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13. Abstract (Maximum 200 words). While sampling at a Nyquist frequency equal to the highest frequency present in the data (critical sampling) is sufficient to prevent aliasing in both the data and the autocorrelation of a bandlimited energy signal, the sampling requirements for the avoidance of aliasing in higher-order correlations and spectra are not the same. Also, there is a difference in aliasing effects depending on whether one samples the original continuous-time signal and calculates the autocorrelation or one samples the continuous-time autocorrelation. This distinction between sampling procedures must be made for correlations of higher order, as well, for which not only the type of aliasing but also the sampling requirements to prevent aliasing differ. In particular, if one samples the continuous-time autocorrelation or autotricorrelation, critical sampling is sufficient to prevent aliasing. In practice, however, it is not usually the continuous-time autocorrelation or autotricorrelation that is sampled. Generally, it is the original continuous-time signal that is sampled and used to calculate the discrete-time autocorrelation or autotricorrelation.			
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Sampling requirements and aliasing for higher-order correlations

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While sampling at a Nyquist frequency equal to the highest frequency present in the data (critical sampling) is sufficient to prevent aliasing in both the data and the autocorrelation of a bandlimited energy signal, the sampling requirements for the avoidance of aliasing in higher-order correlations and spectra are not the same. Also, there is a difference in aliasing effects depending on whether one samples the original continuous-time signal and calculates the autocorrelation or one samples the continuous-time autocorrelation. This distinction between sampling procedures must be made for correlations of higher order, as well, for which not only the type of aliasing but also the sampling requirements to prevent aliasing differ. In particular, if one samples the continuous-time autobicorrelation or autotricorrelation, critical sampling is sufficient to prevent aliasing. In practice, however, it is not usually the continuous-time autobicorrelation or autotricorrelation that is sampled. Generally, it is the original continuous-time signal that is sampled and used to calculate the discrete-time autobicorrelation or autotricorrelation. In this case, to prevent aliasing, the sampling interval for the autobicorrelation must be no greater than two-thirds the interval associated with critical sampling, and no greater than one-half for the autotricorrelation. Numerical calculations of autocorrelation, autobicorrelation, and autotricorrelation zero-lag values corresponding to spectral area and volume as well as bispectral contour plots for two model bandlimited energy signals are presented as demonstrations of these conclusions. Sampling requirements for higher-order correlations of rectified signals are also discussed. If the signal is to be rectified for use in a correlation detector or time delay estimator, then the sampling rate must increase to accommodate the higher frequencies that usually result from the process of rectification. Spectral masking filters, which remove aliasing in higher-order correlations, are defined and a bispectral filter is applied to some representative energy signals. Lag domain convolution filters for the removal of aliasing in the bicorrelation and tricorrelation are also given. These filters assume critical sampling or finer for the original signal to remove aliasing effects totally

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INTRODUCTION

Higher-order correlations and spectra may be used to extract information about signals, which the autocorrelation or energy/power spectrum does not contain, such as phase, deviations from Gaussianity, and the presence of nonlinearities in a signal. In addition, the special properties associated with higher-order correlations and spectra introduce a myriad of uses for them in the areas of signal detection (Nagata, 1978; Hinich, 1982; Dwyer, 1984, 1985; Brockett *et al.*, 1987; Hinich *et al.*, 1989; G. Ioup *et al.*, 1989a,b; Sullivan and Hinich, 1990; Hinich and Wilson, 1990; Pike *et al.*, 1991; Pflug *et al.*, 1992b), time delay estimation (Nikias and Pan, 1988; Pflug *et al.*, 1990, 1993), and phase retrieval (Bartelt *et al.*, 1984; Matsuoka and Ulrych, 1984; Nikias and Raghuveer, 1987), which in turn may be used for target discrimination, localization, classification, etc. For most practical applications, the signal to be studied is first properly digitized, i.e., the signal is generally bandlimited according to the sampling rate and then sampled. For energy signals, direct calculation of

higher-order correlations using a digitized signal requires finer sampling relative to the highest frequency present than the sampling requirements of the autocorrelation to avoid aliasing, in the absence of a masking filter (Pflug *et al.*, 1992a).

In Secs. I–III, definitions and descriptions of aliasing in the autocorrelation, autobicorrelation, and autotricorrelation are presented. A discussion of the differences in sampling requirements for (a) discrete-time correlations from sampled data versus (b) those obtained by sampling continuous-time correlations is given in Sec. IV. In Sec. V, examples of critically sampled energy signals with aliasing present in the autobicorrelation are shown and tables of spectral area and volume, as defined by correlation central ordinate values, are given to demonstrate the sampling requirements for second-, third-, and fourth-order correlations. Higher-order masking filters are defined in Sec. VI and a bispectral masking filter is applied to remove aliasing in the autobicorrelation of the signals introduced in the previous section, which for this application are critically

sampled. Finally, in Sec. VII, the effects of rectification on signal sampling requirements are discussed.

I. ALIASING IN THE AUTOCORRELATION

Let $x(t)$ represent a real continuous-time function with continuous-frequency Fourier transform (Bracewell, 1986)

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-i2\pi ft) dt.$$

The inverse transform is then

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(i2\pi ft) df.$$

Existence of the Fourier transform is discussed by Papoulis (1962) and Bracewell (1986). If $x(t)$ is sampled in time with sampling interval Δt [resulting in the discrete-time Fourier transform (DTFT)], and then a time window is applied and the transform is sampled in frequency with N points, the result is the discrete-time Fourier series (DTFS) pair given by Marple (1987)

$$X(k \Delta f) = \Delta t \sum_{n=0}^{N-1} x(n \Delta t) \exp[-i2\pi(n \Delta t)(k \Delta f)/N]$$

$$(k = -N/2, \dots, -N/2 - 1),$$

with inverse

$$x(n \Delta t) = \Delta f \sum_{k=-N/2}^{N/2-1} X(k \Delta f) \exp[i2\pi(n \Delta t)(k \Delta f)/N]$$

$$(n = 0, \dots, n-1),$$

and $\Delta f = 1/(N \Delta t)$. Although Marple (1987) distinguishes the DTFS from the DFT, for the remainder of this paper these forms will be referred to as the DFT pair, and we use t in place of $n \Delta t$ and f in place of $k \Delta f$ in discrete equations. In practice, a frequency window should be applied before the original time domain signal is sampled. If the signal is bandlimited in frequency and the continuous-time function is oversampled, i.e., $f_N = 1/(2\Delta t)$ is chosen such that $f_N > f_i$, where f_N represents the Nyquist or foldover frequency, and f_i is the top or highest frequency present in the signal, then the centers of the replications of the continuous transform are spaced greater than a distance of $2f_i$ apart, and do not overlap. In this case, the principal replication may be isolated using a rectangular frequency window defined by Bracewell (1986),

$$H(f) = \begin{cases} 1, & |f| < f_i \\ 0, & |f| > f_i \end{cases}$$

for which $\pm f_i$ define the minimum passband filter limits. However, the limits of the filter can be any frequency up to $\pm(2f_N - f_i)$. If the continuous-time function is bandlimited at f_i with the frequency content at $\pm f_i$ finite, and the function is sampled at $f_N = f_i$, then the centers of the replications in the frequency domain are spaced exactly $2f_i$ apart (critical sampling) and only overlap at the endpoints of each replication. No aliasing occurs. However, if the function is not sufficiently sampled, the principal replica-

tion is not an accurate representation of the true continuous-time transform, that is, aliasing has occurred. In other words, if the function is sampled with sampling interval $\Delta t > \Delta t_c$, where Δt_c is the sampling interval necessary for critically sampling a signal, then the transform replications overlap in the principal domain and the transform at each point of overlap will be the sum of the unaliased and overlapping transforms.

To avoid confusion, we must make a clear distinction between the Nyquist sampling frequency, f_N , and the highest or top frequency present in a bandlimited signal, denoted f_i . Whereas f_N is purely a characteristic of the sampling interval, f_i is purely a function of the individual signal. For an essentially naturally bandlimited signal, these values are completely independent of each other. However, for a noise signal with theoretically infinite bandwidth, f_i is dependent on the initial sampling rate, although the signal may be interpolated (or modified with a masking filter as we discuss later) to avoid aliasing when using the digitized signal in higher-order correlation calculations. The calculation of the autocorrelation of a sampled signal requires only the bandwidth of the original signal to avoid aliasing. Therefore, if a discrete energy signal $x(t)$ has autocorrelation,

$$a_1(\tau) = \sum_{t=0}^{N-1} x(t)x(t+\tau)\Delta t,$$

and energy spectrum defined by

$$ES_x(f) = X(f)X^*(f) = \sum_{t=0}^{N-1} a_1(t) \exp(-i2\pi ft) \Delta t,$$

then $x(t)$ must be sampled with sampling interval $\Delta t \leq \Delta t_c = 1/(2f_i)$ to avoid aliasing in the autocorrelation (Bracewell, 1986).

