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Brunel University

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SUMMARIES OF PAPERS

The summaries of papers in this booklet are listed in alphabetical order of first author in three groups:

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Once again Ms Molly Demmar has retyped all the abstracts for the Conference. Her perseverance and attention to detail in doing this is also gratefully acknowledged.

MAFELAP 1993

Eighth Conference on the Mathematics of Finite Elements and Applications

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THE SECOND ZIENKIEWICZ LECTURE

ERROR ESTIMATION AND CONTROL

IN

COMPUTATIONAL FLUID DYNAMICS

by

J Tinsley Oden

This paper describes theory and methods for developing a posteriori error estimates and an adaptive strategy for *hp*-finite element approximations of the incompressible Navier-Stokes equations.

For an error estimation, use is made of a new approach which is based on the work of Ainsworth, Wu and the author. That theory has been shown to produce good results for general elliptic systems and general *hp*-finite element methods. Recently, these techniques have been extended to the Navier-Stokes equations.

A three-step adaptive algorithm is also described which produces reasonable *hp* meshes very efficiently. These techniques are applied to representative transient and steady state problems of incompressible viscous flow.

INVITED LECTURES

I BABUŠKA

Reliability of computational mechanics

The aim of computational mechanics is to provide a reliable description of physical phenomena related to the aim of the analysis. Typically, for example, for preventing a failure, avoiding various engineering experiments, etc.

The framework of computational analysis is a) mathematical modelling; b) numerical solution; and c) interpretation of the results. The errors of all these parts have to be under control to achieve the desired reliability.

The talk will address

a) The modelling of plasticity problems under cyclic loadiing for 5454 aluminium alloy in the H-34 temper.
b) The adaptive modelling of the heat problem in thin domains and a-posteriori estimation of the modelling errors.
c) The task of a-posteriori error estimation in finite elements and the theoretical principles of comparing today's major estimators used in the literature; and the concrete results related to the performance of the estimators in the case of nearly patchwise uniform meshes.

H BERGER, G WARNECKE and W L WENDLAND

Coupling of FEM and BEM for transonic flows

Consider a two-dimensional stationary compressible inviscid transonic flow around a given airfoil. As a mathematical model we use the full potentiial equation of isoenergetic, homentropic flows of an ideal gas. As is well known, this equation degenerates where the local density vanishes and for subsonic travelling velocities the flow develops regions with supersonic velocity separated by down-stream shocks from the down-stream subsonic flow region. For the numerical computation we choose a bounded ring domain around the airfoil and use the generalized finite element least squares method developed by Glowinsky et al.

The flow must contain an appropriate vorticity and the jump of the potential along a slit generated by the trailing edge subject to the Kutta-Joukowski condition. In order to avoid nonphysical solutions we incorporate an appropriate penalty term, in addition to the least squares functional, which provides the right selection principle. Traditionally, the exterior Neumann condition on the boundary of the computational domain is given by the flux generated by an exterior parallel flow. However, if the exterior flow was known, the flux would be given by the corresponding conormal derivative of the exterior potential associated with the continuous Dirichlet boundary data belonging to the interior potential field. The mapping of the Dirichlet data into the exterior flux is given by the nonlocal Steklov-Poincaré operator associated with the exterior field. Replacing the exterior flow by a linear Prandtl-Glavert equation given by the quantities of the parallel flow at infinity, the Steklov-Poincaré operator can explicitly be defined in terms of appropriate boundary integral operators defined on the artificial boundary of the bounded computational ring domain around the airfoil. The numerical approximation of the Steklov-Poincaré operator then can be given in terms of boundary element formulations. If we assume that the exterior model flow problem admits a unique solution satisfying the Kutta-Joukowski condition at the trailing edge having there a subsonic local flow region and under the assumption that the flow speed through the artificial boundary is subsonic, we can show that a corresponding coupled finite element - boundary element optimization problem provides a sequence of approximate solutions which converges to the unique solution of an exterior transmission problem defined by the weak solution of the transonic full potential equation in the ring domain and the exterior Prandtl-Glavert equation. The convergence proof is essentially based on a compactness result by Mandel and Nečas in connection with standard approximation results for finite elements and for boundary elements. The computational results are in good agreement with the aforementioned analysis.

G BUGEDA and E OÑATE

A study of mesh optimality criteria in adaptive finite element anlysis

The evaluation of discretization error and the design of suitable meshes via adaptive mesh refinement (AMR) are nowadays two of the challenging issues in the finite element method (FEM).

In this paper a methodology for deriving AMR procedures is proposed. The basis of the approach is the decoupling of the concepts of error measure and mesh optimality criteria. This leads to the definitions of global and local error parameters from which the element refinement strategy can be simply ob'ained. Moreover, this allows us to identify clearly the convergence rates of the global and local error norms, which strongly influence the expressions of the element refinement parameter. It is shown that an inaccurate evaluation of this important parameter can lead to oscillations in the refinement process.

The methodology proposed is particularized for two mesh optimality criteria using Zienkiewicz-Zhu error estimator [1], First the standard criterion of equal distribution of the global error over all the elements is studied. We will show here that a careful interpretation of the concepts of global error and optimal mesh leads in this case to an enhanced expression of the element refinement parameter slightly different from that typically used in literature.

The second mesh optimality criterion studied is based in the equal distribution of the error per unit area or volume (i.e. the specific error). This strategy allows us to concentrate more and smaller elements in zones where the gradients of the problem unknowns (i.e. stresses in structural problems) are higher, as should be expected from the engineering point of view.

The layout of the paper is the following. First the concepts of error estimator and acceptable solution are briefly revisited. Then a general expression for the element refinement parameter in terms of global and local error parameters is derived. The mesh optimality criteria based on global and specific error distributions are then analyzed and the expressions for the local and global error parameters and that of the element refinement parameter are obtained for each case. Finally, the efficiency of the two AMR methodologies studied is compared in the analysis of plane strain, plate, shell and potential flow problems.

References:
Zienkiewicz, O C and Zhu, J Z, A simple error estimator and adaptive procedure for practical engineering analysis. Int. Num. Meth. Engrg. 24, 337-357, 1987.

M J CROCHET

Finite elements for polymer flow: A comparison

Over the last five years, several finite element methods have been introduced for analyzing viscoelastic flow. In the present paper, we examine a class of finite elements of the mixed type where the stress and velocity components and the pressure are independent variables; they are combined with various methods of upwinding for the constitutive equations. We have selected several benchmark problems; with and without geometrical singularities, with and without change of type of the vorticity equation. We show that it is presently difficult to select "the" optimal method for all flows although some algorithms perform much better than others. We also explain some outstanding problems for the simulation of viscoelastic flow.

C JOHNSON

Adaptive finite element methods in computational mechanics

We give an overview of our recent work on adaptive finite element methods for problems in solid and fluid mechanics including elasticity and plasticity problems, and incompressible and compressible fluid flow. We present new a-posteriori error estimates for the indicated classes of problems and show that the corresponding adaptive algorithms for quantitative error control are reliable and efficient. We discuss the theoretical principles underlying the a-posteriori error estimates and present the results of numerical experiments.

V MAZ'YA

Approximate approximations

The starting idea is to represent an arbitrary function as a linear combination of basis functions, which, in contrast to splines, form an approximate partition of unity. The approximations obtained do not converge as the mesh size tends to zero. The lack of the convergence is compensated for, first of all, by the flexibility in the choice of the basis functions and by the simplicity of the generalization to the multidimensional case. Another, and probably the most important advantage, is the possibility to obtain explicit formulae for values of various integral and pseudodifferential operators of mathematical physics applied to the basis functions.

This property, when used for solving elliptic boundary value problems, leads to a new approach to the discretization of boundary integral equations, which is not based upon the decomposition of the boundary into elements. In this approach the coefficients of the resulting algebraic system depend only on the coordinates of a finite number of boundary points; hence the name Boundary Point Method (BPM) seems quite natural.

The first step of the BPM consists of the approximation of the boundary data and of the potential densities by a linear combination of basis functions, each of them being either concentrated near a particular boundary point or decreasing rapidly with the distance from this point. In the second step the calculation of the potentials, whose densities are the basis functions, is carried out. I discuss only one concrete variant of the BPM which leads to an algebraic system with coefficients that are explicitly expressed - up to a simple quadrature.

The approximation method can be employed also to solve various non-stationary problems. Its effectiveness is demonstrated by the numerical solution of Cauchy problem for quasilinear parabolic equations. An important feature of the algorithms is that they are both explicit and stable under much milder restrictions to the time step, depending on the size of the space grid, in comparison with usual explicit difference schemes.

K W MORTON and E SÜLI

A priori and a posteriori error analysis of finite volume methods

Over the last twenty years, finite volume methods have enjoyed great popularity in the computational aerodynamics community and have been widely used for the numerical simulation of transonic and supersonic flows. The basic idea behind their construction is to exploit the divergence form of the conservation laws governing the flows by integrating them over finite volumes, and converting the volume integrals into contour integrals which are then discretised.

However, despite their practical success, their theoretical foundation is still unsatisfactory and their stability and accuracy properties are not well undestood.

The aim of this lecture is to present an overview of some recent theoretical developments concerning finite volume approximations of hyperbolic problems, and parabolic problems, with dominant hyperbolic behaviour. Operating within the framework of non-conforming Petrov-Galerkin methods, we develop the analysis with particular emphasis on the cell-vertex scheme.

A key ingredient of the a priori error analysis is a discrete Gårding inequality which ensures coercivity in a generalised sense. Hence we derive optimal error bounds in various mesh-dependent norms and investigate the effects of mesh distortion on the accuracy of the scheme.

Through a decomposition of the global error into its local and propagating part, we derive residual-based local a posteriori error estimates and indicate their relevance in adaptive mesh-refinement.

Our theoretical results will be illustrated by numerical experiments.

S SHAW, M K WARBY and J R WHITEMAN

Numerical techniques for problems of quasistatic and dynamic viscoelasticity

The problem of linear viscoelastic stress analysis of solid bodies is defined in terms of the governing momentum equation and two alternative hereditary constitutive laws. Initially it is assumed that the deformation takes place under quasistatic conditions and the inertial term in the momentum equation is neglected, providing an equilibrium problem at each time level. Weak formulations of this problem are given for each of the constitutive relationships. These are (semi) discretised in time using, variously, numerical quadrature and finite difference replacements to produce the two fully discrete formulations, and error estimates for the solutions of these in the H^1-norm are stated. Two independent numerical algorithms result from the fully discrete formulations. Numerical results arising from these are illustrated for model problems and for a problem involving a nylon material. These results illustrate the range of loadings and times over which the linear viscoelastic constitutive models are valid. Some discussion of extensions to nonlinear models is given.

Finally the case where the neglecting of the acceleration term in the momentum equation is not justified, is discussed. In this context the formulation is in terms of an integrodifferential equation of either hyperbolic or parabolic type, depending on which constitutive law is adopted. For each of these numerical schemes are presented together with numerical results.

Mary F WHEELER and N HARDING

A characteristics-mixed method for modelling bioremediation in groundwater

Transport of contaminants in groundwater is frequently described by advective-diffusion equations. Numerical procedures based on the coupling of characteristic methods with Galerkin finite elements or point centered finite difference methods for treating the diffusion (Eulerian Lagrangian, modified method of characteristics (MMOC). ELLAM) have been applied with much success. One of the main advantages of these approaches is that large time steps may be employed for a given accuracy. Disadvantages of some of these schemes include difficulties of implementation in three dimensions and nonconservation.

Here we formulate and analyze in three spatial dimensions a conservative characteristics mixed algorithm for linear advection problems. In this procedure the time derivative and the advection terms are combined as a directional derivative and the spatial derivatives are treated as in a mixed finite element method. The concentration approximating space is assumed to contain piecewise discontinuous constants defined over the triangulation.

In our presentation we will discuss application to a bioremediation problem involving a three dimensional six component model. We have employed the lowest order Raviart-Thomas spaces in our numerical experiments. Concentrations and their dispersive fluxes are obtained and the latter are used to construct a higher order approximation (discontinuous trilinears) for concentration in each cell. A slope-limited scheme is employed when necessary to prevent overshoot and undershoot. The kinetics are treated by operator splitting.

ALL PAPERS IN LECTURE SESSIONS

J AALTO, J LOUHIVIRTA and J R WHITEMAN

On gradient smoothing in grids with singularity element patches

One possibility for dealing with corner singularities in the finite element analysis of boundary value problems is to use a special subgrid of curved elements, called a singularity element patch, around each corner point. This is done by introducing a suitable geometrical mapping between the parametric u,v-plane and the physical x,y-plane. In the u,v-plane the geometry of the elements is linear. In the x,y-plane the elements are distorted according to the equations of mapping.

This paper shows how typical gradient smoothing techniques can be modified so that they work properly in grids with singularity element patches. The idea of the modification is based on reasonable smoothed approximation for the gradient $\bar{g} = \nabla\phi$. Within the elements outside the singularity element patches the components $g_x \equiv \partial\phi/\partial x$ and $g_y \equiv \partial\phi/\partial y$ of the gradient are represented typically as

$$\bar{g}_x = \sum_{i=1}^{m} N_i a_{xi}\,,\quad \bar{g}_y = \sum_{i=1}^{m} N_i a_{yi}\,. \tag{1}$$

In the vicinity of the singularity point, however, the gradient components g_x and g_y are singular. Therefore it is not reasonable to use approximations (1), as such, within the elements inside a singularity element patch. Instead of these, the components $g_u \equiv \partial\phi/\partial u$ and $g_v \equiv \partial\phi/\partial v$, which are regular, are represented as

$$\bar{g}_u = \sum_{i=1}^{m} N_i a_{ui}\,,\quad \bar{g}_v = \sum_{i=1}^{m} N_i a_{vi}\,. \tag{2}$$

In terms of these the components $\bar{g}_x$ and $\bar{g}_y$ are obtained using known relations between the gradient components g_x, g_y and g_u, g_v.

Such modifications for three well known smoothing procedures: Nodal Averaging (NA), Global Least Squares smoothing (GLS) and the Zienkiewicz-Zhu Superconvergent Patch Recovery technique (SPR) are dealt with. Numerical examples of both (a) the Poisson's equation and (b) the plane elasticity problem are given.

R T ACKROYD, C R E de OLIVEIRA and A J H GODDARD

Time-dependent finite element solution of the Boltzmann equation for photon transport

A method of solving the time-dependent Boltzmann equation for particle transport in the case of material-radiation intercation is described. Spatial dependence of the solution is treated by the finite element method. A discontinuous variational formulation is employed in order to cope with the abrupt changes of the physical characteristics of the system with time. Particle scattering is treated by means of a spherical harmonic expansion for particle direction. The method is illustrated by the solution of the nonlinear equation for photon transport in a reactive atmosphere by presenting solutions for the classical Marshak thermal wave problem. The results show that the method is capable of accurately predicting the correct speed of the thermal wave even in the case of very coarse meshes.

R L ACTIS and B A SZABÓ

Stresses computed from hierarchic plate models

Stress distributions near boundaries, discontinuities and, in the case of laminated plates, at interfaces are generally very different from the stress distribution in the interior regions of structural plates. Boundary layer effects are normally present, and the problem in those regions is essentially three-dimensional. Hiearchic models for laminated plates make it possible to approximate the three-dimensional problem through the solution of two-dimensional problems without the expense of a fully three-dimensional analysis.

Higher order models can be selected in regions of local discontinuities or near boundaries, while low-order models are used in the smooth interior regions of the domain. Improved transverse stresses can be obtained for any model of the hierarchy by using an extraction technique. In particular, the integration of the equilibrium equations has shown substantial improvement over the direct computation for the transverse shear and normal stresses. Examples are presented which show that hierarchic models are capable of approximating the fully three-dimensional solution.

M AINSWORTH

A posteriori error estimation for finite element approximation of Stokes' problem

A new approach to a posteriori error estimation for linear elasticity and Stokes problems, and which is applicable to steady state Navier Stokes equations, is presented. The approach extends earlier work [1] to problems involving an incompressibility constraint. The method is valid for non-uniform, irregular and adaptively designed *h-p* meshes including cases where the polynomial degree need not be uniform.

The approach makes use of duality arguments and is based on the element residual method (ERM), in which one solves small local problems to obtain error estimators on each element in the mesh. Key features of the approach are the development of a systematic approach to the derivation of boundary conditions for the local error residual problems. Significantly, the local problem need not in fact be a local Stokes type problem, a point which is of great importance in simplifying the approximation of the local problems.

In the talk, the theoretical foundations of the method are outlined and results of its applications to several representative problems are presented.

1. M Ainsworth and J T Oden, A unified approach to a posteriori error estimation based on element residual methods. Numerische Mathematik, To appear (1992).

Boundary integral equations for the exterior acoustic problem

The boundary integral solution of the Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0\,, \quad p \in D_+ \tag{1}$$

in an unbounded domain, D_+, exterior to a smooth C^∞ boundary Γ, satisfying a general boundary condition on Γ and an appropriate radiation condition at infinity is considered. By a careful application of the Green's second theorem the unique solution of this problem can be obtained in the form

$$\phi(p) = \int_\Gamma \phi(q) \frac{\partial G_k}{\partial n_q}(p,q) d\Gamma_q - \int_\Gamma G_k(p,q) \frac{\partial \phi}{\partial n_q}(q) d\Gamma_q\,,\ p \in D_+. \tag{2}$$

In 2-D, the Green's function $G_k(p,q) = i/4\, H_0^{(1)}(k\,|\,p - q\,|\,)$. The Helmholtz integral representation formula (2) requires the Cauchy data $(\phi, \partial\phi/\partial n)$. The boundary condition on Γ gives either ϕ or $\partial\phi/\partial n$ or a relationship between the two in the case of "spring like" scatterer. We need another boundary relationship between ϕ and $\partial\phi/\partial n$. This is usually obtained by taking the limit as $p_+ \in D_+ \to p \in \Gamma$ in (2). We obtain the so-called "Surface Helmholtz Equation"

$$-\frac{1}{2}\phi(p) + \int_\Gamma \phi(q) \frac{\partial G_k}{\partial n_q}(p,q) d\Gamma_q = \int_\Gamma G_k(p,q) \frac{\partial \phi}{\partial n_q}(q) d\Gamma_q. \tag{3}$$

The operators on either side of equation (3) can be shown to be singular for a countable set of real positive values of k which correspond to the standing wave solutions of respective interior problems. The following formulation due to Burton and Miller (1971) overcomes this non-uniqueness problem and is based on coupling (3) with its derivative in the direction of the normal to Γ at p,

$$\left(-\frac{1}{2}I + M_k + i\eta N_k\right)\phi(p) = \left(L_k + i\eta\left(\frac{1}{2}I + M_k'\right)\right)\frac{\partial \phi}{\partial n}(p),\quad p \in \Gamma. \tag{4}$$

In (4), L_k and M_k are the single and double layer potentials with M_k' and N_k as their respective derivatives. In appropriate Sobolev spaces $H^r(\Gamma)$ it can be shown that L_k, M_k, $M_k' : H^r \to H^{r+1}$ are smoothing operators of order -1 whilst $N_k : H^{r+1} \to H^r$ is principally an order +1 differentiation operator.

