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# NAVAL POSTGRADUATE SCHOOL Monterey, California



## THESIS

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THE RESONANT INTERACTION OF A  
SUBMARINE'S WAKE WITH A  
STRATIFIED FLUID

by

Joseph A. Kowalewski

June 1993

Thesis Advisor:

C. L. Frenzen

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The Resonant Interaction of a  
Submarine's Wake With  
a Stratified Fluid

by

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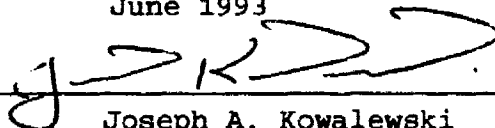
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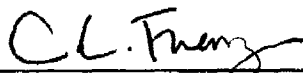
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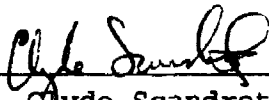
  
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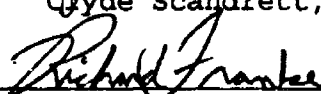
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## ABSTRACT

Through the use of asymptotic and perturbation methods, this thesis presents a theoretical study of the flow of a stratified fluid over variable topography as a model of the resonant interaction of a submarine's wake with a stratified fluid. Such resonant interactions may be able to produce significant upstream disturbances. The long time solution obtained in our model exhibits growth in time for the resonant case, indicating that perhaps nonlinear effects, balanced by dispersion, could cause significant upstream disturbances. These disturbances could allow the possibility of detecting a submarine by the internal waves its wake generates in the resonant case.

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## I. INTRODUCTION

The nonacoustic detection of submarines is an area of intensive and well funded research. Although, it is not clear whether devices based on nonacoustic means of detection will become as useful as those based on acoustic means, the military importance of such devices demands pursuing such avenues of technological research. One possible nonacoustic means of detection is to attempt to detect the submarine optically through the use of a laser by examining the way the submarine reflects or absorbs blue-green light. Another more promising method is to try to detect the submarine by the heat it emits. However, an even more promising technique is the detection of submarines by the internal waves they generate.

In the ocean a stable density stratification generally exists. As the submarine moves through the density-stratified ocean, it leaves behind a wake capable of generating internal waves. These internal waves result from the stabilizing effect of the stratification: as a water particle is displaced, it experiences a buoyant restoring force in the opposite direction from which it was displaced.

Many theoretical and experimental studies have been conducted on the subject of the flow of a stratified fluid over a localized topography. For the two-dimensional flow of an inviscid, incompressible stratified fluid, most studies

have been confined to steady flow or linearized theory. Earlier studies concerned themselves only with a description of the downstream stationary lee-wave field. A question that has generated some controversy is the phenomenon of upstream influence. To what extent do disturbances generated by the topography through transient and nonlinear processes propagate upstream?

McIntyre (1972) showed that the only significant upstream disturbances were weak second order long-wave motions generated by nonlinear interactions in the lee-wave. McIntyre (1972) extended the earlier theories of Benjamin (1970) and Keady (1971) by performing an expansion using as the small parameter the ratio of the height of the topography to the total fluid depth. Baines (1977,1979), on the other hand, in experimentally examining the flow of a continuously stratified fluid over topography, found upstream disturbances that were first order in the topography height. These upstream disturbances were not only generated by nonlinear processes over the topography but also travelled at the long-wave speed. Baines also observed that these upstream disturbances were very strong when the fluid approached resonance. Here, resonance is defined as the near coincidence of the basic flow speed and one of the free long-wave speeds. Grimshaw and Smyth (1986) presented a theoretical study of stratified fluid flow over localized topography for the two-dimensional case when the basic flow is near resonance. One interesting



feature of their results is the appearance of significant upstream disturbances as observed by Baines (1977,1979). Grimshaw and Smyth showed that solitary waves were formed when the localized forcing over the topography produced a positive disturbance upstream, and an oscillatory wavetrain was formed when the disturbance had negative polarity.

Motivated by these results, this thesis presents a theoretical study of the flow of a stratified fluid over localized topography for the three dimensional case.

## II. PROBLEM FORMULATION

### A. DIMENSIONAL PROBLEM

We consider the three-dimensional flow of an inviscid, incompressible stratified fluid. Figure 1 (see page 5) provides a sketch of the coordinate system. We assume that the basic state has a constant horizontal velocity,  $\bar{V}$ , from left to right with equilibrium pressure and density respectively as  $\bar{p}_o(\bar{z})$  and  $\bar{\rho}_o(\bar{z})$ . Here and throughout this paper barred variables represent dimensional variables. We introduce  $\bar{\zeta}$  as the vertical particle displacement from some initial position at time  $\bar{t} = 0$  to the position at time  $\bar{t} = \bar{T}$ . Using the Lagrangian representation  $\bar{\zeta}(\bar{x}, \bar{y}, \bar{z}, \bar{T}) = \bar{z}_{\bar{T}} - \bar{z}_o$ , the density,  $\bar{\rho}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ , becomes  $\bar{\rho}_o(\bar{z} - \bar{\zeta}(\bar{x}, \bar{y}, \bar{z}, \bar{t}))$ . Therefore, the equations of motion are

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} + \bar{w}_{\bar{z}} = 0, \quad (2.1)$$

$$\bar{\rho}_o(\bar{z} - \bar{\zeta}) \frac{D\bar{u}}{D\bar{t}} + \bar{p}_{\bar{x}} = 0, \quad (2.2)$$

$$\bar{\rho}_o(\bar{z} - \bar{\zeta}) \frac{D\bar{v}}{D\bar{t}} + \bar{p}_{\bar{y}} = 0, \quad (2.3)$$

$$\bar{\rho}_o(\bar{z} - \bar{\zeta}) \frac{D\bar{w}}{D\bar{t}} + \bar{\rho}_o(\bar{z} - \bar{\zeta})g + \bar{p}_{\bar{z}} = 0, \quad (2.4)$$

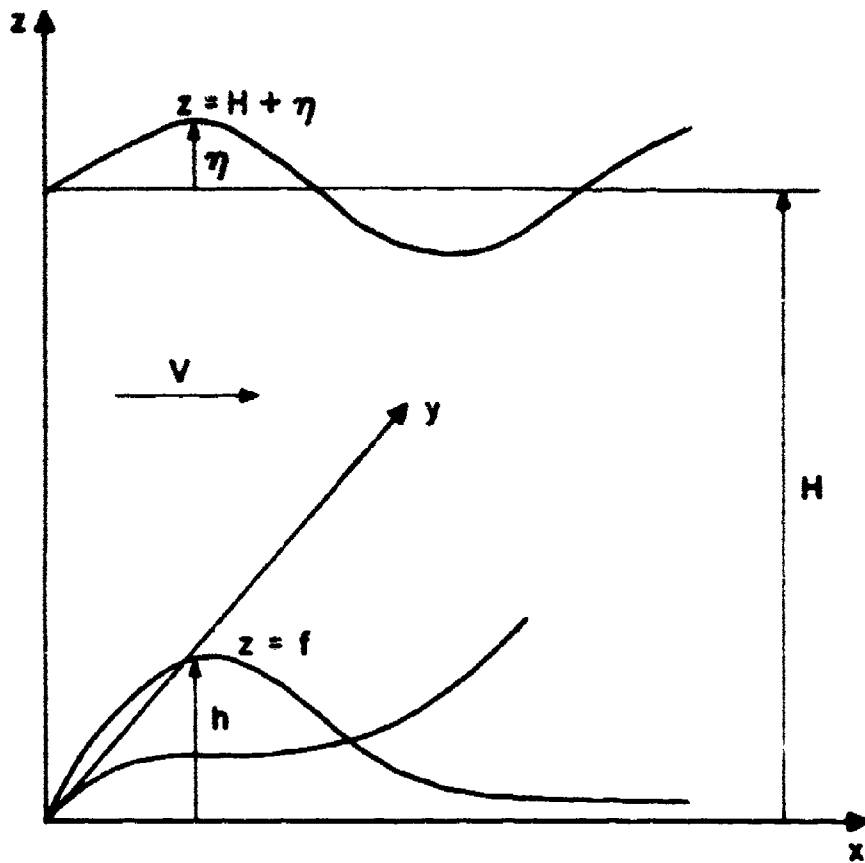


Figure 1. The Coordinate System

$$\bar{w} = \frac{D\bar{z}}{D\bar{t}}, \quad (2.5)$$

where

$$\frac{D}{D\bar{t}} = \frac{\partial}{\partial \bar{t}} + (\bar{v} + \bar{u}) \frac{\partial}{\partial \bar{x}} + \bar{v} \frac{\partial}{\partial \bar{y}} + \bar{w} \frac{\partial}{\partial \bar{z}}. \quad (2.6)$$

Here,  $\bar{u}$ ,  $\bar{v}$ , and  $\bar{w}$  are the velocity components relative to the basic state which has velocity  $(V, 0, 0)$  with respect to a right-handed Cartesian coordinate system.

