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FUZZY HYPERGRAPHS AND FUZZY INTERSECTION GRAPHS

by William L. Craine, Major, USAF

August 1993, 73 pages.

Doctor of Philosophy with a Major in Mathematics

University of Idaho

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In Chapter 2 we pay particular attention to the family of fuzzy transversals of a fuzzy hypergraph. We give an *algorithmic method* to construct fuzzy transversals. We also generalize the vertex coloring lemma of Berge, providing a characterization of the family of all minimal fuzzy transversals of a fuzzy hypergraph. In Chapter 3 we use similar methods to define and characterize the family of vertex colorings of a fuzzy hypergraph.

In Chapter 4 we use the max and min operators to define the fuzzy intersection graph of a family of fuzzy sets. We show that every fuzzy graph without loops is the intersection graph of some family of fuzzy sets. We show that the Gilmore and Hoffman characterization of interval graphs extends naturally to fuzzy interval graphs, but the Fulkerson and Gross characterization does not. We conclude with a number of alternate edge strength functions that are related to recent developments in crisp intersection graph theory.

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[1] C. Berge (1989), *Hypergraphs, Combinatorics of Finite Sets.* North-Holland, Amsterdam.

[2] M.L. McAllister (1988), Fuzzy Intersection Graphs. *Comput. Math. Applic.* 15 (10), 871-886.

[3] A. De Luca and S. Termini (1972), A Definition of a non-probabilistic entropy in the setting of fuzzy sets theory. *Inf Control* 20, 301-312.

[4] D. Dubois and H. Prade (1980), *Fuzzy Sets and Systems, Theory and Applications.* Academic Press, New York.

[5] H.J. Zimmermann (1985), *Fuzzy Set Theory and its Applications.* Kluwer-Nijhoff Publishing, Boston.

[6] A. Rosenfeld (1975), Fuzzy Graphs. In: L.A. Zadeh, K.S. Fu, M. Shimura, Eds., *Fuzzy Sets and their Applications.* Academic Press, New York.

[7] G.J. Klir and T.A. Folger (1988), *Fuzzy Sets, Uncertainty, and Information.* Prentice Hall, New Jersey.

[8] F. Roberts (1976), *Discrete Mathematical Models.* Prentice Hall, New Jersey.

[9] M.S. Jacobson, F.R. McMorris and E.R. Scheinerman (1991), General Results on Tolerance Intersection Graphs. *Journal of Graph Theory* 15 (6), 573-577.

[10] T.A. McKee (1991), Foundations of Intersection Graph Theory. *Utilitas Mathematica* 40, 77-86.

[11] P.C. Gilmore and A.J. Hoffman (1964), A Characterization of Comparability Graphs and of Interval Graphs. *Canadian Journal of Mathematics* 16, 539-548.

[12] P.C. Fishburn (1985), *Interval Orders and Interval Graphs.* John Wiley & Sons, New York.

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FUZZY HYPERGRAPHS AND FUZZY INTERSECTION GRAPHS

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by

Major William L. Craine

August 1993

Major Professor: Roy Goetschel, Jr., Ph.D.

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This dissertation of Major William L. Craine, submitted for the degree of Doctor of Philosophy with a major in Mathematics and titled "Fuzzy Hypergraphs and Fuzzy Intersection Graphs," has been reviewed in final form, as indicated by the signatures and dates given below. Permission is now granted to submit final copies to the College of Graduate Studies for approval.

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To my wife Debra,

and to my children

James and Megan.

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CHAPTER 1. General Concepts

This chapter introduces fuzzy hypergraphs and investigates some of their basic properties. In general we follow the hypergraph definitions and notation given by Berge [1]. Our primary references for fuzzy set theory are Dubois and Prade [4], Klir and Folger [7], and Zimmermann [5]. For this paper each *vertex set* X and each *edge set* E or $\mathcal{E}$ is required to be finite.

Section 1. Fuzzy Hypergraphs

A *hypergraph* is a pair H = (X,E) where X is a finite set and E is a finite family of nonempty subsets of X whose union is X. Equivalently, the edge set E may be defined by a finite set of *characteristic functions* $\chi_A : X \longrightarrow \{0,1\}$ where $\chi_A(x) = 1$ if and only if (iff) $x \in A$. A hypergraph is *simple* if there are no repeated edges and no edge properly contains another. Hypergraphs are often defined by an *incidence matrix* with columns indexed by the edge set and rows indexed by the vertex set; the x,A entry being $\chi_A(x)$.

Given only an edge set E, the vertex set is understood to be $X = \bigcup\{A \mid A \in E\}$. Therefore one can use the edge set E or pair H = (X,E) interchangeably to define a hypergraph. Sometimes to avoid confusion we use V(H) and E(H) to denote the vertex set and edge set of H, respectively.

A *fuzzy set* on a set X is a mapping $\mu : X \longrightarrow [0,1]$. We define the *support of* μ by $\text{supp } \mu = \{x \in X \mid \mu(x) \neq 0\}$ and say μ is *nontrivial* if supp μ is nonempty. The *height of* μ is $h(\mu) = \max \{\mu(x) \mid x \in X\}$. We say μ is *normal* if $h(\mu) = 1$. If μ and ν are fuzzy sets on X we use the max and min operators to define new fuzzy sets on X by $\mu \vee \nu = \max \{\mu, \nu\}$ and $\mu \wedge \nu = \min \{\mu, \nu\}$. The fuzzy sets $\mu \vee \nu$ and $\mu \wedge \nu$ are common definitions of fuzzy union and fuzzy intersection, respectively. We write $\mu \leq \nu$ (fuzzy subset) if $\mu(x) \leq \nu(x)$ for each $x \in X$. If $\mu \leq \nu$ and $\mu(x) < \nu(x)$ for some $x \in X$ we say ν properly contains μ and write $\mu < \nu$. We will use $\mathcal{F}$ or $\mathcal{E}$ to denote a family of fuzzy sets and $\mathcal{F}(X)$ to denote the family of all fuzzy subsets of X.

For emphasis we often refer to traditional sets as *crisp sets*. Clearly the characteristic function of a crisp set defines a fuzzy set. As is common in fuzzy set theory we identify a crisp set with its characteristic function; when the context is clear we use the two concepts interchangeably. The reader may verify that the fuzzy definitions given throughout this paper agree with the usual crisp set definitions when applied to characteristic functions of crisp sets.

DEFINITION 1.1. Let X be a finite set and let $\mathcal{E}$ be a finite family of nontrivial fuzzy sets on X such that $X = \bigcup_{\mu\in\mathcal{E}} \operatorname{supp} \mu$. Then the pair $\mathcal{H} = (X,\mathcal{E})$ is a *fuzzy hypergraph* on X; $\mathcal{E}$ is the family of *fuzzy edges* of $\mathcal{H}$ and X is the (crisp) vertex set of $\mathcal{H}$. We let $h(\mathcal{H})$ denote the *height of* $\mathcal{H}$; that is, $h(\mathcal{H}) = \max\{h(\mu) \mid \mu \in \mathcal{E}\}$. The *order* of $\mathcal{H}$ (number of vertices) is denoted by $|X|$ and the number of edges is denoted by $|\mathcal{E}|$. The *rank* is the maximal column sum of the incidence matrix and the *antirank* is the minimal column sum. We say $\mathcal{H} = (X,\mathcal{E})$ is a *uniform fuzzy hypergraph* if and only if $\operatorname{rank}(\mathcal{H}) = \operatorname{antirank}(\mathcal{H})$.

DEFINITION 1.2. Let σ be a fuzzy subset of X and $\mathcal{E}$ a collection of fuzzy subsets of X such that for each $\mu \in \mathcal{E}$ and $x \in X$, $\mu(x) \le \sigma(x)$. Then the pair $(\sigma,\mathcal{E})$ is a *fuzzy hypergraph on the fuzzy set* σ.

The fuzzy hypergraph $(\sigma,\mathcal{E})$ is also a fuzzy hypergraph on $X = \operatorname{supp} \sigma$; the fuzzy set σ defines a condition for membership in the edge set $\mathcal{E}$. This condition can be stated separately, so without loss of generality (WOLOG) we restrict attention to fuzzy hypergraphs on crisp vertex sets.

EXAMPLE 1.3. Radio coverage network. Let X be a finite set of radio receivers (vertices); perhaps a set of representative locations at the centroid of a geographic region. For each of m radio transmitters we define the fuzzy set "listening area of station j" where $\mu_j(x)$ represents the "quality of reception of station j at location x." Membership values near 1 could signify "very clear reception on a very poor radio" while values near 0 could signify "very poor reception on even a very sensitive radio."

Since ͜graphy affects signal strength, each "listening area" is a fuzzy set. Also, for a fixed radio the reception will vary between different stations. This model thus utilizes the full definition of a fuzzy hypergraph. The model could be used to determine station programming or marketing strategies (who is my significant competition?) or in establishing an emergency broadcast network (is there a minimal subset of stations that reaches every radio with at least strength c?). Further variables could relate signal strength to changes in time of day, weather and other conditions.

DEFINITION 1.4. A fuzzy set $\mu: X \longrightarrow [0,1]$ is an *elementary fuzzy set* if μ is single valued on supp μ; or equivalently, if there is a nonzero constant c such that $\mu = c \cdot \chi_E$ where E = supp μ. We can thus use $c \cdot E$ to specify an elementary fuzzy set μ with $h(\mu) = c$ and supp $\mu = E$. An *elementary fuzzy hypergraph* $\mathcal{H} = (X, \mathcal{E})$ is a fuzzy hypergraph whose fuzzy edges are all elementary.

EXAMPLE 1.5. We explore the sense in which a fuzzy graph is a fuzzy hypergraph. A *fuzzy graph* on a set X is a pair $\mathcal{G} = (X, \mu)$ where μ is a symmetric fuzzy subset of XxX. That is, $\mu: XxX \longrightarrow [0,1]$ and for each x and y in X we have $\mu(x,y) = \mu(y,x)$. A *fuzzy graph on a fuzzy subset* $\sigma \in X$ is a pair $\mathcal{G} = (\sigma, \mu)$ where the symmetric mapping $\mu: XxX \longrightarrow [0,1]$ satisfies $\mu(x,y) \le \min \{\sigma(x), \sigma(y)\}$. Since μ is well defined, a fuzzy graph has no multiple edges. An edge is nontrivial if $\mu(x,y) \neq 0$. A *loop at* x is represented by $\mu(x,x) \neq 0$.

Alternately, a nontrivial edge (or loop) represents an elementary fuzzy subset of X with two (or one) element support. That there are no multiple edges is equivalent to the property that each pair of edges have distinct supports. A fuzzy graph without loops is defined by an anti-reflexive relation; or equivalently, by not allowing fuzzy subsets with single element support. Therefore a fuzzy graph (fuzzy graph with loops) is an elementary fuzzy hypergraph for which edges have distinct two vertex (or one element) supports.

Directed fuzzy graphs (*fuzzy digraphs)* on a set X or a fuzzy subset σ of X are similarly defined in terms of a mapping $\delta: X \times X \longrightarrow [0,1]$ where $\delta(x,y) \le \min \{\sigma(x), \sigma(y)\}$. Since δ is well-defined, a fuzzy digraph has at most two edges (which must have opposite orientation) between any two vertices. Therefore fuzzy graphs and fuzzy digraphs are special cases of fuzzy hypergraphs.

A *fuzzy multigraph* is a multivalued symmetric mapping $\mathcal{M}: X \times X \longrightarrow [0,1]$. A fuzzy multigraphs can be considered to be the "disjoint union" or "disjoint sum" of a collection of simple fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore these structures can be considered as "disjoint unions" or "disjoint sums" of fuzzy hypergraphs.

The preceding discussion motivates the following definitions.

DEFINITION 1.6. A fuzzy hypergraph $\mathcal{H} = (X, \mathcal{E})$ is *simple* if $\mu, \nu \in \mathcal{E}$ and $\mu \le \nu$ imply $\mu = \nu$. In particular, a (crisp) hypergraph $H = (X,E)$ is *simple* if $A, B \in E$ and $A \subseteq B$ imply that $A = B$.

DEFINITION 1.7. A fuzzy hypergraph $\mathcal{H} = (X, \mathcal{E})$ is *support simple* if $\mu, \nu \in \mathcal{E}$, supp μ = supp ν, and $\mu \le \nu$ imply that $\mu = \nu$.

DEFINITION 1.8. A fuzzy hypergraph $\mathcal{H} = (X, \mathcal{E})$ is *strongly support simple* if $\mu, \nu \in \mathcal{E}$ and supp μ = supp $\nu \in \mathcal{E}$ imply that $\mu = \nu$.

OBSERVATION 1.9. These definitions all reduce to familiar definitions in the special case where $\mathcal{H}$ is a crisp hypergraph. The fuzzy definition of simple is identical to the crisp definition of simple. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges.

For fuzzy hypergraphs all three concepts imply no multiple edges. Simple fuzzy hypergraphs are support simple and strongly support simple fuzzy hypergraphs are support simple. Simple and strongly support simple are independent concepts.

EXAMPLE 1.10. A fuzzy hypergraph $\mathcal{H} = (X,\mathcal{E})$ is a fuzzy graph (with loops) if and only if $\mathcal{H}$ is elementary, support simple and each edge has two (or one) element support.

EXAMPLE 1.11. Let $H = (X,E)$ be a crisp hypergraph on X. Since E is a subset of $\mathcal{P}(X)$ one could define the edge set ε of a fuzzy hypergraph on X as a fuzzy subset of $\mathcal{P}(X)$, that is, $\varepsilon:\mathcal{P}(X) \longrightarrow [0,1]$. However by a simple transformation this structure satisfies the definition of an elementary fuzzy hypergraph given above, with

$$\mathcal{E} = \{\mu_A \mid A \in \mathcal{P}(X)\} \text{ where } \mu_A(x) = \begin{cases} \varepsilon(A), & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Since ε is well-defined, no two edges have the same support. Thus a fuzzy subset of $\mathcal{P}(X)$ corresponds to an elementary, strongly support simple fuzzy hypergraph on X.

LEMMA 1.12. Let $\mathcal{H} = (X,\mathcal{E})$ be an elementary fuzzy hypergraph. Then $\mathcal{H}$ is support simple if and only if $\mathcal{H}$ is strongly support simple.

Proof. Suppose that $\mathcal{H}$ is elementary, support simple and that $\operatorname{supp} \mu = \operatorname{supp} \nu$. We assume WOLOG that $h(\mu) \le h(\nu)$. Since $\mathcal{H}$ is elementary it follows that $\mu \le \nu$, and since $\mathcal{H}$ is support simple that $\mu = \nu$. Therefore $\mathcal{H}$ is strongly support simple.

Conversely, by Observat on 1.9 it follows that if $\mathcal{H}$ is strongly support simple then $\mathcal{H}$ is supp rt simple. □

The complexity of a fuzzy hypergraph depends in part on how many edges it has. The natural question arises: is there an upper bound on the number of edges of a fuzzy hypergraph of order n?

OBSERVATION 1.13. Let $\mathcal{H} = (X,\mathcal{E})$ be a simple fuzzy hypergraph of order n. Then there is no upper bound on $|\mathcal{E}|$.

Proof. Let $X = \{x,y\}$, and define $\mathcal{E}_N = \{\mu_i | i = 1,2,\cdots,N\}$

$$\text{where } \mu_i(x)=\frac{1}{i+1} \quad \text{and} \quad \mu_i(y)=1-\frac{1}{i+1}$$

Then $\mathcal{H}_N = (X,\mathcal{E}_N)$ is a simple fuzzy hypergraph with N edges. □

OBSERVATION 1.14. Let $\mathcal{H} = (X,\mathcal{E})$ be a support simple fuzzy hypergraph of order n. Then there is no upper bound on $|\mathcal{E}|$.

Proof. The class of support simple fuzzy hypergraphs contains the class of simple fuzzy hypergraphs, thus the result follows from Observation 1.13. □

OBSERVATION 1.15. Let $\mathcal{H} = (X,\mathcal{E})$ be a strongly support simple fuzzy hypergraph of order n. Then $|\mathcal{E}| \le 2^n-1$, with equality if and only if

$$\{\text{supp } \mu | \mu \in \mathcal{E}\} = \mathcal{P}(X)\backslash\emptyset.$$

Proof. Each nontrivial $A \subseteq X$ can be the support of at most one $\mu \in \mathcal{E}$ so $|\mathcal{E}| \le 2^n-1$. The second statement is clear. □

OBSERVATION 1.16. Let $\mathcal{H} = (X,\mathcal{E})$ be an elementary, simple fuzzy hypergraph of order n. Then $|\mathcal{E}| \le 2^n-1$, with equality if and only if $\{\text{supp } \mu | \mu \in \mathcal{E},\ \mu \ne 0\} = \mathcal{P}(X)\backslash\emptyset$.

Proof. Since $\mathcal{H}$ is elementary and simple, each nontrivial $A \subseteq X$ can be the support of at most one $\mu \in \mathcal{E}$. Therefore $|\mathcal{E}| \le 2^n-1$. To show there exists an elementary, simple $\mathcal{H}$ with $|\mathcal{E}| = 2^n-1$, let $\mathcal{E} = \{\mu_A | A \subseteq X\}$ be the set of functions defined by

$$\mu_A(x) = \begin{cases} \frac{1}{|A|} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Then each one element set has height 1, each two element set has height .5 and so on. $\mathcal{H}$ is elementary and simple, and $|\mathcal{E}| = 2^n-1$. □

Section 2. The Fundamental Sequence of a Fuzzy Hypergraph

The concept of cut-level sets has played a crucial role in fuzzy set theory. Essentially we construct a sequence of crisp sets that helps us visualize how a fuzzy set's structure changes at various "strengths" between zero and one. Of particular interest are relationships between properties of the cut-level sets and properties of the associated fuzzy set.

DEFINITION 2.1. Given $c \in [0,1]$ and a fuzzy set μ, we define the *c cut-level set of* μ to be the crisp set $\mu_c = \{x \in \text{supp } \mu | \mu(x) \geq c\}$. Conversely, a fuzzy set is often defined by specifying a family of cut-level sets, and letting $\mu(x) = \sup \{c = (0,1] | x \in \mu_c\}$. If $\mathcal{F}$ is a family of fuzzy sets we let $\mathcal{F}_c$ denote the family of nonempty crisp sets

$$\mathcal{F}_c = \{\mu_c | \mu \in \mathcal{F} \text{ and } \mu_c \neq \emptyset\}.$$

Therefore if $\mathcal{H} = (X,\mathcal{E})$ is a fuzzy hypergraph the *c level hypergraph of* $\mathcal{H}$ is well-defined as $\mathcal{H}_c = (X_c,\mathcal{E}_c)$ where

$$\mathcal{E}_c = \{\mu_c | \ \mu \in \mathcal{E}, \ \mu_c \neq \emptyset\} \text{ and } X_c = \bigcup_{\mu\in\mathcal{E}} \mu_c.$$

To formalize the association of a fuzzy hypergraph with a sequence of (crisp) hypergraphs we require the following notation and definitions. If $\mathcal{A}$ and $\mathcal{B}$ are families of crisp (fuzzy) sets we write $\mathcal{A} \preceq \mathcal{B}$ in case for each $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $A \subseteq B$ ($A \leq B$). If $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, we write $\mathcal{A} \prec \mathcal{B}$.

OBSERVATION 2.2. While the relation $\preceq$ is reflexive and transitive it is not in general antisymmetric. Therefore $\preceq$ induces a preorder rather than a partial order. In fact it is possible for $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ with $\mathcal{A} \neq \mathcal{B}$. If, however, $\preceq$ is applied to a collection of simple families of sets the relation is antisymmetric and $\preceq$ induces a partial ordering on the family.

OBSERVATION 2.3. If $s \geq t$ and μ is a fuzzy set, then $\mu_s \subseteq \mu_t$. Thus for cut-level hypergraphs of $\mathcal{H}$, $\mathcal{E}_s \subsetneq \mathcal{E}_t$. Distinct edges of $\mathcal{E}$ could produce the same c cut, and so the c level hypergraph $\mathcal{H}_c$ could be considered multiedged. However we do not take this approach; we will assume that $\mathcal{H}_c$ is without repeated edges.

For any fuzzy hypergraph $\mathcal{H} = (X,\mathcal{E})$ we associate a finite sequence of real numbers called the *fundamental sequence* of $\mathcal{H}$. This sequence is well-defined since both X and $\mathcal{E}$ are finite.

DEFINITION 2.4. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph and for each $c \in (0.1]$ let $\mathcal{H}_c = (X_c,\mathcal{E}_c)$ be the c level hypergraph of $\mathcal{H}$. The sequence of real numbers $r_1, r_2, \cdots, r_n$, with $1 \geq r_1 > r_2 > \cdots > r_n > 0$ and having the properties:

(i) If $1 \geq s > r_1$ then $\mathcal{E}_s = \emptyset$

(ii) If $r_i \geq s > r_{i+1}$ then $\mathcal{E}_s = \mathcal{E}_{r_i}$ (let $r_{n+1} = 0$)

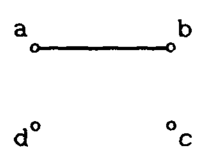
(iii) $\mathcal{E}_{r_i} \subsetneq \mathcal{E}_{r_{i+1}}$

is called the *fundamental sequence* of $\mathcal{H}$ and is denoted by $fs(\mathcal{H})$. The corresponding sequence of r_i level hypergraphs $H_{r_1} \subsetneq H_{r_2} \subsetneq \cdots \subsetneq H_{r_n}$ is called the $\mathcal{H}$ *induced fundamental sequence* and is denoted by $I(\mathcal{H})$. The r_n level is called the *support level* of $\mathcal{H}$ and the hypergraph H_{r_n} is called the *support hypergraph* of $\mathcal{H}$.

EXAMPLE 2.5. Let $\mathcal{H}$ be given by the incidence matrix of Figure 1.

$$M = \begin{array}{c c} & \begin{array}{ccc} \lambda & \mu & \nu \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} .8 & .7 & .6 \\ .8 & .7 & .6 \\ 0 & .6 & 0 \\ 0 & 0 & .6 \end{bmatrix} \end{array}$$

$\mathcal{H}_{.8}$ =



$\mathcal{H}_{.6}$ =

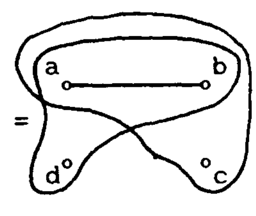


Figure 1. *The $\mathcal{H}$ induced fundamental sequence of Example 2.5*

By examining the cut-level hypergraphs of $\mathcal{H}$ we see $\mathcal{H}_{.8} = \mathcal{H}_{.7}$ and so $fs(\mathcal{H}) = \{.8,.6\}$. Notice that $\mu(a) = .7$ is not an element of $fs(\mathcal{H})$.

Further, $\mu_{.8} \neq \mu_{.7}$ shows that while the edge sets remain constant between fundamental sequence cut-levels, a particular edge's cut-levels may vary. This example illustrates the "repeat edges" of Observation 2.3.

DEFINITION 2.6. Let $\mathcal{H}$ be a fuzzy hypergraph with $fs(\mathcal{H}) = \{r_1, \cdots, r_n\}$ and let $r_{n+1} = 0$. Then $\mathcal{H}$ is *sectionally elementary* if for each edge $\tau \in \mathcal{E}$, each $i \in \{1, \cdots, n\}$, and each $c \in (r_{i+1}, r_i]$, we have $\tau_c = \tau_{r_i}$.

Clearly $\mathcal{H}$ is sectionally elementary if and only if $\mu(x) \in fs(\mathcal{H})$ for each $\mu \in \mathcal{E}$ and each $x \in X$.

DEFINITION 2.7. Given $\mathcal{H} = (X, \mathcal{E})$ and $\mathcal{E}' \subseteq \mathcal{E}$ we call $\mathcal{H}' = (X', \mathcal{E}')$, where $X' = \bigcup \{\text{supp } \mu | \mu \in \mathcal{E}'\}$, the *partial fuzzy hypergraph generated by* $\mathcal{E}'$. If $\mathcal{H}_1$ is a partial fuzzy hypergraph of $\mathcal{H}_2$ we write $\mathcal{H}_1 \leq \mathcal{H}_2$; we write $\mathcal{H}_1 < \mathcal{H}_2$ if $\mathcal{H}_1 \leq \mathcal{H}_2$ and $\mathcal{H}_1 \neq \mathcal{H}_2$ (that is $\mathcal{H}_1 \subset \mathcal{H}_2$). Of course the definition applies to the special case when $\mathcal{H}_1$ and $\mathcal{H}_2$ are crisp hypergraphs.

OBSERVATION 2.8. The relation "partial fuzzy hypergraph of" is a partial ordering on the collection of all fuzzy hypergraphs. The proof is a straight-forward application of the fact that $\subseteq$ is a partial order on the (crisp) families of edges. If H' is a partial fuzzy hypergraph of H, then $H' \not\leq H$. The terms of a $\mathcal{H}$ induced fundamental sequence may not be partial hypergraphs of $\mathcal{H}_{r_n}$.

