (41), and (42). Suppose further that S is a subset of R^n consisting of all finite sequences $\{x_1, x_2, \cdots, x_n\}$ such that $x_i \neq x_j$ whenever $i \neq j$. Suppose finally that the mapping $F : S \rightarrow R^n$ is defined by the formula*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} \int_a^b \omega(x)\sigma_1(x)dx \\ \int_a^b \omega(x)\sigma_2(x)dx \\ \vdots \\ \int_a^b \omega(x)\sigma_n(x)dx \end{pmatrix}. \tag{52}$$

Then $\{x_1, x_2, \cdots, x_n\}$ are the Gaussian nodes with respect to the system of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ if and only if

$$F(x_1, \cdots, x_n) = 0. \tag{53}$$

4.2 Computation of Gaussian Quadratures

In this subsection, we observe that the modified Newton method in Subsection 3.2 assumes a particularly simple form when it is applied to the system of equations (53), and establish Theorem 4.5, the principal numerical tool of this paper. Theorem 4.5 shows that an extremely simple iterative scheme converges quadratically for the system of equations (53).

Theorem 4.4 below provides an analytical expression for the Jacobian matrix of the mapping F defined by the formula (52). Theorem 4.3 is the consequence of Lemmas 4.2 and 4.3, and will be used in the proof of Theorem 4.4. Theorem 4.5 follows immediately form Lemma 3.3, Corollary 4.2, and Theorem 4.4.

Lemma 4.2 *Suppose that functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ form an extended Hermite system with $\varphi_i \in C^3[a, b]$ for all $i = 1, 2, \cdots, 2n$. Suppose also that*

$$x_1, x_2, \cdots, x_n \tag{54}$$

are n distinct points on the interval $[a, b]$, and functions

$$\sigma_1, \sigma_2, \cdots, \sigma_n, \eta_1, \eta_2, \cdots, \eta_n \tag{55}$$

are determined by the set of points (54) via formulae (39) and (40). Suppose further that l is an integer such that $1 \leq l \leq n$, and δ is a real number such that

$$x_1, \cdots, x_{l-1}, x_l + \delta, x_{l+1} \cdots, x_n \tag{56}$$

are n distinct points on interval $[a, b]$. Suppose finally that the functions

$$\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_n, \tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_n \tag{57}$$

are determined by the set of points (56) via formulae (39) and (40).

Then there exist coefficients α_{il} and β_{il} with $i = 1, 2, \cdots, n$, such that

$$\tilde{\sigma}_i(x) = \sigma_i(x) + \alpha_{il} \cdot \sigma_l(x) + \beta_{il} \cdot \eta_l(x) \tag{58}$$

for all $i \neq l$, $i = 1, 2, \cdots, n$, and

$$\tilde{\sigma}_l(x) = \alpha_{ll} \cdot \sigma_l(x) + \beta_{ll} \cdot \eta_l(x). \tag{59}$$

Proof: Due to Lemma 4.1, there exist unique coefficients α_{ij}, β_{ij} with $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, n$, such that for all $i = 1, 2, \cdots, 2n$,

$$\tilde{\sigma}_i(x) = \sum_{j=1}^{n} (\alpha_{ij} \cdot \sigma_j(x) + \beta_{ij} \cdot \eta_j(x)) \tag{60}$$

for any $x \in [a, b]$. Differentiating (60), we have

$$\tilde{\sigma}_i'(x) = \sum_{j=1}^{n} \left(\alpha_{ij} \cdot \sigma_j'(x) + \beta_{ij} \cdot \eta_j'(x)\right). \tag{61}$$

For the functions (57), the conditions (39), (40) assume the form

$$\begin{cases} \tilde{\sigma}_i(x_l + \delta) &= 0, \\ \tilde{\sigma}_i'(x_l + \delta) &= \delta_{il}, \end{cases} \tag{62}$$

and

$$\begin{cases} \tilde{\sigma}_i(x_k) &= 0, \\ \tilde{\sigma}_i'(x_k) &= \delta_{ik} \end{cases} \tag{63}$$

for all $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots, n$, and $k \neq l$.

For any $k \neq l$, evaluating (60) at x_k and applying conditions (39), (40), (63), we obtain

$$\beta_{ik} = 0 \tag{64}$$

for all $i = 1, 2, \cdots, n$.

Similarly, for any $k \neq l$, evaluating (61) at x_k and applying conditions (39), (40), (63), we have

$$\alpha_{ik} = \delta_{ik} \tag{65}$$

for all $i = 1, 2, \cdots, n$.

Now, (58), (59) immediately follow from (60), (64), (65). ■

Lemma 4.3 *Suppose that under the assumptions of Lemma 4.2, the coefficients α_{il} and β_{il} are defined via formulae (58) and (59) for all $i = 1, 2, \cdots, n$, Then*

$$\alpha_{il} = -\delta \cdot \sigma_i''(x_l) + O(\delta^2), \tag{66}$$

$$\beta_{il} = O(\delta^2) \tag{67}$$

for all $i \neq l$, $i = 1, 2, \cdots, n$, *and*

$$\alpha_{ll} = 1 - \delta \cdot \sigma_l''(x_l) + O(\delta^2)), \tag{68}$$
$$\beta_{ll} = \delta + O(\delta^2). \tag{69}$$

Proof: Expanding the functions

$$\sigma_i, \eta_i, \sigma_i', \eta_i',$$

into the Taylor series at x_l, we have

$$\begin{cases} \sigma_i(x_l + \delta) &= \sigma_i(x_l) + \delta \cdot \sigma_i'(x_l) + O(\delta^2), \\ \eta_i(x_l + \delta) &= \eta_i(x_l) + \delta \cdot \eta_i'(x_l) + O(\delta^2), \end{cases} \tag{70}$$

$$\begin{cases} \sigma_i'(x_l + \delta) &= \sigma_i'(x_l) + \delta \cdot \sigma_i''(x_l) + O(\delta^2), \\ \eta_i'(x_l + \delta) &= \eta_i'(x_l) + \delta \cdot \eta_i''(x_l) + O(\delta^2), \end{cases} \tag{71}$$

for all $i = 1, 2, \cdots, n$.

Evaluating (58) and (59) at $x = (x_l + \delta)$ and using the conditions (39), (40), (62), (70), we obtain

$$\beta_{il} = -\alpha_{il} \cdot \delta + O(\delta^2) \tag{72}$$

for all $i = 1, 2, \cdots, n$.

Differentiating (58) at $x = (x_l + \delta)$ and using conditions (39), (40), (62), (71), we have

$$\begin{aligned} \alpha_{ll} &= \left(1 + \delta \cdot \sigma_l''(x_l)\right)^{-1} \left(1 + O(\delta^2)\right), \\ &= 1 - \delta \cdot \sigma_l''(x_l) + O(\delta^2). \end{aligned} \tag{73}$$

Similarly, differentiating (59) at $x = (x_l + \delta)$ and applying the conditions (39), (40), (62), and (71), we get

$$\alpha_{il} = -\delta \cdot \sigma_i''(x_l) + O(\delta^2) \tag{74}$$

for all $i = 1, 2, d \cdots, n$, and $i \neq l$

Finally, combining (72) with (73), (74), we have

$$\beta_{ll} = \delta + O(\delta^2), \tag{75}$$
$$\beta_{il} = O(\delta^2) \tag{76}$$

for all $i = 1, 2, \cdots, n$, and $i \neq l$ ■

Combining Lemmas 4.2, 4.3, we now obtain the following theorem.

Theorem 4.3 *Under the assumptions of Lemma 4.2,*

$$\tilde{\sigma}_l(x) = \sigma_l(x) - \delta \cdot \sigma_l''(x_l) \cdot \sigma_l(x) - \delta \cdot \eta_l(x) + O(\delta^2) \tag{77}$$
$$\tilde{\sigma}_i(x) = \sigma_i(x) - \delta \cdot \sigma_i''(x_l) \cdot \sigma_l(x) + O(\delta^2) \tag{78}$$

for any $i = 1, 2, \cdots, n$, *and* $i \neq l$.

The following theorem is an immediate consequence of Theorem 4.3. It provides a simple expression for the Jacobian of the mapping (52), showing that the latter is nearly diagonal in the vicinity of the solution of the equation (53).

Theorem 4.4 *Suppose that functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ form an extended Hermite system, and $\varphi_i \in C^3[a,b]$ for $i = 1, 2, \cdots, 2n$. Suppose further that x_1, x_2, $\cdots$, x_n are n distinct point on the interval $[a,b]$, and functions*

$$\sigma_1, \sigma_2, \cdots, \sigma_n, \eta_1, \eta_2, \cdots, \eta_n$$

are determined by formulae (39) and (40). Then the Jacobian $DF(\mathrm{x})$ of the mapping F defined by (52) is given by the formula

$$DF(\mathrm{x}) = -\begin{pmatrix} \int_a^b \omega(x)\eta_1(x)dx & 0 & \cdots & 0 \\ 0 & \int_a^b \omega(x)\eta_2(x)dx & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \int_a^b \omega(x)\eta_n(x)dx \end{pmatrix} - \mathrm{E}(\mathrm{x}) \qquad (79)$$

where $\mathrm{x} = (x_1, x_2, \cdots, x_n)^T$, and matrix $E(\mathrm{x})$ is given by the formula

$$E(\mathrm{x}) = \begin{pmatrix} \sigma_1''(x_1) & \cdots & \sigma_1''(x_n) \\ \vdots & & \vdots \\ \sigma_n''(x_1) & \cdots & \sigma_n''(x_n) \end{pmatrix} \begin{pmatrix} \int_a^b \omega(x)\sigma_1(x)dx & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \int_a^b \omega(x)\sigma_n(x)dx \end{pmatrix}. \qquad (80)$$

Proof: Suppose that l is an integer such that $1 \le l \le n$, and δ is a real number such that

$$x_1, \cdots, x_{l-1}, x_l + \delta, x_{l+1} \cdots, x_n \qquad (81)$$

are n distinct points on the interval $[a,b]$. Suppose further that the functions

$$\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_n, \tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_n \qquad (82)$$

are determined by the set of points (81) via formulae (39) and (40).