The bottom topography is given by  $\bar{z} = \bar{f}(\bar{x}, \bar{y}, \bar{t})$ , where initially  $\bar{f}(\bar{x}, \bar{y}, 0) = 0$ . For large times the topography attains a steady value  $\bar{g}(\bar{x}, \bar{y})$  and hence we assume that

$$\bar{f}(\bar{x}, \bar{y}, \bar{t}) \rightarrow \bar{g}(\bar{x}, \bar{y}) \text{ as } \bar{t} \rightarrow \infty. \quad (2.7)$$

We also assume that at all times the topography is localized where  $\bar{f}(\bar{x}, \bar{y}, \bar{t}) \rightarrow 0$  as  $\bar{x}^2 + \bar{y}^2 \rightarrow \infty$ . The kinematic bottom boundary condition is then

$$\bar{\zeta} = \bar{f} \quad \text{at} \quad \bar{z} = \bar{f}. \quad (2.8)$$

We are assuming that this "topography" is created by the submarine's wake, which when viewed from the moving submarine, actually has a steady shape. In this thesis, we investigate the interaction of the wake with internal waves. Källén (1987) has investigated this type of interaction in a linear, steady state model and comments in detail on the validity of assuming that the wake acts as rigid topography. Furthermore, from experimental results, Lin and Pao (1979) show the mean flow occurs at the wake boundaries implying that part of the flow travels around rather than through the wake. It is this flow around the wake that generates the internal waves. Our model, in fact, does not depend critically on the assumed form of "topography" the submarine creates and our purpose here is to investigate a time dependent model.

The upper boundary, assumed to be a free surface, is located at  $\bar{z} = \bar{H} + \bar{\eta}(\bar{x}, \bar{y}, \bar{t})$ , where  $\bar{H}$  is the vertical distance from the surface to the submarine and  $\bar{\eta}(\bar{x}, \bar{y}, \bar{t})$  is the free surface displacement. One must remember to include not only the kinematic boundary condition at the upper boundary but the dynamic boundary condition as well. The upper boundary conditions take the form:

$$\bar{\zeta} = \bar{\eta} \quad \text{at} \quad \bar{z} = \bar{H} + \bar{\eta}, \quad (2.9a)$$

$$\bar{p}_o(\bar{H} + \bar{\eta}) + \bar{p}_p(\bar{H} + \bar{\eta}) = \bar{p}_o(\bar{H}) \quad \text{at} \quad \bar{z} = \bar{H} + \bar{\eta}. \quad (2.9b)$$

Here,  $\bar{p}$ , is the perturbation to the pressure  $\bar{p}_0$  of the basic state.

## B. DIMENSIONAL SCALING

The solution of practical problems in fluid mechanics requires both theoretical developments and experimental results. By grouping significant dimensional quantities into dimensionless parameters it is possible to present results in a compact form which is applicable to all similar situations.

In the problem, the characteristic horizontal length scale  $L$  is determined by the ratio  $U/N$ , where  $U$  is a characteristic horizontal velocity and  $N$  is the Brunt-Väisälä frequency at the location of the submarine. We shall take the characteristic horizontal velocity  $U$  to be given by  $N_1 H$ , ( $U = N_1 H$ ), where  $N_1$  is the maximum Brunt-Väisälä frequency and  $H$ , the characteristic vertical length, is the depth of the submarine. Typically  $N_1$  is  $10^{-2} \text{ sec}^{-1}$  and  $H$  is 200 meters (m), so  $U$  is 2 m/sec. This is a typical order of magnitude patrolling speed for a submarine at that depth. Of course,  $O(1)$  variations in  $N_1$  and  $H$  can be made without changing our analysis.

From the mass conservation equation and with the imposed horizontal length and velocity scales, we see that the vertical velocity scale,  $W$ , is  $UH/L$ . The horizontal length and velocity scales are also used in determining the characteristic time scale,  $T$ , where  $T$  is defined by  $T = L/U =$

1/N. Lastly, the characteristic pressure scale is defined as  $p = \rho_1 g H$  where  $\rho_1$  is a characteristic density. These scales define the parameter  $\beta = N_1^2 H / g$ , which is a dimensionless measure of the ratio of the buoyancy force to the gravitational force.

### C. NONDIMENSIONALIZATION

By using the various dimensional scales introduced in section B, we shall nondimensionalize the equations of motion (2.1-2.6) along with the boundary conditions (2.8-2.9b). However, before deriving the nondimensional equations of motion and boundary conditions, we will consider several related dimensional equations that will assist in simplifying the nondimensional derivation of equations (2.1-2.6) and boundary conditions (2.8-2.9b). We first introduce the following nondimensional variables:

$$\begin{aligned}
 p_o &= \frac{\bar{p}_o}{\rho_1 g H} , & z &= \frac{\bar{z}}{H} , & \rho_o &= \frac{\bar{\rho}_o}{\rho_1} , \\
 N &= \frac{\bar{N}}{N_1} , & \zeta &= \frac{\bar{\zeta}}{H} .
 \end{aligned}
 \tag{2.10}$$

To nondimensionalize the hydrostatic equation, we begin with

$$\frac{d\bar{\rho}_o}{dz} = -g\bar{\rho}_o. \quad (2.11)$$

Using (2.10) yields

$$p_o'(z) = -\rho_o. \quad (2.12)$$

The dimensional Brunt-Väisälä frequency is defined as

$$\bar{N}^2 = -\frac{g}{\rho_o} \frac{d\bar{\rho}_o}{dz} \quad (2.13)$$

where  $d\bar{\rho}_o/dz < 0$ . Using (2.10), we obtain

$$\frac{N_1^2 H}{g} N^2 = -\frac{1}{\rho_o} \frac{d\rho_o}{dz} \quad (2.14)$$

but  $\beta = N_1^2 H/g$ , therefore (2.14) simplifies to

$$\rho_o \beta N^2 = -\frac{d\rho_o}{dz}. \quad (2.15)$$



In nondimensionalizing the Lagrangian representation of the particle's vertical displacement, we start with

$$\bar{z}_o = \bar{z} - \bar{\zeta}(\bar{x}, \bar{y}, \bar{z}, \bar{t}). \quad (2.16)$$

Using (2.10) yields

$$z_o = z - \zeta(x, y, z, t). \quad (2.17)$$

The last equation needed before considering the equations of motion and boundary conditions is the dimensional pressure equation:

$$\bar{p}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \rho_1 g H p_o(z) + \rho_1 g H \beta q(x, y, z, t) \quad (2.18)$$

where  $\bar{p}_p = \rho_1 g H \beta q(x, y, z, t)$  and  $\beta$  is defined after (2.14). Thus,  $q(x, y, z, t)$  is the dimensionless perturbation to the hydrostatic pressure  $p_o$ .

Before deriving the nondimensional equations of motion and their associated boundary conditions, we first introduce the following nondimensional variables:

$$\begin{aligned}
x &= \frac{\bar{x}}{L}, & y &= \frac{\bar{y}}{L}, & z &= \frac{\bar{z}}{H}, & u &= \frac{\bar{u}}{U}, \\
v &= \frac{\bar{v}}{U}, & V &= \frac{\bar{V}}{U}, & w &= \frac{\bar{w}}{W}, & \rho_0 &= \frac{\bar{\rho}_0}{\rho_1}, \\
t &= \frac{\bar{t}}{T}, & \zeta &= \frac{\bar{\zeta}}{H}, & \eta &= \frac{\bar{\eta}}{H}, & f &= \frac{\bar{f}}{h}, \\
\frac{\bar{H}}{H} &= 1,
\end{aligned}
\tag{2.19}$$

where  $L$ ,  $H$ ,  $U$ ,  $W$ ,  $\rho_1$ , and  $T$  are the characteristic dimensional scales defined in section B and  $h$  is a characteristic amplitude of the topography.

We begin with the mass conservation equation (2.1). Using (2.19) and remembering that  $W = UH/L$ , (2.1) becomes

$$u_x + v_y + w_z = 0. \tag{2.20}$$

To assist in deriving the nondimensional form of the x-component of the momentum equations, given by (2.2), we shall now derive the nondimensional form for the convective derivative  $D/D\bar{t}$ :

$$\frac{D}{D\bar{t}} = \frac{\partial}{\partial \bar{t}} + (\bar{v} + \bar{u}) \frac{\partial}{\partial \bar{x}} + \bar{v} \frac{\partial}{\partial \bar{y}} + \bar{w} \frac{\partial}{\partial \bar{z}}. \tag{2.21}$$

Using (2.19), (2.21) becomes

$$\frac{D}{Dt} = \frac{1}{T} \frac{\partial}{\partial t} + \frac{U}{L} (V+u) \frac{\partial}{\partial x} + \frac{U}{L} v \frac{\partial}{\partial y} + \frac{U}{L} w \frac{\partial}{\partial z} \quad (2.22)$$

but  $T = L/U$ , therefore (2.22) simplifies to

$$\frac{D}{Dt} = \frac{1}{T} \frac{D}{Dt}, \quad (2.23)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (V+u) \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \quad (2.24)$$

Substituting (2.17), (2.18), and (2.23) into (2.2) and using (2.19), we obtain

$$\rho_1 \rho_o (z-\zeta) \frac{U}{T} \frac{Du}{Dt} + \frac{\rho_1 g H \beta}{L} \alpha_x = 0 \text{ or,} \quad (2.25)$$

$$\rho_o (z-\zeta) \frac{Du}{Dt} + \alpha_x = 0.$$

A similar analysis for the y-component of the momentum equations (2.3) yields

$$\rho_o(z-\zeta) \frac{Dv}{Dt} + q_y = 0. \quad (2.26)$$

To derive the nondimensional equation for the z-component of the momentum equations (2.4), we begin with

$$\bar{\rho}_o(\bar{z}-\bar{\zeta}) \frac{D\bar{w}}{D\bar{t}} + \bar{\rho}_o(\bar{z}-\bar{\zeta})g + \bar{p}_z = 0. \quad (2.27)$$

