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FORCE-FREE MAGNETIC FIELDS, CURL EIGENFUNCTIONS, AND THE SPHERE IN TRANSFORM SPACE, WITH APPLICATIONS TO FLUID DYNAMICS AND ELECTROMAGNETIC THEORY

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13. ABSTRACT *(Maximum 200 words)*
The mathematical foundation of a new description of force-free magnetic fields (FFMFs) is given using Moses' curl eigenfunctions, in preparation for an investigation of solar magnetic clouds and their interaction with the Earth's magnetosphere and perturbation of the radiation belts. Constant-α FFMFs are defined completely on the unit hemisphere in Fourier transform space, reducing the three-dimensional physical space problem to a two-dimensional transform space problem. A scheme for classifying these fields by the dimensionality, symmetry, and complexity of their supporting sets in transform space is sketched. The fields corresponding to the simplest 0-, 1-, and 2-dimensional transform sphere sets are exhibited. Four applications illustrate the technique: (1) the constant-α FFMF vector potential is shown to be unimodal; (2) α is identified with a normalized magnetic helicity; (3) the helicity hierarchy for Trkalian fluids is shown to depend only on α and the mean kinetic energy; (4) the Maxwell equations are reduced to an FFMF problem, providing a new point of view for electromagnetic theory. Speculative applications to turbulence and the laboratory modeling of astrophysical FFMFs are mentioned. Future directions for development are indicated, and extensive connections to related work are documented.

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Force-Free Magnetic Fields, Curl Eigenfunctions, and the Sphere in Transform Space, with Applications to Fluid Dynamics and Electromagnetic Theory

1. INTRODUCTION

More than two decades ago Moses[1] developed representations for eigenfunctions of the curl operator, applying them to a number of physical problems of interest. We employ these functions here to construct a new basis for the representation of force-free magnetic fields that will facilitate their application to the solution of geophysical and astrophysical problems.

The concept of the force-free magnetic field (FFMF) has proven very useful in modeling a number of physical systems of interest since its introduction by Lundquist[2], the initial development by Lust and Schlüter[3], and the appearance of the useful mathematical description by Chandrasekhar and Kendall[4]. While physical applications have ranged from stellar astrophysics to controlled thermonuclear fusion, the dominant topic of papers published has been the modeling of solar magnetic fields and their effects on the upper atmosphere of the sun[5]. The magnetic clouds ejected from the sun which have produced the major perturbations to the Earth's radiation belts during the satellite era seem to possess FFMFs which have budded from the solar magnetic field (Burlaga and Lepping[6], Klein and Burlaga[7], Lepping, Jones, and Burlaga[8]). The present work, which is motivated by the desire to describe these solar magnetic clouds, their origins, and their interactions with the Earth's magnetosphere, introduces a new viewpoint for the investigation of these fields and indicates its broader application in fluid dynamics and electromagnetics.

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In spite of the numerous papers which have dealt competently with various parts of the theory, the mathematical description of FFMFs remains incomplete with many ramifications unexplored. (As a cogent example, note the recent appearance of a modern analysis of properties of the curl operator: Yoshida and Giga[9].) The purpose of the present paper is threefold: to introduce a new representation of these fields which offers both practical computational advantages and a change in theoretical viewpoint, to suggest a scheme for classifying FFMFs, and to indicate the broad applicability of the new representation to fluid dynamics and electromagnetics.

2. THE GENERAL FORCE-FREE MAGNETIC FIELD

The concept of a force-free magnetic field arose naturally in describing the equilibrium states of a non-resistive plasma. Considering the particular limiting case in which the hydrostatic pressure and the Lorentz force are both zero

$$\nabla p = \mathbf{J} \times \mathbf{B} = 0 , \tag{1}$$

Lundquist[2] realized that the requirement that the current density satisfy the static Maxwell equation

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \tag{2}$$

amounted to a constraint on the magnetic field

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \tag{3}$$

which may be equivalently stated

$$(\nabla \times \mathbf{B}) = \alpha \mathbf{B} . \tag{4}$$

Thus, for the magnetic field to be force-free, the magnetic field vorticity must be parallel to the field itself with a proportionality that is in general a function of the spatial coordinates:

$$\alpha = \alpha(\mathbf{x}) \tag{5}$$

The type of functional dependence of α partitions the class of force-free magnetic fields into two subclasses: variable α characterizes the so-called nonlinear subclass, while constant α characterizes the other subclass. We restrict our attention here to this latter case, as have most previous investigators.

The investigation presented here is mathematical; the results obtained are thus independent of the degree to which the FFMF model is applicable to any given system and thus applicable to all appropriate systems. See Roberts[10], Chapters 1 and 4, for a helpful discussion of the physical idealizations and approximations commonly used.

3. THE CONSTANT-α FORCE-FREE MAGNETIC FIELD

3.1 The Mathematical Boundary Value Problem

The mathematical problem addressed here is the determination of the magnetic field, **B**, which satisfies the first order partial differential equation system

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \qquad (6)$$
$$\nabla \cdot \mathbf{B} = 0$$

(with α constant) together with appropriate boundary conditions.

A thorough discussion of the boundary value problem (BVP) is complex and lengthy, and would divert us from the main point of the report; it is deferred to a later report. The question of the proper posing of the BVP may be perceived as an undercurrent running through many of the published papers, especially those dealing with the numerical modeling of the fields. The early intuitive comments of Grad and Rubin[11] have frequently been used to justify the approaches taken by those later authors who have confronted the problem directly. The literature here is extensive; consult the review by Sakurai[12] for orientation, and the following papers for a small sampling of the variety of approaches: Saks[13], Boström[14], Semel[15], Aly[16], and Yan et al.[17]. The series of papers by Kress[18,19,20] present a careful formulation of certain FFMF BVPs in terms of integral equations.

