AD-A265 960
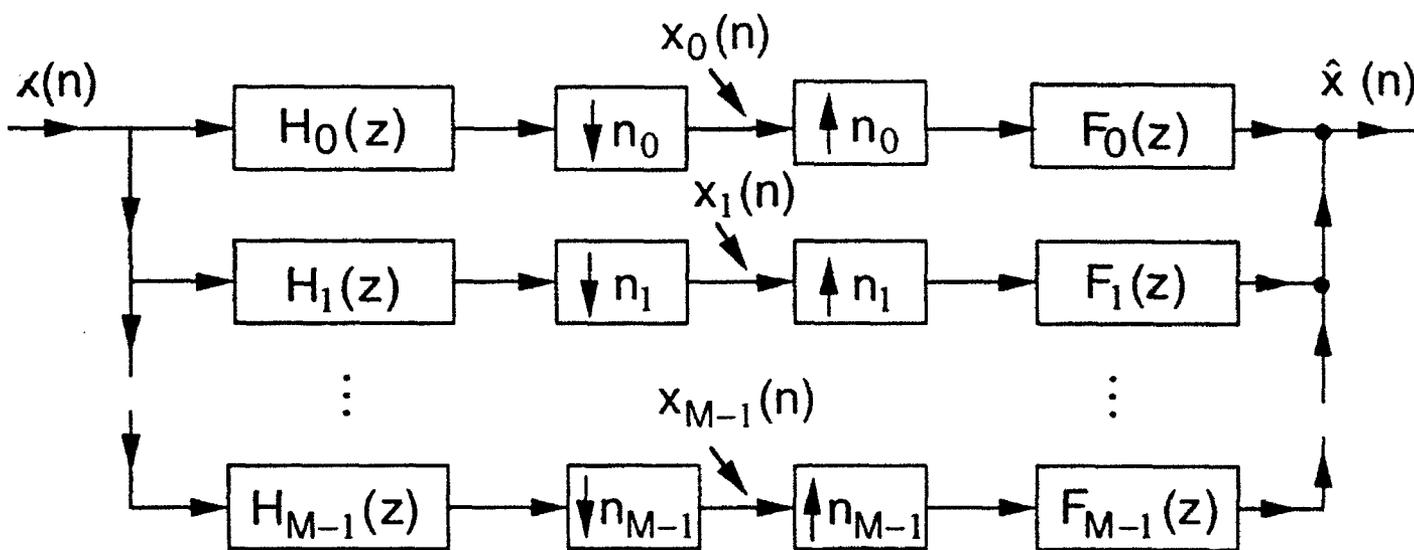
March '93

RESULTS ON BIORTHONORMAL

FILTER BANKS

Igor Djokovic and P. P. Vaidyanathan

Orthonormalization of a biorthonormal
filter bank

California Institute of Technology

Pasadena, CA 91125

93-13418

93 6 15 120

RESULTS ON BIORTHONORMAL FILTER BANKS

Igor Djokovic, student member, IEEE and P. P. Vaidyanathan. Fellow. IEEE

Abstract. For a maximally decimated nonuniform filter bank. the perfect reconstruction (PR) property is equivalent to biorthonormality. We start from this result and derive a number of properties of PR filter banks. For example, no two integer decimators in a biorthonormal system can be coprime; moreover if all analysis and synthesis filters have unit energy, then perfect reconstruction is equivalent to orthonormality. We also generalize the Nyquist and power complementary properties of orthonormal filter banks, for the biorthonormal case. We then show that whenever the decimation ratios are such that biorthonormality is possible with rational filters. it is in particular possible to obtain orthonormality with rational filters. This is done by developing an orthonormalization procedure. While reminiscent of the Gram-Schmidt approach. the procedure converges in a finite number of steps and furthermore preserves the filter-bank like form of the basis functions. We then modify the orthonormalization procedure for the application of subband decorrelation. It will be demonstrated that mere decorrelation of subband signals does not necessarily optimize the coding gain of a system. Finally we consider the problem of alias cancellation. and obtain a generalization of a previously known necessary condition called compatibility.

---

1. INTRODUCTION

Fig. 1(a) shows an M channel filter bank with integer decimation ratios n_k. The input signal $x(n)$ is split into M signals which are passed through the analysis filters $H_0(z), H_1(z), \ldots, H_{M-1}(z)$ and decimated by n_k (integers) for $k = 0, 1, \ldots, M - 1$. At the synthesis end, these signals are expandeded, passed through synthesis filters $F_0(z), F_1(z), \ldots, F_{M-1}(z)$ and added. When $\hat{x}(n) = cx(n - n_0)$, this system achieves perfect reconstruction (PR). In this paper the PR property corresponds only to $\hat{x}(n) = x(n)$, as this eliminates some inconvenient notations without much loss of generality.

When $\sum_k 1/n_k = 1$, we have a maximally decimated filter bank. A special case is when $n_k = M$ for all k. We call it the uniform filter bank. Every nonuniform maximally decimated filter bank can be equivalently represented by a "larger" uniform filter bank as in Fig. 1(b) (see [1], [2] and [3]). The theory of uniform filter banks is well developed and such a system is shown in Fig. 2(a). The analysis and synthesis filters can be expressed in polyphase form as

$$H_i(z) = \sum_{k=0}^{M-1} z^{-k} E_{ik}(z^M) \quad \text{and} \quad F_i(z) = \sum_{k=0}^{M-1} z^k R_{ki}(z^M). \tag{1.1}$$

With each filter represented like this, the system can be drawn as in Fig. 2(b) where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are, respectively, the polyphase matrices of the analysis and synthesis banks. This system has the PR property $\hat{x}(n) = x(n)$ if and only if $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$. There are different ways to design a uniform filter bank that achieves PR, so the existence of rational filters (i.e., transfer functions which are ratios of two polynomials) satisfying the PR property is trivially guaranteed. But in the nonuniform case, it is not always possible to achieve PR with rational filters [1] (block decimation [3] is not considered in this paper). Notice, however, that ideal filters (non rational, with possibly complex impulse response) can always be found such that the PR property holds for any set $\{n_k\}$ satisfying $\sum_k 1/n_k = 1$. So, whenever we discuss existence of PR systems, the discussion pertains *only to rational filters*.

A set of *necessary and sufficient* conditions on the set $\{n_k\}$ for PR to be possible is not known. On the other hand, we know some sufficient conditions. If the numbers $\{n_k\}$ are coming from a

tree structure, for example, then we can have PR with rational filters [4], [5]. Not all decimation ratios allowing PR, allow it with a tree structure. [†]For example consider $M = 23$ and the set

$$\{6, 10, 15, \underbrace{30, \ldots, 30}_{20 \text{ times}}\}.$$

This set satisfies satisfies $\sum_{k=0}^{22} 1/n_k = 1$. The filters that achieve PR are $H_i(z) = \tilde{F}_i(z) = z^{-l_i}$ where the set of l_i's is $\{0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 19, 20, 22, 23, 25, 26, 27, 28, 29\}$. For this, note that the output of the ith decimator is $x(mn_i - l_i)$. We want every input sample to go through one and only one branch, which is equivalent to saying that $mn_i - l_i \neq kn_j - l_j$ for $i \neq j$ and any choice of m and k. On the other hand, since $\gcd(n_0, n_1, \ldots, n_{22}) = 1$, these numbers cannot come from a tree structure (if there were a tree, the decimation ratio at the first level of the tree would be a factor of this gcd). Because of such possibilities, we will not assume that $\{n_k\}$'s come from a tree. Before we discuss these issues in greater detail, let us explain some conventions and definitions in this paper.

All our signals are in l_2 space (i.e., the space of finite energy sequences). The inner product is defined as

$$< x(n), y(n) > = \sum_{n=-\infty}^{\infty} x(n)y^*(n). \qquad (1.2)$$

and the norm $||x(n)||_2$ will be defined according to $||x(n)||_2^2 = < x(n), x(n) >$.

Filter bank-like systems

The reconstructed signal $\hat{x}(n)$ at the output can be written as

$$
\begin{aligned}
\hat{x}(n) &= \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_i(n_i k - m)x(m)f_i(n - n_i k) \\
&= \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} x_i(k)f_i(n - n_i k).
\end{aligned}
\qquad (1.3)
$$

where

$$x_i(k) = \sum_{m=-\infty}^{\infty} h_i(n_i k - m)x(m) \qquad \text{(decimated subband signals).} \qquad (1.4)$$

[†] The authors would like to thank Tsuhan Chen for pointing out this example.

Equation (1.3) is an expansion of $\hat{x}(n)$ in terms of sequences $\{f_i(n - n_i k)\}_{i=0}^{M-1}$, $\forall k \in \mathcal{Z}$ ($\mathcal{Z}$ is the set of all integers). Here $x_i(k)$'s are the "filter bank transform coefficients" (i.e., the decimated subband signals). The set of sequences

$$\eta_{ik}(n) = \{f_i(n - n_i k)\}_{i=0}^{M-1}, \qquad k \in \mathcal{Z} \tag{1.5}$$

is made of a special form of functions namely shifts (by integer multiples of n_i) of a finite number of sequences. We refer to (1.5) as a filter bank-like system. The set of sequences $f_i(n - n_i k)$ will be referred to as the n_i-shifted versions of the synthesis filters $f_i(n)$. Later on it will be shown that this system is a Riesz basis for l_2 space. [This is subtle because, in an infinite-dimensional Hilbert space, completeness and independence (see Sec. 2.2. for the definition) of a set of vectors is *not* sufficient to conclude that these vectors form a Riesz or unconditional basis; see Corollary 4, Sec. 2.2.]

Birothonormality

Definition 1.1. A system of sequences $\{h_i(n - mn_i), f_l(n - kn_l)\}$, $0 \leq i, l \leq M - 1$ for all $m, k \in \mathcal{Z}$ is called a biorthonormal system if

$$< h_i(n - mn_i), f_l^*(-n + kn_l) > = \delta(i - l)\delta(k - m) \qquad \text{(biorthonormality)}. \tag{1.6}$$

In the special case of orthonormal filter banks, the perfect reconstruction property is achieved by setting $f_k(n) = h_k^*(-n)$. In this case, the biorthonormality reduces to

$$< h_i(n - mn_i), h_l(n - kn_l) > = \delta(i - l)\delta(k - m) \qquad \text{(orthonormality)}. \tag{1.7}$$

If the above equation holds for some i and l, we often say that "the two filters $H_i(z)$ and $H_l(z)$ are orthonormal". It should be borne in mind that the acutal meaning depends on n_i and n_l. It will be shown that perfect reconstruction (PR) in a maximally decimated filter banks implies biorthonormality. [†] In Section 2.1., we will show that the most general form of PR for a maximally

---

[†] This is a fairly subtle fact, holding only because of *maximal decimation*. For example, consider a two-channel undecimated system with filters $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1}$, $F_0(z) = F_1(z) = 1/2$. Then we have PR, but it can readily be verified that biorthonormality is not satisfied.

decimated filter bank is a biorthonormal *basis* for l_2 space, formed by the analysis and synthesis filters. This issue has come up in earlier work. but has not been shown or proved this way. The relation between filter banks and wavelets. and the role of orthonormality has been discussed in [6], [7], [8], [9] and [4].

From this result we can obtain useful conclusions. For example we will conclude that there must be *no coprime pairs* in the set of decimation ratios $\{n_k\}_{k=0}^{M-1}$ in order for PR to be possible with rational filters (ideal perfect reconstruction filters can always be defined for any maximally decimated filter bank). Thus, the existence of rational filters achieving PR is a nontrivial question in the nonuniform case. By contrast, in the uniform filter bank theory. we can design analysis filters and just invert the polyphase matrix $\mathbf{E}(z)$ (as long as its determinant is not identically zero) in order to obtain PR (with possibly unstable or noncausal synthesis filters).

Orthonormality

A *second* question of interest in nonuniform filter bank theory is the following: suppose the integers $\{n_k\}_{k=0}^{M-1}$ are such that a biorthonormal PR system (with biorthonormal. rational filters) exists. Does it mean that an orthonormal PR system also exists? (Again. for the uniform case. the existence is trivially guaranteed simply by constraining $\mathbf{E}(z)$ to be paraunitary.) We will show by construction that for a given set of integer decimation ratios $\{n_k\}_{k=0}^{M-1}$. the existence of biorthonormal systems implies the existence of orthonormal PR systems as well (Sec. 3.).

The procedure to convert the biorthonormal system to an orthonormal one is reminiscent of the Gram-Schmidt (GS) procedure, but is not the same. for a variety of reasons. First, the orthonormalization of the basis is required to preserve the filter bank-like form of the basis: conventional GS procedure would not give us this. Furthermore. using z-domain analysis and the special form of our system, we will be able to do the orthonormalization process in a *finite number of steps* (even though l_2 is an infinite dimensional space). This is another point of departure from the traditional GS technique.

At this point, the reader should be warned that this orthonormalization procedure is mostly

4

of theoretical importance. The filters resulting from the orthonormalization not only are IIR in general, but also have huge orders; the proposed orthonormalization is not an alternative design technique for filter banks (after all we do not have biorthonormal filters to start the orthonormalization with). For the purpose of subband coding, there exists a simple scheme to generate inexpensive orthonormal filter banks, based on the so-called power symmetric filters (pp. 204 [10]). These can also be used in a tree structure to obtain a subclass of nonuniform IIR orthonormal systems.

1.1. Paper outline

In Sec. 2. we discuss the detailed reasons why biorthonormality and perfect reconstruction (PR) are identical concepts for *maximally* decimated filter banks[†]. Several corollaries of this result are derived in Sec. 2.2. For example, we show that for PR to be possible, no two decimation ratios can be relatively prime. We also show that if a perfect reconstruction system is such that all the analysis and synthesis filters have unit energy, then the system becomes *orthonormal* (paraunitary in the uniform case).

In Sec. 3. we show that whenever the decimation ratios $\{n_i\}$ of a maximally decimated system are such that perfect reconstruction is possible (i.e.. such that there exist biorthonormal filters), then in particular, there exists an orthonormal filter bank. The proof is constructive. that is. given a set of biorthonormal filters we show how to find a set of orthonormal filters starting from these. Numerical examples are included. In general, the resulting orthonormal filters turn out to be IIR even if we start with an FIR biorthonormal system. However, the IIR filters are guaranteed to be free from poles on the unit circle. This means that, should they turn out to be unstable, a noncausal implementation can be found which is stable [12], [13].