DEFINITION 2.9. A sequence of hypergraphs $H_i = (X_i, \mathcal{E}_i)$, $1 \leq i \leq n$, is said to be *ordered* if the sequence of hypergraphs is linearly ordered by the relation "partial hypergraph," that is, if $H_1 < H_2 < \cdots < H_n$. The sequence is *simply ordered* if it is ordered and if whenever $E \in \mathcal{E}_{i+1} \setminus \mathcal{E}_i$, then $E \nsubseteq X_i$ ($\exists\, x \in E \ni x \notin X_i$).

DEFINITION 2.10. A fuzzy hypergraph $\mathcal{H}$ is *ordered* if the $\mathcal{H}$ induced fundamental sequence of hypergraphs is ordered. The fuzzy hypergraph $\mathcal{H}$ is *simply ordered* if the $\mathcal{H}$ induced fundamental sequence of hypergraphs is simply ordered.

OBSERVATION 2.11. If $\mathcal{H} = (X,\mathcal{E})$ is an elementary fuzzy hypergraph, then $\mathcal{H}$ is ordered. Also, if $\mathcal{H} = (X,\mathcal{E})$ is an ordered fuzzy hypergraph with simple support hypergraph, then $\mathcal{H}$ is elementary.

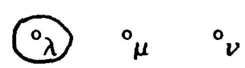
Section 3. Further Extensions of Hypergraph Definitions

DEFINITION 3.1. The *dual* of a fuzzy hypergraph $\mathcal{H} = (X,\mathcal{E})$ is a fuzzy hypergraph $\mathcal{H}^D = (\mathcal{E},X^D)$ whose vertex set is the edge set of $\mathcal{H}$ and with edges $x^D:\mathcal{E}^D \longrightarrow [0,1]$ by $x^D(\mu^D) = \mu(x)$. $\mathcal{H}^D$ is a fuzzy hypergraph whose incidence matrix is the transpose of the incidence matrix of $\mathcal{H}$, thus $\mathcal{H}^{DD} = \mathcal{H}$.

EXAMPLE 3.2. The dual of the fuzzy hypergraph of Example 2.5 and Figure 1 is shown in Figure 2. Notice that $fs(\mathcal{H}) \neq fs(\mathcal{H}^D)$ and while $\mathcal{H}$ had no repeat edges, $\mathcal{H}^D$ has the repeat edge $a^D = b^D$. Also $\mathcal{H}$ is simple and support simple while $\mathcal{H}^D$ is neither.

$$M^T = \begin{array}{c} \\ \lambda \\ \mu \\ \nu \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} .8 & .8 & 0 & 0 \\ .7 & .7 & .6 & 0 \\ .6 & .6 & 0 & .6 \end{bmatrix} \end{array}$$

$\mathcal{H}_{.8}$ =



$\mathcal{H}_{.7}$ = $\mathcal{H}_{.6}$ =

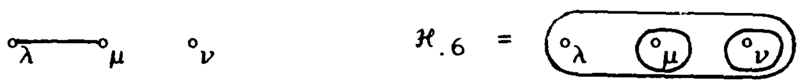


Figure 2. *The dual of the fuzzy hypergraph of Example 2.5*

DEFINITION 3.3. Let $\mathcal{H} = (X,\mathcal{E})$ and $\mathcal{H}' = (X',\mathcal{E}')$ be fuzzy hypergraphs. Then $\mathcal{H}' = (X',\mathcal{E}')$ is a *sub-fuzzy hypergraph* of $\mathcal{H}$ if there exists a fuzzy set $\nu \in \mathcal{F}(X)$ where $X' = \bigcup_{\mu\in\mathcal{E}} \text{supp}\,(\nu \wedge \mu)$ and $\mathcal{E}' = \{\nu \wedge \mu | \mu \in \mathcal{E},\ \nu \wedge \mu \neq 0\}$. The fuzzy hypergraph $\mathcal{H}' = (X',\mathcal{E}')$ is called the *sub fuzzy hypergraph induced by the fuzzy vertex set* ν.

OBSERVATION 3.4. A sub-fuzzy hypergraph can be interpreted as a method of defining a fuzzy hypergraph on a <u>fuzzy set</u> ν. The min operator then represents the condition that the edge strength at a vertex x is no greater than the vertex strength $\nu(x)$. When defining $\mathcal{H}$ by an incidence matrix one can specify ν in an additional column of the matrix.

LEMMA 3.5. The relation "sub-fuzzy hypergraph of" is a partial ordering on the collection of all fuzzy hypergraphs.

Proof. Let $\nu(x) = 1$ for all $x \in X$. Then $\mathcal{H} = (X,\mathcal{E})$ is a sub-fuzzy hypergraph of $\mathcal{H} = (X,\mathcal{E})$ and the relation is reflexive. Suppose that

$$\mathcal{E}'' = \{\nu' \wedge \mu' \,|\, \mu' \in \mathcal{E}',\ \nu' \wedge \mu' \neq 0\} \quad \text{and}$$

$$\mathcal{E}' = \{\nu \wedge \mu \,|\, \mu \in \mathcal{E},\ \nu \wedge \mu \neq 0\}.$$

Then $\nu' \wedge \mu' = \nu' \wedge (\nu \wedge \mu) = (\nu' \wedge \nu) \wedge \mu$ and the relation is transitive.

We prove anti-symmetry by contradiction. Let

$$\mathcal{E}' = \{\nu \wedge \mu \,|\, \mu \in \mathcal{E},\ \nu \wedge \mu \neq 0\}$$

$$\text{and } \mathcal{E} = \{\nu' \wedge \mu' \,|\, \mu' \in \mathcal{E}',\ \nu' \wedge \mu' \neq 0\}.$$

Suppose that $\mathcal{E} \neq \mathcal{E}'$; so WOLOG there exists a $\alpha' \in \mathcal{E}'$ with $\alpha' \notin \mathcal{E}$. Then $\alpha' = \nu \wedge \alpha$ for some $\alpha \in \mathcal{E}$. By definition of the min operator, $\alpha'(x) \leq \alpha(x)$ for all $x \in X$. By hypothesis $\alpha' \neq \alpha$, so there exists a $y \in X$ with $\alpha'(y) < \alpha(y)$. Thus $\alpha'(y) = \nu(y)$ which implies that each $\mu' \in \mathcal{E}'$ satisfies $\mu'(y) \leq \nu(y) < \alpha(y)$. However, this contradicts $\alpha = \nu' \wedge \mu'$ for a $\mu' \in \mathcal{E}'$. Specifically $\mu'(y) < \alpha(y)$ for each $\mu' \in \mathcal{E}'$. □

If $\nu(\mathcal{H})$ is a sub-fuzzy hypergraph of $\mathcal{H}$, then $\nu(\mathcal{H}) \preceq \mathcal{H}$. The terms of a $\mathcal{H}$ induced fundamental sequence may not be sub-hypergraphs of $\mathcal{H}_{r_n}$.

DEFINITION 3.6. For $x \in X$, let the *star* $\mathcal{H}(x)$ *with center* x to be the partial fuzzy hypergraph formed by the edges with nonzero membership value for x. Let the *c star with center* x to be the set of all $\mu \in \mathcal{E}$ such that $\mu(x) \geq c$. Define the *degree of* x *in* $\mathcal{H}$ as $\deg(x) = \sum_{\mu\in\mathcal{E}} \mu(x)$ (incidence matrix row sum) and the *maximum degree of* $\mathcal{H}$ as $\Delta(\mathcal{H}) = \max_{x\in X} \deg(x)$. A fuzzy hypergraph in which all vertices have the same degree is said to be *regular*.

As a straight generalization of a hypergraph theorem [1] we have $\Delta(\mathcal{H}) = \operatorname{rank}(\mathcal{H}^D)$ and the dual of a regular fuzzy hypergraph is uniform.

CHAPTER 2. Fuzzy Transversals of Fuzzy Hypergraphs

Transversals of crisp hypergraphs are used in coloring problems and a variety of other applications. Of particular interest is the set of all minimal transversals and the hypergraph formed with minimal transversals as the edge set. This chapter defines and studies the basic properties of fuzzy transversals and the fuzzy hypergraph formed with minimal fuzzy transversals as the edge set.

Section 1. Definition of Fuzzy Transversal

For a crisp hypergraph H = (X,E) a *transversal of* H is any subset T of X with the property that for each $A \in E$, $T \cap A \neq \emptyset$. A transversal T of H is a *minimal transversal of* H if no proper subset of T is a transversal of H. Clearly a transversal always contains a minimal transversal. The collection of minimal transversals of H can be considered the edge set of a hypergraph where the vertex set is a (perhaps proper) subset of X. Both the set of all minimal transversals of H and the hypergraph defined by this set will be denoted by Tr(H).

Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph on X, and τ be a fuzzy subset of X such that $\tau \wedge \mu \neq 0$ for each $\mu \in \mathcal{E}$. Then $\tau/2 \wedge \mu \neq 0$ for each $\mu \in \mathcal{E}$ as well. Thus a strict analog of a crisp transversal is of little use, for we cannot define a minimal transversal. However, if for each c level cut of a transversal of $\mathcal{H}$ is a transversal for the c level hypergraph $\mathcal{H}_c$, we obtain a suitable definition.

DEFINITION 1.1. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph and recall the height of μ is $h(\mu) = \max\{\mu(x) \mid x \in X\}$. A *fuzzy transversal* of H is a fuzzy set $\tau \in \mathcal{F}(X)$ such that $\tau_{h(\mu)} \cap \mu_{h(\mu)} \neq \emptyset$ for each $\mu \in \mathcal{E}$. A *minimal fuzzy transversal* of $\mathcal{H}$ is a fuzzy transversal τ of H for which $\rho < \tau$ implies ρ is not a fuzzy transversal of H. The set of all minimal fuzzy transversals of $\mathcal{H}$ (and the fuzzy hypergraph formed by this set) will be denoted $Tr(\mathcal{H})$.

OBSERVATION 1.2. Let $\mathcal{H} = (X, \mathcal{E})$ be a fuzzy hypergraph. An immediate consequence of Definition 1.1 is that the following statements are equivalent.

(i) τ is a fuzzy transversal of $\mathcal{H}$

(ii) For each $\mu \in \mathcal{E}$ and each c with $0 < c \le h(\mu)$, $\tau_c \cap \mu_c \neq \emptyset$

(iii) For each c with $0 < c \le r_1$, τ_c is a transversal of $\mathcal{H}_c$

We show in Lemma 1.10 that each fuzzy transversal contains a minimal fuzzy transversal.

EXAMPLE 1.3. Although τ may be a minimal fuzzy transversal of $\mathcal{H}$, τ_c may not be a minimal transversal of $\mathcal{H}_c$ for each c. Also, the set of minimal fuzzy transversals of $\mathcal{H}$ may not form a hypergraph on X. Define $\mathcal{H}$ by the incidence matrix of Figure 3. The remainder of Section 1 will justify the claim that $\mathcal{H}$ has only two minimal fuzzy transversals. Both have zero strength for a and d while neither has a .4 cut that is minimal in $\mathcal{H}_{.4}$.

$$\mathcal{H} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccc} \lambda & \mu & \nu & \rho \end{array} \\ \begin{bmatrix} .5 & 0 & 0 & 0 \\ .5 & .8 & 0 & 0 \\ .4 & 0 & 0 & .7 \\ 0 & .4 & .7 & 0 \\ 0 & 0 & 0 & .7 \\ 0 & 0 & .8 & .4 \end{bmatrix} \end{array} \qquad \mathrm{Tr}(\mathcal{H}) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cc} \tau_1 & \tau_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ .8 & .8 \\ .7 & 0 \\ 0 & 0 \\ 0 & .7 \\ .8 & .8 \end{bmatrix} \end{array}$$

$\mathcal{H}_{.4} =$

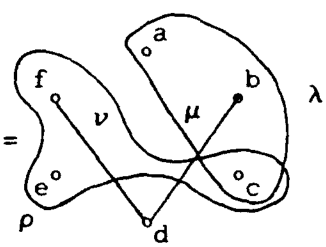


Figure 3. *Minimal fuzzy transversals that are not locally minimal*

We saw in Example 1.3 that a cut-level set of a minimal fuzzy transversal need not be a minimal transversal of the cut-level hypergraph. Fuzzy transversals that have this property are of interest and so we provide the following definition.

DEFINITION 1.4. If τ is a fuzzy set with the property that τ_c is a minimal transversal of $\mathcal{H}_c$ for each $c \in (0,1]$, then τ is called a *locally minimal fuzzy transversal of* $\mathcal{H}$. The set of all locally minimal fuzzy transversals on $\mathcal{H}$ is denoted by $\mathrm{Tr}^*(\mathcal{H})$.

We now provide some basic properties of fuzzy transversals that justify the claim of Example 1.3. In Section 2 we provide a characterization of $\mathrm{Tr}(\mathcal{H})$. Then in Section 3 we explore conditions under which $\mathrm{Tr}^*(\mathcal{H})$ is nonempty and under which $\mathrm{Tr}^*(\mathcal{H}) = \mathrm{Tr}(\mathcal{H})$.

LEMMA 1.5. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph with $fs(\mathcal{H}) = \{r_1,\cdots,r_n\}$. If τ is a fuzzy transversal of $\mathcal{H}$, then $h(\tau) \geq h(\mu)$ for each $\mu \in \mathcal{E}$. If τ is minimal, then $h(\tau) = \max\,\{h(\mu) \mid \mu \in \mathcal{E}\} = r_1$.

Proof. Trivial consequence of the definitions. □

LEMMA 1.6. For each $\tau \in \mathrm{Tr}(\mathcal{H})$ and for each $x \in X$, $\tau(x) \in fs(\mathcal{H})$. Therefore the fundamental sequence of $\mathrm{Tr}(\mathcal{H})$ is a (possibly proper) subset of $fs(\mathcal{H})$.

Proof. Let $\tau \in \mathrm{Tr}(\mathcal{H})$ and $\tau(x) \in (r_{i+1},r_i]$. Define φ by

$$\varphi(y) = \begin{cases} r_i & \text{if } y = x \\ \tau(y) & \text{otherwise} \end{cases}$$

By definition of φ, $\varphi_{r_i} = \tau_{r_i}$. By definition of $fs(\mathcal{H})$, $\mathcal{H}_c = \mathcal{H}_{r_i}$ for each $c \in (r_{i+1},r_{r_i}]$. Therefore φ_{r_i} is a transversal of $\mathcal{H}_c$ for each $c \in (r_{i+1}, r_{r_i}]$. Since τ is a fuzzy transversal and $\varphi_c = \tau_c$ for each $c \notin (r_{i+1}, r_{r_i}]$, φ is a fuzzy transversal as well. Now $\varphi \leq \tau$ and the minimality of τ implies $\varphi = \tau$; hence $\tau(x) = \varphi(x) = r_i$. It follows that for each $\tau \in \mathrm{Tr}(\mathcal{H})$ and for each $x \in X$ we have $\tau(x) \in fs(\mathcal{H})$. Therefore $fs(\mathrm{Tr}(\mathcal{H})) \subseteq fs(\mathcal{H})$. □

LEMMA 1.7. $\mathrm{Tr}(\mathcal{H})$ is sectionally elementary (see Definition 1.2.6).

Proof. Let $fs(\mathrm{Tr}(\mathcal{H})) = \{r_1,\cdots,r_n\}$. Assume there exists some $\tau \in \mathrm{Tr}(\mathcal{H})$ and some $c \in (r_{i+1},r_i]$ such that τ_{r_i} is properly contained in τ_c. Since $\mathrm{Tr}(\mathcal{H})_{r_i} = \mathrm{Tr}(\mathcal{H})_c$, there exists some $\varphi \in \mathrm{Tr}(\mathcal{H})$ such that $\varphi_{r_i} = \tau_c$. Then $\tau_{r_i} \subset \varphi_{r_i}$ implies the fuzzy set α defined by

$$\alpha(x) = \begin{cases} c & \text{if } x \in \varphi_{r_i} \setminus \tau_{r_i} \\ \varphi(x) & \text{otherwise} \end{cases}$$

is a fuzzy transversal of $\mathcal{H}$. Now $\alpha < \varphi$ contradicts the minimality of φ. □

LEMMA 1.8. For each $\tau \in \mathrm{Tr}(\mathcal{H})$ the top cut of τ, τ_{r_1}, is a minimal transversal of $\mathcal{H}_{r_1}$.

Proof. If not, there exists a minimal transversal T of $\mathcal{H}_{r_1}$ such that $T \subset \tau_{r_1}$. Define the fuzzy set φ where

$$\varphi(x) = \begin{cases} r_2 & \text{if } x \in \tau_{r_1} \setminus T \\ \tau(x) & \text{otherwise} \end{cases}$$

By Observation 1.2 φ is a fuzzy transversal of $\mathcal{H}$, contradicting the minimality of τ. □

Studies of crisp hypergraphs have shown that finding Tr(H) is a NP complete problem. However for hypergraphs with few vertices there are methods that are sufficiently effective [1, p 53]. We show that determining $\mathrm{Tr}(\mathcal{H})$ for fuzzy hypergraphs is only slightly more difficult.

Construction 1.9 gives an algorithm for finding $\mathrm{Tr}(\mathcal{H})$.

CONSTRUCTION 1.9. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph with $I(\mathcal{H}) = \{\mathcal{H}_{r_1}, \mathcal{H}_{r_2}, \cdots, \mathcal{H}_{r_n}\}$. We construct a minimal fuzzy transversal τ of $\mathcal{H}$ by a recursive process:

(i) Find a (crisp) minimal transversal T_1 of $\mathcal{H}_{r_1}$.

(ii) Find a transversal T_2 of $\mathcal{H}_{r_2}$ that is minimal with respect to the property that $T_1 \subseteq T_2$. Equivalently, construct a new hypergraph H_2 with edge set E_{r_2} augmented by a loop at each $x \in T_1$; that is, $E(H_2) = E_{r_2} \cup \{\{x\} | x \in T_1\}$. Let T_2 be any minimal transversal of H_2.

(iii) Continue recursively, letting T_j be a transversal of $\mathcal{H}_{r_j}$ that is minimal with respect to the property $T_{j-1} \subseteq T_j$.

(iv) For $1 \le j \le n$, let τ_j be the elementary fuzzy set with support T_j and height r_j. Then $\tau = \vee\{\tau_j | 1 \le j \le n\}$ is a minimal fuzzy transversal of $\mathcal{H}$.

To see why this algorithm is valid first note that Observation 1.2 implies τ is a fuzzy transversal of $\mathcal{H}$. If $\gamma < \tau$ there exists an $x \in X$ such that $\gamma(x) < \tau(x)$. But then x is not an element of the $\tau(x)$ level cut of γ, so $\gamma_{\tau(x)} \subset \tau_{\tau(x)} = T_{\tau(x)}$; therefore $\gamma_{\tau(x)}$ is not a transversal of $\mathcal{H}_{\tau(x)}$.

Thus τ is a minimal fuzzy transversal of $\mathcal{H}$. The construction clearly shows whether τ is also a member of $Tr^*(\mathcal{H})$.

Finding $Tr(\mathcal{H})$ involves determining all possible branchings at each step in the algorithm. Suppose $Tr(\mathcal{H}_{r_1})$ has m members. Then at the r_2 level one must compute $Tr(H_2)$ for as many as m distinct hypergraphs. Continuing recursively we develop m trees; each rooted with a minimal transversal of $\mathcal{H}_{r_1}$ and with branching at each level corresponding to the number of choices available when applying step (ii) of Construction 1.9.

The same algorithm can be used to construct $Tr(\mathcal{H}^*)$. However we have seen that for $s > t$ there may be no way to extend a minimal transversal of $\mathcal{H}_s$ to a minimal transversal of $\mathcal{H}_t$. This "top down" construction may *therefore* be less efficient than a "bottom up" algorithm when finding $Tr^*(\mathcal{H})$.

LEMMA 1.10. Let α be a fuzzy transversal of a fuzzy hypergraph $\mathcal{H}$. Then there exists a minimal fuzzy transversal τ of $\mathcal{H}$ such that $\tau \leq \alpha$.

Proof. A slight modification of Construction 1.9 provides the proof. By Lemma 1.8, α_{r_1} is a minimal transversal of $\mathcal{H}_{r_1}$. Recursively define T_j as a transversal of $\mathcal{H}_{r_j}$ that is minimal with respect to the property that $T_{j-1} \subseteq T_j \subseteq \alpha_{r_j}$. Let τ_j be the elementary fuzzy set with support T_j and height r_j. Then $\tau = \vee\{\tau_j | 1 \leq j \leq n\}$ is a minimal fuzzy transversal of $\mathcal{H}$ with $\tau \leq \alpha$.

Section 2. The Vertex-Coloring Lemma and Tr($\mathcal{H}$)

While the set of minimal transversals of H forms the edge set of a hypergraph (which we also denote by Tr(H)), the vertex set may not be all of X. The same holds true for fuzzy transversals. We will obtain a characterization of when x is a vertex of Tr(H) and construct a partial hypergraph of $\mathcal{H}$ that aids in the computation of Tr(H). Repeating the process for fuzzy hypergraphs we obtain a simple, elementary fuzzy hypergraph $\mathcal{H}^*$ for which $Tr(\mathcal{H}) = Tr(\mathcal{H}^*)$. We also generalize the "vertex-colori ꞡ lemma" of Berge [1].

THEOREM 2.1. Let $H = (X,E)$ be a crisp hypergraph and let $x \in X$ be a vertex of H. Then x is an element of *some minimal* transversal of H if and only if x is an element of some edge that does not properly contain another edge. In particular, if H is a simple hypergraph then Tr(H) is a hypergraph on X.

Proof. WOLOG assume H has no repeated edges. Suppose that A is an edge that does not properly contain another edge of H. For each edge $B \in E \backslash A$ there exists an element $x_B \in B \backslash A$. Let $T' \subseteq \{x_B | B \neq A \text{ and } B \in E\}$ be a minimal transversal for the edge set $E \backslash A$. Then $x \notin T'$ and $T' \cap A = \emptyset$ implies that $T = T' \cup \{x\}$ is a minimal transversal of H.

Conversely, suppose that x is a vertex of H that belongs only to edges which properly contain another edge. Then for each edge A with $x \in A$, there exists an edge $B \subset A \backslash \{x\}$. Any minimal transversal T of H must contain a vertex $y \neq x$ of $B \subset A$. The minimality of $T \in Tr(H)$ then implies that $x \notin T$. □

OBSERVATION 2.2. By purging the edge set E of H of all edges that properly contain another edge, the remaining set E^* forms the edge set of a simple partial hypergraph called the *transversal core of* H denoted by $H^* = (X^*,E^*)$ and satisfying

(i) $Tr\ H^* = Tr(H)$

(ii) $\bigcup Tr(H) = X^*$

(iii) $X \backslash X^*$ is exactly the set of vertices of H that belong to no member of Tr(H).

LEMMA 2.3 (Vertex Coloring Lemma of Berge [1, p44]). Let $H = (X,E)$ be a hypergraph and $H' = (X',E')$ be a simple hypergraph with $X' \subseteq X$. Then $H' = Tr(H)$ if and only each partition (Y,Z) of X satisfies exactly one of the following conditions:

(i) there exists $A \in E$ such that $A \subseteq Y$.

(ii) there exists $B \in E'$ such that $B \subseteq Z$.

COROLLARY 2.4 [1, p45]. If H is simple then $Tr(Tr(H)) = (H)$.

COROLLARY 2.5. Any hypergraph H satisfies $Tr(Tr(H)) \subseteq (H)$.

Proof. By Observation 2.2 there exists a partial hypergraph H^* of H that is simple and for which $Tr\ H^* = Tr(H)$. Corollary 2.4 then implies $Tr(Tr(H)) = Tr(Tr\ H^*) = H^* \subseteq (H)$. □

We now proceed with a characterization of $Tr(\mathcal{H})$ for fuzzy hypergraphs. We first provide fuzzy analogs of Theorem 2.1 and the Vertex Coloring Lemma. We then construct the fuzzy transversal core.

LEMMA 2.6. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph and suppose $\tau \in Tr(\mathcal{H})$. If $x \in \text{supp}\ \tau$ then there exists a fuzzy edge μ of $\mathcal{H}$ for which

(i) $\mu(x) = h(\mu) = \tau(x) > 0$

(ii) $\tau_{h(\mu)} \cap \mu_{h(\mu)} = \{x\}$.

Proof. Suppose that $\tau(x) > 0$ and let A be the set of all fuzzy edges of $\mathcal{H}$ where for each $\alpha \in A$, $\alpha(x) \geq \tau(x)\}$. Since $\tau_{\tau(x)}$ is a transversal of $\mathcal{H}_{\tau(x)}$ and $x \in \tau_{\tau(x)}$, this set is nonempty. Further, each $\alpha \in A$ satisfies $h(\alpha) \geq \alpha(x) \geq \tau(x)$. If (2.6.i) is false then for each $\alpha \in A$ we have $h(\alpha) > \tau(x)$ and there exists $x_\alpha \neq x$ with $x_\alpha \in \alpha_{h(\alpha)} \cap \tau_{h(\alpha)}$ (definition of transversal). Define the new fuzzy set φ by

$$\varphi(y) = \begin{cases} \tau(y) & \text{if } y \neq x \\ \max\ \{h(\mu) \mid h(\mu) < \tau(x) & \text{if } y = x \end{cases}.$$

Clearly φ is a fuzzy transversal of $\mathcal{H}$ and $\varphi < \tau$. This contradicts the minimality of τ, so some $\mu \in \mathcal{E}$ satisfies (2.6.i).