The constant parameter α plays a central role in the development below. Although it can be viewed merely as a scale factor in physical space (as is evident on inspection of the system (6)), the choice of this scale factor has significant implications, as will be seen below. It was recognized early that α is intimately involved with the boundary conditions imposed on the field (Lüst and Schlüter[3], Chandrasekhar[21]), but even today this relationship is not well understood. Were we to address this here it would involve us deeply with the boundary value problem. In keeping with the decision to defer the BVP discussion, we do not follow this path here.

The mathematical basis for the approach taken here may be found in the works of Ehrenpreis[22] and Palamodov[23], in which Fourier transform methods are applied to the solution of systems of partial differential equations with constant coefficients. We assume that the general solutions derived below can be fitted to appropriate boundary values using the inverse Fourier transform, such as exemplified by Sneddon[24]. Since our primary concern is the elucidation of a new approach to FFMFs, no direct application of this technique is presented here.

Remark. There are interesting ties between the current approach and a number of areas that are worth mentioning *en passant*. The BVP as treated by Chandrasekhar and Kendall[4] invokes the second order vector Helmholtz equation obtained by applying the curl to the first of Eqs. (6) above. The field is then obtained by constructing a vector solution from solutions to the *scalar* Helmholtz equation in analogy with the vector waves found by Hansen[25] and developed by Elsasser[26]. The sphere in transform space is as fundamental a concept for the scalar Helmholtz equation as it is for the curl eigenfunction representation (Miller[27], Newton[28]). Algebraic group operators restricted to the sphere (the transform images of differential group operators in x-space) can probably be used to investigate the separability of the FFMF equation in various coordinate systems in a manner completely analogous to that expounded by Miller[27]. In this connection the FFMF group structure derived by Yemets' and Kovbasenko[29] should be helpful. While a number of investigators have implicitly recognized the role of the sphere in transform space (e.g., Meyer[30], Dombre et al.[31]), and others have approached the use of curl eigenfunctions (McLaughlin and Pironneau[32], Constantin and Majda[33]), it is the explicit use of the Moses curl eigenfunctions, with their orthogonality, completeness, and associated vector Fourier transforms, that has made possible the analysis described here.

3.2. Curl Eigenfunction Representation of Force-Free Magnetic Fields

The system (6) reveals that an FFMF must possess vorticity and no divergence; it is a solenoidal, rotational field. Following Moses' extension of the Helmholtz theorem, such a field may be represented as a sum over the rotational curl eigenfunctions (see the Appendix for a summary of properties of the curl eigenfunctions):

$$\mathbf{B}(\mathbf{x}) = \sum_{\lambda}' \int d^{3}k \; \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k}) \, b_{\lambda}(\mathbf{k}) \tag{7}$$

where the prime on the summation index indicates the omission of the $\lambda = 0$ (irrotational) eigenfunction, restricting the summation to the values $\lambda = \pm 1$. Substitution of this representation into the defining equation (1) and making use of the eigenproperties of $\boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k})$ leads immediately to the equation

$$\sum_{\lambda}' \int d^{3}k \; (\lambda k - \alpha) \, \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k}) \, b_{\lambda}(\mathbf{k}) = 0 \, . \tag{8}$$

Taking the spatial scalar product of this with $\boldsymbol{\chi}^{*}_{\lambda}(\mathbf{x} \mid \mathbf{k}')$ and using the orthogonality of the curl eigenfunctions transforms the relation to

$$\sum_{\lambda}{}' \int d^3k\, (\lambda\, k - \alpha)\, \delta(\mathbf{k} - \mathbf{k}')\, \delta_{\mu\lambda}\, b_\lambda(\mathbf{k}) = 0 \quad \rightarrow \quad (\mu\, k' - \alpha)\, b_\mu(\mathbf{k}') = 0 \tag{9}$$

with solution (remembering $\mu = 1/\mu$ for $\mu = \pm 1$)

$$b_\mu(\mathbf{k}') = \frac{\delta(k' - \mu\, \alpha)}{k'^2}\, s_\mu(\boldsymbol{\kappa}') \tag{10}$$

where we have made use of the radial delta function (see Barton[34]), $\boldsymbol{\kappa}'$ is a vector drawn to a point on the unit sphere, and $s_\mu(\boldsymbol{\kappa}')$ is a function yet to be determined. This is the central result which provides the basis for the rest of the report. There are two immediate consequences of this relation:

(1) an arbitrary constant-α force-free magnetic field is defined entirely in terms of the values of its transform on the sphere $k' = \alpha/\mu = \alpha\,\mu$;

(2) since k' must be positive (as the magnitude of the real vector $\mathbf{k}'$), only one eigenfunction contributes to the field: the one whose sign matches the sign of α: $\mu = \mathrm{sgn}(\alpha)$.

Reconsidering the defining equation (4), we see that in the constant-α case, α is merely a (signed) reciprocal length unit, and the subsequent formulation can be simplified by introducing the non-dimensional variable $\boldsymbol{\xi}$:

$$\boldsymbol{\xi} \equiv |\alpha|\, \mathbf{x} \tag{11}$$

which, reverting to λ as the designator of the curl eigenvector component, changes Eqs. (6) to the form

$$\begin{aligned} \nabla' \times \mathbf{B} &= \lambda\, \mathbf{B} \\ \nabla' \cdot \mathbf{B} &= 0 \end{aligned} \tag{12}$$

where the prime indicates differentiation with respect to $\boldsymbol{\xi}$, with the result that the unit sphere in transform space assumes a dominant position in the theory. The expression for the FFMF now simplifies to:

$$
\begin{aligned}
\mathbf{B}_{\lambda}(\boldsymbol{\xi}) &= \int k^{2} dk\, d\Omega \, \frac{\delta(k - \lambda\, \alpha)}{k^{2}} \, s_{\lambda}(\boldsymbol{\kappa})\, \boldsymbol{\chi}_{\lambda}(\boldsymbol{\xi} \mid \boldsymbol{\kappa}) \\
&= \int d\Omega \, \boldsymbol{\chi}_{\lambda}(\boldsymbol{\xi} \mid \boldsymbol{\kappa})\, s_{\lambda}(\boldsymbol{\kappa}) \qquad (13) \\
&= (2\pi)^{-3/2} \int d\Omega \, e^{i \boldsymbol{\kappa} \cdot \boldsymbol{\xi}} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa})\, s_{\lambda}(\boldsymbol{\kappa}).
\end{aligned}
$$

where $d\Omega = \sin\eta \, d\eta \, d\psi$ denotes the surface element on the unit sphere in terms of the polar angle η and the azimuthal angle ψ :

$$
\mathbf{k} \cdot \mathbf{x} = k \, \boldsymbol{\kappa} \cdot \frac{\boldsymbol{\xi}}{|\alpha|} = \lambda \frac{\alpha}{|\alpha|} \boldsymbol{\kappa} \cdot \boldsymbol{\xi} = \boldsymbol{\kappa} \cdot \boldsymbol{\xi} \qquad (14)
$$

Remark. It has been known for some time that the general solution to the scalar Helmholtz equation may be cast in terms of a Fourier integral over the unit sphere in transform space; see Miller[27], as well as the recent detailed investigation of Newton[28]. The essential difference between the FFMF and Helmholtz solutions is the appearance of the $\mathbf{Q}_{\lambda}(\boldsymbol{\kappa})$ matrices in the FFMF transform expressions.

Thus the general three-dimensional constant-$\boldsymbol{\alpha}$ FFMF is defined entirely by the variation of the scalar function $s_{\lambda}(\boldsymbol{\kappa}) = s_{\lambda}(\eta, \psi)$ in conjunction with the curl eigenfunction $\boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k})$ over the unit sphere in transform space, a reduction of the 3-D problem in physical space to a 2-D problem for a function of compact support in transform space.

An additional simplification appears on examining the requirement that the FFMFs be real. For this to be so, the field must be equal to its conjugate:

$$
\mathbf{B}_{\lambda}(\boldsymbol{\xi}) = \mathbf{B}_{\lambda}^{*}(\boldsymbol{\xi}) \qquad (15)
$$

or

$$
\int d\Omega \, e^{i \boldsymbol{\kappa} \cdot \boldsymbol{\xi}} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa})\, s_{\lambda}(\boldsymbol{\kappa}) = \int d\Omega \, e^{-i \boldsymbol{\kappa} \cdot \boldsymbol{\xi}} \, \mathbf{Q}_{\lambda}^{*}(\boldsymbol{\kappa})\, s_{\lambda}^{*}(\boldsymbol{\kappa}) . \qquad (16)
$$

Inverting $\boldsymbol{\kappa}$ on the right-hand side, multiplying by $e^{-i \boldsymbol{\kappa}' \cdot \boldsymbol{\xi}}$ and integrating over $\boldsymbol{\xi}$ and $\boldsymbol{\kappa}'$ leads to

$$
\mathbf{Q}_{\lambda}(\boldsymbol{\kappa}')\, s_{\lambda}(\boldsymbol{\kappa}') = \mathbf{Q}_{\lambda}^{*}(-\boldsymbol{\kappa}')\, s_{\lambda}^{*}(-\boldsymbol{\kappa}') , \qquad (17)
$$

which, making use of the inversion relation between the $\mathbf{Q}_\lambda$ (see Appendix),

$$\mathbf{Q}_\lambda{}^*(-\boldsymbol{\kappa}) = -\left(\frac{\kappa_1 - i\,\lambda\,\kappa_2}{\kappa_1 + i\,\lambda\,\kappa_2}\right)\mathbf{Q}_\lambda(\boldsymbol{\kappa}) \tag{18}$$

leads to the restriction on $s_\lambda(\boldsymbol{\kappa})$:

$$s_\lambda(\boldsymbol{\kappa}) = -\left(\frac{\kappa_1 - i\,\lambda\,\kappa_2}{\kappa_1 + i\,\lambda\,\kappa_2}\right)s_\lambda{}^*(-\boldsymbol{\kappa}) \tag{19}$$

or, equivalently,

$$s_\lambda(-\boldsymbol{\kappa}) = -\left(\frac{\kappa_1 - i\,\lambda\,\kappa_2}{\kappa_1 + i\,\lambda\,\kappa_2}\right)s_\lambda{}^*(\boldsymbol{\kappa}) \tag{20}$$

This means that *the complex FFMF transform need only be defined on a hemisphere of the unit sphere*, since the transform on the "lower" half-space is defined when the conjugate function is known on the "upper" half-space. This is precisely what Moses found in the general vector field case[1].

The general expression for the FFMFs may now be simplified by using the above relation to restrict the transform to a hemisphere in transform space. Consider the general expression:

$$\mathbf{B}_\lambda(\boldsymbol{\xi}) = (2\pi)^{-3/2}\int d\Omega\; e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\,\mathbf{Q}_\lambda(\boldsymbol{\kappa})\,s_\lambda(\boldsymbol{\kappa}) \tag{21}$$

For ease of calculation, we orient the z-axis along $\boldsymbol{\xi}$, defining the polar angle η as the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$:

$$\cos\eta \equiv \boldsymbol{\kappa}\cdot\boldsymbol{\xi}/\xi \tag{22}$$

and we split the integration over the sphere into that over the +z and −z hemispheres:

$$
\begin{aligned}
\mathbf{B}_{\lambda}(\xi) = (2\pi)^{-3/2}\int_{0}^{2\pi} d\psi \Big[& \int_{0}^{\pi/2} \sin\eta \, d\eta \, e^{i\xi\cos\eta} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa}) \, s_{\lambda}(\boldsymbol{\kappa}) \\
& + \int_{\pi/2}^{\pi} \sin\eta \, d\eta \, e^{i\xi\cos\eta} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa}) \, s_{\lambda}(\boldsymbol{\kappa}) \Big]
\end{aligned}
\tag{23}
$$