Combining (77), (78) with Definition 3.1 of Jacobian matrix, and the definition (52) of the mapping F, we immediately obtain

$$\begin{aligned} (DF(\mathrm{x}))_{il} &= \frac{\partial}{\partial x_l}\int_a^b \omega(x)\sigma_i(x)dx \\ &= \lim_{\delta\to 0} \frac{\int_a^b \omega(x)\tilde{\sigma}_l(x)dx - \int_a^b \omega(x)\sigma_l(x)dx}{\delta} \\ &= -\sigma_l''(x_l)\int_a^b \omega(x)\sigma_l(x)dx - \int_a^b \omega(x)\eta_l(x)dx \end{aligned} \qquad (83)$$

for any $i \neq l$, $i = 1, 2, \cdots, n$, and

$$\begin{aligned}
(DF(\mathrm{x}))_{ll} &= \frac{\partial}{\partial x_l}\int_a^b \omega(x)\sigma_l(x)dx \\
&= \lim_{\delta \to 0} \frac{\int_a^b \omega(x)\tilde{\sigma}_i(x)dx - \int_a^b \omega(x)\sigma_i(x)dx}{\delta} \\
&= -\sigma_i''(x_l)\int_a^b \omega(x)\sigma_l(x)dx. \qquad (84)
\end{aligned}$$

■

Corollary 4.1 follows immediately from Theorem 2.1 and Theorem 4.4, and corollary 4.2 is the consequence of Corollary 4.1 and Theorem 4.4.

Corollary 4.1 *Suppose that under the assumptions of Theorem 4.4, the function $F : R^n \to R^n$ is defined by (52). Then there exists a unique $\mathrm{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in R^n$ such that*

$$F(\mathrm{x}^*) = 0, \qquad (85)$$

and the Jacobian matrix

$$DF(\mathrm{x}^*) = -\begin{pmatrix} \int_a^b \omega(x)\eta_1(x)dx & 0 & \cdots & 0 \\ 0 & \int_a^b \omega(x)\eta_2(x)dx & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \int_a^b \omega(x)\eta_n(x)dx \end{pmatrix} \qquad (86)$$

is nonsingular, where the functions $\eta_1, \eta_2, \cdots, \eta_n$ are determined by the set of points $x_1^*, x_2^*, \cdots, x_n^*$ via the formula (40).

Corollary 4.2 *Suppose that under the assumptions of Theorem 4.4, the function $F : R^n \to R^n$ is defined by (52), and x^* is the unique zero of F. Then F is continuously differentiable, and there exist three positive real numbers r, β and γ such that*

$$||DF(\mathrm{x}^*)^{-1}|| \leq \beta, \qquad (87)$$

and

$$||DF(\mathrm{x}) - DF(\mathrm{y})|| \leq \gamma||\mathrm{x} - \mathrm{y}|| \qquad (88)$$

for any x and y such that $||\mathrm{x} - \mathrm{x}^*|| < r$, $||\mathrm{y} - \mathrm{x}^*|| < r$.

The following theorem is the principal numerical tool of this paper. It shows that an extremely simple iterative scheme converges quadratically for the system of equations (53), and is an immediate consequence of Lemma 3.3, Corollary 4.2, and Theorem 4.4.

Theorem 4.5 *Suppose that functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ form an extended Hermite system, and the mapping $G : R^n \to R^n$ is defined by the formula*

$$y_i = x_i + \frac{\int_a^b \omega(x)\sigma_i(x)dx}{\int_a^b \omega(x)\eta_i(x)dx}, \tag{89}$$

with $i = 1, 2, \cdots, n$, and the functions

$$\sigma_1, \sigma_2, \cdots, \sigma_n, \eta_1, \eta_2, \cdots, \eta_n$$

defined by the points $x_1, x_2, \cdots, x_n$ via the formulae (39) and (40). Suppose further that $\varphi_i \in C^3[a, b]$ for all $i = 1, 2, \cdots, 2n$, and the function $F : R^n \to R^n$ is defined by (52). Suppose finally that x^* is the unique zero of F, that x_0 is an arbitrary point in R^n, and the sequence $\mathrm{x}_1, \mathrm{x}_2, \cdots$, of points in R^n is defined by the formula

$$\mathrm{x}_{k+1} = G(\mathrm{x}_k) \tag{90}$$

for all $k = 0, 1, 2, \cdots$.

Then there exists $\epsilon > 0$ and $\alpha > 0$ such that the sequence $\mathrm{x}_1, \mathrm{x}_2, \cdots$, generated by (90) converges to x^*, and

$$||\mathrm{x}_{k+1} - \mathrm{x}^*|| \leq \alpha ||\mathrm{x}_k - \mathrm{x}^*||^2 \tag{91}$$

for any initial point x_0 such that $||\mathrm{x}_0 - \mathrm{x}^*|| < \epsilon$.

Remark 4.1 *In Theorem 4.5, we impose the condition*

$$\varphi_i \in C^3[a, b] \tag{92}$$

for all $i = 1, 2, \cdots, 2n$. However, it can be easily observed that the condition (92) is excessively restrictive, and a somewhat more involved proof shows that as long as φ_i are continuously differentiable, and φ_i' satisfy the Lipschitz condition for all $i = 1, 2, \cdots, 2n$, the modified Newton method (90) will still converge quadratically.

5 Integration of Singular Functions

In this section, the theory of the generalized Gaussian quadrature rules established in Section 4 will be generalized to a variety of functions with end-point singularities. We will first introduce the concepts of Chebyshev and extended Hermite systems in the case of singular functions. Then we will prove Theorems 5.1 and Theorem 5.2 for the construction of Gaussian quadrature rules, providing effective numerical construction for Gaussian quadratures for functions with end-point singularities.

Definition 5.1 *(Chebyshev System)*

Suppose that $X \subset R^1$ is either (a, b), or $[a, b)$, or $(a, b]$. Then a finite sequence of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ will be referred to as a Chebyshev system on X if and only if it constitutes a Chebyshev system on every closed subinterval $[c, d] \subset X$ (see Definition 2.1).

Definition 5.2 *(Extended Hermite System)*

Suppose that $X \subset R^1$ is either (a,b), or $[a,b)$, or $(a,b]$. Then a finite sequence of functions $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ will be referred to as an extended Hermite system on X if and only if it constitutes an extended Hermite system on every closed subinterval $[c,d] \subset X$ (see Definition 2.3).

The following is an important example of extended Hermite systems.

Example 5.1 *For any natural n, the functions*

$$1, x^{\alpha}, x, x^{1+\alpha}, x^2, x^{2+\alpha}, \cdots, x^{n-1}, x^{n-1+\alpha} \tag{93}$$

constitute an extended Hermite system on the interval $(0,1]$ with α an arbitrary non-integer real number. (see Example 2.3 and Definition 5.2)

Theorem 5.1 *Suppose that functions $\{\varphi_1, \varphi_2, \cdots, \varphi_{2n}\}$ are all integrable on $[a,b]$, and constitute a Chebyshev system on $(a,b]$. Suppose further that on the interval $[a+\delta, b]$, the n-point Gaussian quadrature (7) is given by the nodes*

$$x_1^{(\delta)}, x_2^{(\delta)}, \cdots, x_n^{(\delta)}, \tag{94}$$

and the weights

$$w_1^{(\delta)}, w_2^{(\delta)}, \cdots, w_n^{(\delta)} \tag{95}$$

for any $\delta \in (0, b-a)$.

Then for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$,

$$\left| \int_a^b \omega(x)\varphi_i(x)dx - \sum_{j=1}^{n} w_j^{(\delta)} \varphi_i(x_j^{(\delta)}) \right| < \epsilon. \tag{96}$$

for all $i = 1, 2, \cdots, 2n$.

Proof: Due to the Definition 2.4 of the Gaussian Quadratures, we have

$$\int_{a+\delta}^b \omega(x)\varphi_i(x)dx = \sum_{j=1}^{n} w_j^{(\delta)} \varphi_i(x_j^{(\delta)}) \tag{97}$$

for all $i = 1, 2, \cdots, n$. Subtracting

$$\int_a^b \omega(x)\varphi_i(x)dx$$

from both sides of (97), we obtain

$$\begin{aligned} \int_a^b \omega(x)\varphi_i(x)dx - \sum_{j=1}^{n} w_j^{(\delta)} \varphi_i(x_j^{(\delta)}) &= \int_a^b \omega(x)\varphi_i(x)dx - \int_{a+\delta}^b \omega(x)\varphi_i(x)dx \\ &= \int_a^{a+\delta} \omega(x)\varphi_i(x)dx. \end{aligned} \tag{98}$$

Due to the assumption that φ_i are integrable on the interval $[a,b]$ for all $= 1, 2, \cdots, 2n$, for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$,

$$\left| \int_a^{a+\delta} \omega(x)\varphi_i(x)dx \right| < \epsilon \tag{99}$$

for all $= 1, 2, \cdots, 2n$.

Now, (96) follows immediately from (98), (99). ■

Theorem 5.2 *Suppose that under the assumptions of Theorem 2.2 and Theorem 5.1,*

$$\varphi_1 \equiv 1. \tag{100}$$

Then there exists a unique n-point Gaussian quadrature (7) with respect to the functions

$$\varphi_1, \varphi_2, \cdots, \varphi_{2n} \tag{101}$$

such that all the nodes $x_1, x_2, \cdots, x_n$ lie in the open interval (a,b), and all the weights $w_1, w_2, \cdots, w_n$ are positive. Furthermore, for all $i = 1, 2, \cdots, n$,

$$\lim_{\delta \to 0} x_i^{(\delta)} = x_i \tag{102}$$

and

$$\lim_{\delta \to 0} w_i^{(\delta)} = w_i \tag{103}$$

where the nodes $x_i^{(\delta)}$ and the weights $w_i^{(\delta)}$ are defined in Theorem 5.1.