Now suppose each $\alpha \in A$ satisfies (2.6.i) and also contains an $x_\alpha \neq x$ with $x_\alpha \in \alpha_{h(\alpha)} \cap \tau_{h(\alpha)}$. Repeating the argument of (2.6.i) provides a contradiction and completes the proof. □

THEOREM 2.7. Suppose that $\mathcal{H} = (X,\mathcal{E})$ is a fuzzy hypergraph and $x \in X$. Then there exists $\tau \in Tr(\mathcal{H})$ with $x \in \operatorname{supp} \tau$, if and only if there exists a $\mu \in \mathcal{E}$ satisfying

(i) $\mu(x) = h(\mu)$

(ii) for each $\nu \in \mathcal{E}$ such that $h(\nu) > h(\mu)$, the $h(\nu)$ level cut of ν is not a subset of the $h(\mu)$ level cut of μ.

(iii) the $h(\mu)$ level cut of μ does not properly contain any other edge of $\mathcal{H}_{h(\mu)}$.

Under the conditions above, τ may be *chosen* so that $\tau(x) = h(\mu) = \mu(x)$.

Proof. Suppose that $\tau \in Tr(H)$ and $\tau(x) > 0$. Condition (2.7.i) follows from Lemma 2.6. For (2.7.ii), assume for each μ satisfying (2.7.i), there exists $\nu \in \mathcal{E}$ such that $h(\nu) > h(\mu)$ and $\nu_{h(\nu)} \subseteq \mu_{h(\mu)}$. Then there exists $y \neq x$ with $y \in \nu_{h(\nu)} \cap \tau_{h(\nu)} \subseteq \mu_{h(\mu)} \cap \tau_{h(\mu)}$, contradicting Lemma 2.6.

For (2.7.iii), suppose that for each μ satisfying (2.7.i) and (2.7.ii) there exists $\nu \in \mathcal{E}$ such that $\emptyset \subset \nu_{h(\mu)} \subset \mu_{h(\mu)}$. Since $\nu_{h(\mu)} \neq \emptyset$ and by (2.7.ii), we have $h(\nu) = h(\mu) = \mu(x)$. If $\nu(x) = h(\nu)$ our hypothesis provides $\nu' \in \mathcal{E}$ such that $\emptyset \subset \nu'_{h(\mu)} \subset \nu_{h(\mu)} \subset \mu_{h(\mu)}$. Continuing recursively the chain must end in finitely many steps so WOLOG assume $\nu(x) < h(\nu)$. However this implies there exists $y \neq x$ such that $y \in \nu_{h(\mu)} \cap \tau_{h(\mu)} \subseteq \mu_{h(\mu)} \cap \tau_{h(\mu)}$, contradicting Lemma 2.6. Therefore there exists some $\mu \in \mathcal{E}$ that satisfies (2.7.i), (2.7.ii) and (2.7.iii).

Sufficiency of the conditions is shown using Construction 1.9. Let the pair (x,μ) satisfy conditions (2.7.i), (2.7.ii) and (2.7.iii). By condition (2.7.i), $h(\mu) = r_i$ for some member of the fundamental sequence. By conditions (2.7.ii) and (2.7.iii) there exists $y_\nu \in \nu_{h(\nu)} \setminus \mu_{h(\mu)}$ for each $\nu \in \mathcal{E}$ such that $\nu \neq \mu$ and $h(\nu) \geq h(\mu)$. Let A be the set of all such vertices, so $A \cap \mu_{h(\mu)} = \emptyset$. Construct the initial sequence of transversals so that $T_j \subseteq A$ for each $1 \leq j < i$, and $T_i \subseteq A \cup \{x\}$. Clearly this is possible and for each i, $x \in T_i$. Continuation of Construction 1.9 will produce a suitable minimal fuzzy transversal τ with $\tau(x) = \mu(x) = h(\mu)$. □

THEOREM 2.8. Fuzzy analog of the Vertex Coloring Lemma. Let $\mathcal{H} = (X, \mathcal{E})$ and $\mathcal{H}' = (X', \mathcal{E}')$ be fuzzy hypergraphs. Then $\mathcal{H}' = \mathrm{Tr}(\mathcal{H})$ if and only if $\mathcal{H}'$ is simple, $X' \subseteq X$, $h(\alpha) = h(\mathcal{H})$ for each $\alpha \in \mathcal{E}'$, and for each fuzzy set $\beta \in \mathcal{F}(X)$ exactly one of the following conditions is satisfied:

(i) there exists $\alpha \in \mathcal{E}'$ such that $\alpha \le \beta$ or

(ii) there exist $\mu \in \mathcal{E}$ and $c \in (0, h(\mu)]$ such that $\mu_c \cap \beta_c = \emptyset$; that is, β is not a fuzzy transversal of $\mathcal{H}$.

Proof. Let $\mathcal{H}' = \mathrm{Tr}(\mathcal{H})$. By definition $\mathrm{Tr}(\mathcal{H})$ is a simple fuzzy hypergraph on $X' \subseteq X$. By Lemma 1.5, each edge of $\mathrm{Tr}(\mathcal{H})$ has height $h(\mathcal{H})$. Let $\beta \in \mathcal{F}(X)$. If β is a fuzzy transversal of $\mathcal{H}$, then by Lemma 1.10 there exists a minimal fuzzy transversal contained in β. Thus condition (2.8.i) holds and condition (2.8.ii) does not.

If β is not a fuzzy transversal of $\mathcal{H}$, then there exists $\mu \in \mathcal{E}$ such that $\beta_{h(\mu)} \cap \nu_{h(\mu)} = \emptyset$. Clearly $\alpha \le \beta$ implies $\alpha_{h(\mu)} \cap \nu_{h(\mu)} = \emptyset$ as well, so condition (2.8.ii) is satisfied and condition (2.8.i) is not.

Conversely, let $\mathcal{E}'$ have the properties above and let $\alpha \in \mathcal{E}'$. By setting $\alpha = \beta$ we have $\alpha \le \alpha$ and α violates condition (2.8.ii). Thus α is a fuzzy transversal of $\mathcal{H}$. If $\tau \le \alpha$ and τ is a fuzzy transversal of $\mathcal{H}$, τ violates condition (2.8.ii) and there exists $\beta \in \mathcal{E}'$ with $\beta \le \tau \le \alpha'$. However $\mathcal{H}'$ simple implies $\beta = \tau = \alpha$; hence $\mathcal{E}' \subseteq \mathrm{Tr}(\mathcal{H})$. Similarly $\tau \in \mathrm{Tr}(\mathcal{H})$ violates condition (2.8.ii) and there exists $\alpha \in \mathcal{E}'$ with $\alpha \le \tau$. But α is a fuzzy transversal of $\mathcal{H}$ so the minimality of τ implies $\alpha = \tau$. Therefore $\mathcal{E}' = \mathrm{Tr}(\mathcal{H})$ as required. □

OBSERVATION 2.9. Theorem 2.8 was stated in a simplified form. By Lemma 1.5 any fuzzy transversal τ satisfies $h(\tau) \ge h(\mathcal{H})$. By Lemma 1.6 and Lemma 1.7 the fuzzy hypergraph $\mathrm{Tr}(\mathcal{H})$ is sectionally elementary with

$fs(\mathrm{Tr}(\mathcal{H})) \subseteq fs(\mathcal{H})$. Using these properties as additional hypothesis the set of β's (test fuzzy sets) could be reduced to those $\beta \in \mathcal{F}(X)$ such that $h(\beta) = h(\mathcal{H})$ and for each $x \in X$, $\beta(x) \in fs(\mathcal{H})$. An obvious advantage to using this collection of fuzzy sets is that the cardinality is finite.

LEMMA 2.10. Let $\mathcal{H}$ be a fuzzy hypergraph with $fs(\mathcal{H}) = \{r_1,\cdots,r_n\}$ and $r_1 \cdot \mathcal{H}_{r_1}$ be the elementary fuzzy hypergraph where ν is an edge of $r_1 \cdot \mathcal{H}_{r_1}$ if and only if $h(\nu) = r_1$ and supp ν is an edge of $\mathcal{H}_{r_1}$. Then $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))$ is a partial fuzzy hypergraph of $r_1 \cdot \mathcal{H}_{r_1}$.

Proof. By Lemma 1.8 and Construction 1.9, the r_1 level hypergraph of $\mathrm{Tr}(\mathcal{H})$ is $\mathrm{Tr}(\mathcal{H}_{r_1})$; that is, $(\mathrm{Tr}(\mathcal{H}))_{r_1} = \mathrm{Tr}(\mathcal{H}_{r_1})$. Let $\alpha \in \mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))$. By Theorem 2.7, $\alpha(x) > 0$ implies there exists $\tau \in \mathrm{Tr}(\mathcal{H})$ with $\alpha(x) = h(\tau)$. By Lemma 1.5, $h(\tau) = r_1$ for each minimal transversal τ. Therefore α is elementary with height r_1. Since supp $\alpha = \alpha_{r_1}$, Lemma 1.8 implies supp α is a minimal transversal of $(\mathrm{Tr}(\mathcal{H}))_{r_1}$ and hence supp $\alpha \in \mathrm{Tr}(\mathrm{Tr}(\mathcal{H}_{r_1}))$. By Corollary 2.5 supp α is an edge of $\mathcal{H}_{r_1}$, so α is an edge of $r_1 \cdot \mathcal{H}_{r_1}$. □

COROLLARY 2.11. If $\mathcal{H}$ is a fuzzy hypergraph with $\mathcal{H}_{r_1}$ simple, then $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) = r_1 \cdot \mathcal{H}_{r_1}$ where $r_1 \cdot \mathcal{H}_{r_1}$ is defined as in Lemma 2.10.

Proof. By Lemma 2.10 $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) \le r_1 \cdot \mathcal{H}_{r_1}$. Let α be elementary with $h(\alpha) = r_1$ and supp $\alpha \in \mathcal{H}_{r_1}$. As in the proof of Lemma 2.10 supp α is a minimal transversal of $\mathrm{Tr}(\mathcal{H}_{r_1})$ and hence supp α is a minimal transversal of $(\mathrm{Tr}(\mathcal{H}))_{r_1}$. Since each minimal transversal of $\mathrm{Tr}(\mathcal{H})$ is elementary, Construction 1.9 terminates at the r_1 level and $\alpha \in \mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))$. Therefore $r_1 \cdot \mathcal{H}_{r_1} \le \mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))$ and $\mathcal{H}_{r_1} = \mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))$. □

It is evident from Lemma 2.6 and from Theorem 2.7 that the set of all minimal fuzzy transversals of $\mathcal{H}$ is completely determined by the top cut-level sets of those edges of $\mathcal{H}$ that satisfy the three conditions of Theorem 2.7. We proceed to generalize Observation 2.2 by the following two constructions.

CONSTRUCTION 2.12. We obtain a subsequence of $fs(\mathcal{H})$ denoted $fs^*(\mathcal{H})$ and a set of partial hypergraphs from $I(\mathcal{H})$ denoted $\hat{I}(\mathcal{H})$ as follows:

(i) Obtain the partial hypergraph $\hat{\mathcal{H}}_{r_1}$ of $\mathcal{H}_{r_1}$ by eliminating all edges that properly contain another edge of $\mathcal{H}_{r_1}$. By Observation 2.2 $\mathrm{Tr}(\hat{\mathcal{H}}_{r_1}) = \mathrm{Tr}(\mathcal{H}_{r_1})$.

(ii) Obtain the partial hypergraph $\hat{\mathcal{H}}_{r_2}$ of $\mathcal{H}_{r_2}$ by eliminating all edges that either properly contain another edge of $\mathcal{H}_{r_2}$ (property 2.7.iii) or contain (properly or improperly) an edge of $\hat{\mathcal{H}}_{r_1}$ (property 2.7.ii). By Theorem 2.7, $\hat{\mathcal{H}}_{r_2}$ is nonempty if and only there exists a $\tau \in \mathrm{Tr}(\mathcal{H})$ and an $x \in X_{r_2}$ such that $\tau(x) = r_2$.

(iii) Continue recursively obtaining $\hat{\mathcal{H}}_{r_i}$ from $\mathcal{H}_{r_i}$ by eliminating all edges that either properly contain another edge of $\mathcal{H}_{r_i}$ or contain an edge of $\hat{\mathcal{H}}_{r_1}$ or $\hat{\mathcal{H}}_{r_2}$ or $\circ\circ\circ$ $\hat{\mathcal{H}}_{r_{i-1}}$. By Theorem 2.7, $\hat{\mathcal{H}}_{r_i}$ is nonempty if and only there exists $\tau \in \mathrm{Tr}(\mathcal{H})$ and an $x \in X_{r_i}$ such that $\tau(x) = r_i$.

Collecting those nonempty partial hypergraphs and corresponding cut-levels $r_1^* = r_1 > r_2^* > \circ\circ\circ > r_m^*$ we define

$$fs^*(\mathcal{H}) = \{r_1^*, \circ\circ\circ, r_m^*\} \quad \text{and} \quad \hat{I}(\mathcal{H}) = \{\hat{\mathcal{H}}_{r_1^*}, \circ\circ\circ, \hat{\mathcal{H}}_{r_m^*}\}$$

OBSERVATION 2.13. By construction, the fundamental sequence of $\mathrm{Tr}(\mathcal{H})$ is $fs^*(\mathcal{H})$. Further, T is an edge of $\hat{\mathcal{H}}_{r_i^*}$ if and only if T is the top cut of an edge of $\mathcal{H}$ with height r_i^*, satisfying the last two conditions of Theorem 2.7.

We have now identified the fundamental attributes of $\mathrm{Tr}(\mathcal{H})$. Using these attributes we construct an elementary (thus ordered) fuzzy hypergraph called the fuzzy transversal core of $\mathcal{H}$.

CONSTRUCTION 2.14. Let $\mathcal{H}$ be a fuzzy hypergraph where $fs^*(\mathcal{H})$ and $\hat{I}(\mathcal{H})$ are defined as in Construction 2.12. Define the *fuzzy transversal core of $\mathcal{H}$* as the elementary fuzzy hypergraph $\mathcal{H}^* = (X^*, \mathcal{E}^*)$ in which

(i) $fs^*(\mathcal{H})$ is the fundamental sequence of $\mathcal{H}^*$

(ii) $\mu^* \in \mathcal{E}^*$ has height $r_j^* \in fs^*(\mathcal{H})$ iff $\text{supp}(\mu^*)$ is an edge of $\hat{\mathcal{H}}_{r_j^*}$

OBSERVATION 2.15. The following statements are trivial consequences of previous definitions and results. The height of each edge in $\mathcal{E}^*$ is a member of $fs^*(\mathcal{H})$ and $fs(\mathcal{H}^*) = fs^*(\mathcal{H}) = fs(\text{Tr}(\mathcal{H}))$. Further, $\mathcal{H}^*$ is elementary (thus ordered) and is support simple. The sequence $I(\mathcal{H}^*) = \{\mathcal{H}^*_{r_1^*}, \circ\circ\circ, \mathcal{H}^*_{r_m^*}\}$ is obtained recursively from $\hat{I}(\mathcal{H})$ by letting $\mathcal{E}^*_c = \bigcup_{i=1}^{j} \hat{\mathcal{E}}_{r_i^*}$ where r_j^* is minimal with respect to $r_j^* \geq c$. The vertex set of $\mathcal{H}^*$ is exactly the vertex set of $\text{Tr}(\mathcal{H})$

THEOREM 2.16. For every fuzzy hypergraph $\mathcal{H}$, $\text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{H}^*)$.

Proof. Let $\tau \in \text{Tr}(\mathcal{H})$ and $\mu^* \in \mathcal{E}^*$. By definition of $\mathcal{H}^*$ we have $h(\mu^*) = r_i^*$ for some $r_i^* \in fs^*(\mathcal{H})$ and $\mu^*_{r_i^*}$ is an edge of $\hat{\mathcal{H}}_{r_i^*}$. However $\hat{\mathcal{H}}_{r_i^*}$ is a partial hypergraph of $\mathcal{H}_{r_i^*}$ and τ_{r_i} is a transversal of $\mathcal{H}_{r_i^*}$ so $\mu^*_{r_i^*} \cap \tau_{r_i} \neq \emptyset$. Therefore τ is a (possibly nonminimal) fuzzy transversal of $\mathcal{H}^*$.

Let $\tau^* \in \text{Tr}(\mathcal{H}^*)$ and $\mu \in \mathcal{E}$. By definition of the fundamental sequence of $\mathcal{H}$, $\mu_{h(\mu)} \in \mathcal{H}_{r_j}$ for some $r_j \in fs(\mathcal{H})$, where $r_j \geq h(\mu)$. By definition of the sequence $\hat{I}(\mathcal{H})$, there is some edge T of $\hat{\mathcal{H}}_{r_k^*}$ where $T \subseteq \mu_{h(\mu)}$ and $r_k^* \geq r_j \geq h(\mu)$. Let μ^* be the edge of $\mathcal{E}^*$ with support T and height r_k^* so $T = \mu^*_{r_k} \subseteq \mu_{h(\mu)}$. Since $\tau^* \in \text{Tr}(\mathcal{H}^*)$ there exists a $y \in \mu^*_{r_k} \cap \tau^*_{r_k}$. Then $y \in \ \subseteq \mu_{h(\mu)} \cap \tau^*_{h(\mu)}$ and τ^* is a transversal of $\mathcal{H}$.

For minimality, recall $\tau \in \text{Tr}(\mathcal{H})$ implies τ is a transversal of $\mathcal{H}^*$. Then there exists $\tau^* \in \text{Tr}(\mathcal{H}^*)$ such that $\tau^* \leq \tau$. However τ^* is a transversal of $\mathcal{H}$ so minimality of τ (in $\mathcal{H}$) implies $\tau^* = \tau$. Therefore $\text{Tr}(\mathcal{H}) \subseteq \text{Tr}(\mathcal{H}^*)$. A similar argument shows $\text{Tr}(\mathcal{H}^*) \subseteq \text{Tr}(\mathcal{H})$ and completes the proof. □

Section 3. Locally Minimal Fuzzy Transversals

In Example 1.3 we saw that each cut-level of a minimal fuzzy transversal may not be a minimal transversal of the corresponding cut-level hypergraph. We thus gave Definition 1.4 naming locally minimal fuzzy transversals and $Tr^*(\mathcal{H})$. Conversely, a transversal of a cut-level hypergraph need not be the cut-level set of any fuzzy transversal. In this section we explore some of these issues. Of particular interest will be conditions for which $Tr^*(\mathcal{H})$ is nonempty and for which $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$.

In Construction 2.14 we defined $\mathcal{H}^*$ (the transversal core of $\mathcal{H}$) and in Theorem 2.16 we showed that $Tr(\mathcal{H}) = Tr(\mathcal{H}^*)$. In this section we show that $Tr^*(\mathcal{H}) \subseteq Tr^*(\mathcal{H}^*) \subseteq Tr(\mathcal{H}^*) = Tr(\mathcal{H})$ and that the inclusions may be proper. We also give a condition which implies equality. We begin with a crisp result.

LEMMA 3.1. Suppose $\hat{H} = (Y,F)$ is a (crisp) partial hypergraph of a (crisp) hypergraph $H = (X,E)$. Then

(i) If T is a minimal transversal of H, then there exists a minimal transversal $\hat{T}$ of $\hat{H}$ such that $\hat{T} \subseteq T$.

(ii) Let $\hat{H}$ and H be simply ordered and $\hat{T}$ be a minimal transversal of $\hat{H}$. Then there exists a minimal transversal T of H such that $\hat{T} \subseteq T$.

Proof. Let T be any minimal transversal of H. Since each edge of $\hat{H}$ is an edge of H, $T \cap Y$ is a transversal of $\hat{H}$. The $\hat{T}$ required for part (i) then clearly exists. For part (ii) let $G = \{A \in E \mid A \notin F\}$ and let $\hat{T}$ be a minimal transversal of $\hat{H}$. If G is empty we are done. Otherwise, since $\hat{H}$ and H are simply ordered, for each $A \in G$ there exists $x_A \in A \backslash Y$. Then $\hat{T} \cup \{x_A \mid A \in G\}$ is a transversal of H that contains a minimal transversal T of H. Clearly $\hat{T}$ is contained in T. □

EXAMPLE 3.2. Consider the ordered hypergraphs $\hat{H}$ and H defined by Figure 4.

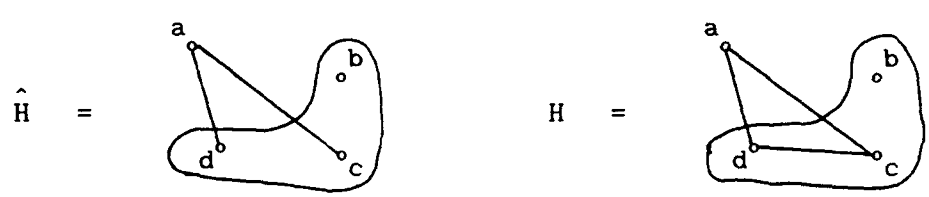


Figure 4. *Extending a minimal transversal of a partial hypergraph*

Notice that $\hat{T} = \{a,b\}$ is a minimal transversal of $\hat{H}$ and that no extension of $\hat{T}$ is a minimal transversal of H. Therefore Lemma 3.1.ii requires a stronger condition than ordered.

THEOREM 3.3. Suppose that $\mathcal{H}$ is an ordered fuzzy hypergraph with $fs(\mathcal{H}) = \{r_1, \cdots, r_n\}$. If T_j is a minimal transversal of $\mathcal{H}_{r_j}$ then there exists $\tau \in Tr(\mathcal{H})$ such that $\tau_{r_j} = T_j$ and for all $i < j$, τ_{r_i} is a minimal transversal of $\mathcal{H}_{r_i}$. In particular, if T_n is a minimal transversal of $\mathcal{H}_{r_n}$ then there exists a locally minimal fuzzy transversal τ such that $\tau_{r_n} = T_n$. Therefore if $\mathcal{H}$ is ordered then $Tr^*(\mathcal{H})$ is nonempty.

Proof. Let T_j be a minimal transversal of $\mathcal{H}_{r_j}$. Since $\mathcal{H}$ is ordered, $\mathcal{H}_{r_{j-1}}$ is a partial hypergraph of $\mathcal{H}_{r_j}$. By Lemma 3.1.i, there exists a minimal transversal T_{j-1} of $\mathcal{H}_{r_{j-1}}$ with $T_{j-1} \subseteq T_j$. Continuing recursively we obtain a sequence $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_j$ where each T_i is a minimal transversal of the corresponding $\mathcal{H}_{r_i}$.

Following the method of Construction 1.9 we obtain minimal extensions $T_j \subseteq T_{j+1} \subseteq \cdots \subseteq T_n$ to transversals of $\mathcal{H}_{r_j} < \mathcal{H}_{r_{j+1}} < \cdots < \mathcal{H}_{r_n}$. Let τ_i be the elementary fuzzy set with height r_i and support T_i. Then defining $\tau(x) = \max\{\tau_i(x) \mid 1 \le i \le n\}$ produces the required minimal fuzzy transversal of $\mathcal{H}$. Clearly if $j = n$, τ is a locally minimal fuzzy transversal and $Tr^*(\mathcal{H})$ is nonempty. □

A consequence of Theorem 3.3 is that for ordered fuzzy hypergraphs, each minimal transversal T of a cut-level hypergraph is a cut-level set of some minimal fuzzy transversal τ. However by Example 3.2 τ may not be a locally minimal fuzzy transversal and $Tr^*(\mathcal{H})$ may be properly contained in $Tr(\mathcal{H})$. For simply ordered fuzzy hypergraphs we get a slightly stronger result.

THEOREM 3.4. Suppose that $\mathcal{H}$ is a simply ordered fuzzy hypergraph with $fs(\mathcal{H}) = \{r_1, \cdots, r_n\}$. If T_j is a minimal transversal of $\mathcal{H}_{r_j}$, then there exists $\tau \in Tr^*(\mathcal{H})$ such that $\tau_{r_j} = T_j$.

Proof. Suppose that $\mathcal{H}$ is simply ordered and let T_j be a minimal transversal of $\mathcal{H}_{r_j}$. By Lemma 3.1.i and 3.1.ii there exists a sequence

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T_j \subseteq \cdots \subseteq T_n$$

where each T_i is a minimal transversal of $\mathcal{H}_{r_i}$. Let τ_i be the elementary fuzzy set with height r_i and support T_i. Then $\tau = \max\{\tau_i | 1 \leq i \leq n\}$ is a locally minimal fuzzy transversal of $\mathcal{H}$ with $\tau_{r_j} = T_j$. $\square$

We now provide notation and definitions that will aid in considering the question of when $Tr(\mathcal{H}) = Tr^*(\mathcal{H})$.

DEFINITION 3.5. An ordered pair (H,K) of crisp hypergraphs is T-*related* if the conditions P_K is a transversal of K, T_H is a minimal transversal of H, and $T_H \subseteq P_K$, imply that there exists a minimal transversal T_K of K such that $T_H \subseteq T_K \subseteq P_K$.