Substituting (2.17), (2.18), and (2.23) into (2.27) and using (2.19), we obtain

$$\rho_o(z-\zeta) \frac{W}{T} \frac{Dw}{Dt} + \rho_o(z-\zeta)g + gp_\alpha + g\beta q_z = 0. \quad (2.28)$$

Using  $T = L/N_1H$  and  $W = N_1H^2/L$ , (2.28) simplifies to

$$N_1^2 H \left(\frac{H}{L}\right)^2 \rho_o(z-\zeta) \frac{Dw}{Dt} + \rho_o(z-\zeta)g + gp_\alpha + g\beta q_z = 0. \quad (2.29)$$

Using (2.12) and the definition of  $\beta$ ,  $\beta = N_1^2 H/g$ , (2.29) simplifies to

$$e^2 \rho_o(z-\zeta) \frac{Dw}{Dt} + \frac{1}{\beta} (\rho_o(z-\zeta) - \rho_o(z)) + q_z = 0, \quad (2.30)$$

where

$$e = \frac{H}{L}. \quad (2.31)$$

Lastly, we derive the nondimensional vertical velocity from its definition (2.5). Using (2.19) yields

$$w = \frac{D\zeta}{Dt}. \quad (2.32)$$

We now consider the kinematic bottom boundary condition

$$\bar{\zeta} = \bar{f} \quad \text{at} \quad \bar{z} = \bar{f}. \quad (2.33)$$

Using (2.19), we obtain

$$\zeta = \alpha f \quad \text{at} \quad z = \alpha f, \quad (2.34)$$

where

$$\alpha = \frac{h}{H}. \quad (2.35)$$

For the free surface kinematic boundary condition,

$$\bar{\zeta} = \bar{\eta} \quad \text{at} \quad \bar{z} = \bar{H} + \bar{\eta}. \quad (2.36)$$

We use (2.19) to obtain

$$\zeta = \eta \quad \text{at} \quad z = 1 + \eta. \quad (2.37)$$

Next, we nondimensionalize the free surface dynamic boundary condition in (2.9b):

$$\bar{p}_o(\bar{H} + \bar{\eta}) + \bar{p}_p(\bar{H} + \bar{\eta}) = \bar{p}_o(\bar{H}) \quad \text{at} \quad \bar{z} = \bar{H} + \bar{\eta}. \quad (2.38)$$

Using (2.19) and (2.18), (2.38) becomes

$$p_o(1 + \eta) + \beta q = p_o(1) \quad \text{at} \quad z = 1 + \eta. \quad (2.39)$$

To recapitulate our nondimensional versions of equations (2.1-2.6) and the surface boundary conditions (2.8-2.9b), the nondimensional equations of motion are

$$u_x + v_y + w_z = 0, \quad (2.40)$$

$$\rho_o(z-\zeta) \frac{Du}{Dt} + q_x = 0, \quad (2.41)$$

$$\rho_o(z-\zeta) \frac{Dv}{Dt} + q_y = 0, \quad (2.42)$$

$$\varepsilon^2 \rho_o(z-\zeta) \frac{Dw}{Dt} + \frac{1}{\beta} (\rho_o(z-\zeta) - \rho_o(z)) + q_z = 0, \quad (2.43)$$

$$w = \frac{D\zeta}{Dt}, \quad (2.44)$$

where  $\varepsilon = H/L$  and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (V+u) \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \quad (2.45)$$

with boundary conditions

$$\zeta = \alpha f \quad \text{at} \quad z = \alpha f, \quad (2.46)$$

$$\zeta = \eta \quad \text{at} \quad z = 1 + \eta, \quad (2.47a)$$

$$p_o(1+\eta) + \beta q = p_o(1) \quad \text{at} \quad z = 1 + \eta, \quad (2.47b)$$

where  $\alpha = h/H$ .

An exact analytical solution of these nonlinear partial differential equations is not available. However, we shall

obtain approximate solutions for the case when  $\epsilon$  defined by (2.31) and  $\alpha$  defined by (2.35) are both small positive constants. Since  $\epsilon = H/L = N/N_1$ , the condition  $\epsilon \ll 1$  is satisfied when the density changes rapidly across a thin thermocline. However, there is some ambiguity in the definition of the thermocline. We will consider the thermocline to be the region where density variations are determined chiefly by temperature variations. This means that the submarine is usually submerged to a depth below the thermocline. Is this a valid assumption? To avoid acoustic detection the submarine submerges to depths below the mixed layer or layer depth. Depending on the thickness of the thermocline the submarine can be either in the thermocline or below it. We shall consider only the case for which the submarine is below the thermocline in which the parameter  $\alpha$ , the ratio of the characteristic amplitude of the topography to the submarine's depth defined by (2.35), is also small in this situation.



### III. THE ASYMPTOTIC EXPANSION OF THE DEPENDENT VARIABLES

To study the behavior of the internal waves excited by a small amplitude disturbance in the presence of a uniform oncoming flow, as when a submarine moves through the water at constant velocity  $V$ , we perform an asymptotic expansion on the variables  $\zeta$ ,  $q$ ,  $u$ ,  $v$ , and  $\eta$  in terms of the small parameter  $\alpha$ . This is reasonable since the small parameter  $\alpha$  enters the bottom boundary condition in an analytic way and since the solutions arise by analytic processes, it is natural to expect the solution to exhibit an analytic dependence on the parameter  $\alpha$ . Since the disturbance is  $O(\alpha)$ , we shall initially suppose that the response is also  $O(\alpha)$ . Therefore, we introduce the following asymptotic expansions

$$\begin{aligned}\zeta &= \alpha \zeta_0(x, y, z, t) + \alpha^2 \zeta_1 + \dots \\ q &= \alpha q_0(x, y, z, t) + \alpha^2 q_1 + \dots \\ u &= \alpha u_0(x, y, z, t) + \alpha^2 u_1 + \dots \\ v &= \alpha v_0(x, y, z, t) + \alpha^2 v_1 + \dots \\ \eta &= \alpha \eta_0(x, y, z, t) + \alpha^2 \eta_1 + \dots\end{aligned}\tag{3.1}$$

in which we consider the limit process as  $\alpha \rightarrow 0$ .

Substituting (3.1) into the continuity equation (2.40), we obtain

$$\frac{\partial}{\partial x} [\alpha u_0] + \frac{\partial}{\partial y} [\alpha v_0] + \frac{\partial}{\partial z} \left[ \frac{D}{Dt} (\alpha \xi_0) \right] + O(\alpha^2) + \dots = 0. \quad (3.2)$$

Dividing by  $\alpha$  and taking the limit as  $\alpha \rightarrow 0$ , (3.2) simplifies to

$$u_\alpha + v_\alpha + w_\alpha = 0, \quad (3.3)$$

where

$$w_\alpha = \frac{D_0 \xi_0}{D_0 t}, \quad (3.4)$$

and

$$\frac{D_0}{D_0 t} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}. \quad (3.5)$$

To show that  $D_0/D_0 t$  is the  $O(1)$  part of the convective derivative given in (2.45), we collect only the  $O(1)$  terms in the expansion for  $D/Dt$  given in (3.6) below:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (v + \alpha u_0) \frac{\partial}{\partial x} + \alpha v_0 \frac{\partial}{\partial y} + O(\alpha^2) + \dots \quad (3.6)$$

Substituting (3.1) into the x-component of the momentum equations (2.41), we obtain

$$\rho_o(z-\zeta) \frac{D_o}{D_o t} (\alpha u_o) + \alpha q_{ax} + O(\alpha^2) + \dots = 0. \quad (3.7)$$

Expanding  $\rho_o(z-\zeta)$  in a Taylor Series and using (3.1) yields

$$\rho_o(z-\zeta) = \rho_o(z) - \rho_o'(z) [\alpha \zeta_o] + O(\alpha^2) + \dots \quad (3.8)$$

Substituting (3.8) into (3.7), we obtain

$$\alpha \rho_o(z) \frac{D_o u_o}{D_o t} + \alpha q_{ax} + O(\alpha^2) + \dots = 0. \quad (3.9)$$

Dividing by  $\alpha$  and taking the limit as  $\alpha \rightarrow 0$ , (3.9) becomes

$$\rho_o(z) \frac{D_o u_o}{D_o t} + q_{ax} = 0. \quad (3.10)$$

A similar analysis for the y-component of the momentum equations (2.42) yields

$$\rho_o(z) \frac{D_o v_o}{D_o t} + q_{\alpha} = 0. \quad (3.11)$$

For the z-component of the momentum equations (2.43), we substitute (3.1) into (2.43) to obtain

$$\alpha \varepsilon^2 \frac{D_o^2 \zeta_o}{D_o t^2} + \frac{1}{\beta} [\rho_o(z-\zeta) - \rho_o(z)] + \alpha q_{\alpha} + O(\alpha^2) + \dots = 0. \quad (3.12)$$

Expanding  $\rho_o(z-\zeta)$  in a Taylor Series as in (3.8) and substituting into (3.12), we obtain

$$\alpha \varepsilon^2 \frac{D_o^2 \zeta_o}{D_o t^2} - \frac{\alpha}{\beta} \rho_o'(z) \zeta_o + \alpha q_{\alpha} + O(\alpha^2) + \dots = 0. \quad (3.13)$$