Proof: Due to Theorem 2.2, there exists a unique Gaussian quadrature (7) with respect to the functions (101) on the interval $[a,b]$ such that all the nodes $x_1, x_2, \cdots, x_n$ lie in the open interval (a,b), and all the weights $w_1, w_2, \cdots, w_n$ are positive.

On the other hand, due to Theorem 2.1, for any $\delta \in (0, b-a)$, there exists a unique Gaussian quadrature (7) with respect to the functions (101) on the interval $[a+\delta, b]$ such that the nodes $x_i^{(\delta)} \in (a+\delta, b)$, and the weights

$$w_i^{(\delta)} > 0 \tag{104}$$

for all $i = 1, 2, \cdots, n$. Combining (100) with Definition 2.4 of the Gaussian quadratures, we obtain

$$\int_{a+\delta}^{b} \omega(x)\varphi_1(x)dx = \sum_{i=1}^{n} w_i^{(\delta)}. \tag{105}$$

Now, for any $\delta \in (0, b-a)$, we will define two vectors $\mathrm{x}_\delta, \mathrm{w}_\delta \in R^n$ via the formulae

$$\mathrm{x}_\delta = (x_1^{(\delta)}, x_2^{(\delta)}, \cdots, x_n^{(\delta)})^T, \tag{106}$$
$$\mathrm{w}_\delta = (w_1^{(\delta)}, w_2^{(\delta)}, \cdots, w_n^{(\delta)})^T, \tag{107}$$

where $x_i^{(\delta)}$ and $w_i^{(\delta)}$ are the Gaussian nodes and weights with respect to functions (101) on the interval $[a+\delta, b]$.

Clearly, for any $\delta \in (0, b-a)$, we have

$$\|\mathrm{x}_\delta\| \le c \cdot \sqrt{n} \tag{108}$$

with $c = \max(|a|, |b|)$.

Combining (105) and (104), we obtain

$$\|\mathrm{w}_\delta\| \le d \cdot \sqrt{n} \tag{109}$$

for any $\delta \in (0, b-a)$, with d given by the formula

$$d = \int_a^b \omega(x)dx. \tag{110}$$

We will show that for all $\delta \in (0, b-a)$, there exists only one limit point for the set of vectors x_δ, and only one limit point for the set of vectors w_δ.

Suppose that there exists a sequence of positive real numbers $\delta_1, \delta_2, \cdots$, such that $\lim_{k\to\infty} \delta_k = 0$,

$$\lim_{k\to\infty} \mathrm{x}_{\delta_k} = \mathrm{y}, \tag{111}$$

$$\lim_{k\to\infty} \mathrm{w}_{\delta_k} = \mathrm{v}, \tag{112}$$

with $\mathrm{y} = (y_1, y_2, \cdots, y_n)^T$, and $\mathrm{v} = (v_1, v_2, \cdots, v_n)^T$.

Due to Theorem 5.1, Definition 2.4 of the Gaussian Quadratures, and conditions (111), (112), we have

$$\begin{aligned}\int_a^b \omega(x)\varphi_i(x)dx &= \lim_{k\to\infty} \int_{a+\delta_k}^b \omega(x)\varphi_i(x)dx \\ &= \lim_{k\to\infty} \sum_{j=1}^n w_j^{(\delta_k)} \varphi_i(x_j^{(\delta_k)}) \\ &= \sum_{j=1}^n v_j \varphi_i(y_j) \end{aligned} \tag{113}$$

for all $i = 1, 2, \cdots, 2n$.

Due to (113) and the uniqueness of the Gaussian quadrature (7) with respect to the functions (101) on the interval $[a, b]$, we have

$$y_j = x_j \tag{114}$$

$$v_j = w_j \tag{115}$$

for all $j = 1, 2, \cdots, n$, where $x_1, x_2, \cdots, x_n$ are the nodes and $w_1, w_2, \cdots, w_n$ are the weights of the Gaussian Quadrature (7) with respect to the functions (101) on $[a, b]$. In other words, the set of vectors x_δ for all $\delta \in (0, b-a)$ has a unique limit point

$$\mathrm{x}_0 = (x_1, x_2, \cdots, x_n)^T,$$

and the set of vectors $\mathbf{w}_\delta$ for all $\delta \in (0, b-a)$ has a unique limit point

$$\mathbf{w}_0 = (w_1, w_2, \cdots, w_n)^T.$$

Now, the formulae (102) and (103) follow from (108), (109), and the fact that each of the two sets of the vectors $\mathbf{x}_\delta$ and $\mathbf{w}_\delta$ possesses only one limit point in R^n. ■

Remark 5.1 *Clearly, the condition $\varphi_1 \equiv 1$ in Theorem 5.2 can be relaxed. To insure the boundedness of the Gaussian weights, we only need to impose the condition that there exists a real $\epsilon > 0$, and $2n$ real numbers α_1, α_2, $\cdots$, α_{2n} such that*

$$\sum_{i=1}^{2n} \alpha_i \cdot \varphi_i(x) \geq \epsilon \tag{116}$$

for all $x \in (a, b)$.

6 The Numerical Algorithm

We can now compute Gaussian quadratures for both smooth functions and functions with end-point singularities using the numerical apparatus developed in Sections 4 and 5. The Gaussian quadrature rules for an extended Hermite system can be obtained by solving a system of non-linear equations (53). Due to Theorems 4.5 and 5.2, the modified Newton's method defined by the formula (90) converges quadratically when it is applied to the system of equations (53).

As is well-known, the Newton method is sensitive to the choice of the initial approximation $\mathbf{x}_0$, and we use the continuation method (see Subsection 3.3 above) to obtain the latter. More specifically, given an extended Hermite system

$$\varphi_1, \varphi, \cdots, \varphi_{2n}, \tag{117}$$

on the interval $[a, b]$, we construct a family of extended Hermite systems

$$\varphi_1^t, \varphi_2^t, \cdots, \varphi_{2n}^t, \tag{118}$$

with $t \in [0, 1]$, and such that

$$\varphi_i^0(x) = x^{i-1}, \tag{119}$$

$$\varphi_i^1(x) = \varphi_i(x), \tag{120}$$

for all $x \in [a, b]$, and $i = 1, 2, \cdots .2n$. For each $t \in [0, 1]$, we construct the system of equations (53) corresponding to the extended Hermite system (118) via the formulae (39), (40). Clearly, for the extended Hermite system (119), the solution of the system of equations (53) is known (see Remark 2.2), and we use the continuation method (see Subsection 3.3) to obtain the solution of (53) for the Hermite system (117).

In the numerical examples of the following section, the one-parameter families of Hermite systems are constructed as follows.

1. For Hermite systems of the form

$$J_0, J_1, \cdots, J_{2n-1}, \tag{121}$$

the one-parameter family of systems is

$$\varphi_i^t(x) = (1-t)\cdot x^{i-1} + t\cdot \varphi_i(x) \tag{122}$$

for $i = 0, 1, \cdots, 2n-1$.

2. For Hermite systems of the form

$$1, \ln x, x, x\ln x, x^2, x^2\ln x, \cdots, x^{n-1}, x^{n-1}\ln x, \tag{123}$$

the one-parameter family of systems is given by (122).

3. For Hermite systems of the form

$$1, s(x), x, xs(x), x^2, x^2 s(x), \cdots, x^{n-1}, x^{n-1}s(x), \tag{124}$$

with

$$s(x) = x^{\alpha}, \tag{125}$$

and α an arbitrary non-integer real number, the one-parameter family of systems is

$$1, s(t,x), x, x\cdot s(t,x), x^2, x^2\cdot s(t,x), \cdots, x^{n-1}, x^{n-1}\cdot s(t,x), \tag{126}$$

with

$$s(t,x) = x^{t\cdot\alpha}. \tag{127}$$

Remark 6.1 *The necessary number m of steps in the continuation process (see Subsection 3.3) is significantly reduced if, prior to the application of the above procedure, the original system (117) is orthonormalied (for example, via the Gram-Schmidt process). To do that, we discretize the original functions φ_i at nested Chebyshev nodes (see Subsection 3.1), and perform the Gram-Schmidt procedure on the obtained finite-dimensional representations.*

The following is the formal description of the numerical algorithm (excluding the continuation process).

Initialization
Comment [Build the structure for integration and interpolation.]

Step 1

do
 Subdivide $[a, b]$ into subintervals clustering near end-points (see Figure 3.1).
enddo

Orthogonalization (optional)
Comment [Perform Gram-Schmidt orthogonalization on the given set of functions.]

Step 2

do $i = 1, 2, \cdots, 2n$
  do $j = 1, 2, \cdots, i-1$
    Orthogonalize the i-th function φ_i with respect to j-th function φ_j
  endo
enddo

Nested Chebyshev Approximation (optional)
Comment [Generate the nested Chebyshev expansions for the orthogonalized functions (see Subsection 3.1).]

Step 3

do $i = 1, 2, \cdots, 2n$
  Construct the local Chebyshev expansion of the i-th function φ_i based on Observation 3.1.
enddo

Newton's Iteration
Comment [Conduct Newton's iteration to find Gaussian nodes and weights.]

Step 4

do
  Construct functions σ_i and η_i for $i = 1, 2, \cdots, n$ via formulae (39) and (40).
enddo

Step 5

do
  Adjust Gaussian nodes $\{x_i\}$ via formulae (90).
enddo

Step 6

do
  Compute $error = ||\mathrm{x}_{k+1} - \mathrm{x}_k||$.
  If $error > \epsilon$, Go to Step 4.
enddo

Remark 6.2 *The procedure described above requires the construction of the functions* σ_i, η_i, $i = 1, 2, \cdots, n$, *given the functions* φ_i, $i = 1, 2, \cdots, 2n$. *The latter is possible for any extended Hermite systems, and is equivalent to inverting the matrix (3). Obviously, for many choices of functions* $\varphi_1, \varphi_2, \cdots, \varphi_{2n}$, *the matrix (3) will be ill-conditioned, including the numerical examples given in the following section. Thus, in order to obtain the double precision results presented in this paper, the authors have performed all computations in extended precision (REAL *32).*

7 Numerical Results

We have implemented the numerical algorithm described in Section 6 for the computation of Gaussian quadrature formulae, and tested it on various examples.