The orthogonalization technique will then be used in Sec. 4. for a different purpose. namely decorrelation of subband signals of a *uniform* filter bank. In other words, referring to Fig. 2(a), imagine that we are given a wide sense stationary input signal $x(n)$. We will show how to find the analysis filters $\{H_k(z)\}$ such that the decimated subband signals $x_k(n)$ and $x_i(m)$ are uncorrelated

---

[†] A brief sketch of some of the results has been presented in [11]

for all n, m (for $k \neq i$). While this might appear to be similar to the Karhunen-Loeve transform (KLT) [14], there is a fundamental difference. Namely, the KLT decorrelates $x_k(n)$ and $x_i(m)$ only for $n = m$. It is of course true that we can decorrelate $x_k(n)$ and $x_i(m)$ for all n, m trivially by use of ideal non overlapping filters. But we will construct finite order (rational) filters, tuned to the statistics of the WSS input $x(n)$ with the power spectrum $S_{xx}(z)$ which is assumed to be a rational function.

As a consequence of the analysis in Sec. 4., we learn that if a uniform filter bank is orthonormal [paraunitary $\mathbf{E}(z)$], then the subband signals cannot be decorrelated in this way, unless the power spectrum $S_{xx}(e^{j\omega})$ of $x(n)$ has the form $F(e^{j\omega M})$ (i.e., has a period of $2\pi/M$ rather than 2π). Whether this is a disadvantage of orthonormal systems is arguable because, decorrelation of the subband signals does not necessarily maximize the coding gain of the system, again *unlike* in transform coding! We will demonstrate this with an example. This section is restricted to the uniform case, mainly because of a technical difficulty: even though the subband signals of a filter bank are WSS for WSS input $x(n)$, they are in general not *jointly* wide sense stationary in the nonuniform case; many of the standard second order statistical tools cannot then be applied.

In Sec. 5. we derive some further necessary conditions on the decimation ratios $\{n_k\}$, for perfect reconstructibility. These can be regarded as generalizations of the compatibility condition given in [1] and [p. 285 of 10]. Some of the technical details which arise in the proofs have been moved to the Appendices (A-C) to provide a smoother reading.

1.2. Notations and conventions

1. The quantities $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ stand for transposition and transpose conjugation of the matrix $\mathbf{A}$. The notation $\tilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(1/z^*)$. Thus $\tilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(z)$ on the unit circle.

2. $\mathcal{Z}$ represents the set of all integers $0, \pm 1, \pm 2, \ldots$

3. *Special integers.* The integer M denotes the number of channels of the nonuniform system (Fig. 1(a)). The integer $L = \text{lcm}\ (n_0, n_1, \ldots n_{M-1})$. Also, $g_{ij} = \text{gcd}\ (n_i, n_j)$ throughout the

paper. The integers $\{k_i\}$ are numbers that satisfy

$$L = k_0 n_0 = k_1 n_1 = \cdots = k_{M-1} n_{M-1}. \tag{1.8}$$

4. $W_L = \epsilon^{-2\pi j/L}$. The subscript L is omitted whenever it is clear from the context. W is the $L \times L$ DFT matrix. It has elements $[\mathbf{W}]_{mn} = W_L^{mn}$. Note that $\mathbf{W}^\dagger \mathbf{W} = L\mathbf{I}$.

5. The AC (alias-component) matrix for analysis filters (defined for uniform L-channel filter banks) is the one with components $[\mathbf{H}(z)]_{mn} = H_n(zW^m)$. For the synthesis filters we define a similar matrix: $[\mathbf{F}(z)]_{mn} = F_n(zW^m)$.

6. A delay chain is represented by $\mathbf{e}(z) = [\begin{matrix} 1 & z^{-1} & \ldots & z^{-M+1} \end{matrix}]^T$ (see, e.g., Fig.2(b)).

7. The M-fold decimator has input-output relation $y(n) = x(Mn)$, or in the z-domain

$$Y(z) = \frac{1}{M} \sum_{i=0}^{M-1} X(z^{1/M} W_M^i). \tag{1.9}$$

The M-fold expander has the input-output relation

$$y(n) = \begin{cases} x(n/M), & n = \text{mul. of } M \\ 0 & \text{otherwise.} \end{cases} \tag{1.10a}$$

or in the z-domain

$$Y(z) = X(z^M). \tag{1.10b}$$

The notations $\downarrow_M$ and $\uparrow_M$ are used to denote these operations. Thus $a(n)\!\downarrow_M$ denotes the decimated version $a(Mn)$, and $A(z)\!\downarrow_M$ is the corresponding z-transform that is z-transform of $a(n)\!\downarrow_M$.

8. For uniform filter banks, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the polyphase matrices of the analysis and synthesis filter banks (Fig. 2(b)).

9. Perfect reconstruction (PR) means $\hat{x}(n) = x(n)$. In the uniform case it is equivalent to $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ (see Fig. 2(b)).

10. The so-called noble identity for multirate systems [10] can be stated, for our purpose, as follows

$$\left(A(z^{m_1}) B(z) \right)\Big|_{\downarrow m_1 m_2} = \left(A(z) \left(B(z) \right)\Big|_{\downarrow m_1} \right)\Big|_{\downarrow m_2}. \tag{1.11}$$

2. EQUIVALENCE OF BIORTHOGONALITY AND PR PROPERTY

For the study and design of uniform filter banks. there exist powerful tools such as the polyphase formulation and the AC matrix formulation. In order to use them in a nonuniform filter bank. we have to transform it into the equivalent uniform one [1], [2]. This is shown in Fig. 1(b). There are L branches (where L is the lcm of $\{n_i\}$), and each of them has the same decimation ratio. The analysis filters are numbered as

$$S_0(z), S_1(z), \ldots, S_{k_0-1}(z), S_{k_0}(z), \ldots \qquad (2.1)$$

and similarly for the synthesis filters $Q_i(z)$. Thus the analysis and synthesis filters are

$$S_i(z) = z^{pn_i} H_l(z), \qquad Q_k(z) = z^{-rn_m} F_m(z). \qquad (2.2)$$

where

$$i = p + \sum_{j=0}^{l-1} k_j, \quad 0 \le p \le k_l - 1, \qquad k = r + \sum_{j=0}^{m-1} k_j, \quad 0 \le r \le k_m - 1. \qquad (2.3)$$

Here $k_j = L/n_j$ and its meaning is clear; we just made k_i delayed filters from each original filter $H_i(z)$, i.e. each new filter comes from one of M original filters. The biorthonormality (1.6) means

$$\sum_n h_i(n - mn_i)f_l(-n + kn_l) = \delta(i - l)\delta(m - k), \quad 0 \le i,l \le M - 1. \quad m,k \in \mathcal{Z}. \qquad (2.4)$$

Eqn. (2.4) can also be rewritten as [4]

$$\sum_n h_i(n)f_l(mg_{il} - n) = \delta(i - l)\delta(m), \quad 0 \le i,l \le M - 1, \quad m \in \mathcal{Z} \qquad \text{(biorthonormality)}. \quad (2.5)$$

where $g_{il} = \gcd(n_i, n_l)$. This infinite set of conditions can be compactly written as a finite set of conditions in the z-domain

$$\left(H_i(z)F_l(z) \right) \Big\downarrow_{g_{il}} = \delta(i - l), \quad 0 \le i,l \le M - 1 \qquad \text{(biorthonormality)}. \qquad (2.6)$$

Again, if $F_i(z) = \tilde{H}_i(z)$. then the above property is called the orthonormal property. and can be written as

$$\left(H_i(z)\tilde{H}_l(z) \right) \Big\downarrow_{g_{il}} = \delta(i - l) \qquad \text{(orthonormality)}. \qquad (2.7)$$

8

or equivalently as

$$\left(F_i(z)\tilde{F}_l(z)\right)\Big|_{g_{il}} = \delta(i - l) \qquad \text{(orthonormality)}. \tag{2.8}$$

A filter bank with analysis and synthesis filters satisfying these biorthonormal (orthonormal) conditions will be called a biorthonormal (orthonormal) filter bank. We sometimes say that two filters $H_i(z)$ and $F_l(z)$ are biorthonormal if they satisfy (2.6). It should be noticed that the biorthonormality definition involves n_i and n_l.

In this section, we will show that the most general form of PR for a maximally decimated filter bank is a biorthonormal basis for l_2 space, formed by the analysis and synthesis filters. This issue has come up in earlier work, but has not been shown or proved this way. This will be followed by the derivation of a number of corollaries.

2.1. PR Implies Biorthogonality of Analysis and Synthesis Filters

Theorem 2.1. Let the system in Fig. 1(a) be a maximally decimated filter bank with decimation ratios $\{n_k\}_{k=0}^{M-1}$. If the filter bank has the perfect reconstruction property, the filters form a biorthonormal system, that is satisfy (2.6).

Remarks.

1. This is a fairly subtle fact, holding *only because of maximal decimation*. For example, consider a two-channel undecimated system with filters $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1}$, $F_0(z) = F_1(z) = 1/2$. Then we have PR, but not biorthonormality.

2. The converse of the above theorem also holds; see Appendix B of [15].

Proof. Let $S_i(z)$ and $Q_k(z)$ be analysis and synthesis filters of the eqivalent uniform filter bank. They are related to $H_l(z)$ and $F_m(z)$ by (see Fig. 1(b))

$$S_i(z) = z^{pn_l}H_l(z), \qquad Q_k(z) = z^{-rn_m}F_m(z), \tag{2.9}$$

where

$$i = p + \sum_{j=0}^{l-1} k_j, \quad 0 \leq p \leq k_l - 1, \qquad k = r + \sum_{j=0}^{m-1} k_j, \quad 0 \leq r \leq k_m - 1. \tag{2.10}$$

9

Let $\mathbf{S}(z)$ denote the AC matrix of the above analysis bank $S_i(z)$, that is $[\mathbf{S}(z)]_{mn} = S_n(zW^m)$ (Sec. 1.2.) and let $\mathbf{E}(z)$ be the corresponding polyphase matrix. These are related as [10, pp. 234]

$$\mathbf{S}(z) = \mathbf{W}^\dagger \mathbf{\Lambda}(z) \mathbf{E}^T(z^L),$$ (2.11)

where $\mathbf{\Lambda}(z) = \mathrm{diag}(1, z^{-1}, \ldots, z^{-(L-1)})$. Similarly for the synthesis bank define $\mathbf{Q}(z)$ such that $[\mathbf{Q}(z)]_{mn} = Q_n(zW^m)$, and let $\mathbf{R}(z)$ be the polyphase matrix as defined in Sec. 1. We have

$$\mathbf{Q}(z) = \mathbf{W}\mathbf{\Lambda}^{-1}(z)\mathbf{R}(z^L).$$ (2.12)

Then

$$\mathbf{S}(z)\mathbf{Q}^T(z) = \mathbf{W}^\dagger \mathbf{\Lambda}(z) \overbrace{\left(\mathbf{R}(z^L)\mathbf{E}(z^L)\right)}^{=\mathbf{I}}{}^T \mathbf{\Lambda}^{-1}(z)\mathbf{W} = \mathbf{W}^\dagger \mathbf{W} = L\mathbf{I}.$$ (2.13)

This is because $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, due to PR property. So we get [since $\mathbf{S}(z)$ and $\mathbf{Q}(z)$ are square matrices]

$$\mathbf{Q}(z)\mathbf{S}^T(z) = \mathbf{S}(z)\mathbf{Q}^T(z) = L\mathbf{I}.$$ (2.14)

Notice that without the assumption of maximal decimation, the matrix $\mathbf{\Lambda}(z)$ would not be square, $\mathbf{\Lambda}^{-1}(z)$ would not exist and we would not have $\mathbf{S}^T(z)\mathbf{Q}(z) = \mathbf{Q}(z)\mathbf{S}^T(z) = L\mathbf{I}$ (which is an important step in the proof!).

The condition $\mathbf{S}^T(z)\mathbf{Q}(z) = L\mathbf{I}$ implies, in view of the definitions of $\mathbf{S}(z)$, $\mathbf{Q}(z)$ and (1.9), that

$$\left(S_i(z)Q_k(z)\right)\Big\downarrow_L = \delta(i - k).$$ (2.15)

Now suppose that i, k are such that $l \neq m$ in (2.3), i.e. that $S_i(z)$ and $Q_k(z)$ do not come from the same original branch. Then (2.15) can be written as

$$\left(z^{pn_l - rn_m} H_l(z)F_m(z)\right)\Big\downarrow_L = 0.$$ (2.16)

for $0 \leq p \leq k_l - 1$ and $0 \leq r \leq k_m - 1$. With

$$n_l = b_{lm}g_{lm} \qquad \text{and} \qquad n_m = b_{ml}g_{lm}.$$ (2.17)

10

where $g_{lm} = \gcd(n_l, n_m)$, this becomes

$$\left(z^{(pb_{lm} - rb_{ml})g_{lm}} H_l(z) F_m(z) \right) \Big\downarrow_L = 0. \tag{2.18}$$

By multiplying (2.18) by z^d, we get

$$z^d \left(\left(z^{(pb_{lm} - rb_{ml})g_{lm}} H_l(z) F_m(z) \right) \Big\downarrow_L \right) = 0, \qquad \forall d \in \mathcal{Z}. \tag{2.19}$$

or using $L = k_m n_m = k_m b_{ml} g_{ml}$

$$\left(z^{(pb_{lm} - rb_{ml} + dk_m b_{ml})g_{lm}} H_l(z) F_m(z) \right) \Big\downarrow_L = 0. \tag{2.20}$$

It is shown in Appendix A that $(pb_{lm} - rb_{ml} + dk_m b_{ml})$ can take any integer value a, under the conditions $0 \le p \le k_l - 1$, $0 \le r \le k_m - 1$, and $d \in \mathcal{Z}$. Then

$$\left(z^{ag_{lm}} H_l(z) F_m(z) \right) \Big\downarrow_{k_l b_{lm} g_{lm}} = 0 \qquad \forall a \in \mathcal{Z}. \tag{2.21}$$

This is equivalent to

$$\left(z^a \left((H_l(z) F_m(z)) \Big\downarrow_{g_{lm}} \right) \right) \Big\downarrow_{k_l b_{lm}} = 0 \qquad \forall a \in \mathcal{Z}. \tag{2.22}$$

Since this holds for all integers a, we can rewrite it as

$$\left(H_l(z) F_m(z) \right) \Big\downarrow_{g_{lm}} = 0. \tag{2.23}$$

Let now $S_i(z)$ and $Q_k(z)$ come from the same branch, i.e. $l = m$. Then (2.15) means

$$\left(z^{(p-r)n_m} H_m(z) F_m(z) \right) \Big\downarrow_L = \delta(p - r). \tag{2.24}$$

that is,

$$\left(z^{(p-r)} \left((H_m(z) F_m(z)) \Big\downarrow_{n_m} \right) \right) \Big\downarrow_{k_m} = \delta(p - r). \tag{2.25}$$

Now $p - r$ can reach any integer in $[-k_m + 1, k_m - 1]$, so that the last equation is equivalent to

$$(H_m(z) F_m(z)) \Big\downarrow_{n_m} = 1. \tag{2.26}$$

11

Together with (2.23), this implies biorthonormality (2.6).

$$\nabla \nabla \nabla$$

2.2. Corollaries

Corollary 1. No two decimators can be coprime.