II. THE BICORRELATION

A. Aliasing in the bicorrelation

The autobicorrelation and corresponding bispectrum of random signals are defined in Brillinger and Rosenblatt (1967a) and Hasselmann *et al.* (1963). The autobicorrelation of an energy signal is defined as a natural extension of the autocorrelation of an energy signal (Bracewell, 1986; G. Ioup *et al.*, 1989a; Pflug *et al.*, 1992a,b), and for continuous-time $x(t)$ is given by

$$a_2(\tau_1, \tau_2) = \int_{-\infty}^{\infty} x(t)x(t+\tau_1)x(t+\tau_2)dt,$$

and for discrete-time $x(t)$ by

$$a_2(\tau_1, \tau_2) = \sum_{t=0}^{N-1} x(t)x(t+\tau_1)x(t+\tau_2)\Delta t.$$

Its two-dimensional continuous-time Fourier transform, the bispectrum, is defined by (Pflug *et al.*, 1992a)

$$B(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_2(\tau_1, \tau_2) \times \exp[-i2\pi(\tau_1 f_1 + \tau_2 f_2)] d\tau_1 d\tau_2,$$

and in discrete form,

$$B(f_1, f_2) = \sum_{\tau_1=0}^{N-1} \sum_{\tau_2=0}^{N-1} a_2(\tau_1, \tau_2) \exp[-i2\pi(\tau_1 f_1 + \tau_2 f_2)] \Delta\tau_1 \Delta\tau_2.$$

The bispectrum of the discrete-time autocorrelation is replicated with period $2f_N$ in the f_1 - f_2 plane; however, the principal domain or principal replication may be defined by restricting f_1 and f_2 such that $-f_N \leq f_1 \leq f_N$ and $-f_N \leq f_2 \leq f_N$.

It has been shown that the bispectrum of a real energy transient may be expressed as a product of one-dimensional Fourier transforms (Pflug *et al.*, 1992a), that is,

$$B(f_1, f_2) = X(f_1)X(f_2)X^*(f_1 + f_2)$$

or

$$B(f_1, f_2) = X(f_1)X(f_2)X(f_3),$$

where $f_3 = -f_1 - f_2$. The three-index notation was originally introduced by Brillinger and Rosenblatt (1967b) and discussed in Rosenblatt (1985) for stationary random processes. However, it applies to finite length energy transients as well (G. Ioup *et al.*, 1989a,b, 1990; Pflug *et al.*, 1992a,b). A derivation of this expression for energy signals is given by Pflug *et al.* (1992a). The total unaliased domain (TUD) of the autobispectrum of a sampled signal is the result of the overlap of the domains of the two factors $X(f_1)X(f_2)$ and $X(-f_1 - f_2)$ (G. Ioup *et al.*, 1989a, 1990; Pflug *et al.*, 1989, 1992a). When the autocorrelation is calculated from discrete-time data, the factors in the above equation for $B(f_1, f_2)$ must be replicated infinitely with period $2f_N$ in the bifrequency plane, i.e.,

$$B(f_1, f_2) = \left(\sum_{k_1=-\infty}^{\infty} X(f_1 + 2k_1 f_N) \right) \times \left(\sum_{k_2=-\infty}^{\infty} X(f_2 + 2k_2 f_N) \right) \times \left(\sum_{k_3=-\infty}^{\infty} X^*(f_1 + f_2 + 2k_3 f_N) \right).$$

This result is derived in Appendix A for the bispectrum of the bicoherence calculated from discrete-time energy signals. The corresponding result for discrete-time stationary stochastic signals is derived in Appendix B.

The total domain of the bispectrum results from the overlap of each individual term in each summation of the above equation, as shown for a small part of the total bifrequency domain by the shaded regions in Fig. 1 for a signal with bottom frequency $f_b = 0$ and $f_N = f_t$. The analysis of the overlap is facilitated by considering the domains of the replications of $X(f_1)X(f_2)$ separately from the domains of the replications of $X(-f_1 - f_2)$ (G. Ioup *et al.*, 1989a; Pflug *et al.*, 1992a). In Fig. 1 the factors that contribute to the base replication are shaded, with the principal replication of the TUD outlined. The area in the principal replication outside the TUD of the bispectrum in Fig.

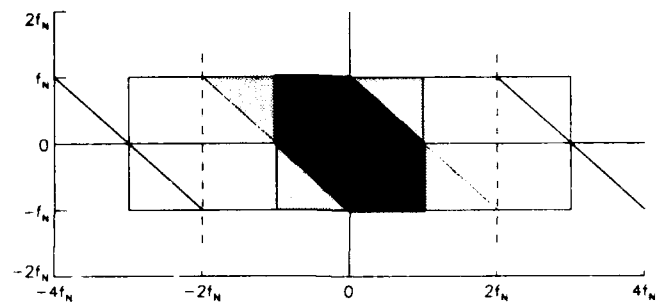


FIG. 1. Domain of the bispectral factors which overlap to form the TUD and TAD of the principal replication of the bispectrum when a signal is sampled such that $f_N = f_t$.

1 is the total aliased domain (TAD). This aliasing in the bispectrum is a result of neighboring diagonal factors $X(-f_1 - f_2 \pm 2f_N)$ overlapping the principal replication, similar to aliasing in the ordinary spectrum.

Figure 1 shows aliasing in the bispectrum as a result of critically sampling the original $x(t)$. While the type of aliasing caused by overlap of the factor $X(f_1)X(f_2)$ and the replications of the factor $X(-f_1 - f_2)$ to the immediate left and right of the principal diagonal replication is removable via a masking filter (which is discussed in more detail later), sampling below the critical rate results in additional overlap of the factor $X(f_1)X(f_2)$ with $X(f_1 \pm 2f_N)X(f_2 \pm 2f_N)$ as well as the diagonal factors, causing aliasing that is not removable. To avoid aliasing of either kind in the bispectrum of a sampled signal, the centers of the replications must be a distance at least $3f_t$ apart. For the energy spectrum, if f_t is the top frequency present in a signal, then $\Delta t_C \leq 1/(2f_t)$; but for the bispectrum the sampling interval Δt_B must be (Pflug *et al.*, 1992a)

$$\Delta t_B \leq 1/(3f_t) = (2/3)\Delta t_C.$$

If the signal is sampled such that $f_t \leq f_N \leq 3f_t/2$, then the aliasing can be removed by using a masking filter, as discussed in Sec. VI.

B. Symmetries in the bispectrum

Permuting arguments in the expression $B(f_1, f_2) = X(f_1)X(f_2)X(-f_1 - f_2)$ leaves the value of the bispectrum unchanged. This results in six lines of symmetry which intersect the principal replication of the bispectrum (Brillinger and Rosenblatt, 1967b):

$$\begin{aligned} f_1 &= f_2, & 2f_1 + f_2 &= 0, & f_1 + 2f_2 &= 0, \\ f_1 &= 0, & f_2 &= 0, & f_1 &= -f_2. \end{aligned}$$

The last three symmetry lines involve a complex conjugate operation. The six symmetry lines divide the TUD into 12 triangular domains, each containing the same bispectral values. From the more general expression given above for the bispectrum of a bicoherence calculated from sampled data, we find four more symmetry lines which intersect the principal replication of the bispectrum, that is, for $k_1 = k_2 = 0$ and $k_3 = \pm 1$, the aliased parts of the bispectrum are divided by the lines

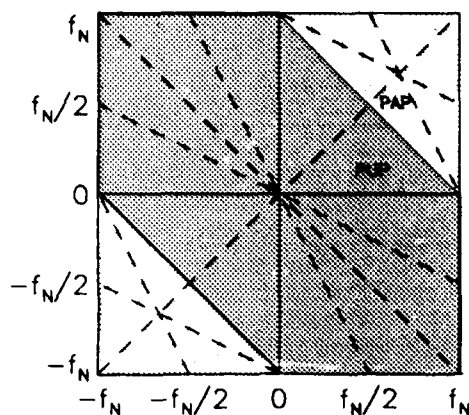


FIG. 2. Symmetry lines in the TUD and TAD of the bispectrum calculated from a discrete signal sampled such that $f_N = f_i$. Symmetry lines in the TUD apply to the bispectrum of a continuous-time autocorrelation as well.

$$2f_1 + f_2 = \pm 2f_N$$

and

$$f_1 + 2f_2 = \pm 2f_N.$$

Like the TUD, the total aliased domain is divided into 12 regions. The axes and the dashed lines shown in Fig. 2 are the lines of symmetry present in the bispectrum when a signal is sampled such that $f_N = f_i$ ($f_b = 0$ in this figure). The entire bispectrum may be mapped from values in the principal unaliased polygon (PUP) and the principal aliased polygon (PAP) using these symmetries (Nikias and Raghuvver, 1987; Hinich and Wolinsky, 1988; Pflug *et al.*, 1992a).