Example 7.1 *Gaussian Quadratures with respect to the Bessel Functions*

$$J_0, J_1, \cdots, J_{2n-1} \tag{128}$$

on $[0, 10]$ *are given in Table 1, and tested on selected functions in Table 12.*

Example 7.2 *Gaussian Quadratures with respect to the Bessel Functions (128) on* $[0, 10]$ *with the weight function*

$$\omega(x) = \frac{1}{\sqrt{x}} \tag{129}$$

are given in Table 2, and tested on selected functions in Table 13.

Example 7.3 *Gaussian Quadratures with respect to the system of functions*

$$1, \ln x, x, x \ln x, x^2, x^2 \ln x, \cdots, x^{n-1}, x^{n-1} \ln x \tag{130}$$

on $[0, 1]$ *are given in Table 3, and tested on selected functions in Table 14.*

Example 7.4 *Gaussian Quadratures with respect to the systems of functions*

$$1, x^{\alpha}, x, x^{1+\alpha}, x^2, x^{2+\alpha}, \cdots, x^{n-1}, x^{n-1+\alpha} \tag{131}$$

on $[0, 1]$ *are given respectively in Tables 4-11 for*

$$\alpha = \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3},$$

and tested on selected functions in Tables 15-22 respectively.

Remark 7.1 *Systems of the form (128) are often encountered in physics. It turns out that the system of functions (128) on the interval* $[0, B]$ *is an extended Hermite system only for certain combinations of* B *and* n. *A somewhat subtle analysis shows that the system (128) is an extended Hermite system on the interval* $[0, B]$ *as long as* $\frac{B}{2} \leq n \leq B$.

8 Conclusions

A numerical algorithm has been presented for the construction of the generalized Gaussian quadrature rules, introduced in [6]. The quadrature rules of this paper possess most of the desirable properties of the classical Gaussian integration formulae, such as positivity of the weights, rapid convergence, mathematical elegance, etc. The algorithm is applicable to a wide class of functions, including smooth functions (not necessarily polynomials), as well as functions with end-point singularities, such as those encountered in the solution of integral equations, complex analysis, potential theory, and several other areas.

References

[1] B. Alpert, *Rapidly-Convergent Quadrature for Integral Operators with Singular Kernels* Lawrence Berkeley Laboratory, University of California at Berkeley, December 1990.

[2] P. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, Inc., Orlando, Florida, 1984.

[3] J. E. Dennis, and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1983.

[4] F. A. Ficken, The Continuation Method for Functional Equations, *Comm. Pure Appl. Math.* 4(1951), pp435-456.

[5] D. Gottlieb, and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, Regional Conference Series in Applied Mathematics, SIAM, 1977.

[6] S. Karlin and W. Studden, *Tchebycheff Systems with Applications in Analysis and Statistics*, John Wiley (Interscience), New York, 1966.

[7] V. Rokhlin, End-Point Corrected Trapezoidal Quadrature Rules for Singular Functions, *Computers Math. Applic.* Vol.20, No.7, pp.51-62, 1990.

[8] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1983.

[9] A. H. Stroud and D. Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1966.

Table 1: Gaussian Quadrature for Bessel Functions of the First Kind

$$\int_0^{10} J_{k-1}(x)dx = \sum_{i=1}^{N} w_i J_{k-1}(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

N	Nodes x_i	Weights w_i
5	0.469238675868960E+00	0.117179089779279E+01
	0.223157952970870E+01	0.224136849121358E+01
	0.473407702933183E+01	0.266214797121592E+01
	0.735478272434508E+01	0.247001516625585E+01
	0.940238197915203E+01	0.145415103193520E+01
10	0.130535696170244E+00	0.333260223918652E+00
	0.672886387019932E+00	0.742501752317741E+00
	0.159143208236292E+01	0.108005279782693E+01
	0.280041847052746E+01	0.132028812483455E+01
	0.419499640127942E+01	0.144971022336955E+01
	0.566066246651666E+01	0.146169692583314E+01
	0.707810341253441E+01	0.135292016838366E+01
	0.832621870954264E+01	0.112355472887245E+01
	0.928767980905348E+01	0.782143625547308E+00
	0.986125239933237E+01	0.353871429096005E+00

Table 2: Gaussian Quadrature for Bessel Functions with Weight Function

$$\int_0^{10} \frac{1}{\sqrt{x}} J_{k-1}(x)dx = \sum_{i=1}^{N} w_i J_{k-1}(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

N	Nodes x_i	Weights w_i
5	0.221525014168133E+00	0.185798994260858E+01
	0.181515943791217E+01	0.162483079218151E+01
	0.434757211490782E+01	0.132302463248861E+01
	0.710741692330460E+01	0.990971542854905E+00
	0.933190129335545E+01	0.527439013034203E+00
10	0.586135127137856E-01	0.966145917958189E+00
	0.517839232816169E+00	0.939662007448660E+00
	0.138739079186343E+01	0.889891823703254E+00
	0.258047692418516E+01	0.821395494863014E+00
	0.398718764645530E+01	0.737970249083253E+00
	0.548586592652336E+01	0.641493179529387E+00
	0.694907018937755E+01	0.531905946105306E+00
	0.824679794056492E+01	0.408087704961759E+00
	0.925206018774378E+01	0.269503595251694E+00
	0.985409749058602E+01	0.118499400771494E+00

Table 3: Gaussian Quadrature for Products of Polynomials and Logarithmic function

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

$$\text{where } \{\varphi_i\} = \{1, \ln x, x, x \ln x, \cdots, x^{N-1}, x^{N-1} \ln x\}$$

N	Nodes x_i	Weights w_i
5	0.565222820508010E-02	0.210469457918546E-01
	0.734303717426523E-01	0.130705540744447E+00
	0.284957404462558E+00	0.289702301671314E+00
	0.619482264084778E+00	0.350220370120399E+00
	0.915758083004698E+00	0.208324841671986E+00
10	0.482961710689630E-03	0.183340007378985E-02
	0.698862921431577E-02	0.134531223459918E-01
	0.326113965946776E-01	0.404971943169583E-01
	0.928257573891660E-01	0.818223696589036E-01
	0.198327256895404E+00	0.129192342770138E+00
	0.348880142979353E+00	0.169545319547259E+00
	0.530440555787956E+00	0.189100216532996E+00
	0.716764648511655E+00	0.177965753961471E+00
	0.875234557506234E+00	0.133724770615462E+00
	0.975245698684393E+00	0.628655101770325E-01
15	0.105784548458629E-03	0.403217724648460E-03
	0.156624383616782E-02	0.306297843478700E-02
	0.759521890320709E-02	0.978421211876615E-02
	0.228310673939862E-01	0.215587522255813E-01
	0.523886301568200E-01	0.383230673708892E-01
	0.100758685201213E+00	0.588981990263004E-01
	0.170740768849943E+00	0.811170299392595E-01
	0.262591206118993E+00	0.102122101972069E+00
	0.373536505184558E+00	0.118789059030401E+00
	0.497746358414533E+00	0.128210316446694E+00
	0.626789031392373E+00	0.128163327417093E+00
	0.750516103461408E+00	0.117489465888492E+00
	0.858255335207861E+00	0.963230185695904E-01
	0.940141291212346E+00	0.661345398318934E-01
	0.988401595986342E+00	0.296207140035355E-01

Table 3: Gaussian Quadrature for Products of Polynomials and Logarithmic function

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for} \quad k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, \ln x, x, x\ln x, \cdots, x^{N-1}, x^{N-1}\ln x\}$

N	Nodes x_i	Weights w_i
20	0.352330453033401E-04	0.134499676467758E-03
	0.526093982517410E-03	0.103477692295062E-02
	0.258751954058141E-02	0.337726367723322E-02
	0.793447194838041E-02	0.767355619359468E-02
	0.186828881374457E-01	0.142054962855420E-01
	0.370976733697505E-01	0.229844384632086E-01
	0.653124886740214E-01	0.337363605577136E-01
	0.105048504711551E+00	0.459147630734522E-01
	0.157359691819002E+00	0.587404799428040E-01
	0.222430062767455E+00	0.712650131611020E-01
	0.299443765654100E+00	0.824518089775832E-01
	0.386542446943882E+00	0.912682015163873E-01
	0.480876453826790E+00	0.967797159091613E-01
	0.578747932205507E+00	0.982381433400897E-01
	0.675835475840038E+00	0.951553030540297E-01
	0.767482460872564E+00	0.873556504104574E-01
	0.849025253970320E+00	0.750027772122717E-01
	0.916133703241664E+00	0.585972958082337E-01
	0.965135427900256E+00	0.389472505496114E-01
	0.993303536456954E+00	0.171372052681059E-01

Table 3: Gaussian Quadrature for Products of Polynomials and Logarithmic function

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, \ln x, x, x\ln x, \cdots, x^{N-1}, x^{N-1}\ln x\}$