In this paper we study the spectral properties of the four Helmholtz integral operators. The choice of the coupling parameter η and also the convergence analysis of collocation and discrete collocation solutions of the strongly elliptic pseudo-differential operator equation (4) will be discussed. We also report on several 2-grid iterative methods for our boundary element equations.

B ANDERSSON

On the modelling errors in fracture analysis of delaminated plates

Composite materials are frequently used in engineering applications. One critical failure mechanism for this class of materials is related to the delamination of sets of laminate layers. For safe design using these materials, reliable methods for prediction of growth and arrest of delamination damage are needed. Delaminations have been observed to propagate in very complex patterns. Initially circular delamination fronts may under compressive loading (so-called buckling-driven delamination) form "worm"-like shapes or "telephone cord"-shapes. The numerical models that attempt to simulate this type of delamination growth are generally based on nonlinear plate theory and fracture mechanics. The reasons for using plate models are that the delaminations appear to be "thin" and that a three-dimensional analysis is considered to be too costly. A critical question in this context is, under what circumstances do plate models yield accurate solutions?

The growth of delamination fronts is assumed to be governed by a local growth criterion of the type $f(K_I(y), K_{II}(y), K_{III}(y)) = 0$, where $K_\alpha(y)$ are the stress intensity functions and y is a coordinate along the delamination front. It is of special importance to separate the individual fracture modes α, since experiments indicate significant differences in energy release rates in opening and sliding/tearing modes. The reliability of solutions obtained from this type of numerical model depend critically on the modelling error, the discretisation error, and the extraction procedures used to derive the stress intensity functions $K_\alpha(y)$. For plate models, bending moments and shear forces are determined. With known moments and forces, the stress intensity functions are estimated using analytical solutions (split-beam solutions) for a two-dimensional domain representing the delamination-tip-region.

In the present paper, the modelling errors obtained using the Kirchoff and the Reissner-Mindlin plate models are studied. Laminates of isotropic, bi-material isotropic and orthotropic material having quadratic, elliptic, or (some) real-life shaped delamination fronts are considered. Three-dimensional almost exact reference solutions (i.e. maximum relative pointwise errors in $K_\alpha(y)$ of the order 10^{-3}) are derived using the *h-p* version of FEM and a novel computational procedure for extracting edge stress intensity functions at 3D interfacial delaminations in orthotropic laminates. The solutions for the Kirchoff and the Reissner-Mindlin plate models are derived using the *p* version of FEM. Modelling errors, thus isolated from discretization errors, are determined virtually exactly for different plate models, fracture mechanics models, material combinations, delamination shapes, and delamination/thicknesses ratios for a simple loading case.

Since boundary layers are very different for different models, so are calculated intensity functions. For delaminations with a diameter/thickness ratio of $\kappa = 20$ and simple shapes of the delamination front (i.e. quadratic or elliptic shape), the modelling errors $(K_\alpha^{3D} - K_\alpha^{Plate})/K_\alpha^{3D}$ are of the order 5-15%. For near-circular delaminations with $\kappa = 20$ and a "wavy" boundary (as sometimes observed in practice), the modelling error may be as large as 25-40%. The conclusion is that since energy release rates G_α are proportional to K_α^2, plate model solutions often do not provide reliable quantitative results when applied to real-life delaminations with complex boundaries.

P ANDERSON, J C ELLIOTT and S E P DOWKER

A finite difference method of determination of local coefficients in dental enamel and other inhomogeneous permeable media

Methods to study transport processes in teeth (composed of inhomogeneous, permeable materials) are important in the understanding of carious, destructive and repair processes. In normal and diseased enamel, and many other inhomogeneous materials, the local porosity and tortuosity, and therefore the diffusion coefficient vary with position. To measure diffusion coefficients in homogeneous permeable materials (e.g. a glass frit), a solute with high X-ray absorbance (e.g. 1 mole L^{-1} KI) is allowed to diffuse from a constant source at one surface into water contained within its pores. The concentration of KI at any point along the 1-dimensional diffusion path can be calculated from X-ray attenuation measurements. Repeated measurements along the diffusion path yield a time series of concentration-distance profiles from which the diffusion coefficient for KI can be calculated using the Crank-Nicolson (CN) algorithm for Fick's 2nd equation. the aim of this study is to extend this method to measure the local diffusion coefficient (D_{local}) in inhomogeneous materials. Experimental data was simulated for the diffusion of a solute within a permeable material of predetermined inhomogeneity using the CN algorithm with very small distance and time steps. Using this data, D_{local} was calculated at many positions using a modification of the CN algorithm. Systematic errors due to the approximations required for the finite difference method were investigated. Errors associated with intrinsic uncertainties due to X-ray photon counting statistics were studied using Monte-Carlo simulations. Improvement in accuracy by increasing count-time is constrained by the change in concentration with time. However, KI can be repeatedly diffused in and out of the permeable solid and an averaging algorithm used which effectively increases the count-time for any particular concentration determination without introducing systematic errors.

T APEL

Anisotropic finite elements and application to edge singularities

The classical local interpolation error estimates (see e.g. Ciarlet 1978) were derived under an assumption which is in two dimensions known as Zlámal's minimal angle condition. This condition was weakened by different authors (Jamet 1976, Babuška/Aziz 1976, Křížek 1989) to a maximal angle condition. But the possible advantage of using mesh sizes with different asymptotics in different directions which leads to small angles, was not extracted.

In this paper, anisotropic interpolation error estimates in two and three dimensions, proved by Dobrowolski and Apel (1992) are reviewed. Here, one derives benefit from the different asymptotics of the mesh sizes.

The results are applied to the finite element approximation of elliptic equations on domains with edges. Practical computations show that the proposed finite element meshes are well suited for elliptic problems with solution of r^{λ}-type.

D N ARNOLD

The boundary layer for the Reissner-Mindlin plate model

The solution to the Reissner-Mindlin plate model, in contrast to that of the biharmonic model, exhibits a complex dependence on the thickness of the plate. For thin plates there may be a boundary layer, the existence and structure of which depends on the boundary conditions, the plate geometry, and the solution component considered. For example, the transverse displacement variable does not exhibit any edge effect, but the rotation vector exhibits a boundary layer except for very special data or plate geometry. The bending moment tensor and shear force vector have more pronounced boundary layers. The boundary layer is stronger for a free or simply-supported plate than for a clamped one. In this talk we describe recent results which furnish a complete asymptotic description of the dependence of the solution on plate thickness. The techniques used to obtain the asymptotic expansions and rigorous error bounds for them are described and illustrated by concrete examples. Applications to finite element analysis are presented.

A AUGE and G LUBE

A stablized Galerkin finite element method for solving the Boussinesq equations

We consider the Boussinesq approximation of the stationary, incompressible and non-isothermal Navier-Stokes equations. Spurious numerical solutions of standard mixed Galerkin finite element formulations may be generated by

(a) inappropriate combinations of velocity/pressure interpolation functions and/or
(b) the presence of (locally) dominating convective terms.

As a remedy we extend the least-squares stabilization approach [FF] to the Boussinesq equations. In particular, we add weighted residuals of the basic equations to the Galerkin discretization in order to stabilize the method and to allow arbitrary finite element approximations.

For a linearized problem of Oseen type, we prove asymptotic error estimates on a non-uniform mesh and give an explanation of the parameter design which is frequently used in CFD. As a basic tool we take advantage of a modified Babuska-Brezzi condition. Furthermore, we extend existence and convergence results to the case of regular solutions of the original problem using Banach's fixed point theorem [LA]

We conclude with numerical results for some benchmark problems.

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S AZIMI

A new elemental formulation for the dynamic analysis of structures

Use of a new "Dynamic Mode Shape Function" (DMSF) based on the dynamic behaviour of every structural element is under study. Using this DMSF the new elemental stiffness and mass matrices are obtained. The solution of the problem for the free vibration results in exact frequencies and mode shapes of the structures. As an application the method has been applied to a bar finite element. Some examples of vibration of bar systems have been considered for clarification and the results are compared with the previous published results.

C BAILEY, M CROSS and P CHOW

A finite volume procedure to predict deformation and residual stress in castings

The foundry/casting industry plays a major part within the manufacturing base of the United Kingdom. For a new cast design the present practice is to undertake numerous trials and tests to find the optimum design options. This practice is very wasteful in both materials and lost man hours.

During the casting process hot liquid metal is poured into a mould. Once the mould is filled with liquid metal, the casting together with the running system and feeders will cool and solidify. The development of a code to completely simulate all the macroscopic processes involved in castings will greatly aid the foundry engineer in reducing lead times and producing sound castings. The developed code must include the following types of analysis:

1) Fluid flow analysis for mould filling.
2) Thermal flow analysis of solidification and the evolution of latent heat.
3) Fluid Flow analysis to represent thermal convection.
4) Deformation, hence stress analysis to predict geometrical soundness of cast.
5) Macroscopic porosity formation.

There is a high degree of coupling between the above phenomena, for example the convective currents are dependent on the temperature changes. These in turn are dependent on the geometric deformation as heat losses are restricted across the cast/mould interface due to gap formation. Obviously it would be of great benefit to solve all the above using a single code.

The route taken in this research is to discretise the conservation equations, representing the above phenomena, using Finite Volume procedures. This presentaiton will concentrate on the methods used for deformation and stress and how these methods have been coupled to the flow and solidification calculations to simulate some simple castings.

M J BAINES

Adaptive methods for grid generation and finite element solutions of time-dependent partial differential equations

A direct variational approach has been used to generate algorithms which determine optimal discontinuous piecewise linear and constant L_2 fits with adjustable nodes to a continuous function [1]. The process generates data-dependent grids. The algorithms are fast and robust, the mesh cannot tangle, and convergence can be proved under certain conditions.

It can be shown that one iterate of such an algorithm corresponds to an explicit time step of the Moving Finite Element (MFE) procedure for the solution of a particular time-dependent differential equation [2]. When the MFE procedure is run to steady state for this equation it follows a path leading to the best fit, with adjustable nodes, of the steady state operator to the zero function. The result generalises to a wider class of differential equations and operators.

For first order partial differential equations it is known [3] that the MFE procedure approximates the method of characteristic strips. The above result shows that there also exists a superimposed speed component arising from the L_2 projection which is "best fit seeking".

An alternative moving grid strategy is proposed based on a two pass method [4]. The first pass is a fixed grid predictor: this is followed by one or two iterations of the best fit algorithm with adjustable nodes. A second pass of the solver is then made on the moving grid, the motion being provided by the above node adjustment.

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A BALOCH, P TOWNSEND and M F WEBSTER

Viscoelastic effects in expansion flows

In this work we investigate expansion flows in both two-dimensional and three-dimensional domains. We are particularly interested in comparing numerical solutions for Newtonian and viscoelastic fluids over a range of inertial and elasticity values. A comparison is made of our numerical solutions with experimental results, and some conclusions are reached with respect to vortex enhancement, and presence and origin of vortices. For the two-dimensional study a 40:3 ratio geometry is considered in line with the equivalent experiments and solutions are tested for accuracy and convergence on more than one mesh. In the three-dimensional case the 40:3 expansion ratio is extended into the third dimension to provide a 40:3:3 geometry and both Newtonian and viscoelastic solutions are computed.

The numerical scheme is a time-stepping Taylor-Galerkin finite element method that involves fractional stages at each time step through the additional consideration of a pressure-correction scheme. Such an algorithmic approach is implemented in a *semi-implicit* fashion, and is accurate and robust when solving Newtonian flows. However, for the viscoelastic regime more sophistication is required to overcome nonlinear instabilities that arise. This necessitates the further consideration of a consistent streamline upwind weighted residual approach. Also the constitutive models are selected from a generalised class of Phan-Thien/Tanner fluids which exhibit shear-thinning and a finite extensional viscosity. Comments are made as to the suitability of these particular choices.

A BALOCH, P TOWNSEND and M F WEBSTER

Influence of Extensional Viscosity on Vortex Development in Complex Geometries

In this study we are concerned with the influence of strain thickening and inertia upon the steady laminar incompressible flow of non-Newtonian fluids. We concentrate on flows through circular contraction geometries and chose to work at contraction ratios of four to one. A generalised Newtonian material description is adopted with an extensional viscosity depending on second and third invariants of the rate of deformation tensor. Particular attention is paid to the effect of the abrupt reentrant corner by introducing in contrast a rounded corner geometry. Under such circumstances this work casts light on the formation of vortices, their origin and subsequent growth with increase in flow rate and elasticity number. Both salient and lip vortices are observed under different conditions.

The numerical scheme employed in the simulations is a semi-implicit time-stepping procedure that combines the merits of a Taylor-Galerkin weighted residual method with those of a pressure-correction method. The resulting procedure is a fractional staged scheme taken over each time step, that has already proved its flexibility in solving a range of complex incompressible viscous flows. The required extensional behaviour is incorporated through an appropriate functional form for the viscosity. Our numerical solutions are compared with other numerical predictions in the literature, and with some experimental data.

M BERNADOU

On the approximation of junctions between thin shells by curved C^1-finite element methods

Many thin plate problems are formulated on curved boundary plane domains. Likewise, by using a mapping technique, thin shell problems can be formulated on such curved boundary plane domains. Such mapping techniques can be extended to the junction between thin shells. This means that for all these cases, we need to approximate a fourth order problem set on curved boundary plane domains. These approximations by conforming finite element methods require the use of curved C_1-finite elements.

In this talk, we will present some results obtained in the analysis of curved C^1-finite elements and on their applications to solve the above mentioned problems.

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4. Bernadou, M and Boisserie, J M: Curved finite elements of class C^1: implementation and numerical experiments. Part 1: Construction and numerical tests on the interpolation properties, (to appear in Comp. Meth. Appl. Mech. Engng.).
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M BERZINS and J WARE

Reliable finite volume methods for time-dependent PDEs

In the numerical solution of time-dependent pde's the accuracy is influenced by the spatial discretisation method used, the spatial mesh and the method of time integration. In particular the spatial discretisation method and position of the spatial mesh points should ensure that the spatial error is controlled to meet the user's requirements. It is then desirable to integrate the o.d.e. system in time with sufficient accuracy so that the temporal error does not corrupt the spatial accuracy or the reliability of the spatial error estimates. However, in most existing methods for time dependent p.d.e.'s either a stability control and a specialised time integration method is employed or an o.d.e. solver is used which makes use of local time error control that is unrelated to the spatial error.

This paper is concerned with combining the spatial and temporal error balancing approach of Berzins, [5], with the adaptive mesh algorithms of Berzins et al. [1], [3]. This combination of error control strategies provides an algorithm that adapts the space mesh in order that the spatial error is controlled and then, accordingly, adjusts the local error tolerance used in the o.d.e. integrator so that the two errors are related in some way. This not only ensures that the main error present is that due to spatial discretisation but offers the tantalising possibility of overall error control.

In the error balancing procedure the local time error is controlled so that it is a fraction of the growth in the spatial error over the timestep. An analysis is presented to show that although the method attempts to control accuracy a Courant-like stability condition is also satisfied. Numerical experiments show that the error control strategy appears to offer an effective method of balancing the spatial and temporal errors.

The unstructured triangular mesh spatial discretisation method, see [1] and [4], is a cell-centered, finite volume scheme that is shown to achieve second-order accuracy by use of a six triangle stencil around each edge. The growth of the spatial error is measured locally in time by using the computed solution as the high-order solution and constructing an error estimate by calculating a computationally inexpensive low-order solution. Thus local extrapolation in space is effectively being used. These spatial error estimates are shown experimentally to mimic the behaviour of the global spatial error. Adaptive unstructured mesh techniques e.g. [1] or[3], are used to rezone the mesh at times chosen by the time integration algorithm. Selection of appropriate times is made by using a combination of present estimated errors and predicted future errors. The time integration technique used in this paper is a new adaptive Theta Method, [2], suitably modified to use the new error control approach.

The implementation of the algorithm raises a number of issues. For instance, since the accuracy tolerance for the time integration over the next time step after spatial remeshing depends partially on the error incurred prior to spatial remeshing, this tolerance must be modified according to the expected reduction in the spatial discretisation error.

The present algorithm thus goes some way towards providing a reliable solution method for two-dimensional time-dependent flows. The numerical results show that this can be achieved for both simple convection type problems and for Euler flows in two space dimensions.

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I BOCK and J LOVISEK

On the discretization of an optimal design problem for an unilaterally supported viscoelastic plate

The discretizaiton of an optimal design problem for a visco-elastic plate with respect to its variable thickness is solved. The middle surface of the plate is identified with an open bounded plane domain G with a Lipschitz boundary $\partial G = \Gamma_u \cup \Gamma_K$. The plate is clamped on Γ_u and unilaterally supported on Γ_K. Let $K \subset H^2(\Omega)$ be the closed convex set of admissible deflections. The deflection $w(e,t,x)$ is a solution of the initial-boundary value problem for a pseudoparabolic variational inequality.

$$w(e,t) \in K, \quad t \in [0,T] \tag{1}$$

$$\int_G \int e^3(x)\left[A_{ijkl}^{(0)} \frac{\partial}{\partial t} w_{,ij} + A_{ijkl}^{(1)} w_{,ij}\right] [v - w(e,t,x)]dx$$
$$\geq \langle f(e,t), v - w(e,t)\rangle \quad \text{for all} \quad v \in K, \; t \in [0,T] \tag{2}$$

$$w(e,0) = w_0(e) \in K \,. \tag{3}$$

We connect with (1), (2), (3) the design problem

$$\|Dw(\bar{e},T) - z_d\|_X^2 + j(\bar{e}) = \min_{e \in U_{ad}} \left\{\|Dw(e,T) - z_d\|_X^2 + j(e)\right\}, \tag{4}$$

where $U_{ad} \subset C^{0,1}(\bar{G})$ is the admissible set of thickness-functions, X is any Hilbert space, $D : X \to H^2(G)$ is a linear bounded operator. Let $K_{\varkappa}$ and $U_{\varkappa}^{ad}$ be finite element approximations of the sets K and U_{ad} respectively, $d = T/N$. Using finite differences with the step d with respect to t we approximate the problem (1) - (4) by N elliptic optimal control problems with admissible deflections and thickness-functions from K_h and U_{ad} respectively.

L BRUSA and F RICCIO

A hybrid direct-iterative method for the solution of finite element linear systems with positive definite matrices

The major cost of many finite element analyses arises from the solution of large sparse sets of linear equations which can often be performed only by using vector and parallel computers. Direct methods, based on Gaussian elimination, are efficient but they can easily require prohibitively large amounts of storage, which hamper their practical application for complex engineering analyses. This drawback is avoided by using iterative schemes, mainly the conjugate gradient method, but their performance is often unsatisfactory because of the large number of iterations required for convergence, due to the ill-conditioning of the matrices.