The symmetries in the bispectrum are a function of the sampling, i.e., a function of f_N , regardless of the f_i for the individual signal. The bispectral symmetries are shown in Fig. 3 for a signal with $0 < f_b < f_i/2$ and sampled with f_N chosen to be greater than f_i . If the signal is sampled such that $f_N = f_i$, then the bispectral domain is slightly different, as shown in Fig. 4. In this figure, PAP is a four-sided polygon, not a triangle as shown in Fig. 3 with $f_N > f_i$.

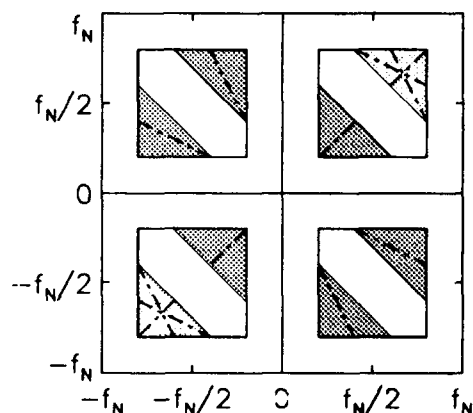


FIG. 3. Domain of the bispectrum of a discrete signal with $0 < f_b < f_i/2$, sampled such that $f_N > f_i$.

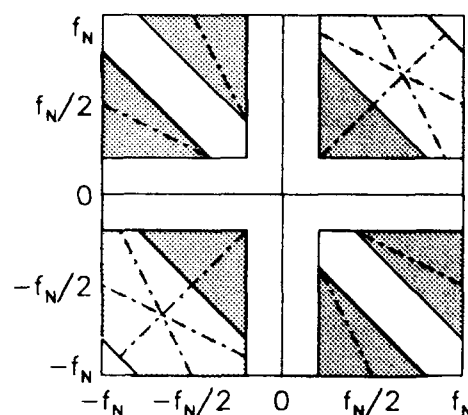


FIG. 4. Domain of the bispectrum of a discrete signal with $0 < f_b < f_i/2$, sampled such that $f_N = f_i$.

The TUD disappears for $f_b > f_i/2$, in general. If f_N is chosen such that $3f_b/2 < f_N < 3f_i/2$ and $f_b > f_i/2$, then the TUD of the bispectrum disappears while the TAD is still present (Pflug *et al.*, 1992a).

III. THE TRICORRELATION

A. Aliasing in the tricorrelation

To discuss aliasing in the tricorrelation, we follow reasoning similar to that given for the bicoherence. The continuous-time autotricorrelation $a_3(\tau_1, \tau_2, \tau_3)$ and its three-dimensional Fourier transform are given by (Brillinger and Rosenblatt, 1967a; G. Ioup *et al.*, 1989a)

$$a_3(\tau_1, \tau_2, \tau_3) = \int_{-\infty}^{\infty} x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)dt$$

and

$$T(f_1, f_2, f_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_3(\tau_1, \tau_2, \tau_3) \times \exp[-i2\pi(\tau_1 f_1 + \tau_2 f_2 + \tau_3 f_3)] d\tau_1 d\tau_2 d\tau_3,$$

and the discrete-time autotricorrelation and trispectrum are given by

$$a_3(\tau_1, \tau_2, \tau_3) = \sum_{t=0}^{N-1} x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)\Delta t$$

and

$$T(f_1, f_2, f_3) = \sum_{\tau_1=0}^{N-1} \sum_{\tau_2=0}^{N-1} \sum_{\tau_3=0}^{N-1} a_3(\tau_1, \tau_2, \tau_3) \times \exp[-i2\pi(\tau_1 f_1 + \tau_2 f_2 + \tau_3 f_3)] \Delta \tau_1 \Delta \tau_2 \Delta \tau_3.$$

The trispectrum of a real finite energy transient may equivalently be expressed as (Brillinger and Rosenblatt, 1967b; Dalle Molle and Hinich, 1989; Pflug, 1990; Pflug *et al.*, 1992a)

$$T(f_1, f_2, f_3) = X(f_1)X(f_2)X(f_3)X^*(f_1 + f_2 + f_3)$$

or

$$T(f_1, f_2, f_3) = X(f_1)X(f_2)X(f_3)X(f_4),$$

where $f_4 = -f_1 - f_2 - f_3$. For an autotricorrelation calculated from a sampled signal, each factor is replicated infinitely in the trifrequency plane, i.e.,

$$T(f_1, f_2, f_3) = \left(\sum_{k_1=-\infty}^{\infty} X(f_1 + 2k_1 f_N) \right) \left(\sum_{k_2=-\infty}^{\infty} X(f_2 + 2k_2 f_N) \right) \left(\sum_{k_3=-\infty}^{\infty} X(f_3 + 2k_3 f_N) \right) \times \left(\sum_{k_4=-\infty}^{\infty} X^*(f_1 + f_2 + f_3 + 2k_4 f_N) \right).$$

The TUD of the trispectrum calculated from a sampled signal is defined by the overlap of the nonzero domain of the two factors $X(f_1)X(f_2)X(f_3)$ and $X^*(f_1 + f_2 + f_3)$ in the principal replication as discussed by Pflug *et al.* (1992a). The remaining area in the principal replication is a result of the overlap of the factor

$$X(f_1)X(f_2)X(f_3)$$

from the principal replication and

$$X(-f_1 - f_2 - f_3 - 2k_4 f_N)$$

with $k_4 = \pm 1$, and comprises the TAD of the trispectrum.

To avoid the kind of aliasing caused by sampling a signal with $f_N = f_i$, the centers of the trispectral replications must be a minimum distance of $4f_i$ apart. Hence, the sampling interval Δt_T must be (Pflug *et al.*, 1992a)

$$\Delta t_T \leq 1/(4f_i) = (1/2)\Delta t_C.$$

If the signal $x(t)$ is sampled such that $f_i \leq f_N \leq 2f_i$, then the aliasing in the trispectrum can be removed with a masking filter as discussed below.

B. Symmetries in the trispectrum

Symmetry planes in the trispectrum are derived by permuting the arguments in the above expression for the trispectrum of a trispectrum calculated from sampled data. When $k_1 = k_2 = k_3 = k_4 = 0$, the symmetry planes are (Brillinger and Rosenblatt, 1967b; Dalle Molle and Hinich, 1989; Pflug, 1990; Pflug *et al.*, 1992a)

$$f_1 = f_2, \quad f_2 = f_3, \quad f_1 = f_3, \quad 2f_1 + f_2 + f_3 = 0,$$

$$f_1 + 2f_2 + f_3 = 0, \quad f_1 + f_2 + 2f_3 = 0,$$

$$f_1 = -f_2, \quad f_1 = -f_3, \quad f_2 = -f_3,$$

$$f_1 = 0, \quad f_2 = 0, \quad f_3 = 0.$$

These planes divide the TUD of the trispectrum into two types of unaliased principal polyhedral domains, PUP1 and PUP2, each replicated 48 times. PUP1, as defined by Pflug *et al.* (1992a), disappears for $f_b > f_i/3$. The TAD

also contains two types of principal aliased polyhedra, PAP1 and PAP2, a result of the intersection of the symmetry planes

$$2f_1 + f_2 + f_3 = \pm 2f_N,$$

$$f_1 + 2f_2 + f_3 = \pm 2f_N,$$

$$f_1 + f_2 + 2f_3 = \pm 2f_N,$$

with the TAD. Both PAP1 and PAP2 are each replicated 48 times in the TAD. Detailed discussion and figures depicting the TUD, TAD, PUP, and PAP of the trispectrum can be found in Pflug *et al.* (1992a).