N	Nodes x_i	Weights w_i
25	0.148805205646328E-04	0.568460660250201E-04
	0.223091159576411E-03	0.439997585768285E-03
	0.110464364905333E-02	0.145071890475698E-02
	0.341946946887918E-02	0.334401873816821E-02
	0.815052929502022E-02	0.630809954735095E-02
	0.164289374947747E-01	0.104488723103430E-01
	0.294459835598307E-01	0.157795036631243E-01
	0.483575697078870E-01	0.222157908473636E-01
	0.741870939196732E-01	0.295777024140889E-01
	0.107732955883456E+00	0.375970456071727E-01
	0.149486638258006E+00	0.459308515949728E-01
	0.199566730959199E+00	0.541797236656935E-01
	0.257673355831231E+00	0.619100915223039E-01
	0.323066266406625E+00	0.686790748928476E-01
	0.394568512069093E+00	0.740604961651057E-01
	0.470596049553319E+00	0.776705045127697E-01
	0.549212146433099E+00	0.791912877674640E-01
	0.628203942243359E+00	0.783914525088262E-01
	0.705177201069257E+00	0.751418416138657E-01
	0.777664184415768E+00	0.694258212157761E-01
	0.843238762138540E+00	0.613433919048328E-01
	0.899632416106158E+00	0.511088512980136E-01
	0.944844733405702E+00	0.390421895640177E-01
	0.977242575226688E+00	0.255554713626385E-01
	0.995647215456440E+00	0.111503547267104E-01

Table 3: Gaussian Quadrature for Products of Polynomials and Logarithmic function

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for} \quad k = 1, 2, \cdots, 2N$$

$$\text{where } \{\varphi_i\} = \{1, \ln x, x, x \ln x, \cdots, x^{N-1}, x^{N-1} \ln x\}$$

N	Nodes x_i	Weights w_i
30	0.732379743551900E-05	0.279892154036191E-04
	0.110044700353982E-03	0.217365526303388E-03
	0.546918325703179E-03	0.720703585941237E-03
	0.170185751774368E-02	0.167446096386701E-02
	0.408386360682336E-02	0.319128240452640E-02
	0.830004117175941E-02	0.535378831096153E-02
	0.150229781480879E-01	0.820962136493378E-02
	0.249539236043865E-01	0.117680292095091E-01
	0.387833861559789E-01	0.159981435010663E-01
	0.571508984622219E-01	0.208290410897408E-01
	0.806057414498694E-01	0.261515976575626E-01
	0.109570394208134E+00	0.318220682660372E-01
	0.144308372971936E+00	0.376672559968915E-01
	0.184897949395368E+00	0.434910622609641E-01
	0.231213001514211E+00	0.490821532269183E-01
	0.282911960162101E+00	0.542224286606307E-01
	0.339435481155556E+00	0.586959442912318E-01
	0.400013113074849E+00	0.622979181160659E-01
	0.463678856906249E+00	0.648434457091797E-01
	0.529295142710295E+00	0.661755598539193E-01
	0.595584395297873E+00	0.661722952804487E-01
	0.661167040372616E+00	0.647524589265887E-01
	0.724604528155535E+00	0.618798583538578E-01
	0.784445734718383E+00	0.575658036673855E-01
	0.839274951465986E+00	0.518697691962532E-01
	0.887759597608297E+00	0.448981784990110E-01
	0.928695795957382E+00	0.368013625032992E-01
	0.961050059184175E+00	0.277688703796847E-01
	0.983995703519948E+00	0.180238737856617E-01
	0.996945958679506E+00	0.782767019615502E-02

Table 4: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{2}{3}}$

N	Nodes x_i	Weights w_i
5	0.111388121461113E-01	0.350341916241438E-01
	0.989954782999841E-01	0.152986023564027E+00
	0.325462965706881E+00	0.293439234461264E+00
	0.650376177175503E+00	0.329423482895757E+00
	0.923830141383311E+00	0.189117067454808E+00
10	0.106075936690850E-02	0.342465634725548E-02
	0.105957835374351E-01	0.179697406786380E-01
	0.419569285287569E-01	0.471712377689652E-01
	0.109001951813403E+00	0.883592422887387E-01
	0.219861071532861E+00	0.132980363859702E+00
	0.372071233678686E+00	0.168929398195588E+00
	0.550779272731362E+00	0.184191248381816E+00
	0.730779332479715E+00	0.170661619172872E+00
	0.881954366270197E+00	0.126951038006216E+00
	0.976639980362546E+00	0.593614553002092E-01
15	0.241818436310427E-03	0.785216839155443E-03
	0.247904422657297E-02	0.428667428787457E-02
	0.102231505109504E-01	0.119847435540932E-01
	0.280798632789153E-01	0.245432960884266E-01
	0.608339281242860E-01	0.416404983681140E-01
	0.112443594490995E+00	0.619571219761760E-01
	0.185105777705090E+00	0.833277026033653E-01
	0.278545764961840E+00	0.103032984392016E+00
	0.389652905586201E+00	0.118190780445325E+00
	0.512532528698514E+00	0.126187019062355E+00
	0.638982003700951E+00	0.125082141394901E+00
	0.759334800695944E+00	0.113931212165169E+00
	0.863560355211660E+00	0.929687707138682E-01
	0.942468575671709E+00	0.636311469832558E-01
	0.988861955722520E+00	0.284506911259040E-01

Table 4: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{2}{3}}$

N	Nodes x_i	Weights w_i
20	0.822258720897910E-04	0.267576170194858E-03
	0.851231173182477E-03	0.148254407742955E-02
	0.356411730804281E-02	0.424221866790033E-02
	0.999446520349904E-02	0.897068317712191E-02
	0.222308791404550E-01	0.158666804741713E-01
	0.424339835363049E-01	0.248722570057384E-01
	0.725741080670503E-01	0.356656880963368E-01
	0.114174415866984E+00	0.476793965125403E-01
	0.168083972212092E+00	0.601415054233364E-01
	0.234302815001518E+00	0.721375449972604E-01
	0.311876663295677E+00	0.826870082942195E-01
	0.398872872080255E+00	0.908280936995649E-01
	0.492442140014574E+00	0.957031990037015E-01
	0.588962927640006E+00	0.966376154468787E-01
	0.684258212417279E+00	0.932044202436777E-01
	0.773867742847468E+00	0.852697379326592E-01
	0.853353935948184E+00	0.730142442020013E-01
	0.918616478895035E+00	0.569289535722086E-01
	0.966190178403627E+00	0.377866171625958E-01
	0.993508545383756E+00	0.166140158404627E-01

Table 5: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{3}}$

N	Nodes x_i	Weights w_i
5	0.970916313338209E-02	0.314958290433846E-01
	0.927420088040289E-01	0.147817740145233E+00
	0.315872313916462E+00	0.292773974169340E+00
	0.643182477910772E+00	0.334349276188739E+00
	0.921965110615521E+00	0.193563180453303E+00
10	0.901742772555592E-03	0.299828120481279E-02
	0.966072992118868E-02	0.168386395659664E-01
	0.396093898716370E-01	0.455491829065261E-01
	0.105011991918026E+00	0.868038128143013E-01
	0.214610971190650E+00	0.132106151126701E+00
	0.366460914978464E+00	0.169114219381655E+00
	0.545885024355929E+00	0.185393787355447E+00
	0.727418879329945E+00	0.172422600578352E+00
	0.880346704943949E+00	0.128574309018165E+00
	0.976306802645093E+00	0.601990160480740E-01
15	0.203617338486320E-03	0.680684768275793E-03
	0.223725122307619E-02	0.397246938629008E-02
	0.954788886412147E-02	0.114344447894424E-01
	0.267548306484035E-01	0.238083807101952E-01
	0.587263787087472E-01	0.408326185452397E-01
	0.109550941121055E+00	0.612195391729371E-01
	0.181570447192808E+00	0.828014703688452E-01
	0.274636464058843E+00	0.102824724623773E+00
	0.385717152459535E+00	0.118348384119426E+00
	0.508930876164803E+00	0.126687910323023E+00
	0.636017805747573E+00	0.125836213297194E+00
	0.757194044089115E+00	0.114797651013883E+00
	0.862273903642310E+00	0.937832798881199E-01
	0.941904608637681E+00	0.642380897499318E-01
	0.988750439427061E+00	0.287341392434250E-01

Table 5: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i\varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{3}}$

N	Nodes x_i	Weights w_i
20	0.688907338392845E-04	0.230763814351417E-03
	0.764138398652809E-03	0.136612455217214E-02
	0.331021381556300E-02	0.402291522792898E-02
	0.946809336603785E-02	0.864643822538178E-02
	0.213344721137020E-01	0.154553358601049E-01
	0.410964086262251E-01	0.244080922533623E-01
	0.707645876791935E-01	0.351941337584140E-01
	0.111910522350377E+00	0.472505923470994E-01
	0.165432926252467E+00	0.598034836852591E-01
	0.231376114514019E+00	0.719298383077259E-01
	0.308818842285795E+00	0.826354136333313E-01
	0.395845853335002E+00	0.909411778827062E-01
	0.489607131209166E+00	0.959709232198866E-01
	0.586462066987943E+00	0.970322101197066E-01
	0.682198172145692E+00	0.936833837251139E-01
	0.772307249294583E+00	0.857805916759546E-01
	0.852296694398426E+00	0.735004796732940E-01
	0.918010354508321E+00	0.573364747582598E-01
	0.965932757966444E+00	0.380699389274024E-01
	0.993458519536807E+00	0.167416883525447E-01

Table 6: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for} \quad k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{3}}$

N	Nodes x_i	Weights w_i
5	0.831174531456776E-02	0.279778782123048E-01
	0.863966362795308E-01	0.142398935990482E+00
	0.305943516943443E+00	0.291943668689807E+00
	0.635656558720652E+00	0.339447240627363E+00
	0.920004207024857E+00	0.198232276480043E+00
10	0.751897878625465E-03	0.259000938099737E-02
	0.874666094268371E-02	0.157085175605557E-01
	0.372684363855664E-01	0.438972012175236E-01
	0.100986759928202E+00	0.851979518885884E-01
	0.209274931334610E+00	0.131185462315937E+00
	0.360730157291068E+00	0.169279428891491E+00
	0.540868549812902E+00	0.186612247123971E+00
	0.723966527445448E+00	0.174225330076043E+00
	0.878692655294032E+00	0.130242598410538E+00
	0.975963746745065E+00	0.610612531343552E-01
15	0.168121349224655E-03	0.582014429388243E-03
	0.200427399422478E-02	0.366362217710688E-02
	0.888445820257933E-02	0.108842247988913E-01
	0.254380748331252E-01	0.230661756216959E-01
	0.566163366479501E-01	0.400108265603416E-01
	0.106639742234206E+00	0.604643908138577E-01
	0.177998880856086E+00	0.822581407444590E-01
	0.270675783247184E+00	0.102603879506980E+00
	0.381720959785030E+00	0.118500791181315E+00
	0.505267785653412E+00	0.127191992481750E+00
	0.632999186646096E+00	0.126600784497532E+00
	0.755011893747991E+00	0.115679059680403E+00
	0.860961671393287E+00	0.946133646757434E-01
	0.941329076190074E+00	0.648572778769515E-01
	0.988636608057791E+00	0.290234549535849E-01