DEFINITION 3.6. Let $\mathcal{H}$ be a fuzzy hypergraph $\mathcal{H}$ with $fs(\mathcal{H}) = \{r_1, \cdots, r_n\}$. Then $\mathcal{H}$ is T-*related* if as $1 \leq i \leq n-1$, each successive ordered pair $(\mathcal{H}_{r_i}, \mathcal{H}_{r_{i+1}})$ is T-related. If the fundamental sequence of $\mathcal{H}$ is a singleton, $\mathcal{H}$ is considered (vacuously) to be T-related.

THEOREM 3.7. Let $\mathcal{H}$ be a T-related fuzzy hypergraph. Then $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$.

Proof. Since all fuzzy hypergraphs satisfy $Tr^*(\mathcal{H}) \subseteq Tr(\mathcal{H})$ we need only show $Tr(\mathcal{H}) \subseteq Tr^*(\mathcal{H})$. Let $\tau \in Tr(\mathcal{H})$ and suppose there exists $r_j \in fs(\mathcal{H})$ where τ_{r_j} is a nonminimal transversal of $\mathcal{H}_{r_j}$. We assume WOLOG that r_j is the greatest such value and note that by Lemma 1.8 $r_1 > r_j$. By the definition of T-related there exists a minimal transversal T of $\mathcal{H}_{r_j}$ such that $\tau_{r-1} \subset T \subset \tau_{r_j}$. (the inclusions are proper). We now define the fuzzy set α where

$$\alpha(x) = \begin{cases} \tau(x) & \text{if } \tau(x) \geq r_j \text{ and } x \in T \\ r_{j+1} & \text{if } \tau(x) \geq r_j \text{ and } x \in \tau_{r_j} \backslash T \\ \tau(x) & \text{if } \tau(x) < r_j \end{cases}$$

However by Observation 1.2, α is a fuzzy transversal of $\mathcal{H}$ and by definition α is properly contained in τ. This contradicts the assumption that τ is a minimal fuzzy transversal *and hence* $\tau \in Tr^*(\mathcal{H})$. □

EXAMPLE 3.8. The fuzzy hypergraph of Figure 5 provides an example where the converse of Theorem 3.7 is false. The set $\{d,e\}$ is a minimal transversal of $\mathcal{H}_{.8}$ and the set $\{d\}$ is a minimal transversal of $\mathcal{H}_{.4}$. Thus no minimal transversal of $\mathcal{H}_{.4}$ can contain $\{d,e\}$ and the pair $(\mathcal{H}_{.8}, \mathcal{H}_{.4})$ is not T-related.

$$\mathcal{H} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccc} \lambda & \mu & \nu \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ .8 & .8 & .4 \\ .4 & .4 & .8 \end{bmatrix} \end{array} \qquad Tr(\mathcal{H}) = Tr^*(\mathcal{H}) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccc} \tau & \varphi & \omega \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 5. $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$ *with* $\mathcal{H}$ *not T-related*

We now show there is a partial converse of Theorem 3.7 that applies to a large class of fuzzy hypergraphs.

THEOREM 3.9. Let $\mathcal{H}$ be an ordered fuzzy hypergraph. Then $\mathcal{H}$ is T-related if and only if $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$.

Proof. The forward implication is Theorem 3.7. Conversely, suppose that $\mathcal{H}$ is ordered and that $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$. Let $T_j \in Tr(\mathcal{H}_{r_j})$ and let T be a fuzzy transversal of $\mathcal{H}_{r_{j+1}}$ with $T_j \subseteq T$. Each edge of $\mathcal{H}_{r_j}$ is also an edge of $\mathcal{H}_{r_{j+1}}$, hence if T_j is a transversal of $\mathcal{H}_{r_{j+1}}$ then T_j is necessarily minimal and we are done. Otherwise there exists T_{j+1} where $T_j \subseteq T_{j+1} \subseteq T$ and T_{j+1} is a minimal extension of T_j to a transversal of $\mathcal{H}_{j+1}$. As in the proof of Theorem 3.3 there then exists $\tau \in Tr(\mathcal{H})$ with $\tau_{r_j} = T_j$ and $\tau_{r_{j+1}} = T_{j+1}$. We assumed $Tr(\mathcal{H}) = Tr^*(\mathcal{H})$ so τ is locally minimal and T_{j+1} is a minimal transversal of $\mathcal{H}_{r_{j+1}}$ as required. □

COROLLARY 3.10. We note the following relationships between a fuzzy hypergraph $\mathcal{H}$ and the corresponding transversal core $\mathcal{H}^*$:

(i) $\mathrm{Tr}(\mathcal{H}) = \mathrm{Tr}^*(\mathcal{H}^*)$ if and only if $\mathcal{H}^*$ is T-related

(ii) $\mathrm{Tr}(\mathcal{H}^*) = \mathrm{Tr}^*(\mathcal{H}^*)$ if and only if $\mathcal{H}^*$ is T-related

Proof. By Observation 2.15 $\mathcal{H}^*$ is ordered. Apply Theorems 2.16 and 3.9. □

THEOREM 3.11. For each fuzzy hypergraph $\mathcal{H}$, $\mathrm{Tr}^*(\mathcal{H}) \subseteq \mathrm{Tr}^*(\mathcal{H}^*)$. The inclusion may be proper.

Proof. We use the notation of Construction 2.12 through Theorem 2.16 extensively. Let $\mathcal{H}$ be a fuzzy hypergraph and $\tau \in \mathrm{Tr}^*(\mathcal{H})$ a locally minimal fuzzy transversal of $\mathcal{H}$. By Observation 2.15

$$fs(\mathrm{Tr}(\mathcal{H})) = fs(\mathcal{H}^*) = fs^*(\mathcal{H}) = \{r_1^*, \cdots, r_m^*\} .$$

By definition of locally minimal, $\tau_{r_j^*}$ is a minimal transversal of $\mathcal{H}_{r_j^*}$ for each r_j^*. Then by Construction 2.12 and Observation 2.13 $\tau_{r_j^*}$ is a minimal transversal of $\mathcal{H}^*_{r_j^*}$ for each r_j^*. Therefore $\tau \in \mathrm{Tr}^*(\mathcal{H}^*)$ as required.

For the second statement recall by Example 1.3 $\mathrm{Tr}^*(\mathcal{H})$ may be empty. However $\mathcal{H}^*$ is ordered hence $\mathrm{Tr}^*(\mathcal{H}^*)$ is nonempty by Theorem 3.3. □

We summarize a number of results in the following corollary. The proof of each statement follows immediately from previous theorems.

THEOREM 3.12. Let $\mathcal{H}$ be a fuzzy hypergraph then

(i) $\mathrm{Tr}^*(\mathcal{H}) \subseteq \mathrm{Tr}^*(\mathcal{H}^*) \subseteq \mathrm{Tr}(\mathcal{H}^*) = \mathrm{Tr}(\mathcal{H})$; inclusions may be proper.

(ii) If $\mathcal{H}$ is T-related, then $\mathrm{Tr}^*(\mathcal{H}) = \mathrm{Tr}^*(\mathcal{H}^*) = \mathrm{Tr}(\mathcal{H}^*) = \mathrm{Tr}(\mathcal{H})$.

(iii) $\mathrm{Tr}^*(\mathcal{H}^*) = \mathrm{Tr}(\mathcal{H}^*) = \mathrm{Tr}(\mathcal{H})$ if and only if $\mathcal{H}^*$ is T-related.

(iv) If $\mathrm{Tr}^*(\mathcal{H}) = \mathrm{Tr}(\mathcal{H})$, then $\mathcal{H}^*$ is T-related.

(v) If $\mathcal{H}$ is T-related, then $\mathcal{H}^*$ is T-related as well.

Proof. Clearly the set of locally minimal fuzzy transversals is contained in the set of all fuzzy transversals. The other inclusions of (3.12.i) follow from Theorems 2.16 and 3.11. Part (3.12.ii) follows immediately from Theorems 3.7 and 3.12.i. By Observation 2.15, $\mathcal{H}^*$ is ordered;

therefore (3.12.iii) follows from Theorem 3.9. For (3.12.iv) we show the contrapositive. If $\mathcal{H}^*$ is not T-related then Theorem 3.12.iii implies $Tr^*(\mathcal{H}^*) \neq Tr(\mathcal{H}^*)$. Then Theorem 3.12.i implies $Tr^*(\mathcal{H}) \neq Tr(\mathcal{H})$. Part (3.12.v) follows from Theorem 3.7; if $\mathcal{H}$ is T-related then $Tr^*(\mathcal{H}) = Tr(\mathcal{H})$. Then Theorem 3.12.iv completes the proof. □

The reader may verify that Example 3.8 provides a fuzzy hypergraph for which $\mathcal{H}^*$ is T-related and $\mathcal{H}$ is not. Therefore the converse of Theorem 3.12.v fails. The converse of Theorem 3.12.iv also fails by the next example.

EXAMPLE 3.13. Let $\mathcal{H}$ be defined by the incidence matrix in Figure 6. The only minimal transversal of the 1 cut-level of $\mathcal{H}$ is the set {a,b}, which is also a (nonminimal) transversal of $\mathcal{H}_{.5}$. Therefore the edge set of $\mathcal{H}^*$ consists of the characteristic functions of the sets {a} and {b}; $Tr(\mathcal{H})$ consists of the characteristic function of the set {a,b}. The .5 level hypergraph has minimal transversals {a}, {b} and {c}. Therefore $Tr^*(\mathcal{H})$ is empty. Since $Tr^*(\mathcal{H}^*) = Tr(\mathcal{H}^*)$, $\mathcal{H}^*$ is T-related by Theorem 3.12.iii.

$$\mathcal{H} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} \lambda & \mu & \nu \end{array} \\ \begin{pmatrix} 1 & .5 & .5 \\ 0 & 1 & .5 \\ .5 & 0 & .5 \end{pmatrix} \end{array} \qquad Tr(\mathcal{H}) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \tau \\ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

Figure 6. $\mathcal{H}^*$ *is T-related while* $Tr^*(\mathcal{H}) \neq Tr(\mathcal{H})$

CHAPTER 3. Vertex Colorings

This chapter introduces the notion of a vertex coloring of a fuzzy hypergraph. We first provide some basic results and then define a (crisp) hypergraph that simplifies the calculation of the chromatic number. We also provide a fuzzy hypergraph that serves the same purpose. In the second section we pay particular attention to the class of ordered fuzzy hypergraphs.

Section 1. Vertex Coloring of a Fuzzy Hypergraph

Let $H = (X,E)$ be a crisp hypergraph and let $k \geq 2$ be an integer. A *k-coloring* of the vertex set is a function $\Delta: X \longrightarrow \{1,2,\cdots,k\}$ such that for each nonloop edge $A \in E$, Δ is nonconstant on A. The function Δ induces a partition of X into the *color classes* $\{S_1,S_2,\cdots,S_k\}$ where $S_i = \Delta^{-1}(i)$. A vertex coloring may be equivalently defined by this partition of X where each nonloop edge has nonempty intersection with at least two distinct color classes.

An edge is said to be *monochromatic* if it is contained in a single color class. The *chromatic number* of H, denoted $\chi(H)$, is the minimal integer k for which there exists a k-coloring of H.

In a vertex coloring of H the only monochromatic edges are loops. Conversely, loops need not be considered when constructing vertex colorings.

OBSERVATION 1.1. In Observation 2.2.2 we defined the transversal core of a hypergraph $H = (X,E)$. We now define a similar *color core* of H to be the simple partial hypergraph $\hat{H} = (\hat{X},\hat{E})$ where $\hat{E}$ is formed by purging from E all loops and all edges that properly contain another nonloop edge. Formally we let

$$\hat{E} = \{A \in E \mid |A| \geq 2 \text{ and } (A' \in E,\ A' \subseteq A,\ A' \neq A) \Rightarrow |A'| = 1\}.$$

$$\text{and} \qquad \hat{X} = \bigcup A \in \hat{E}$$

Any coloring of $\hat{H}$ may be extended to a coloring of H by assigning an arbitrary coloring to the vertices of $X \setminus \hat{X}$. Thus $\hat{H}$ completely determines the possible colorings of H and $\chi(\hat{H}) = \chi(H)$.

Recall that the existence of a loop at vertex x requires $x \in T$ for any transversal T. However loops have no effect on vertex colorings. Therefore the color core and transversal core are not in general equal.

CONSTRUCTION 1.2. Minimal transversals of a hypergraph H may be used to define a coloring of H by the following construction:

(i) WOLOG assume H has no loops. Find a minimal transversal T_1 of H. Clearly no edge of H is contained in $X \setminus T_1$.

(ii) Let H_2 be the partial hypergraph of H induced by those (nonloop) edges of H that are contained in T_1. If H_2 is empty, the partition $(X \setminus T_1) \cup T_1$ is a two coloring of H. If not, let T_2 be a minimal transversal of H_2. Clearly no edge of H is contained in $T_1 \setminus T_2$.

(iii) Continue recursively where if H_k is empty, then

$$(X \setminus T_1) \cup (T_1 \setminus T_2) \cup \cdots \cup (T_{k-1} \setminus T_k)$$

is a k + 1 coloring of H. Since for any minimal transversal T of a hypergraph (without loops) H, $|T| < |X|$, the process must stop in at most $k = |X| - 1$ steps.

EXAMPLE 1.3. Construction 1.2 will not always produce a minimal coloring of H, even if we pick minimal transversals with minimal cardinality at each step. The partition $\{a,b,c,e\} \cup \{d,f,g\}$ is a 2-coloring of the graph G in Figure 7. By choosing the minimal transversal $T_1 = \{d,e\}$, Construction 1.2 produces the 3-coloring $\{a,b,c,f,g\} \cup \{d\} \cup \{e\}$.

G =

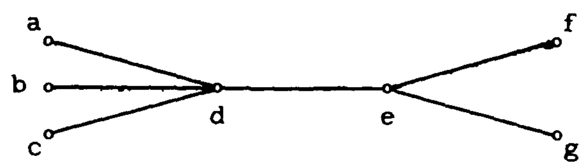


Figure 7. *Construction 1.2 may induce a nonminimal vertex coloring*

Given a fuzzy hypergraph $\mathcal{H} = (X, \mathcal{E})$, we are interested in when vertex colorings are also colorings for the cut-level hypergraphs of $\mathcal{H}$. We show by example that a coloring at one cut-level may not be consistent with a coloring at either a higher or lower value in $fs(\mathcal{H})$. Even if there exist consistent colorings, a minimal coloring at one level may not induce a minimal coloring at a higher or lower level.

EXAMPLE 1.4. Consider the fuzzy hypergraph $\mathcal{H}$ defined by the sequence of cut-level hypergraphs given in Figure 8. Since H_{r_1} and H_{r_2} have the same vertex set with $\chi(H_{r_1}) < \chi(H_{r_2})$, no minimal coloring of H_{r_1} can be extended to a coloring of H_{r_2}. Also, $\chi(H_{r_2}) > \chi(H_{r_3})$, so no minimal coloring of H_{r_3} can induce a coloring of H_{r_2}. Finally, although $\chi(H_{r_1}) = \chi(H_{r_3}) = 2$, *no* 2-coloring of H_{r_1} is compatible with a 2 coloring of H_{r_3}.

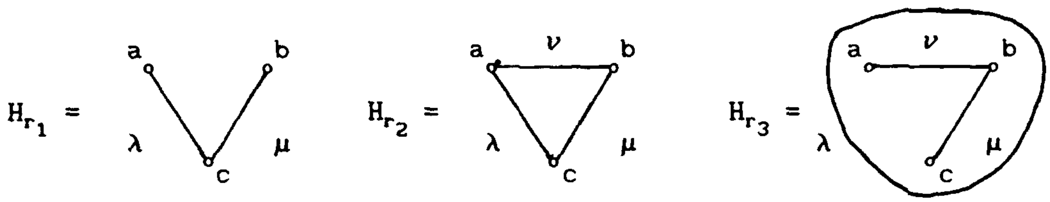


Figure 8. *A coloring of a cut-level hypergraph may not induce a coloring of another level*

OBSERVATION 1.5. Clearly a coloring Δ of $H_{r_{i+1}}$ is a coloring of H_{r_i} if and only if no μ_{r_i} is contained in a single color class of Δ. Similarly, a coloring Δ_i of H_{r_i} is extendible to a coloring Δ_{i+1} of $H_{r_{i+1}}$ if and only if no $\mu_{r_{i+1}}$ is contained in a single color class of Δ_i. In particular, if $\mathcal{H}$ is simply ordered, then for all $r_i \in fs(\mathcal{H})$ and for all colorings Δ_i of H_{r_i}, there exists an extension Δ_{i+1} of Δ_i to a coloring of H_{r+1}.

DEFINITION 1.6. Let $\mathcal{H} = (X, \mathcal{E})$ be a fuzzy hypergraph and let k be an integer ≥ 2. A k-*coloring* of the vertex set is a function $\Delta: X \longrightarrow \{1, 2, \cdots k\}$ such that for each $c \in fs(\mathcal{H})$ and for each nonloop edge $\mu_c \in E_c$, Δ is nonconstant on μ_c. The *chromatic number* of $\mathcal{H}$, denoted $\chi(\mathcal{H})$, is the minimal integer k for which there exists a k-coloring of $\mathcal{H}$.

As in the crisp case a k-coloring may be defined equivalently as a partition of X into k color classes where each nonloop cut-level edge μ_c has nonempty intersection with two or more color classes. Since the vertex set X_c may be properly contained in X we allow $X_c \cap S$ to be empty for a color class S.

Finding a coloring of a fuzzy hypergraph may be reduced to finding a coloring of a crisp hypergraph with the following construction.

DEFINITION 1.7. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph with fundamental sequence $fs(\mathcal{H})$. The (crisp) *color core of* $\mathcal{H}$ denoted $\mathbb{C}(\mathcal{H}) = (Y,E)$ is defined as follows. Let $\{\mu_c | \mu \in \mathcal{E} \text{ and } c \in fs(\mathcal{H})\}$ be the family of all sets that are an edge in some cut-level hypergraph of $\mathcal{H}$. The edge set E of $\mathbb{C}(\mathcal{H})$ is formed from this family by deleting all loops and all sets that properly contain another nonloop set of the family. Formally, we let E be the family of sets where $A \in E$ if and only if A satisfies the two conditions:

(i) There exist $\mu \in \mathcal{E}$ and $c \in (0,1]$ with $A = \mu_c$ and $|\mu_c| \geq 2$

(ii) $\nu \in \mathcal{E}$, $\nu_t \subseteq \mu_c$, and $\nu_t \neq \mu_c$ implyy $|\nu_t| = 1$.

The vertex set Y of $\mathbb{C}(\mathcal{H})$ is formed from the union of its edges. Clearly $Y \subseteq X$.

LEMMA 1.8. Δ is a k-coloring of $\mathcal{H}$ if and only if Δ is an (extended) k-coloring of $\mathbb{C}(\mathcal{H})$. Therefore, $\chi(\mathcal{H}) = \chi(\mathbb{C}(\mathcal{H})$.

Proof. Suppose that Δ is a k-coloring of $\mathcal{H}$ and A is an edge of $\mathbb{C}(\mathcal{H})$. By definition of $\mathbb{C}(\mathcal{H})$, $A = \mu_c$ is not a loop for some $\mu \in \mathcal{E}$ and $c \in fs(\mathcal{H})$. Since Δ is a k-coloring of $\mathcal{H}$, Δ is nonconstant on $\mu_c = A$ as required.

Suppose that Δ is a coloring of $\mathbb{C}(\mathcal{H})$ which has been extended to X by an arbitrary coloring the vertices of X\Y. Let μ_c be an arbitrary nonloop edge of an arbitrary cut-level of $\mathcal{H}$. By definition of $\mathbb{C}(\mathcal{H})$, there exists $A \in \mathbb{C}(\mathcal{H})$ such that $A \subseteq \mu_c$. Since Δ is nonconstant on A, Δ is nonconstant on μ_c as well. □

OBSERVATION 1.9. It follows immediately from the definitions that each fuzzy hypergraph satisfies the relation $\chi(\mathcal{H}) \geq \max\{\chi(H_c) \mid c \in fs(\mathcal{H})\}$. By the example defined in Figure 9 the inequality may be strict.

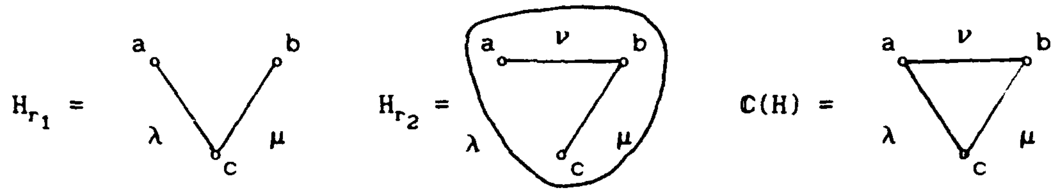


Figure 9. *The chromatic number of $\mathcal{H}$ may exceed that of any cut-level*

Section 2. Fuzzy Vertex Colorings

The notion of studying cut-levels of a hypergraph is vital because it allows one a measure of how significant an edge is in the hypergraph, and of how significant a vertex is in a particular edge. However edge strengths are not considered in vertex colorings. The color core of Definition 1.7 thus retains very little of the structure of the original fuzzy hypergraph. In applications using colorings, it may be useful to have a measure of how significant a color class is, or of the significance of a vertex in a color class.

For example, fundamental sequence values may represent how "likely" an event is. Given a particular coloring, it may be the case that a color class need not be considered unless some "improbable" event occurs. Therefore a nonminimal coloring with few nonempty color classes at "fairly certain" levels may be preferable to a second coloring that has fewer total color classes but many classes at "fairly certain" levels.

In other applications one may want an "almost even" distribution of vertices between color classes. Again, considering higher level sets "more important" than lower level sets can induce preferences between colorings. We reintroduce some of the original fuzzy hypergraph structure by defining a fuzzy coloring of the vertex set.

DEFINITION 2.1. Let $\mathcal{H} = (X, \mathcal{E})$ be a fuzzy hypergraph and $\mathcal{F} = \{\sigma_1, \cdots \sigma_k\}$ be a family of nontrivial fuzzy sets on X. Then $\mathcal{F}$ partitions X into k *fuzzy color classes* (or $\mathcal{F}$ is a *fuzzy k-coloring*) if $\mathcal{F}$ satisfies the conditions

(i) $\sigma_i \wedge \sigma_j = 0$ for $i \neq j$

(ii) for each $c \in (0,1]$, $X_c = \bigcup_{i=1}^{k} (\sigma_i)_c$

(iii) for each $c \in (0,1]$ and each nonloop $\mu_c \in \mathcal{E}_c$, μ_c has nonempty intersection with two or more color classes $(\sigma_i)_c$.

OBSERVATION 2.2. There exists a correspondence between the family of (crisp) vertex colorings of $\mathcal{H}$ and the family of fuzzy vertex colorings of $\mathcal{H}$. Given a partition of X into the color classes $\{A_1, \cdots A_k\}$, let α_i be the fuzzy set defined by $\alpha_i(x) = h_x \cdot \chi_{A_i}(x)$ where

$$h_x = \max\{c \in (0,1] \mid \text{there exists } \mu \in \mathcal{E} \text{ such that } x \in \mu_c\}.$$

Clearly the family $\{\alpha_1, \cdots, \alpha_k\}$ satisfies the conditions of Definition 2.1. Also this family is the only fuzzy coloring where supp $\alpha_i = A_i$ for each i.

For a fuzzy hypergraph $\mathcal{H}$ we now define a simple, elementary fuzzy hypergraph that has the same set of fuzzy vertex colorings as $\mathcal{H}$.

DEFINITION 2.3. Let $\mathcal{H} = (X, \mathcal{E})$ be a fuzzy hypergraph and let $\mathbb{C}(\mathcal{H}) = (Y, E)$ be the color core of $\mathcal{H}$ in Definition 1.7. For each $A \in E$, let $h \cdot A$ be the elementary fuzzy set with support A and height h, where h is the largest member of $fs(\mathcal{H})$ such that there exists $\mu \in \mathcal{E}$ with $\mu_h = A$. Define the *fuzzy color core of* $\mathcal{H}$ to be the elementary fuzzy hypergraph $\mathfrak{C}(\mathcal{H}) = (Y, \mathfrak{E})$ where $\mathfrak{E} = \{h \cdot A \mid A \in E\}$.

OBSERVATION 2.4. It is a trivial consequence of Observation 2.2 and Lemma 1.8 that a family of fuzzy sets $\mathcal{F}$ is a fuzzy k-coloring of $\mathcal{H}$ if and only if it is an extended fuzzy k coloring of $\mathfrak{C}(\mathcal{H})$. As in the crisp case, the vertices of $\mathcal{H}$ which are not vertices of $\mathfrak{C}(\mathcal{H})$ are essentially colored arbitrarily.

Section 3. Vertex Colorings of Ordered Fuzzy Hypergraphs

Chromatic number theory of crisp hypergraphs is well developed. By Lemma 1.8 the chromatic number of a fuzzy hypergraph is completely determined by its (crisp) color core. Therefore we largely refer the reader to the literature; Berge [1, Chapter 4] is an excellent reference.