Now invert $\boldsymbol{\kappa}$ in the $-z$ integral, setting

$$
\boldsymbol{\kappa} = -\boldsymbol{\kappa}', \quad \eta = \pi - \eta', \quad \psi = \pi + \psi' \tag{24}
$$

to obtain

$$
\begin{aligned}
\mathbf{B}_{\lambda}(\xi) = (2\pi)^{-3/2}\int_{0}^{2\pi} d\psi \Big[& \int_{0}^{\pi/2} \sin\eta \, d\eta \, e^{i\xi\cos\eta} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa}) \, s_{\lambda}(\boldsymbol{\kappa}) \\
& + \int_{0}^{\pi/2} \sin\eta' \, d\eta' \, e^{-i\xi\cos\eta'} \, \mathbf{Q}_{\lambda}(-\boldsymbol{\kappa}') \, s_{\lambda}(-\boldsymbol{\kappa}') \Big]
\end{aligned}
\tag{25}
$$

and make use of the reality conditions to find

$$
\mathbf{Q}_{\lambda}(-\boldsymbol{\kappa}') \, s_{\lambda}(-\boldsymbol{\kappa}') = \left(\frac{\kappa'_1 + i\lambda\kappa'_2}{\kappa'_1 - i\lambda\kappa'_2} \right) \mathbf{Q}_{\lambda}^{*}(\boldsymbol{\kappa}') \left(\frac{\kappa'_1 - i\lambda\kappa'_2}{\kappa'_1 + i\lambda\kappa'_2} \right) s_{\lambda}^{*}(\boldsymbol{\kappa}') = \mathbf{Q}_{\lambda}^{*}(\boldsymbol{\kappa}') \, s_{\lambda}^{*}(\boldsymbol{\kappa}') \tag{26}
$$

leading to the expression

$$
\begin{aligned}
\mathbf{B}_{\lambda}(\xi) = (2\pi)^{-3/2}\int_{0}^{2\pi} d\psi \Big[& \int_{0}^{\pi/2} \sin\eta \, d\eta \, e^{i\xi\cos\eta} \, \mathbf{Q}_{\lambda}(\boldsymbol{\kappa}) \, s_{\lambda}(\boldsymbol{\kappa}) \\
& + \int_{0}^{\pi/2} \sin\eta' \, d\eta' \, e^{-i\xi\cos\eta'} \, \mathbf{Q}_{\lambda}^{*}(\boldsymbol{\kappa}') \, s_{\lambda}^{*}(\boldsymbol{\kappa}') \Big]
\end{aligned}
\tag{27}
$$

which is obviously real, since the second integral is clearly the conjugate of the first, and taken only over the limits of the upper z-hemisphere.

Thus we have shown that *a general constant-α force-free magnetic field is defined completely by its variation on a unit hemisphere in Fourier tranform space.*

4. APPLICATIONS OF THE CURL EIGENFUNCTION REPRESENTATION

We now demonstrate that the representation introduced above readily simplifies and clarifies certain FFMF concepts and is capable of generalized application to fluid dynamics and electromagnetics.

4.1. Vector Potentials for the Force-Free Magnetic Field

The current density and the magnetic field must both be nonzero within the region considered in order for an FFMF to exist. For nonzero α, an FFMF must be described by a vector potential, since otherwise the system (6) becomes inconsistent. In the curl eigenfunction representation, the vector potential is in general a sum of two rotational modes analogous to Eq. (7) (equivalent to adopting the Lorentz gauge in the absence of an electric potential):

$$\mathbf{A}(\mathbf{x}) = \sum_{\lambda}{}' \int d^{3}k\, \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k})\, a_{\lambda}(\mathbf{k}) \tag{28}$$

If this is now equated to an FFMF mode

$$\mathbf{B}_{\lambda}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = \sum_{\lambda}{}' \lambda \int d^{3}k\, k\, \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k})\, a_{\lambda}(\mathbf{k}) \tag{29}$$

then the spatial scalar product leads to the expected relation between the field and vector potential transforms :

$$b_{\lambda}(\mathbf{k}) = \lambda\, k\, a_{\lambda}(\mathbf{k}) = \alpha\, a_{\lambda}(\mathbf{k}) . \tag{30}$$

The modes must match. On transforming to x-space, we find the corresponding relation

$$\mathbf{B}_{\lambda}(\mathbf{x}) = \int d^{3}k\, b_{\lambda}(\mathbf{k})\, \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k}) = \alpha \int d^{3}k\, a_{\lambda}(\mathbf{k})\, \boldsymbol{\chi}_{\lambda}(\mathbf{x} \mid \mathbf{k}) = \alpha\, \mathbf{A}_{\lambda}(\mathbf{x}) \tag{31}$$

As Moses found in the general case, there is no mixing of modes; i.e., a magnetic field described by an eigenfunction with $\lambda = +1$ must be derived from a vector potential also described by an eigenfunction with $\lambda = +1$. For an FFMF this becomes a more stringent result: the only curl eigenfunction component needed to describe the vector potential of an FFMF is the one whose eigenvalue matches that of the magnetic field. After all, a constant-α FFMF is fundamentally a curl eigenfunction, and *vice versa.*

4.2. Magnetic Helicity of the Force-Free Magnetic Field

Several papers by Chandrasekhar and Woltjer[35] and by Woltjer[36,37,38] (see also the recent work of Laurence and Avellaneda[39]) were devoted to exploring the energy and stability of an FFMF by examining the behavior of a pseudoscalar quantity which we denote by M:

$$M \equiv \int d^3x \, \mathbf{A}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{32}$$

where the integral is taken over the space occupied by the field. This quantity, termed the *magnetic helicity* by Moffatt[40], has a fundamental importance for any magnetic field (see the discussion by Rañada[41] and references therein, and Berger and Field[42]), but takes on a simple form for a constant-α FFMF in the curl eigenfunction representation,where it takes the form

$$M_\lambda \equiv \int d^3x \, \mathbf{A}^*_\lambda(\mathbf{x}) \cdot \mathbf{B}_\lambda(\mathbf{x}) = \alpha \int d^3x \left| A_\lambda(\mathbf{x}) \right|^2 \tag{33}$$

or, inserting the transformed expression, we find:

$$M_\lambda = \alpha \int d^3k \, \left| a_\lambda(\mathbf{k}) \right|^2 \tag{34}$$