Table 6: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{3}}$

N	Nodes x_i	Weights w_i
20	0.565910606963981E-04	0.196275471368437E-03
	0.680854395380959E-03	0.125263851254803E-02
	0.306270110956997E-02	0.380561564887333E-02
	0.894917200513315E-02	0.832217213031223E-02
	0.204442383726108E-01	0.150414196754775E-01
	0.397611592475325E-01	0.239388575134562E-01
	0.689512869857525E-01	0.347155654274889E-01
	0.109635234283660E+00	0.468137478429227E-01
	0.162762389006638E+00	0.594574979240510E-01
	0.228422459407150E+00	0.717153419416687E-01
	0.305728258549570E+00	0.825791227715473E-01
	0.392782689780531E+00	0.910522631915573E-01
	0.486735430814252E+00	0.962395518442346E-01
	0.583926798874029E+00	0.974303994336976E-01
	0.680108431067843E+00	0.941680397096348E-01
	0.770723438587922E+00	0.862983560861100E-01
	0.851223225892569E+00	0.739938065555806E-01
	0.917394745239490E+00	0.577502214287376E-01
	0.965671257181204E+00	0.383577114718547E-01
	0.993407695194012E+00	0.168713954188783E-01

Table 7: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{4}}$

N	Nodes x_i	Weights w_i
5	0.762794972812507E-02	0.262298035906717E-01
	0.831895488095584E-01	0.139587984773339E+00
	0.300844037453126E+00	0.291460348074849E+00
	0.631759839813656E+00	0.342065172592006E+00
	0.918985115815450E+00	0.200656690969134E+00
10	0.680661832514276E-03	0.239309679083200E-02
	0.829805126031941E-02	0.151439816123742E-01
	0.361008488950617E-01	0.430595363165574E-01
	0.989606158324035E-01	0.843751810440922E-01
	0.206573413000166E+00	0.130706769333825E+00
	0.357817634747240E+00	0.169354243901417E+00
	0.538312396928998E+00	0.187227677434183E+00
	0.722204307474990E+00	0.175143061430482E+00
	0.877847435011894E+00	0.131094401625182E+00
	0.975788344331792E+00	0.615020505110549E-01
15	0.151429644289982E-03	0.534953080073061E-03
	0.189120816621145E-02	0.351128622129104E-02
	0.855735146816403E-02	0.106091654389163E-01
	0.247829453316383E-01	0.226922601463664E-01
	0.555603949382247E-01	0.395945355599134E-01
	0.105176990238943E+00	0.600799922753622E-01
	0.176199068882164E+00	0.819798274477196E-01
	0.268675523469233E+00	0.102488560198090E+00
	0.379699417736514E+00	0.118574970046103E+00
	0.503412401869408E+00	0.127445268951420E+00
	0.631468760396525E+00	0.126987144614300E+00
	0.753904757831583E+00	0.116125573735615E+00
	0.860295553378594E+00	0.950344508261170E-01
	0.941036823465374E+00	0.651716225755041E-01
	0.988578794355859E+00	0.291703888832093E-01

Table 7: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{\frac{1}{4}}$

N	Nodes x_i	Weights w_i
20	0.508401144727475E-04	0.179921229090492E-03
	0.640671409304506E-03	0.119702077067209E-02
	0.294139627824379E-02	0.369773784996860E-02
	0.869257229367654E-02	0.816003283674897E-02
	0.200014921598775E-01	0.148334720914000E-01
	0.390944230688757E-01	0.237022859923791E-01
	0.680431707412333E-01	0.344735758559312E-01
	0.108493186848796E+00	0.465922232807169E-01
	0.161419592745879E+00	0.592814319750155E-01
	0.226935224532033E+00	0.716054742949001E-01
	0.304170319054039E+00	0.825491665123296E-01
	0.391237157031920E+00	0.911070366015320E-01
	0.485285420936866E+00	0.963742175415786E-01
	0.582645888835788E+00	0.976308832970205E-01
	0.679052102488331E+00	0.944125656226918E-01
	0.769922539292854E+00	0.865599052197165E-01
	0.850680233407198E+00	0.742432057258809E-01
	0.917083282994612E+00	0.579594958287601E-01
	0.965538933432666E+00	0.385033140388101E-01
	0.993381975142296E+00	0.169370334348568E-01

Table 8: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{4}}$

N	Nodes x_i	Weights w_i
5	0.382423605041850E-02	0.159917283936684E-01
	0.634597428923166E-01	0.121052296479220E+00
	0.268079430001204E+00	0.287406177837817E+00
	0.606207314918122E+00	0.358886538601875E+00
	0.912243119495094E+00	0.216663258687420E+00
10	0.311664421263886E-03	0.132408316145357E-02
	0.573840166641843E-02	0.117668861542949E-01
	0.291436456086289E-01	0.378545434895510E-01
	0.865989640195454E-01	0.791317874471543E-01
	0.189850316327027E+00	0.127550269062920E+00
	0.339616983180453E+00	0.169682299097476E+00
	0.522237733374721E+00	0.191015829563274E+00
	0.711076275414628E+00	0.180901690009248E+00
	0.872496278362019E+00	0.136476978227289E+00
	0.974676373102776E+00	0.642956337873400E-01
15	0.671537172803988E-04	0.286236608068052E-03
	0.126348368805257E-02	0.262818581079834E-02
	0.666285677053233E-02	0.895931713383214E-02
	0.208996392447129E-01	0.204062530697822E-01
	0.492092869824728E-01	0.370154044724167E-01
	0.962910504069878E-01	0.576707391266183E-01
	0.165187187294757E+00	0.802098718885372E-01
	0.256372388322078E+00	0.101723142015267E+00
	0.367215812312105E+00	0.118989956384134E+00
	0.491920272820932E+00	0.128983991417795E+00
	0.621967797979428E+00	0.129367050472822E+00
	0.747019986367819E+00	0.118892374347340E+00
	0.856148272384472E+00	0.976520449023468E-01
	0.939215799595110E+00	0.671291997815284E-01
	0.988218403767777E+00	0.300862325687140E-01

Table 8: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{4}}$

N	Nodes x_i	Weights w_i
20	0.221820235538413E-04	0.946625535057872E-04
	0.420668156388107E-03	0.879580435485498E-03
	0.224897834059227E-02	0.306154371288216E-02
	0.719410590426075E-02	0.718677600572896E-02
	0.173785665899570E-01	0.135708222783955E-01
	0.351053065795165E-01	0.222536266859334E-01
	0.625706338467844E-01	0.329813226328586E-01
	0.101573342538239E+00	0.452169532360338E-01
	0.153248856878474E+00	0.581794488048750E-01
	0.217855193592975E+00	0.709075209058289E-01
	0.294632990834409E+00	0.823427811731510E-01
	0.381755192469280E+00	0.914245608123939E-01
	0.476373817974157E+00	0.971877635212259E-01
	0.574762338493956E+00	0.988547376185605E-01
	0.672543338723645E+00	0.959126372029696E-01
	0.764983189265488E+00	0.881690173465919E-01
	0.847329128952469E+00	0.757803722977461E-01
	0.915160094733383E+00	0.592508853437043E-01
	0.964721591676480E+00	0.394024605588481E-01
	0.993223076818039E+00	0.173425268732812E-01

Table 9: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{3}}$

N	Nodes x_i	Weights w_i
5	0.325694880051994E-02	0.143436825038591E-01
	0.600861836255002E-01	0.117635462047981E+00
	0.262200689766926E+00	0.286500581347155E+00
	0.601527262010120E+00	0.361906424437754E+00
	0.910998040648555E+00	0.219613849663250E+00
10	0.261010670878138E-03	0.116634009622830E-02
	0.533570406879994E-02	0.112056554482295E-01
	0.279923598777842E-01	0.369538395786576E-01
	0.845001067179312E-01	0.782012224225048E-01
	0.186967540598809E+00	0.126971878207205E+00
	0.336449186747933E+00	0.169715037099809E+00
	0.519422386872603E+00	0.191665171997203E+00
	0.709119391752190E+00	0.181908136234217E+00
	0.871552940481438E+00	0.137424134774237E+00
	0.974480099088773E+00	0.647885841417101E-01
15	0.559212686135403E-04	0.250636999997100E-03
	0.116764741757783E-02	0.248629682453753E-02
	0.635876913141328E-02	0.868418049215750E-02
	0.202601345786029E-01	0.200174031540757E-01
	0.481469972090806E-01	0.365707842279019E-01
	0.947893055727041E-01	0.572506624316614E-01
	0.163312504831184E+00	0.798969454587593E-01
	0.254266678317515E+00	0.101582504459800E+00
	0.365070714120304E+00	0.119053915578673E+00
	0.489939651507906E+00	0.129244090759107E+00
	0.620326692664507E+00	0.129774994114533E+00
	0.745828824368843E+00	0.119369417740098E+00
	0.855429899278146E+00	0.981047682668204E-01
	0.938900129707038E+00	0.674683579975495E-01
	0.988155905432853E+00	0.302450414943277E-01