IV. SAMPLING THE DATA VERSUS SAMPLING THE CORRELATION

The characterization of aliasing for functionals such as the correlation contains a subtlety that is often unrecognized. That is, the nature of the aliasing and even the requirements to prevent it can depend on whether the correlation is calculated from data that are already sampled or the continuous-time correlation itself is sampled. Following the reasoning of Kanasewich (1973) the expression for the transform of the autocorrelation, which describes the effects of aliasing for a sampled continuous-time autocorrelation, is

$$\tilde{E}S_x(f) = \sum_{k=-\infty}^{\infty} |X(f - 2kf_N)|^2,$$

where

$$\tilde{X}(f) = \sum_{k=-\infty}^{\infty} X(f - 2kf_N)$$

is the aliased Fourier transform. The tilde is used to indicate aliased functions and transforms. This result does indeed describe the aliasing if the continuous-time autocorrelation itself is sampled. If, however, the discrete-time autocorrelation is calculated from data $\tilde{x}(t)$, which have already been sampled such that $\tilde{x}(t)$ contains aliasing, then the appropriate transform domain expression corresponding to the aliasing of the autocorrelation is

$$\tilde{E}S_x(f) = \left| \sum_{k=-\infty}^{\infty} X(f - 2kf_N) \right|^2.$$

The difference is that the unaliased transform in the first equation is replicated after squaring whereas the unaliased $X(f)$ in the second equation is replicated before squaring, giving rise to interference due to phase. The requirement to prevent aliasing, namely that $f_N \geq f_i$, with f_i the highest frequency present in the unsampled data $x(t)$, is the same for both cases, however.

The characterization of aliasing for higher-order correlations contains an additional striking feature. The unsampled continuous-time autobicorrelation transform, $X(f_1)X(f_2)X^*(f_1 + f_2)$, has a nonzero domain which can be determined from the overlap of the nonzero domains of the individual factors. This method has been used previously by G. Ioup *et al.* (1989a,b and 1990), and Pflug *et al.* (1992a). The nonzero domains of continuous-time

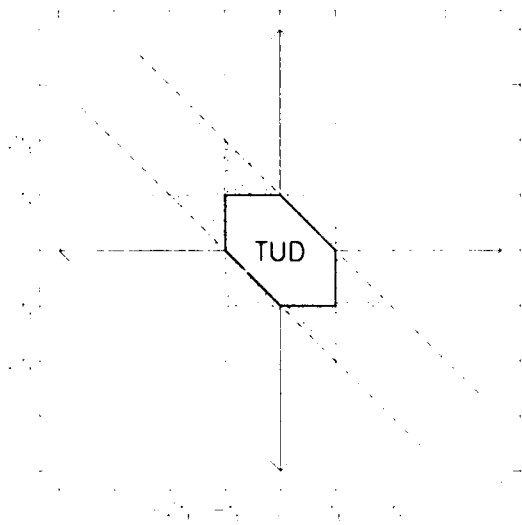


FIG. 5. TUD of the bispectrum of a continuous-time autocorrelation.

transforms $X(f_1)$ and $X(f_2)$ are the infinite strips defined by $-f_i \leq f_1 \leq f_i$ and $-f_i \leq f_2 \leq f_i$, respectively, and the nonzero domain of $X^*(f_1 + f_2)$ is the infinite strip defined by $-f_i \leq f_1 + f_2 \leq f_i$. These are shown in Fig. 5. The overlap of these three nonzero domains is the heavily shaded region in Fig. 5, which corresponds to the TUD pictured in Figs. 1 and 2. All the nonzero autobispectrum (the transform of the continuous-time autocorrelation) is contained in a square that is $2f_i$ on a side, so the sampling required to prevent aliasing if the continuous-time autocorrelation itself is sampled is $f_N \geq f_i$, the same as that required for the function and the autocorrelation. This result has been derived independently by Nielsen (1992). Pflug *et al.* (1992a) have shown that the sampling of the data needed to prevent aliasing for the discrete-time autocorrelation calculated from sampled data is $f_N \geq 3f_i/2$. An extension of this argument can be made for sampling the continuous-time autotricorrelation directly, for which the requirement to prevent aliasing is also $f_N \geq f_i$. If sampled data are to be used, the requirement is $f_N \geq 2f_i$. Note that there will be no TAD region if a continuous-time higher-order correlation is sampled directly.

It is important to consider the applicability of these two types of models. Most calculations of the correlation involve the use of sampled data, so the models for aliasing in the correlations due to calculations from sampled data are most often applicable. The sampling of the correlation directly occurs much less frequently and principally when the correlation itself is the result of a physical measurement. For these cases the model for directly sampling the continuous-time correlation applies.

The obvious difference between sampling the continuous-time correlation and calculating the bicorrelation from sampled data is the presence in the latter situation of what is referred to in this paper as the TAD. Aliasing in the bicorrelation and tricorrelation due to sampling such that $f_i \leq f_N < 3f_i/2$ and $f_i \leq f_N < 2f_i$, respectively, is removable via a masking filter, while sampling such that $f_N < f_i$ results in aliasing that is not generally

removable. One might choose not to refer to the TAD of the bispectrum and trispectrum as the aliased regions since they are completely removable and only exist when the higher-order correlation is calculated from data that are undersampled for this purpose. Hinich *et al.* (Hinich and Wolinsky, 1988; Hinich *et al.*, 1989; Hinich and Wilson, 1990) do not refer to the contents of this region as aliasing, but instead call the principal domain of this region the OT. Hasselmann *et al.* (1963) use the term aliasing to refer to this region of the bispectrum, and Brillinger and Rosenblatt (1967b) discuss these regions of the bispectrum and trispectrum in the section of their paper on aliasing. Our reasons for using the name aliasing for the TAD are given in the following discussion.

Aliasing is the lack of sufficient sampling to represent correctly the underlying continuous-time function caused by higher frequency content folded over onto, and masquerading as, lower frequency content. Sampling data such that $f_i \leq f_N < 3f_i/2$ for a bicorrelation calculation or such that $f_i \leq f_N < 2f_i$ for a tricorrelation calculation causes the resulting discrete-time correlations to be inaccurate representations of the continuous-time correlations, due to distortion of the Fourier transforms in their two- or three-dimensional principal domains, which result from the replication of the transforms produced by sampling the continuous-time data, as is the case for ordinary aliasing. Moreover, the filtering, which is normally done to reconstruct the continuous-time two- or three-dimensional function, i.e., excluding all frequencies outside the principal domain or replication, does not remove the TAD. It is possible to argue that a special filter should be used to reconstruct the higher-order correlation, indeed, the filter that we discuss, but this is not in keeping with general reconstruction for arbitrary two- or three-variable functions. An additional important feature is that while the autocorrelation calculated from data sampled such that $f_N = f_i$ is correct at the sample points, the higher-order autocorrelations calculated from the same data are incorrect at the sample points. For these important reasons, the regions of the bispectrum and trispectrum defined by sampling such that $f_i \leq f_N < 3f_i/2$ and $f_i \leq f_N < 2f_i$, respectively, are referred to as aliased domains.

V. EXAMPLES OF ALIASING IN THE CORRELATIONS OF SAMPLED SIGNALS

Two model energy signals are used to show the occurrence of aliasing in higher-order correlations and spectra due to sampling a signal at $f_N = f_i$. The first is a model of an amplitude and frequency modulated signal for a 20-Hz finback whale signal fitted by J. Ioup *et al.* (1988; G. Ioup *et al.*, 1989a) and modeled after a real signal shown in Watkins *et al.* (1987), with frequencies equal to and above 32-Hz set to zero for simplicity. The signal and its Fourier magnitude spectrum are shown in Fig. 6. Notice that amplitudes for frequencies below about 10 Hz are quite small. Hence the signal has a narrow-band character with f_i not much larger than $2f_b$, if f_b is considered to be near 10 Hz. Thus we expect the volume under its bispectrum to be small (Pflug *et al.*, 1992a). Applying the conditions de-

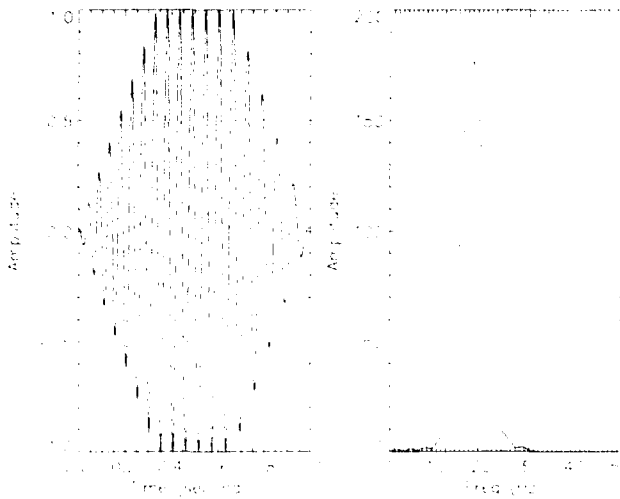


FIG. 6. Simulated whale transient and its Fourier magnitude spectrum.

scribed in earlier sections, the signal should be sampled with $f_N=32$ Hz to avoid aliasing in the autocorrelation, with $f_N=48$ Hz to avoid aliasing in the autobicorrelation, and with $f_N=64$ Hz to avoid aliasing in the autotricorrelation. For the whale signal of 1-s duration, this corresponds to 64 sample points to avoid aliasing in the autocorrelation, 96 sample points for the autobicorrelation, and 128 points for the autotricorrelation.