If any two n_i's are relatively prime, then their gcd is 1 and we cannot satisfy the conditions for PR with rational filters. This is because (2.6) now implies $H_l(z)F_m(z) = 0$ for $l \neq m$ and this cannot be satisfied with rational filters.

Corollary 2. Completeness.

Definition 2.1. A set of vectors $\{\mathbf{x}_i\}_{i=0}^{\infty}$ in an infinte-dimensional Hilbert space is said to be complete if the zero vector is the only vector orthogonal to all of $\mathbf{x}_i$'s (pp. 6. [16]).

Assuming that the filter bank has the perfect reconstruction property, the completeness of the filter bank follows immediately. To see this, let us write (1.3) [with $\hat{x}(n) = x(n)$] in a different way

$$x(n) = \sum_{i=0}^{M-1} \sum_{k} \langle x(-m), h_i^*(m + n_i k) \rangle f_i(n - n_i k). \qquad (2.27)$$

Assume there is a nonzero input $x(n)$ such that $x(-n)$ is orthogonal to all the analysis filters and their n_i-shifted versions. Then the above sum would be zero and the system would not be a PR system. Similar conclusion can be made for the synthesis filters if we interchange the analysis and synthesis filters (because PR is not violated by such an interchange).

Corollary 3. Linear independence.

Definition 2.2. A set of vectors $\eta_{ik}(n) = \{f_i(n - n_i k)\}_{i=0}^{M-1}$, $k \in \mathcal{Z}$ in an infinte-dimensional space is said to be linearly independent (or minimal) if none of $\eta_{JK}(n)$'s lie in the closure of the linear span of $\{\eta_{lm}(n)\}_{l=0}^{M-1}$ $m \neq K$ for $l = J$ (see pp. 28. [16]).

Since the synthesis filters form a biorthonormal system, it can be proved [16] that the set of sequences $\eta_{ik}(n)$ is linearly independent. We will often say "filters $F_i(z)$ and $F_j(z)$ are lin-

early independent", meaning that the corresponding time sequences and their shifts are linearly independent in the above sense. Using z-domain techniques, we have another useful manifestation of linear independence. Let $A_i(z)$ be rational functions, then the linear independence of $\{F_0(z), F_1(z), \ldots, F_{M-1}(z)\}$ implies

$$\sum_{i=0}^{M-1} A_i(z^{n_i})F_i(z) = 0 \implies A_i(z) \equiv 0 \text{ for } i = 0, 1, \ldots, M-1. \tag{2.28}$$

To see this, just take inverse z-transform of (2.28). The result is

$$\sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} a_i(k)f_i(n - kn_i) = \{0\},$$

where $\{0\}$ denotes the zero sequence. Taking inner product of both sides with $h_i(n - n_i k)$ for $i = 1, 2, \ldots, M-1$ and $\forall k \in \mathcal{Z}$, we get (2.28).

Corollary 4. Basis property.

In a Hilbert space, completeness and independence of a set of vectors is not sufficient to conclude that these vectors form a basis[†]. However, by using the further assumption that the synthesis and analysis filters are stable (i.e. $\sum_n |h_i(n)| < \infty$ and $\sum_n |f_i(n)| < \infty$ or $F_i(e^{j\omega})$, $H_i(e^{j\omega})$ exist and are upper bounded by a finite constant), we show that $\{f_i(n - mn_i)\}_{i=0}^{M-1}$ and $\{h_i(n - mn_i)\}_{i=0}^{M-1}$ $\forall m \in \mathcal{Z}$ are bases for l_2 space. For this, we will invoke Theorem 9, pp. 32, [16]. Since completeness and existence of a complete biorthonormal sequence have been established earlier, it is sufficient, according to the above theorem, to show

$$\sum_{i=0}^{M-1} \sum_m | < x(n), f_i^*(mn_i - n) > |^2 < \infty \quad \text{and} \quad \sum_{i=0}^{M-1} \sum_m | < x(n), h_i^*(mn_i - n) > |^2 < \infty. \tag{2.29}$$

for any $x(n) \in l_2$. In this discussion, all summations with unindicated limits are from $-\infty$ to ∞. We will show that

$$\sum_{i=0}^{M-1} \sum_m | < x(n), f_i^*(mn_i - n) > |^2 \le C\|x(n)\|_2^2. \tag{2.30}$$

[†] In this paper, the term 'basis' stands for the Riesz or unconditional basis, as defined in [16] pp. 31, or [17] pp. 71. Other kinds of bases, the Schauder and Hamel bases [16] do not concern us here.

for any sequence $x(n) \in l_2$. Thus

$$\sum_{i=0}^{M-1} \sum_m | < x(n), f_i^*(mn_i - n) > |^2 = \sum_{i=0}^{M-1} \sum_m \left| \sum_n x(n) f_i(mn_i - n) \right|^2 \qquad (2.31)$$

$$= \sum_{i=0}^{M-1} \sum_m |z_i(m)|^2,$$

where $z_i(n)$ is the n_i-fold decimation of the convolution $x(n) * f_i(n)$. Using Parseval's relation the above can be rewritten as

$$\sum_{i=0}^{M-1} \sum_{m=-\infty}^{\infty} | < x(n), f_i^*(mn_i - n) > |^2 = \sum_{i=0}^{M-1} \frac{1}{2\pi} \int_0^{2\pi} |Z_i(e^{j\omega})|^2 d\omega$$

$$\leq \sum_{i=0}^{M-1} \frac{1}{2\pi} \int_0^{2\pi} |X(e^{j\omega}) F_i(e^{j\omega})|^2 d\omega \qquad (2.32)$$

$$\leq \|x(n)\|_2^2 \sum_{i=0}^{M-1} \sup_{\omega \in [0, 2\pi]} |F_i(e^{j\omega})|^2 < \infty$$

as long as $x(n) \in l_2$.

The first inequality follows because the energy of a decimated sequence is no greater than that of the undecimated version. The proof for the analysis filters is similar. So, we really have a biorthonormal basis formed by the set of n_i-shifted versions of the synthesis and analysis filters.

Corollary 5. Unit energy implies orthonormality!

Consider the maximally decimated system [Fig. 1(a)]. Suppose the following two properties are satisfied:

1. Perfect reconstruction property, and

2. All analysis and synthesis filters have unit energy, i.e., $\sum_n |h_i(n)|^2 = \sum_n |f_i(n)|^2 = 1$, for $0 \leq i \leq M - 1$.

Then the synthesis filters satisfy orthonormality. In other words, eqn. (2.7) holds. (This does not, of course, imply that a biorthonormal system can be orthonormalized simply by scaling the filters to have unit energy.) To prove this, note that the perfect reconstruction property implies biorthnormality (Theorem 2.1.), so that, in particular,

$$\sum_n h_i(n) f_i(-n) = 1. \qquad (2.33)$$

14

Now, Cauchy-Schwarz inequality says

$$\sum_n |h_i(n)|^2 \sum_n |f_i(n)|^2 \geq \left| \sum_n h_i(n)f_i(-n) \right|^2. \tag{2.34}$$

The right hand side is unity, by (2.33). The left hand side is also unity if the analysis and synthesis filters have unit energy. But equality in Cauchy-Schwarz inequality implies $h_i(n) = e^{j\theta_i} f_i^*(-n)$ for some θ_i. Substituting in (2.5) we readily conclude that $\theta_i = 0$ and that the set of synthesis filters (equivalently, the set of analysis filters) satisfies orthonormality.

Corollary 6. Generalization of Nyquist and power complementary properties.

For uniform filter banks ($n_i = M$ for all i) it is well-known that if the system is orthonormal (paraunitary), the filters $H_i(z)$ and $F_i(z)$ are spectral factors of Nyquist(M) filters (or Mth band filters) [10, page 297]. In the more general (nonuniform, and biorthonormal case), this property is replaced with the property that $H_i(z)F_i(z)$ is a Nyquist(n_i) filter. We can readily see this from the biorthonormality condition (2.6) by setting $i = l$

$$\Big(H_i(z)F_i(z) \Big) \Big\downarrow_{n_i} = 1. \tag{2.35}$$

Next, for the uniform paraunitary filter bank, it is well known that the anaysis filters are power complementary, and so are the synthesis filters [10, p. 296]. For the general case (nonuniform, biorthonormal) we have

$$\sum_i \frac{1}{n_i} H_i(z)F_i(z) = 1, \tag{2.36}$$

(see [15]) which reduces to the power complementary property

$$\sum_i H_i(z)\tilde{H}_i(z) = M \tag{2.37}$$

in the uniform paraunitary case.

3. ORTHONORMALIZATION OF BIORTHONORMAL FILTER BANKS

From the second section, we know that if the integers $\{n_k\}_{k=0}^{M-1}$ are such that PR is possible, the analysis and synthesis filters (and their shifted versions) form a *biorthonormal* basis. Under this condition, does there exist a PR system with *orthonormal* filters? The answer to this question is in the affirmative; we will present an orthonormalization process which preserves the filter bank-like form of the system $\{h_i(n - mn_i), f_l(n - kn_l)\}$ for $0 \leq i, l \leq M - 1$ and $m, k \in \mathcal{Z}$. If one wants just to orthonormalize some set of vectors, the Gram-Schmidt technique is one way of doing this, but we want more, namely to preserve the filter bank-like form of the system. We now show how to achieve this aim. Our procedure is reminiscent of the Gram-Schmidt technique, but it converges in a finite number of steps even though the space has infinite dimension.

3.1. Normalization condition or Nyquist condition

Let $F_k(z)$ be a rational transfer function. Define

$$G_k(z) = \alpha_k(z^{n_k})F_k(z). \tag{3.1}$$

Then $G_k(z)\widetilde{G}_k(z) = \alpha_k(z^{n_k})\widetilde{\alpha}_k(z^{n_k})F_k(z)\widetilde{F}_k(z)$, so that

$$\left(\alpha_k(z^{n_k})\widetilde{\alpha}_k(z^{n_k})F_k(z)\widetilde{F}_k(z)\right)\Big\downarrow_{n_k} = \alpha_k(z)\widetilde{\alpha}_k(z)\left(F_k(z)\widetilde{F}_k(z)\right)\Big\downarrow_{n_k}. \tag{3.2}$$

Now if we choose $\alpha_k(z)$ such that

$$\alpha_k(z)\widetilde{\alpha}_k(z) = \frac{1}{\left(F_k(z)\widetilde{F}_k(z)\right)\Big\downarrow_{n_k}}. \tag{3.3}$$

we get

$$\left(G_k(z)\widetilde{G}_k(z)\right)\Big\downarrow_{n_k} = 1. \tag{3.4}$$

A function $G_k(z)$ with this property will be called *normalized*. This is different from the usual meaning of normalization of vectors in l_2. In the time domain, the above normalization condition

16

means that the n_k-shifted versions of $g_k(n)$ (i.e., $\{g_k(n-in_k)\}$) form an orthonormal set. Equivalently, $\widetilde{G}_k(z)G_k(z)$ is a Nyquist(n_k) filter [10, p. 151]; that is its impulse response coefficients $h(n)$ satisfy $h(n_k i) = 0$ for $i \neq 0$.

The existence of $\alpha_k(z)$ satisying (3.3) is assured because of the following. We have

$$\left(F_k(z)\widetilde{F}_k(z)\right)\Big|_{n_k} = \frac{1}{n_k} \sum_{l=0}^{n_k-1} F_k(z^{1/n_k} W^l)\widetilde{F}_k(z^{1/n_k} W^l). \tag{3.5}$$

Since $F_k(z)\widetilde{F}_k(z) \geq 0$ on the unit circle, each term in the above expression remains nonnegative. Thus whenever $F_k(z)$ is rational, the function $\left(F_k(z)\widetilde{F}_k(z)\right)\Big|_{n_k}$ is rational and nonnegative on the unit circle. Such functions can always be written as a product $a(z)\widetilde{a}(z)$. The spectral factor $a(z)$ (a rational function, not unique) can be obtained by standard spectral factorization techniques. Now take $\alpha_k(z) = 1/a(z)$ so that

$$\alpha_k(z)\widetilde{\alpha}_k(z) = \frac{1}{a(z)\widetilde{a}(z)} = \frac{1}{\left(F_k(z)\widetilde{F}_k(z)\right)\Big|_{n_k}}. \tag{3.6}$$

and (3.3) is satisfied. The function $\alpha(z)$ can be chosen to have no poles outside the unit circle (by choosing $a(z)$ to have minimum phase), but what if $a(z)$ has a zero on the unit circle? Then $\alpha_k(z)$ will have a pole on the unit circle! This potential instability will be handled later on.

If two filters $F_i(z)$ and $F_k(z)$ are orthogonal, will that property be preserved by the above operation? Let $G_i(z)$ and $G_k(z)$ be the normalized versions of $F_i(z)$ and $F_k(z)$. Then

$$\left(G_i(z)\widetilde{G}_k(z)\right)\Big|_{g_{ik}} = \alpha_i(z^{n_i/g_{ik}})\widetilde{\alpha}_k(z^{n_k/g_{ik}}) \underbrace{\left(F_i(z)\widetilde{F}_k(z)\right)\Big|_{g_{ik}}}_{=0} = 0 \tag{3.7}$$

showing that orthogonality is preserved (notice that n_k/g_{ik} and n_i/g_{ik} are integers). Summarizing, if we have a set of orthogonal filters $\{F_i(z)\}_{i=0}^{M-1}$ then the above normalization can be used to obtain a set of orthonormal filters $\{G_i(z)\}_{i=0}^{M-1}$.