Hybrid methods, based on the use both of direct and iterative techniques, have been proposed in an attempt to exploit the advantages of the two approaches and to mitigate their drawbacks. The starting point of the methods considered in the paper is the substructure technique which arranges the elements in groups (or substructures) so that two classes of variables can be defined, i.e. the internal veriables relevant to nodes within substructures, and the interface variables relevant to nodes belonging to two or more substructures.

The internal variables are decoupled and are eliminated by applying a direct method to factorize the related matrices. The reduced system obtained after this step involves only the *interface variables and is* solved by means of the preconditioned conjugate gradient method (PCGM). In this way the efficiency of Gaussian elimination is exploited in the solution of medium-sized problems, while PCGM, applied to solve the reduced system, deals with a matrix which is better conditioned than the original one. This feature of the reduced system, as far as we know, has been experimentally verified and demonstrated only for special matrices.

The paper presents a proof that the condition number of the Schur complement (the matrix of the reduced system) of a positive definite matrix is lower than the one relevant to the global matrix. On the basis of this result, indications are given to build preconditioners for the reduced system, starting from those defined for the original problem. Some of these preconditioners are tested for the solution of structural mechanics problems and the performances are compared with those obtained by applying PCGM to the complete problem. The numerical experimentation has been performed on the parallel shared memory multiprocessor ALLIAN FX/80 and future work will concern the implementation of the method on a distributed memory hardware platform.

T BULENDA

Hybrid Arnoldi-Newton Algorithm

Applying a predictor-corrector scheme to the augmented set of nonlinear equilibrium equations in statics $\mathbf{G}(\mathbf{u},\lambda) = \mathbf{0}$, we get for an inexact Newton-Raphson iteration in the corrector phase (superscripts stand for the iteration number):

$$\begin{bmatrix} {}^{i}K_T & -P \\ {}^{i}f_u^T & {}^{i}f_\lambda \end{bmatrix} \begin{bmatrix} {}^{i+1}\Delta u \\ {}^{i+1}\Delta\lambda \end{bmatrix} = \begin{bmatrix} -{}^{i}R = -F({}^{i}u) + {}^{i}\lambda P \\ -f({}^{i}u, {}^{i}\lambda) \end{bmatrix} + {}^{i}r = -G + {}^{i}r \text{ with } \frac{|{}^{i}r|}{|G|} \leq {}^{i}\delta$$

(1)

with the tangential stiffness matrix ${}^{i}\mathbf{K}_T = {}^{i}(\partial\mathbf{R}/\partial\mathbf{u})$, the residuum vector $\mathbf{R}$, the vector of inner forces $\mathbf{F}$ (dependent on the vector of displacements $\mathbf{u}$), the reference load vector $\mathbf{P}$ and the scaling factor for the loads λ. The equation $\mathbf{f} = 0$ describes the surface of iteration and ${}^{i}\mathbf{f}_u = {}^{i}(\partial\mathbf{f}/\partial\mathbf{u})$, ${}^{i}\mathbf{f}_\lambda = (\partial\mathbf{f}/\partial\lambda)$. The inexact Newton method requires some coupling of the linear equation solver and the Newton iteration. This is achieved via the term ${}^{i}\delta$ which sets the residuum ${}^{i}\mathbf{r}$ of the linear equation in relation to that of the Newton process ${}^{i}\mathbf{G}$. The value of ${}^{i}\delta$ is chosen as a forcing sequence. In the present work we use: ${}^{i}\delta = \min(10^{-i}, 0.1 \, \| {}^{i}\mathbf{G} \| / {}^{i}\lambda)$.

To get an approximate solution of the linear equations, we need a proper equation solver. In this work the Arnoldi algorithm is used. It can be regarded as a generalization of the well known Lanczos algorithm to unsymmetric system matrices. The solution of our linear system is thus reduced to the repeated computation of (system-matrix)-vector products. Via preconditioning and reorthogonalization respectively the algorithm's performance and stability is improved. As the Arnoldi algorithm can handle unsymmetric matrices, we solve directly the augmented system (1) instead of using the widely known partitioning method of Batoz and Dhatt (1979).

The results of our computations show great savings in the total number of Arnoldi iterations we get by working with the augmented system instead of the partitioning method. Whether this is a saving in computer time also, depends on the structure of K_T and the cost relation for building the matrix-vector product in each step of the Arnoldi process and performing one Arnoldi step. If K_T is unsymmetric, the augmented system is favourable anyway. Then the additional effort for solving an $n+1$-dimensional system can be neglected compared with solving a second n-dimensional system when using the partitioning method. If K_T is symmetric, the decision, which way to choose, depends on the above mentioned cost relation. The higher the cost for the matrix-vector product, the more favourable is the augmented system. For any strategy the inexact Newton method additionally produces great savings.

M BUONSANTI

A FEM Approach for the Analysis of Composite Elements

The study of the behaviour under load of heterogeneously built solids is treated by analysing the contact area of two different constituent details. The problem of unilateral contact with or without friction as well as the deformative adaptation that results from the application of the loads is studied. The disjunction among elements, both in the particular and bi-dimensional case is probed. In each case we suggest a FEM model for numerical applications.

P BURDA

Cubic finite elements in elliptic problems with interfaces and singularities

In applications of the finite element method, essentially two approaches are most widely used. First one makes use of some kind of refinement of the mesh near the singularity. Second one uses additional trial functions in the neighbourhood of the singularity. In our paper we give an alternative to these techniques, based on using cubic polynomials in the FEM.

P CARTRAUD, C WIELGOSZ and O DEBORDES

Computation of the homogenized behaviour of a gasket

The present study is devoted to the determination of mechanical behaviour of a cylinder-head gasket. We consider here a gasket made of an heterogeneous material, with an iron core (perforated sheet) and a paper (non asbestos material) which is clamped by rolling on both sides of the sheet. This gasket exhibits an elastoplastic behaviour and its effective properties are computed by an homogenzation technique.

The homogenization method implies a good description of the microstructure, and especially the characteristics of the gasket's components. At this stage, the main difficulty is to model the paper behaviour. This behaviour, displayed with compression tests, looks like that of soil.

We therefore use an elastoplastic model, including non-linear and anisotropic elasticity, plasticity with two yield surfaces and isotropic hardening. One homogeneous test is used to determine the material parameters based on an analytical solution. We then compute the numerical solution of another compression test, with the finite element code SIC (Université Technologique de Compiègne), which incorporates specific developments for the numerical integration of elastoplastic constitutive laws for geomaterials. We obtain good agreement between the model and experimental results.

One can then compute the three dimensional homogenized behaviour of the gasket. We first make use of the planar periodicity of the gasket material (due to the sheet perforation) to reduce the study to that of a representative volume element (r.v.e.). The homogenized behaviour of the gasket is then obtained from numerical simulation of elementary mechanical tests on a finite element model of this r.v.e. Here again we use the finite element code, SIC, in which a specific module of homogenization has been developed. In practice, the r.v.e. is submitted to an average strain, increasing in time, and we compute the associated average stresses. The effective properties of the gasket define the relations between the averages of strains and stresses. One can thus construct the elastoplastic evolution law of the gasket, by considering successive strains in different directions. These computations compare well with experimental tests on the gasket.

These results will enable us to incorporate the gasket behaviour more effectively into the finite element computation of an engine.

J C CAVENDISH, C A HALL and T A PORSCHING

A complementary volume approach for modelling three-dimensional Navier-Stokes equations using dual Delaunay/Voronoi tessellations

In this paper we described a new mathematical approach to deriving and solving co-volume models of three-dimensional, incompressible Navier-Stokes flow equations. The approach integrates three technical components into a single modelling algorithm:

1. *Automatic Grid Generation.* An algorithm previously used for three-dimensional finite element mesh generation applications is used to automatically discretize the flow domain into a Delaunay tetrahedronization and a Voronoi tessellation. These two geometric constructs provide 3-D complementary volume discretizations.
2. *Co-volume Equation Generation.* Using the tetrahedronization and dual tessellation, a finite volume method is applied to derive a system of simple, second-order convergent, finite difference equations to model velocity and pressure dependent variables satisfying the Navier-Stokes equations.
3. *Dual Variable Reduction.* A graph theoretic analysis technique is used to transform the finite difference system into an equivalent banded system which is considerably smaller than the finite difference system. Unlike most Navier-Stokes solution algorithms, the velocity field is *a priori* discretely divergence free.

We display a nonsymmetric flow duct of rectangular cross section in which flow enters through one inlet port and exits through two outlets. A Delaunay tetrahedronization of the flow domain (boundary faces only) into 1320 tetrahedra is produced and the dual Voronoi tessellation which contains 542 convex Voronoi polytopes is obtained. The derived covolume difference system (which we call the *primitive system*) is 6003 x 6003 while the dual variable system is approximately one third as large. Numerical calculations will be compared with flow visualization results for water flow through the (transparent plastic) duct.

P CHAUSSECOURTE, B MÉTIVET and G NICOLAS

Adaptive mesh refinement for the 3D magnetic code TRIFOU

For some years, we have been working to develop TRIFOU, a finite element code devoted to 3D magnetic field problems. One application is the modelling of eddy current non destructive testing. In this problem, we must build a mesh for cracked bodies. This should be done carefully: an accurate representation of the induced current around the crack is necessary to achieve a good representation of the control signals. But forecasting the shape of the currents and building an appropriate mesh are often difficult. Therefore, we decided to implement an adaptive mesh refinement strategy in TRIFOU.

The Maxwell equations for time periodic situations lead to solving the following problem: find the complex magnetic field h satisfying $i\,\omega\,\mu\,h + \mathrm{curl}\,(\mathrm{curl}(h)/\sigma) = 0$, where ω, μ and σ are calculation parameters. The variational formulation of this problem can be seen as: find h in $H = \{h \in \mathbb{L}^2(E), \mathrm{curl}(h) \in \mathbb{L}^2(E)\}$ such that $a(h,h') = \langle b,h' \rangle$, for all h' in H, where b belongs to the dual of H. The approximate solution, h_k, is sought in the finite dimensional space, H_k, which is connected to the mesh, M_k. We can define the residual of the discrete formulation, R_k, by: $\langle R_k,h' \rangle = \langle b,h' \rangle - a(h_k,h')$, for all h' in H. The natural norm of R_k is given by the maximum of $| < R_k,h' > |$ for all h' in the unit sphere of H. This value is a norm of the global error, but its computation is difficult. Moreover, this value does not allow us to say where and how the mesh should be refined. Nevertheless, we shall use this idea for our adaptive mesh refinement, which can be seen as an optimisation algorithm: the successive meshes are built up in order to minimise this residual.

The mesh M_{k+1} is obtained by refining some carefully chosen areas of the mesh M_k. We define a unit refinement as the lowest increase in the approximation space H_k. An adaptive mesh iteration is divided into two stages. First, we test all the possible unit refinements of the mesh: for each of these, we enrich the approximation space H_k temporarily with the few new functions related to this virtual refinement; we evaluate the efficiency of this. In the second stage of the process, we begin by selecting the most efficient unit refinements. We then build up the H_{k+1} space, adding all the selected unit refinements to H_k.

To evaluate the efficiency of a virtual unit refinement, we consider the highest value of $| \langle R_k,h' \rangle | / \| h' \|$ for h' in the set of the new functions. We have to solve a small linear system; its rank is approximately the number of degrees of freedom which are introduced by the unit refinement.

The results of the first applications of this technique are promising, and the refinement occurs in the immediate surroundings of the cracks or the singularities.

H CHEN

Richardson extrapolation for the streamline diffusion finite element method

The Richardson extrapolation is a classical technique for increasing the accuracy in numerical analysis and has been applied to the standard finite element method, c.f. Blum, Lin and Rannacher [1]. The purpose of this paper is the study of extrapolation methods for the streamline diffusion finite element method (the SD method) that has been developed by Hughes and Brooks [2] for solving convection-dominated convection-diffusion problems numerically. The theoretical analysis of the SD method has been started by Johnson and Nävert in [3]. Global and local error estimates in L^2-norm of order $O(h^{k+1/2})$ have been derived. These estimates are optimal for general quasi-uniform meshes. In Johnson, Schatz and Wahlbin [4] the pointwise error behaviour was analysed for the linear finite element and an error estimate of order $O(h^{5/4} \mid \log h \mid^{3/2})$ was proved. An improved error estimate of order $O(h^{11/8} \mid \log h \mid)$ has been obtained in Niijima [6]. Up to now, this estimate is the best pointwise error estimate. In this paper we derive error asymptotic expansions with remainders of order up to $O(h^{3+1/2} \mid \log h \mid^{1/2})$ for the SD method. These are the theoretical approach of the extrapolation methods for increasing accuracy. Our methods require that the exact solution is sufficiently locally smooth. In the meantime, a quasi-optimal pointwise error estimate of order $O(h^{2/3} \mid \log h \mid^{1/2})$ will be derived for locally uniform meshes by making use of the known superconvergence results for the finite element interpolation.

As a model problem, let us consider the following boundary value problem:

$$\begin{cases} -\rho u_{xx} - \epsilon u_{yy} + u_x + u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & , \end{cases} \tag{1}$$

where $\Omega \in R^2$ is a bounded smooth domain and ρ, ε are small positive parameters.

Let $S_h \subset H_0^1(\Omega)$ be a piecewise linear finite element space constructed on a quasi-uniform mesh T_h. The so-called streamline diffusion finite element approximation $u^h \in S_h$ to the problem (1) is determined through the relation

$$B_h(u^h,\chi) = (f,\chi + \rho_h \chi_x) \,, \quad \forall\, \chi \in S_h \,, \tag{2}$$

where

$$\begin{aligned} B_h(u^h,\chi) &= \rho(u_x^h,\chi_x) + \epsilon_h(u_y^h,\chi_y) + (u_x^h + u^h + \rho_h \chi_x) \\ &= (\rho_h + \rho)(u_x^h,\chi_x) + \epsilon_h(u_y^h,\chi_y) + (1-\rho_h)(u_x^h,\chi) + (u^h,\chi) \end{aligned}$$

and $\rho_h = ch$, $\varepsilon_h = ch^{3/2}$. For simplicity, we assume that $\varepsilon \le \varepsilon_h$, $\rho \le \rho_h$.

Then we will derive the following error asymptotic expansions

$$u^h = u + \epsilon_{hE_1} + \rho_h \rho E_2 + h^2 E_3 + \epsilon_h \rho_{hE_4} + \epsilon_h^2 E_5 + \rho_h^2 \rho E_6 + h^2 \rho_{hE_7} + O(\epsilon + h^{3+1/2} |\log h|^{1/2}).$$

Based on this expansion, we design some extrapolation methods to gain higher accuracy.

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S H CHEN

A new model of jointed rock masses reinforced by passive, fully-grouted bolts

Rock bolts have been used as general reinforcement measure to improve the strength of jointed rock masses in recent decades. Several useful models have been proposed by some researchers.

This paper presents a new model for the jointed rock masses reinforced by passive, fully-grouted bolts. The model is based on an "equivalent continuum" approach. On one hand, the increment of load is shared among the bolts, rock materials, and joints respectively. On the other hand, the strain increments of the reinforced jointed rock masses are given as the sum of the increment strains of the reinforced rock materials and reinforced joints. The proposed model has an advantage in comparison to previous "equivalent continuum" models - it can calculate the bolt stresses separately in joints and rock materials. Because their deformations are different, this is a more reasonable approach to the real work state of bolts in the jointed rock masses. A computer program of the Finite Element Method has been compiled, and a typical case study of the rock slope stability problem has been made.

Z CHEN, B COCKBURN, J W JEROME and C-W SHU

Runge-Kutta local projection discontinuous Galerkin methods for conservation laws and applications to the hydrodynamic device model

In this paper we introduce and analyze a class of nonlinearly stable Runge-Kutta local projection discontinuous Galerkin (RKDG) finite element methods for the hyperbolic conservation law

$$u_t + \operatorname{div} f(u) = 0, \quad \text{in } (0,T) \times \Omega \tag{1.1}$$

with suitable initial-boundary conditions, where $\Omega \subset \mathbb{R}^d$, $u = (u_1, \ldots, u_m)$, and f is such that any real combination of the Jacobian matrices $\sum_{\ell=1}^{d} \xi_\ell \frac{\partial f_\ell}{\partial u}$ has m real eigenvalues and a complete set of eigenvectors. The main difficulty in solving (1.1) is that solutions may contain discontinuities even if the initial-boundary data are smooth. Among the successful numerical methods for solving (1.1), we mention the nonoscillatory conservative finite difference methods such as total variation diminishing, total variation bounded, and essentially nonoscillatory schemes, and the nonstandard finite element methods such as streamline diffusion and characteristic Galerkin approaches. One distinctive feature of the methods under consideration is a local projection limiting, which comes from the successful nonoscillatory finite difference methodology, guarantees total variation boundedness for one-dimensional nonlinear scalar equations and linear systems, and yields a local maximum principle for multi-dimensional scalar equations. The main advantages of these methods, over most finite difference methods, are that they can easily handle complicated geometries and boundary conditions, and that, over most other finite element methods, they use high order total variation diminishing Runge-Kutta type time discretizations, which renders them explicit, fully parallelizable, and computationally efficient. In this paper, the analysis of the RKDG methods defined on general triangulations is carried out for the two-dimensional system (1.1), $d = 2$ and $m > 1$. It is shown that these methods satisfy local maximum principles, are uniformly high-order accurate, and converge to a weak solution.

After the analysis above, the application of the RKDGR methods to the hydrodynamic model for semiconductor devices is presented:

$$\begin{aligned} n_t + \operatorname{div}(nv) &= 0, \\ p_t + v \operatorname{div} p + p \cdot \nabla v &= -enE - \nabla(nT) + (p_t)_c \\ w_t + \operatorname{div}(vw) &= env \cdot E - \operatorname{div}(vnT) + \nabla T) + (wn)_c, \end{aligned}$$

where n is the electron density, v is the velocity, p is the momentum density, $e(> 0)$ is the electronic charge, E is the electric field, T is the temperature, w is the energy density, k is the heat conduction coefficient, and the index c indicates collision terms. It is assumed that the energy bands are parabolic as

$$\mathbf{p} = mn\mathbf{v},\ w = \frac{3}{2}nT + \frac{1}{2}mn|\mathbf{v}|^2 ,$$

where m is the effective electron mass. The electric field satisfies the Poisson's equation

$$\mathrm{div}(\epsilon\nabla\phi) = -e(N_D - N_A - n),\ E = -\nabla\phi$$

where N_D and N_A are the densities of donors and acceptors, respectively. This model defines the variables n, $\mathbf{p}$, w, and ϕ, treats electron flow in a semiconductor device via the Euler equations of gas dynamics, with the addition of a heat conduction term, and is a coupled system of hyperbolic, parabolic, and elliptic equations. The nonlinear hyperbolic equations support shock waves.