Ideally, if there is no aliasing in a correlation calculated from sampled data, finer sampling in a fixed time period will not change the zero-lag value, i.e., the area beneath the energy/power spectrum (Kanasewich, 1973). Other lag values could be used to demonstrate these properties. The whale signal is sampled with various numbers of points and the area beneath the energy spectrum and volumes beneath the bispectrum and trispectrum are calculated by zero-lag correlation value. Results to five significant digits are shown in Table I. The zero lag value of the rectified autobicorrelation is also given, and will be dis-

cussed later. While the autocorrelation requires only approximately 64 sample points, as predicted, for the effects of aliasing to become negligible to five significant digits, the autobicorrelation and autotricorrelation each require more sample points. The autobicorrelation requires the predicted 96 sample points before the aliasing disappears, and the hypervolume beneath the autotricorrelation is constant to five significant digits if a minimum of 104 points is used, somewhat fewer than predicted.

A color contour plot of the magnitude bispectrum of the whale signal sampled with 64 points is shown in Fig. 7 with the bispectral symmetry lines distinguishing the unaliased and aliased regions (compare to Fig. 2). As we predict, the bispectrum contains aliasing. However, the energy spectrum with a nonzero amplitude at a maximum of 32 Hz (see Fig. 6 for the Fourier magnitude spectrum) contains no aliasing at this rate of sampling. If 128 sample points are used, the aliasing in the bispectrum disappears, as shown in Fig. 8, which has a different normalization from that used for Fig. 7.

In contrast to the narrow-band whale signal, the second example is a narrow pulse (narrow in time, broadband in frequency) shown with its Fourier magnitude in Fig. 9. For this signal of 1-s duration, we expect that aliasing will not exist in the autocorrelation if the signal is sampled such that $f_N=f_t=256$ Hz, or with 512 sample points. To avoid aliasing in the autobicorrelation, the signal must be sampled with at least 768 points, and to avoid aliasing in the autotricorrelation, the signal must be sampled with at least 1024 points. We see in Table II that the zero lag value of the autocorrelation is constant to five significant digits if a minimum of 512 sample points is used. The autobicorrelation central ordinate is constant to five significant digits using a minimum of 704 points, and the autotricorrelation central ordinate is constant using a minimum of 832 points. Using the predicted 768 points or more for the

TABLE I. Time-domain calculations of correlation central ordinate values of the whale transient with various sampling rates.

Number of points	Whale transient			
	Autocorrelation (amp units) ² · s	Autobicorrelation (amp units) ³ · s	Autotricorrelation (amp units) ⁴ · s	Rectified autobicorrelation (amp units) ³ · s
32	2.2296×10^{-2}	3.4944×10^{-5}	1.0268×10^{-3}	4.6164×10^{-3}
48	0.27676	3.3260×10^{-2}	0.17467	0.21102
64	0.27708	3.0329×10^{-3}	0.16384	0.20488
72	0.27708	6.1817×10^{-4}	0.16094	0.20364
80	0.27708	6.8907×10^{-4}	0.16549	0.20585
88	0.27708	6.8659×10^{-4}	0.17463	0.21190
96	0.27708	6.8661×10^{-4}	0.17413	0.21093
104	0.27708	6.8661×10^{-4}	0.17414	0.21051
112	0.27708	6.8661×10^{-4}	0.17414	0.21129
120	0.27708	6.8661×10^{-4}	0.17414	0.21211
128	0.27708	6.8661×10^{-4}	0.17414	0.21161
136	0.27708	6.8661×10^{-4}	0.17414	0.21136
200	0.27708	6.8661×10^{-4}	0.17414	0.21143
300	0.27708	6.8661×10^{-4}	0.17414	0.21141
400	0.27708	6.8661×10^{-4}	0.17414	0.21142
500	0.27708	6.8661×10^{-4}	0.17414	0.21142
600	0.27708	6.8661×10^{-4}	0.17414	0.21142

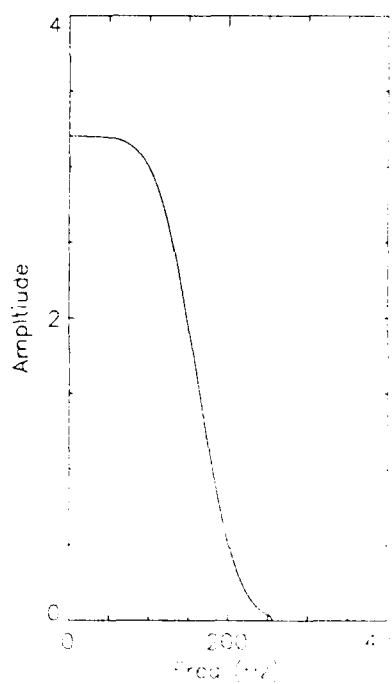
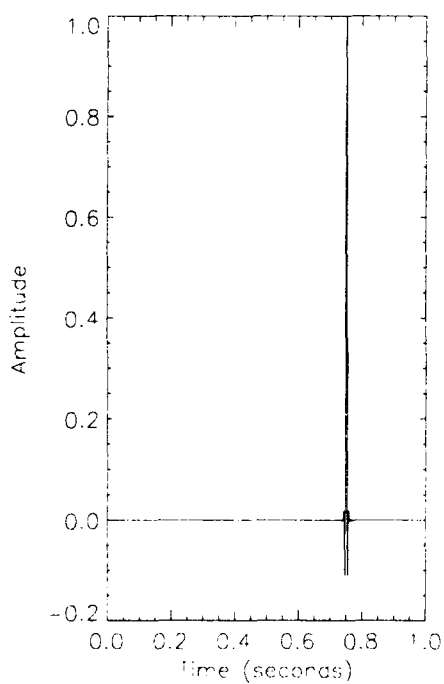
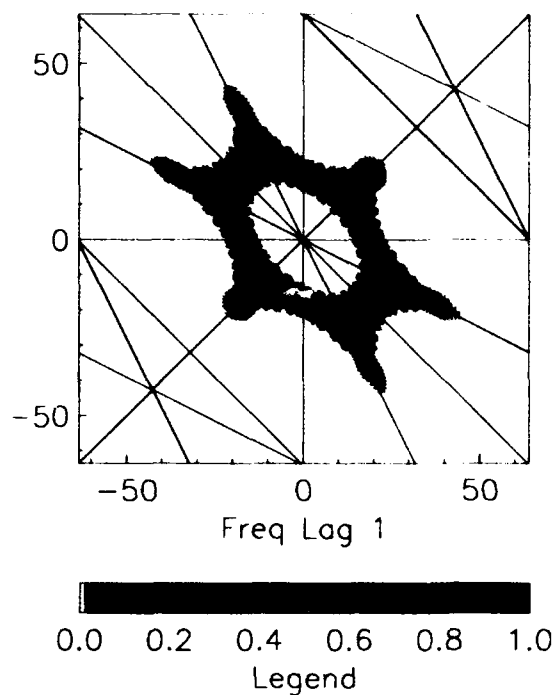
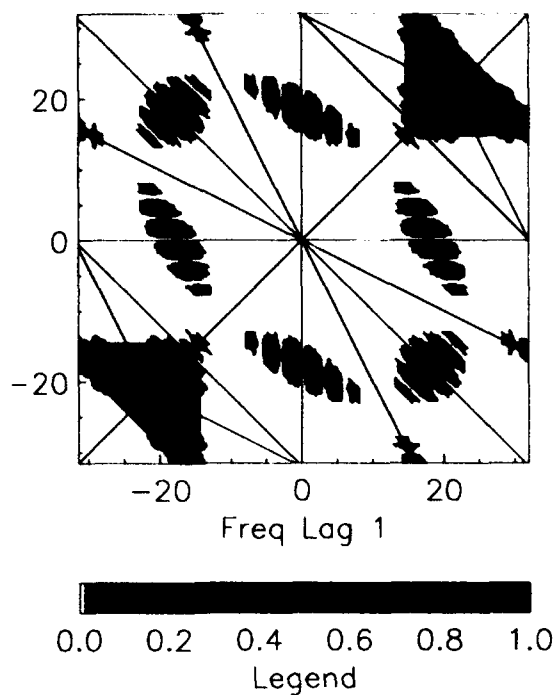


FIG. 9. Narrow pulse and its Fourier magnitude spectrum.