Table 9: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{3}}$

N	Nodes x_i	Weights w_i
20	0.184194907598179E-04	0.826457102725699E-04
	0.387587458269692E-03	0.829421040741860E-03
	0.213955259577947E-02	0.295732155968210E-02
	0.695120073310895E-02	0.702434283841401E-02
	0.169467664404234E-01	0.133575962599994E-01
	0.344417452000176E-01	0.220068985356573E-01
	0.616535209242469E-01	0.327254003368527E-01
	0.100407215362434E+00	0.449795404557805E-01
	0.151866019233245E+00	0.579877214786053E-01
	0.216313281828314E+00	0.707843838119479E-01
	0.293009074582624E+00	0.823037488978054E-01
	0.380137222473773E+00	0.914756213036679E-01
	0.474850532149502E+00	0.973244581548574E-01
	0.573412898781505E+00	0.990625442531322E-01
	0.671427980742799E+00	0.961685890414821E-01
	0.764136026657879E+00	0.884443478050775E-01
	0.846753983774620E+00	0.760438602249420E-01
	0.914829857705509E+00	0.594724986507429E-01
	0.964581196697193E+00	0.395568714306716E-01
	0.993195777887590E+00	0.174121882096675E-01

Table 10: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i\varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{2}}$

N	Nodes x_i	Weights w_i
5	0.220055532702321E-02	0.111142584286221E-01
	0.532526444285811E-01	0.110450910249386E+00
	0.250000000000000E+00	0.284444444444444E+00
	0.591721954534264E+00	0.368177760249980E+00
	0.908380401265687E+00	0.225812626627567E+00
10	0.170217313506295E-03	0.869843410720295E-03
	0.455197375232787E-02	0.100832309490842E-01
	0.256945562245545E-01	0.351184957693976E-01
	0.802601948484878E-01	0.762838816843756E-01
	0.181103722710989E+00	0.125764125553643E+00
	0.329978061692620E+00	0.169760099161110E+00
	0.513655588977735E+00	0.192982837625621E+00
	0.705104124523579E+00	0.183967866746584E+00
	0.869615340441312E+00	0.139368118201496E+00
	0.974076745830678E+00	0.658015008979678E-01
15	0.360449058720925E-04	0.184634499540016E-03
	0.983656825228959E-03	0.220691172454993E-02
	0.576031032998391E-02	0.813303209689366E-02
	0.189863966971689E-01	0.192316020292443E-01
	0.460162191690594E-01	0.356670580668843E-01
	0.917631475619828E-01	0.563926955430610E-01
	0.159522718866145E+00	0.792541212080791E-01
	0.250000000000000E+00	0.101289120962781E+00
	0.360716812863579E+00	0.119177364119033E+00
	0.485914494639546E+00	0.129768304472501E+00
	0.616988391777598E+00	0.130602147750110E+00
	0.743404128057339E+00	0.120339075896910E+00
	0.853966893740411E+00	0.990261883702783E-01
	0.938257049225935E+00	0.681591357635582E-01
	0.988028562926358E+00	0.305686074965773E-01

Table 10: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i\varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{1}{2}}$

N	Nodes x_i	Weights w_i
20	0.118040372897702E-04	0.605164515048587E-04
	0.324505506016988E-03	0.731395632734516E-03
	0.192569889609293E-02	0.275022407735182E-02
	0.647083718891321E-02	0.669890718081941E-02
	0.160868754200355E-01	0.129282095841639E-01
	0.331142308281794E-01	0.215082324328235E-01
	0.598127724451431E-01	0.322066292668297E-01
	0.980610158280346E-01	0.444969640416654E-01
	0.149078672924258E+00	0.575967453908003E-01
	0.213200816542450E+00	0.705318509111265E-01
	0.289727337675948E+00	0.822215362195993E-01
	0.376864524065904E+00	0.915762410818034E-01
	0.471767104543454E+00	0.975991452767166E-01
	0.570679774395970E+00	0.994820091823469E-01
	0.669167911554694E+00	0.966862995286949E-01
	0.762418781880186E+00	0.890019102330765E-01
	0.845587809011132E+00	0.765778343958853E-01
	0.914160127147419E+00	0.599218242567572E-01
	0.964296432783931E+00	0.398700341676524E-01
	0.993140403222385E+00	0.175534906876473E-01

Table 11: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for } k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{2}{3}}$

N	Nodes x_i	Weights w_i
5	0.127054140407448E-02	0.798095300782248E-02
	0.462681413226638E-01	0.102702832369157E+00
	0.237083298080667E+00	0.282009089112635E+00
	0.581208702007414E+00	0.374825379887486E+00
	0.905561736397846E+00	0.232481745622901E+00
10	0.945849186124229E-04	0.599207947722056E-03
	0.379603281250484E-02	0.895661938070323E-02
	0.233952614888275E-01	0.332290268550819E-01
	0.759462672091217E-01	0.742813877203913E-01
	0.175082164491838E+00	0.124481407759041E+00
	0.323295565014521E+00	0.169777375189087E+00
	0.507673537823808E+00	0.194331900792034E+00
	0.700934090961697E+00	0.186099994751647E+00
	0.867600470065010E+00	0.141387747529539E+00
	0.973657038209960E+00	0.668553320747537E-01
15	0.197855962731515E-04	0.125556576654582E-03
	0.809581023980278E-03	0.193257660045135E-02
	0.517321860359321E-02	0.757902277249522E-02
	0.177160194789544E-01	0.184325783933937E-01
	0.438708712542835E-01	0.347412411366044E-01
	0.886977166657579E-01	0.555083582856597E-01
	0.155667671624679E+00	0.785867325255755E-01
	0.245646859906903E+00	0.100978745423403E+00
	0.356264972533447E+00	0.119295279382622E+00
	0.481792161982642E+00	0.130299495385019E+00
	0.613565444840732E+00	0.131446824433812E+00
	0.740915815910203E+00	0.121332382739473E+00
	0.852464602157936E+00	0.999716176822775E-01
	0.937596443363357E+00	0.688685445138014E-01
	0.987897722986291E+00	0.309010441487584E-01

Table 11: Gaussian Quadrature for Products of Polynomials and Fractional Powers

$$\int_0^1 \varphi_k(x)dx = \sum_{i=1}^{N} w_i \varphi_k(x_i) \quad \text{for} \quad k = 1, 2, \cdots, 2N$$

where $\{\varphi_i\} = \{1, s(x), x, xs(x), \cdots, x^{N-1}, x^{N-1}s(x)\}$ with $s(x) = x^{-\frac{2}{3}}$

N	Nodes x_i	Weights w_i
20	0.644080669471020E-05	0.408973476604909E-04
	0.265385497209493E-03	0.636150271129477E-03
	0.171795135681289E-02	0.254437715881698E-02
	0.599650565966228E-02	0.637191713688041E-02
	0.152297986688364E-01	0.124939147170132E-01
	0.317830016969808E-01	0.210015095348042E-01
	0.579590115876318E-01	0.316775070055125E-01
	0.956908382982334E-01	0.440030511092655E-01
	0.146256150019543E+00	0.571949651553830E-01
	0.210043256963358E+00	0.702705143585106E-01
	0.286393214270049E+00	0.821337066503373E-01
	0.373535741616497E+00	0.916751595069523E-01
	0.468627945387644E+00	0.978762191044640E-01
	0.567895206083029E+00	0.999075512797006E-01
	0.666863963449413E+00	0.972128770925840E-01
	0.760667402163021E+00	0.895698602397595E-01
	0.844398042013510E+00	0.771222579836698E-01
	0.913476675531666E+00	0.603802128097188E-01
	0.964005785910473E+00	0.401896290968344E-01
	0.993083879404685E+00	0.176977224410030E-01

Table 12: Integration by the Gaussian Quadrature of Bessel Functions in TABLE 1

$$\int_0^{10} f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

$$f(x) = \sin(x)$$

$\int_0^{10} f(x)dx = 0.183907152907645E+01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.183911205770184E+01	0.40529E-04	0.22038E-04
10	0.183907152907645E+01	0.15543E-14	0.84516E-15

$$f(x) = -\cos(x)$$

$\int_0^{10} f(x)dx = 0.544021110889370E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.544074430126847E+00	0.53319E-04	0.98010E-04
10	0.544021110889362E+00	0.77716E-14	0.14285E-13

$$f(x) = -x$$

$\int_0^{10} f(x)dx = 0.500000000000000E+02$			
N	Computed Integral	Absolute Error	Relative Error
5	0.499933635432992E+02	0.66365E-02	0.13273E-03
10	0.499999999999999E+02	0.14921E-12	0.29843E-14

Table 13: Integration by the Gaussian Quadrature of Bessel Functions in TABLE 2

$$\int_0^{10} \frac{1}{\sqrt{x}} f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(x)$

$\int_0^{10} \frac{1}{\sqrt{x}} f(x)dx = 0.152512353028332E+01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.152515304377709E+01	0.29513E-04	0.19352E-04
10	0.152512353028335E+01	0.33307E-13	0.21839E-13

$f(x) = 1 - \cos(x)$

$\int_0^{10} \frac{1}{\sqrt{x}} f(x)dx = 0.522924912143076E+01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.522898009347896E+01	0.26903E-03	0.51447E-04
10	0.522924912142827E+01	0.24931E-11	0.47676E-12

$f(x) = x$

$\int_0^{10} \frac{1}{\sqrt{x}} f(x)dx = 0.210818510677892E+02$			
N	Computed Integral	Absolute Error	Relative Error
5	0.210781199174431E+02	0.37312E-02	0.17698E-03
10	0.210818510677890E+02	0.19895E-12	0.94371E-14

Table 14: Integration by the Gaussian Quadrature for the Products of Polynomials and Logarithmic Functions in TABLE 3