There are some interesting results concerning relationships between chromatic numbers of cut-level hypergraphs. Clearly any coloring of the vertex set that produces a coloring for each cut-level hypergraph will provide a coloring for both the hypergraph $\mathcal{H}_{r_1}$ and the support hypergraph $\mathcal{H}_{r_n}$. In turn we explore conditions under which every coloring of $\mathcal{H}_{r_1}$ or every coloring of $\mathcal{H}_{r_n}$ will induce a coloring on $\mathcal{H}$.

OBSERVATION 3.1. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph with fundamental sequence $fs(\mathcal{H}) = \{r_1,r_2,\cdots,r_n\}$. Then a coloring Δ of $\mathcal{H}_{r_i}$ may be extended to a coloring of $\mathcal{H}_{r_{i+1}}$ if and only if no (nonloop) edge of $\mathcal{H}_{r_{i+1}}$ is contained in a *single color* class of Δ. In particular, if $\mathcal{H}$ is simply ordered, then every coloring of $\mathcal{H}_{r_1}$ may be extended to a coloring of $\mathcal{H}$.

LEMMA 3.2. Let $\mathcal{H} = (X,\mathcal{E})$ be a fuzzy hypergraph with fundamental sequence $fs(\mathcal{H}) = \{r_1,r_2,\cdots,r_n\}$, and let $\hat{H}_{r_n}$ be the color core of $\mathcal{H}_{r_n}$. Then every k-coloring of $\mathcal{H}_{r_n}$ is a k-coloring of $\mathcal{H}$ if and only if for each $c \in fs(\mathcal{H})$, and for each $\mu \in \mathcal{E}$ for which μ_c is not a loop, there exists $A \in \hat{H}_{r_n}$ such that $A \subseteq \mu_c$.

Proof. The implication is shown by contrapositive. Suppose that there exists $c \in fs(\mathcal{H})$ and $\mu \in \mathcal{E}$, such that $|\mu_c| \geq 2$ and for each $A \in \hat{H}_{r_n}$, $A \nsubseteq \mu_c$. Let the vertices of μ_c define a color class. Construct a new hypergraph K that is the sub-hypergraph of $\mathcal{H}$ formed by deleting μ_c from the vertex set of $\hat{H}_{r_n}$. Thus the edge set of K is $\{A\backslash\mu_c | A \in \hat{H}_{r_n}\}$. Since each nonloop $\nu_{r_n} \in \mathcal{H}_{r_n}$ (including μ_{r_n}) contains some $A \in \hat{H}_{r_n}$, each nonloop ν_{r_n} has nonempty intersection with the vertex set of K. Let $\{S_2,S_3,\cdots,S_k\}$ be a coloring of K. Then $\{\mu_c, S_2, S_3, \cdots, S_k\}$ is a coloring of $\mathcal{H}_{r_n}$, with μ_c monochrome. Therefore there exists a coloring of $\mathcal{H}_{r_n}$ that is not a coloring of $\mathcal{H}$.

Conversesely suppose that for each $c \in fs(\mathcal{H})$ and each $\mu \in \mathcal{E}$ where $|\mu_c| \geq 2$, there exists $A \in \hat{H}_{r_n}$ such that $A \subseteq \mu_c$. Let c and μ be arbitrary but fixed, and Δ a coloring of $\mathcal{H}_{r_n}$. Since Δ is nonconstant on A, Δ is nonconstant on μ_c, Therefore Δ is a coloring of $\mathcal{H}$. □

COROLLARY 3.3. Let $\mathcal{H}$ be ordered. Then every k-coloring of $\mathcal{H}_{r_n}$ is a k-coloring of $\mathcal{H}$ and $\chi(\mathcal{H}_{r_1}) \leq \cdots \leq \chi(\mathcal{H}_{r_1})$.

Proof. Any edge of $\mathcal{H}_c$ is by definition an edge of $\mathcal{H}_{r_n}$. □

COROLLARY 3.4. There is a partial converse to Corollary 3.3. If each k-coloring of $\mathcal{H}_{r_n}$ is a k-coloring of $\mathcal{H}$ and if $\mathcal{H}_{r_n}$ is simple, then $\mathcal{H}$ is ordered.

Proof. If $\mathcal{H}$ is not ordered there exists $\mu \in \mathcal{E}$ and $c \in fs(\mathcal{H})$ such that $\mu_c \notin E_{r_n}$. Since μ_c is properly contained in μ_{r_n} and $\mathcal{H}_{r_n}$ is simple, no edge of $\mathcal{H}_{r_n}$ is contained in μ_c. Apply Lemma 3.2. □

CHAPTER 4. Fuzzy Intersection Graphs

Intersection graphs and, in particular, interval graphs have been used extensively in mathematical modeling. Roberts [8] cites applications in archaeology, developmental psychology, mathematical sociology, organization theory, and ecological modeling. These disciplines all have components that are ambiguously defined, require subjective evaluation, or are satisfied to differing degrees; thus these areas can benefit from an application of fuzzy methods. In fact Klir [7] cites these disciplines as among the most active areas of application of fuzzy methods.

In this paper we define the fuzzy intersection graph of a family of fuzzy sets and explore the extent to which crisp characterizations can be generalized to fuzzy graphs. We provide a fuzzy analog of Marczewski's Theorem by showing that every fuzzy graph without loops is the intersection graph of some family of fuzzy sets. We also show that the natural generalization of the Fulkerson and Gross characterization of interval graphs fails. We then provide a natural generalization of the Gilmore and Hoffman characterization. We conclude by defining a variety of edge strength functions that are related to recent developments in intersection graph theory.

Although essentially all of the definitions and theorems of the paper can be extended to infinite sets, we will restrict our attention to graphs with finite vertex sets.

Section 1. Fuzzy Intersection Graphs

A *fuzzy set* α on a set X is a mapping $\alpha : X \longrightarrow [0,1]$. We let the *support of* α be $\text{supp}\ \alpha = \{x \in X | \alpha(x) \neq 0\}$ and say that α is *nontrivial* if $\text{supp}\ \alpha \neq \emptyset$. The *height of* α is $h(\alpha) = \max\{\alpha(x) | x \in X\}$; α is *normal* if $h(\alpha) = 1$. If μ and ν are fuzzy sets on X we use the max and min operators to define new fuzzy sets on X by $\mu \vee \nu = \max\{\mu,\nu\}$ and $\mu \wedge \nu = \min\{\mu,\nu\}$.

The fuzzy sets $\mu \vee \nu$ and $\mu \wedge \nu$ are the most common definitions of fuzzy union and fuzzy intersection, respectively. We write $\alpha \le \beta$ (fuzzy subset) if $\alpha(x) \le \beta(x)$ for each $x \in X$ and write $\alpha < \beta$ if $\alpha \le \beta$ and $\alpha(x) < \beta(x)$ for some $x \in X$. $\mathcal{F}(X)$ will denote the family of all fuzzy subsets of X.

As is common in fuzzy set theory, we identify a crisp set with its corresponding characteristic function; when the context is clear we use the two concepts interchangeably. The reader may verify that the fuzzy definitions, when restricted to characteristic functions of crisp sets, coincide with the usual crisp definitions. For example, crisp graphs and digraphs are special cases of fuzzy graphs and digraphs, respectively.

A (crisp) *graph* $G = (X,E)$ consists of a finite *set of vertices* X and a *set of edges* E. The set E is a symmetric subset of the Cartesian product XxX. We identify $(x,y) \in E$ with $(y,x) \in E$ and emphasize this symmetry with the notation $\{x,y\} \in E$. A *directed graph* $D = (X,A)$ is an ordered pair where A is an arbitrary subset of XxX. The set A is called the *arc set* of D in order to emphasize the fact that symmetry is not required.

A graph $H = (K,F)$ is *complete* if $\{x,y\} \in F$ for each $x,y \in K$. A *clique* of $G = (X,E)$ is a maximal (with respect to set inclusion) complete subgraph of G. Clearly a complete graph is determined by its vertex set, so we adopt the convention of naming a clique by its vertex set.

A *fuzzy graph* on a finite set X is a pair $\mathcal{G} = (X,\mu)$ where μ is a symmetric fuzzy subset of XxX; that is, $\mu: X\text{x}X \longrightarrow [0,1]$ and $\mu(x,y) = \mu(y,x)$ for all x and y in X. A *fuzzy graph on a fuzzy subset* σ of X is a pair $\mathcal{G} = (\sigma,\mu)$ where $\sigma: X \longrightarrow [0,1]$ and $\mu: X\text{x}X \longrightarrow [0,1]$ is a symmetric mapping such that $\mu(x,y) \le \min\{\sigma(x),\sigma(y)\}$. The fuzzy set σ is called the *fuzzy vertex set* of $\mathcal{G}$. We refer to the pair $(x,\sigma(x))$ as a *fuzzy vertex of* $\mathcal{G}$ or say the vertex x has vertex strength $\sigma(x)$. The fuzzy set μ is called the *fuzzy edge set* of $\mathcal{G}$. By a *fuzzy edge of* $\mathcal{G}$ we mean the two pairs $((x,y),\mu(x,y))$

and $((y,x),\mu(x,y))$; we also say the edge $\{x,y\}$ has edge strength $\mu(x,y)$. When $\mu(x,y) = 0$, we say $\{x,y\}$ is a *trivial edge* of $\mathcal{G}$.

A *fuzzy digraph* on a finite set X is a pair $\mathcal{D} = (X,\delta)$ where $\delta: XxX \longrightarrow [0,1]$ is an arbitrary mapping. A *fuzzy digraph on a fuzzy subset* σ of X is a pair $\mathcal{D} = (\sigma,\delta)$ where $\sigma: X \longrightarrow [0,1]$, $\delta: XxX \longrightarrow [0,1]$, and $\delta(x,y) \le \min \{\sigma(x),\sigma(y)\}$. We call the pairs $((x,y),\delta(x,y))$ *fuzzy arcs* to emphasize symmetry is not required. When convenient we refer to a fuzzy arc as the arc (x,y) with arc strength $\delta(x,y)$.

Often a fuzzy graph (or fuzzy digraph) is defined by an *adjacency matrix* where the rows and columns are indexed by the vertex set X and the x,y entry is $\mu(x,y)$ (or $\delta(x,y)$). A column may be added to indicate vertex strength with the x,σ entry being $\sigma(x)$.

A fuzzy graph $\mathcal{G}' = (\tau,\nu)$ is called a *partial fuzzy subgraph* of $\mathcal{G} = (\sigma,\mu)$ if for all $x,y \in X$, $\tau(x) \le \sigma(x)$ and $\nu(x,y) \le \mu(x,y)$. For any fuzzy subset τ of σ, the *fuzzy subgraph of* (σ,μ) *induced by* τ is the fuzzy graph (τ,ν) where $\nu(x,y) = \min\{\tau(x),\tau(y),\mu(x,y)\}$. The *complement* of the fuzzy graph $\mathcal{G} = (X,\mu)$ is the fuzzy graph $\mathcal{G}^c = (X,\mu^c)$ where $\mu^c(x,y) = 1 - \mu(x,y)$.

Given $c \in [0,1]$ and a fuzzy set α, we define the *c cut-level set of* α to be the crisp set $\alpha_c = \{x \in \text{supp } \alpha | \alpha(x) \ge c\}$. The c-level graph of $\mathcal{G}$ is defined as the (crisp) graph $\mathcal{G}_c = (\sigma_c,\mu_c)$. For a family $\mathcal{F}$ of fuzzy sets we let the c-level family of $\mathcal{F}$ be the family of crisp sets $\mathcal{F}_c = \{\alpha_c | \alpha \in \mathcal{F}\}$. Conversely a fuzzy set α is often defined by specifying a family of level sets and then defining $\alpha(x) = \sup \{c \in [0,1] | x \in \alpha_c\}$.

The *intersection graph* of a family (perhaps with repeated members) of crisp sets F is the graph G = (F,E) where for each A_i and A_j in F, $(A_i,A_j) \in E$ if and only if $A_i \cap A_j$ is nonempty. Generally loops are suppressed; that is, $i \ne j$ is required when forming the intersections. If the family F is the edge set of a hypergraph H, then the intersection graph of F is called the *line graph* of H.

Given a finite family of fuzzy sets $\mathcal{E}$, McAllister [2] defines two structures which together are called a fuzzy intersection graph. The first is essentially a fuzzy hypergraph with edge set consisting of all nonempty intersections of two distinct members of $\mathcal{E}$. The second is a fuzzy graph with a crisp vertex set (essentially $\mathcal{E}$) where the edge strength of a pair (α,β) given by a "measure of fuzzyness of $\alpha \wedge \beta$". Both structures are represented by incidence matrices with the α,β entry being the membership furction $\alpha \wedge \beta$ or the edge strength of $\{\alpha,\beta\}$, respectively. McAllister's main concern was to explore when linear algebra methods could be used to study eigenvalues, stability, or other properties of these matrices. We note that neither of McAllister's structures agree with the usual definition of intersection graphs when applied to families of crisp sets.

We take a different approach in defining the fuzzy intersection graph of a finite family of fuzzy sets. Our structure is a fuzzy graph where the fuzzy vertex and fuzzy edge sets are based on the max and min operators.

DEFINITION 1.1. Let $\mathcal{F} = \{\alpha_1, \cdots, \alpha_n\}$ be a finite family of fuzzy sets on a set X and consider $\mathcal{F}$ as a crisp vertex set. The *fuzzy intersection graph* of $\mathcal{F}$ is the fuzzy graph $\mathrm{Int}(\mathcal{F}) = (\sigma,\mu)$ where

$$\sigma:\mathcal{F}\longrightarrow[0,1] \text{ by } \sigma(\alpha_i) = h(\alpha_i) \text{ and}$$

$$\mu:\mathcal{F}\times\mathcal{F}\longrightarrow[0,1] \text{ is defined by } \mu(\alpha_i,\alpha_j) = \begin{cases} h(\alpha_i \wedge \alpha_j) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

An edge $\{\alpha_i,\alpha_j\}$ has zero strength if and only if $\alpha_i \wedge \alpha_j$ is the zero function (empty intersection) or $i = j$ (no loops).

OBSERVATION 1.2. If $\mathcal{F} = \{\alpha_1, \cdots, \alpha_n\}$ is a family of fuzzy sets and if $c \in [0,1]$, then $\mathrm{Int}(\mathcal{F}_c) = (\mathrm{Int}(\mathcal{F}))_c$. The graph $\mathrm{Int}(\mathcal{F}_c)$ has one vertex for each $\alpha_i \in \mathcal{F}$ such that $(\alpha_i)_c \neq \emptyset$ or equivalently such that $h(\alpha_i) \geq c$. The pair $\{(\alpha_i)_c, (\alpha_j)_c\}$ is an edge of $\mathrm{Int}(\mathcal{F}_c)$ if and only if (iff) $i \neq j$ and $(\alpha_i)_c \cap (\alpha_j)_c \neq \emptyset$; equivalently iff $h(\alpha_i \wedge \alpha_j) \geq c$. Similarly, the graph $(\mathrm{Int}(\mathcal{F}))_c$ has one vertex for each $\alpha_i \in \mathcal{F}$ such that $h(\alpha_i) \geq c$. The pair $\{\alpha_i, \alpha_j\}$ is an edge of $\mathrm{Int}(\mathcal{F}_c)$ iff $i \neq j$ and $\mu(\alpha_i, \alpha_j) = h(\alpha_i \wedge \alpha_j) \geq c$. As graphs, the two structures are equivalent.

In particular, if $\mathcal{F}$ is a family of characteristic functions of crisp subsets of X, the fuzzy intersection graph and crisp intersection graph definitions coincide.

THEOREM 1.3. Let $\mathcal{G} = (\sigma, \mu)$ be a fuzzy graph without loops. Then there exists a family of fuzzy sets $\mathcal{F}$ where $\mathcal{G} = \mathrm{Int}(\mathcal{F})$.

Proof. The proof is a generalization of Marczewski's [8] crisp result [8]. Let $G = (\sigma, \mu)$ be a fuzzy graph with fuzzy vertex set $\sigma: X \longrightarrow [0,1]$ and symmetric edge membership function $\mu: X \times X \longrightarrow [0,1]$. We must find a family of fuzzy sets $\mathcal{F} = \{\alpha_x | x \in X\}$ where

(i) for each $x \in X$, $h(\alpha_x) = \sigma(x)$

(ii) for each $x \neq y \in X$, $h(\alpha_x \wedge \alpha_y) = \mu(x,y)$.

For each $x \in X$ define the symmetric fuzzy set $\alpha_x: X \times X \longrightarrow [0,1]$ by

$$\alpha_x(y,z) = \begin{cases} \sigma(x) & \text{if } y = x \text{ and } z = x \\ \mu(x,z) & \text{if } y = x \text{ and } z \neq x \\ \mu(y,x) & \text{if } y \neq x \text{ and } z = x \\ 0 & \text{if } y \neq x \text{ and } z \neq x \end{cases}$$

We show that $\mathcal{F} = \{\alpha_x | x \in X\}$ is the desired family of fuzzy sets. Fix $x \in X$ and let $y \in X$ and $z \in X$ be arbitrary. By the definition of a fuzzy graph, $\sigma(x) \geq \mu(x,y) \geq 0$ for each $y \in X$. Therefore $\sigma(x) \geq \alpha_x(y,z)$. Computing $\alpha_x(x,x) = \sigma(x)$ we have $h(\alpha_x) = \sigma(x)$ as required.

Let $x \neq y$ be fixed elements of X, z and w be arbitrary elements of X and consider the value of $(\alpha_x \wedge \alpha_y)(z,w) = \alpha_x(z,w) \wedge \alpha_y(z,w)$. If $x \neq z$ and $x \neq w$, then $\alpha_x(z,w) = 0$. Similarly $y \neq z$ and $y \neq w$ implies $\alpha_y(z,w) = 0$. Therefore a nonzero value is possible only if $x = z$ and $y = w$ (or $y = z$ and

$x = w$). By definition

$$(\alpha_x \wedge \alpha_y)(x,y) = \alpha_x(x,y) \wedge \alpha_y(x,y) = \mu(x,y).$$

Thus $h(\alpha_x \wedge \alpha_y) = \mu(x,y)$ as required. □

Section 2. Fuzzy Interval Graphs

The families of sets most often considered in connection with intersection graphs are families of intervals of a linearly ordered set. This class of *interval graphs* is central to applications in many areas of graph theory. In this section we define a fuzzy interval and examine some of their basic properties.

In developing terminology and giving examples we sometimes refer to two characterizations of (crisp) interval graphs. Theorem 3.3 gives the Fulkerson and Gross characterization and Theorem 4.11 provides the Gilmore and Hoffman characterization.

DEFINITION 2.1. Let X be a linearly ordered set. A *fuzzy interval* $\mathcal{I}$ on X is a normal, convex fuzzy subset of X. That is, there exists an $x \in X$ such that $\mathcal{I}(x) = 1$ and for all $w \leq y \leq z$ in X, $\mathcal{I}(y) \geq \mathcal{I}(w) \wedge \mathcal{I}(z)$. We call X the *host interval* of $\mathcal{I}$ and call elements of X the *points* of X. A *fuzzy interval graph* is the fuzzy intersection graph of a finite family of fuzzy intervals. A *fuzzy number* is a real fuzzy interval.

OBSERVATION 2.2. In both the crisp and fuzzy cases, two distinct families of sets can have the same intersection graph. For example, the intersection properties of a finite family of real intervals can be defined in terms of the ordering of the finite set of interval endpoints. Conversely, any finite interval can be considered a real interval by linear extension.

Similarly, the intersection properties of a finite family of fuzzy numbers can be fully represented by a family of fuzzy sets with finite support (essentially the set of all endpoints of all fundamental sequence cut-level intervals). Conversely a fuzzy set with finite linearly ordered support can be used to define a fuzzy number.

Therefore we can with complete generality restrict our attention to host intervals with finitely many points. We take this approach in order to explore the relationship between the points of the host interval and the cliques of the corresponding interval graph.

OBSERVATION 2.3. If $\mathcal{G}$ is a fuzzy interval graph, then the vertex set of $\mathcal{G}$ is crisp. This follows since by definition $h(\alpha) = 1$ for each fuzzy interval in the family, and therefore $\sigma(\alpha) = 1$ for each vertex of $\mathcal{G}$. □

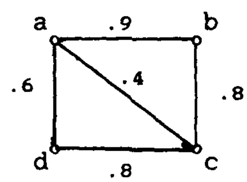
THEOREM 2.4. Let $\mathcal{G} = \text{Int}(\mathcal{F})$ be a fuzzy interval graph. Then for each $c \in (0,1]$, $\mathcal{G}_c$ is an interval graph.

Proof. Let $\mathcal{F} = \{\alpha_1, \cdots, \alpha_n\}$ be a family of fuzzy intervals and let $\mathcal{G} = \text{Int}(\mathcal{F})$. Since each $\alpha_i \in \mathcal{F}$ is convex, it follows that for each $c \in (0,1]$, $\alpha_{i_c} \in \mathcal{F}_c$ is a crisp interval. By Observation 1.2.,

$$\mathcal{G}_c = (\text{Int}(\mathcal{F}))_c = \text{Int}(\mathcal{F}_c).$$

Therefore $\mathcal{G}_c$ is an interval graph. □

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & .9 & .4 & .6 \\ .9 & 0 & .8 & 0 \\ .4 & .8 & 0 & .8 \\ .6 & 0 & .8 & 0 \end{bmatrix} \end{array}$$

$\mathcal{G} =$ 

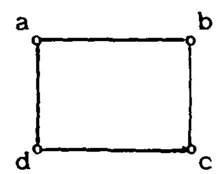
$G_{.6} =$ 

Figure 10. *A fuzzy graph that is not a fuzzy interval graph*

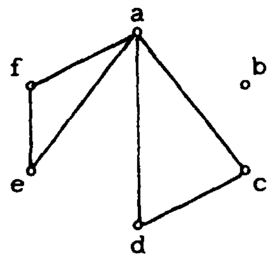
EXAMPLE 2.5. Let $\mathcal{G}$ be the fuzzy graph with vertex set {a,b,c,d} defined by the adjacency matrix A in Figure 10. Then $\mathcal{G}$ is not a fuzzy interval graph, since $\mathcal{G}_{.6}$ is a cycle of length 4 which by the crisp result (Theorem 4.11) given below is not an interval graph.

EXAMPLE 2.6. The converse of Theorem 2.4 is false. We provide a fuzzy graph $\mathcal{G}$ where for each $c \in (0,1]$, $\mathcal{G}_c$ is an interval graph, but $\mathcal{G}$ is not a fuzzy interval graph. Let $\mathcal{G}$ be defined by adjacency matrix A in Figure 11.

Figure 11 also shows the fundamental sequence of cut-level graphs of $\mathcal{G}$ and an interval representation for each cut-level graph.

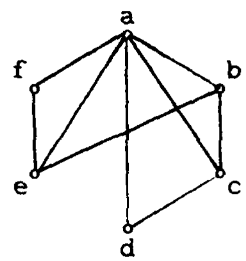
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & r_2 & r_1 & r_1 & r_1 & r_1 \\ r_2 & 0 & r_2 & r_3 & r_2 & r_3 \\ r_1 & r_2 & 0 & r_1 & 0 & 0 \\ r_1 & r_3 & r_1 & 0 & 0 & 0 \\ r_1 & r_2 & 0 & 0 & 0 & r_1 \\ r_1 & r_3 & 0 & 0 & r_1 & 0 \end{bmatrix} \end{array}$$

$\mathcal{G}_{r_1} =$



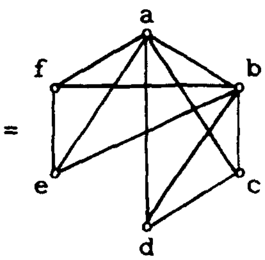
$a \longmapsto [1,5]$
$b \longmapsto [0]$
$c \longmapsto [1,2]$
$d \longmapsto [1]$
$e \longmapsto [4,5]$
$f \longmapsto [5]$

$\mathcal{G}_{r_2} =$



$a \longmapsto [1,5]$
$b \longmapsto [2,4]$
$c \longmapsto [1,2]$
$d \longmapsto [1]$
$e \longmapsto [4,5]$
$f \longmapsto [5]$

$\mathcal{G}_{r_3} =$



$a \longmapsto [1,5]$
$b \longmapsto [1,5]$
$c \longmapsto [1,2]$
$d \longmapsto [1]$
$e \longmapsto [4,5]$
$f \longmapsto [5]$

Figure 11. *A fuzzy graph that is not a fuzzy interval graph but each cut level graph is an interval graph*

We show $\mathcal{G}$ is not a fuzzy intersection graph by contradiction. Suppose that $\mathcal{F}$ is a family of fuzzy intervals defined on an interval I and assume $\mathcal{G} = \mathrm{Int}(\mathcal{F})$. Let the vertex x of $\mathcal{G}$ correspond to the fuzzy interval x of $\mathcal{F}$. Since $h(c \wedge e) = 0$, we can assume WOLOG that supp(c) lies strictly to the left of supp(e). Note that {a,c,d} defines a clique of $\mathcal{G}_{r_1}$. By a well-known interval graph theorem, there exists an x_1 such that $x_1 \in a_{r_1} \cap c_{r_1} \cap d_{r_1}$. Therefore $a(x_1) \wedge c(x_1) \wedge d(x_1) \geq r_1$. Similarly, the set {a,e,f} forms a clique of $\mathcal{G}_{r_1}$ and so there exists an x_5 such that $a(x_5) \wedge e(x_5) \wedge f(x_5) \geq r_1$. Note that $h(b \wedge d) = r_3$ with $d(x_1) \geq r_1$, implies $b(x_1) \leq r_3$ and that $h(b \wedge f) = r_3$ with $f(x_5) \geq r_1$ implies $b(x_5) \leq r_3$.