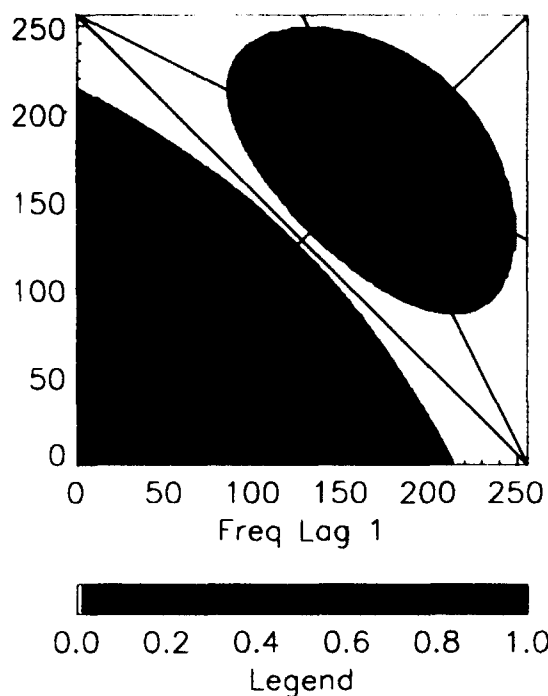


FIG. 10. Contour plot of the first quadrant of the magnitude bispectrum of the narrow pulse sampled with 512 points so that aliasing is present. The amplitudes have been normalized to unit height.

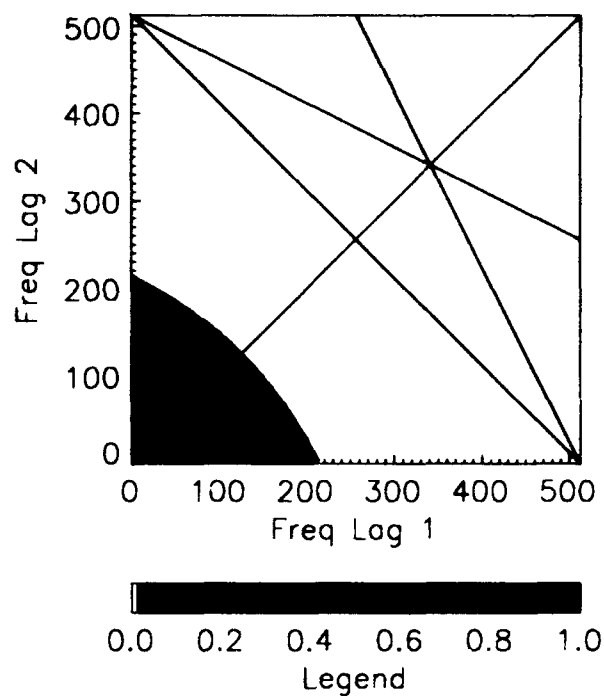


FIG. 11. Contour plot of the first quadrant of the magnitude bispectrum of the narrow pulse sampled with 1024 points so that no aliasing is present. The amplitudes have been normalized to unit height.

TABLE II. Time-domain calculations of correlation central ordinate values of the narrow pulse transient with various sampling rates.

Number of points	Narrow pulse			
	Autocorrelation ($\times 10^{-3}$) (amp units) $^2 \cdot s$	Autobicoorrelation ($\times 10^{-3}$) (amp units) $^3 \cdot s$	Autotricorrelation ($\times 10^{-3}$) (amp units) $^4 \cdot s$	Rectified autobicoorrelation ($\times 10^{-3}$) (amp units) $^3 \cdot s$
256	2.3263	1.7867	1.3766	1.7868
320	2.5966	2.2879	2.0588	2.2892
384	2.6770	2.4089	2.3176	2.4170
448	2.6875	2.3150	2.2257	2.3308
512	2.6880	2.2331	2.0731	2.2441
576	2.6880	2.2055	1.9787	2.2100
640	2.6880	2.2019	1.9426	2.2093
704	2.6880	2.2017	1.9324	2.2188
768	2.6880	2.2017	1.9305	2.2108
832	2.6880	2.2017	1.9303	2.2094
896	2.6880	2.2017	1.9303	2.2094
960	2.6880	2.2017	1.9303	2.2103
1024	2.6880	2.2017	1.9303	2.2104
2000	2.6880	2.2017	1.9303	2.2101
3000	2.6880	2.2017	1.9303	2.2100
4000	2.6880	2.2017	1.9303	2.2100
5000	2.6880	2.2017	1.9303	2.2100
6000	2.6880	2.2017	1.9303	2.2100
7000	2.6880	2.2017	1.9303	2.2100
8000	2.6880	2.2017	1.9303	2.2100

autobocorrelation and a double precision calculation, the central ordinate value is constant to ten significant digits, and the autotricorrelation central ordinate is constant to 14 significant digits using the predicted 1024 points or more. In each case, the central value is constant to ten significant digits using no more than the predicted number of sample points. Comparing the first quadrant of the magnitude bispectrum of the narrow pulse sampled with 512 points, shown in Fig. 10, and the Fourier magnitude spectrum of the original signal shown in Fig. 9, we see that aliasing is present in the bispectrum if only 512 points are used even though the energy spectrum contains no aliasing if sampled such that $f_N \geq 256$ Hz. If 1024 points are used, the bispectrum no longer shows any aliasing, as shown in Fig. 11.

VI. HIGHER-ORDER MASKING FILTERS

Aliasing in higher-order correlations due to sampling such that $f_N \geq f_t$ may be removed using one of two techniques. The first is to operate on the original discrete-time signal. The discrete-time signal may be interpolated to include a sufficient number of points within the signal duration to avoid aliasing in the autobocorrelation or autotricorrelation. This is most often done by zero-padding its Fourier transform.

The second technique involves operating on the frequency domain autobispectrum or autotrispectrum by applying a higher-order masking filter or on the aliased autobocorrelation or autotricorrelation with a convolutional masking filter. A bispectral masking filter in the frequency domain is defined by (Pflug *et al.*, 1992a)

$$H(f_1, f_2) = \begin{cases} 1, & |f_1 + f_2| < f_t, \\ 0, & |f_1 + f_2| > f_t \end{cases}$$

or in the lag domain,

$$h(\tau_1, \tau_2) = 8f_t^3 \sum_{t=-\infty}^{\infty} \text{sinc}(2f_t t) \text{sinc}[2f_t(t + \tau_1)] \\ \times \text{sinc}[2f_t(t + \tau_2)] \Delta t,$$

where the time domain filter $h(\tau_1, \tau_2)$ is the autobocorrelation of the sinc function. If instead of the continuous-frequency filter, the time domain equivalent of a discrete-frequency filter is sought, the above sum becomes a finite sum over digital sines (Marple, 1987). If $f_N \geq f_t$, then any value between f_t and f_N may be used as the bandlimit in the filter. The unaliased bispectrum is $B(f_1, f_2) = B(f_1, f_2)H(f_1, f_2)$ and the corresponding unaliased autobocorrelation $a_2(\tau_1, \tau_2) = a_2(\tau_1, \tau_2) * h(\tau_1, \tau_2)$. Frequency-domain multiplications are more commonly used since they are faster than time-domain convolutions. Indeed, convolving the two-dimensional time domain filter $h(\tau_1, \tau_2)$ with the aliased autobocorrelation could be prohibitive.

A bispectral filter $H(f_1, f_2)$ was applied to the whale transient sampled with 64 points, a number just sufficient to prevent aliasing in the autocorrelation. As shown in Table I, the volume beneath the bispectrum is 3.0329×10^{-3} (amp units)³·s without the filter, a 341.7%

error from the expected unaliased value of 6.8661×10^{-4} (amp units)³·s. However, the volume is reduced to 6.6208×10^{-4} (amp units)³·s when the filter is applied, resulting in a 3.7% error. For the narrow pulse sampled with 512 points, a masking filter reduced the aliased volume from 2.2331×10^{-3} (amp units)³·s, a 1.4% error, to 2.2017×10^{-3} (amp units)³·s, which has zero error up to five significant digits (see Table II).