Thus the magnetic helicity for the constant-α FFMF is proportional to the mean square magnitude of the vector potential taken over x-space or to the mean square vector potential transform taken over k-space with a proportionality constant equal to α (known in differential geometry as the *abnormality* of the vector field **B** (or **A**, since they satisfy the same equation for the given gauge choice): see Ericksen[43] and Truesdell[44]). The abnormality, α, may be defined to be the *normalized magnetic helicity*, M_λ:

$$\alpha = M_\lambda \equiv \frac{\int d^3x \, \mathbf{A}^*_\lambda(\mathbf{x}) \cdot \mathbf{B}_\lambda(\mathbf{x})}{\int d^3x \, \mathbf{A}^*_\lambda(\mathbf{x}) \cdot \mathbf{A}_\lambda(\mathbf{x})} \tag{35}$$

The magnetic helicity is but one element of an infinite helicity hierarchy whose individual terms can be very simply expressed in terms of the curl eigenfunctions in the constant-α case. We consider this hierarchy, couched in terms of fluid dynamics, in the following section.

4.3. Trkalian Motion in Fluid Dynamics

It is well known that the general force-free magnetic field is equivalent to the general Beltrami flow in fluid dynamics, while the constant-α FFMF is equivalent to the Trkalian subset of the Beltrami flow (Beltrami[45], Trkal[46], Truesdell[44], Bjørgum[47], Bjørgum and Godal[48], Godal[49]). Although the development above is couched in terms of the magnetic field problem, everything discussed there can be interpreted in fluid dynamical terms with the velocity of the fluid acting as the counterpart of the magnetic field. In particular, the Trkalian problem can be formulated entirely in terms of the behavior of the velocity transform on the unit sphere in transform space. Note, however, that the stability analysis for the fluid case must be distinguished from the analogous magnetohydrodynamic case, as pointed out by Moffatt [50,51].

In this section we consider a sequence of integrals that are second-degree functions of the fluid velocity and which we call, for want of a better term, the *helicity hierarchy*. In keeping with the restriction of this paper to constant-α fields, the hierarchy is considered for Trkalian fluids, though the formal application to general Beltrami flows is straightforward and will be addressed in a later report.

In analogy to the definition of helicity, we introduce the helicity hierarchy as the following set of integrals of the scalar product of the various interated curls of the fluid velocity:

$$H_{mn} = \int d^3x \left[\nabla\times \right]^m \mathbf{u}(\mathbf{x}) \cdot \left[\nabla\times \right]^n \mathbf{u}(\mathbf{x}) \tag{36}$$

where (m, n) are taken to be (positive) integers. The (kinematic) *helicity* is immediately recognizable as H_{01}. Three other low order elements of this sequence occur more or less frequently in the literature: the (incompressible) *kinetic energy per unit mass*, H_{00}; the *enstrophy*, H_{11}, and the *superhelicity*, H_{12} (Hide[52]). A given member of the set, H_{mn}, can be described in statistical fluid dynamic terms as the correlation of the mth iterated curl of the velocity with the nth interated curl, and we note that the matrix of the H_{mn} is symmetric. We now consider the form which this hierarchy takes for Trkalian fluids.

A Trkalian fluid possesses a velocity, $\mathbf{u}$, that satisfies the system (6) above:

$$
\begin{aligned}
\nabla \times \mathbf{u} &= \alpha\, \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
\qquad (37)
$$

hence the analysis of Section 3 goes through unchanged and we can write

$$
\mathbf{u}_\lambda(\mathbf{x}) = \int d\Omega\; S_\lambda(\Omega)\, \boldsymbol{\chi}_\lambda(\boldsymbol{\xi} \mid \boldsymbol{\kappa}) \qquad (38)
$$

In view of the properties of the curl eigenfunctions, we can immediately write down the expression for the *n*th interated curl of the velocity:

$$
\begin{aligned}
\left[\nabla\times \right]^m \mathbf{u}_\lambda(\mathbf{x}) &= \lambda^m \int d^3k\; k^m\; U_\lambda(\mathbf{k})\, \boldsymbol{\chi}_\lambda(\mathbf{x} \mid \mathbf{k}) \\
&= \alpha^m \int d\Omega\; S_\lambda(\Omega)\, \boldsymbol{\chi}_\lambda(\boldsymbol{\xi} \mid \boldsymbol{\kappa})
\end{aligned}
\qquad (39)
$$

enabling us to evaluate the helicity hierarchy defined above as the sequence of integrals

$$
\begin{aligned}
H_{mn} &= \alpha^{m+n-3} \int d\Omega\; S_\lambda^*(\boldsymbol{\kappa}) \int d\Omega'\; S_\lambda(\boldsymbol{\kappa}') \int d^3\xi\; \boldsymbol{\chi}_\lambda^*(\boldsymbol{\xi} \mid \boldsymbol{\kappa})\, \boldsymbol{\chi}_\lambda(\boldsymbol{\xi} \mid \boldsymbol{\kappa}') \\
&= \alpha^{m+n-3} \int d\Omega \left| S_\lambda(\boldsymbol{\kappa}) \right|^2
\end{aligned}
\qquad (40)
$$

This is a remarkable result. It says that for a Trkalian velocity field the elements of the set of fluid correlation functions defined by Eq. (36) are all reducible to H_{00}, the kinetic energy per unit mass and a power of the velocity field's abnormality (or normalized helicity). The magnitude of H_{mn} grows or diminishes with the order $(m+n)$ according as α is greater or less than unity. The extreme simplicity of the helicity hierarchy for this case emphasizes both the very special nature of this type of field and its intimate connection to the concept of helicity.