Dividing by  $\alpha$  and taking the limit as  $\alpha \rightarrow 0$ , (3.13) simplifies to

$$-\frac{1}{\beta} \rho_o'(z) \zeta_o + q_{\alpha} = 0. \quad (3.14)$$

Notice that the  $\varepsilon^2$  term has been neglected. As previously discussed, in order for our theory to work we require  $\varepsilon \ll 1$ . Therefore, we see the  $\varepsilon^2$  term is small. The fact that the  $\varepsilon^2$

term is small by itself is not justification in dropping this term from (3.13). In order to justify the dropping of the  $\varepsilon^2$  term, we shall assume that  $\alpha\varepsilon^2 = o(\alpha)$  as  $\alpha \rightarrow 0$ , or that  $\varepsilon^2 = o(1)$  as  $\alpha \rightarrow 0$ . Now as one takes the limit  $\alpha \rightarrow 0$  in (3.13), we clearly see that  $\varepsilon^2 \rightarrow 0$ . In continuing our derivation of the z-component of the momentum equations, we substitute (2.15) in for  $d\rho_0/dz$ , so that (3.14) becomes

$$q_{\alpha} + \rho_0 N^2 \zeta_0 = 0. \quad (3.15)$$

Next, we consider the surface boundary conditions. We begin by substituting (3.1) into the kinematic bottom boundary condition (2.46) to obtain

$$\alpha \zeta_0 = \alpha f + O(\alpha^2) + \dots \quad \text{at} \quad z = \alpha f. \quad (3.16)$$

Taking the limit as  $\alpha \rightarrow 0$ , we find

$$\zeta_0 = f \quad \text{at} \quad z = 0. \quad (3.17)$$

Note that the  $O(\alpha)$  bottom boundary condition is applied at  $z = 0$ . Substituting (3.1) into the free surface kinematic boundary condition (2.47a) results in

$$\alpha \zeta_o = \alpha \eta_o + O(\alpha^2) + \dots \quad \text{at} \quad z = 1 + \alpha \eta_o. \quad (3.18)$$

Taking the limit as  $\alpha \rightarrow 0$ , (3.18) becomes

$$\zeta_o = \eta_o \quad \text{at} \quad z = 1. \quad (3.19)$$

Again the  $O(\alpha)$  free surface condition is applied at  $z = 1$ .

The last boundary condition to consider is the free surface dynamic boundary condition (2.47b). Substituting (3.1) into (2.47b), we obtain

$$p_o(1+\eta) + \alpha \beta q_o + O(\alpha^2) + \dots = p_o(1). \quad (3.20)$$

Expanding  $p_o(1+\eta)$  in a Taylor Series and using (3.1) gives

$$p_o(1+\eta) = p_o(1) + \alpha p_o'(z) \eta_o + O(\alpha^2) + \dots \quad (3.21)$$

Substituting (3.21) into (3.20), we find

$$p_o(1) + \alpha p_o'(1) \eta_o + \alpha \beta q_o + O(\alpha^2) + \dots = p_o(1). \quad (3.22)$$

Canceling the  $p_0(1)$  terms, dividing by  $\alpha$ , and taking the limit as  $\alpha \rightarrow 0$ , (3.22) becomes

$$p'_0(z)\eta_0 + \beta q_0 = 0 \quad \text{at} \quad z = 1 \quad (3.23)$$

but using (2.12) simplifies the free surface dynamic boundary condition to

$$\beta q_0 = \rho_0 \eta_0 \quad \text{at} \quad z = 1. \quad (3.24)$$

In summary, we see from substituting the asymptotic expansion (3.1) into (2.40-2.45), (2.46), and (2.47a,b), that the  $O(\alpha)$  problem is:

$$u_{\alpha x} + v_{\alpha y} + w_{\alpha z} = 0, \quad (3.25)$$

$$\rho_0(z) \frac{D_0 u_0}{D_0 t} + q_{\alpha x} = 0, \quad (3.26)$$

$$\rho_0(z) \frac{D_0 v_0}{D_0 t} + q_{\alpha y} = 0, \quad (3.27)$$

$$q_{\alpha z} + \rho_0 N^2 \zeta_0 = 0, \quad (3.28)$$

$$w_0 = \frac{D_0 \zeta_0}{D_0 t}, \quad (3.29)$$

where

$$\frac{D_0}{D_0 t} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}, \quad (3.30)$$

with boundary conditions

$$\zeta_0 = f \quad \text{at} \quad z = 0, \quad (3.31)$$

$$\zeta_0 = \eta_0 \quad \text{at} \quad z = 1, \quad (3.32a)$$

$$\beta \alpha_0 = \rho_0 \eta_0 \quad \text{at} \quad z = 1. \quad (3.32b)$$

Combining (3.32a) and (3.32b) gives the single boundary condition at the free surface:

$$\beta \alpha_0 = \rho_0 \zeta_0 \quad \text{at} \quad z = 1. \quad (3.33)$$



#### IV. DEVELOPMENT OF THE TWO-DIMENSIONAL WAVE EQUATION

##### A. EIGENVALUE PROBLEM

In order to solve (3.25-28), (3.31), and (3.33), we look for solutions of the form

$$\begin{aligned}u_o &= Ae^{i(KX + LY - \Omega T)} \phi'(z) \\v_o &= Be^{i(KX + LY - \Omega T)} \phi'(z) \\z_o &= De^{i(KX + LY - \Omega T)} \phi(z) \\q_o &= Dc^2 e^{i(KX + LY - \Omega T)} \rho_o \phi'(z)\end{aligned}\tag{4.1}$$

where  $\kappa = (K, L)$  is the horizontal wave vector,  $\Omega$  is the frequency, and  $c$  is the long-wave phase speed relative to the basic state at rest. Substituting (4.1) into (3.25), (3.26), and (3.27) and solving the system of linear equations, we find the dispersion relationship to be

$$\Omega = c|\kappa| \quad \text{where} \quad |\kappa| \neq 0.\tag{4.2}$$

Furthermore, substituting (4.1) into (3.28), (3.31), and (3.33) results in

$$(\rho_0 \phi'(z))' + \frac{\rho_0 N^2}{c^2} \phi(z) = 0, \quad (4.3a)$$

with boundary conditions

$$\phi(z) = 0 \quad \text{at} \quad z = 0, \quad (4.3b)$$

$$\phi(z) = \beta c^2 \phi'(z) \quad \text{at} \quad z = 1. \quad (4.3c)$$

The equations (4.3a-c) formally pose an eigenvalue problem where  $c^2$  is the eigenvalue and  $\phi$  the eigenfunction. Note that the dispersion relationship in (4.2) was obtained for wave motion in a stratified fluid at rest where  $V = 0$  and  $D_0/D_0 t = \partial/\partial t$ . In the problem we are considering, the dispersion relationship would differ from that found in (4.2) but the eigenvalue problem would remain the same. Hence, our purpose here is to show the existence and character of the solutions that define the eigenvalue problem. The solutions have the horizontal character of plane waves travelling in the  $(x, y)$  plane with a vertical structure defined by the eigenfunctions,  $\phi(z)$ . We shall see that the general solution to (3.25-3.28), (3.31), (3.33) can be obtained in terms of a superposition of normal modes whose depth dependence is determined by the  $\phi_n(z)$ , the eigenfunctions of the following eigenvalue problem:

$$(\rho_o \phi_x)' + \frac{\rho_o N^2}{c_r^2} \phi_s = 0, \quad (4.4a)$$

$$\phi_s(z) = 0 \quad \text{at} \quad z = 0, \quad (4.4b)$$

$$\phi_s(z) = \beta c_r^2 \phi_x(z) \quad \text{at} \quad z = 1. \quad (4.4c)$$

Here  $c_r^2$  is the eigenvalue, where  $c_r$  is the long-wave phase speed relative to the basic state at rest.

#### B. ORTHOGONALITY

The eigenfunctions  $\phi_s(z)$  satisfy the orthogonality condition

$$c_r^2 \int_0^1 \rho_o \phi_x \phi_\pi dz = \delta_{r\pi} I_r, \quad (4.5)$$

where  $I_r$  is defined by the left side of (4.5) for  $r = s$ . To prove (4.5), we let  $\lambda_s$  and  $\lambda_\pi$  be eigenvalues with corresponding eigenfunctions  $\phi_s$  and  $\phi_\pi$ , where  $\lambda_s = c_s^2$  and  $\lambda_\pi = c_\pi^2$ . The differential equations satisfied by the eigenfunctions are

$$(\rho_o \phi_x)' + \lambda_s \rho_o N^2 \phi_s = 0, \quad (4.6a)$$

$$(\rho_o \phi_\pi)' + \lambda_\pi \rho_o N^2 \phi_\pi = 0. \quad (4.6b)$$

In addition, both  $\phi_s$  and  $\phi_r$  satisfy the same set of homogeneous boundary conditions defined in (4.4b,c). Multiplying (4.6a) by  $\lambda_r \phi_r$  and (4.6b) by  $\lambda_s \phi_s$  yields

$$\lambda_r (\rho_o \phi_x)' \phi_r + \lambda_r \lambda_s \rho_o N^2 \phi_s \phi_r = 0, \quad (4.7a)$$

$$\lambda_s (\rho_o \phi_x)' \phi_s + \lambda_r \lambda_s \rho_o N^2 \phi_s \phi_r = 0. \quad (4.7b)$$