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(17x)$

$\int_0^1 f(x)dx = 0.750096081206822\text{E-}01$			
N	Computed Integral	Absolute Error	Relative Error
5	-0.445245704584552E+00	0.52026E+00	0.69358E+01
10	0.759657791361190E-01	0.95617E-03	0.12747E-01
15	0.750095713818829E-01	0.36739E-07	0.48979E-06
20	0.750096081208711E-01	0.18893E-12	0.25188E-11
25	0.750096081206832E-01	0.10131E-14	0.13506E-13
30	0.750096081206823E-01	0.11102E-15	0.14801E-14

$f(x) = x^{45}$

$\int_0^1 f(x)dx = 0.217391304347826\text{E-}01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.397077526634515E-02	0.17768E-01	0.81734E+00
10	0.206816335417969E-01	0.10575E-02	0.48645E-01
15	0.217347730977134E-01	0.43573E-05	0.20044E-03
20	0.217391290710919E-01	0.13637E-08	0.62730E-07
25	0.217391304347564E-01	0.26222E-13	0.12062E-11
30	0.217391304347826E-01	0.69389E-17	0.31919E-15

$f(x) = -x^{45} \ln x$

$\int_0^1 f(x)dx = 0.472589792060492\text{E-}03$			
N	Computed Integral	Absolute Error	Relative Error
5	0.349440397064675E-03	0.12315E-03	0.26058E+00
10	0.554396948210119E-03	0.81807E-04	0.17310E+00
15	0.473519740800504E-03	0.92995E-06	0.19678E-02
20	0.472590360153710E-03	0.56809E-09	0.12021E-05
25	0.472589792079166E-03	0.18675E-13	0.39515E-10
30	0.472589792060492E-03	0.37947E-18	0.80296E-15

Table 15: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{\frac{2}{3}}$ in TABLE 4

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.205829062722746E+00	0.21922E-01	0.11920E+00
10	0.183906457937759E+00	0.69497E-06	0.37789E-05
15	0.183907152908513E+00	0.86728E-12	0.47158E-11
20	0.183907152907645E+00	0.19429E-15	0.10565E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.3846153846153 85E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.261006235736059E-01	0.12361E-01	0.32138E+00
10	0.384350080122515E-01	0.26530E-04	0.68979E-03
15	0.384615380725112E-01	0.38903E-09	0.10115E-07
20	0.384615384615385E-01	0.69389E-16	0.18041E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.377358490566038E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.261006235736059E-01	0.12361E-01	0.32138E+00
10	0.384350080122515E-01	0.26530E-04	0.68979E-03
15	0.384615380725112E-01	0.38903E-09	0.10115E-07
20	0.384615384615385E-01	0.69389E-16	0.18041E-14

Table 16: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{\frac{1}{2}}$ in TABLE 5

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.205245178663791E+00	0.21338E-01	0.11603E+00
10	0.183906379065431E+00	0.77384E-06	0.42078E-05
15	0.183907152908629E+00	0.98405E-12	0.53508E-11
20	0.183907152907645E+00	0.27756E-16	0.15092E-15

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.384615384615385E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.253967007271653E-01	0.13065E-01	0.33969E+00
10	0.384311761588644E-01	0.30362E-04	0.78942E-03
15	0.384615379738251E-01	0.48771E-09	0.12681E-07
20	0.384615384615383E-01	0.20817E-15	0.54123E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.377358490566038E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.243848070841558E-01	0.13351E-01	0.35380E+00
10	0.377002264605833E-01	0.35623E-04	0.94400E-03
15	0.377358483110791E-01	0.74552E-09	0.19756E-07
20	0.377358490566035E-01	0.22898E-15	0.60681E-14

Table 17: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{\frac{1}{3}}$ in TABLE 6

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$$f(x) = \sin(10x)$$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.203627012649713E+00	0.19720E-01	0.10723E+00
10	0.183906319056362E+00	0.83385E-06	0.45341E-05
15	0.183907152908719E+00	0.10739E-11	0.58396E-11
20	0.183907152907644E+00	0.86042E-15	0.46786E-14

$$f(x) = x^{25}$$

$\int_0^1 f(x)dx = 0.384615384615385E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.246599205561149E-01	0.13802E-01	0.35884E+00
10	0.384268064253340E-01	0.34732E-04	0.90303E-03
15	0.384615378508424E-01	0.61070E-09	0.15878E-07
20	0.384615384615384E-01	0.11102E-15	0.28866E-14

$$f(x) = x^{25}s(x)$$

$\int_0^1 f(x)dx = 0.380228136882129E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.246599205561149E-01	0.13802E-01	0.35884E+00
10	0.384268064253340E-01	0.34732E-04	0.90303E-03
15	0.384615378508424E-01	0.61070E-09	0.15878E-07
20	0.384615384615384E-01	0.11102E-15	0.28866E-14

Table 18: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{\frac{1}{4}}$ in TABLE 7

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.202354038388179E+00	0.18447E-01	0.10031E+00
10	0.183906300136278E+00	0.85277E-06	0.46370E-05
15	0.183907152908747E+00	0.11020E-11	0.59920E-11
20	0.183907152907645E+00	0.19429E-15	0.10565E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.3846153846153855E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.242788857020719E-01	0.14183E-01	0.36875E+00
10	0.384243967690490E-01	0.37142E-04	0.96568E-03
15	0.384615377784612E-01	0.68308E-09	0.17760E-07
20	0.384615384615385E-01	0.41633E-16	0.10825E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.380228136882129E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.242788857020719E-01	0.14183E-01	0.36875E+00
10	0.384243967690490E-01	0.37142E-04	0.96568E-03
15	0.384615377784612E-01	0.68308E-09	0.17760E-07
20	0.384615384615385E-01	0.41633E-16	0.10825E-14

Table 19: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{-\frac{1}{4}}$ in TABLE 8

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.185981461566628E+00	0.20743E-02	0.11279E-01
10	0.183906491673941E+00	0.66123E-06	0.35955E-05
15	0.183907152908381E+00	0.73541E-12	0.39988E-11
20	0.183907152907644E+00	0.80491E-15	0.43767E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.3846153846153385E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.218060230285103E-01	0.16656E-01	0.43304E+00
10	0.384060797524605E-01	0.55459E-04	0.14419E-02
15	0.384615371307963E-01	0.13307E-08	0.34599E-07
20	0.384615384615383E-01	0.19429E-15	0.50515E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.380228136882129E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.218060230285103E-01	0.16656E-01	0.43304E+00
10	0.384060797524605E-01	0.55459E-04	0.14419E-02
15	0.384615371307963E-01	0.13307E-08	0.34599E-07
20	0.384615384615383E-01	0.19429E-15	0.50515E-14

Table 20: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{-\frac{1}{3}}$ in TABLE 9

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.181420301850517E+00	0.24869E-02	0.13522E-01
10	0.183906605077266E+00	0.54783E-06	0.29788E-05
15	0.183907152908166E+00	0.52114E-12	0.28337E-11
20	0.183907152907645E+00	0.30531E-15	0.16601E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.3846153846153385E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.213608283352790E-01	0.17101E-01	0.44462E+00
10	0.384022552100793E-01	0.59283E-04	0.15414E-02
15	0.384615369753817E-01	0.14862E-08	0.38640E-07
20	0.384615384615384E-01	0.69389E-16	0.18041E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.389105058365759E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.213608283352790E-01	0.17101E-01	0.44462E+00
10	0.384022552100793E-01	0.59283E-04	0.15414E-02
15	0.384615369753817E-01	0.14862E-08	0.38640E-07
20	0.384615384615384E-01	0.69389E-16	0.18041E-14

Table 21: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{-\frac{1}{2}}$ in TABLE 10

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E{+}00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.170313857697349E+00	0.13593E-01	0.73914E-01
10	0.183906941803763E+00	0.21110E-06	0.11479E-05
15	0.183907152907514E+00	0.13078E-12	0.71114E-12
20	0.183907152907644E+00	0.14155E-14	0.76970E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.384615384615385E{-}01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.204389144887852E-01	0.18023E-01	0.46859E+00
10	0.383937885473486E-01	0.67750E-04	0.17615E-02
15	0.384615366085782E-01	0.18530E-08	0.48177E-07
20	0.384615384615386E-01	0.11102E-15	0.28866E-14

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.392156862745098E{-}01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.214450816964760E-01	0.17771E-01	0.45315E+00
10	0.391572846318390E-01	0.58402E-04	0.14892E-02
15	0.392156850446440E-01	0.12299E-08	0.31362E-07
20	0.392156862745100E-01	0.16653E-15	0.42466E-14

Table 22: Integration by the Gaussian Quadrature for the Products of Polynomials and Fractional Power $s(x) = x^{-\frac{2}{3}}$ in TABLE 11

$$\int_0^1 f(x)dx = \sum_{i=1}^{N} w_i f(x_i)$$

$f(x) = \sin(10x)$

$\int_0^1 f(x)dx = 0.183907152907645E+00$			
N	Computed Integral	Absolute Error	Relative Error
5	0.156179751994470E+00	0.27727E-01	0.15077E+00
10	0.183907466630775E+00	0.31372E-06	0.17059E-05
15	0.183907152906471E+00	0.11742E-11	0.63847E-11
20	0.183907152907646E+00	0.10825E-14	0.58859E-14

$f(x) = x^{25}$

$\int_0^1 f(x)dx = 0.384615384615385E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.194693376078146E-01	0.18992E-01	0.49380E+00
10	0.383840696841997E-01	0.77469E-04	0.20142E-02
15	0.384615361512299E-01	0.23103E-08	0.60068E-07
20	0.384615384615379E-01	0.52042E-15	0.13531E-13

$f(x) = x^{25}s(x)$

$\int_0^1 f(x)dx = 0.395256916996047E-01$			
N	Computed Integral	Absolute Error	Relative Error
5	0.194693376078146E-01	0.18992E-01	0.49380E+00
10	0.383840696841997E-01	0.77469E-04	0.20142E-02
15	0.384615361512299E-01	0.23103E-08	0.60068E-07
20	0.384615384615379E-01	0.52042E-15	0.13531E-13