3.2. Orthogonalization

Let $\{F_k(z)\}_{k=0}^{M-1}$ be the set of rational synthesis transfer functions for a maximally decimated PR filter bank with decimation ratios $\{n_k\}_{k=0}^{M-1}$. We now describe a procedure to get a new set of

17

rational transfer functions $\{G_k(z)\}_{k=0}^{M-1}$ which are mutually orthogonal, i.e. satisfy

$$\left(G_k(z)\widetilde{G}_l(z)\right)\Big\downarrow_{g_{kl}} = 0 \qquad \text{for } k, l = 0, 1, \ldots, M-1 \text{ and } k \neq l. \tag{3.8}$$

We start by making $G_0(z) = F_0(z)$ and $G_1(z)$ orthogonal to $G_0(z)$. For this, let us look for $G_1(z)$ of the form

$$G_1(z) = F_1(z) - \beta_{01}(z^{g_{01}})G_0(z). \tag{3.9}$$

We want

$$\left(G_1(z)\widetilde{G}_0(z)\right)\Big\downarrow_{g_{01}} = 0 \qquad \text{(orthonormality)}, \tag{3.10}$$

or, using (3.9)

$$\left(F_1(z)\widetilde{G}_0(z)\right)\Big\downarrow_{g_{01}} - \beta_{01}(z)\left(G_0(z)\widetilde{G}_0(z)\right)\Big\downarrow_{g_{01}} = 0. \tag{3.11}$$

This can be satisfied if we choose $\beta_{01}(z)$ as

$$\beta_{01}(z) = \frac{(F_1(z)\widetilde{G}_0(z))\downarrow_{g_{01}}}{(G_0(z)\widetilde{G}_0(z))\downarrow_{g_{01}}}. \tag{3.12}$$

Clearly $\beta_{01}(z)$ is a rational transfer function. Then, $G_1(z)$ as in (3.9) remains a rational transfer function. This is how we start this orthogonalization process. Now assume that we have made $G_0(z), G_1(z), \ldots, G_{s-1}(z)$ orthogonal to each other in the sense of (3.8). In the sth step we want to make $G_s(z)$ orthogonal to $G_0(z), G_1(z), \ldots, G_{s-1}(z)$. Assume $G_s(z)$ in the form

$$G_s(z) = F_s(z) - \sum_{i=0}^{s-1} \beta_{is}(z^{g_{si}})G_i(z). \tag{3.13}$$

Let

$$L = g_{s0}c_{s0} = g_{s1}c_{s1} = \cdots = g_{s,s-1}c_{s,s-1}. \tag{3.14}$$

After expanding $\beta_{is}(z)$ into c_{si}-fold polyphase components we get

$$\beta_{is}(z^{g_{si}}) = \sum_{l=0}^{c_{si}-1} z^{-lg_{si}}\beta_{isl}(z^L), \tag{3.15}$$

so $G_s(z)$ is of the form

$$G_s(z) = F_s(z) - \sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1} \beta_{isl}(z^L)z^{-lg_{si}}G_i(z). \tag{3.16}$$

18

We want to make $G_s(z)$ orthogonal to $G_k(z)$ for $k = 0, 1, \ldots, s - 1$. In other words we want

$$\left(G_s(z)\widetilde{G}_k(z)\right)\Big\downarrow_{g_{sk}} = \left(F_s(z)\widetilde{G}_k(z)\right)\Big\downarrow_{g_{sk}} - \sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1}\left(\beta_{isl}(z^L)z^{-lg_{si}}G_i(z)\widetilde{G}_k(z)\right)\Big\downarrow_{g_{sk}} = 0. \quad (3.17)$$

It is easily verified that a rational function $A(z)$ satisfies $(A(z))\downarrow_g = 0$ if and only if $(z^{mg}A(z))\Big\downarrow_{L=cg} = 0$ for $m = 0, 1, \ldots, c - 1$. Then, (3.17) can be written as

$$\sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1}\beta_{isl}(z)\left(z^{mg_{sk}-lg_{si}}G_i(z)\widetilde{G}_k(z)\right)\Big\downarrow_L = \left(z^{mg_{sk}}F_s(z)\widetilde{G}_k(z)\right)\Big\downarrow_L. \quad (3.18)$$

for $m = 0, 1, \ldots, c_{sk} - 1$ and $k = 0, 1, \ldots, s - 1$. So we have $\sum_{i=0}^{s-1} c_{si}$ unknowns $\beta_{isl}(z)$, and the same number of linear equations. If the determinant of the system (which is a rational function of z) is not identically zero, we can solve the system of linear equations for $\beta_{isl}(z)$'s. If this is not the case, we can keep decreasing the number of unknowns until we have a determinant that is not identically zero (see Appendix C). After solving it, we see that $\beta_{is}(z)$'s are rational functions, so $G_s(z)$ will remain a rational transfer function. At the end of this process, we have a new set of rational transfer functions $\{G_k(z)\}_{k=0}^{M-1}$ satisfying (3.8).

3.3. Stability

In this subsection we show that if the transfer functions resulting from the orthonormalization have poles outside the unit circle, they can be moved inside, preserving the orthonormal and PR property. We also show that in the process of orthogonalization and normalization described in Sec. 3.2. and 3.1., the poles will *automatically* be excluded from being on the unit circle.

First assume that after the orthonormalization we got $\{G'_k(z)\}_{k=0}^{M-1}$ with some poles outside the unit circle. For example, let z_0 be a pole outside the unit circle. Construct the FIR filter

$$\prod_{l=0}^{n_k-1}(1 - z^{-1}z_0 W_{n_k}^l) \quad (3.19)$$

(Recall $W_N = e^{-j2\pi/N}$ as usual). This has zeros at

$$z_0 W_{n_k}^l, \quad 0 \le l \le n_k - 1. \quad (3.20)$$

19

Simplifying (3.19) we obtain the form $(1 - z^{-n_k} z_0^{n_k})$. Define the product

$$Q_k(z^{n_k}) = (1 - z^{-n_k} z_0^{n_k})(1 - z^{-n_k} z_1^{n_k}) \ldots \qquad (3.21)$$

where $z_0, z_1 \ldots$ are the poles of $G'_k(z)$ outside the unit circle, and construct the allpass function

$$\gamma_k(z^{n_k}) = \frac{Q_k(z^{n_k})}{\widetilde{Q}_k(z^{n_k})}. \qquad (3.22)$$

This has all poles inside the unit circle. Now form a new set of functions as

$$G_k(z) = \gamma_k(z^{n_k}) G'_k(z). \qquad (3.23)$$

Then $G_k(z)$ has no poles outside the unit circle. The new set satisfies orthogonality because, for $m \neq k$,

$$\begin{aligned}
\left(G_k(z) \widetilde{G}_m(z) \right) \Big\downarrow_{g_{km}} &= \left(\gamma_k(z^{n_k}) G'_k(z) \widetilde{\gamma}_m(z^{n_m}) \widetilde{G}'_m(z) \right) \Big\downarrow_{g_{km}} \\
&= \gamma_k(z^{n_k/g_{km}}) \widetilde{\gamma}_m(z^{n_m/g_{km}}) \underbrace{\left(G'_k(z) \widetilde{G}'_m(z) \right) \Big\downarrow_{g_{km}}}_{=0} = 0.
\end{aligned} \qquad (3.24)$$

(Recall that n_k/g_{km} and n_m/g_{km} are integers.) Normality is preserved too since

$$\left(G_k(z) \widetilde{G}_k(z) \right) \Big\downarrow_{n_k} = \underbrace{\gamma_k(z) \widetilde{\gamma}_k(z)}_{=1} \underbrace{\left(G'_k(z) \widetilde{G}'_k(z) \right) \Big\downarrow_{n_k}}_{=1} = 1. \qquad (3.25)$$

So, we have shown how to replace the poles outside the unit circle with poles inside, without destroying orthonormality.

Avoiding poles on the unit circle.

Let us repeat (3.13) below, but call it $P_s(z)$ for notational convenience.

$$P_s(z) = F_s(z) - \sum_{i=0}^{s-1} \beta_{is}(z^{g_{si}}) G_i(z). \qquad (3.26)$$

This function, in general, can have both poles and zeros on the unit circle. First, assume that it has a pole of order r at $z_p = e^{j\omega_p}$. It will be shown that this will be cancelled in the process of normalization. Recall that the normalized function $G_s(z)$ is constructed according to

$$G_s(z) = \alpha_s(z^{n_s}) P_s(z), \qquad (3.27)$$

20

where

$$\alpha_s(z)\widetilde{\alpha}_s(z) = \frac{1}{\left(P_s(z)\widetilde{P}_s(z)\right)\Big\downarrow_{n_s}}. \qquad (3.28)$$

It is shown in Appendix B that if $P_s(z)$ has a pole of order r on the unit circle, then $\alpha_s(z^{n_s})$ defined as per (3.28) will have a zero of order at least r at the same point. This zero will cancel out the pole of $P_s(z)$, so that the normalized $G_s(z)$ will not have any pole at that point. We see that $G_s(z)$ cannot have any pole on the unit circle coming from $P_s(z)$.

The other possibility is that $\alpha_s(z^{n_s})$ itself has a pole on the unit circle, i.e., $\left(P_s(z)\widetilde{P}_s(z)\right)\Big\downarrow_{n_s}$ has a *zero* on the unit circle. Assume that $\alpha_s(z^{n_s})$ has a pole of order r at $z_0 = e^{j\omega_0}$, and hence at $z_0 W_{n_s}^k$, $0 \leq k \leq n_s - 1$. We have

$$\alpha_s(z^{n_s})\widetilde{\alpha}(z^{n_s}) = \frac{1}{\left(\left(P_s(z)\widetilde{P}_s(z)\right)\Big\downarrow_{n_s}\right)\Big\uparrow_{n_s}}. \qquad (3.29)$$

From this equation we conclude that $\alpha_s(z^{n_s})$ can have a pole of order r at some point, on the unit circle, if and only if $\left(\left(P_s(z)\widetilde{P}_s(z)\right)\Big\downarrow_{n_s}\right)\Big\uparrow_{n_s}$ has a zero of order $2r$ at that point. For this to happen $P_s(z)$ must have zeros of order at least r at $z = z_0 W_{n_s}^k$ for $k = 0, 1, \ldots, n_s - 1$ (for the proof see appendix B). These zeros will cancel with the above mentioned poles of $\alpha_s(z^{n_s})$ when $G_s(z)$ is formed. From this we can conclude that $G_s(z)$ cannot have any poles on the unit circle. Together with the fact that poles outside the unit circle can be moved inside, we conclude that the described procedure leads to stable filters.

3.4. Numerical Examples

Example 3.1. Uniform system. As an example of the above described procedure, we orthonormalized an uniform, four-channel filter bank. The filters that we started with were all FIR, linear-phase, obtained from a two-level tree of two-channel filter banks. Each filter in the two-channel module has length 10 ([18]). The resulting orthonormal filters are IIR and their numerator degrees are $28, 44, 140, 380$ and denominator degrees $25, 41, 77, 377$ respectively. We see that the orders of the filters increase rapidly as we proceed with the orthonormalization process. The magnitude

21

responses [see Fig. 3(a) and 3(b)] are more or less the same before and after orthonormalization. Most of the polynomial coefficients after orthonormalization are very small and can be discarded without harming the frequency response, but it deteriorates the orthonormality property.

Example 3.2. Nonuniform system. We orthonormalized a three-channel filter bank with decimation ratios 4,4 and 2. The filters that we started with were all FIR with lengths 28,28 and 10. After the orthonormalization, we got IIR filters with numerator degrees 100,28,10 and denominator degrees 33,25,9 respectively. Again their magnitude responses [shown in Fig. 4(a) and 4(b)] do not differ much.

The examples show that, while the above procedure is of theoretical interest, the resulting filters are far from being efficient. The main aim of the section is to emphasize the existence of orthonormal systems for nonuniform filter banks where biorthonormal systems exist, and then demonstrate the orthonormalization technique.

3.5. Numerical Considerations

In actually implementing the orthogonalization algorithm, one faces the problem of decimating IIR trasfer functions [(3.6), (3.12), etc.]. Theoretically, we could expand the rational transfer function into partial fractions, then expand each of them into a power series in z and retain every n_i^{th} term. This does not yield numerically accurate results. There are several ways to avoid the factorization of polynomials.

The first one is based on a state-space manifestation of the decimation. Namely, if $\mathbf{A}$ is a state transition matrix in some realization of $H(z)$, then $\mathbf{A}^{n_i}$ is a transition matrix of the decimated system $H(z) \downarrow_{n_i}$. Now in order to get the denominator of the decimated system, we need to find the characteristic polynomial of $\mathbf{A}^{n_i}$. We see that this already may be a rather big numerical problem, especially if the order of the system is big.

Another method, again system theoretical, relies on the fact that a rational transfer function (with no common factors in the numerator and denominator) of order N can be determined from the first $2N + 1$ impulse response coefficients (it can be shown that the determinant of the system

is nonsingular [19]). The impulse response coefficients of the decimated system can be obtained from the impulse response of the original system, which can be easily obtained from the difference equation described by that transfer function. The problem with this approach is that the matrix of the system of linear equations, even though nonsingular, is typically ill-conditioned.

The third method is based on the frequency manifestation of the decimation. Namely, we know that

$$\left(H(z)\right)\bigg|_{\downarrow n_i} = (1/n_i) \sum_{k=0}^{n_i-1} H(z^{1/n_i} W^{-k}). \tag{3.30}$$

Now if we write $H(z)$ as a ratio $H(z) = N(z)/D(z)$, the denominator of the decimated system can be written as

$$D_d(z) = n_i \prod_{k=0}^{n_i-1} D(z^{1/n_i} W^{-k}). \tag{3.31}$$

So in order to get the denominator of the decimated system, we have to find FFT's of the modulated denominators of the original system $D(zW^{-k})$, multiply them, stretch n_i times (i.e. decimate by n_i) and multiply by n_i. The inverse FFT of the result will give us $D_d(z)$. Notice that here we have a product of polynomials, which is appropriately implemented using the FFT. The critical factor is the number of terms in the product. It depends on n_i only, not on the order of the filter.