Mixed finite element methods are used for the approximation of the electric field. The reason for including the mixed methods is that the transport equations depend on the potential only through this field and these methods provide better approximation of it than more standard Galerkin approaches would give. After the Lagrange multipliers are introduced for the mixed methods, the scheme, a combination of the RKDG and mixed methods, if fully parallelizable. Extensive numerical simulations have been carried out on the Cray-2 at the Minnesota Supercomputer Center and the Army High Performance Computing Research Center in USA.

E R CHRISTENSEN

Structural dynamic finite element analyses of space shuttle propulsion components

This paper describes three finite element structural dynamic analyses of the space shuttle main engine and solid rocket boosters which have recently been conducted for the NASA Marshall Space Flight Center in Huntsville, Alabama, USA, in support of the Advanced Turbopump Development (ATD) and Advanced Solid Rocket Motor (ASRM) programs. These analyses were performed using the Engineering Analysis Language (EAL) and COSMIC NASTRAN finite element codes. In the first analysis, three dimensional, solid finite element models (FEM) of the ATD fuel pump and LOX pump first stage turbine blades were constructed and analyzed using EAL. Since modal analysis results indicated possible resonance conditions, the transient response of the blades due to the operating pressure loads was calculated. Several transient analyses were conducted for various operating conditions with the results indicating the possibility of fatigue life problems. The second analysis involved an investigation of the fluid-structure interaction between liquid hydrogen fuel and toroidal shaped inlet structure on the high pressure fuel turbopump. A FEM representing both the fluid and the structure was constructed and a modal analysis was performed. The results indicated the presence of a large number of "spurious" fluid modes occurring at nonzero frequencies. These spurious modes are a direct result of the displacement formulation of the EAl fluid elements which does not impose the constraint of irrotationality on the fluid. When these constraints were enforced through the use of a global penalty function, however, it was found that the spurious modes could be removed from the solution. The final analysis involved determining the effect of internal pressure on the dynamic characteristics of the ASTM which is currently being developed as a replacement for the solid rocket motors presently used on the shuttle. Several new NASTRAN elements have been developed to model the pressure stiffness which is a nonlinear effect due to the change in direction of the pressure force as the structure deforms. Neglecting this effect can lead to errors of up to 48% in the computed frequency of the pitch/roll mode. Prediced frequencies obtained using the new NASTRAN elements, however, agree with modal test results to within 5%.

R K COOMER and I G GRAHAM

Domain decomposition techniques for massively parallel machines

In this talk we describe massively parallel domain decomposition methods for solving large ill-conditioned linear systems arising from discretisations of symmetric elliptic PDEs. The method has particular power when the ill-conditioning is made worse by the presence of discontinuous coefficients in the PDE. Methods based on mapping subdomains to an array of processors and local elimination of interior unknowns are reviewed. The resulting Schur complement system is solved by the preconditioned conjugate gradient method with preconditioner (in 2D) based on local solves in vertex and/or edge spaces, plus a coarse grid solve to simulate global interaction of nodes. This is an "additive Schwarz" type method, and its analysis has been recently investigated by M Dryja, O Widlund and B Smith. It is explained why this method yields an optimal algorithm (i.e. its convergence rate is independent of the number of degrees of freedom), but in general its convergence rate is affected by the jumps of the coefficients of the PDE across subdomain boundaries. On the other hand, neglecting the vertex solves yields a weakly suboptimal algorithm but with convergence rate unaffected by these jumps. The method is implemented for a class of model problems on the MasPar MP-1, a SIMD machine with 1024 processors using inner iterations to solve the preconditioning problems. Numerical results which support the theoretical properties of the algorithms are reported.

A CRAIG

p-version preconditioning for the mass matrix

The p-version of the finite element method has become more popular over the last few years. The increase in popularity is due to several factors; including the greater return in accuracy for problems with singularities on the mesh and the ease of design of p-adaptive codes. Here, instead of refining the mesh, we increase the accuracy by using higher degree basis functions.

There is an observed problem which causes computational difficulties: the discrete operator from the finite element method is highly ill-conditioned.

Widlund, Pasciak *et al.* and Babuška *et al.*, to name but a few, have all given experimental results which reduce the order of the condition number for h and p-versions of the finite element stiffness matrix from polynomial to log-squared in the number of degrees of freedom; also, numerical results suggest a similar method for the hybrid *h-p*-version.

An additional problem arises in the *p*-version. We observe that hierarchical bases, while being very natural bases for the *p*-version stiffness matrix, are very unnatural bases for the mass matrix. This can be seen by noting that the growth in the condition number is exponential in p, the degree of the polynomial on each element.

We often need to solve problems using the mass matrix, in particular, time-dependent problems, but with our adaptive codes and the use of hierarchical bases, we observe that we cannot use mass-lumping to remove our ill-conditioning.

Using methods derived from those for the stiffness matrix, we shall show that numerically, and in one case analytically, we can control this ill-conditioning at little expense.

J DALIK

An approximation of the two-dimensional convection-diffusion problem with dominating convection

Let us consider the following singularly perturbed boundary value problem:

$$\begin{aligned} -\epsilon\Delta u + p\nabla u &= f(x,y) \quad \text{in} \quad \Omega , \\ u &= u^0 \quad \text{on} \quad \partial\Omega . \end{aligned} \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ and ε, $p = (p_1 , p_2)$ are constant functions such that $0 < \varepsilon << |p|$.

We present a numerical method relating an approximate solution u_h of (1) to any triangulation T_h of Ω with the discretization step h greater than ε . This method can be described as a variant both of the finite-difference method and of the Petrov-Galerkin method.

We formulate the following two results of theoretical analysis: First, the approximate solution u_h does not oscillate. Second, the discrete L^∞-norm of $u - u_h$ on certain subregions $S \subseteq \Omega$ is bounded by $Ch^2 + K\varepsilon$. This local error estimate is proved for strongly regular triangulations only and is uniform whenever no layers of u intersect S.

By means of graphical illustrations of approximate solutions u_h of the problem (1) and of its nonstationary equivalent, we indicate that the breadths of layers of u_h depend on the Reynolds number $|p|/\varepsilon$ essentially and artificial diffusion has no influence on them.

L DEMKOWICZ and J T ODEN

New developments on application of hp-adaptive BE/FE methods to elastic scattering

Following a short summary of the ideas and results described in [1,2,3] two new aspects of the application of *hp* approximations to the solution of elastic scattering problems are presented.

1. Anisotropic h-refinement techniques and the related issues of constrained approximation.
2. A better understanding of the meaning of the asymptotic convergence in the context of the problem considered.

A number of typical numerical examples conclude the presentation.

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P DRÁBEK

Nonhomgeneous degenerate eigenvalue problem

Let us consider the nonhomogeneous eigenvalue problem

$$- \operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) = \lambda b(x,u)|u|^{p-2}u \quad \text{in} \quad \Omega , \tag{1a}$$

$$u = 0 \qquad\qquad \text{on} \quad \partial\Omega . \tag{1b}$$

We assume that $p > 1$ is the arbitrary real number and the Carathéodory functions $a(x,s), b(x,s)$ satisfy the following hypotheses. Let $w \in L^1_{loc}(\Omega)$, $\frac{1}{w} \in L^{\frac{1}{p-1}}_{loc}(\Omega)$ and

$$\begin{gathered} \frac{w(x)}{c} \le a(x,s) \le cw(x)g(|s|) , \\ 0 < b(x,s) \le a(x) \end{gathered} \tag{2}$$

for a.e. $x \in \Omega$ and for all $s \in R$, where $c > 0$ is a constant, $a(x) \in L^\infty(\Omega)$ and $g : R^+ \to [1, \infty)$ is a nondecreasing function.

We say that λ is the *eigenvalue* and $u \in W_0^{1,p}(w,\Omega)$ is the corresponding *eigenfunction* of the eigenvalue problem (1) if the identity

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\nabla\phi dx = \lambda\int_\Omega b(x,u)|u|^{p-2}u\phi dx$$

holds for any $\phi \in W_0^{1,p}(w,\Omega)$ (here $W_0^{1,p}(w,\Omega)$ is the *weighted Sobolev space*).

THEOREM For a given real number r > 0 there exists the least eigenvalue $\lambda_r > 0$ and a corresponding eigenfunction $u_r \geq 0$ (a.e. in Ω) of the eigenvalue problem (1) such that

$\|u_r\|_{L^q(\Omega)} = r$.

Let us note that neither global results for nonlinear eigenvalue problems, nor Ljusternik-Schnirelmann theory, can be used, since the operators are not, in general, potential operators.

Let us note also that the principal part of (1) may contain the *degeneration* (or *singularity*) and it may depend on the modulus of the function u in a very general way (see (2)).

M G EDWARDS

A flux continuous approximation of the pressure equation for h-adaptive grids

A new flux continuous discretisation of the reservoir simulation pressure equation at an h-adaptive grid interface is presented. For regular grids it is well known that a use of a harmonic mean of the permeabilities to define the discrete pressure equation coefficients preserves flux continuity and is important in the case of discontinuous coefficients. This property is generalised to h-adaptive grid interfaces.

The standard cell centred finite volume discretisation of the pressure equation leads to an 0(1/h) pointwise local truncation error at the adaptive grid interface. For arbitrary grid aspect ratio Ar, and interface ratio Gr, it has been shown that the coefficient of this error is proportional to the product Ar*Gr, and this error has a severe impact on results for large Ar, [1].

A correction implemented in [1] leads to a non-symmetric matrix, conditional diagonal dominance and increased support of the scheme resulting in increased complexity in the connectivity of the pressure matrix. These difficulties are handled by implementing the correction explicitly following [2].

In this paper a new correction is presented which removes the local leading 0(1/h) error. The new correction uses the same support as the standard scheme and symmetry of the pressure matrix is maintained. The correction is constructed such that the flux is continuous at all adaptive grid interfaces, which in turn results in an unconditionally diagonally dominant pressure matrix for isotropic coefficients. The simplicity of the new correction enables a fully implicit and consistent implementation of the scheme.

Finally the benefits of the new correction are demonstrated with results for the general case of a dynamically varying h-adaptive grid which tracks a moving high resolution shock front.

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H C ELMAN

Parallel implementation of the hp-version of the finite element method on a shared-memory computer

We study the costs of solving two-dimensional elliptic partial differential equations discretized by the *hp*-version of the finite element method, on a shared memory parallel computer. The computational algorithm consists of elimination of "high order" unknowns associated with element interiors, followed by preconditioned conjugate gradient solution of the global problem on element boundaries, using the submatrix associated with nodal unknowns as a preconditioner. We systematically examine costs in CPU time of the individual steps of the computation, including construction of the local stiffness matrices. Our general observations are that costs of the "naturally" parallel computations associated with local elements are significantly higher than any global computations, so that the latter do not represent a significant bottleneck to parallel efficiency. However, in order to avoid inefficiency caused by hierarchical computer memories, it is necessary to organize the local operations into blocks of computations. We show how to do this effectively using the Level 3 BLAS linear algebra kernels.

R FALK

Numerical approximation and asymptotic properties of some two dimensional plate models

The finite element approximation of some two dimensional plate models derived from mixed variational principles is studied. A crucial element in the derivation of error estimates is to first understand the asymptotic properties of the models. In particular, we consider the boundary layer behaviour and the dependence of the regularity of the solution on the plate thickness. Differences between minimum energy models and complementary energy models are highlighted.

G N GATICA and G C HSIAO

The uncoupling of boundary integral and finite element methods for nonlinear boundary value problems

Recently, the usual coupling of boundary integral and finite element methods has been modified to yield simpler variational formulations for linear exterior boundary value problems. This simplifed procedure is based on choosing the artificial coupling boundary as a circle or a sphere, which allows us to invert the boundary integral operators exactly. This leads to the so called *uncoupling technique*, by means of which the resulting weak formulations reduce to almost the same as those arising from Neumann boundary value problems on bounded domains. In fact, the only boundary integral operator appearing in the formulations is the one provided by the *simple layer potential*.

The purpose of this paper is to apply the uncoupling procedure to study the weak solvability of certain nonlinear exterior boundary value problems. As a model we consider a nonlinear second order elliptic equation in divergence form in a bounded inner region, which becomes the Laplace equation in the corresponding unbounded exterior region. We provide sufficient conditions for the nonlinear coefficients from which existence, uniqueness and approximation results are established. In particular, nonlinear equations yielding both monotone and non-monotone operators are analyzed. Also, the effect of numerical integration, and the polygonal approximation of the coupling boundary, are taken into account to perform the error analysis of the discrete scheme.

L GAVETE CORVINOS, C MANZANO DEL MORAL and A RUIZ PEREA

New high order infinite elements with different decay types

A class of problems of interest in mathematical physics and engineering is characterized by partial differential equations on unbounded domains. For example, the irrotational flow of an incompressible fluid exterior to a body in R^3 is described by Laplace's equation in the exterior domain, Ω, with no flow through the body boundary, Γ, and uniform flow at infinity. Similarly, in acoustics and electromagnetism, Helmholtz's equation describes the exterior scattering from a body in R^3. In this case, the "boundary" condition designates the decay farfield rate of the solution at infinity. Existence and uniqueness of solutions have been obtained by several authors using various techniques, but the treatment of unbounded domains has presented problems for the finite element analyst. Discretizations are typically extended to large distances in attempts to minimize inaccuracies. Unfortunately, coverage of this extensive domain results in CPU costs and storage penalities.

The focus of the present work is the numerical solution of partial differential equations by using infinite elements instead of finite domain truncation. Infinite elements are becoming increasingly popular as an economical means of extending the finite element method to deal with unbounded domains. Various approaches have been adopted in the extension of the method, and these techniques tend to merge into each other. In this paper, new high order mapped infinite elements are investigated.

We conclude that in order to solve efficiently a partial differential equation with an unbounded exterior domain, it is necessary to consider the following:

(a) The type of infinite element to be used and its control.
(b) The location of the interface between finite and infinite elements.
(c) Alternative approaches to Finite Element Formulation (only in some cases).

These questions can be solved by using high order mapped infinite elements with the appropriate decay type. Also we consider the case that it is necessary to employ a Petrov-Galerkin method. The advantages of using the new approach proposed in this paper are shown with different examples of solutions of partial differential equations on unbounded domains.

R GHANEM and J E AKIN

A stochastic finite element analysis method

The finite element method is (ɿ appl.ed to partial differential equations that involve constitutive relations where th aterial constants are expected to have random variations. Certain applications are e. pected to have large random spatial variations in the constitutive parameters. These would include the structural or electrical response of biological materials or large-scale environmental problems, such as flow through porous media. This paper presents the extension of classical finite element computational procedures to the corresponding complete and reliable stochastic finite element code.

The random aspect of the various parameters in the system can be accounted for through the introduction of an additional dimension on the formulation of the problem. Various spectral expansions along this dimension are given by the Karhunen-Loeve expansion and the Polynomial Chaos expansion. These expansions are well known in applied mathematics, and have been successfully implemented in the context of structural engineering and structural mechanics. These expansions are incorporated into spatial discretizations of the domain in accordance with standard finite element procedures. The Karhunen-Loeve expansion is a mean-square convergent optimal expansion of a random process in terms of a denumerable set of orthogonal random variables. We provide an explicit expression for the solution process as a multidimensional surface in terms of orthogonal polynomials, the coefficients of which are obtained through a Galerkin projection scheme coupled with a Polynomial Chaos expansion. Such an expression is superior, both in accuracy and incomputational efficiency, to currently used first- and second-order Taylor expansions. From such an expansion, the probability distribution function of the solution can be obtained.

The numerical implementation of the deterministic projection follows the standard guidelines as those for the deterministic finite element method. The treatment in this paper will, therefore, be confined to the implementation of the projection in the space of random functions. The first step in implementing the proposed method consists of expanding the random process representing the spatial randomness of the system parameters by using the Karhunen-Loeve expansion of the process. Then, the computations leading to the expansion consist of employing an integral eigenvalue equation that can usually be obtained analytically. The next step in the process consists of implementing the Karhunen-Loeve expansion by using Polynomial Chaos. The size of the resulting system of equations is proportional to the number of Polynomial Chaoses used.

The technique develops a fully stochastic finite element formulation that completely incorporates the stochastic nature of the problem. The outcome is an accurate representation of the solution process via a convergent expansion in orthogonal stochastic polynomials. Unlike first- and second-order expansions, this representation affords accurate estimates of the statistics of the solution beyond the second-order moments. These statistics can be either in the form of higher-order moments, or as convergent approximations to the probability distribution function of the solution. Highly accurate approximations of the probability distribution function of the solution process can also be obtained.

D GIVOLI and J B KELLER

Non-reflecting finite elements

Wave problems in domains extending to infinity are very often encountered in various fields of application such as acoustics and structural acoustics, geophysics, meteorology and fluid dynamics. Standard solution procedures of such problems include the boundary element method, the coupled finite element-boundary element method, and the use of infinite elements. Another common procedure consists of the following three steps: (a) first truncate the infinite domain by introducing an *artificial boundary B*, thus making the computational domain finite, (b) then choose a certain *boundary condition* to be imposed on B, (c) finally solve the problem in the finite computational domain by the *finite element method*. This procedure as well as others are described in the recent monograph [1].

It is well known (see the review paper [2]) that a poor choice of the artificial boundary condition on B may give rise to *spurious reflection* of waves, and thus to large inaccuracies in the finite element solution. An effective boundary condition must have the property that outgoing waves hitting B are transmitted through B without any reflection. It is shown in [2] that a robust non-reflecting boundary condition, effective in the general case and not only in some specific cases, must be either *nonlocal* or of sufficiently *high order*.

Exact nonlocal boundary conditions were devised in [3] and [4] for exterior time-harmonic acoustic waves and elastic waves. The artificial boundary B was chosen to be a circle in two dimensions and a sphere in three dimensions. The use of the nonlocal boundary conditions leads to a non-standard finite element formulation. The infinite domain outside the computational domain can be regarded as a "super finite element" with interacting degrees of freedom on B. The finite element scheme converges to the exact solution with the optimal rate of convergence (see [2] and [5-7]).

Local high-order non-reflecting boundary conditions were proposed, e.g. by Engquist and Majda [8], Bayliss and Turkel [9] and Givoli and Keller [4]. The Bayliss-Turkel conditions were used in a finite element scheme in [10]. The use of boundary conditions on B which involve high-order spatial derivatives is problematic from the finite element viewpoint, since standard conforming C^0 finite elements cannot be used in that case in the layer of elements adjacent to the boundary B.

In the present paper we consider both nonlocal and local non-reflecting finite elements (NRFEs), and discuss the differences between the two approaches from various perspectives. We devise three new NRFEs:

1. *A nonlocal NRFE, used with an elliptic artificial boundary.* This NRFE is particularly useful in solving scattering problems with slender obstacles.
2. *A nonlocal NRFE for 2D problems in geophysics.* The geometry in this case is semi-infinite, and the construction of the NRFE turns out to be significantly more difficult than in the case of exterior problems.