Subtracting (4.7b) from (4.7a) results in

$$\lambda_r (\rho_o \phi_x)' \phi_r - \lambda_s (\rho_o \phi_x)' \phi_s = 0. \quad (4.8)$$

Through application of the product rule

$$\lambda_r (\rho_o \phi_x) \phi_r = \lambda_r [(\rho_o \phi_x \phi_r)' - \rho_o \phi_x \phi_{rx}], \quad (4.9)$$

(4.8) becomes

$$\lambda_r (\rho_o \phi_x \phi_r)' - \lambda_r \rho_o \phi_x \phi_{rx} - \lambda_s (\rho_o \phi_x \phi_s)' + \lambda_s \rho_o \phi_x \phi_{sx} = 0. \quad (4.10)$$

Integrating (4.10) yields

$$(\lambda_s - \lambda_r) \int_0^1 \rho_o \phi_{sz} \phi_{rz} dz = \lambda_s \rho_o \phi_{sz} \phi_r \Big|_0^1 - \lambda_r \rho_o \phi_{sz} \phi_s \Big|_0^1. \quad (4.11)$$

Evaluating the right hand side of (4.11) and remembering  $\phi_s(1) = \beta \lambda_s^{-1} \phi_{sz}(1)$ ,  $\phi_r(1) = \beta \lambda_r^{-1} \phi_{rz}(1)$ , and  $\phi_s(0) = \phi_r(0) = 0$ , (4.11) becomes

$$(\lambda_s - \lambda_r) \int_0^1 \rho_o \phi_{sz} \phi_{rz} dz = 0. \quad (4.12)$$

If  $\lambda_s \neq \lambda_r$ , it follows that

$$\int_0^1 \rho_o \phi_{sz} \phi_{rz} dz = 0. \quad (4.13)$$

In other words, the z derivatives of the eigenfunctions ( $\phi_r$  and  $\phi_s$ ) corresponding to different eigenvalues ( $\lambda_r$  and  $\lambda_s$ ) are orthogonal with weight  $\rho_o$ . Therefore, the normal modes do satisfy the orthogonality condition in (4.5).

### C. TWO-DIMENSIONAL WAVE EQUATION

Since the eigenfunctions,  $\phi_n(z)$ , form a complete set, we can represent  $u_o$ ,  $v_o$ ,  $\zeta_o$ , and  $q_o$  by a generalized Fourier series expansion. From (4.1) we know the general form of the solutions and hence we put

$$\zeta_o = \sum_0^{\infty} A_s(x, y, t) \phi_s(z), \quad (4.14a)$$

$$q_o = \rho_o(z) \sum_0^{\infty} c_s^2 B_s(x, y, t) \phi_{sz}(z), \quad (4.14b)$$

$$u_o = -\sum_0^{\infty} D_s(x, y, t) \phi_{sz}(z), \quad (4.14c)$$

$$v_o = -\sum_0^{\infty} E_s(x, y, t) \phi_{sz}(z). \quad (4.14d)$$

To derive a relationship between  $A_s$  and  $B_s$ , we begin with

$$\zeta_o = \sum_0^{\infty} A_s(x, y, t) \phi_s(z). \quad (4.15)$$

We take the derivative of (4.15) with respect to  $z$ , multiply by  $\rho_o c_s^2 \phi_{sz}(z)$  and use the orthogonality condition in (4.5) to obtain

$$I_s A_s = c_s^2 \int_0^1 \rho_o \zeta_{oz} \phi_{sz} dz. \quad (4.16)$$

Integrating (4.16) by parts then gives

$$I_s A_s = c_s^2 \rho_o \zeta_o \phi_{sz} \Big|_0^1 - c_s^2 \int_0^1 \zeta_o (\rho_o \phi_{sz})' dz. \quad (4.17)$$

Using (4.4a), (4.17) becomes

$$I_s A_s = c_s^2 \rho_o \phi_{sz} \Big|_0^1 + \int_0^1 \rho_o \zeta_o N^2 \phi_s dz. \quad (4.18)$$

Now, we multiply (4.14b) by  $\phi_z$  and from the orthogonality condition (4.5), we obtain

$$I_s B_s = \int_0^1 q_o \phi_{sz} dz. \quad (4.19)$$

Integrating (4.19) by parts, we obtain

$$I_s B_s = q_o \phi_s \Big|_0^1 - \int_0^1 q_{oz} \phi_s dz. \quad (4.20)$$

Using (3.28), (4.20) becomes

$$I_s B_s = q_o \phi_s \Big|_0^1 + \int_0^1 \rho_o \zeta_o N^2 \phi_s dz. \quad (4.21)$$

Combining (4.18) and (4.21) yields

$$c_s^2 \rho_o \int_0^1 \phi_{sz} dz + I_s B_s - q_o \int_0^1 \phi_s dz = I_s A_s. \quad (4.22)$$

Substituting (3.33) and (4.4c) for  $q_o(1)$  and  $\phi_s(1)$  respectively and remembering that  $\int_0^1 \phi_s dz = f(x, y, t)$ , (4.22) simplifies to

$$B_s = A_s + F_s, \quad (4.23)$$

where

$$I_s F_s = c_s^2 \rho_o(0) \phi_{sz}(0) f(x, y, t). \quad (4.24)$$

Note that  $F_s(x, y, t)$  is proportional to the bottom topography  $f(x, y, t)$ .

Next, we substitute (4.14a, c, d) into the continuity equation (3.25) to obtain

$$-\sum_0^{\infty} \frac{\partial D_s}{\partial x} \phi_{sz} - \sum_0^{\infty} \frac{\partial E_s}{\partial y} \phi_{sz} + \sum_0^{\infty} \frac{D_s A_s}{D_o t} \phi_{sz} = 0. \quad (4.25)$$

Multiplying (4.25) by  $c_s^2 \rho_o \phi_{sz}$  and applying the orthogonality condition (4.5) then gives



$$\frac{D_o A_s}{D_o t} = \frac{\partial D_s}{\partial x} + \frac{\partial E_s}{\partial y}. \quad (4.26)$$

where  $D_o/D_o t$  is given in (3.30).

To derive the two-dimensional wave equation, we use the x and y components of the momentum equations (3.26) and (3.27). Substituting (4.14b,c) and (4.14b,d) into (3.26) and (3.27) respectively, we obtain

$$\rho_o \frac{D_o}{D_o t} \left[ -\sum_0^{\infty} D_s \phi_{xz} \right] + \rho_o \frac{\partial}{\partial x} \left[ \sum_0^{\infty} c_s^2 B_s \phi_{xz} \right] = 0, \quad (4.27a)$$

$$\rho_o \frac{D_o}{D_o t} \left[ -\sum_0^{\infty} E_s \phi_{xz} \right] + \rho_o \frac{\partial}{\partial y} \left[ \sum_0^{\infty} c_s^2 B_s \phi_{xz} \right] = 0. \quad (4.27b)$$

Multiplying (4.27a) and (4.27b) by  $c_s^2 \phi_{xz}$  and applying the orthogonality condition (4.5) then yields

$$-\frac{D_o D_s}{D_o t} + c_s^2 \frac{\partial B_s}{\partial x} = 0, \quad (4.28a)$$

$$-\frac{D_o E_s}{D_o t} + c_s^2 \frac{\partial B_s}{\partial y} = 0. \quad (4.28b)$$

We next take the x derivative of (4.28a) and the y derivative of (4.28b) and add the two results to obtain

$$-\frac{D_o}{D_o t} \left[ \frac{\partial D_s}{\partial x} + \frac{\partial E_s}{\partial y} \right] + c_s^2 \nabla^2 B_s = 0. \quad (4.29)$$

Substituting (4.23) and (4.26) into (4.29) finally yields

$$\frac{D_o^2 A_s}{D_o t^2} = c_s^2 \nabla^2 [A_s + F_s], \quad (4.30)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (4.31)$$

and

$$\frac{D_o^2}{D_o t^2} = \left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right]^2. \quad (4.32)$$

Equation (4.30) is the inhomogeneous two-dimensional wave equation whose solution will be obtained in Chapter V.

To summarize, we see that substituting (4.14a-d) into (3.25-3.27) implies

$$E_1 = A_1 + F_1, \quad (4.33a)$$

$$\frac{D_o A_1}{D_o t} = \frac{\partial D_1}{\partial x} + \frac{\partial E_1}{\partial y}, \quad (4.33b)$$

$$\frac{D_o^2 A_1}{D_o t^2} = C_1^2 \nabla^2 [ A_1 + F_1 ], \quad (4.33c)$$

where

$$I_1 F_1 = c_1^2 \rho_o(0) \phi_{z_1}(0) f(x, y, t), \quad (4.33d)$$

$$\frac{D_o}{D_o t} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}, \quad (4.33e)$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (4.33f)$$

Here, since  $f$  vanishes at  $t = 0$ ,  $F_1(x, y, 0) = 0$  and from (2.7) we have

$$F_1(x, y, t) \sim G_1(x, y) \quad \text{as} \quad t \rightarrow \infty, \quad (4.34a)$$

where

$$I, G, = c^2 \rho_0(0) \phi_x(0) g(x, y). \quad (4.34b)$$

The initial conditions for (4.14a-d) are that  $A_1$ ,  $B_1$ ,  $D_1$ , and  $E_1$  all vanish at  $t = 0$ .