Normalizing the H_{mn} analogously to the magnetic helicity of the previous section, we find

$$\widehat{H}_{mn} = \alpha^{m+n} \tag{41}$$

so that the first four named normalized correlations become:

kinetic energy: $\widehat{H}_{00} = 1$;

kinematic helicity: $\widehat{H}_{01} = \alpha$;

enstrophy: $\widehat{H}_{11} = \alpha^2$;

superhelicity: $\widehat{H}_{12} = \alpha^3$.

Thus the energy and enstrophy are positive definite, while the kinematic helicity and superhelicity adopt the sign of α. For negative α, the elements of odd index $(m+n)$ are all negative and the even elements are all positive, while for positive α, all elements are positive.

The analogous magnetic helicity hierarchy is obtained by replacing the velocity by the magnetic field, though the indices differ for the unnormalized elements because the analogue of the fluid velocity in the definition of magnetic helicity is the vector potential, not the magnetic field. Of course, a cross helicity hierarchy can be defined in terms of the convolutions of the fluid and magnetic transforms, but we shall not pursue that here.

4.4 Bivector Form of the Maxwell Equations in the FFMF Representation

The representation developed above for the force-free magnetic field can be applied directly to electromagnetic theory to produce a new and potentially very useful viewpoint, which is introduced here and will be developed in a subsequent report.

At the turn of the century Silberstein[53] introduced a bivector formulation of the Maxwell equations by defining a complex field vector with real and imaginary parts given by the electric field, $\mathbf{E}$, and magnetic field, $\mathbf{B}$, resp. The Maxwell equation system then takes a form that is easily treated by the FFMF formalism. Following Stratton[54], but changing notation slightly to avoid a conflict with that used above, we introduce the field bivector

$$\mathbf{G} \equiv \mathbf{B} + i\,\mathbf{E}/c \tag{42}$$

where we use rationalized MKS units and c is understood to refer to the speed of light in the medium:

$$c^2 = \varepsilon \mu . \tag{43}$$

The Maxwell equations then take the form

$$\nabla \times \mathbf{G} + \frac{i}{c}\frac{\partial \mathbf{G}}{\partial t} = \mu \mathbf{J} , \quad \nabla \cdot \mathbf{G} = i \mu c \rho , \tag{44}$$

where (ρ, **J**) are the charge and current source densities. In the absence of sources they become

$$\nabla \times \mathbf{G} + \frac{i}{c}\frac{\partial \mathbf{G}}{\partial t} = 0 , \quad \nabla \cdot \mathbf{G} = 0 , \tag{45}$$

while for harmonic time dependence (read equivalently: Fourier analysis in time) their formal similarity with the FFMF equations is striking:

$$\nabla \times \mathbf{G} = \alpha_{em} \mathbf{G} , \quad \nabla \cdot \mathbf{G} = 0 , \tag{46}$$

where we have defined the electromagnetic helicity as $\alpha_{em} \equiv \frac{\omega}{c}$. The analysis of Section 3 is fully applicable here, and we find that *the three-dimensional electromagnetic wave problem is completely defined on the sphere in transform space for which*

$$k = \lambda \alpha_{em} \tag{47}$$

with the fundamental difference from the FFMF problem that *the full sphere must be utilized* because we cannot require the field to be real; the real and imaginary parts of the vector **G** are independent, being the magnetic and electric field vectors. Turning this result around, we see that *the real and imaginary parts of an FFMF may be interpreted as the magnetic and electric fields of a corresponding electromagnetic problem.*

This result can be generalized immediately (by an appropriate redefinition of α_{em} and **G**) to encompass chiral media characterized by the Drude-Born-Fedorov constitutive relations, simplifying significantly the equation system obtained recently by Lakhtakia[55] (see also Lakhtakia, Varadan and Varadan[56]).

Moreover, the full BVP for the electromagnetic field should be accessible from this treatment through appropriate Green's function techniques. If further work confirms the utility of this approach, it could have a significant impact on both the theory and practice of electromagnetics.

We leave the mathematical problem at this point, having established the formal connection between the more general electromagnetic theory and the force-free magnetic field formulation, the latter clearly a subset of the former. Although the bivector description of the electromagnetic field has been known for some time (Weber[57], Silberstein[53], Bateman[58], Stratton[54]), the FFMF connection has not been previously pointed out.

5. CLASSIFICATION OF CONSTANT-α FORCE-FREE MAGNETIC FIELDS

Having found the general approach to constant-α force-free magnetic fields outlined above, it is natural to ask whether it can be used to elucidate general properties of these fields. In particular, does it suggest a basis for classifying the various types of FFMFs? Such a classification can be approached in various ways, depending in part on the motivation of the classifier. For example, one type of classification might be based on the generalized multipole expansion of Lüst and Schlüter[3] (see also Lüst, Schlüter & Trefftz[59]), arranging fields into classes according to the parametric dependence of their multipole coefficients. By analogy with electrostatic multipoles, such an approach would be expected to be useful in describing FFMFs that are concentrated in limited regions of space. In contrast to such a scheme, which relies on the characteristics of the fields in coordinate space, the classification we sketch here is based on the dimensionality, symmetry, and complexity of the FFMF transforms on the sphere in transform space. Whether this provides a useful classification of types remains to be proven by subsequent development, but the insight gained in the attempt already seems quite valuable.