3. *A local NRFE with high-order continuity on the boundary B.* This element possesses C^0 or C^2 continuity on one of its sides, and C^0 continuity elsewhere. It is fully compatible with standard finite element architecture. One layer of elements of this type is used near the artificial boundary ***B***.

We will discuss the computational aspects and the implementation of these three elements, and will demonstrate their performance through a number of numerical examples.

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G A GRAVVANIS and E A LIPITAKIS

Using explicit preconditioned schemes for solving initial/boundary value problems

A new class of hybrid time implicit-explicit approximating schemes combined with Approximate Inverse Finite Element Matrix (AIFEM) techniques and explicit preconditioning *semi-direct* methods for the numerical solution of initial/boundary value problems is presented. The AIFEM techniques based on the concept of adaptable LDL^T factorization procedures have been recently introduced for computing explicit pseudoinverses of large sparse symmetric matrices of irregular structure, derived from the FE discretization of Elliptic and Parabolic Partial Differential Equations without inverting the corresponding decomposition factors. This category of equations represents a large class of commonly occurring problems in Mathematical Physics and Engineering.

The AIFEM techniques originated from the known algorithmic procedures of inverting a real (n × n) matrix A, which can be decomposed by adaptive approximate factorization methods (i.e. $A = L_r D_r L_r^T$, where D_r is a diagonal matrix and L_r is a sparse lower triangular matrix of the same profile as A), by computing explicitly the elements of the inverse A^{-1}, without inverting the decomposition factors. The effectiveness of the Explicit Preconditioned iterative methods using these AIFEM techniques for solving numerically initial/boundary value problems is related to the fact that the exact inverse of the original sparse coefficient matrix A (although full) exhibits a similar "fuzzy" structure to A, i.e. the largest elements are clustered around the principal diagonal and m-diagonal where m is the semi-bandwidth of A.

It should be noted that an important feature of the AIFEM algorithmic techniques is the provision of both explicit preconditioning direct and iterative methods for solving partial differential equations in two and three space variables. Additional facilities are provided by the choice of "fill-in" (factorization) and "retention" (pseudoinverse) parameters that allow the best method for a given problem to be selected.

Additionally a class of composite Picard/Newton methods combined with Approximate Inverse Finite Element Matrix (AIFEM) techniques and explicit preconditioning semi-direct methods for the numerical solution of non-linear elliptic partial differential equations is presented.

D V GRIFFITHS

Use of computer algebra systems to generate element stiffness matrices

Finite element matrices such as those for stiffness and mass are usually generated using Gaussian quadrature because it leads to convenient formulations in terms of local coordinates. This paper describes how element matrices can be generated in *closed form* with the help of Computer Algebra Systems (CAS) and how this can lead to improvements in run-times. The CAS approach is also shown to be able to generate matrices which would normally be obtained using reduced or selective reduced integration (SRI).

Consider a square 4-node element in plane strain, fixed on three sides, with a horizontal force *P* applied to the remaining corner. Computer Algebra can be used to compute the horizontal deflection of the corner for various element stiffness integration schemes. The expressions show why locking occurs as the material approaches incompressibility with full integration.

Integration	δ_H	$\delta_H(\nu \to 0.5)$
FULL	$\frac{96(3-4\nu)(1+\nu)(1-2\nu)P}{(9-16\nu)(15-16\nu)E}$	0
SRI(λ,μ)	$\frac{16(2-3\nu)(1+\nu)P}{3(5-6\nu)E}$	$\frac{2P}{E}$
SRI(G,K)	$\frac{144(17-25\nu)(1+\nu)P}{25(43-50\nu)E}$	$\frac{2.16P}{E}$
REDUCED	$\frac{(3-4\nu)(1+\nu)P}{1-\nu)E}$	$\frac{3P}{E}$

C GROSSMANN

Newton's method for obstacle problems

The obstacle problem can be considered as a variational problem with inequality constraints. The discretization by piecewise linear finite elements leads to large scale optimization problems with a special structured objective functional and with simple bounds for the variable as constraints. In this investigation we include the constraints into an auxiliary objective functional by means of a penalty technique. The unconstrained problems obtained are nonlinear variational equations. These can be solved by Newton's method. As a well known fact the auxiliary problems generated in penalty techniques are ill conditioned in the limit. Thus an important question is to estimate the area of contraction of Newton's method in dependence of the penalty and of the discretization parameter.

In the first part we summarize some basic facts on obstacle problems, their discretization and on an adapted penalty technique. Later we derive optimal parameter selection rules to adjust the error caused by the penalty terms to the same magnitude as the discretization error of the finite element approximation. Finally, we investigate the convergence behaviour of Newton's method applied to the variational equality obtained. Here the area of contraction of Newton's method is estimated in dependence of the discretization parameter and the penalty parameters by means of maximum principles for elliptic problems. Finally, we adjust the iterations on each level of discretization and investigate the method on families of grids.

J F GUARNACCIA and G F PINDER

Animation of denser-than-water immiscible fluid migration in the subsurface

Subsurface migration of denser-than-water immiscible contaminants, such as chlorinated hydrocarbons, can be simulated and the output presented via animation. The four non-linear partial-differential equations that describe the physical system can be approximated using orthogonal collocation. The resulting system of equations are efficiently solved using domain decomposition in combination with Picard iteration.

Contaminant saturation and concentration are effectively represented using colour-based computer graphics and animation. By calculating the graphical representation of the computed values of these parameters for each time step of the simulation, and converting the output to television quality, a video tape of the dynamics of the system is generated. Imbibition, drainage and eventual dissolution of the contaminant are represented for different geohydrological situations.

J GWINNER

Boundary element methods for contact problems in linear elastostatics

As witnessed by the monograph of Kikuchi and Oden, [1], contact problems in nonlinear solid mechanics are an interesting field of application of the Finite Element Method. On the other hand, contact problems show much more concern about the boundary behaviour, namely boundary displacements and boundary tractions than about the stress distribution inside the body. Therefore it is natural to ask for the application of Boundary Element Methods, at least for linear materials within an infinitesimal theory of elastostatics. There is recent progress towards this direction.

In this contribution we address the most interesting case from the view of mechanics and the most delicate case from the view of mathematics where the linear elastic body constrained by a rigid foundation is solely subjected to applied boundary forces, not necessarily fixed or with displacements imposed at some boundary part. Using Somigliani's identity and the full Calderon projection including the hypersingular integral operator a boundary integral variational inequality [3] can be derived as a mixed variational formulation for the boundary displacements and boundary tractions. The resulting bilinear form is not symmetric, but nonnegative and semicoercive in the sense that it satisfies a Garding inequality [3].

Use of periodic spline functions as trial and test functions on the boundary leads to Galerkin boundary element methods. Here we consider not only piecewise linear approximations for the boundary displacements and piecewise constants for the boundary tractions, but also approximations of higher order that lead to a nonconforming approximation of the unilateral constraint of the contact problem. Based on a recent descretization theory for semicoercive variational inequalities [2] convergence of those boundary element methods can be established with respect to the energy norm [3].

Further we report on numerical calculations [4] for the well-known Hertz contact problem. By a Schur complement technique we transform the discrete variational inequality into a linear complementarity problem that is formulated only in terms of displacements. As an iterative solver the projected SOR algorithm is investigated.

Moreover in virtue of a recent extension of the discretization theory to variational inequalities of the second kind [5] we can also treat contact problems with given friction. To discretize the nonlinear friction functional various quadrature rules can be employed that can be analysed in a similar way as for the finite element method [6]. Finally we discuss how more realistic friction models can be incorporated in the boundary integral approach.

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S HAMMARLING

LAPACK: High performance software for linear algebra

NAG Ltd and US Scientists have collaborated in the development of LAPACK, a high performance linear algebra package, already proven for shared memory parallel machines, now being extended to distributed memory machines.

I HARARI and T J R HUGHES

Finite element methods for general second-order elliptic partial differential equations: Model problems

Finite element methods for second-order elliptic partial differential equations were initially applied to problems which contain only derivatives of the highest order. In such cases finite elements based on the Galerkin method possess several advantageous attributes, with guaranteed performance on configurations of practical interest. The addition of other terms such as *first-order* derivatives in advective-diffusive problems and *undifferentiated* terms in the Helmholtz equation has potentially destabilizing effects. The concept of Galerkin/least squares (GLS) obtained by appending terms in *least squares form* to the standard Galerkin formulation enables the design of stable methods for such applications, retaining the higher-order accuracy of the Galerkin method for problems governed by the Laplacian, at relatively low added computational cost.

General second-order equations, which are studied in this work, have exact solutions which may differ widely in nature depending on values of the coefficients, with rapidly varying values in domain interiors as well as close to boundaries. An analysis of numerical solutions obtained by the Galerkin method employing linear finite elements indicates that relatively fine meshes may be required to retain an acceptable degree of accuracy for certain ranges of values of the physical coefficients. Numerous criteria for improving accuracy by employing GLS technology are examined. Of these, several are found to provide the lowest error in nodal amplification factors, each in a certain range of ratios of physical coefficients. Guidelines for required mesh resolutions are presented for Galerkin and GLS, showing that in large parts of the parameter space substantial savings may be obtained by employing GLS. Numerical computations bear out these observations.

B HEINRICH

The Fourier-finite element method for elliptic problems with axisymmetric edges

The Fourier-finite element method is a semianalytic technique for the approximate solution of three-dimensional elliptic problems in axisymmetric domains with non-axisymmetric data. The method employs truncated Fourier series with coefficients which are the solutions of a finite number of two-dimensional elliptic problems posed on the meridian plane of the domain and approximated by the finite element method. This approach is often used by engineers for thermal and stress analysis of axisymmetric bodies and can be implemented on a parallel computer.

In this paper we present results on the mathematical justification of the Fourier-finite element approach for solving problems in 3D, especially for Poisson's equation, the description of the properties of the approximation, local mesh refinement in the meridian plane for an appropriate treatment of singularities near re-entrant edges and error estimates in the H^1- and L_2-norms, with respect to the discretization parameters N and h (Fourier and finite element approximation) for solutions belonging to Sobolev spaces. Further, algorithmic aspects and the computer implementation of the method are discussed.

J P HENNART

Nodal finite element methods for partial differential equations

Modern nodal methods were developed in the 1970's in numerical reactor calculation. Roughly speaking, nodal methods are fast and accurate methods which try to combine the attractive features of the finite difference method (FDM) and the finite element method (FEM). From the FEM, they borrow a piecewise continuous, usually (but not always) polynomial, behaviour over a given coarse mesh. With the FDM, they have in common the fact that the final algebraic systems are usually quite sparse and well structured, *at least over domains which are not too irregular such as unions of rectangles.* Nodal methods can thus be viewed as fast solvers, to which techniques of vectorization and (or) parallelism are applicable. They are especially suited to all physical situations which have been traditionally modelled by finite differences, and for which the domain as well as the coefficients of the equation(s) are not known with sufficient accuracy. It is then tempting to use a rectangular grid to discretize the domain and to assume that over each cell of the grid the physical parameters are not known with great accuracy, so that a constant or mean value only is available. This situation is prevailing for instance in groundwater hydrology, oil reservoir simulation, and air pollution modelling problems.

In this paper, we show that most nodal schemes can be viewed as finite element schemes provided the FEM is viewed in a fairly liberal way with moments (and not point values) as principal unknowns, with piecewise polynomials (or quasi-polynomials) as basic functions and with a systematic use of variational crimes like numerical quadrature and nonconformity.

These ideas were applied to the three basic types of PDE's, *elliptic*, *parabolic*, and hyperbolic, with concrete applications (and numerical results) in multidimensional diffusion, space-time kinetics and 2D transport of neutrons in nuclear reactors, as well as for benchmark problems of the types mentioned above, including the Poisson equation and the heat diffusion equation.

I HERRERA

Localized Adjoint Method: An overview

The Localized Adjoint Method (LAM) is a new and promising methodology for discretizing partial differential equations, which is based on Herrera's Algebraic Theory of Boundary Value Problems. Applications have successively been made to ordinary differential equations, for which highly accurate and efficient algorithms were developed, multidimensional steady state problems and optimal spatial methods for advection-diffusion equations. More recently, generalizations of Characteristic Methods known as Eulerian-Langragian Localized Adjoint Method (ELLAM), were developed and many specific applications have already been made. Related work and additional applications are underway.

In the construction of approximate solutions there are two processes, equally important but different. They are:

i) Gathering informaiton about the sought solution; and
ii) Interpolating or, more generally, processing such information.

These two processes are distinct, although in many numerical methods they are not differentiated clearly. The information about the exact solution that is gathered, is determined mainly by the weighting functions, while the manner in which it is interpolated depends on the base functions chosen. Examples have been given for which these processes are not only independent but, they do not need to be carried out simultaneously.

The questions posed by the above comments are quite complex and to explore them in all their generality is very difficult. Localized Adjoint Method is a methodology we have proposed to carry out such analysis and develop new numerical procedures using the insight so gained.

In this paper Localized Adjoint Method is briefly explained and ELLAM procedures, which have been quite successful for treating advection dominated transport, are discussed.

Z KESTŘÁNEK

Finite elements for extended variational formulation in nonlinear problems

Nonlinear problems such as the elastohydrodynamic lubrication problem, involving nonconservative mechanical systems are of considerable importance in the field of hydroand aero-elasticity and space mechanics. These problems involve nonsymmetric operators, and therefore variational formulations in a strict sense are not available. The variational formulation presented [1], [2] is called extended in the following sense: Given a problem $N(u) = 0_v$ with a nonlinear operator $N : D(N) \subset U \rightarrow R(N) \subset V = U^*$, find a functional F, if any, whose critical points are solutions to the problem and vice versa. This implies that for a given operator N an operator N exists such that $\delta F = \langle N(u), \delta u N \rangle$ and the problems $N(u) = 0_v$ and $N(u) = 0_v$ have the same solution. This statement requires only that the critical points should coincide with the solutions, without posing the additional requirement that N must be the gradient of the functional. The problem can be rewritten as $N(u) = N_u'^*(u;KN(u)) = 0$, provided that $D(K) \subset R(N)$, $R(K) \subset D(N_u'^*)$, where K is a linear, invertible and symmetrical operator. The modified operator N satisfies the symmetry condition for the Gâteaux derivative, $N_u' = N_u'^*$ and it follows that there exists a potential functional $F[u] = 1/2 \langle N(u), KN(u) \rangle$ whose gradient is the operator N. It is worth noting that the operator K is not uniquely determined and the practical construction of K for the general case presents some difficulties. In numerical applications, however, K can easily be defined. The functional vanishes when the solution is reached. Moreover if K is positive definite, then F[u] is minimum at the critical point. Functional F can be defined over finite-dimensional subspaces S. The kernel k(s,t) of the linear operator K in integral form is naturally defined over $S \times S$ and may be expressed as $k(s,t) = \{L(s)\}^T[A]\{L(t)\}$, where [A] is a symmetric constant matrix and {L} is a vector of the shape functions. Here the choice of the kernel in a finite subspace, generated by means of functions with local compact support, makes it possible to provide finite element models in a straightforward way. The main novelty in this standard procedure is a complication arising from the integro-differential nature of the basic equation. To find the minimum of the functional F we can use the Newton-like type algorithm. This method for the solution of the inverse problem has been successfully developed for special numerical applications.

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A Q M KHALIQ

A parallel semidiscrete Galerkin splitting method for second-order hyperbolic equations

Second-order hyperbolic partial differential equations (PDEs) in multi-space dimensions occur frequently in viscoelastic theory [3], acoustics [4], and in many geoscience applications, for example seismology [1,2,5]. Computational techniques are proposed by several authors, see for example [2,3,6,7] and references therein, for the numerical solution of the wave equation by finite-element methods. Their investigations have been restricted to mainly, two-dimensional wave equations and implementation on serial machines. Recently an explicit finite-difference scheme has been proppoosed by Mufti, [5], for the numerical solution of large-scale three-dimensional seismic models. Since analytical methods are also not effective because they are restricted to simple geometries with homogeneous structures like the finite-difference methods, there is an ever growing need to develop efficient and stable finite-element methods which may be implemented effectively on modern parallel computers.

A new parallel method for solving hyperbolic PDEs in many space dimensions is presented by applying the finite-element technique, and is unconditionally stable. The proposed method is based on the semidiscrete Galerkin approach utilizing Strang-type splitting techniques in which a multi-dimensional problem is reduced to a series of one-dimensional problems. This approach avoids the difficulty of constructing a finite-element space for multi-dimensional problems. A large scale problem requiring large storage and CPU times is thus solved using only a one-dimensional finite-element method on MIMD parallel machine. This approach is seen to be highly competitive to those presented in [7]. Second order convergence and error estimates of the scheme are given using piecewise linear basis functions. However, it is anticipated that cubic B-splines may also be used as basis functions and their convergence may be proved by semi-group theory. Numerical results are demonstrated on problems from the literature. The proposed parallel algorithm for the numerical solution of the wave equation will provide geophysicists with a very useful and efficient tool for predicting and understanding seismic wave propagation. It will also be useful to engineers because of its efficiency for solving large scale problems in acoustics and in numerical approximation to the solution of the linear elastodynamic equations.

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B KOVÁCS and F J SZABÓ

On the stability of layered circular rings

The purpose of this paper is to find the critical external pressure of layered circular ring bucklings in their planes. The composite ring subjects to normal gas pressure and constant-directional pressure. The stability of an arch is analysed by formulating the governing differential equations and the associated boundary conditions using a variational principle. The cross sections in individual layers remain planes after deformation.

In the case of free rings, the critical external pressures are investigated, using theoretical method, the COSMOS/M and the SYSTUS finite element software packages.

As a theoretical investigation the eigenproblem is solved by a successive approximation based on the Schwarz-method and by the Ritz-method, reducing the problem to a matrix-eigenvalue problem.

For the finite element solution the model of the ring is built up using four-nodes composite shell elements and the numerical results are obtained by the Inverse Power iteration technique in case of COSMOS/M and by the Lanczos algorithm in SYSTUS.

The analysis of the numerical results shows that in the case of free rings the constant-direcitonal pressure gives higher critical values than the hydrostatic pressure and one can find a good agreement between the theoretical and finite element solutions.

B KOVÁCS and F J SZABÓ

Free vibrations of layered circular ring segments

The free vibration eigenvalue problem of circular arc shaped three layered sandwich ring segments was investigated theoretically and numerically in the case of linear elastic layer-materials. The comparison of the different solution methods makes it possible to study the plane bending free vibration behaviour of the curved layered beams more accurately.

The free vibration eigenvalue problem of the ring segments was solved theoretically by using the Schwarz-method, which is a fast and flexible, easy-to-program algorithm. This algorithm can be easily completed by an optimization method and can be applied for some special optimization problems, which cannot be handled using the built-in optimization of the known finite element programs (multiobjective optimization, discrete variables, complicated constraints, etc.).