Similarly, $h(b \wedge c) = r_2$ and $h(b \wedge e) = r_2$ implies there exist x_2 and x_4 with $b(x_2) \geq r_2$ and $b(x_4) \geq r_2$. By the normality of b there exists x_3 such that $b(x_3) = 1$. By the convexity of the fuzzy intervals and the assumption that supp(c) and supp(e) are disjoint, the ordering of these points must be $x_1 < x_2 \le x_3 \le < x_4 < x_5$, with $x_2 < x_4$.

Since a is also a normal, convex fuzzy set and $x_1 \le x_3 \le x_5$, it follows that $a(x_3) \ge r_1$. However $b(x_3) = 1$ implies $h(a \wedge b) \ge r_1$, contradicting $h(a \wedge b) = r_2$. □

Section 3. The Fulkerson and Gross Characterization

The Fulkerson and Gross characterization exploits a correspondence between the set of points in a host interval and the set of cliques in an interval graph. We show that for fuzzy graphs this relationship holds only in one direction.

Recall a *clique* of G = (X,E) is a maximal (with respect to set inclusion) complete subgraph of G and we adopt the convention of naming a clique by its vertex set. Clearly if K defines a clique of G and $z \notin K$ is a vertex of G, then there exists an $x \in K$ such that $\{x,z\} \notin E$.

DEFINITION 3.1. Let $\mathcal{G} = (\sigma,\mu)$ be a fuzzy graph. We say that a fuzzy set $\mathcal{K}$ defines a *fuzzy clique* of $\mathcal{G}$ if for each $c \in (0,1]$, $\mathcal{K}_c$ induces a clique of $\mathcal{G}_c$. We associate with $\mathcal{G}$ a *vertex clique incidence matrix* where the rows are indexed by the domain of σ, the columns are indexed by the family of all fuzzy cliques of $\mathcal{G}$, and the $x,\mathcal{K}$ entry is $\mathcal{K}(x)$.

OBSERVATION 3.2. Suppose that $\mathcal{G}$ is a fuzzy graph with $fs(\mathcal{G}) = \{r_1,\cdots,r_n\}$ and let $\mathcal{K}$ be a fuzzy clique of $\mathcal{G}$. The cut-level sets of $\mathcal{K}$ define a sequence $\mathcal{K}_{r_1} \subseteq \cdots \subseteq \mathcal{K}_{r_m}$ where each $\mathcal{K}_{r_i}$ is a *clique of* $\mathcal{G}_{r_i}$. *Conversely,* any sequence $K_1 \subseteq \cdots \subseteq K_m$ where each K_i is a clique of $\mathcal{G}_{r_i}$ defines a fuzzy clique. Therefore K is a clique of the c-level graph G_c if and only if $K = \mathcal{K}_c$ for some fuzzy clique $\mathcal{K}$.

THEOREM 3.3 (Fulkerson and Gross [12]). A (crisp) graph G is an interval graph if and only if there exists a linear ordering of the cliques of G such that the vertex clique incidence matrix has convex rows.

The condition that the matrix have convex rows is also referred to in the literature as the *consecutive ones property* or by the equivalent condition that the clique ordering be *consecutive*.

Sketch of proof. An interval graph theorem states that any set of intervals that defines a complete subgraph will have a point in common. If one such point is associated with each clique, the linear ordering of these points induces a linear ordering on the set of cliques of G. The resulting vertex clique incidence matrix has convex rows.

Conversely each convex row of such a matrix naturally defines the characteristic function of a subinterval of the linearly ordered set of cliques. The graph G is clearly the intersection graph of the family of these intervals.

THEOREM 3.4 (Fuzzy Analog of Fulkerson and Gross). Let $\mathcal{G} = (X,\mu)$ be a fuzzy graph. Then the rows of the vertex clique incidence matrix of $\mathcal{G}$ define a family of fuzzy sets $\mathcal{F}$ for which $\mathcal{G} = \mathrm{Int}(\mathcal{F})$. Further, if there exists an ordering of the fuzzy cliques of $\mathcal{G}$ such that each row of the vertex clique incidence matrix is convex, then $\mathcal{G}$ is a fuzzy interval graph.

Proof. Let $I = \{\mathcal{K}_1,\cdots,\mathcal{K}_p\}$ be the ordered family of fuzzy cliques of $\mathcal{G}$ and let M be the vertex clique incidence matrix where the columns are given this ordering. For each $x \in X$ define the fuzzy set $\mathcal{I}_x : I \longrightarrow [0,1]$ by $\mathcal{I}_x(\mathcal{K}_i) = \mathcal{K}_i(x)$ and let $\mathcal{F} = \{\mathcal{I}_x | x \in X\}$. We must show that each $\mathcal{I}_x$ is normal and that if $x,y \in X$ and $x \neq y$, then $h(\mathcal{I}_x \wedge \mathcal{I}_y) = \mu(x,y)$. Clearly if we also assume each row is convex, then each $\mathcal{I}_x$ is a fuzzy interval and $\mathcal{G}$ is a fuzzy interval graph.

Let $x \in X$. Since x has vertex strength 1, x is a vertex of some clique K of the 1-level graph of $\mathcal{G}$. By Observation 3.2, K is the 1-level cut of some fuzzy clique $\mathcal{K}_i$ in I. Therefore $\mathcal{I}_x(\mathcal{K}_i) = \mathcal{K}_i(x) = 1$ and $\mathcal{I}_x$ is normal.

Suppose $x, y \in X$ and $x \neq y$. By definition

$$h(\mathcal{I}_x \wedge \mathcal{I}_y) = \max\{(\mathcal{I}_x \wedge \mathcal{I}_y)(\mathcal{K}_i) \mid \mathcal{K}_i \in I\} =$$

$$\max\{\mathcal{I}_x(\mathcal{K}_i) \wedge \mathcal{I}_y(\mathcal{K}_i) \mid \mathcal{K}_i \in I\} = \max\{\mathcal{K}_i(x) \wedge \mathcal{K}_i(y) \mid \mathcal{K}_i \in I\} =$$

$$\max\{c \in [0,1] \mid \text{there exists } \mathcal{K}_i \in I \text{ with } \{x,y\} \subseteq (\mathcal{K}_i)_c\}.$$

For each $c > \mu(x,y)$, $\{x,y\}$ is not an edge of $\mathcal{G}_c$ and no $(\mathcal{K}_i)_c$ contains $\{x,y\}$. Therefore $h(\mathcal{I}_x \wedge \mathcal{I}_y) \leq \mu(x,y)$. If $c = \mu(x,y)$, then $\{x,y\}$ is an edge of $\mathcal{G}_c$ and some clique $(\mathcal{K}_i)_c$ contains $\{x,y\}$. Thus $h(\mathcal{I}_x \wedge \mathcal{I}_y) \geq \mu(x,y)$. It follows that $h(\mathcal{I}_x \wedge \mathcal{I}_y) = \mu(x,y)$ as required. □

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{cccccc} 0 & .8 & .5 & .5 & .8 & 0 \\ .8 & 0 & 1 & .5 & 0 & 0 \\ .5 & 1 & 0 & .8 & .2 & 0 \\ .5 & .5 & .8 & 0 & .2 & .2 \\ .5 & 0 & .2 & .2 & 0 & 1 \\ 0 & 0 & 0 & .2 & 1 & 0 \end{array}\right] \end{array} \qquad M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & \mathcal{K}_5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & .5 & .5 \\ 0 & 0 & .8 & 1 & .5 \\ 0 & .2 & .5 & 1 & .8 \\ .2 & .2 & .5 & .5 & 1 \\ 1 & .8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\mathcal{G}$ =

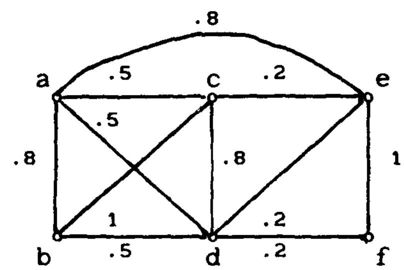
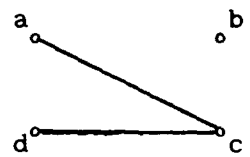


Figure 12. *A vertex clique incidence matrix with convex rows induces a fuzzy interval representation*

EXAMPLE 3.5. Figure 12 shows a fuzzy graph $\mathcal{G}$, a linear ordering of the fuzzy cliques of $\mathcal{G}$ and the corresponding interval representation derived by this method. We emphasize that the $x, \mathcal{K}$ entry represents both $\mathcal{K}(x)$ (the strength of vertex x in clique $\mathcal{K}$) and $\mathcal{I}_x(\mathcal{K})$ (the strength of point $\mathcal{K}$ in the fuzzy interval associated with vertex x). Given distinct vertices x and y, the edge strength of $\{x,y\}$ exceeds a value c if and only if there exists a fuzzy clique $\mathcal{K}_i$ with $\min\{\mathcal{K}_i(x), \mathcal{K}_i(y)\} > c$.

EXAMPLE 3.6. The converse of Theorem 3.4 is false. Let $\mathcal{G}$ be the fuzzy graph defined by the adjacency matrix A in Figure 13. To see that $\mathcal{G}$ is a fuzzy interval graph, let $\mathcal{F}$ be the family of fuzzy subsets of $\{1,2,3,4\}$ defined by the rows of the matrix F of Figure 13. Routine verification shows that $\mathcal{G} = \mathrm{Int}(\mathcal{F})$. Figure 13 also shows the cut-level graphs of $\mathcal{G}$ and a vertex clique incidence matrix M for $\mathcal{G}$.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & .5 & 1 & 0 \\ .5 & 0 & .5 & .5 \\ 1 & .5 & 0 & 1 \\ 0 & .5 & 1 & 0 \end{bmatrix} \end{array} \qquad M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 \end{array} \\ \begin{bmatrix} 1 & .5 & 0 & 0 \\ .5 & 1 & .5 & 1 \\ 1 & .5 & 1 & .5 \\ 0 & 0 & 1 & .5 \end{bmatrix} \end{array} \qquad F = \begin{array}{c} \\ I_a \\ I_b \\ I_c \\ I_d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & .5 & 0 & 0 \\ .5 & .5 & .5 & 1 \\ 1 & 1 & 1 & .5 \\ 0 & 0 & 1 & .5 \end{bmatrix} \end{array}$$

$\mathcal{G}_1 =$ 

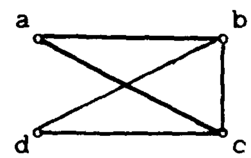
$\mathcal{G}_{.5} =$ 

Figure 13. *A fuzzy interval graph whose vertex clique incidence matrix always has a row that is not convex*

To show that no ordering of the fuzzy cliques will induce convex rows, note that the $\mathcal{K}_1$ and $\mathcal{K}_2$ columns must be adjacent by convexity of row a. However ordering $\mathcal{K}_3$ left of the ordered pair $\mathcal{K}_1,\mathcal{K}_2$ violates the convexity of row d and ordering $\mathcal{K}_3$ right of $\mathcal{K}_1,\mathcal{K}_2$ violates convexity of row b. Similarly no ordering of $\mathcal{K}_3$ is consistent with the ordering $\mathcal{K}_2,\mathcal{K}_1$. Therefore no ordering of the fuzzy cliques produces a vertex clique incidence matrix with convex rows. □

Section 4. The Gilmore and Hoffman Characterization.

We begin by generalizing several graph theory properties that are used in the Gilmore and Hoffman characterization of interval graphs. We then show that this characterization generalizes in an obvious way for fuzzy interval graphs.

A *path* (*simple path*) P *of length* n in a fuzzy graph is a sequence of vertices (distinct vertices) $x_0,\cdots,x_n$ where for each $0 \le i \le n$, $\mu(x_{i-1},x_i) > 0$. The *strength* of P is $\min_{i=1}^{n} \mu(x_{i-1},x_i)$; the minimal weight of the edges contained in P. For paths of length 0 we define the path strength to be the vertex strength $\sigma(x)$. A *cycle of length* n, denoted Z_n, is a path with $n \geq 3$, $x_0 = x_n$, and all other vertices distinct.

A (crisp) graph G is *chordal* if every cycle of length greater than three has a chord. Formally, if $Z_n = x_0, \cdots, x_n$ is a cycle in G there exist integers j and k such that $0 \leq j < k - 1 < n - 1$, either $j \neq 0$ or $k \neq n - 1$, and $\{x_j, x_k\} \in E(G)$. Some authors refer to chordal graphs as *triangulated* graphs.

DEFINITION 4.1. A fuzzy graph $\mathcal{G} = (\sigma, \mu)$ is *chordal* if for each cycle $P = x_0, \cdots, x_n$ with $n \geq 4$, there exist integers j and k such that $0 \leq j < k - 1 < n - 1$, either $j \neq 0$ or $k \neq n - 1$, and $\mu(x_j, x_k) \geq \wedge_{i=1}^{n} \mu(x_{i-1}, x_i)$.

THEOREM 4.2. A fuzzy graph $\mathcal{G} = (\sigma, \mu)$ is chordal if and only if for each $c \in (0,1]$, the c-level graph of $\mathcal{G}$ is chordal.

Proof. $\Rightarrow$ Suppose that $\mathcal{G}$ is chordal, $c \in (0,1]$ and $P = x_0, \cdots, x_n$ defines a cycle in $\mathcal{G}_c$. Then P defines a cycle in $\mathcal{G}$ and there exist integers j and k such that $0 \leq j < k - 1 \leq n - 1$, that $j \neq 0$ or $k \neq n - 1$, and that $\mu(x_j, x_k) \geq \wedge_{i=1}^{n} \mu(x_{i-1}, x_i) \geq c$. Thus $\{x_j, x_k\} \in \mathcal{G}_c$ as required.

$\Leftarrow$ Suppose that for each $c \in (0,1]$, $\mathcal{G}_c$ is chordal and $P = x_0, \cdots, x_n$ defines a cycle in $\mathcal{G}$. Set $c = \wedge_{i=1}^{n} \mu(x_{i-1}, x_i)$. Then P defines a cycle in $\mathcal{G}_c$ and there exist $0 \leq j < k - 1 \leq n - 1$ such that $j \neq 0$ or $k \neq n - 1$ and $\{x_j, x_k\} \in E(\mathcal{G}_c)$. Thus $\mu(x_j, x_k) \geq c$ as required. □

COROLLARY 4.3. If $\mathcal{G}$ is a fuzzy interval graph, then $\mathcal{G}$ is chordal.

Proof. By Theorem 2.4, for each $c \in (0,1]$, $\mathcal{G}_c$ is an interval graph. By the crisp results we include as Theorem 4.11 and Observation 4.13 each interval graph is chordal. The result then follows from Theorem 4.2. □

Recall that the edge set E of a graph $G = (X,E)$ is a symmetric relation on X. An *orientation* A of G is a maximal anti-symmetric subset of

E; that is, A is a relation on X such that for all $x \neq y$ with $\{x,y\} \in E$ either $(x,y) \in A$ or $(y,x) \in A$ but not both. Clearly (X,A) defines a directed graph without cycles of length two that has G as its underlying graph. A complete oriented graph is called a *tournament*.

A graph G is *transitively orientable* if there exists an orientation A of G which is transitive; that is, $(u,v) \in A$ and $(v,w) \in A$ implies $(u,w) \in A$. A transitively orientable graph is often called a *comparability graph*.

Since the cut-level graphs of a fuzzy graph are of primary importance, we define an orientation of the vertex set that applies to both the fuzzy graph and the corresponding fundamental sequence of cut-level graphs.

DEFINITION 4.4. Let A be a tournament on a set X; that is, A is a complete anti-reflexive, anti-symmetric relation on X. For each fuzzy graph $\mathcal{G} = (\sigma,\mu)$ where supp $\sigma = X$, define the *orientation of* $\mathcal{G}$ *by* A to be the fuzzy digraph $\mathcal{G}_A = (\sigma,\mu_A)$ where $\mu_A(x,y) = \begin{cases} \mu(x,y) & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \notin A \end{cases}$.

OBSERVATION 4.5. For each $c \in (0,1]$, $(\mathcal{G}_A)_c = (\mathcal{G}_c)_A$. The c-level graph of $\mathcal{G}_A$ has arc set $\{(x,y) | \mu_A(x,y) \geq c\}$. This is also the arc set of the orientation of $\mathcal{G}_c$ by A. A tournament on X therefore induces an orientation on both a fuzzy graph and each member of its fundamental sequence of cut-level graphs.

DEFINITION 4.6. A fuzzy graph $\mathcal{G} = (\sigma,\mu)$ is *transitively orientable* if there exists a tournament A on X = supp σ for which $\mathcal{G}_A$ is transitive. Formally, if $x,z \in X$ and $x \neq z$, then for each $y \in X$ it follows that $\mu_A(x,y) \wedge \mu_A(y,z) \leq \mu_A(x,z)$.

LEMMA 4.7. A fuzzy graph $\mathcal{G} = (\sigma,\mu)$ is transitively orientable if and only if there exists a tournament A on X such that for each $c \in (0,1]$, A induces a transitive orientation of $\mathcal{G}_c$.

Proof. $\Rightarrow$ Let $\mathcal{G}$ be transitively orientated by A and let $c \in (0,1]$. By Definition 4.6, A induces an orientation of $\mathcal{G}_c$. Assume (x,y) and (y,z) are arcs of $(\mathcal{G}_c)_A$. Then $\mu_A(x,y) \geq c$ and $\mu_A(y,z) \geq c$, which by the transitivity of μ_A implies $\mu_A(x,z) \geq c$. Therefore (x,z) is an arc of $(\mathcal{G}_c)_A$ as required.

$\Leftarrow$ Let A be an orientation of X that induces a transitive orientation of $\mathcal{G}_c$ for each $c \in (0,1]$. Suppose that $x,y,z \in X$, that $x \neq z$, and let $c = \mu_A(x,y) \wedge \mu_A(y,z)$. If $c = 0$ the result is trivial, so WOLOG assume $c > 0$. Then (x,y) and (y,z) are arcs of $(\mathcal{G}_c)_A$ and transitivity implies (x,z) is an arc of $(\mathcal{G}_c)_A$. Therefore $\mu_A(x,z) \geq c$ as required. □

EXAMPLE 4.8. Consider the sequence of transitively orientable subgraphs $G_1 \subseteq G_2 \subseteq G_3$ given in Figure 14. The transitive orientation of G_2 shown does not induce a transitive orientation of G_1, and the transitive orientation of G_2 cannot be extended to a transitive orientation of G_3.

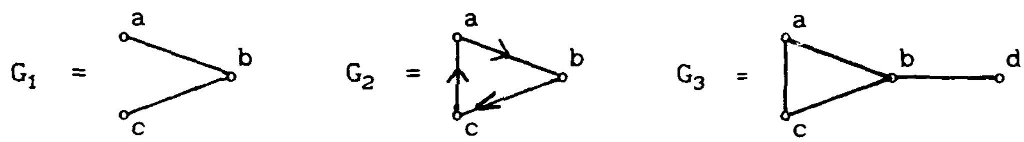


Figure 14. Cut-level *orientations for Example 4.8*

LEMMA 4.9. Suppose that $\mathcal{G} = \mathrm{Int}(\mathcal{F})$ is a fuzzy interval graph. Then there exists an orientation A of $\mathcal{F}$ that induces a transitive orientation of $\mathcal{G}^C$.

Proof. Assume $\{\alpha,\beta\}$ is a nontrivial edge of $\mathcal{G}^C$. Then $h(\alpha \wedge \beta) < r_1 = 1$ and α_{r_1} and β_{r_1} are disjoint. For such pairs let $(\alpha,\beta) \in A$ if and only if α_{r_1} lies strictly to the left of β_{r_1}. Complete the definition of A by an arbitrary assignment of orientation to those $\alpha,\beta \in \mathcal{F}$ for which $h(\alpha \wedge \beta) = \mu(\alpha,\beta) = 1$. Clearly A is well defined, anti-reflexive, anti-symmetric and induces a transitive orientation of $\mathcal{G}^C$.

EXAMPLE 4.10. The fuzzy graph of Example 2.6 and Figure 11 is not an interval graph because there is no transitive orientation of $\mathcal{G}^C$.

Figure 15 shows the cut-level graphs of $\mathcal{G}^C$. Suppose that $(d,e) \in A$ and consider the r_3 level graph of $\mathcal{G}$. Transitivity requires $(d,f) \in A$,

$(c,f) \in A$, and $(c,e) \in A$. Transitivity at the r_2 level then requires $(d,b) \in A$ and $(b,f) \in A$. However, both $(a,b) \in A$ and $(b,a) \in A$ contradict transitivity at the r_1 level. Supposing that $(e,d) \in A$ produces a similar contradiction. Therefore no transitive orientation of $\mathcal{G}^c$ exists.

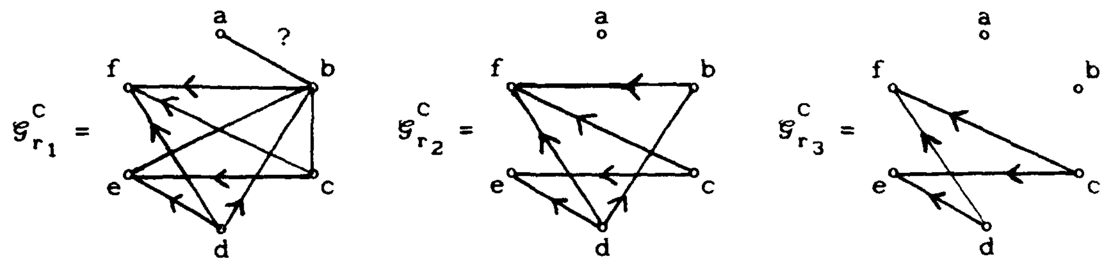


Figure 15. *A fuzzy graph that is not transitively orientable*

We next give a crisp result which states the compliment of an interval graph is transitively orientable. Example 4.10 therefore contains a fuzzy graph $\mathcal{G}^c$ where each cut-level graph of $\mathcal{G}^c$ is transitively orientable while $\mathcal{G}^c$ itself is not.

THEOREM 4.11. (Gilmore and Hoffman [11]) A graph G is an interval graph if and only if it satisfies the two conditions

(i) Each subgraph of G induced by four vertices is chordal

(ii) G^c is transitively orientable

Sketch of proof. Further detail and examples can be found in Roberts [8]. Let G be an interval graph. Condition (i) is a trivial consequence of the definitions of a chordless four cycle and of an interval graph. For condition (ii) orient $\{a,b\} \in G^c$ by $(a,b) \in A$ if and only if the interval corresponding to a lies strictly to the left of the interval corresponding to b.

Methods used in proving the converse of Theorem 4.11 are used extensively in proving the fuzzy analog of this theorem. In the interest of completeness we provide a detailed proof in Construction 4.12.

CONSTRUCTION 4.12. Suppose that $G = (X,E)$ is a chordal graph and A is a transitive orientation of G^c. We define the relation $<$ on the set of all cliques of G as follows. Let $K \neq L$ be cliques of G. Then there exists an $x \in K$ with $x \notin L$; in turn there exists a $y \in L$ such that $\{x,y\} \notin E$ (otherwise $\{x\} \cup L$ induces a complete subgraph of G properly containing L). We define the relation $<$ on the cliques of G by setting $K < L$ if and only if $(x,y) \in A$. To conclude $<$ induces a well ordering of the cliques of G we need only verify that $<$ is well defined and transitive.

We show $<$ is well defined by contradiction. Suppose that there exist $x \in K$ and $y \in L$ with $\{x,y\} \in E^c$ and $(x,y) \in A$, and that there also exist $x' \in K$ and $y' \in L$ with $\{x',y'\} \in E^c$ and $(y',x') \in A$. As K and L are cliques of G, $\{x,x'\}$ and $\{y,y'\}$ are edges of G. Since $\{x,y',y,x'\}$ (Figure 16) is a 4 cycle, by condition (i) we assume WOLOG that $\{x,y'\} \in E^c$. Now consider the orientation of $\{x,y'\}$ by A. If $(x,y') \in A$, transitivity of A implies $(x,x') \in E^c$, a contradiction. Similarly $(y',x) \in A$ implies $\{y',y\} \in E^c$, a contradiction. Therefore $<$ is well defined.

Figure 16. *The clique ordering of Construction 4.12 is well defined*

Transitivity of $<$ is inherited from the transitive orientation A. Let $K < L$ and $L < M$. Then there exist $x \in K$ and $y \in L$ such that $\{x,y\} \notin G$ and $(x,y) \in A$. If $y \in M$ we have $K < M$ as required. If $y \notin M$ there exists $z \in M$ such that $\{y,z\} \notin E$. Then $L < M$ implies $(y,z) \in A$. Transitivity of A gives $(x,z) \in A$; therefore $K < M$ as required.