Viewed at the most general level, any function or distribution, defined on a set of points on the transform sphere, which possesses a curl transform (as we denote the Fourier transform containing the complex vectors $\mathbf{Q}_\lambda$) satisfying the defining system (6) is a valid FFMF. We partition the class of FFMF transforms first of all according to the dimension of the supporting set into point, curve and area transforms, or sets of dimension 0, 1, 2, as well as possibly fractal dimension. The further partitioning of the transforms within these groups is done in terms of complexity, aided by considerations of group symmetry. Only the barest outline of this program can given here; details will be given elsewhere. To provide a hint at the results, examples of the simplest fields in each category are exhibited with brief indications of their properties.

5.1 Point Transform Fields

The simplest point transform, corresponding to a single point, is a delta function characterized by a given amplitude and unit vector drawn to a point on the transform sphere:

$$s_\lambda(\Omega) = s_0 \frac{\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)}{\sin\eta} = s_0 \frac{\delta(\eta - \eta_0)\,\delta(\psi - \psi_0)}{\sin\eta} \qquad (48)$$

leading to a field expression

$$
\begin{aligned}
\mathbf{B}_\lambda(\xi) &= (2\pi)^{-3/2} \int d\Omega \, e^{i \boldsymbol{\kappa} \cdot \xi} \, \mathbf{Q}_\lambda(\boldsymbol{\kappa}) \, s_0 \frac{\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)}{\sin \eta} \\
&= (2\pi)^{-3/2} s_0 \, e^{i \boldsymbol{\kappa}_0 \cdot \xi} \, \mathbf{Q}_\lambda(\boldsymbol{\kappa}_0)
\end{aligned}
\tag{49}
$$

whose real or imaginary part can be recognized as the standing wave that is the well-known simplest FFMF, discussed in Moffatt[40] (with the wave vector $\boldsymbol{\kappa}_0$ oriented along the positive z-axis). An ordered hierarchy of point fields may be constructed using the fundamental expression (49) by simply summing the fields corresponding to different point constellations. In so doing, some surprises are found. While the lines of force of the single point field (49) are straight lines, the lines of force of a 2-point quadrature field are already rather complex, and those of a 3-point quadrature field are in part chaotic (and can be related to the ABC field suggested as a prototypical turbulent velocity field in fluid dynamics; see Arnol'd[60] and Dombre[31]). This is consistent with the viewpoint introduced by Sweet[61], and developed by others (e.g., Cary and Littlejohn[62]), that the magnetic field lines of force are equivalent to the phase-space trajectories of a dynamical system in which the increase in complexity of the dynamical system leads to a progressive loss of integrals of the motion and the appearance of chaos (Zaslavsky *et al*[63]). What new features in complexity may be introduced in higher order members of this hierarchy remain to be discovered.

5.2 Curve Transform Fields

For this group of fields the transform is defined on a continuous curve on the sphere so that our reliance on distributions can give way to the use of ordinary functions. At the same time we must actively take into consideration the variation of the transform amplitude on the defining curves. It seems natural to consider first those transforms that have constant amplitude on the curve, then to expand the treatment to encompass amplitude variation. As an example, consider the simplest family of curve transforms, those for which the amplitude is constant on a circle of constant polar angle η_0:

$$
s_\lambda(\Omega) = s_0 \frac{\delta(\eta - \eta_0)}{\sin \eta}
\tag{50}
$$

After evaluating the integrals, the complex field takes the form

$$
\begin{aligned}
\mathbf{B}_\lambda(\mathbf{x}) = \frac{s_{\lambda 0}}{2\pi^{1/2}} e^{i z/L_z} \Big[& (\cos \eta_0 + 1) J_0(\rho/L_\rho) \, \mathbf{a}_\lambda - (\cos \eta_0 - 1) J_2(\rho/L_\rho) \, e^{i 2 \lambda \varphi} \mathbf{a}_\lambda^* \\
& - \sqrt{2} \, \lambda \, i \sin \eta_0 \, J_1(\rho/L_\rho) \, e^{i \lambda \varphi} \mathbf{k} \Big]
\end{aligned}
\tag{51}
$$

where the constant vector $\mathbf{a}_\lambda$ is defined as

$$\mathbf{a}_\lambda \equiv \frac{\lambda}{\sqrt{2}}\left\{ 1, i\,\lambda, 0 \right\} \tag{52}$$

the length scales are

$$\begin{aligned} L_\rho &\equiv \frac{1}{\alpha \sin \eta_0} \\ L_z &\equiv \frac{1}{\alpha \cos \eta_0} \end{aligned} \tag{53}$$

and (ρ, φ, z) are circular cylindrical coordinates in x-space, and the J_n are the usual Bessel functions. In the limit of $\eta_0 \to 0$, $L_\rho \to \infty$, $L_z \to 1/\alpha$, we recover the point field found above which is periodic in z but has no ρ or φ dependence, while as $\eta_0 \to \pi/2$, $L_\rho \to 1/\alpha$, $L_z \to \infty$, and we find a completely new field without z dependence but which is periodic in φ and quasi-periodic in ρ.

5.3 Area Transform Fields

These fields correspond to transforms defined on 2-dimensional subsets of the sphere. The simplest is constant on the sphere and is equivalent to the Fourier transform of the $\mathbf{Q}_\lambda$:

$$\begin{aligned} \mathbf{B}_\lambda(\mathbf{x}) = {} & (2\pi)^{-1/2}\, b_{\lambda 0} \left[j_0(\alpha r) + i\, j_1(\alpha r)\, P_1(\cos\theta) \right] \mathbf{a}_\lambda - \pi^{-1/2}\, \lambda\, i\, P_1^1(\cos\theta)\, e^{\,i\lambda\varphi}\, j_1(\alpha r)\, \mathbf{k} \\ & - \sqrt{\frac{2}{\pi}} \sum_{n=2}^{\infty} (-i)^n\, (2n+1)\, j_n(\alpha r)\, \frac{(n-2)!}{(n+2)!}\, P_n^2(\cos\theta)\, e^{\,i 2\lambda\varphi}\, \mathbf{a}_\lambda^* \end{aligned} \tag{54}$$

Here r represents the spherical radius and the j_n are the spherical Bessel functions, reflecting the essential spherical nature of this field (and the essential spherical nature of the curl eigenvector component $\mathbf{Q}_\lambda$).