After we get the denominator, getting the numerator is easy. We can again use FFT techniques. Calculate the sampled DFT of $H(zW^{-k})$.

$$H(e^{j\omega_l} W^{-k}) = \frac{P(e^{j\omega_l} W^{-k})}{D(e^{j\omega_l} W^{-k})}. \tag{3.32}$$

where ω_l's are the sampling frequencies (frequencies at which DFT is the sampled Fourier transform of the numerator and denominator). We add all of these, divide by n_i, and stretch n_i times, to get the sampled DFT of $H(z)\downarrow_{n_i} = H_d(z)$. Since we already know the DFT of the denominator of this sampled frequency response, we can get the DFT of the numerator. The product of this sampled DFT and the DFT of $D_d(z)$ will give us

$$P_d(e^{j\omega_l}) = H_d(e^{j\omega_l}) D_d(e^{j\omega_l}). \tag{3.33}$$

23

After doing the inverse DFT of this sequence. we get the numerator polynomial $P_d(z)$. One has to pay attention to the number of points of FFT (sampling density of DFT). to avoid aliasing in the frequency.

The third method yielded much better results than the second one. especially for high order filters and large decimation ratios. This method was actually used for producing all the above examples.

4. COMPLETE DECORRELATION OF THE SUBBAND SIGNALS

4.1. Decorrelation with Orthonormal Filters

4.1.a Uniform Case

In the traditional transform coding, where the polyphase matrices of the corresponding uniform filter bank are just constant unitary matrices (KLT for example). the subband signals $x_i(n)$ are decorrelated for the same time instant. That is. $E\{x_i(n)x_j^*(m)\} = 0$ whenever $i \neq j$ and $m = n$. In the other extreme case where the filters are ideal brick wall filters (the polyphase matrix having infinite order) the subband signals are completely uncorrelated. that is $E\{x_i(n)x_j^*(m)\} = 0$ whenever $i \neq j$, for any choice of n, m.

In this section we will consider the problem of complete decorrelation by use of rational (finite order) filters. We will show that the subband signals cannot be decorrelated in this way. if we use rational paraunitary filter banks (unless the input signal has severely restricted statistical properties: see below).

Consider an uniform system ($n_k = M$ for all k). Let the filter bank input $x(n)$ be WSS with power spectrum $S_{xx}(z)$. assumed to be a rational function of z.

Definition 4.1. For any scalar input signal $x(n)$ we can form a vector signal

$$\mathbf{x}(n) = \begin{pmatrix} x(nM) \\ x(nM-1) \\ \vdots \\ x(nM-M+1) \end{pmatrix}. \tag{4.1}$$

This vector signal is the output of the delay chain in Fig. 2(b) after decimation. and is called the

M-fold blocked version of the input signal $x(n)$. It is known [20] that the power spectral matrix of this vector WSS process is pseudocirculant. Namely

$$\mathbf{S_{xx}}(z) = \begin{pmatrix} S_{xx,0}(z) & S_{xx,1}(z) & \cdots & S_{xx,M-1}(z) \\ z^{-1}S_{xx,M-1}(z) & S_{xx,0}(z) & \cdots & S_{xx,M-2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ z^{-1}S_{xx,1}(z) & z^{-1}S_{xx,2}(z) & \cdots & S_{xx,0}(z) \end{pmatrix}. \tag{4.2}$$

where $S_{xx,i}(z)$ is the ith polyphase component of the autocorrelation function $S_{xx}(z)$. After passing $\mathbf{x}(n)$ through the analysis bank polyphase matrix $\mathbf{E}(z)$, the output signal $y(n)$ has power spectrum

$$\mathbf{S_{yy}}(z) = \mathbf{E}(z)\mathbf{S_{xx}}(z)\widetilde{\mathbf{E}}(z). \tag{4.3}$$

If we want the subband signals to be decorrelated, then $\mathbf{S_{yy}}(z)$ has to be a diagonal matrix. Furthermore, if we use orthonormal systems, the polyphase matrix has to be unitary on the unit circle. Thus, on the unit circle, (4.3) can be regarded as a unitary diagonalization of the Hermitian matrix $\mathbf{S_{xx}}(e^{j\omega})$. Now we recall that a pseudocirculant matrix can be written as [21]

$$\mathbf{S_{xx}}(z) = \frac{\mathbf{\Gamma}(z)\mathbf{W}}{\sqrt{M}}M\mathbf{D}(z)\frac{\mathbf{W}^{\dagger}\mathbf{\Gamma}^{-1}(z)}{\sqrt{M}}. \tag{4.4}$$

where

$$\mathbf{\Gamma}(z) = \text{diag}(1, z^{-1/M}, z^{-2/M}, \ldots, z^{-\frac{M-1}{M}}). \tag{4.5}$$

$\mathbf{W}$ is the DFT matrix, and $\mathbf{D}$ is a diagonal matrix. Since the matrix $\frac{\mathbf{\Gamma}(z)\mathbf{W}}{\sqrt{M}}$ is unitary on the unit circle, it follows that the diagonal matrix $\mathbf{S_{yy}}(e^{j\omega})$ is identical to the diagonal matrix $M\mathbf{D}(e^{j\omega})$ up to rearrangement of the diagonal elements. Ignoring this rearrangement, we get

$$\mathbf{S_{yy}}(e^{j\omega}) = M\mathbf{D}(e^{j\omega}). \tag{4.6}$$

Now assume that $\mathbf{S_{xx}}(e^{j\omega})$ is rational, and that we wish to diagonalize it with the rational paraunitary matrix $\mathbf{E}(e^{j\omega})$. From (4.6) we see that $\mathbf{D}(e^{j\omega})$ has to be rational. Now Eq. (4.4) implies

$$M\mathbf{D}(z)\mathbf{W}^{\dagger} = \mathbf{\Gamma}^{-1}(z)\mathbf{S_{xx}}(z)\mathbf{\Gamma}(z). \tag{4.7}$$

By multiplying out the left hand side, and using the pseudocirculant property of $\mathbf{S_{xx}}(z)$ we conclude that the rationality of $\mathbf{S_{xx}}(z)$ and $\mathbf{D}(z)$ implies that $\mathbf{S_{xx}}(z)$ has the form $C(z)\mathbf{I}$. That is, it is a

diagonal matrix with identical diagonal elements. This means that the power spectrum of the input process $x(n)$ has the form $S_{xx}(z) = C(z^M)$. In other words, the autocorrelation $R(k) = 0$ unless k is a multiple of M.

4.1.b Nonuniform case

The subband signals can be expressed as

$$x_i(n) = \sum_m x(n_i n - m)h_i(m) \qquad \text{and} \qquad x_j(n) = \sum_k x(n_j n - k)h_j(k). \qquad (4.8)$$

Then the crosscorrelation between these two sequences is

$$r_{ij}(n,l) = E\{x_i(n)x_j^*(n-l)\} = \sum_k \sum_m h_i(m)h_j^*(k)E\{x(n_i n - m)x^*(n_j(n-l) - k)\}. \qquad (4.9)$$

Let the autocorrelation of the input signal $x(n)$ (assumed WSS) be $r(k)$. Then

$$r_{ij}(n,l) = \sum_k \sum_m h_i(m)h_j^*(k)r((n_i - n_j)n + n_j l + k - m). \qquad (4.10)$$

We see that $r_{ij}(n,l)$ depends on n if $n_i \neq n_j$, so that $x_i(n)$ and $n_j(n)$ are not jointly WSS even though they are WSS themselves. Because of this we cannot apply the above argument to this problem.

4.2. Biorthogonal Decorrelation

Having shown that an orthonormal filter bank cannot in general be used for decorrelation, we will decorrelate the subband signals using a biorthonormal filter bank. Assume again that $x(n)$ is WSS. For the uniform case ($n_i = M$ for all i) we obtain from (4.10)

$$r_{ij}(l) = r_{ij}(n,l) = \sum_k \sum_m h_i(m)h_j^*(k)r(Ml + k - m).$$

The subband signals are decorrelated if this is zero for $i \neq j$. Equivalently, in the z-domain,

$$S_{ij}(z) = \left(\tilde{H}_j(z)H_i(z)S(z) \right) \Big|_{\downarrow M} = 0 \qquad \text{for} \qquad i \neq j. \qquad (4.11)$$

where $S_{ij}(z)$ is the z-transform of $r_{ij}(l)$.

Given a PR (birothonormal) system with analysis filters $H_i(z)$. we show how to obtain a new set of analysis filters such that the above holds. For this we apply techniques similar to the ones in Section 3. The new analysis filters will be $G_i(z)$. Synthesis filters will be obtained by inverting the polyphase matrix (possibly unstable). Let $G_0(z) = H_0(z)$. In order to make $x_1(n)$ and $x_0(n)$ decorrelated, we look for $G_1(z)$ in the form $G_1(z) = H_1(z) - \beta_{01}(z^M)G_0(z)$. We want

$$\left(G_1(z)\widetilde{G}_0(z)S(z) \right) \Big\downarrow_M = 0. \tag{4.12}$$

For this, we choose

$$\beta_{01}(z) = \frac{\left(H_1(z)\widetilde{G}_0(z)S(z) \right)\Big\downarrow_M}{\left(G_0(z)\widetilde{G}_0(z)S(z) \right)\Big\downarrow_M}. \tag{4.13}$$

If $H_0(z)$ and $H_1(z)$ are linearly independent (in the sense described in Sec. 2.2.). it can be readily verified that $G_0(z)$ and $G_1(z)$ are also linearly independent. If we continue this process. in the sth step we have:

$$G_s(z) = H_s(z) - \sum_{k=0}^{s-1} \beta_{ks}(z^M)G_k(z). \tag{4.14}$$

We do not want $x_s(n)$ to be correlated to any of $x_i(n)$ for $i < s$. For this. we need

$$\sum_{k=0}^{s-1} \beta_{ks}(z) \left(G_k(z)\widetilde{G}_l(z)S(z) \right) \Big\downarrow_M = \left(H_s(z)\widetilde{G}_l(z)S(z) \right) \Big\downarrow_M \quad \text{for} \quad l = 0.1.\ldots.s-1. \tag{4.15}$$

We have s linear equations and that many unknowns. This system has determinant that is not zero identically (see appendix C for the proof in the case of orthogonalization). Otherwise there exist $a_k(z)$, such that

$$\sum_{k=0}^{s-1} a_k(z) \left(G_k(z)\widetilde{G}_l(z)S(z) \right) \Big\downarrow_M = 0 \quad \text{for} \quad l = 0.1.\ldots.s-1. \tag{4.16}$$

Now if we multiply by $\widetilde{a}_l(z)$ and add all these equations. we get[†]

$$\left(\left(\sum_{k=0}^{s-1} a_k(z^M)G_k(z) \right) \left(\sum_{l=0}^{s-1} \widetilde{a}_l(z^M)\widetilde{G}_l(z) \right) S(z) \right) \Big\downarrow_M = 0. \tag{4.17}$$

[†] While it is not obvious, it can be shown. using the theory of Smith-forms [19], [22]. that $a_k(z)$ are polynomials whenever $G_k(z)$ and $S(z)$ are rational.

Since $S(z)$ is a nonnegative function on the unit circle. we can conclude [see the explanation of (C.3)]

$$\sum_{k=0}^{s-1} a_k(z^M)G_k(z) = 0. \tag{4.18}$$

This means that $G_k(z)$ for $k = 0, 1, \ldots, s-1$ are linearly dependent (2.28). It can be shown that this is not possible if $H_i(z)$'s are linearly independent. So the determinant of the system (4.15) is not identically equal to zero. After the system is solved. we have new analysis filters $G_i(z)$'s which decorrelate subband signals. The corresponding synthesis filters can be obtained by inverting the analysis bank polyphase matrix (stability cannot be guaranteed. not even for the analysis filters).

As an example of the above decorrelation procedure. we take a lowpass AR(6) process [14] and a paraunitary two-channel filter bank [23] with FIR filters of order 7 (filter 8A). Fig. 5(a) and 5.(b) show the frequency responses of the original and modified analysis filters that decorrelate subband signals. The resulting analysis filters are FIR of order 7 and 18. The synthesis filters are IIR of orders 18 and 12. To see whether this decorrelation improves the performance of the system. we calculated the coding gain of the system. [†] Since the coding gain depends on the frequency response magnitudes, stability of the filters does not enter into our calculations. The original coding gain with the paraunitary system was $G_{PU} = 3.09$. After decorrelation it is $G_{DC} = 2.8$. This indicates that decorrelation of the subband signals itself does not necessarily increase the coding gain. On the other hand, the coding gain of the ideal brick wall two-channel filter bank is $G_{SBC} = 3.38$ and subband signals *are* fully uncorrelated.

The filters that decorrelate the subband signals depend heavily on the filters that we start with and the coding gain we can achieve with them. The filter banks that decorrelate the subband signals are not paraunitary, so there is nothing to ensure that the coding gain will be greater than one [24].

5. THE COMPATIBILITY CONDITION, AND GENERALIZATIONS

[†] See [14] or [24] for the definitions of coding gains.

28

We now present a different type of necessary conditions for perfect reconstructability in nonuniform filter banks. These can sometimes be used to quickly reject certain sets of integer decimation ratios $\{n_i\}$ from being considered for perfect reconstruction. In all our discussions we assume maximal decimation, that is $\sum_{i=0}^{M-1} 1/n_i = 1$. The discussions of this section do not apply to the case of the so-called block decimation [3].

5.1. Compatibility

In Fig. 1(a), the reconstruced signal $\widehat{X}(z)$ is given by

$$\widehat{X}(z) = \sum_{k=0}^{M-1} F_k(z) \frac{1}{n_k} \sum_{n=0}^{n_k-1} H_k(zW_{n_k}^n) X(zW_{n_k}^n). \tag{5.1}$$

In order for each of the alias terms $X(zW_{n_k}^n)$, $n \neq 0$ to be cancelled, it is necessary that for every k and n there exist $\ell \neq k$ and m such that $W_{n_k}^n = W_{n_\ell}^m$. This requirement is called the compatibility condition and is a necessary condition for alias cancellation (see [1] and page 285 of [10]). If the set of integers $\{n_i\}$ satisfies this, we say that it is a compatible set.

For example, the set $\{2, 3, 6\}$ is not compatible because the quantity $X(zW_6)$ cannot be paired with any of the terms $X(zW_2^\ell)$ or $X(zW_3^\ell)$. On the other hand the set $\{2, 6, 6, 6\}$ is compatible: these decimation ratios come from a tree structure, whose first level has decimators $(2, 2)$ and second level splits the bottom branch using three filters and decimators $(3, 3, 3)$. And any set of decimators that come from a tree is compatible, because we know we can design the tree structure to be alias free. The converse is not true, i.e., not every PR system can be drawn as a tree structure. We saw this in Sec. 1., where a counterexample was presented to demonstrate this point.