A trispectral masking filter can be written as

$$H(f_1, f_2, f_3) = \begin{cases} 1, & |f_1 + f_2 + f_3| < f_t, \\ 0, & |f_1 + f_2 + f_3| > f_t \end{cases}$$

or

$$h(\tau_1, \tau_2, \tau_3) = 16f_t^4 \sum_{t=-\infty}^{\infty} \text{sinc}(2f_t t) \text{sinc}[2f_t(t + \tau_1)] \\ \times \text{sinc}[2f_t(t + \tau_2)] \text{sinc}[2f_t(t + \tau_3)] \Delta t$$

and are applicable in the same way as the bispectral filters. The bi- or tri-spectral masking filter is equivalent to a one-dimensional filter on only the diagonal factor in a bi- or tri-spectral approach to the calculation of the bi- or tri-correlation.

VII. SAMPLING REQUIREMENTS FOR RECTIFIED SIGNALS

Higher-order correlation detectors and time delay estimators applied to certain signals have been shown to benefit from rectification (G. Ioup *et al.*, 1990; Pflug *et al.*, 1992b,c). However, rectification can affect the sampling requirements needed to avoid aliasing.

The process of rectification means simply replacing each negative amplitude in a given sequence with its absolute value, and the unrectified and rectified signal Fourier transforms have a straightforward relationship. If we separate the discrete real sequence $x(t)$ into positive and negative sequences, defining $x_+(t)$ and $x_-(t)$ as

$$x_+(t) = \begin{cases} x(t), & \text{if } x(t) \geq 0, \\ 0, & \text{if } x(t) < 0 \end{cases}$$

and

$$x_-(t) = \begin{cases} 0, & \text{if } x(t) \geq 0, \\ x(t), & \text{if } x(t) < 0, \end{cases}$$

then the Fourier transform of $x(t)$ is equal to the sum of the individual positive and negative transforms,

$$X(f) = X_+(f) + X_-(f)$$

and the energy spectrum is

$$ES_x(f) = [X_+(f) + X_-(f)][X_+(f) + X_-(f)]^* \\ = |X_+(f)|^2 + |X_-(f)|^2 + X_+(f)X_-^*(f) \\ + X_-^*(f)X_+(f).$$

If $x(t)$ is rectified (denote the rectified sequence by x^R) then

$$x^R(t) = \begin{cases} x(t), & \text{if } x(t) \geq 0, \\ -x(t), & \text{if } x(t) < 0, \end{cases}$$

and the transform of the rectified $x(t)$ is simply

$$X^R(f) = X_+(f) - X_-(f),$$

and its energy spectrum is

$$\begin{aligned} ES_x^R(f) &= [X_+(f) - X_-(f)][X_+(f) - X_-(f)]^* \\ &= |X_+(f)|^2 + |X_-(f)|^2 - X_+(f)X_-^*(f) \\ &\quad - X_+^*(f)X_-(f). \end{aligned}$$

Since the zero-lag autocorrelation is the same for the unrectified and rectified signals and using the central ordinate-definite integral (sum) equivalence (Bracewell, 1986)

$$\begin{aligned} \sum_{f=-\infty}^{\infty} ES_x(f) \Delta f &= \sum_{f=-\infty}^{\infty} ES_x^R(f) \Delta f \\ &= \sum_{f=-\infty}^{\infty} \{|X_+(f)|^2 \\ &\quad + |X_-(f)|^2\} \Delta f \end{aligned}$$

and

$$\sum_{f=-\infty}^{\infty} \{X_+(f)X_-^*(f) + X_+^*(f)X_-(f)\} \Delta f = 0,$$

or equivalently,

$$\begin{aligned} \sum_{f=-\infty}^{\infty} \{\text{Re}[X_+(f)]^2 + \text{Re}[X_-(f)]^2 + \text{Im}[X_+(f)]^2 \\ + \text{Im}[X_-(f)]^2\} \Delta f = 0, \end{aligned}$$

and

$$\begin{aligned} \sum_{f=-\infty}^{\infty} \{\text{Re}[X_+(f)]\text{Re}[X_-(f)] \\ + \text{Im}[X_+(f)]\text{Im}[X_-(f)]\} \Delta f = 0. \end{aligned}$$

Although this summation is equal to zero, the individual terms will not be, as they are what defines the difference in the Fourier transform of the unrectified and rectified signals.

The rectification of the whale transient shown in Fig. 6 results in the time series and Fourier magnitude spectrum shown in Fig. 12. Whereas the unrectified whale transient had $f_t = 32$ Hz, the rectified whale transient shows a top frequency of approximately $f_t^R = 150$ Hz. Thus the rectified autocorrelation should require about 450 sample points to avoid aliasing. Indeed, we see in Table I that the volume beneath the bispectrum becomes constant to five significant digits when the signal is sampled with between 300 and 400 points. Similarly, the rectified pulse contains higher frequencies than the unrectified signal with $f_t^R = 1000$ Hz, approximately, and should require about 3000 sample points to prevent aliasing in the autocorrelation. As

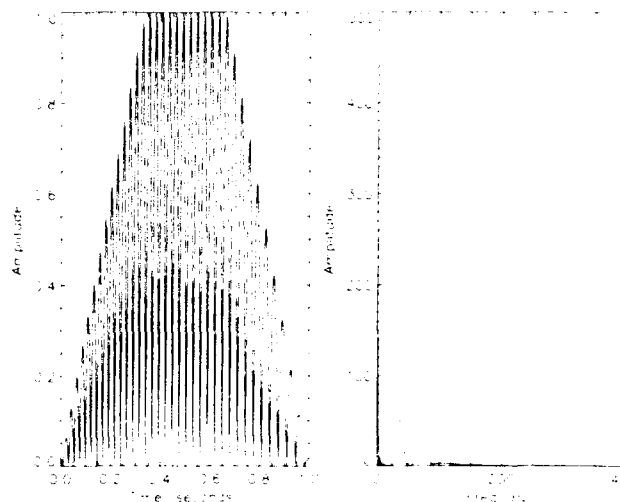


FIG. 12. Rectified whale transient and its Fourier magnitude spectrum.

shown in Table II to five significant digits, the volume beneath the bispectrum becomes constant if a minimum of 3000 sample points is used.

As stated, the area beneath the spectrum is equal to the central ordinate value of the corresponding correlation, and the central ordinate value of the autocorrelation of a rectified signal is the same as the value when the signal is not rectified. The autotricorrelation of a rectified signal also exhibits this property, consequently these values are not shown in Tables I and II since they are repetitive. Therefore, the area and hypervolume beneath the energy spectrum and trispectrum of rectified signals are not appropriate measures of the existence of aliasing. For instance, the rectified energy spectrum of the whale transient obviously contains aliasing when sampled at $f_N = 40$ Hz while the unrectified signal does not (compare Figs. 6 and 12 and see Table II). However, the areas beneath the two energy spectra are equal. The requirements to avoid aliasing in general in the rectified autocorrelation and autotricorrelation are $f_N > f_t^R$ and $f_N > 3f_t^R$, paralleling the requirements for the unrectified correlations.

VIII. SUMMARY

While critical sampling is sufficient to avoid aliasing when sampling the continuous-time autocorrelation, autocorrelation, and autotricorrelation, avoidance of aliasing in calculations of higher order correlations and spectra from discrete-time data requires sampling rates higher than the critical rate for a bandlimited energy signal. In these cases, for the autocorrelation, the sampling interval must be 2/3 the interval used for critical sampling, and for the autotricorrelation, the sampling interval must be 1/2 the critical sampling interval. Examples have been shown in tables for both a broadband and a narrow-band bandlimited transient using the area or volume beneath the energy spectrum, bispectrum, and trispectrum (the correlation central ordinate) as a measure of the presence of aliasing. This measure is also used to exhibit the effective-

ness of a bispectral masking filter on two sample signals. Using rectification as part of a detection or time delay estimation scheme for discrete data requires that the sampling rate be selected or interpolated according to the rectified bandlimit before calculating ordinary or higher-order correlations.

ACKNOWLEDGMENTS

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APPENDIX A: BISPECTRUM FOR A BICORRELATION CALCULATED FROM A DISCRETE-TIME ENERGY SIGNAL

As discussed in the body of this paper, the discrete-time autocorrelation could arise from sampling the continuous-time autocorrelation calculated from a real continuous-time energy signal

$$b_{CT}(\tau_1, \tau_2) = \int_{-\infty}^{\infty} x_{CT}(t) x_{CT}(t + \tau_1) x_{CT}(t + \tau_2) dt.$$

The properties of the corresponding bispectrum are discussed by Nielsen (1992) and in this paper.