A well-known theorem of graph theory states that any complete transitive relation on a set defines a linear ordering of the set. Therefore $<$ linearly orders the cliques of G.

A routine verification shows that the vertex clique incidence matrix with the columns ordered by $<$ has convex rows. Let K and L be cliques and x be a vertex with $K < L$, $x \in K$, and $x \notin L$. Then there exists a vertex $y \in L$ such that $\{x,y\} \notin E$ and $(x,y) \in A$. Since $<$ is well defined and $(y,x) \notin A$, it follows that for each clique M such that $L < M$ we have $x \notin M$. As in Theorem 3.3 the rows of the matrix define a family of intervals which have G as its intersection graph. □

OBSERVATION 4.13. In the language of the original paper the term subgraph always refers to an induced subgraph, and a cycle is defined as a closed path with no chords. Condition (i) states that G does not contain a cycle of length 4. In the interest of clarity we make condition (i) slightly more explicit.

A simple induction argument shows that no interval graph has an induced cycle (without chords) of length $n \geq 4$. Certainly chordal graphs satisfy the weaker condition that each 4-cycle has a chord. Therefore condition i) can be replaced with the condition that G is chordal.

THEOREM 4.14. (Fuzzy analog of the Gilmore and Hoffman characterization). A fuzzy graph $\mathcal{G} = (\sigma,\mu)$ is a fuzzy interval graph if and only if the following conditions hold

(i) for all $x \in \text{supp } \sigma$, $\sigma(x) = 1$ (σ is a crisp set)

(ii) each subgraph of $\mathcal{G}$ induced by four vertices is chordal

(iii) $\mathcal{G}^c$ is transitively orientable

Proof. $\Rightarrow$ If $\mathcal{G}$ is a fuzzy interval graph, the three conditions follow from Observation 2.3, Corollary 4.3, and Lemma 4.9, respectively.

$\Leftarrow$ Because of its length, we divide the proof into a number of lemmas and constructions. Several definitions and examples are also given in the body of the proof. The discussion prior to Construction 4.15 outlines the proof; Construction 4.15 through Lemma 4.23 provide the details.

Suppose that $\mathcal{G} = (X,\mu)$ is a chordal fuzzy graph with crisp vertex set X and that A is a transitive orientation of $\mathcal{G}^C$. For notational convenience we let $\mathcal{K}_{ij}$ denote the r_j cut-level set of the fuzzy set $\mathcal{K}_i$.

We first define a linear ordering < of the fuzzy cliques of $\mathcal{G}$. We then order the columns of the vertex clique incidence matrix of $\mathcal{G}$ by <. By Theorem 3.4 the rows of the vertex clique incidence matrix of $\mathcal{G}$ define a family of fuzzy sets that has $\mathcal{G}$ as its fuzzy intersection graph. If the ordering < has a property called *cut-level consistent*, the rows will be convex and the result follows immediately from Theorem 3.4.

If < is not cut-level consistent, then the rows are not convex. In this case we must modify the vertex clique incidence matrix so that the new rows are convex yet still generate $\mathcal{G}$ as its fuzzy intersection graph. This is done in a "bottom up" construction using the notion of cut-level consistent to determine which columns are retained unchanged and which columns are modified or deleted from the vertex clique incidence matrix. Lemmas 4.22 and 4.23 complete the proof by showing that in the modified matrix each row is normal and convex and that $\mathcal{G}$ is the fuzzy intersection graph of the family of fuzzy intervals defined by the rows.

We begin by defining a linear ordering of the fuzzy cliques of $\mathcal{G}$.

CONSTRUCTION 4.15. Let $\mathcal{G}$ be a fuzzy graph with a crisp vertex set.

2

Suppose that $\mathcal{G}$ has no chordless four cycles and that A is a transitive orientation of $\mathcal{G}^C$. By Theorem 4.2 and Lemma 4.7, for each $c \in (0,1]$ the cut-level graph $\mathcal{G}_c$ satisfies the conditions of Construction 4.12. Therefore each $\mathcal{G}_c$ is an interval graph and Construction 4.12 defines a linear ordering $<_c$ on the family of all cliques of $\mathcal{G}_c$.

Define the relation < on the family of all fuzzy cliques of $\mathcal{G}$ as follows. For each $\mathcal{K} \neq \mathcal{L}$, let $\mathcal{K} < \mathcal{L}$ if and only if $K_c <_c L_c$ where $c \in fs(\mathcal{G}$ is the smallest real number such that $K_c \neq L_c$. Essentially, < is a lexicographic ordering and is clearly well defined, complete and transitive. Therefore $<$ defines a linear ordering on the family of all fuzzy cliques of $\mathcal{G}$.

We now define a relation which is used extensively in the discussion below.

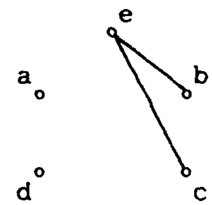
DEFINITION 4.16. Let $\mathcal{G}$ be a fuzzy graph satisfying the conditions of *Theorem* 4.14 and let $<$ be the relation defined in Construction 4.15. Suppose that $c \in fs(\mathcal{G})$ and that $\mathcal{K}$ and $\mathcal{L}$ be fuzzy cliques of $\mathcal{G}$. We say $\mathcal{K}$ and $\mathcal{L}$ are *consistently ordered by* $<$ *at level* c provided $\mathcal{K}_c <_c \mathcal{L}_c$ if and only if $\mathcal{K} < \mathcal{L}$. We say the linear ordering $<$ is cut-level *consistent* if for each pair of fuzzy cliques of $\mathcal{G}$ and for each $c \in fs(\mathcal{G})$ the pair is consistently ordered by $<$ at level c.

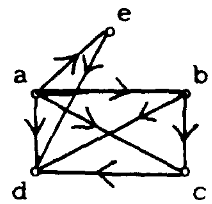
OBSERVATION 4.17. If the linear ordering $<$ is cut-level consistent, we claim that each row of the vertex clique incidence matrix is convex. That $\mathcal{G}$ is a fuzzy interval graph then follows immediately from Theorem 3.4. We proceed by contrapositive, assuming there exists a row that is not convex. Suppose that there exist a vertex $x \in X$ and a sequence of fuzzy cliques $\mathcal{K} < \mathcal{L} < \mathcal{M}$ such that $\mathcal{L}(x) < \min\{\mathcal{K}(x), \mathcal{M}(x)\} \equiv c$. Then $x \in \mathcal{K}_c$, $x \notin \mathcal{L}_c$ and $x \in \mathcal{M}_c$. As in Construction 4.12 there exists $y \in \mathcal{L}_c$ such that $\{x,y\} \notin \mathcal{E}_c$. If $(x,y) \in A$ (the transitive orientation of $\mathcal{G}^C$) then $\mathcal{M}_c <_c \mathcal{L}_c$ with $\mathcal{L} < \mathcal{M}$. If $(y,x) \in A$, then $\mathcal{L}_c <_c \mathcal{K}_c$ with $\mathcal{K} < \mathcal{L}$. In either case the ordering $<$ is not cut-level consistent. □

By Example 3.6 there exist fuzzy interval graphs where no ordering of the fuzzy cliques is cut-level consistent. Therefore in general the construction becomes considerably more complicated. We use Example 4.18 to illustrate an ordering that is not cut-level consistent, and to introduce our method of modifying the vertex clique incidence matrix to correct for rows that are not convex.

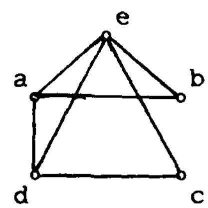
EXAMPLE 4.18. Let $\mathcal{G}$ be the fuzzy graph defined by the adjacency matrix B and the cut-level graphs given in Figure 17. To avoid confusion between subscripts indicating cut-levels and subscripts indicating a linear ordering, we let $fs(\mathcal{G}) = \{s,t\} = \{1,.5\}$. The cliques of $\mathcal{G}_s$ are {a}, {b,e}, {c,e} and {d}. Since {a,b} is oriented (a,b), Construction 4.15 defines $\{a\} <_s \{b,e\}$. Continuing, we see that the ordering of cliques at the s = 1 level is $\{a\} <_s \{b,e\} <_s \{c,e\} <_s \{d\}$. Similarly the transitive orientation of $\mathcal{G}^C$ given defines the t = .5 level clique ordering as $\{a,b,e\} <_t \{a,d,e\} <_t \{c,d,e\}$.

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & .5 & 0 & .5 & .5 \\ .5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & .5 & 1 \\ .5 & 0 & .5 & 0 & .5 \\ .5 & 1 & 1 & .5 & 0 \end{bmatrix} \end{array}$$

$\mathcal{G}_s =$ 

$\mathcal{G}_s^C =$ 

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & \mathcal{K}_5 & \mathcal{K}_6 \end{array} \\ \begin{bmatrix} 1 & .5 & 1 & .5 & 0 & 0 \\ .5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & .5 \\ 0 & 0 & .5 & 1 & .5 & 1 \\ .5 & 1 & .5 & .5 & 1 & .5 \end{bmatrix} \end{array}$$

$\mathcal{G}_t =$ 

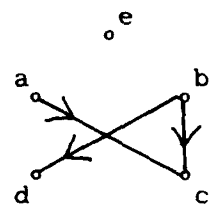
$\mathcal{G}_t^C =$ 

Figure 17. *A fuzzy interval graph and fuzzy clique ordering that is not cut-level consistent*

The vertex fuzzy clique incidence matrix M in Figure 17 has columns ordered by the relation <. By definition, the $x,\mathcal{K}_j$ entry of M is $\mathcal{K}_j(x)$, and each column represents a particular fuzzy clique. We also consider the fuzzy cliques $\mathcal{K}_1$ through $\mathcal{K}_6$ as points of an interval I. The row corresponding to a vertex x then represents a fuzzy subset of I, say I_x, where $I_x(\mathcal{K}_j) = \mathcal{K}_j(x)$.

The ordering < is not cut-level consistent. By definition $\mathcal{K}_2 < \mathcal{K}_3$. However the s = 1 level cut of $\mathcal{K}_2$ is {b,e}, the s = 1 level cut of $\mathcal{K}_3$ is {a}, and $\{a\} <_s \{b,e\}$.

By Theorem 3.4 the rows of M define a family of fuzzy sets $\{I_x | x \in X\}$ which have $\mathcal{G}$ as its fuzzy intersection graph. However the rows headed by a, d and e are not convex, so M fails to provide an interval representation for $\mathcal{G}$.

The set of fuzzy cliques $\mathcal{H} = \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_5, \mathcal{K}_6$ is cut-level consistent and the family of s level cuts $\mathcal{H}_s$ contains each clique of $\mathcal{G}_s$. However the family of t level cuts $\mathcal{H}_t$ does not include the clique $\{a,d,e\}$. Including $\mathcal{K}_3$ or $\mathcal{K}_4$ in $\mathcal{H}$ violates cut-level consistency. We thus modify the "inconsistent" fuzzy cliques $\mathcal{K}_3$ and $\mathcal{K}_4$ at the $s = 1$ cut-level.

Each row of the matrix M would define a fuzzy interval if the "a value in the $\mathcal{K}_3$ column" and "the d value in the $\mathcal{K}_4$ column" were reduced to .5 while "the e value in the $\mathcal{K}_3$ and $\mathcal{K}_4$ columns" were raised to 1. The two "modified" columns would then be equal and one could be deleted. The resulting "modified" matrix would be a fuzzy set-interval point incidence matrix in which the rows do define a family $\mathcal{F}$ of fuzzy intervals for which $\mathcal{G} = \mathrm{Int}(\mathcal{F})$.

Our goal is now to formalize this process by identifying "inconsistent" fuzzy cliques which are then deleted or modified. We first give a technical lemma which serves two purposes. First its proof illuminates the "local" structure of orderings on fuzzy interval graphs that are not cut-level consistent. The lemma is also used to show that Construction 4.20 is well defined.

LEMMA 4.19. Let $\mathcal{G}$ be a fuzzy graph which satisfies the conditions of Construction 4.15 and let $s > t$. Suppose that K_s and L_s are cliques of $\mathcal{G}_s$ that K_t and L_t are cliques of $\mathcal{G}_t$ and that $K_s \subseteq K_t$, $L_s \subseteq L_t$, $L_s <_s K_s$ and $K_t <_t L_t$. Then there exists a clique M of $\mathcal{G}_t$ such that either

(1) $L_s \subseteq M$ and $M <_t L_t$ or

(2) $K_s \subseteq M$ and $K_t <_t M_t$.

Proof. We essentially prove the claim by exhaustion; checking all possible edge configurations. Recall the edge set of the graph $\mathcal{G}_s$ is denoted by $\mathcal{E}_s$. In the figures connected with this proof solid lines denote required edges and dotted lines denote an edge resulting from an assumption which leads to a contradiction. An arrow indicates the orientation of an arc. Those edges that are not shown in a graph or its complement are not essential to the argument.

Each case shares the general conditions shown in Figure 18. By definition of $<_t$, there exist $x \in K_t$ and $y \in L_t$, with $\{x,y\} \notin \mathcal{E}_t$ and $(x,y) \in A$. Similarly, there exist $x' \in K_s$ and $y' \in L_s$, with $\{x',y'\} \notin \mathcal{E}_s$ and $(y',x') \in A$. Clearly either $x \neq x'$ or $y \neq y'$. Then $s > t$ implies $\{x,y\} \notin \mathcal{E}_s$, $\{x,x'\} \in \mathcal{E}_t$ (or $x = x'$) and $\{y,y'\} \in \mathcal{E}_t$ (or $y = y'$). As $<_t$ is well defined, $\{x',y'\} \in \mathcal{E}_t$ and either $\{x,x'\} \notin \mathcal{E}_s$ or $\{y,y'\} \notin \mathcal{E}_s$.

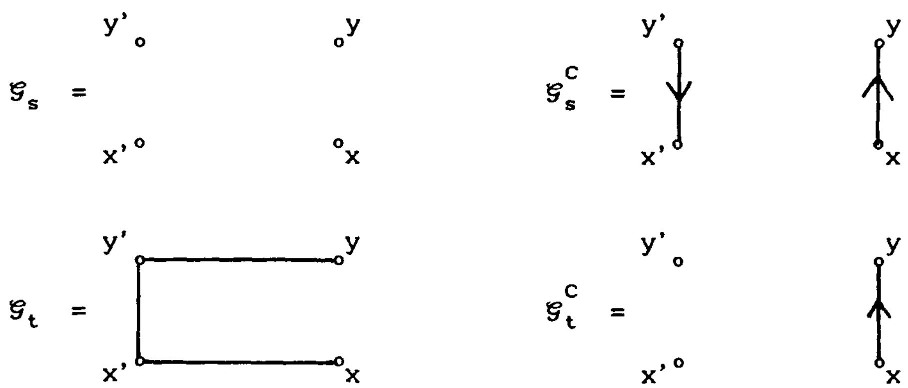


Figure 18. *Basic conditions for inconsistent* cut-level *orderings*

Case 1 (Figure 19). Suppose that $\{x',y\} \notin \mathcal{E}_t$. In this case we allow $x = x'$ or $x \neq x'$. By transitivity at the t level, $(x',y) \in A$. Then $s > t$ implies $\{x',y\} \notin \mathcal{E}_s$ and transitivity at the s level implies $\{y,y'\} \notin \mathcal{E}_s$. We show that $\{x'\} \cup L_s$ is a complete subgraph of $\mathcal{G}_t$; then since $(x',y) \in A$, statement (1) is satisfied. If $L_s = \{y'\}$ we are done so let w be an arbitrary vertex of L_s. If $\{w,x'\} \notin \mathcal{E}_t$, transitivity at the t level implies $(x',w) \in A$. However $s > t$ implies $\{w,x'\} \notin \mathcal{E}_s$ and transitivity at the s level then requires $\{y',w\} \notin \mathcal{E}_s$. This contradicts $\{w,y'\} \subseteq L_s$. Therefore $\{x'\} \cup L_s$ is a complete subgraph of $\mathcal{G}_t$ and is contained in a clique M of $\mathcal{G}_t$. Since $(x',y) \in A$, we have $M <_t L_t$ and statement (1) is satisfied.

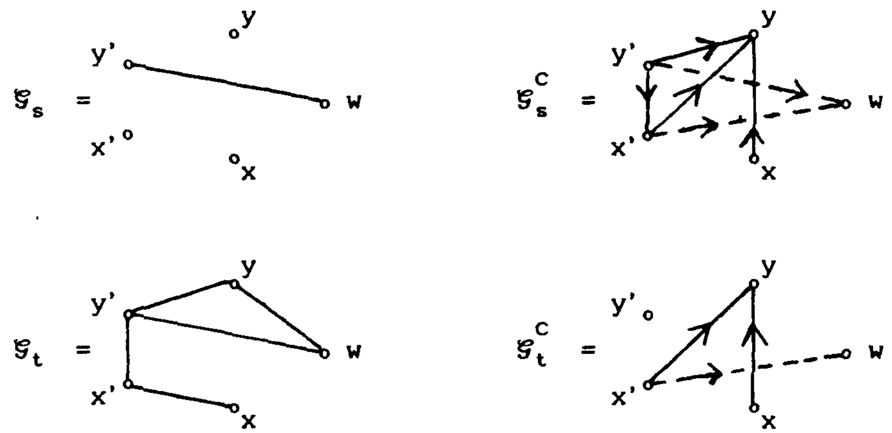


Figure 19. *Case 1 of Lemma 4.19*

Case 2 (Figure 20). Suppose that $\{x,y'\} \notin \mathcal{E}_t$. In this case we allow $y = y'$ or $y \neq y'$. By transitivity at the t level, $(x,y') \in A$. Then $s > t$ implies $\{x,y'\} \notin \mathcal{E}_s$ and transitivity at the s level implies $\{x,x'\} \notin \mathcal{E}_s$. We show that $\{y'\} \cup K_s$ is a clique in $\mathcal{G}_t$; then since $\{x,y'\} \in A$, statement (2) is satisfied.

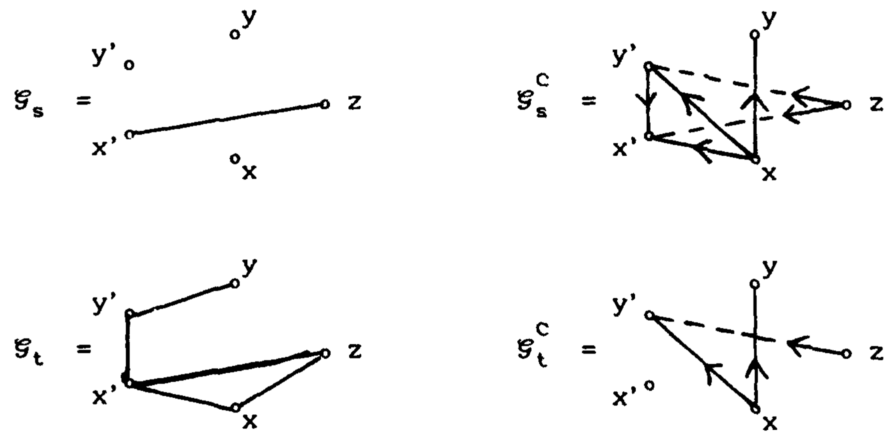


Figure 20. *Case 2 of Lemma 4.19*

If $K_s = \{x'\}$ we are done, so let z be an arbitrary vertex of K_s. If $\{z,y'\} \notin \mathcal{E}_t$, transitivity at the t level implies $(z,y') \in A$. However then

$\{z,y'\} \notin \mathcal{E}_s$ and transitivity at the s level then requires $\{z,x'\} \notin \mathcal{E}_s$, contradicting $\{z,x'\} \subseteq K_s$. Therefore $\{y'\} \cup K_s$ is a complete subgraph of $\mathcal{G}_t$ and is contained in a clique M of $\mathcal{G}_t$. Then $(x,y') \in A$ implies $K_t <_t M$ and statement (2) is satisfied.

Case 3 (Figure 21). Suppose that $\{x,y'\} \in \mathcal{E}_t$ and $\{x',y\} \in \mathcal{E}_t$. Then x, x', y and y' are distinct. We show that $K_s \cup L_s$ is a complete subgraph of $\mathcal{G}_t$. Recall that $\{x',y'\} \in \mathcal{E}_t$, and so if $\{x'\} = K_s$ and $\{y'\} = L_s$ we are done. Let $z \in K_s$ and $w \in L_s$ be arbitrary. We must show that $\{z,w\} \in \mathcal{E}_t$, $\{z,y'\} \in \mathcal{E}_t$ and $\{x',w\} \in \mathcal{E}_t$.

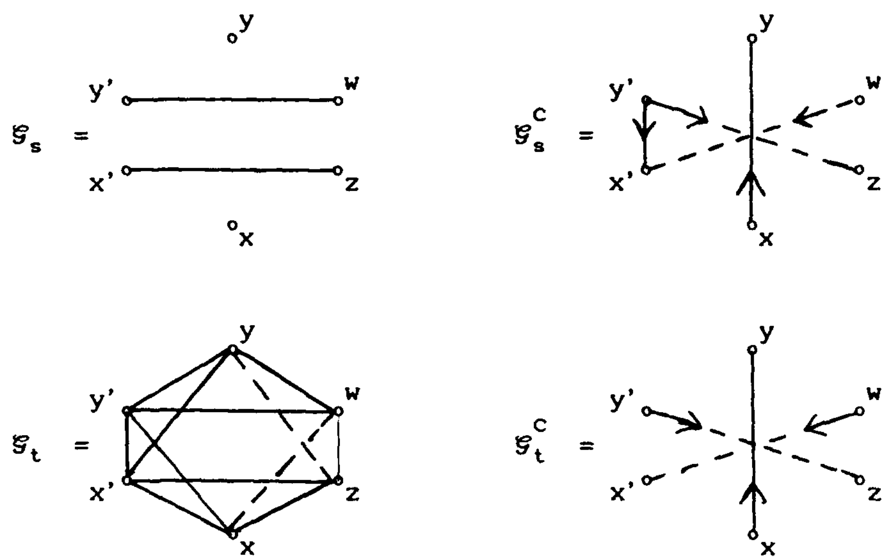


Figure 21. *Case 3 of Lemma 4.19*

Clearly $\{z,w\} \in \mathcal{E}_t$, for if not $L_s <_s K_s$ requires $(w,z) \in A$ and $K_t <_t L_t$ requires $(z,w) \in A$; contradicting the antisymmetry of A. Now if $z = x'$ or $w = y'$ the result follows immediately. Therefore assume that all six vertices are distinct. We show $\{z,y'\} \in \mathcal{E}_t$ by contradiction. Suppose that $\{z,y'\} \notin \mathcal{E}_t$. Then $L_s <_s K_s$ implies $(y',z) \in A$. Also $\{z,y\} \in \mathcal{E}_t$; otherwise $(z,y) \in A$ and $\{y',y\} \in \mathcal{E}_t$ contradict transitivity at the t level. However then $\{z,x,y',y\}$ induces a 4-cycle which has no chords, a contradiction. Similarly, $\{x',w\} \notin \mathcal{E}_t$ and $\{x,w\} \notin \mathcal{E}_t$ contradict

$\{x,x'\} \in K_t$; while $\{x',w\} \notin \mathcal{E}_t$ and $\{x,w\} \in \mathcal{E}_t$ induces a 4-cycle $\{x,x',y,w\}$ with no chords. Thus $\{z,y'\} \in \mathcal{E}_t$ and $\{x',w\} \in \mathcal{E}_t$ as required.

Therefore $K_s \cup L_s$ induces a complete subgraph of $\mathcal{G}_t$ and is contained in some clique M of $\mathcal{G}_t$. If $M <_t K_t <_t L_t$, statement (1) is satisfied. If $K_t <_t M$ statement (2) is satisfied. These three cases completely exhaust all possibilities. □

We now define a directed graph F which in turn defines a linearly ordered family of fuzzy sets that represents our host interval. Some of these fuzzy sets will be fuzzy cliques of $\mathcal{G}$, while others will be lower truncations of fuzzy cliques. The graph theory analogy of a forest with trees allows a good visualization of "vertically growing" cut-level sets which define the required fuzzy sets.

CONSTRUCTION 4.20. Let $\mathcal{G} = (X,\mu)$ be a chordal fuzzy graph with $\mathcal{G}^C$ transitively oriented by A. For each $c \in fs(\mathcal{G}) = \{r_1,r_2,\cdots,r_n\}$ let the cliques of $\mathcal{G}_c$ be linearly ordered by $<_c$ as defined in Construction 4.15. We recursively construct a forest F whose vertex set is the set of all cut-level cliques of $\mathcal{G}$ and which has one tree for each clique of $\mathcal{G}_{r_n}$.

Level r_n: Linearly order the set of all cliques of $\mathcal{G}_{r_n}$ by the relation $<_{r_n}$. Each of these cliques of $\mathcal{G}_{r_n}$ (vertices of F) represent the root of a tree in the forest.

We recursively build the forest by "vertically" adding cut-level cliques as vertices of F and defining a set of arcs between cut-levels. In the recursion let i range from 1 to n - 1.