Test for compatibility.

Assume that the integers n_i are numbered such that

$$n_0 \leq n_1 \leq \ldots \leq n_{M-1}. \tag{5.2}$$

The shifted versions of the input which appear in $\widehat{X}(z)$ are

$$X(zW_{n_k}^n), \quad 1 \leq n \leq n_k - 1. \tag{5.3}$$

29

The compatibility condition is equivalent to the following statement: for every pair of integers

$$(n_i, \ell_i), \quad 1 \le \ell_i \le n_i - 1. \tag{5.4}$$

there must exist a pair of integers

$$(n_m, \ell_m). \quad 1 \le \ell_m \le n_m - 1 \tag{5.5}$$

with $m \ne i$, such that $W_{n_i}^{\ell_i} = W_{n_m}^{\ell_m}$. First consider n_{M-1}, which is the largest. Clearly $W_{n_{M-1}}^{1}$ cannot be equal to $W_{n_i}^{\ell_i}$ for any $n_i < n_{M-1}$. So it is necessary that

$$n_{M-2} = n_{M-1}. \tag{5.6}$$

This also ensures that all the powers of $W_{n_{M-1}}$ have been paired.

Now consider some smaller n_i. Suppose this is a factor of n_{M-1}, that is $n_{M-1} = n_i p_i$. Then $W_{n_i} = W_{n_{M-1}}^{p_i}$ and

$$W_{n_i}^{\ell_i} = W_{n_{M-1}}^{p_i \ell_i}. \tag{5.7}$$

So all the powers of W_{n_i} have been paired. That means that we need not worry about the set of integers which are factors of n_{M-1}.

Next let n_m be some integer which is not a factor of n_{M-1}. Suppose $W_{n_m}^{1}$ can be paired with $W_{n_{M-1}}^{J}$ for some $J < n_{M-1}$. Then

$$\frac{2\pi J}{n_{M-1}} - \frac{2\pi}{n_m} = 2\pi n \tag{5.8}$$

for some integer n. That is, $\frac{J}{n_{M-1}} - \frac{1}{n_m} = n$. But since $0 \le J < n_{M-1}$, this is possible only for $n = 0$. Thus $n_{M-1} = J n_m$ violating the assumption that n_m is not a factor of n_{M-1}. Thus $W_{n_m}^{1} \ne W_{n_{M-1}}^{J}$ for any J in $1 \le J < n_{M-1}$. It cannot therefore be equal to a power of W_{n_i} for any n_i which is a factor of n_{M-1} (because the powers of W_{n_i} form a subset of powers of $W_{n_{M-1}}$).

Summarizing, we can partition the set of decimators $\{n_i\}$ into two classes.

Set 1: Those that are factors of n_{M-1}.

Set 2: Those that are *not* factors of n_{M-1}.

For each n_i in Set 1, $W_{n_i}^{-\ell_i}$ has been paired with a power of $W_{n_{M-1}}$. And none of the n_i's in Set 2 can be paired with any member of Set 1. Thus, the original set of decimators $\{n_i\}$ is compatible *if and only if* Set 2 itself passes the compatibility test.

Statement of the test. Summarizing the above, we can test for compatibility as follows. *Step 1.* Check if $n_{M-2} = n_{M-1}$. If *no*, then $\{n_i\}$ is not compatible. If *yes*, continue. *Step 2.* Form the smaller set by collecting those n_i that are *not* factors of n_{M-1}. Then imagine that this is the given set, and repeat the test, i.e., go to Step 1.

At some point, if the answer is *no* in Step 1, then the original set is not compatible. If we keep getting *yes* in Step 1, then after a finite number of repetitions, the "smaller set" in Step 2 becomes empty. The original set $\{n_i\}$ is then compatible. Thus, the test always gives a decision in a finite number of steps.

To demonstrate the test, consider the set $(2, 6, 10, 12, 12, 30, 30)$. We have $n_{M-2} = n_{M-1} = 30$ so that Step 1 is successful. The smaller set of numbers that are not factors of 30 is given by $(12, 12)$. This set again passes Step 1 successfully. The next smaller set is empty. Thus the test has been completed, and the given set $(2, 6, 10, 12, 12, 30, 30)$ is indeed compatible. It turns out that this set of integers cannot come from a tree structure (binary or otherwise). For, if it did, then the first level of the tree would have to be a two channel system with decimators $(2, 2)$. The second level then splits the lower branch of the first level only, with the decimators $(3, 5, 6, 6, 15, 15)$. Since 3 and 5 do not have common factors, this set of numbers could not have come from a tree-structured connection of uniform filter banks.

It turns out that while the above set is compatible, it is still not consistent with another necessary condition for alias cancellation (hence PR). This statement will be elaborated at the end of the next subsection.

5.2. Generalizations

We can obtain further necessary conditions by looking deeper into the details of alias cancellation. Thus consider the PR condtion, expressed in terms of the L-channel equivalent uniform filter

31

bank (where L is the lcm of n_i's; Sec. 1.2.), it takes the form

$$
\begin{pmatrix}
S_0(z) & S_1(z) & \cdots & S_{L-1}(z) \\
S_0(zW_L) & S_1(zW_L) & \cdots & S_{L-1}(zW_L) \\
\vdots & \vdots & \ddots & \vdots \\
S_0(zW_L^{L-1}) & S_1(zW_L^{L-1}) & \cdots & S_{L-1}(zW_L^{L-1})
\end{pmatrix}
\begin{pmatrix}
Q_0(z) \\
Q_1(z) \\
\vdots \\
Q_{L-1}(z)
\end{pmatrix}
=
\begin{pmatrix}
L \\
0 \\
\vdots \\
0
\end{pmatrix} .
\qquad (5.9)
$$

By substituting for the L pairs of filters $\{S_k(z), Q_k(z)\}$ in terms of the original M pairs filters $\{H_k(z), F_k(z)\}$ using (2.9), we can rewrite this in the form

$$
\mathbf{H}_L(z)\mathbf{f}(z) =
\begin{bmatrix}
L \\
0 \\
\vdots \\
0
\end{bmatrix} ,
\qquad (5.10)
$$

where $\mathbf{f}(z) = [\, F_0(z) \quad \cdots \quad F_{M-1}\,]^T$ and $\mathbf{H}_L(z)$ is an $L \times M$ matrix. The ith column of this matrix has the form

$$
[\, k_i H_i(z) \quad \mathbf{0} \quad k_i H_i(zW_{n_i}) \quad \mathbf{0} \quad k_i H_i(zW_{n_i}^2) \quad \cdots \quad \mathbf{0} \quad k_i H_i(zW_{n_i}^{n_i-1}) \quad \mathbf{0}\,]^T ,
\qquad (5.11)
$$

where $\mathbf{0}$ is a string of $k_i - 1$ zeros. (Recall that k_i are integers such that $k_i n_i = L$). The compatibility condition says that any nonzero row of $\mathbf{H}_L(z)$ should have at least two nonzero entries.

Now notice that the decimation ratios $n_0, n_1, \ldots n_{M-1}$ of the M channel nonuniform filter bank may not all be distinct. Let us relabel them in terms of distinct integers, for convenience of discussion. Thus let the decimators be

$$
\underbrace{n_0, n_0, \ldots n_0,}_{N_0 \text{ times}} \underbrace{n_1, n_1, \ldots n_1,}_{N_1 \text{ times}} \cdots \underbrace{n_{K-1}, n_{K-1}, \ldots n_{K-1},}_{N_{K-1} \text{ times}}
\qquad (5.12)
$$

In this notation, n_i are distinct integers and $N_0 + N_1 + \ldots N_{K-1} = M$.

For example, let us fully understand the 0th column of the matrix $\mathbf{H}_L(z)$. If $N_0 = 2$, then there are two columns (zeroth and first) of the form (5.11), with the same decimation ratio n_0 (and the same k_0). More generally, there are N_0 columns of the form (5.11) with the same n_0 and the same k_0. These N_0 columns have nonzero entries occuring in the same positions, namely 0th, k_0^{th}, $2k_0^{th}$ and so forth. Consider now another column, say the one corresponding to n_i. This has nonzero elements occuring at the locations 0, k_i, $2k_i$, and so forth. Now compare this with the 0th column.

and identify the locations where nonzero elements overlap. With the exception of the 0th location, the first overlap of nonzero elements will occur at the location $\text{lcm}(k_0, k_i)$. Define

$$m_0 = \frac{\min_{i \neq 0} \text{lcm}(k_0, k_i)}{k_0}. \tag{5.13}$$

Then the nonzero elements of the leftmost column in the k_0^{th} $2k_0^{th}$, ... $(m_0 - 1)k_0^{th}$ positions do not overlap with any nonzero elements from any other columns, except of course, columns $1, 2, \ldots N_0$. We can isolate these nonzero elements in the first N_0 columns of equation (5.10), and write

$$\begin{bmatrix} H_0(zW_{n_0}) & H_1(zW_{n_0}) & \ldots & H_{N_0-1}(zW_{n_0}) \\ H_0(zW_{n_0}^2) & H_1(zW_{n_0}^2) & \ldots & H_{N_0-1}(zW_{n_0}^2) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW_{n_0}^{m_0-1}) & H_1(zW_{n_0}^{m_0-1}) & \ldots & H_{N_0-1}(zW_{n_0}^{m_0-1}) \end{bmatrix} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{N_0-1}(z) \end{bmatrix}}_{\mathbf{g}(z)} = \mathbf{0}. \tag{5.14}$$

A necessary condition. We will now prove that, if the number of rows $m_0 - 1 \geq$ number of columns N_0, then perfect reconstruction is not possible! So the condition

$$m_j - 1 < N_j, \quad 0 \leq j \leq K - 1 \tag{5.15}$$

is necessary for perfect reconstruction, where N_j is the number of decimators equal to n_j, and

$$m_j = \frac{\min_{i \neq j} \text{lcm}(k_i, k_j)}{k_j}. \tag{5.16}$$

This is a generalization of the compatibility condition which merely said that any nonzero row of $\mathbf{H}_L(z)$ should have at least two nonzero entries.

Proof of the necessary condition. Eqn. (5.14) implies that the columns of the matrix are linearly dependent [unless all the $F_i(z)$'s in that equation are zero, which is not possible in a maximally decimated perfect reconstruction system]. If $m_0 - 1 \geq N_0$, this means that the *rows* are linearly dependent. Denoting the first row of the matrix in (5.14) as $\mathbf{h}(zW_{n_0})$, the remaining rows are $\mathbf{h}(zW_{n_0}^2) \ldots \mathbf{h}(zW_{n_0}^{m_0-1})$. The linear dependence implies that

$$\mathbf{h}(zW_{n_0}) = \sum_{i=2}^{m_0-1} \alpha_i(z)\mathbf{h}(zW_{n_0}^i). \tag{5.17}$$

33

Since this holds for all z, we can replace z with $zW_{n_0}^{-1}$ to obtain

$$\mathbf{h}(z) = \sum_{i=1}^{m_0-2} \alpha_{i+1}(zW_{n_0}^{-1})\mathbf{h}(zW_{n_0}^i) = \sum_{i=1}^{m_0-2} \beta_i(z)\mathbf{h}(zW_{n_0}^i). \tag{5.18}$$

But eqn. (5.14) says that $\mathbf{h}(zW_{n_0}^i)\mathbf{g}(z) = 0$ for $1 \le i \le m_0 - 1$. Using this in (5.18) we conclude $\mathbf{h}(z)\mathbf{g}(z) = 0$. That is,

$$H_0(z)F_0(z) + H_1(z)F_1(z) + \ldots + H_{N_0-1}(z)F_{N_0-1}(z) = 0. \tag{5.19}$$

But this cannot happen in a maximally decimated perfect reconstruction system. To see this note that the biorthonormality condition (2.6) implies, in particular,

$$\left(H_0(z)F_0(z) + H_1(z)F_1(z) + \ldots + H_{N_0-1}(z)F_{N_0-1}(z) \right)\Big|_{\downarrow n_0} = N_0. \tag{5.20}$$

which is not possible if (5.19) is true! This completes the proof of (5.15) for $j = 0$. The same argument can be used to show that (5.15) is true for $j = 1, 2, \ldots, K - 1$ as well.

This test is strictly stronger than the test for compatibility. To demonstrate, consider the same set $(2, 6, 10, 12, 12, 30, 30)$ from the end of the previous subsection. As shown there, it satisfies the compatibility condition. According to the notation in this subsection, n_i's are distinct numbers and we have $K = 5$, $L = 60$ and

i	0	1	2	3	4
N_i	2	2	1	1	1
n_i	30	12	10	6	2
k_i	2	5	6	10	30
m_i	3	2	1	1	1

Since $m_0 - 1 = 2 = N_0$, we conclude that PR is not possible.

6. CONCLUSION

For a maximally decimated nonuniform filter bank, the perfect reconstruction (PR) property is equivalent to biorthonormality. Using this fact we derived a number of properties of PR filter banks. We then showed that whenever the decimation ratios are such that biorthonormality is possible, it is in particular possible to obtain orthonormality. This was done by developing an

orthonormalization procedure. While reminiscent of the Gram-Schmidt approach, the procedure converges in a finite number of steps and furthermore preserves the filter bank-like form of the basis functions. We then modified the orthonormalization procedure for the application of subband decorrelation. It was demonstrated that mere decorrelation of subband signals does not necessarily optimize the coding gain of a system. Finally we considered the problem of alias cancellation, and obtained a generalization of the so-called compatibility condition which is a necessary condition for perfect reconstruction in maximally decimated systems.

APPENDIX A. REACHING ARBITRARY INTEGERS

In connection with equation (2.20), we will show that the quantity $pb_{lm} - rb_{ml} + dk_m b_{ml}$ can be made to take any integer value by proper choice of the integer d, and the integers p, r in the ranges $0 \leq p \leq k_l - 1$, $0 \leq r \leq k_m - 1$. For this recall the meanings of the integers b_{ml}, b_{lm}, and L, namely, eqns. (1.8), and (2.17). Since $L = k_l n_l = k_m n_m$ by definition, we have $k_l b_{lm} g_{lm} = k_m b_{ml} g_{lm}$. So $k_l b_{lm} = k_m b_{ml}$. Since b_{lm} and b_{ml} are relatively prime by construction, there exist integers $\hat{p}$ and $\hat{r}$ such that

$$\hat{p} b_{lm} - \hat{r} b_{ml} = \text{ any desired integer } a. \qquad (A.1)$$

We can always decompose $\hat{p}$ and $\hat{r}$ as $\hat{p} = p + nk_l$ and $\hat{r} = r + ik_m$, where $0 \leq p \leq k_l - 1$, and $0 \leq r \leq k_m - 1$. Substituting this into (A.1) and rearranging, we get

$$pb_{lm} - rb_{ml} + dk_m b_{ml} = a, \qquad (A.2)$$

where $d = (n - i)$. Thus, we can write any integer a as above where p and r are in the stated range, provided we can assign any integer value to d.