An alternate procedure which produces a discrete-time autocorrelation is its calculation directly from a discrete-time energy signal. For this purpose we make the straightforward substitution of a summation for the integral in the previous expression

$$b(\tau_1, \tau_2) = \sum_{t=-\infty}^{\infty} x(t) x(t + \tau_1) x(t + \tau_2) \Delta t,$$

with t , τ_1 , and τ_2 discrete-time variables sampled at Δt . Time domain functions not labeled CT in this Appendix are discrete-time functions.

It is important to remember that the Fourier transform of the discrete-time signal is given by

$$x(t) \supset \tilde{X}(f) = \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{\Delta t}\right),$$

with $x_{CT}(t) \supset X(f)$, $k/\Delta t = 2kf_N$, and $X(f)$ the discrete-time Fourier transform (DTFT) (Marple, 1987). The DTFT for $b(\tau_1, \tau_2)$ is

$$\begin{aligned} B(f_1, f_2) &= \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} \left(\sum_{t=-\infty}^{\infty} x(t) x(t + \tau_1) x(t + \tau_2) \Delta t \right) \exp[-i2\pi(f_1\tau_1 + f_2\tau_2)] \Delta\tau_1 \Delta\tau_2 \\ &= \sum_{t=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} x(t) x(t + \tau_2) \Delta t \Delta\tau_2 \exp(-i2\pi f_2\tau_2) \left(\sum_{\tau_1=-\infty}^{\infty} x(t + \tau_1) \exp(-i2\pi f_1\tau_1) \Delta\tau_1 \right) \\ &= \sum_{t=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} x(t) x(t + \tau_2) \Delta t \Delta\tau_2 \exp(-i2\pi f_2\tau_2) \tilde{X}(f_1) \exp(+i2\pi f_1 t) \\ &= \sum_{t=-\infty}^{\infty} x(t) \Delta t \exp(+i2\pi f_1 t) \left(\sum_{\tau_2=-\infty}^{\infty} x(t + \tau_2) \exp(-i2\pi f_2\tau_2) \Delta\tau_2 \right) \tilde{X}(f_1) \\ &= \sum_{t=-\infty}^{\infty} x(t) \Delta t \exp(+i2\pi f_1 t) [\tilde{X}(f_2) \exp(+i2\pi f_2 t)] \tilde{X}(f_1) \\ &= \sum_{t=-\infty}^{\infty} x(t) \exp[+i2\pi(f_1 + f_2)t] \Delta t \tilde{X}(f_1) \tilde{X}(f_2) \end{aligned}$$

and for $x(t)$ real

$$= \left(\sum_{t=-\infty}^{\infty} x(t) \exp[-i2\pi(f_1 + f_2)t] \Delta t \right)^* \tilde{X}(f_1) \tilde{X}(f_2)$$

$$\begin{aligned}
&= \tilde{X}^*(f_1 + f_2) \tilde{X}(f_1) \tilde{X}(f_2) \\
&= \left(\sum_{k_1=-\infty}^{\infty} X(f_1 + 2k_1 f_N) \right) \\
&\quad \times \left(\sum_{k_2=-\infty}^{\infty} X(f_2 + 2k_2 f_N) \right) \\
&\quad \times \left(\sum_{k_3=-\infty}^{\infty} X^*(f_1 + f_2 + 2k_3 f_N) \right).
\end{aligned}$$

It is the transform replications in this expression that produce the aliasing in the TAD of the transform of the discrete-time autocorrelation calculated from discrete-time energy signals as described in this paper.

APPENDIX B: BISPECTRUM FOR A BICORRELATION CALCULATED FROM A DISCRETE TIME STOCHASTIC POWER SIGNAL

A result corresponding to that of Appendix A can be derived for stationary stochastic power signals following the derivation of Hinich (1990) [his Eq. (3.4) and subsequent discussion]. Hinich's derivation is for continuous- and discrete-time autocorrelations (equivalent to autocovariances for his zero mean assumption) calculated from continuous-time stationary stochastic signals and for their spectra. Hinich's Eq. (3.4) is

$$\begin{aligned}
&E\{A_x(f)A_x(g)A_x(h)\} \\
&= E\left\{\int \int \int x(t)x(r)x(s) \exp[-i2\pi(f t + g r + h s)] dt dr ds \right. \\
&\quad \left. + \int \int c_{xxx}(\tau, \theta) \exp[-i2\pi(f \tau + g \theta)] d\tau d\theta \right. \\
&\quad \left. \times \int \exp[-i2\pi(f + g + h)t] dt \right\} \\
&= B_x(f, g) \delta(f + g + h),
\end{aligned}$$

for which

$$A_x = \int_{-\infty}^{\infty} x(t) \exp(-i2\pi f t) dt$$

(A_x a complex random variable),

$$c_{xxx}(\tau, \theta) = E\{x(t)x(t+\tau)x(t+\theta)\},$$

and $B_x(f, g)$ is the autobispectrum of $x(t)$. Hinich goes on to say, "It follows from (3.4) that $\int B_x(f, g) df dg = E\{\int A_x(f)A_x(g)A_x(h) df dg\}$ when $h = -f - g$. But since $A_x(f) = 0$ for $f > f_0$, then $B_x(f, g) = 0$ for $f + g > f_0$." He then discusses the consequences of sampling the continuous-time $c_{xxx}(\tau, \theta)$. The results correspond in terms of aliasing to those given in this paper for sampling the continuous-time autocorrelation of energy signals, and agree with those of Nielson (1992) as well.

If instead an analogous result is desired for discrete-time autocorrelations calculated from discrete-time stationary stochastic power signals, the corresponding derivation is as follows. Let $x_{DT}(t)$ be the discrete-time stochastic signal, and use a tilde to represent quantities calculated from it. [Contrary to Appendix A, we let the unsubscripted $x(t)$ be the continuous-time function to agree with Hinich's notation.] Then

$$x_{DT}(t) \supset \tilde{A}_x(f) = \sum_{k=-\infty}^{\infty} A_x(f - 2k f_N),$$

and

$$\tilde{c}_{xxx}(\tau, \theta) = E\{x_{DT}(t)x_{DT}(t+\tau)x_{DT}(t+\theta)\}.$$

Therefore

$$\begin{aligned}
&E\{\tilde{A}_x(f)\tilde{A}_x(g)\tilde{A}_x(h)\} \\
&= E\left\{\sum_{t=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} x_{DT}(t)x_{DT}(r)x_{DT}(s) \right. \\
&\quad \left. \times \exp[-i2\pi(f t + g r + h s)] \Delta t \Delta r \Delta s \right\} \\
&= \sum_{\tau=-\infty}^{\infty} \sum_{\theta=-\infty}^{\infty} \tilde{c}_{xxx}(\tau, \theta) \exp[-i2\pi(f \tau + g \theta)] \\
&\quad \times \Delta \tau \Delta \theta \sum_{t=-\infty}^{\infty} \exp[-i2\pi(f + g + h)t] \Delta t \\
&= \tilde{B}_x(f, g) \delta(f + g + h).
\end{aligned}$$

From this result, $\tilde{B}_x(f, g)$ corresponds to $E\{\tilde{A}_x(f) \times \tilde{A}_x(g)\tilde{A}_x(-f-g)\}$. But

$$\tilde{A}_x(f) = \sum_{k_1=-\infty}^{\infty} A_x(f - 2k_1 f_N),$$

$$\tilde{A}_x(g) = \sum_{k_2=-\infty}^{\infty} A_x(g - 2k_2 f_N),$$

and

$$\tilde{A}_x(-f-g) = \sum_{k_3=-\infty}^{\infty} A_x(-f-g-2k_3 f_N).$$

One can no longer assert that since $A_x(-f-g)$ is zero in the domain outside $(f+g) > f_N$, the autobispectrum will be zero in that domain for the autocorrelation calculated from discrete-time stationary stochastic power signals. Instead one must look at $\tilde{A}_x(f-g)$, which is not zero for $(f+g) > f_N$. Therefore the PAP (the OT of Hinich) and TAD will be nonzero for this bicorrelation even for stationary signals, if $\Delta t = 1/(2f_i)$, with f_i the highest frequency present in $x(t)$. Only if $\Delta t \geq 1/(3f_i)$ will the PAP and TAD be zero. This result is in contrast to that for sampling the continuous-time autocorrelation, and the corresponding Hinich-Wolinsky (1988) test for stationarity, for which the PAP (OT) is zero if the autocorrelation is sampled at $\Delta t = 1/(2f_i)$.

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