The results of the theoretical solution were compared with those obtained by a finite element analysis made in the SYSTUS finite element program which runs on a VAX 3100 workstation under VMS. For the eigenvalue solution the LANCZOS method was used. Two different finite element models were built up:

- The first model contains eight-nodes hexahedron brick elements and this makes it possible to apply several elements along the thickness, which increases the accuracy of the model.
- The second model consists of four-nodes composite shell elements, which results in a very easy-to-solve finite element model which needs much less storing and computation capacity than the previous one.

Numerical results of the theoretical and finite element analysis were made for the circular arc of radius varied in the ranges between 100mm to 200mm, with the same thickness and width data, in the case of two different core materials.

(Although both of the first four bending eigenvalues shows a very close agreement between the theoretical and finite element results, which is better in case of long (length of 150mm or more) beams. In the case of very short (length of 50mm or less) beams there may be a considerable difference between the results, so for these structures it could be necessary to investigate the problem experimentally, too. Comparing the two different finite element models, one can say that the model containing hexahedron elements needs more computing and storing capacity but gives more accurate results than composite shell elements. Because of the possibly long calculation time in case of complicated, long solution procedures (nonlinear or transient dynamic analysis containing many time cards) the composite shell elements are more suitable.

M KŘÍŽEK

Finite element approximation of a nonlinear anisotropic heat conduction problem

We prove the existence and uniqueness of a solution of the following second order quasilinear elliptic problem of a divergence form with nonlinear mixed boundary conditions:

$$\operatorname{div}(A(x,u)\operatorname{grad} u) = F(x,u) \qquad \text{in } \Omega,$$

$$u = \bar{u} \qquad \text{on } \Gamma_1,$$

$$n^T A(s,u)\operatorname{grad} u + G(s,u) = 0 \qquad \text{on } \Gamma_2,$$

where $\Omega \subset R^d$, $d \in \{1,2,...\}$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, n is the outward unit normal to the boundary, Γ_1, Γ_2 are relatively open and disjoint subsets of $\partial\Omega$ and $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$, $\bar{u} \in W_2^1(\Omega)$ and $A = (a_{ij})_{i,j=1}^d$ is a uniformly positive definite matrix function. We further assume that $F(x,.)$ and $G(s,.)$ are nondecreasing functions for almost every $x \in \Omega$ and $s \in \Gamma_2$ (in particular, they can be independent of u). We show that the problem *is nonpotential in general* and also that the theory of monotone operators cannot be applied. Sufficient conditions from [1] which guarantee the uniqueness of the classical and weak solutions are introduced. Note that there exist examples of non-unique solutions if the equation is not in divergence form (see e.g. [2]). We also give a finite element approximation of the problem and present some results concerning the existence and uniqueness of the Galerkin solution. We then introduce two convergence theorems of [1]. A practical example of computing a temperature field in the magnetic cores of transformers is shown.

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M S KUCZMA and J R WHITEMAN

A variational inequality formulation for flow theory plasticity

We consider the quasi-static deformation process of an elasto-plastic body, under the assumption of small strains. The elasto-plastic behaviour of the material is assumed to be governed by the von Mises yield function with linear strain hardening (softening). The yield function is written in terms of strains, and we define the incremental problem as the system that consists of (a) a variational equation being the equilibrium condition for the body and (b) a variational inequality expressing the unilateral character of the plastic strain-rate multiplier. An iterative method for solving this non-linear system is proposed, and results of some numerical experiments based on the FEM-approximation are provided.

G KUHN

A BE-formulation for finite strain elastoplasticity

An extension of the boundary-element-method to finite strains and to constitutive relations based on the concept of an intermediate configuration is presented. Within this concept material behaviour is described in terms of a hyperelastic relation relative to the intermediate configuration together with evolution equations for this configuration and further internal variables. In the context of numerical solution techniques this enables an operator split leading to a global hyperelastic problem and local time integration schemes for the internal variables. The proposed BEM-approach allows for arbitrary hyperelastic relations relative to the intermediate configuration and leads to a nonlinear set of equations with displacement gradients as basic unknowns. As a Total-Lagrange scheme is adopted the expensive computation of the system matrices has to be done only once in the initial configuration. Nevertheless the nonlinear set of equations can be consistently linearised with respect to the local integration schemes. Constitutive equations, integration schemes and linearisation techniques used in FEM can be directly applied. The BEM-formulation is presented for associated rate independent elastoplasticity using a yield condition in strain space. Some numerical results will be shown.

G KUNERT

Adaptive mesh refinement in the finite element method

The rate of convergence of the finite element method can be improved by applying adaptive mesh refinement. This has been investigated for different kinds of error indicators, and numerical experiments were performed for the Laplace operator.

R LEVKOVITZ and G MITRA

The factorization of unstructured sparse symmetric positive definite matrices on distributed memory MIMD parallel computers

The efficient solution of unstructured large sparse symmetric positive definite equations on a distributed memory computer using direct methods is an important and challenging area of parallel computing research. We concentrate on the design and implementation of a parallel Cholesky factorization algorithm and focus the discussion on the distribution of data, efficient distribution of the computation and the aggregation of the communication. We show that the accumulation of non-zeros in certain areas of the

Cholesky factor lends itself to a hybrid sparse-dense computation paradigm and demonstrate how to extend this approach to a distributed memory parallel MIMD computer.

J MACKENZIE, T SONAR and E SÜLI

Adaptive finite volume methods for hyperbolic problems

We develop an a posteriori error analysis of finite volume methods. Local a posteriori error estimates are derived in various norms which provide local bounds for the global error in terms of a computable residual. For hyperbolic problems with smooth solutions the error and the residual are measured in local L^2-norms; however, when the solution is discontinuous, as is typically the case for nonlinear hyperbolic conservation laws, we use a local negative Sobolev norm, instead.

We apply these residual-based a posteriori error estimates to two finite volume methods. The first one is the cell vertex finite volume method on structured quadrilateral meshes [1]. The numerical investigation of the performance of the error indicator for a model problem of two-dimensional advection demonstrates its reliability and efficiency [2]. The error indicator is then applied to the transonic Euler equations where it is contrasted with classical techniques based on calculating gradients of flow variables.

The performance of one of the residual-based error estimators for a two-dimensional linear advection problem solved by the cell vertex method is shown. Very good agreement can be found between the error estimate and the local error.

The second adaptive finite volume method is constructed on a general triangulation, and it is based on the Riemann solver of Osher and Solomon [3]. A MUSCL recovery technique is employed to increase the accuracy through barycentric subdivisions. The adaptive algorithm uses point insertion techniques. Again, the finite element residual serves as an error indicator to control the adaption process. A triangle K of the triangulation is refined if $h\|r^h\|_{L^2(K)} > TOL$ or, alternatively, if $\|r^h\|_{H^{-1}(K)} > TOL$.

Our numerical results indicate the potential of residual-based a posteriori error estimates for finite volume approximations of compressible flow problems and highlight their superiority when compared with classical methods of adaptive error control.

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H MAISCH, H KARL and G LEHNER

The application of the Finite Element Method to the solution of nonlinear partial differential equations and a comparison to the Finite Difference Method.

Solitons are special solutions of certain nonlinear partial differential equations with increasing importance in plasma physics, hydrodynamics, electrical engineering and other fields. In this paper we have studied the applicability of the finite element method to solve soliton equations. As the most important and famous example we have solved the Korteweg-de Vries equation (KdV) using Hermite elements. Here we have studied the interaction of solitons among themselves and also the generation of solitons from arbitrary initial conditions. The results agree very well with theoretical predictions. In the case of soliton generation there are often no analytical results available, therefore we have checked the three lowest order conservations laws While for the Korteweg-de Vries equation the continuity of the first derivative is absolutely necessary, for most other equations continuity conditions are less stringent and Lagrange type elements are sufficient. Some numerical results for the regularized long wave equation (RLW), the Sine-Gordon equation (SG), nonlinear Schrödinger equation (NLS) and the Zakharov equations are also presented briefly. The numerical solution of two dimensional soliton equations is now also possible due to increasing computer power. We have chosen the two dimensional Sine-Gordon equation as an example. For comparison we solved the Korteweg-de Vries equation and the two-dimensional Sine-Gordon equation also by using finite differences, especially the method of lines. It turned out, that the more flexible finite element method definitively performed better in both cases.

J MANDEL

Balancing domain decomposition preconditioners and discontinuous coefficients

This talk will present the formulation and experience with several preconditioners based on solving, in each iteration, a local problem on each subdomain coupled with a coarse level problem. This provides for an efficient global propagation of the error and guarantees that the possibly singular local problems, solved in every iteration, are consistent.

The preconditioner is based on solving in each iteration possibly singular problems based on the local stiffness matrices of the subdomains. Is it proved that the condition number remains bounded for arbitrary size jumps of coefficients between subdomains in any number of dimensions, is bounded independently of the number of subdomains, and grows only as the logarithm of subdomain size. Computational experiments confirm the theory and show that the method is remarkably robust and performs very well for general strongly dicontinuous coefficients as well as unstructured subdomains. Possible modifications for shells and plates will be also discussed.

J MANDEL

Fast iterative solvers for p-version solids, plates, and shells

We present the formulation of and experience with several methods for the iterative solution of large-scale systems of algebraic linear equations arising from the p-version finite element method in three dimensions. The basic idea of the method is to use domain decomposition approaches with each element considered to be a subdomain. The resulting iterative methods have been shown to be superior to state-of-the-art direct solvers for real-world problems with severely distorted solid elements, such as engine crankshaft, complete aircraft fuselage, and stiffened shell panel with a window rim. The data have been generated by the packages MSC/PROBE and STRIPE.

The method is essentially block preconditioning, with one block based on selected lower degree functions and other blocks associated with vertices, edges, and faces of the elements.

A bound on the condition number independent of the number of elements is proved, and reasonably low growth of the condition number has been observed in practice, including thin elements, i.e., plates and shells. The selection of the specific preconditioner adapts itself to the local characteristics of the problem with no user intervention required.

I MAREK

Approximations of the principal eigenelements of a class of transport operators

Under some reasonable restrictions on the scattering indicatrix and the cross sections a class of transport equations with boundary conditions for incoming particles is investigated. The simplest approximate spaces are proposed in which the approximating operators preserve certain positivity properties of the operators approximated. Appropriate error bounds are established.

A K MOHAMMED, M H BALUCH and S T GOMAA

Finite element modelling of deep beams using a new refined theory

A recent refined theory that incorporates effects of transverse shear, transverse normal stress and transverse normal strain is used in the finite element formulation for flexure of deep beams. The finite element equations for both beam bending and inplane problems were developed using the Galerkin finite element method. Several problems of uniformly loaded deep beams with different support conditions were solved to evaluate the performance of the proposed finite element model. The results are compared with closed form and elasticity solutions whenever they exist.

A K MOHAMMED, M H BALUCH and S T GOMAA

Finite element modelling of thick isotropic plates using a new refined theory.

A recent refined theory that incorporates effects of transverse shear, transverse normal stress and transverse normal strain is used in the finite element formulation of flexure of isotropic plates. The finite element equations for both plate bending and inplane problems were developed using Galerkin finite element methods. Several problems of uniformly loaded thick plates with different support conditions were solved to evaluate the performance of the proposed finite element model. The results are compared with Mindlin/Reissner and elasticity solutions whenever they exist.

M R MOKHTARZADEH-DEHGHAN, D J STEPHENS and L YU

Finite element analysis of turbulent flow through an orifice meter with a concentric obstruction of conical shape at its centre

Orifice meters provide a simple and cheap method of measuring flow rates. The problem of flow through an annular constriction, namely a simple orifice plate modified by a concentric plug, arises in flow meters which are designed to adjust the flow area with variations in the flow rate. Such adjustment is done by allowing the plug to move in the flow direction and therefore change the available area through which the fluid flows. In a simple orifice plate the relationship between the flow rate and the pressure difference across the orifice is non-linear and therefore the measured pressure difference, and the accuracy of the meter, decreases rapidly as the flow rate decreases. If the flow area is allowed to change, by careful design of the plug profile, it is possible to obtain a linear relationship between the flow rate and the pressure drop and maintain the accuracy over a wider range of flow rates. The disadvantage in this case is the increase in cost and higher permanent pressure loss.

The paper describes a finite element study of turbulent flow through a simplified model, consisting of an orifice plate with a concentric conical plug at its centre. The mathematical model is based on the time-averaged equations of the continuity and momentum in a cylindrical polar coordinate system and the k - ε turbulence model. The computer program FIDAP has been used as the computational tool.

The velocity and pressure fields, the pressure variation along the pipe wall and the pressure losses across the device and the corresponding discharge coefficient are presented and discussed. Comparisons are made with the results obtained for a simple orifice meter.

P B MONK and K PARROTT

A dispersion analysis of finite element methods for Maxwell's equations

A variety of different finite element methods are currently in use for the solution of Maxwell's equations. R L Lee and N K Madsen (1990) amongst others have made comparisons between time domain methods for these equations. An important indicator of the merit of a given time domain method is the accuracy with which it propagates plane electromagnetic waves. This is best established through the calculation of the method's discrete dispersion relation. We have carried this calculation out in two dimensions on uniform triangulations, consisting of either equilateral triangles or uniform right triangles; (see Monk, 1992, for similar, but less detailed, results on quadrilateral elements).

On the basis of this dispersion analysis, it is clear that some combinations of finite element spaces are not very attractive (linear-constant, edge(1)-Edge(1), edge(2)-constant) and some methods need more analysis before becoming truly reliable (non-conforming-constant). Two methods, the linear-linear scheme and the edge(1)-constant scheme have significant advantages and disadvantages. The linear-linear scheme has excellent dispersion properties on both meshes studied here and can be mass lumped (at the cost of a decrease in accuracy). However, it is complicated to deal with boundary conditions, and the divergence conditions on the field are only approximated. On the other hand, the edge(1)-constant method is more sensitive to the mesh showing fourth order convergence on the equilateral mesh, but only second order on the right-angled mesh. Furthermore the method has unexpected parasitic modes. Mass Lumping of the edge-constant scheme is possible provided the mesh contains only strictly acute triangles. On the other hand the method deals with boundary conditions and divergence conditions in a natural way. These results accord with the suggestion by G Mur (1985, 1992) that optimal approximations can be obtained by selecting between these two families of elements, using appropriate criteria reflecting geometry and material property considerations.

Computation of the dispersion relations were carried out using Mathematica. These results are applicable in three dimensions if prismatic elements are used to discretize Maxwell's equations. The more interesting case of tetrahedral elements remains to be analyzed.

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R MÜCKE and J R WHITEMAN

Remarks on a-posteriori error estimation for finite elasticity

Several methods of a-posteriori estimation of the discretization error in finite element solutions are well established and widely used in engineering practice for linear boundary value problems. In contrast we are concerned here with finite elasticity. The main features of finite elasticity are first summarized. Using the residuals in the equilibrium conditions (Babuška approach) an error measure in the energy norm is proposed and physically interpreted. The effe^ts arising from the nonlinear displacement-strain mapping and the accuracy of the proposed error measure are demonstrated by a numerical example. The estimated error distribution is then used to control an h-adaptive finite element procedure.

J NEDOMA

Finite element analysis in nuclear safety

In order to produce criteria for secure siting, location and protection of operation of high level radioactive waste repositories (HLRWRS) and nuclear power plants (NPPS) mathematical simulations must be undertaken of geodynamic and geomechanic processess in both the global and the local areas, where HLRWRS or NPPS, RESP, will be situated. In the contribution the finite element analyses of mathematical models based on the contact problems of Signorini type in thermoelasticity and on the nonstationary incompressible thermo-Bingham problem will be discussed. The methods for monitoring the stress-strain field in situ, of the determination of the optimal design, of the measuring tensometric probes and of the postprocessing of measured data based on the FEM will also be briefly discussed. Existence and convergence theorems will be proved, and the algorithms will be presented.

J NEDOMA

FEM analysis of artificial substitutes of human joints and their optiml design

The actual situation in human joints under loading is much more complicated than that described with classical methods of biomechanics such as Bombelli (1983). Some new ideas of biomechanics of human joints and of their artificial substitutes were given in Nedoma and Stehlik (1989), Nedoma (1991). These are based on contact problems in elasticity and thermoelasticity (plasticity).

In the contribution mechanical processes which take place during static burdening in the contact area of the acetabulum and the head of the hip and of their artificial substitutes - the artificial acetabulum and the head of the endoprosthesis - will be discussed. The mathematical problem introduced in the contribution represents a simulation of the function of human joints and their total substitutes by total endoprostheses (TEP). The model problems lead to coercive and semicoercive contact problems in quasi-coupled thermoelasticity. The FEM is used for numerical analyses of such model problems. The optimal design of the TEPs based on the FEM analyses of the contact problems in thermoelasticity is also briefly discussed and existence and uniqueness of the solutions of the model problems and a convergence theorem will be proved.

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P NEITTAANMÄKI and V RIVKIND

Finite element method for model problems of changing phase

We consider the model of multiphase flow described through full coupled Navier-Stokes and Stephan equations with special boundary conditions. Proof of the existence and the uniqueness is given (with an assumption on the limitation of initial parameters of the problem). A finite element method for the numerical solution is given. The existence and uniqueness of the approximate solutions is proved (with assumptions of the uniqueness and smoothness of the exact solution). Proofs of convergence and of the rate of convergence of the approximate solution to the exact one are given. Several iterative methods such as the full and partial linearization, Newton's method, and the domain decomposition method are proposed. The case of many media, including media with large relations between parameters of sub-media, is discussed.

H ÖRS

Numerical simulation of flow circulation in the golden horn

A numerical model of the flow in the Golden Horn is presented. The model is based on shallow water theory. Considering the complex geometry of the domain of interest, the governing set of partial differential equations, with appropriate boundary conditions, is solved by the finite element method. Given the inner sea nature of the Marmara Sea with minimal tidal effects, the steady state case is considered. The computational domain is discretized into 140 triangular elements with 111 node points. Different interpolation functions are tried in the context of the Galerkin approach. The final system of equations is solved by a Cholesky factorization method. Flow field values thus obtained are compared with on site measurement values. The ultimate goal of the project, supported by Boğaziçi University Research Fund, being to study the pollution mechanisms of the Golden Horn and ways of cleaning it, a second code is written to compute the dispersion rates of the pollutants. Values of the velocity field previously obtained are used in incorporating convection terms. Different wind force distributions and pollution sources are also tried. Computations are carried out on a HP 720 workstation and results are presented graphically.

F PENZEL

Error estimates for discretized boundary element methods for three-dimensional crack problems

The displacement in an unbounded elastic solid with a given internal crack under normal traction can be calculated with the aid of boundary element methods.