## V. SOLUTION TO THE TWO-DIMENSIONAL WAVE EQUATION

### A. METHOD OF DESCENT

To solve (4.33c) we first introduce the following change of variables:

$$\begin{aligned}\zeta &= x - vt, \\ y &= y, \\ \sigma &= t.\end{aligned}\tag{5.1a}$$

Derivatives with respect to  $x$ ,  $y$ ,  $t$  can be expressed as derivatives in the new variables  $\zeta$ ,  $y$ ,  $\sigma$  as follows:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial \zeta}, \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \sigma} - v \frac{\partial}{\partial \zeta}.\end{aligned}\tag{5.1b}$$

Substituting (5.1a,b) into (4.33c) yields

$$\frac{\partial^2 A_s}{\partial \sigma^2} - c_s^2 \left[ \frac{\partial^2 A_s}{\partial \zeta^2} + \frac{\partial^2 A_s}{\partial y^2} \right] = f_s(\zeta + v\sigma, y, \sigma),\tag{5.2}$$

where

$$f,(\xi+V\sigma, y, \sigma) = c^2 \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2} \right] F,(\xi+V\sigma, y, \sigma). \quad (5.3)$$

We shall solve (5.2) by beginning with the solution to the following three-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right], \quad (5.4a)$$

$$u(x, y, z, 0) = \Phi(x, y, z), \quad (5.4b)$$

$$u_t(x, y, z, 0) = \Psi(x, y, z). \quad (5.4c)$$

Using Kirchoff's formula (see Strauss 1992), the solution to (5.4a-c) is

$$\begin{aligned} u(x_0, y_0, z_0, t_0) = & \frac{1}{4\pi c^2 t_0} \iint_b \Psi(x, y, z) dS \\ & + \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2 t_0} \iint_b \Phi(x, y, z) dS \right], \end{aligned} \quad (5.5)$$

where

$$D = [ (x, y, z) : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = c^2 t_0^2 ]. \quad (5.6)$$

Note that the value of  $u(x_0, y_0, z_0, t_0)$  depends only on the values of  $\Psi(x, y, z)$  and  $\Phi(x, y, z)$  on the spherical surface  $D$ .

By the method of descent, we now obtain the solution to the following two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right], \quad (5.7a)$$

$$u(x, y, 0) = \Phi(x, y), \quad (5.7b)$$

$$u_t(x, y, 0) = \Psi(x, y). \quad (5.7c)$$

We regard the solution  $u(x, y, t)$  of (5.7a-c) as a solution to the three-dimensional problem (5.4a-c) with initial data independent of  $z$ . By Kirchoff's formula (5.7), we have

$$u(0, 0, t) = \frac{1}{4\pi c^2 t} \iint_{D_t} \Psi(x, y) dS + \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \iint_{D_t} \Phi(x, y) dS \right], \quad (5.8)$$

where

$$D_t = \{ (x, y, z) : x^2 + y^2 + z^2 = c^2 t^2 \}. \quad (5.9)$$

This is the solution to (5.7a-c) but it can be simplified. To simplify (5.8), we must express the surface integral as an ordinary double integral. It is twice the integral over the top hemisphere  $z = (c^2 t^2 - x^2 - y^2)^{1/2}$  where  $(x, y)$  ranges over the region  $D_t = \{ (x, y) : x^2 + y^2 \leq c^2 t^2 \}$ . By definition

$$dS = |R_x \times R_y| dx dy, \quad (5.10a)$$

where

$$R = \langle x, y, \sqrt{(c^2 t^2 - x^2 - y^2)} \rangle, \quad (5.10b)$$

and

$$|R_x \times R_y| = \frac{ct}{\sqrt{(c^2 t^2 - x^2 - y^2)}}. \quad (5.10c)$$

Substituting (5.10a-c) into (5.8) yields



$$\begin{aligned}
u(0,0,t) &= \frac{1}{2\pi c} \iint_{D_t} \frac{\Psi(x,y)}{\sqrt{c^2 t^2 - x^2 - y^2}} dx dy \\
&+ \frac{\partial}{\partial t} \left[ \frac{1}{2\pi c} \iint_{D_t} \frac{\Phi(x,y)}{\sqrt{c^2 t^2 - x^2 - y^2}} dx dy \right]
\end{aligned} \tag{5.11}$$

This is the solution at the point  $(0,0,t)$ . Therefore, the solution at a general point is

$$\begin{aligned}
u(x_0, y_0, t_0) &= \frac{1}{2\pi c} \iint_{D_t} \frac{\Psi(x,y)}{\sqrt{c^2 t_0^2 - (x-x_0)^2 - (y-y_0)^2}} dx dy \\
&+ \frac{\partial}{\partial t_0} \left[ \frac{1}{2\pi c} \iint_{D_t} \frac{\Phi(x,y)}{\sqrt{c^2 t_0^2 - (x-x_0)^2 - (y-y_0)^2}} dx dy \right]
\end{aligned} \tag{5.12}$$

where

$$D_t = \left[ (x,y) : (x-x_0)^2 + (y-y_0)^2 \leq c^2 t^2 \right]. \tag{5.13}$$

Note that (5.12) shows that the value  $u(x_0, y_0, t_0)$  depends on the values of  $\Phi(x,y)$  and  $\Psi(x,y)$  inside the disc defined in (5.13). This illustrates the well known fact that Huygens' principle fails in two dimensions. We shall use (5.12) in the solution to the nonhomogeneous two-dimensional wave equation discussed in the next section.

## B. DUHAMEL'S PRINCIPLE

We now consider the following nonhomogeneous two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] = g(x, y, t), \quad (5.14a)$$

with homogeneous initial conditions

$$u(x, y, 0) = 0, \quad (5.14b)$$

$$u_t(x, y, 0) = 0. \quad (5.14c)$$

By Duhamel's principle, the solution to (5.14a-c) is given by

$$u(x, y, t) = \int_0^t U(x, y, t, s) ds \quad (5.15)$$

where for each  $s \geq 0$ ,  $U(x, y, t, s)$  is the solution of

$$\frac{\partial^2 U}{\partial t^2} - c^2 \left[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right] = 0 \quad \text{for } t \geq s, \quad (5.16a)$$

with initial data prescribed on the plane  $t = s$  such that

$$U(x, y, s, s) = 0, \quad (5.16b)$$

$$U_t(x, y, s, s) = g(x, y, s). \quad (5.16c)$$

Since the wave operator is invariant under translations in time  $t$  (see Strauss 1992), we can shift the time  $t$  by a constant and still have a solution. We let  $V(x, y, t, s) = U(x, y, t+s, s)$ , so that  $V(x, y, t, s)$  is a solution to

$$V_{tt} - c^2(V_{xx} + V_{yy}) = 0 \quad \text{for } t \geq 0, \quad (5.17a)$$

with initial data prescribed on the plane  $t = 0$  such that

$$V(x, y, 0, s) = U(x, y, s, s) = 0, \quad (5.17b)$$

$$V_t(x, y, 0, s) = U_t(x, y, s, s) = g(x, y, s). \quad (5.17c)$$

Using Kirchoff's formula (5.12) to obtain a solution to (5.17a-c) then yields

$$V(x, y, t, s) = \frac{1}{2\pi c} \iint_b \frac{g(\bar{x}, \bar{y}, s)}{\sqrt{c^2 t^2 - (\bar{x} - x)^2 - (\bar{y} - y)^2}} d\bar{x} d\bar{y} \quad (5.18)$$

where  $D_t = \{ (\bar{x}, \bar{y}) : (\bar{x}-x)^2 + (\bar{y}-y)^2 \leq c^2 t^2 \}$ . Since our purpose here is to solve the nonhomogeneous two-dimensional wave equation in (5.14a-c) by Duhamel's principle (5.15), we can again shift the time  $t$  by a constant so that  $U(x, y, t, s) = V(x, y, t-s, s)$ . This last expression for  $U(x, y, t, s)$  together with (5.18) then implies

$$U(x, y, t, s) = \frac{1}{2\pi c} \iint_{D_t} \frac{g(\bar{x}, \bar{y}, s)}{\sqrt{c^2(t-s)^2 - (\bar{x}-x)^2 - (\bar{y}-y)^2}} d\bar{x} d\bar{y} \quad (5.19)$$

where

$$D_t = [ (\bar{x}, \bar{y}) : (\bar{x}-x)^2 + (\bar{y}-y)^2 \leq c^2(t-s)^2 ]. \quad (5.20)$$

Now using Duhamel's principle (5.15), we obtain

$$u(x, y, t) = \frac{1}{2\pi c} \int_0^t \left[ \iint_{D_t} \frac{g(\bar{x}, \bar{y}, s)}{\sqrt{c^2(t-s)^2 - (\bar{x}-x)^2 - (\bar{y}-y)^2}} d\bar{x} d\bar{y} \right] ds \quad (5.21)$$

the solution to (5.14a-c) where  $D_t$  is defined in (5.20).