APPENDIX B. POLES ON THE UNIT CIRCLE

We will show that when a biorthonormal filter bank is orthonormalized, the resulting filters will naturally be free from poles on the unit circle. We will do this in two parts.

Observation 1.

35

Let $A(z)$ be a rational function with a pole on the unit circle, at $z_0 = e^{j\omega_0}$. Let r_a be the order of this pole. Then, in the neighbourhood of z_0, the function $\widetilde{A}(e^{j\omega})A(e^{j\omega})$ behaves as

$$\widetilde{A}(e^{j\omega})A(e^{j\omega}) \sim \frac{c_a}{(e^{j\omega} - e^{j\omega_0})^{r_a}(e^{-j\omega} - e^{-j\omega_0})^{r_a}}. \tag{B.1}$$

This is the behavior of a pole of order $2r_a$. Since $\widetilde{A}(z)A(z) \geq 0$ on the unit circle, we have $c_a > 0$. Now let $B(z)$ be another rational function with a possible pole at the same point z_0, with order r_b. Then $\widetilde{B}(e^{j\omega})B(e^{j\omega})$ can be expressed in a similar way. So

$$\widetilde{A}(e^{j\omega})A(e^{j\omega}) + \widetilde{B}(e^{j\omega})B(e^{j\omega}) \sim \frac{c_a}{(e^{j\omega} - e^{j\omega_0})^{r_a}(e^{-j\omega} - e^{-j\omega_0})^{r_a}} + \frac{c_b}{(e^{j\omega} - e^{j\omega_0})^{r_b}(e^{-j\omega} - e^{-j\omega_0})^{r_b}}. \tag{B.2}$$

Since $c_a, c_b > 0$, we see that there can be no cancellations, and as ω approaches ω_0, the result behaves like a pole of order $= \max(2r_a, 2r_b)$. Similarly, if we have a sum of several nonnegative functions having poles of various orders on the unit circle, the sum behaves like a pole of order equal to the largest one.

Consequence of observation 1. Now consider the normalization step (3.29). The denominator of $\alpha_s(z^{n_s})\widetilde{\alpha}_s(z^{n_s})$ can be written as

$$\left(\left(P_s(z)\widetilde{P}_s(z) \right) \downarrow_{n_s} \right) \uparrow_{n_s} = \frac{1}{n_s} \sum_{k=0}^{n_s-1} P_s(zW_{n_s}^k)\widetilde{P}_s(zW_{n_s}^{-k}). \tag{B.3}$$

Each term on the righthand side is nonnegative on the unit circle. So if $P_s(z)$ has a pole of order r at $z_0 = e^{j\omega_0}$ then the above summation still has this pole, with order $\geq 2r$. As a result, $\alpha_s(z^{n_s})$ has a *zero* of order r. This means that when we form the normalized filter $G_s(z) = \alpha_s(z^{n_s})P_s(z)$, this unit-circle pole will be completely cancelled.

Observation 2.

From (3.29) we see that $\alpha_s(z^{n_s})$ will have a pole of order r at $z_0 = e^{j\omega_0}$ if and only if $\left(\left(P_s(z)\widetilde{P}_s(z) \right) \downarrow_{n_s} \right) \uparrow_{n_s}$ has a zero of order $2r$ at $z = z_0 = e^{j\omega_0}$. Now consider (B.3). Each term in this sumation is nonnegative. Suppose the function $P_s(zW_{n_s}^k)$ has a zero of order r_k at z_0. Then

$$P_s(e^{j\omega}W_{n_s}^k)\widetilde{P}_s(e^{j\omega}W_{n_s}^k) = (e^{j\omega} - e^{j\omega_0})^{r_k}(e^{-j\omega} - e^{-j\omega_0})^{r_k} \times \text{(nonnegative function)} \tag{B.4}$$

36

on the unit circle. If the summation in (B.3) has the factor $(e^{j\omega} - e^{j\omega_0})^r (e^{-j\omega} - e^{-j\omega_0})^r$ it is therefore necessary that $r_k \geq r$ for each k. That is, each of the quantities $P_s(zW^k)$ has to have a zero of order $\geq r$ at z_0. In particular, therefore, $P_s(z)$ has a zero of order $\geq r$ at z_0.

The conclusion is that, if $a_s(z^{n_s})$ has a pole of order r at $z_0 = e^{j\omega_0}$, then $P_s(z)$ has a zero of order at least r at z_0.

APPENDIX C. ELIMINATING REDUNDANT VARIABLES

If the system of equations (3.18) has a determinant that is identically zero, we can reduce the size of the problem as follows. In this case, there exists $a_{mk}(z)$, with at least one of them different from zero, such that

$$\sum_{k=0}^{s-1} \sum_{m=0}^{c_{sk}-1} a_{mk}(z) \left(z^{mg_{sk}-lg_{si}} G_i(z) \widetilde{G}_k(z) \right) \Big\downarrow_L = 0. \qquad (C.1)$$

for $i = 0, 1, \ldots, s-1$ and $l = 0, 1, \ldots, c_{si} - 1$. While it is not obvious that there are polynomials $a_{mk}(z)$ satisfying (C.1), this can be verified to be the case, by use of the Smith-McMillan decomposition for rational functions [19], [22]. The previous equation can be rewritten as

$$\left(z^{-lg_{si}} G_i(z) \sum_{k=0}^{s-1} \sum_{m=0}^{c_{sk}-1} a_{mk}(z^L) z^{mg_{sk}} \widetilde{G}_k(z) \right) \Big\downarrow_L = 0 \qquad \text{for all} \quad \begin{cases} i = 0, 1, \ldots, s-1 \\ l = 0, 1, \ldots, c_{si} - 1 \end{cases}. \qquad (C.2)$$

After multiplying each of these equations by $\tilde{a}_{li}(z)$, and summing them with respect to l and i, we get

$$\left(\left(\sum_{i=0}^{s-1} \widetilde{A}_i(z^{g_{si}}) G_i(z) \right) \left(\sum_{i=0}^{s-1} A_i(z^{g_{si}}) \widetilde{G}_i(z) \right) \right) \Big\downarrow_L = 0. \qquad (C.3)$$

where $A_i(z) = \sum_{m=0}^{c_{si}-1} z^m a_{mi}(z^{c_{si}})$. Now, for any rational $P(z)$, the equation $\left(\widetilde{P}(z) P(z) \right) \Big\downarrow_L \equiv 0$ implies $P(z) \equiv 0$. This is because $\widetilde{P}(e^{j\omega}) P(e^{j\omega}) \geq 0$ on the unit circle, and the decimated version cannot otherwise be identically zero. So (C.1) implies

$$\sum_{i=0}^{s-1} \widetilde{A}_i(z^{g_{si}}) G_i(z) = 0. \qquad (C.4)$$

There is at least one $a_{mi}(z) \not\equiv 0$, say $a_{Jk}(z) \not\equiv 0$. Then (C.4) implies

$$z^{-Jg_{sk}} G_k(z) = \frac{-1}{\tilde{a}_{Jk}(z^L)} \sum_{\substack{i=0}}^{s-1} \sum_{\substack{m=0 \\ m \neq J, \text{ for } i=k}}^{c_{si}-1} \tilde{a}_{mi}(z^L) z^{-mg_{si}} G_i(z) \qquad (C.5)$$

37

From here we see that the form (3.16) of $G_s(z)$ is redundant and we can just drop the term $\beta_{ksJ}(z^L)z^{-Jg_{sk}}G_k(z)$ and form a smaller linear system like (3.17). We can keep doing this till the determinant is not identically zero.

REFERENCES

[1] Hoang, P. -Q., and Vaidyanathan, P. P. "Non-uniform multirate filter banks: theory and design," *Proc. IEEE Int. Symp. on Circuits and Systems*, pp. 371-374, Portland, Oregon, May 1989.

[2] Kovačević, J., and Vetterli, M. "Perfect reconstruction filter banks with rational sampling rate changes," *Proc. IEEE Int. Conf. on Acoustics, Speech, and Signal Proc.*, pp. 1785-1788, Toronto, Canada, May 1991.

[3] Nayebi, K., Barnwell, III, T. P. and Smith, M. J. T. " Nonuniform filter banks: a reconstruction and design theory," *IEEE Trans. on Signal Proc.*, vol. SP-41, June 1993.

[4] Soman, A. K. and Vaidyanathan, P. P. "On orthonormal wavelets and paraunitary filter banks," *IEEE Trans. on Signal Processing*, vol. SP-41, March 1993.

[5] Smith, M. J. T., and Barnwell III, T. P. "Exact reconstruction techniques for tree-structured subband coders," *IEEE Trans. on Acoustics, Speech and Signal Proc.*, pp. 434-441, June 1986.

[6] Vetterli, M., and Herley, C. "Wavelets and filter banks," *IEEE Trans. on Signal Processing*, vol. SP-40, 1992.

[7] Rioul, O., and Vetterli, M. "Wavelets and signal processing," *IEEE Signal Processing Magazine*, pp. 14-38, Oct. 1991.

[8] Gopinath, R. A., and Burrus, C. S. "Wavelet transform and filter banks," In *Wavelets and applications*, edited by C. H. Chui, Academic Press, 1991.

[9] Daubechies, I. *Ten lectures on wavelets*, SIAM, CBMS series, April 1992.

[10] Vaidyanathan, P. P. *Multirate systems and filter banks*, Prentice Hall, Inc., Englewood Hills, NJ, 1993.

[11] Igor, D., and Vaidyanathan, P. P. "Biorthonormal filter banks: some necessary conditions, and orthonormalization," *Proc. IEEE Int. Symp. on Circuits and Systems*, Chicago, May 1993.

[12] Oppenheim, A. V., and Schafer, R. W. *Discrete-time signal processing*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1989.

[13] Ramstad, T. A. "IIR filterbank for subband coding of images." *Proc. IEEE Int. Symp. on Circuits and Systems*, pp. 827-830, Espoo, Finland, June 1988.

[14] Jayant, N. S., and Noll, P. *Digital coding of waveforms: principles and applications to speech and video*, Prentice-Hall Signal Processing Series, Englewood Cliffs, NJ, 1984.

[15] Vaidyanathan, P. P. "Orthonormal and biorthonormal filter-banks as convolvers, and convolutional coding gain," *IEEE Trans. on Signal Processing*, vol. SP-41, June 1993.

[16] Young, R. M. *An introduction to nonharmonic Fourier series*, Academic Press, New York 1980.

[17] Chui, C. K. *An introduction to wavelets*, Academic Press, New York, 1992.

[18] Nguyen, T. Q., and Vaidyanathan, P. P. "Two-channel perfect-reconstruction FIR QMF structures which yield linear-phase analysis and synthesis filters," *IEEE Trans. on Acoustics, Speech and Signal Proc.*, vol. ASSP-37, pp. 676-690, May 1989.

[19] Kailath, T. *Linear systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[20] Sathe, V. P., and Vaidyanathan, P. P. "Effects of multirate systems on the statistical properties of random signals," *IEEE Trans. on Signal Processing*, vol. SP-41, March 1993.

[21] Vaidyanathan, P. P., and Mitra, S. K. "Polyphase networks, block digital filtering, LPTV systems and alias-free QMF banks: a unified approach based on pseudocirculants," *IEEE Trans. on Acoustics, Speech and Signal Proc.*, vol. ASSP-36, pp. 381-391, March 1988.

[22] Gantmacher, F. R. *The theory of matrices*, vol. 1, Chelsa Publ. Co., N.Y., 1959.

[23] Vaidyanathan, P. P., and Hoang, P. -Q. "Lattice structures for optimal design and robust implementation of two-channel perfect reconstruction QMF banks," *IEEE Trans. on Acoustics, Speech and Signal Proc.*, vol. ASSP-36, pp. 81-94, January 1988.

[24] Soman, A., and Vaidyanathan, P. P. "Coding gain in paraunitary analysis/synthesis systems," *IEEE Trans. Signal Processing*, vol. SP-41, May 1993.

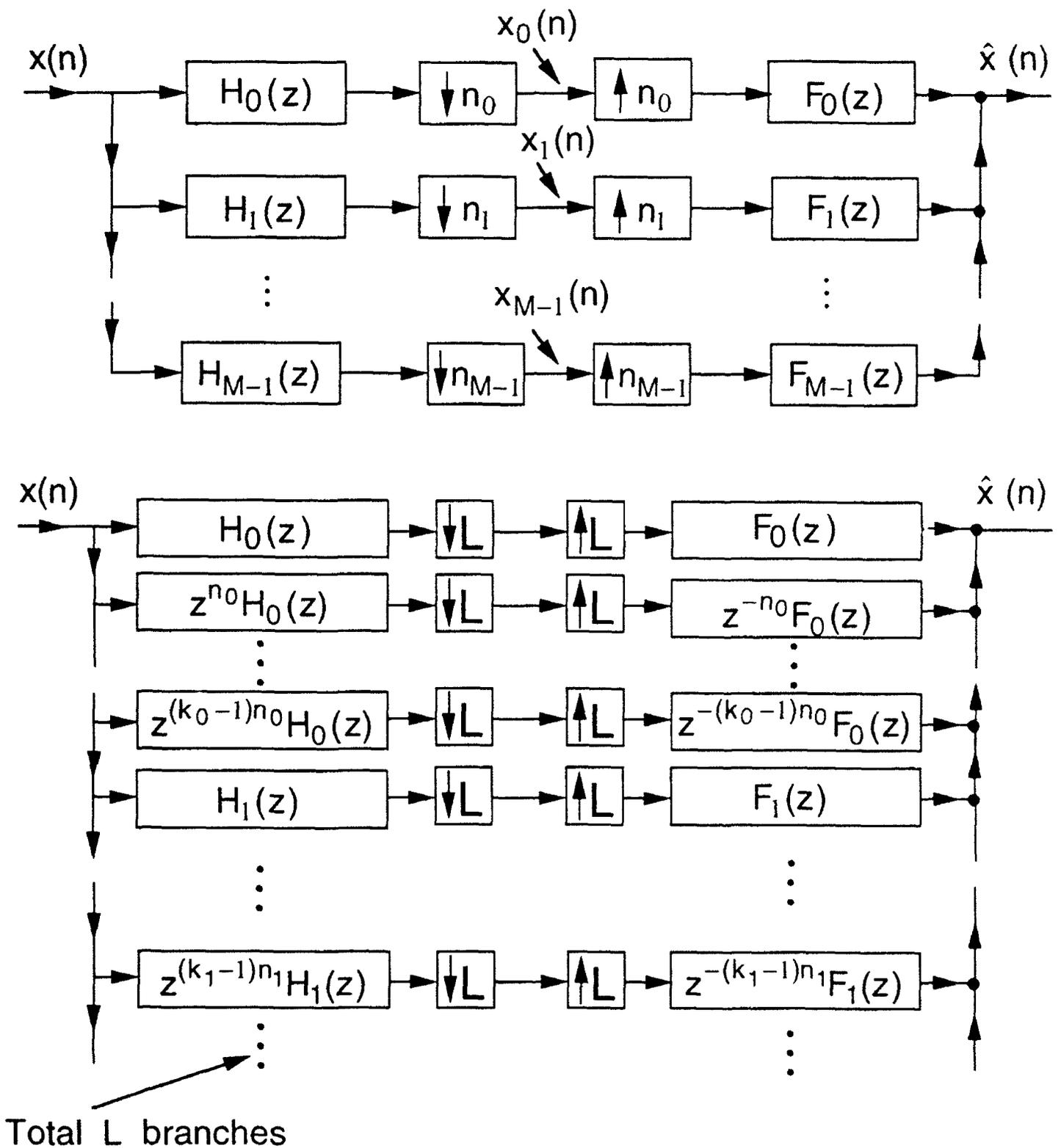## LIST OF FIGURES

.