Let us assume that the crack surface $\Gamma \subset \{(x_1,x_2,0) : (x_1,x_2) \in \mathbb{R}^2\}$ is a polygon. The crack opening displacement $u \in [H^{1/2}(\Gamma)]^3$ is known to be a solution of the hypersingular boundary integral equation $Du = v$, $v \in [H^{-1/2}(\Gamma)]^3$ is the known traction vector and D is a well-known pseudodifferential operator of order $+1$.
The COD u is approximated by a Galerkin scheme using a regular triangulation of Γ and piecewise-linear finite elements. Let h be the supremum of the diameters of the elements in a fixed triangulation. If the traction data v is a sufficiently smooth function, then it is known that the solution u_h of the Galerkin scheme converges to u with an order of $h^{1/2-\eta}$, for $\eta > 0$ chosen arbitrarily small.

Here we present a fully discretized Galerkin scheme obtained by use of a quadrature formula in the definition of the scalar products. We reduce the analysis of convergence of the fully discretized Galerkin method for the hypersingular integral equation $Du = v$ to the analysis of the weakly singular integral operator -1. Under the assumption that the quadrature error is sufficiently small we prove convergence of order $h^{1/2-\eta}$ of the solution of the fully discretized Galerkin method. By use of explicitly determined error bounds we formulate restrictions on the quadrature formulas which are sufficient to ensure this convergence result.

E PRIOLO and G SERIANI

Low- and high-order FEM in seismic modelling: Experience and perspectives.

In this paper we describe some experience in improving the computational efficiency of a finite element code based on a global approach, used for seismic modelling in the framework of geophysical oil exploration. Applications in this area are characterized by models that have complex geometries and heterogeneous structures, and can be successfully handled by the finite element method. Boundary conditions such as a free surface can easily be taken into account. The main drawbacks of a classical approach based on the assembly of global matrices are the low computational efficiency and the possible appearance of spurious effects.

Results coming from runs of different methods and models on a mini super workstation APOLLO DN10000 are reported and compared. Low- and high-order elements are used to solve both the 2D-acoustic and 2D-elastic wave equations. Time integration is performed with implicit Newmark two-step method. The use of compressed storage is shown to greatly reduce not only memory requirement but also computing time. With Chebyshev spectral elements great accuracy can be reached with almost no numerical artifacts. For spectral elements the static condensation of internal nodes reduces memory requirements and CPU time, and is well suited to parallel computing in a distributed environment. With this technique the solution of the linear system is performed with a hybrid method by using a preconditioned conjugate gradient for the Shur complement and a direct method for the internal node matrices. A speed-up of about 3 and a saving in memory of about 25% is reached in this way. The matrix-by-vector product of both sparse and full matrices is a fundamental operation for which an efficient programming and storage technique is decisive.

The size of 2D-geological models that can be handled in a reasonable time on computers of this kind is nowadays hardly sufficient, whereas 3D modelling is completely unfeasible. However, the introduction of new FEM algorithms (Legendre spectral elements, element-by-element approach, and domain decomposition) coupled to the use of new emerging computer architectures are encouraging for the future.

E D PROVIDAS

A simple and efficient solution for the facet approximation of shells

A method for adding drilling rotation degrees of freedom to the constant strain triangle is presented. The method is free from any deficiencies and does not alter the strain energy of the constant strain triangle. Superposition with the displacement version of the hybrid bending element yields the required six degrees of freedom per vertex. Several standard shell problems are solved. Attention is focussed on testing the capacity of the element to simulate inextensional bending of real arbitrary shells. For this purpose a displacement prescribed patch test and a matrix procedure are employed. The element, in contrast to other similar elements, is proven to respond satisfactorily to this type of deformation modes.

W RACHOWICZ

Anisotropic h-adaptive finite element method for hypersonic viscous flows with shock-boundary layer interaction

Viscous hypersonic flows are analysed using second order Taylor-Galerkin scheme. An h-adaptive version of the finite element method is used to approximate the solutions. The method supports anisotropic h-refinements, i.e. the possibility of breaking of quadrilateral elements in one of the two possible directions. The strategy of selecting elements to be refined and the direction of the refinement is based on minimization of the interpolation error and it involves estimation of second order derivatives of the solution. The anisotropic strategy is especially effective in boundary layers where solutions are almost one-dimensional. The method allows for accurate resolution of aerothermal loads with considerably smaller computational effort than with the standard adaptivity. Numerical simulation of a shock-boundary layer interaction problem illustrates the approach.

A RAMAGE and A J WATHEN

On preconditioning for finite element equations on irregular grids

One efficient algorithm for solving Galerkin finite element equations is the preconditioned conjugate gradient method. A large number of preconditioning strategies have been proposed and these are generally analysed and compared using model problems: simple discretisations of Laplacian operators on regular computational grids, generally in two space dimensions. Many of these techniques have attractive theoretical convergence estimates on such model problems and it is principally for geometrically irregular (non-model) problems that the applicability and economy of preconditioned conjugate gradient methods is most useful. This is particularly true for problems on irregular unstructured three-dimensional grids.

Developing rigorous theory for irregular problems has proved to be more difficult than in analogous regular cases and, as a result, many aspects of working with unstuctured finite element grids are a lot less well understood. Here we present some theoretical results which apply to any finite element grid regardless of irregularity. We extend the work of the second author to find easily computed *eigenvalue* bounds for a class of *finite element* matrices on irregular grids. In addition, we obtain (weak) bounds on the interior *eigenvalues* of a general *finite element* matrix. These are useful in the analysis of various preconditioners for irregular finite element problems. We also present a result concerning the effect of a certain class of element-based preconditioners on the matrix condition number, which is again relevant to the rate of PCG convergence.

A RANJBARAN

A precis of developments in modelling embedded reinforcement for the finite element analysis of reinforced concrete structures.

To date three alternatives are available for modelling of reinforcement in reinforced concrete elements, i.e. discrete, smeared and embedded.

In this paper various techniques of modelling reinforcement are critically reviewed and their merits and shortcomings are highlighted. The embedded models are put into two general categories, i.e. locally embedded model (LEM) and globally embedded model (GEM). When the trajectory of a reinforcement in isoparametric coordinates of concrete is assumed as *a priori*, the model is called LEM. On the other hand, if the trajectory in global coordinates is known the term GEM is used. A totally general GEM is proposed. The relevant mathematical formulation for computation of stiffness and load contribution of a reinforcing bar to those of concrete are derived. The two- and three-dimensional problems are treated in a unified manner. Moreover, it is shown that all previous embedded models, available in the literature, can be considered as special cases of the proposed model. The proposed model is implemented in a finite element program. Numerical examples are included to verify the efficacy of the model.

E RANK

A zooming-technique using a hierarchical hp-version of the finite element method

The hp-version of the finite element method combines local mesh refinement with an increase of the polynomial order of the shape functions. It has been shown theoretically and in many numerical examples, that exponential rate of convergence in energy norm can be obtained for linear elliptic boundary value problems. The hp-version has also been applied successfully to more general problems like reaction diffusion equations or nonlinear Navier-Stokes equations. Recently a variant of the hp-version as a combination of a high order approximation with a domain decomposition method has been suggested [1] by the author. This 'hp-version' is very similar to Fish's 's-version' [2] using a superposition of independent finite element meshes.

The basic idea can be explained as follows. In a first step of the analysis a pure p-version approximation is performed on a coarse finite element mesh. Controlled by user interaction or by an a posteriori error estimation the coarse mesh is then covered partially by a geometrically independent fine mesh. On this second mesh a low order approximation is performed and the global approximation is defined as the *hierarchical sum* of the p-approximation on the coarse mesh and the h-approximation on the fine mesh. Global continuity of the finite element solution can be guaranteed by imposing homogeneous conditions at the fine mesh boundary. The hierarchical nature of the approximation also reflects in the structure of the arising linear equation system and can be used in an efficient solution algorithm. As an example a composed mesh and contour lines for a potential flow around a well will be presented. On the coarse mesh polynomial degree $p = 4$ is used, the fine mesh approximation is performed with linear elements.

The paper discusses algorithmic details of the suggested approach and addresses the question of how this domain decomposition can be implemented without major modifications into existing finite element codes. An extension of the method offers also the possibility of 'zooming' areas of critical solution behaviour. It is possible to compute the global p-version approximation with 'averaged' material parameters, and only locally to resolve the structure in detail. It will be shown how the hierarchical approach offers the possibility of a consistent modelling of such local-global behaviour. In several numerical examples the efficiency and accuracy of the method will be demonstrated.

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H-G ROOS

Nonconforming exponentially fitted elements

We consider the singularly perturbed boundary value problem

$$- \epsilon \Delta u + b \nabla u + cu = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Omega \qquad (1)$$

under the assumptions $c - 1/2 \, div \, b > 0$, $| b | \neq 0$. There are a lot of special finite element techniques to discretize (1) in the convection dominated case: upwinding (Griffiths, Mitchell), symmetrization (Morton), streamline diffusion (Hughes). But only based on some exponential fitting is it possible to show uniform convergence with respect to the singular perturbation parameter. A first result in this direction is due to O'Riordan and Stynes using tensor products of usual 1D-L-splines for both, the trial and the test space. Under the restrictive assumption $b = (b_1, b_2)$ and $b = (b_1(x), b_2(y))$ on a rectangular domain they proved uniform $O(h^{1/2})$ convergence.

For the general case $b = (b_1(x,y), b_2(x,y))$ it seems to be adequate to use exponentially fitted splines which are no longer globally continuous. Therefore it is necessary to investigate the corresponding *nonconforming* Petrov-Galerkin methods.

First we will present some abstract convergence theorems for nonconforming Petrov-Galerkin methods which generalize the well known theorems of Strang.

Further we will show that these abstract results are a suitable tool to handle nonconforming exponentially fitted elements. We study the model problem

$$- \epsilon u'' + b(x) u' + c(x) u = f(x)$$
$$u(0) = u(1) = 0$$

and choose the following combinations for the trial and the test spaces:

(i) conforming linear elements/approximated (nonconforming) L^*-splines
(ii) approximated L-splines/approximated L^*-splines.

In both cases we are able to prove that the discrete problems admit unique solutions. The key of the error analysis consists in establishing a stability condition

$$\exists \psi \in X_h : \; b_h(u_h, \psi) \geq m_h \| u_h \| \, \| \psi \| \quad (m_h > 0)$$

for the corresponding bilinear form

$$b_h(u_h, \phi) = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \epsilon u_h' \phi' + \frac{1}{2} b_i u_h' \phi - \frac{1}{2} b_i u_h \phi' + (c_i - \frac{1}{2} b_i') u_h \phi \, dx.$$

Finally we prove the optimal error estimations

(i) $\| u^I - u_h \| \leq C h^{1/2}$

(ii) $\| u - u_h \| \leq C h^{1/2}$

(C is independent of ϵ) with respect to the adapted energy norm.

S RIPPA and B SCHIFF

Improving the accuracy of finite-element solutions to two-dimensional elliptic problems

We present a method for improving the accuracy of a finite-element solution of a two-dimensional elliptic problem over a triangular mesh by modifying the triangulation whilst keeping the mesh points unaltered. The triangulation is modified by a process of "local optimisation" in which each quadrilateral formed by two triangles sharing a common edge is considered in turn. The value of the energy integral over the two triangles is compared with the integral over the two triangles which would be obtained had the given quadrilateral been divided by the opposite diagonal If the latter integral is less than the former, we modify the triangulation by replacing the diagonal of the given quadrilateral by the oposite diagonal. As the remainder of the triangulation is unaltered, the value of the energy integral over the whole domain has decreased. After completing a sweep of local optimisation over all of the quadrilaterals, the finite element equations may be solved again over the new triangulation to obtain new nodal values. Alternatively, before re-solving, additional sweeps of local optimisation can be carried out using the same nodal values. In either case, once the equations have been solved over the new triangulation, local optimisation is found to lead to no significant improvement in the solution. A third possibility is to terminate the procedure without re-solving at all.

The method has been tested for the Poisson equation using linear elements over each triangle, and for the biharmonic equation using (C^1) cubic macroelements of Clough-Tocher and reduced Clough-Tocher (normal derivative linear over the external edges of the macro-element) type. Various problems with known solutions were solved, and the process often led to considerable improvement in the results, particularly when the solution has a marked directional dependence. For the biharmonic equation, two problems with a boundary singularity were also solved, the determination of the Airy stress-function for a rectangular elastic plate containing an edge crack under uniform tension, and the Stokes flow in a square domain with one of the four walls moving with constant velocity (the "driven cavity" problem). Comparison with existing solutions obtained for these two problems by a variety of other methods shows that the "local optimisation" technique results again in a considerable improvement of the original finite-element solution.

C SCHWAB and M SURI

The numerical resolution of boundary layers and locking effects in the Reissner-Mindlin plate model

The Reissner-Mindlin plate model is characterized by both boundary layers and locking as the thickness t tends to zero. The boundary layers have the form $\exp(-\lambda p/t)$, where $\lambda > 0$ is a known constant, and p is the distance to the boundary. For t small, the approximation of such lkayers can be quite unsatisfactory unless the discretization parameter is sufficiently small. Locking effects can also cause the approximations to deteriorate when t is small. These occur due to the *Kirchhoff constraint* being imposed on both the true and the approximate solutions in the limiting case, as t tends to zero. In order to accurately approximate such models numerically, it is essential that the numerical method under consideration be designed in such a way that both these effects are resolved uniformlym, i.e. with errors independent of the thickness.

In this talk, we present various results for the approximation of boundary layers by the *h*, *p* and *h-p* finite element method. For the *h* version we show how exponentially graded meshes towards the boundary yield an optimal convergence rate independent of t. We also show that by a suitable combination of mesh refinement and degree selection (the *h-p* method), one can obtain uniform exponential rates of convergence.

Next, we discuss the locking and robustness properties of various *h* and *p* type finite element methods. We quantify the exact degree of locking that can be expected if various t riangular and rectangular meshes are used. We show that the *p* version is free of locking for this problem.

K SEGETH

Numerical experience with grid adjustment based on a posteriori error estimators.

Recently, a variety of techniques for a posteriori error estimation have been theoretically developed and practically applied. A posteriori estimates can serve as a means for grid adjustment ensuring the optimal number and optimal distribution of grid points in the finite element method. We, however, need a solution of the problem on some grid to construct a new, optimal grid. This approach, therefore, is very suitable e.g. for solving parabolic partial differential equations by the method of lines. The analysis of the approximate solution at a fixed time level then yields a new grid to be used for the time step leading to the next time level.

In the paper, some approaches using the idea of monitor equidistribution, with monitors based on a posteriori error bounds as well as with those independent of error estimators, are presented together with numerical experience.

D SILVESTER and A J WATHEN

Efficient preconditioners for incompressible flow

Having an efficient Stokes solver is an important requisite of many Navier-Stokes solution algorithms. The Stokes operator is self-adjoint so iterative solution methods are an attractive approach for large problems. The only impediment to efficiency is the indefinite nature of the discrete system.

In this talk, some new theoretical results will be presented showing the effect of preconditioning discrete Stokes systems. Our analysis covers a variety of preconditioners, including simple diagonal scaling, incomplete factorisations and multilevel preconditioning. Our results apply to stable approximations of the Stokes operator, typically based on staggered grids, and also to recently developed *stabilised* approximations: equal order finite element methods or finite difference methods on unstaggered grids.

M SLODIČKA

Numerical solution of a parabolic equation with a weakly singular positive type memory term

There exist many models from heat conduction in materials with memory, population dynamics and visco-elasticity, which can be described by integro-differential equations. Usually, the memory term depends on the solution u or on its spatial derivatives. But there exist cases where this memory term can depend on the time derivative of u. Integro-differential equations of this nature appear in describing diffusion models for fractured media (c.f. Hornung-Showalter [1]).

We discuss the numerical solution of the following problem

$$u_t(t) - \Delta u(t) + \int_0^t a(t-s)u_s(s)ds = f(t,u(t)) \quad \text{in} \quad \Omega, \quad t > 0,$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \quad t > 0,$$
$$u(0) = v \quad \text{in} \quad \Omega,$$

where Ω *is a bounded convex polyhedral domain in* $\mathbb{R}^d$ ($d \geq 1$), $v \in \mathring{H}^1(\Omega)$. Our integral kernel (in the memory term is supposed to be weaking singular and positive type.

We use the backward Euler method for discretization in time and the Galerkin finite element method for discretization in space. We prove the convergence of our approximation scheme in some functional spaces, the existence and uniqueness of the solution, and produce some regularity results for exact solution.

Reference

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E STEIN and F-J BARTHOLD

Design optimization of rubber elastic structures

Sensitivity analysis of physical and geometrical nonlinear behaviour is an outstanding research field within design optimization. In this lecture sensitivity analysis of rubber elastic structures and its efficient implementaiton into a design optimization procedure is outlined.

Based on continuum mechanics at finite strains, essential remarks on the finite element method for nearly incompressible rubber elastic materials and on the efficient use of adaptive mesh refinement are discussed. Then the sensitivity analysis for rubber elastic materials with respect to changes of geometry and material parameters is described using a parametric model consisting of space coordinates, time and design variables. Due to large strains occurring in technical applications of rubber materials a current configuration approach is used both in structural and sensitivity analysis.

The concept of a design optimization procedure and integrated algorithms for Computer Aided Design (CAD), Finite Element Method (FEM), Mathematical

Programming and Sensitivity Analysis is explained. A highly modular subprogram and data base concept for these methods is used to implement efficiently the analytical derived sensitivities into the FE research program INA-OPT (INelastic Analysis and OPTimization) developed at our institute. Applications of the optimization procedure to material parameter identification problems in tyre layout are shown.

The methodology for the approximate solution of the inverse problem was used for the identification of material parameters in the Ogden material equation which can also be understood as a validation of the constitutive model for the tyre material investigated.

An optimal design of steel cords in a tyre layout based on a minimal internal energy equilibrium state is derived using the optimization procedure described.

R STENBERG

On linear and bilinear elements for Reissner-Mindlin plates

We consider the following elements:

1. The MITC4 analyzed by Brezzi and Bathe in the MAFELAP 1984 Conference for the case of rectangular elements. We give the error analysis for the more general class of meshes for which the "Q_1-P_0" Stokes element was analyzed by Pitkäranta and Stenberg in the same conference.
2. We consider the linear triangular MITC element in which the deflection and rotation are linear and the reduction operator is given by the lowest order rotated Raviart-Thomas element. We give an error analysis for the case when the triangular mesh is obtained from a quadrilaterial one by drawing the two diagonals in each quadrilateral. The error analysis can now be performed for the same quadrilateral meshes as for the MITC4. Our analysis seems to explain the good behaviour reported by Hughes and Taylor in MAFELAP 1981 for a very similar element. As a byproduct we also give an error analysis of the well known linear constant strain triangle with a "criss-cross" mesh for an incompressible material.
3. We show that both of the above elements can be modified so that they are optimally convergent without any restrictions on the meshes.