In deriving the solution to the nonhomogeneous two-dimensional wave equation (5.14a-c), we have solved (5.2). We change the names of the dependent and independent variables in (5.21) to obtain the solution of (5.2):

$$A_s(\xi, \eta, \sigma) = \frac{1}{2\pi c_s} \int_0^\sigma \left[ \iint_{D_s} \frac{f_s(\bar{x}+Vs, \bar{y}, s)}{\sqrt{c_s^2(\sigma-s)^2 - (\bar{x}-\xi)^2 - (\bar{y}-\eta)^2}} d\bar{x} d\bar{y} \right] ds \quad (5.22)$$

where  $D_s = \{ (\bar{x}, \bar{y}) : (\bar{x}-\xi)^2 + (\bar{y}-\eta)^2 \leq c_s^2(\sigma-s)^2 \}$ . Letting  $\xi = x - Vt$  where  $\sigma = t$ , (5.22) becomes

$$A_s(x, y, t) = \frac{1}{2\pi c_s} \int_0^t \left[ \iint_{D_t} \frac{f_s(\bar{x}+Vs, \bar{y}, s)}{\sqrt{c_s^2(t-s)^2 - (\bar{x}-(x-Vt))^2 - (\bar{y}-y)^2}} d\bar{x} d\bar{y} \right] ds \quad (5.23)$$

where  $D_t = \{ (\bar{x}, \bar{y}) : (\bar{x}-(x-Vt))^2 + (\bar{y}-y)^2 \leq c_s^2(t-s)^2 \}$ .

To derive the solution for the nonhomogeneous two-dimensional wave equation, we began by solving the homogeneous three-dimensional wave equation using Kirchoff's formula (5.5). Next, we used the method of descent to solve the homogeneous two-dimensional wave equation by reducing Kirchoff's surface integral formula to an ordinary double integral (5.12). We developed the solution to the homogeneous two-dimensional wave equation in order to solve the nonhomogeneous two-dimensional wave equation by applying Duhamel's principle (5.21). This gave the solution of the wave equation in (5.2) in the form given above in (5.23). In

Chapter VI we will examine the solution to (5.23) for long time.

## VI. LONG TIME SOLUTION

### A. $C_s^2 \neq V^2$

Before looking at the behavior of (5.23) as  $t \rightarrow \infty$  (where the topography  $f_1(x+Vs, y, s)$  has a general form), we shall first introduce a specific moving point source for the topography, so that

$$f_1(x, y, s) = \delta(x-x_0) \delta(y-y_0) g(s). \quad (6.1a)$$

Consequently

$$f_1(x+Vs, y, s) = \delta(x+Vs-x_0) \delta(y-y_0) g(s), \quad (6.1b)$$

where  $g$  is a non-negative function satisfying

$$g(0) = 0 \quad \text{and} \quad g(s) = 1 \quad \text{for} \quad s \geq s_0. \quad (6.1c)$$

If we can solve (5.23) by substituting (6.1b) in for  $f_1(x+Vs, y, s)$ , then we can solve (5.23) for any form of topography by applying the principle of superposition.

To solve (5.23), we begin by substituting (6.1b) into (5.23) to obtain

$$A_2(x, y, t) =$$

$$\frac{1}{2\pi c_1} \int_0^t \left[ \iint_{D_s} \frac{\delta(\bar{x} + Vs - x_0) \delta(\bar{y} - y_0) g(s)}{\sqrt{c_1^2 (t-s)^2 - [\bar{x} - (x - Vt)]^2 - (\bar{y} - y)^2}} d\bar{x} d\bar{y} \right] ds. \quad (6.2)$$

We let  $X = (\bar{x} + Vs - x_0)$  and  $Y = (\bar{y} - y_0)$  and substitute into (6.2) to obtain

$$A_2(x, y, t) =$$

$$\frac{1}{2\pi c_1} \int_0^t \left[ \iint_{D_s} \frac{\delta(X) \delta(Y) g(s) dX dY}{\sqrt{c_1^2 (t-s)^2 - [X + V(t-s) - (x - x_0)]^2 - [Y - (y - y_0)]^2}} \right] ds \quad (6.3)$$

where

$$D_s = \left[ (X, Y) : [X + V(t-s) - (x - x_0)]^2 + [Y - (y - y_0)]^2 \leq c_1^2 (t-s)^2 \right]. \quad (6.4)$$

In order to solve (6.3) we must determine those  $s$  for which (6.4) is satisfied. Letting  $\tau = (t-s)$  for  $(t-s) \geq 0$ , (6.4) simplifies to

$$(c_1^2 - V^2) \tau^2 + 2V(x - x_0) \tau - [(x - x_0)^2 + (y - y_0)^2] \geq 0. \quad (6.5)$$



Note that X and Y are not included in (6.5) since the integration in (6.3) with respect to X and Y makes both X and Y become zero. Solving (6.5) for  $\tau$  yields

$$\tau_{\pm} = \frac{-V(x-x_0) \pm \sqrt{c_s^2(x-x_0)^2 + (c_s^2 - V^2)(y-y_0)^2}}{c_s^2 - V^2}. \quad (6.6)$$

The values of  $\tau_+$  and  $\tau_-$  will depend on whether  $c_s^2 > V^2$  or  $c_s^2 < V^2$ . We shall examine both cases in the following sections.

1.  $c_s^2 > V^2$

When the long-wave phase speed is greater than the velocity of the submarine, we see from (6.6) that  $\tau_+ > 0$  and  $\tau_- < 0$ . Factoring (6.5) yields

$$(c_s^2 - V^2)(\tau - \tau_+)(\tau - \tau_-) \geq 0 \quad (6.7)$$

where  $\tau_+$  and  $\tau_-$  are defined in (6.6). Since the left hand side of (6.7) must be greater than or equal to zero,  $\tau$  must be greater than or equal to  $\tau_+$  ( $\tau \geq \tau_+$ ). Hence,  $s$  must be less than or equal to  $(t - \tau_+)$ . Therefore, the upper limit of  $s$  in (6.3) becomes  $(t - \tau_+)$ . Now, integrating (6.3) with respect to X and Y yields

$$A_s(x, y, t) = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_0^{t-\tau_s} \frac{g(s) ds}{\sqrt{((t-\tau_s)-s)((t-\tau_s)-s)}} \quad (6.8)$$

where  $\tau_s$  and  $\tau$  are defined in (6.6). Using (6.1c), we break (6.8) into the following parts

$$A_s(x, y, t) = I_1 + I_2, \quad (6.9a)$$

where

$$I_1 = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_0^s \frac{g(s) ds}{\sqrt{((t-\tau_s)-s)((t-\tau_s)-s)}}, \quad (6.9b)$$

and

$$I_2 = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_s^{t-\tau_s} \frac{ds}{\sqrt{((t-\tau_s)-s)((t-\tau_s)-s)}}. \quad (6.9c)$$

We first look at the behavior of  $I_1$  as  $t \rightarrow \infty$ . We see from (6.9b) that for long time,  $I_1$  is  $O(1/t)$ . Therefore, we can neglect this portion of  $A_s(x, y, t)$ . Now in evaluating  $I_2$  for long time, we let  $\tau = (t-s)$  so that  $I_2$  simplifies to

$$I_2 = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_{\tau_-}^{\tau_+} \frac{d\tau}{\sqrt{(\tau - \tau_+) (\tau - \tau_-)}}. \quad (6.10)$$

We complete the square of the denominator of (6.10) to obtain

$$I_2 = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_{\tau_-}^{\tau_+} \frac{c^2 \tau}{\sqrt{(\tau - a)^2 - b^2}}. \quad (6.11)$$

Here  $a = (\tau_+ + \tau_-)/2$ , and  $b^2 = (\tau_+ - \tau_-)^2/4$ . Letting  $bu = (\tau - a)$  and substituting into (6.11) then yields

$$I_2 = \frac{1}{2\pi c_s (c_s^2 - v^2)^{1/2}} \int_{\frac{\tau_- - a}{b}}^{\frac{\tau_+ - a}{b}} \frac{du}{\sqrt{u^2 - 1}}. \quad (6.12)$$

Integrating  $I_2$ , we find

$$I_2 = \frac{1}{2\pi c_s (c_s^2 - V^2)^{1/2}} \left[ \ln \left| \frac{(t-s_0-a)}{b} + \left[ \frac{(t-s_0-a)^2}{b^2} - 1 \right]^{1/2} \right| \right] \quad (6.13)$$

$$+ \frac{-1}{2\pi c_s (c_s^2 - V^2)^{1/2}} \left[ \ln \left| \frac{(\tau_+ - a)}{b} + \left[ \frac{(\tau_+ - a)^2}{b^2} - 1 \right]^{1/2} \right| \right].$$

For long time  $t$ ,  $I_2$  is  $O(\ln t)$ .

In the case where  $c_s^2 > V^2$ , we see that the long time solution for  $A_s(x, y, t)$  is  $O(\ln t)$ . Next we examine the case for which  $c_s^2 < V^2$ .

## 2. $c_s^2 < V^2$

When the long-wave phase speed is less than the velocity of the submarine, we see from (6.5) that  $x$  must be greater than  $x_0$ , since  $\tau$  is greater than or equal to zero. We must reexamine (6.6) to see if a positive real value for  $\tau$  is possible. Rearranging (6.6) for  $c_s^2 < V^2$  yields

$$\tau_{\pm} = \frac{V(x-x_0) \pm \sqrt{c_s^2(x-x_0)^2 - (V^2 - c_s^2)(y-y_0)^2}}{V^2 - c_s^2}. \quad (6.14)$$

From (6.14) we see that a real solution for  $\tau$  exists only if the following inequality holds

$$c_s^2(x-x_0)^2 - (V^2 - c_s^2)(y-y_0)^2 \geq 0. \quad (6.15)$$

Hence, a solution exists only within the region defined by

$$x \geq x_0 + \left[ \frac{V^2}{c_s^2} - 1 \right]^{1/2} (y - y_0) \quad \text{for } y > y_0, \quad (6.16a)$$

and

$$x \geq x_0 + \left[ \frac{V^2}{c_s^2} - 1 \right]^{1/2} (y_0 - y) \quad \text{for } y < y_0. \quad (6.16b)$$

Since  $\tau_+$  and  $\tau_-$  are both positive, we see from (6.7) that  $\tau$  must be between  $\tau_+$  and  $\tau_-$ , i.e.,  $(\tau_- \leq \tau \leq \tau_+)$ . Therefore,  $s$  must be less than or equal to  $(t - \tau_-)$ .