Level r_{n-i}: Let $s = r_{n-i}$ and $t = r_{n-i+1}$; so $s > t$. Linearly order the set of all cliques of $\mathcal{G}_s$ by the relation $<_s$. Let E_s be any set of arcs that satisfy the conditions:

(i) each clique K_s of $\mathcal{G}_s$ is a vertex of exactly one arc of E_s

(ii) if $(K_t,K_s) \in E_s$ then K_t is a clique of $\mathcal{G}_t$, K_s is a clique of

$\mathcal{S}_s$, and $K_s \subseteq K_t$. Thus an arc joins two cuts level sets of (some) fuzzy clique.

(iii) For each pair of arcs $(K_t, K_s) \in E_s$ and $(L_t, L_s) \in E_s$ we have $K_s <_s L_s$ or $K_s = L_s$ if and only if $K_t <_t L_t$ or $K_t = L_t$. Thus when viewed as cut-levels of a family of fuzzy cliques, the s level ordering is cut-level consistent with all "lower" levels.

We show in Observation 4.25 that there can be a number of arc sets that satisfy these conditions. We now use Lemma 4.19 to demonstrate the existence of at least one such forest. Let K_s be the minimal (with respect to $<_s$) clique of $\mathcal{S}_s$. Clearly there exists a minimal (with respect to $<_t$) clique K_t of $\mathcal{S}_t$ where $K_s \subseteq K_t$. Let $(K_t, K_s) \in E_s$.

Next let L_s be the successor of K_s (with respect to $<_s$) and let L_t be minimal (with respect to $<_t$) such that $L_s \subseteq L_t$ and $K_t <_t L_t$ or $K_t = L_t$. Clearly L_t exists, for if not let L be maximal (with respect to $<_t$) with $L_s \subseteq L$. Now $K_s <_s L_s$ and $L <_t K_t$ are the conditions of Lemma 4.19. However statement (1) contradicts the minimality of K_t and statement (2) contradicts the maximality of L. Therefore let $(L_t, L_s) \in E_s$.

Continuing recursively we add one arc for each clique of $\mathcal{S}_s$. By construction this family of arcs satisfies the three conditions. It may be that for some clique M_t of $\mathcal{S}_t$, there is no arc from M_t. We shall call such a clique a *dead branch* and require no arc of F originate at M_t.

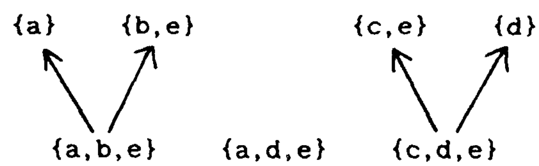


Figure 22. *An ordered forest with a dead branch*

Figure 22 shows the forest F associated with the fuzzy graph of Example 4.18. In this example F is unique, the only possible arc to {b,e} is from {a,b,e}, which is not consistent with the arc ({a,d,e},{a}). Thus there must be an arc from {a,b,e} to {a}. Similarly F must contain ({c,d,e},{c,e}) and ({c,d,e},{d}) and cannot contain ({a,d,e},{d}). The clique {a,d,e} is a dead branch.

The result of recursively constructing the arc set $F_{r_{n-i}}$ for each $i \in \{1,\cdots,n-1\}$ is a forest F with arc set $\bigcup_{i=1}^{n-1} F_{r_{n-i}}$. We use the arcs of F to define a linearly ordered family of fuzzy sets, one for each path from a root to a dead branch or a r_1 level clique. The set of paths of length n define a cut-level consistent family of fuzzy cliques of $\mathcal{G}$. The paths to dead branches represent "inconsistent" fuzzy cliques that must be modified.

CONSTRUCTION 4.21. Let $\mathcal{G}$ be a fuzzy graph satisfying the conditions of Theorem 4.14 and F be a forest for $\mathcal{G}$ as defined in Construction 4.20. Let P be a simple path in F which begins at a root of F (clique of $\mathcal{G}_{r_n}$) and ends at a clique of $\mathcal{G}_{r_1}$ or at a dead branch. We allow trivial paths (a root is also a dead branch). Associated with P define the fuzzy set $\mathcal{P}$ on the vertex set of $\mathcal{G}$ by

$$\mathcal{P}(x) = \max \{s \in fs(\mathcal{G}) \mid x \text{ is an element of the s level clique of P}\}.$$

Clearly for a path of length n, the associated fuzzy set is a fuzzy clique of $\mathcal{G}$. For a path ending at a dead branch, the fuzzy set is the lower truncation of one or more fuzzy cliques of $\mathcal{G}$.

Let I be the set of all such fuzzy sets. As in Construction 4.15 the family of cut-level orderings lexicographically defines a linearly ordering of I. We now construct the *vertex forest matrix of* $\mathcal{G}$ where the rows are indexed by the vertex set of $\mathcal{G}$, the columns by the (ordered) fuzzy sets of I and the $x,\mathcal{P}_i$ entry is $\mathcal{P}_i(x)$. By Observation 4.17 the sub-matrix formed using the columns that define fuzzy cliques has convex rows.

Figure 23 shows the vertex forest matrix C corresponding to Example 4.18. The columns $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_4$, and $\mathcal{P}_5$ represent the fuzzy cliques $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_5$, and $\mathcal{K}_6$ of Figure 17, respectively. The column $\mathcal{P}_3$ corresponds to the dead branch {a,d,e} of Figure 22 and the "inconsistent" fuzzy cliques $\mathcal{K}_3$ and $\mathcal{K}_4$ of Figure 17.

$$C = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 \end{array} \\ \begin{bmatrix} 1 & .5 & .5 & 0 & 0 \\ .5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .5 \\ 0 & 0 & .5 & .5 & 1 \\ .5 & 1 & .5 & 1 & .5 \end{bmatrix} \end{array} \qquad D = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 \end{array} \\ \begin{bmatrix} 1 & .5 & .5 & 0 & 0 \\ .5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .5 \\ 0 & 0 & .5 & .5 & 1 \\ .5 & 1 & 1 & 1 & .5 \end{bmatrix} \end{array}$$

Figure 23. *A vertex forest matrix and a vertex interval matrix*

A routine verification shows the rows of C define a family of fuzzy sets that has $\mathcal{G}$ as its fuzzy intersection graph. However this matrix does not have convex rows since the dead branch of column 3 produces too low a value for $\mathcal{P}_3(e)$.

We may create convex rows by increasing vertex values in those columns associated with dead branches. We thus define the *vertex interval matrix of* $\mathcal{G}$ by increasing values in columns associated with dead branches just enough to create convex rows.

More formally, for each fuzzy set $\mathcal{P}_j$ define a corresponding fuzzy set ρ_j as follows. Let x be a vertex of $\mathcal{G}$ and $\mathcal{P}_j$ a fuzzy set defining a column of the vertex forest matrix. If there is no pair of columns $\mathcal{P}_i$ and $\mathcal{P}_k$ with $i < j < k$ and $\mathcal{P}_j(x) < \mathcal{P}_i(x) \wedge \mathcal{P}_k(x))$, then we define $\rho_j(x) = \mathcal{P}_j(x)$. If there is such a pair, we define

$$\rho_j(x) = \max \{\mathcal{P}_i(x) \wedge \mathcal{P}_k(x) \mid i < j < k \text{ and } \mathcal{P}_j < \mathcal{P}_i(x) \wedge \mathcal{P}_k(x)\}.$$

Rephrasing in terms of cliques, if $\mathcal{P}_j$ is a fuzzy clique then $\rho_j = \mathcal{P}_j$. If $\mathcal{P}_j$ is not a fuzzy clique, there is a cut-level t where $\mathcal{P}_{jt}$ is a dead branch. For $s \leq t$ let $\rho_{js} = \mathcal{P}_{js}$. For $s > t$, let $\rho_{js} = \mathcal{P}_{is} \cap \mathcal{P}_{ks}$ where i is the largest index with $i < j$ and $\mathcal{P}_{is}$ a clique of $\mathcal{G}_s$, and k is the smallest index with $j < k$ and $\mathcal{P}_{ks}$ a clique of $\mathcal{G}_s$. The fuzzy set ρ_j is then defined in the usual way by the cut-level sets.

In Figure 23 matrix D is the vertex interval matrix for Example 4.18. Notice that for s = 1 we have $\rho_{3s} = \mathcal{P}_{2s} \cap \mathcal{P}_{4s} = \{e\}$.

Let J denote the family of fuzzy sets defined by the columns and $\mathcal{F}$ denote the family of fuzzy sets defined by the rows of the vertex interval matrix. We now complete the proof of Theorem 4.14 by showing that each member of $\mathcal{F}$ is normal and convex (a fuzzy interval) and that $\mathcal{G} = \mathrm{Int}(\mathcal{F})$.

LEMMA 4.22. We assume the conditions and notation above. For each vertex x of $\mathcal{G}$ define $\mathcal{I}_x : J \longrightarrow [0,1]$ by $\mathcal{I}_x(\rho_i) = \rho_i(x)$; that is, $\mathcal{I}_x$ is defined by the row associated with x in the vertex interval matrix. Then $\mathcal{I}_x$ is a fuzzy interval.

Proof. Let x be a vertex of $\mathcal{G}$. Then x is a vertex in some clique of $\mathcal{G}_{r_1}$, say K. By Constructions 4.20 and 4.21 K is the $r_1 = 1$ level cut of some fuzzy set ρ in J. Therefore $\mathcal{I}_x(\rho) = \rho(x) = 1$ and $\mathcal{I}_x$ is normal.

To show each $\mathcal{I}_x$ is convex, we must show that i < j < k implies $\min\{\mathcal{I}_x(\rho_i), \mathcal{I}_x(\rho_k)\} \le \mathcal{I}_x(\rho_k)$; or equivalently that $\rho_i(x) \wedge \rho_k(x) \le \rho_j(x)$. However, Construction 4.21 clearly provides these conditions. If ρ_i, ρ_j and ρ_k are all fuzzy cliques, the result follows immediately from Observation 4.17. Otherwise, the result follows by definition of the fuzzy sets ρ_i, ρ_j and ρ_k. □

LEMMA 4.23. Given the definitions and conditions of Theorem 4.14 through Lemma 4.22, $\mathcal{G} = \mathrm{Int}(\mathcal{F})$.

Proof. There is a clear correspondence between the crisp vertex set X and the family of fuzzy intervals $\mathcal{F}$. Let $x \neq y$ be elements of X. We must show $\mu(x,y) = h(\mathcal{I}_x \wedge \mathcal{I}_y)$. By definition

$$\begin{aligned} h(\mathcal{I}_x \wedge \mathcal{I}_y) &= \max\{\mathcal{I}_x(\rho_j) \wedge \mathcal{I}_y(\rho_j) \mid \rho_j \in J\} \\ &= \max\{\rho_j(x) \wedge \rho_j(y) \mid \rho_j \in J\} \\ &= \max\{s \in fs(\mathcal{G}) \mid \{x,y\} \subseteq \rho_{js}\}. \end{aligned}$$

Suppose that $\{x,y\} \subseteq \rho_{js}$, the s level cut of ρ_j. By construction, ρ_{js} is either a clique of $\mathcal{G}_s$ or is contained in one. In either case $\{x,y\}$

is an edge of $\mathcal{G}_s$ and $\mu(x,y) \geq s$. Therefore $\mu(x,y) \geq h(\mathcal{I}_x \wedge \mathcal{I}_y)$. Conversely, $\{x,y\}$ is clearly an edge of the $\mu(x,y) = t$ cut-level graph of $\mathcal{G}$. Therefore $\{x,y\}$ is contained in some clique K of G_t. By Constructions 4.20 and 4.21 $K = \rho_{jt}$ for some $\rho_j \in J$. Therefore $\mu(x,y) \leq h(\mathcal{I}_x \wedge \mathcal{I}_y)$. □

We have now completed the proof of Theorem 4.14.

The reader may verify that this construction produces the interval representations of Example 3.6 and Example 4.18. We now give a complete example of Theorem 4.14.

EXAMPLE 4.24. Consider the fuzzy graph $\mathcal{G}$ defined by the incidence matrix B in Figure 24 where $fs(\mathcal{G}) = \{s,t,q\} = \{1,.8,.5\}$. Figure 24 also shows the cut-level graphs of $\mathcal{G}$ and a transitive orientation A of $\mathcal{G}^c$.

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & .8 & .5 & .8 & .5 \\ .8 & 0 & .5 & 0 & 1 \\ .5 & .5 & 0 & .5 & .8 \\ .8 & 0 & .5 & 0 & .5 \\ .5 & 1 & .8 & .5 & 0 \end{bmatrix} \end{array}$$

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & \mathcal{K}_5 & \mathcal{K}_6 \end{array} \\ \begin{bmatrix} .8 & 1 & .5 & 1 & .5 & .5 \\ 0 & 0 & 0 & .8 & 1 & .5 \\ .5 & .5 & 1 & .5 & .5 & 1 \\ 1 & .8 & .5 & 0 & 0 & 0 \\ .5 & .5 & .8 & .5 & 1 & .8 \end{bmatrix} \end{array}$$

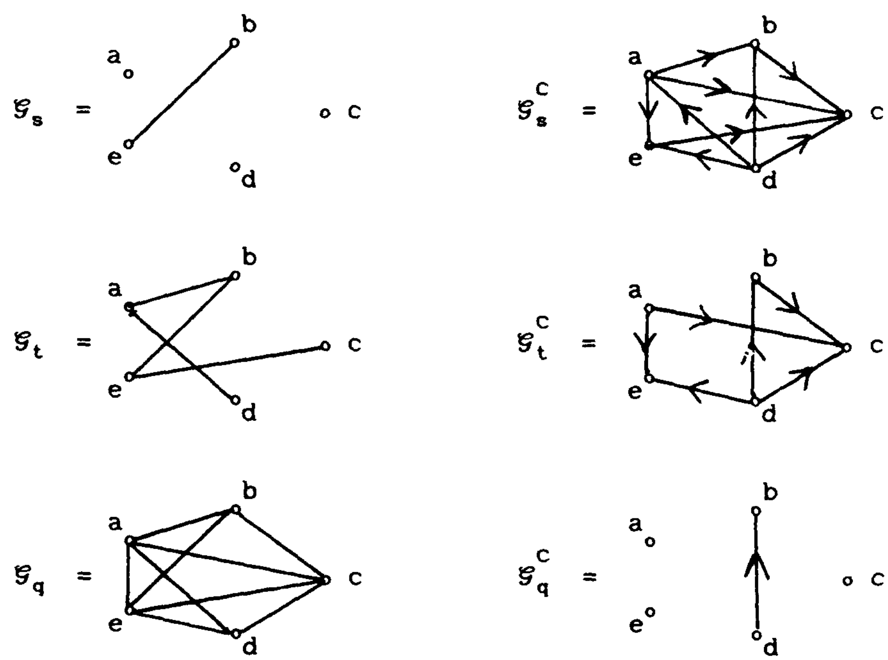


Figure 24. *A chordal fuzzy graph with transitive orientation of* $\mathcal{G}^c$

Applying Construction 4.15 we obtain the linear orderings of the families of cut-level cliques:

$$s = 1 \qquad \{d\} <_s \{a\} <_s \{e,b\} <_s \{c\}$$

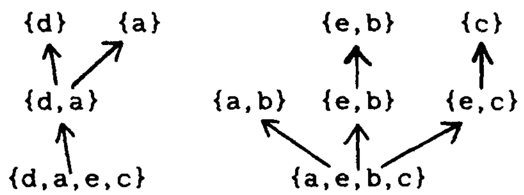
$$t = .8 \qquad \{d,a\} <_t \{a,b\} <_t \{e,b\} <_t \{e,c\}$$

$$q = .5 \qquad \{d,a,e,c\} <_q \{a,e,b,c\}$$

By Observation 3.2 there are six fuzzy cliques of $\mathcal{G}$; subscripts indicate the order induced by Construction 4.15. The matrix M of Figure 24 is the associated vertex clique incidence matrix for $\mathcal{G}$. The rows indexed by a, c and e are not convex; therefore the orientation A induces a fuzzy clique ordering that is not cut-level consistent.

Following Construction 4.20 gives the forest F of Figure 25. The only dead branch in F is the t = .8 level cut of $\mathcal{P}_3$ which is the clique {a,b}. The paths P_1, P_2, P_4 and P_5 correspond to the fuzzy cliques $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_5$ and $\mathcal{K}_6$, respectively. The path $\mathcal{P}_3$ which terminates at the dead branch {a,b} corresponds to a lower truncation of the fuzzy clique $\mathcal{K}_4$. The fuzzy clique $\mathcal{K}_3$ is deleted all together.

The vertex forest matrix C of Figure 25 has convex rows and so C is also the vertex interval matrix of $\mathcal{G}$. In terms of cut-level clique orderings, $\mathcal{P}_{2s} \cap \mathcal{P}_{4s} = \emptyset$ and $\mathcal{P}_3$ need not be modified. Therefore the matrix C provides a fuzzy interval representation for $\mathcal{G}$.



$$C = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 \end{array} \\ \begin{bmatrix} .8 & 1 & .8 & .5 & .5 \\ 0 & 0 & .8 & 1 & .5 \\ .5 & .5 & .5 & .5 & 1 \\ 1 & .8 & 0 & 0 & 0 \\ .5 & .5 & .5 & 1 & .8 \end{bmatrix} \end{array}$$

Figure 25. *A fuzzy interval representation for Example 4.24*

OBSERVATION 4.25. The interval representation of a fuzzy graph $\mathcal{G}$ is not in general unique. The construction depends heavily on the transitive orientation of $\mathcal{G}^C$ and so different orientations can produce different vertex interval matrices.

In addition, slight modifications of Construction 4.20 can also produce different vertex interval matrices. We favored a "left to right construction of arcs while building the forest F. We could have instead specified a "right to left" construction. Figure 26 gives a forest F_2 a[illegible] associated matrix C_2 which defines a second interval representation for [illegible] fuzzy graph of Example 4.24.

$$F_2 = \begin{tikzcd}[column sep=small] \{d\} & \{a\} & \{e,b\} & \{c\} \\ \{d,a\} \arrow[u] & \{a,b\} \arrow[u] & \{e,b\} \arrow[u] & \{e,c\} \arrow[u] \\ \{d,a,e,c\} \arrow[u] & & \{a,e,b,c\} \arrow[ul, dash] \arrow[u] \arrow[ur] & \end{tikzcd} \qquad C_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 \end{array} \\ \begin{bmatrix} .8 & 1 & .8 & .5 & .5 \\ 0 & .8 & .8 & 1 & .5 \\ .5 & .5 & .5 & .5 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ .5 & .5 & .5 & 1 & .8 \end{bmatrix} \end{array}$$

$$F_3 = \begin{tikzcd}[column sep=small] \{d\} & \{a\} & \{a\} & \{e,b\} & \{c\} \\ \{d,a\} \arrow[u] & & \{a,b\} \arrow[u] & \{e,b\} \arrow[u] & \{e,c\} \arrow[u] \\ \{d,a,e,c\} \arrow[u] & & & \{a,e,b,c\} \arrow[ul, dash] \arrow[u] \arrow[ur] & \end{tikzcd} \qquad C_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 \end{array} \\ \begin{bmatrix} .8 & 1 & 1 & .5 & .5 \\ 0 & 0 & .8 & 1 & .5 \\ .5 & .5 & .5 & .5 & 1 \\ 1 & .8 & 0 & 0 & 0 \\ .5 & .5 & .5 & 1 & .8 \end{bmatrix} \end{array}$$

Figure 26. *Alternate fuzzy interval representations for Example 4.*

Construction 4.20 also specified each cut-level clique be the terminal node of only one arc. We could allow a cut-level clique to be terminal node of a number of arcs, as long as cut-level consistency is maintained. Figure 26 gives a forest F_3 with two copies of the clique [illegible] at the s = 1 level. Clearly F_3 has no dead branches. The associated vertex interval matrix C_3 provides a third interval representation for [illegible] fuzzy graph of Example 4.24.

Section 5. Alternate Edge Strength Functions

Depending on the application, one may wish to use alternate measure of vertex strength or of edge strength. The following provide some examples of alternate edge strength functions. It is important to note, that none of these examples reduce to crisp intersection graphs in the ca of crisp families of sets.

EXAMPLE 5.1. McAllister suggests in [2] that m(i,j) be a "measure of fuzzyness" of $\mu_i \wedge \mu_j$. However since any crisp set has "measure of fuzzyness" equal to zero (see [3], [4] and [5] for definitions and examples), the fuzzy intersection graph is empty for crisp families of sets.

EXAMPLE 5.2. Let $m(i,j) = |\mu_i \wedge \mu_j|/|\mu_i \vee \mu_j|$ where $|\alpha| = \Sigma\alpha(x)$ is the cardinality of the fuzzy set α. Note that if $\mu_i = \mu_j$, then $m(i,j) = 1$ an if $\mu_i \wedge \mu_j = 0$, then $m(i,j) = 0$. As a fuzzy relation on $\mathcal{E}$, m is symmetri and reflexive, but is in general not transitive (see [6]). For crisp families of sets, this relation induces a weighted graph that models $A \cap$ as a percentage of $A \cup B$.

EXAMPLE 5.3. Let $m(i,j) = 1 - |\mu_i \triangledown \mu_j|/|\mu_i \wedge \mu_j|$ where $\triangledown$ is the symmetric difference relation of [4], $|a|$ denotes absolute value and $\mu_i \triangledown \mu_j(x) = |\mu_i(x) - \mu_j(x)|$. If $\mu_i = \mu_j$, then $m(i,j) = 1$ and if $\mu_i \wedge \mu_j = 0$, then $m(i,j) = 0$. As a fuzzy relation on $\mathcal{E}$, m is symmetric reflexive, but in general is not transitive. For crisp families of sets this relation induces a weighted graph that models $A \cap B$ as a percentage $A \cup B$.

For crisp sets $A \cap B = (A \cup B)\backslash(A\backslash B \cup B\backslash A)$, therefore Examples 5.2 and 5.3 produce the same membership function. With fuzzy sets however, there is no simple relation between intersection and symmetric differenc and the membership functions of Examples 5.2 and 5.3 are distinct.

EXAMPLE 5.4. Let $m(i,j) = 1 - |\mu_i \ominus \mu_j|/|\mu_i|$ where $\ominus$ is the bounded difference relation of [4] defined by $\mu_i \ominus \mu_j(x) = \max\{0, \mu_i(x) - \mu_j(x)\}$. Note that if $\mu_i \le \mu_j$, then $m(i,j) = 1$ and if $\mu_i \wedge \mu_j = 0$, then $m(i,j) = 0$. As a fuzzy relation on $\mathcal{E}$, m is reflexive, but is in general not symmetric or transitive; m induces a fuzzy directed graph.

EXAMPLE 5.5. Let $m_p(i,j) = \begin{cases} 1 & \text{if} \quad |\mu_i \wedge \mu_j| \ge p \quad \text{and } i \ne j \\ 0 & \text{if} \quad |\mu_i \wedge \mu_j| < p \quad \text{or } i = j \end{cases}$

A crisp graph is induced by this relation which is symmetric, anti-reflexive, and in general not transitive. This relation induces a *p-intersection graph* which was recently introduced for crisp families of se by Jacobson, McMorris and Scheinerman [9].

EXAMPLE 5.6. Let $m(i,j) = |\mu_i \wedge \mu_j|$ for $i \ne j$. The resulting graph can have edge weights greater than 1, and so m defines a fuzzy multigraph. A variety of recent papers discuss properties of crisp intersection multigraphs [10]. The relation m is symmetric, anti-reflexive, but in general is not transitive.

[1] C. Berge (1989), *Hypergraphs, Combinatorics of Finite Sets*. North-Holland, Amsterdam.

[2] M.L. McAllister (1988), Fuzzy Intersection Graphs. *Comput. Math. Applic.* 15 (10), 871-886.

[3] A. De Luca and S. Termini (1972), A Definition of a non-probabilistic entropy in the setting of fuzzy sets theory. *Inf Control* 20, 301-312.

[4] D. Dubois and H. Prade (1980), *Fuzzy Sets and Systems, Theory and Applications*. Academic Press, New York.

[5] H.J. Zimmermann (1985), *Fuzzy Set Theory and its Applications*. Kluwer-Nijhoff Publishing, Boston.

[6] A. Rosenfeld (1975), Fuzzy Graphs. In: L.A. Zadeh, K.S. Fu, M. Shimura, Eds., *Fuzzy Sets and their Applications*. Academic Press, New York.

[7] G.J. Klir and T.A. Folger (1988), *Fuzzy Sets, Uncertainty, and Information*. Prentice Hall, New Jersey.

[8] F. Roberts (1976), *Discrete Mathematical Models*. Prentice Hall, New Jersey.

[9] M.S. Jacobson, F.R. McMorris and E.R. Scheinerman (1991), General Results on Tolerance Intersection Graphs. *Journal of Graph Theory* 15 (6), 573-577.

[10] T.A. McKee (1991), Foundations of Intersection Graph Theory. *Utilitas Mathematica* 40, 77-86.

[11] P.C. Gilmore and A.J. Hoffman (1964), A Characterization of Comparability Graphs and of Interval Graphs. *Canadian Journal of Mathematics* 16, 539-548.

[12] P.C. Fishburn (1985), *Interval Orders and Interval Graphs*. John Wiley & Sons, New York.