A TESSLER, H R RIGGS and S C MACY

Application of a variational method for computing smooth stresses, stress gradients and error estimation in finite element analysis

The displacement finite element method is known to be deficient as far as the accuracy of strain and stress predictions is concerned. This aspect is particularly regrettable since accurate predictions of strains and stresses are necessarily needed in structural design and failure predictions. Moreover, in the process of adaptively refining the finite element discretization to achieve converged solutions in an automated and systematic way, *a-posteriori* estimation of errors is required. Error estimation is commonly pursued by way of computing improved stress solutions which are used to measure the goodness of the finite element solution.

The source of the computational difficulty in recovering strains and stresses is well established. The displacement field is approximated with piecewise C^0-continuous shape functions ensuring continuity of displacements along element boundaries. The strains (and stresses) are the dependent quantities that are generally represented by first gradients of the displacements; hence, they are only C^{-1} continuous, exhibiting nonphysical discontinuity along element boundaries. Reasonably accurate strain (stress) results can generally be recovered at the limited number of discrete locations within a finite element, these being *optimal* (or Barlow) points. Since these points are in the element interior, and for the lowest-order elements there is only a single centroidal point, reliable stgrain (stress) values on the element boundaries are generally not available.

Recently, Tessler and co-workers developed a smoothing formulation based on a variational principle which combines weighted least-squares and penalty-constraint functionals in a single variational form and uses, in the general three-dimensional case, four independent field variables - the smoothed quantity (strain or stress) and its three orthogonal gradients. The unique characteristic of this approach is that the resulting stress field is globally C^1-continuous.

The mathematical formulation can be briefly stated as follows:

Let $\hat{\sigma}_p = \hat{\sigma}(x_p)$ $(x_p = (x_p^1, x_p^2, x_p^3),\ p = 1,2,\ldots,N)$ represent a set of discrete data, such as Barlow-point strains (stresses), and let $\Omega = \{x = (x^1,x^2,x^3) \subset \Re^3\}$ denote the domain of the finite element discretization defined in the three-dimensional orthogonal frame. We seek a smooth, scalar-valued continuous (stress) function $\sigma(x)$ which can best represent the discrete data by casting the problem as a minimization of the following least-squares/penalty-constraint functional

$$\delta\Phi = 0$$

with Φ given as

$$\Phi = \sum_{p=1}^{N} [\hat{\sigma}_p - \sigma(x_p)]^2 + \lambda \int_\Omega [(\sigma_{,1} - \theta_1)^2 + (\sigma_{,2} - \theta_2)^2 + (\sigma_{,3} - \theta_3)^2] d\Omega \tag{1}$$

where N is the total number of data points, x_p denotes the position vector of the data point, λ is a large *penalty* number, θ_1, θ_2, and θ_3 are independent continuous functions also defined on Ω, and $\sigma_{,i} = \partial/\partial x_i\, \sigma$ (i = 1,2,3). The minimization of Φ is performed with respect to the coefficients in σ and θ_i which serve as the unknowns in this problem. The first term in (1) represents a discrete least-squares functionall with the term

$$e_p = \hat{\sigma}_p - \sigma(x_p) \tag{2}$$

representing the *error* in $\hat{\sigma}_p$ as compared with the smooth solution $\sigma(x_p)$. The second term in (1) is a penalty constraint functional which, for $\lambda \rightarrow \infty$, yields the following constraint relations

$$\sigma_{,i} \rightarrow \theta_i \quad (i=1,2,3) \text{ on } \Omega\ . \tag{3}$$

This means that, for all practical purposes as long as λ is very large, θ_i (i = 1,2,3) represent the first order gradients of σ with respect to the orthogonal coordinates.

The present paper revists the smoothing variational formulation for the purpose of demonstrating its effectiveness and robustness in the three applicaiton areas: (1) recovery of accurate, globally C^1 continuous stresses (2) computations of ply-level transverse shear stresses in laminated composite shell analysis, and (3) computations of reliable, exact local-equilibrium error estimators needed in adaptive mesh refinement analyses. The success of the analyses (2) and (3) is related directly to the effectiveness of the smoothing method to recover stress (strain) gradients from a conventional, displacement-based finite element analysis.

J L TORRES

Finite element solution of unsteady-state mass transfer through a stationary liquid

Mass transfer by diffusion of one liquid component through another can be modelled by the expression:

$$-\nabla \cdot cD_{AB}\nabla y_A + \nabla \cdot c_A v + \frac{\partial c_A}{\partial t} - R_A = 0 \,. \tag{1}$$

If constant liquid density and constant diffusivity are assumed, and if the diffusion occurs through a stationary layer, the expression reduces to the so-called Fick's Second Law of Diffusion for non-reacting systems:

$$\frac{\partial c_A}{\partial t} = D_{AB}\nabla^2 c_A \,. \tag{2}$$

The calculation of a concentration profile as a function of time and position in such systems is usually carried out using equation (2), mainly because the rigorous application of (1) can lead to computational complications.

The assumptions that lead to equation (2) are usually appropriate, with one important exception. When the diffusion takes place in a stationary liquid layer, in many cases the diffusivity is a strong function of concentration; in these cases equation (1) would have to be used, with its non-linear terms. The calculations cannot be carried out using a conventional finite-difference formulation, because, in essence, the system becomes non-isotropic almost immediately, due to the rapidly varying diffusivity coefficient.

In this paper, an alternative finite element method for solving problems with high concentration dependence in D_{AB} is described. The problem requires a re-statement of the domain properties at every time step. The calculation of concentration-dependent diffusivity was carried out using a cubic spline model. The paper discusses accuracy vs. computational effort, and also a possible application in the experimental measurement of diffusivities.

F van KEULEN

A family of triangular plate bending elements, as a basis for geometrically nonlinear shallow shell elements

A class of shell elements can be formulated as a combination of plate bending and membrane elements. Generally, this results in facet elements. Some well-known examples are the facet elements based on the constant bending moment triangle, the Discrete Kirchhoff Triangle and on the Discrete Shear Triangle. In order to account for initially curved geometries and to improve numerical efficiency, shallow shell terms can be considered. This results in shallow shell elements, which were originally facet elements.

Here our attention is concentrated mainly on a family of plate bending elements which is very suitabale as a basis of the formulation of shallow shells. This family consists of low-order as well as higher-order plate bending elements. The family has members that are based on the Kirchhoff-Love assumptions and members that are based on first order shear deformation theory. The independent kinematic degrees of freedom are the nodal displacements, the rotations about the element sides and additionally, the transverse shear strains along the element sides.

Some numerical examples are shown, which demonstrate the features of the plate bending elements under consideration.

Finally, the way towards geometrically nonlinear shallow shell elements based on this family of plate bending elements is outlined. The formulation of efficient shallow shell elements is not completely straightforward. In particular, the higher-order elements will require more attention, in order to avoid membrane locking and to achieve a correct description of rigid body motions.

H van LENGEN, B MAY, F-G BUCHOLZ and H A RICHARD

A substructured elastic-plastic fracture analysis programme for parallel processing on transputer networks

In recent years the Finite-Element-Method became a powerful and excellent computational tool for the analysis of complex problems in engineering science. In order to enable the solution of large problems on low cost computers with small high speed memory (RAM) and slow peripheral storage media (Hard disk) special techniques have been developed. The two most important of these approaches are the substructuring techniques and the frontal solution method /1,2/. Using substructuring techniques means to subdivide the whole FE-Model into smaller sub-regions with two classes of variables: the internal variables and the external variables related to the nodes on main net level. Using such substructured models most steps of the FE-calculation can be done without synchronisation in parallel on different processors. The implicit parallelism of this method in combination with the frontal solution method provides an excellent basis for the utilisation of modern hardware developments for massive parallel computing [3].

This paper presents a finite element programme for complex elastic-plastic fracture analysis on transputer networks. In those networks each transputer has access to 4-16Mb RAM-memory and is connected through a maximum four links to other transputers of the network. The parallel version of PLASTPARA has been implemented under the HELIOS operating system both on a small PC-host based network and on the Supercluster SC320 of PC^2 (Paderborn Centre of Parallel Computing). The algorithms are designed and optimised with respect to the distributed-memory model of transputer hardware, so that the required I/O transfer between the processors and to the Hard disk is minimised. All sources are written in FORTRAN.

The programme PLASTPARA is split into a master programme and several worker programmes running on different transputers. So every substructure can be calculated in parallel up to the step of reducing all variables of the substructures to the external ones by the frontal solution method. In the next step the master programme will assemble and solve these subequations on one transputer and compute the displacements on main net level. In the last step the internal displacements are calculated by parallel backward substitution on subnet level. By this method a high degree of parallel processing is already achieved, and will be improved by a parallel main-net solver in future.

As an example for the efficency of this approach 2D and 3D FE-analyses will be discussed. The effect of the number of substructures and transputers will be given for both implementations of PLASTPARA.

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M VANMAELE

On an external finite element method for a second-order eigenvalue problem on a concave 2D-domain with Dirichlet boundary conditions

We consider a finite element method for a second-order elliptic eigenvalue problem in a concave bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$, where homogeneous Dirichlet boundary conditions are imposed. In the method a 'variational crime' is committed, the approximate eigenvalue problem being formulated on a domain $\Omega_h \not\subset \Omega$. Hence the finite element approximation space $V_h \not\subset V$, $V = H_0^1(\Omega)$.

We restrict ourselves to a linear triangular finite element mesh (h mesh parameter). The convergence of the method and optimal error estimates for both the eigenvalues and the eigenfunctions are obtained in [1]. The proofs of the estimates in that paper rest heavily upon the $O(h^2)$-estimate for the error of the elliptic projection in the $L_2(\Omega_h)$-norm, see [1, Theorem 4.8]. In the present paper we prove this estimate in an alternative way, proceeding similarly as in the proof of the well-known Aubin-Nitsche trick. Moreover in this paper we obtain an optimal, $O(h^2)$,estimate (from above) for the eigenvalues under weaker conditions on $\partial\Omega$ than in [1, Theorem 5.5].

1. M Vanmaele and A Ženíšek, External finite element approximations of eigenvalue problems, *RAIRO M²AN* (accepted)

K P WANG and J C BRUCH, Jr

An efficient iterative parallel finite element computational method

An implementation of using parallel computation along with adaptive mesh finite element analysis will be discussed. In the h-refinement of the finite element analysis, the number of nodes and elements will be increased after a finite element mesh has been refined. The computation load for the finite element system also increases. This situation can be resolved by using a multi-processor computer so that the computation load can be shared by many processors after an appropriate load balancing operation. A very efficient parallel iterative scheme for the finite element system was utilized to meet the requirements necessary to solve the system on a multi-processor (parallel/concurrent) computer. In addition, different refinement schemes were employed to investigate the effectiveness of the adaptive mesh finite element analysis procedure. The model problem considered was a free surface seepage problem formulated using a fixed domain method. This imposed a restriction on the numerical iterative approach. Numerical results were obtained on an iPSC/2 Hypercube concurrent computer.

M K WARBY and J R WHITEMAN

The computational modelling of the thermoforming process for the creation of axisymmetric container structures

This paper is concerned with the computational modelling of the thermoforming process which is a process used in the packaging industry to create polymeric container structures by forcing thin sheets into moulds. In our model only axisymmetric geometries are considered and the sheet, although generally multilayered, deforms to a reasonable approximation like a membrane. When the containers to be formed are "shallow" it is usually satisfactory to use pressure alone as the forcing action and this case has been considered by several authors in the recent literature. In this context, the term satisfactory usually refers to the thickness distribution of the formed container (pot). In our model the process starts with a sheet of uniform thickness and as the deformation proceeds, the thickness distribution becomes non uniform. When the thickness distribution becomes highly non uniform or more specifically when any part of the sheet becomes too thin then we would describe the container as being unsatisfactory. This is the situation that occurs when pressure alone is used and the pot is deep. In the manufacturing process deep pots are instead created by the combined actions of a rigid plug and an applied pressure. The use of the plug leads to a different deformation history before contact is made with the mould than is the case when pressure alone is used and, since the material is viscoelastic, this has the potential of leading to a thickness distribution which is different from that obtained with pressure alone. However the main reason for the more satisfactory distribution of the thickness being obtained is as a result of the slight cooling effect that the plug has on the sheet. Specifically the plug, which is cooler than the sheet, has the effect of slightly cooling and hence slightly stiffening the part of the sheet to which it makes contact. The purpose of constructing a computational model is to be able to quantify this effect for different plug shapes, depths and temperatures. This requires a model which takes account of the following effects.

The model must take account of the large deformation and hence the equations are nonlinear. There is a contact problem as the sheet slides on the plug and for the part of the sheet in contact with the plug there is the nonisothermal cooling effect. When the sheet is inflated against the mould there is a simpler contact problem as the sheet sticks to the mould. For the constitutive model for the polymers only standard Mooney Rivlin elastic and simple viscoelastic relations have been considered as no more better experimentally based information was available. Fortunately the model involved only one space dimension and thus the finite element discretization is straightforward. In our model cubic Hermite elements using a carefully chosen moving mesh is used in order to capture the main features at relatively low cost. A form of Newton's method is used at each stage to solve the nonlinear system of equations. Results will be presented for the intermediate membrane profiles and the final thickness distribution predicted using the model.

A J WATHEN

Element-by-element preconditioning

Element-by-element preconditioning methods were introduced by T J R Hughes and co-workers in 1983 as efficient and highly vectorisable techniques which arise out of and are highly consistent with natural finite element datastructures.

In this talk we present a description and analysis of such 'EBE' preconditioning methods for self-adjoint elliptic partial differential equations in 2- and 3-dimensional domains. As well as analysis for model Poisson problems, we present results for problems with discontinuous variable coefficients and for constant coefficient problems discretised using elements with large aspect ratios.

Some of this work is joint with Han-Chow Lee.

J WEISZ

A posteriori error indicator and estimator for time-space FE discretization of parabolic problems

In the area of adaptive numerical methods for parabolic partial differential equations there are two traditional directions: In the first one the equation is discretized in space and then the corresponding system of ordinary differential equations is solved by usual methods using the adaptive control of the time step. The second possibility is to discretize the equation in time and then to solve the corresponding elliptic equations adaptively.

The simultaneous time-space discretization [1] is a generalization of the above methods. The equation is discretized in time and space simultaneously using time-space finite element spaces. We obtain an equation which is "elliptic" and finite-dimensional. Our aim is to derive a-posteriori error indicators and estimators for such discretization of the linear parabolic initial-boundary value problem. Using the technique of [2] the error of the finite element approximation is estimated in terms of the discretization parameter, data of the problem and the approximate solution.

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H WERN

Finite-element solutions for mechanical drilling methods

The principle for evaluating the unknown depth distributions of residual stresses from drilling methods relaxed strain data is viewed as an Integral Method [1-3]. The mechanical hole drilling [4,5] as well as the ring-core-method [6,7] are analyzed using finite elements. The influence of the notch sensitivity is discussed for different hole-, slot- and sample geometries. Temperature effects caused by the drilling procedure are considered. From the calculated strain relaxation data, possible integral operators for the relaxation process are discussed.

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N-E WIBERG and F ABDULWAHAB

Improved patch stress recovery procedure by equilibrium and boundary conditions

The cornerstone of an adaptive FE-analysis is a cheap and accurate calculation of the local and global distribution of the error, so that successively improved solutions can be obtained. The Zienkiewicz-Zhu error estimate is undoubtedly the most practical *a posteriori* error estimator capable of estimating both local and global error of discretization. This error estimator, for instance in the energy norm, is asymptotically exact if the recovered derivatives are superconvergent. Very recently two important recovery procedures that recover superconvergent gradients have been developed. Namely the Superconvergent Patch Recovery (developed by Zienkiewicz and Zhu) and patch recovery based on superconvergence and equilibrium (developed by the authors). Both methods are based on local patches of elements. Their performance is excellent for recovering derivatives for points in inner patches. Generally speaking the quality of recovered derivatives at or near boundaries is however inferior compared to that at interior points.

Both methods do not take into consideration the boundary conditions which can be either prescribed displacements or prescribed tractions. In this paper, a postprocessing technique for points on or near boundaries is described. We introduce a weighted least-squares polynomial fit in an attempt to 'force' the recovered derivative field to satisfy the prescribed boundary conditions. In the literature, the least square methods are deemed to be insensitive to different weighting functions over a wide range. In spite of this, we assume that higher values of weighting functions for prescribed boundary conditions would improve the recovered derivatives near boundaries. Several numerical examples illustrate the very good performance of the proposed recovery technique at or near boundaries.

J R WILLIAMS and K AMARATUNGA

Matrix and image decomposition using wavelets

This paper demonstrates how the wavelet transform may be advantageously applied to matrix and image data. The resulting decomposition yields a reduced set of data which retains the essential properties of the original data. As a result, the wavelet decomposition enables us to capture the behaviour of a physical system in a greatly simplified mathematical model.

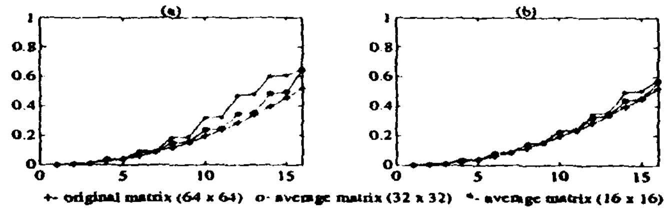

Figure 1: Eigenvalues of a stiffness matrix and wavelet reduced matrices using (a) D6 and (b) D20

Figure 1 compares the eigenvalues of the stiffness matrix for a spring-mass system with those of the average matrices of 1/4 and 1/16 the size derived using the Daubechies D6 and D20 wavelet systems. The figure shows that good estimates of the lower eigenvalues of the stiffness matrix may be obtained from the averaged matrices. This result has significant implications in modal analysis, where the lower eigenfrequencies are often the most critical of all.

Similarly, in the wavelet-Galerkin method for solving PDE's, the resulting wavelet differential operator matrix may be decomposed using the wavelet transform, leading to an averaged differential operator matrix and a reduced set of equations. By solving the reduced set of equations, a good estimate of the solution may be obtained with relatively little work.

The averaging property of wavelets is attractive in itself for the rapid solution of engineering problems. The appeal of wavelets is greatly enhanced by their hierarchical properties, which permits the subsequent addition of detail to the solution, thereby providing the capability for progressive refinement of the solution.

Z P WU and D PHILLIPS

A general bond-slip formulation for embedded reinforcing bar elements.

This is a generalized formulation developed for isoparametric elements including embedded reinforcing bars and bond-slip effects, in which the representation of the bond-slip behaviour is evaluated by prescribing bond characteristics at a set of artificial "nodes" introduced along the bar. The embedded bar element and the parent concrete element are now general finite elements removing all the restraints from the conventional approaches. The condition of compatibility for the system is obtained by only condensing the middle nodes of the bond-slip while the equilibrium condition is satisfied by assembling the stiffness matrix of the reinforcing bar and bond slip into that of the parent concrete element at element level. Several examples are employed to illustrate the performance of the formulation, i.e. pull-out test, transfer test and beam-column joints. The numerical results are compared with experimental ones where good agreements have been achieved.