To solve  $A_1(x, y, t)$  for  $c_s^2 < V^2$  as  $t \rightarrow \infty$ , we start with (6.3). Integrating (6.3) with respect to  $X$  and  $Y$  yields

$$A_1(x, y, t) = \frac{1}{2\pi c_s (V^2 - c_s^2)^{1/2}} \int_0^{t-\tau_+} \frac{g(s) ds}{\sqrt{((t-\tau_+) - s)((t-\tau_-) - s)}} \quad (6.17)$$

where  $\tau_+$  and  $\tau_-$  are defined in (6.14). Using (6.1c), we break (6.17) into the following parts

$$A_1(x, y, t) = I_3 + I_4, \quad (6.18a)$$

where

$$I_3 = \frac{1}{2\pi c_s (V^2 - c_s^2)^{1/2}} \int_0^s \frac{g(s) ds}{\sqrt{((t-\tau_+) - s)((t-\tau_-) - s)}}, \quad (6.18b)$$

and

$$I_4 = \frac{1}{2\pi c_s (c_s^2 - V^2)^{1/2}} \int_s^{t-\tau_-} \frac{ds}{\sqrt{((t-\tau_+) - s)((t-\tau_-) - s)}}. \quad (6.18c)$$

Looking at the behavior of  $I_3$  as  $t \rightarrow \infty$ , we see from (6.18b) that  $I_3$  is  $O(1/t)$ . Therefore, we can neglect this portion of  $A_1(x, y, t)$ . The evaluation of  $I_4$  is similar to the analysis as developed for  $I_2$ :

$$I_4 = \frac{1}{2\pi c_s (V^2 - c_s^2)^{1/2}} \left[ \ln \left| \frac{(t-s_0-a)}{b} + \left[ \frac{(t-s_0-a)^2}{b^2} - 1 \right]^{1/2} \right| \right] \quad (6.19)$$

$$+ \frac{-1}{2\pi c_s (V^2 - c_s^2)^{1/2}} \left[ \ln \left| \frac{(\tau_- - a)}{b} + \left[ \frac{(\tau_- - a)^2}{b^2} - 1 \right]^{1/2} \right| \right].$$

Here  $a = (\tau_+ + \tau_-)/2$  and  $b^2 = (\tau_+ - \tau_-)^2/4$ . For long time,  $I_4$  is  $O(\ln t)$ .

In the case where  $V^2 > c_s^2$ , we see that the long time solution for  $A_s(x, y, t)$  is  $O(\ln t)$ . Next, we examine the resonant case where  $c_s^2 = V^2$ .

#### B. $c_s^2 = V^2$

Previously, we developed the long time solution for the cases when  $c_s^2 > V^2$  and  $c_s^2 < V^2$  with the interesting result that in both cases the solution had growth of  $O(\ln t)$ . The reason for this growth is the failure of Huygens' principle in two dimensions. Though  $\ln(t)$  tends to infinity for long time, it is a function of extremely slow growth, and for the times we are interested in, it can be considered as a bounded function.

In the resonant case, we are concerned with the behavior of  $A_s(x, y, t)$  as  $t \rightarrow \infty$  for the case when the speed of the submarine is equal to the long-wave phase speed ( $c_s^2 = V^2$ ).

To determine if a solution to (6.3) is possible for  $c^2 = V^2$ , we must reexamine (6.5). From (6.5) we see that a solution exists only if  $s$  satisfies the following inequality

$$s \leq t - \frac{(x-x_0)^2 + (y-y_0)^2}{2V(x-x_0)} \quad \text{for} \quad V(x-x_0) > 0. \quad (6.20)$$

Integrating (6.3) with respect to  $X$  and  $Y$  yields

$$A_s(x, y, t) = \frac{1}{2\pi c} \int_0^A \frac{g(s) ds}{\sqrt{2V(x-x_0)(t-s) - [(x-x_0)^2 + (y-y_0)^2]}}. \quad (6.21)$$

Here  $A = [t - ((x-x_0)^2 + (y-y_0)^2) / (2V(x-x_0))]$ . Using (6.1c), we break (6.21) into the following parts

$$A_s(x, y, t) = I_3 + I_6, \quad (6.22a)$$

where



$$I_3 =$$

$$\frac{1}{2\pi c_s} \int_0^t \frac{g(s) ds}{\sqrt{2V(x-x_0)(t-s) - [(x-x_0)^2 + (y-y_0)^2]}} \quad (6.22b)$$

and

$$I_6 =$$

$$\frac{1}{2\pi c_s} \int_0^A \frac{ds}{\sqrt{2V(x-x_0)(t-s) - [(x-x_0)^2 + (y-y_0)^2]}} \quad (6.22c)$$

Here  $A = [t - ((x-x_0)^2 + (y-y_0)^2) / (2V(x-x_0))]$ .

Looking at the behavior of  $I_3$  as  $t \rightarrow \infty$ , we see from (6.22b) that for long time,  $I_3$  is  $O(1/t^{1/2})$ . Therefore we can neglect this portion of  $A_s(x, y, t)$ . Integrating  $I_6$  directly, we obtain

$$I_6 =$$

$$\frac{1}{2\pi c_s} \frac{\sqrt{2V(x-x_0)(t-s_0) - [(x-x_0)^2 + (y-y_0)^2]}}{V(x-x_0)} \quad (6.23)$$

From (6.23) we see that  $I_6$  is  $O(t^{1/2})$  as  $t \rightarrow \infty$ . Therefore, we see that the solution for  $A_s(x, y, t)$  at resonance grows like  $O(t^{1/2})$  for long time.

To derive the long time solution ( $t \rightarrow \infty$ ) for  $A_1(x,y,t)$ , we began by introducing a moving point source for the topography,  $f_1(x+Vs,y,s)$ , with the knowledge that if a solution was attainable by using (6.1c), we could solve (5.23) for any form of the topography by the principle of superposition. We first looked at the case where  $c_1^2 > V^2$  and determined that the long time solution for  $A_1(x,y,t)$  was  $O(\ln t)$ . Next, we looked at the case where  $c_1^2 < V^2$  and saw that a solution existed only within a certain region defined by (6.16a,b). The long time solution for  $A_1(x,y,t)$  in this case was also  $O(\ln t)$ . We attribute the  $O(\ln t)$  growth to the failure of Huygens' principle in two dimensions. For the resonant case where  $c_1^2 = V^2$ , we found the solution for  $A_1(x,y,t)$  to be  $O(t^{1/2})$  for long time. From this analysis, we see in this problem that if a submarine patrols at or near a long-wave phase speed a significant resonant could occur for long times.

## VII. CONCLUSIONS

This thesis has presented a theoretical study of the flow of a stratified fluid over variable topography as a model of the resonant interaction of a submarine's wake with a stratified fluid. We considered the case in which a submarine patrols at a speed near the long-wave phase speed of an internal wave. Internal waves are generated by the flow around the submarine's wake.

The long time solution obtained in our model for the cases when  $c_1^2 > V^2$  and  $c_1^2 < V^2$  had growth of  $O(\ln t)$ . (see (6.13) and (6.19)). We attribute the  $O(\ln t)$  growth to the failure of Huygens' principle in two dimensions. For the resonant case where  $c_1^2 = V^2$ , we found the solution to be  $O(t^{1/2})$  for long time. (see (6.23)). It is important to note that for the time scales we are concerned with, both long time solutions for the nonresonant cases are bounded and have extremely slow growth. Therefore it is the resonant case which is important.

The growth in time occurring in the resonant case indicates that perhaps nonlinear effects, balanced by dispersion, could cause significant upstream disturbances. From these disturbances, the possibility exists of detecting a submarine by the internal waves its wake generates.

Further studies in this problem should begin by rescaling the asymptotic expansions to balance the leading order quadratic nonlinear terms with the topographic forcing. This will achieve a balance between nonlinearity and dispersion. We expect the solution to this problem to show that significant upstream disturbances could occur when the submarine patrols at a speed at or near a long-wave phase speed.

If internal waves were generated only by underwater moving objects then the problem of detecting the submarine by the internal waves its wake generates would be made simpler, but this is not the case. Internal waves are a natural occurring phenomenon in the oceans. This complicates the problem because in order to detect the submarine by the internal waves that it generates, the ambient internal wave patterns at the location of the submarine must be known. The problem of discerning ambient internal waves from internal waves generated by the submarine's wake is similar to the problem of detecting a submarine passively (such as with a tail). The sonar operator filters out the ambient noises or background noises of the ocean from the incoming signal to detect the acoustic signature of the submarine. From further understanding of the ocean's naturally occurring internal waves, we may one day be able to discern internal waves generated by a submarine from those of the ocean.

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