41

FOOTNOTES

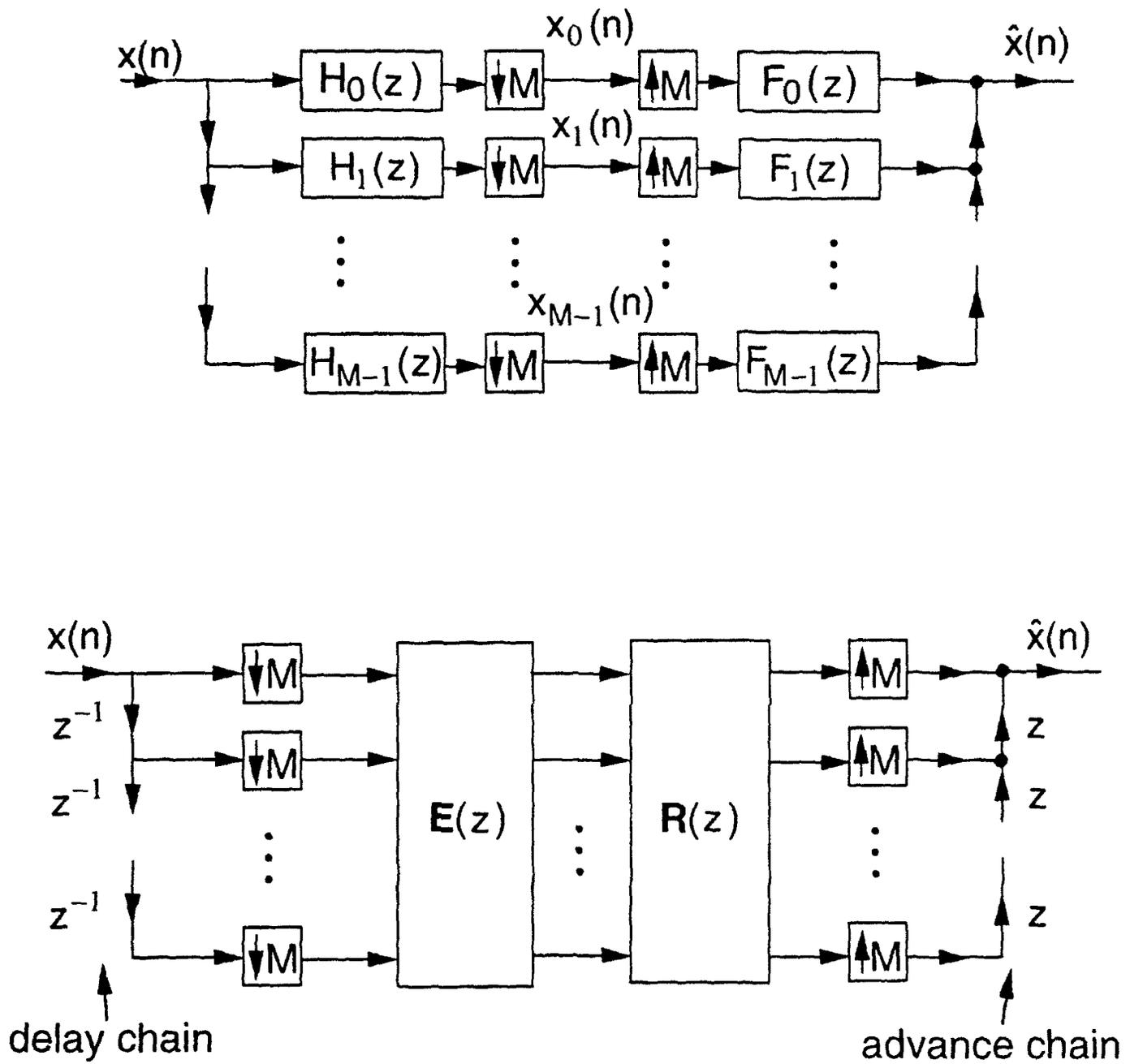Fig. 1. (a) A nonuniform filter bank and (b) an equivalent uniform bank.

Fig. 2. (a) An uniform filter bank and (b) its polyphase decomposition.
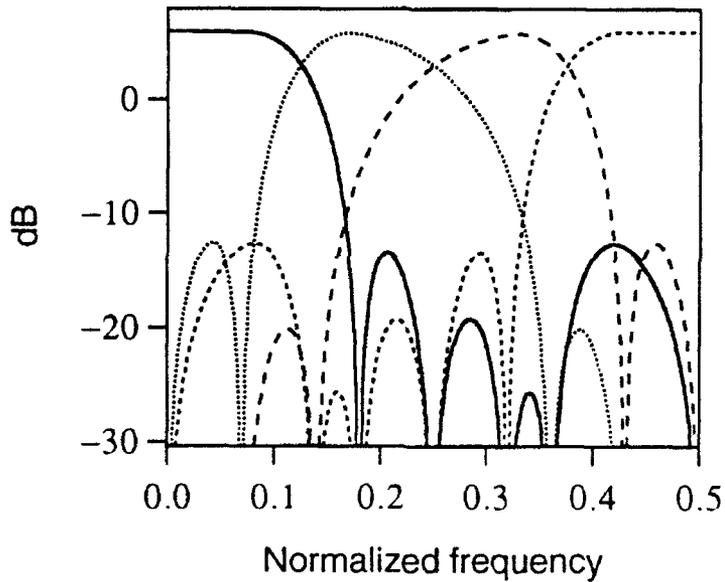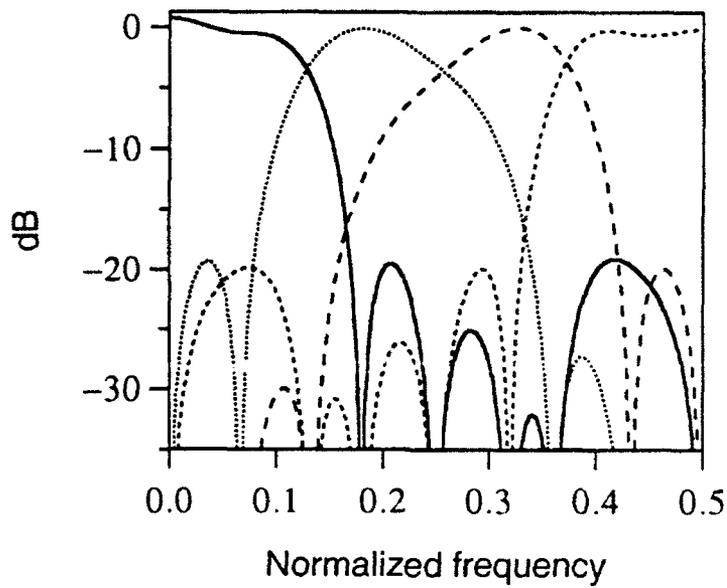
Fig. 3. Example 3.1. Magnitude responses of analysis filters,
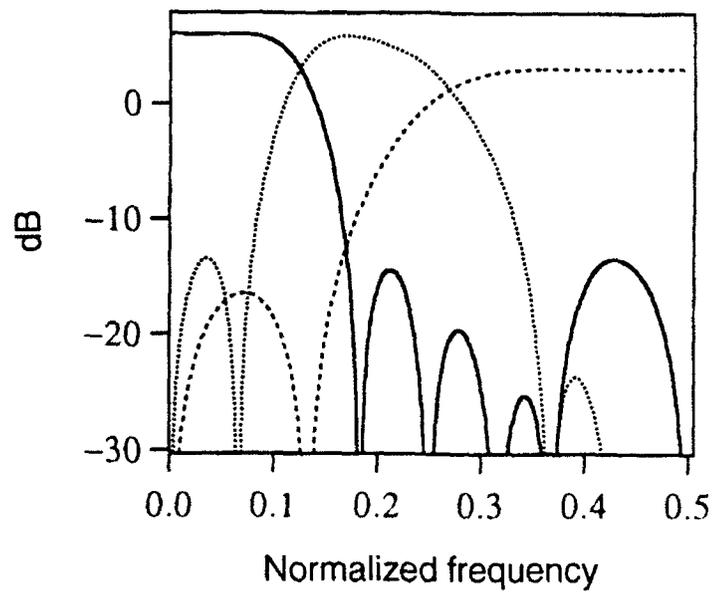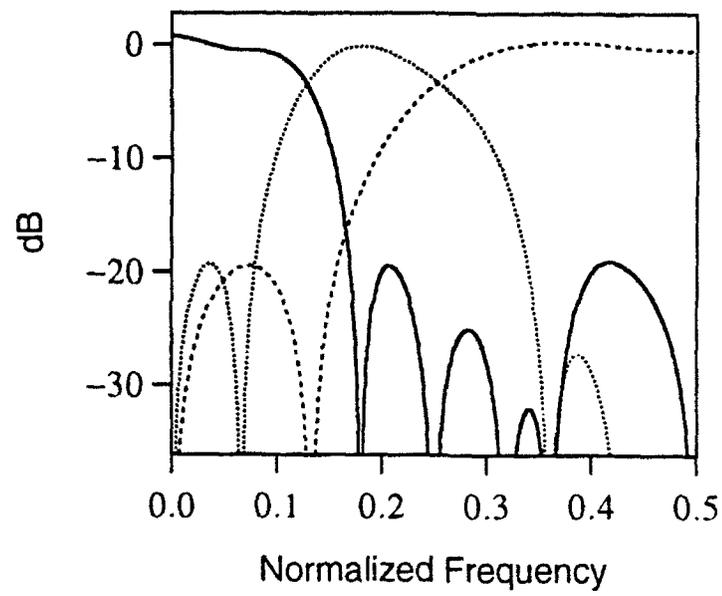(a) before orthonormalization, (b) after orthonormalization.

Fig. 4. Example 3.2. Magnitude resonses of analysis filters,
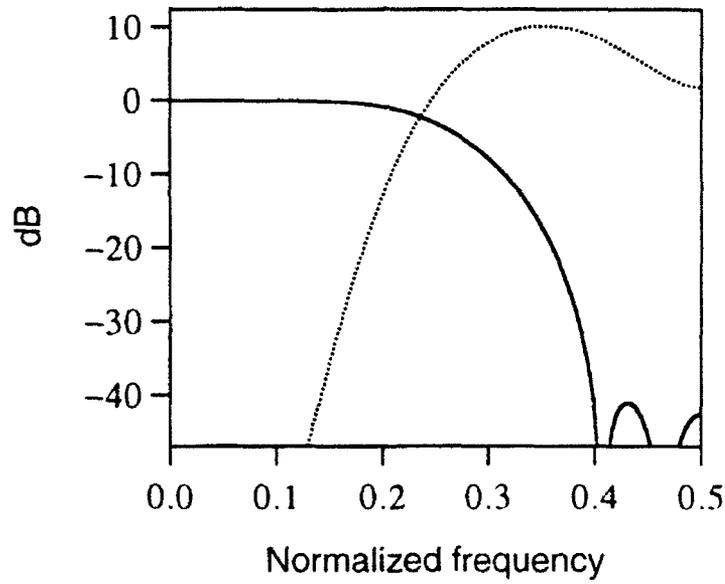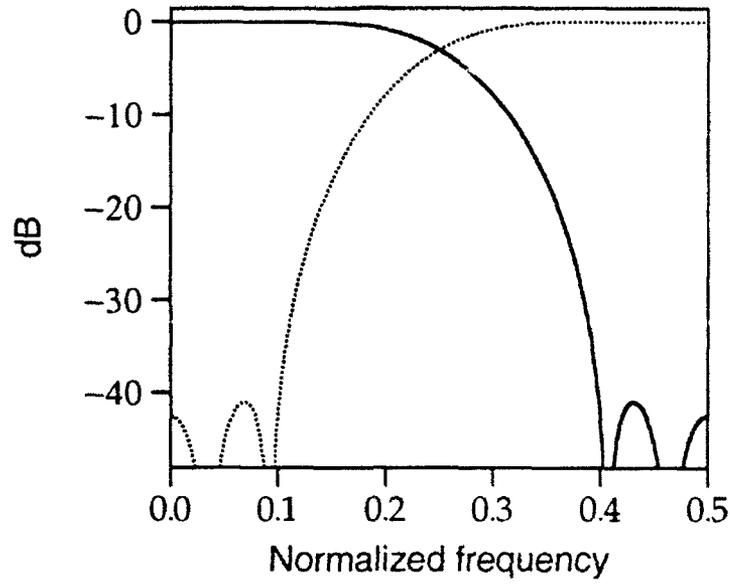(a) before orthonormalization, (b) after orthonormalization.

Fig 5. Magnitude responses of analysis filters,
(a) before decorrelation, (b) after decorrelation.