To handle other area fields, it seems expeditious to expand the transform in spherical harmonics, since any function defined on a sphere can be so represented (Courant and Hilbert[64]). Other functions are also available, such as vector spherical harmonics (Moses[1]) and Heun functions (Kalnins, Miller, & Tratnik[65]) which could prove more useful to represent functions with specific symmetry properties.

5.4 Fractal Transform Fields

The previous three sections have exhibited FFMFs whose defining sets on the transform sphere have integral dimensions of 0, 1 and 2. Other sets possessing non-integral dimensions are easily imagined within the context of modern fractal geometry. Mandelbrot's definition of fractal is "a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension" (Mandelbrot[66]; see also Edgar[67]). Fractal support sets have recently been introduced for dynamo magnetic field models (Finn and Ott[68]). Whether such sets have an important role to play in the further development of FFMF theory and practice remains to be seen. Their construction and curl transform seem non-trivial and are not attempted here, but the possibility of their existence must be explicitly acknowledged.

6. SUMMARY, SPECULATIONS, AND FUTURE DIRECTIONS

6.1 Summary

By introducing the Moses curl eigenfunctions to describe constant-α force-free magnetic fields, we have been able to show that such fields are defined entirely by the value of their curl transform on the unit hemisphere in transform space. This change of viewpoint enables an orderly exploration and classification of their properties, an introduction to which was briefly sketched, exhibiting the simplest fields defined on 0, 1, and 2-dimensional sets on the transform sphere. The new representation was applied to the determination of several quantities of interest: (1) the magnetic vector potential, showing no mode mixing; (2) the magnetic helicity, demonstrating that the parameter α can be interpreted as a normalized magnetic helicity; (3) the helicity hierarchy (correlation sequence) for a Trkalian fluid, proving that the members of the hierarchy are defined entirely in terms of the mean kinetic energy and powers of the parameter α. Applying the representation to electromagnetics, we showed that the bivector formulation of Maxwell's equations reduces to a force-free magnetic field problem in which the unit sphere in transform space plays an essential role, opening the way to a new approach to electromagnetic theory and applications.

6.2 Speculations

The geometrical viewpoint employed above suggests further immediate developments and applications of the curl transform representation, several of which are being actively pursued (see §6.3). We mention briefly here connections to two current problems that are more speculative in nature.

Turbulence: Lagrangian, Trkalian and magnetic. The form of the helicity hierarchy derived above undoubtedly has significant implications for Trkalian turbulence (if it exists!), making the determination of α from the boundary conditions a high priority item. While a connection has not been attempted here between the helicity hierarchy and direct or inverse cascades of energy, enstrophy, helicity or any of the other fluid parameters, it bears looking into (cf. Moffatt[69] and references therein, especially: André and Lesieur[70]; Kraichnan[71]; Frisch et al.[72]). The corresponding effects in magnetohydrodynamics and their influence on the generation, maintenance and decay of dynamo fields through the α-effect of Steenbeck, Krause and Rädler[73] should also prove interesting, if not crucial (see also: Krause and Rädler[74]). The time dependence of Trkalian flows, if similar to the corresponding FFMFs, may be simply exponential (Lundquist[2], Chandrasekhar and Kendall[4], Moffatt[40]; see also Marris and Wang[75], and Marris[76,77]). If more complex time dependences can be found, a substantial insight into the cascades and other characteristics of turbulence and dynamo theory may result. The question of stability of FFMFs bears on this and has generated a long trail of papers in the literature. A substantial fraction of these deals with the generation of current sheets, largely in the context of solar flare mechanisms (sampling: Syrovatskii[78], and a long series of papers by Parker: see Parker[79] for earlier references). Do similar localized velocity sheets or jets exist in Trkalian/Beltrami flows and do they contribute in an essential way to turbulent mechanisms?

Laboratory simulation of astrophysical magnetic fields. The most difficult part of simulating many astrophysical processes in the laboratory is getting the scaling of the essential parameters correct. If this is not done, the physical processes which can be supported and studied in the laboratory environment may bear little relation to those of the real universe. Force-free magnetic fields may turn out to be easier to model in the laboratory than expected because of the dependence on the single parameter α, which is itself a length scale. The fixing of α through the specification of appropriate boundary conditions will probably be the key to such modeling. We note that transient force-free magnetic field configurations have been reported recently in the laboratory by Stenzel and Urrutia[80].

6.3 Future Directions

The current work provides the foundation for a number of applications, first being the investigation and classification of the FFMFs touched on in Section 5 and their relation to problems of current interest, especially the description of solar magnetic clouds and their interaction with the Earth's magnetosphere. A complete solution to the FFMF problem can only be given when the influence of boundary conditions on the field form is known, so that an investigation of the full boundary value problem must be high on the agenda. The enlargement of the fluid dynamic investigation to establish connections between the Trkalian helicity hierarchy and Lagrangian/Trkalian turbulence holds out hope of near-term progress. Reconsidering some of the classic electromagnetic problems, such as the proper radiation condition, multipole expansions, and the description of polarization, in terms of the FFMF formulation of the bivector Maxwell equations has the potential of providing new insights to old problems. On a more distant time scale, the application of the present formulation to the dual fluid velocity-magnetic field description involved in the dynamo problem would seem to present advantages due to the natural and explicit representation of the essential effects of vorticity and helicity.

References

1. Moses, H.E. (1971) Eigenfunctions of the Curl Operator, Rotationally Invariant Helmholtz Theorem, and Applications to Electromagnetic Theory and Fluid Dynamics, *SIAM J. Appl. Math.*, **21**: 114-144.

2. Lundquist, S. (1950) Magneto-hydrostatic fields, *Ark. Fysik*, **2**: 361-365.

3. Lüst, R., and Schlüter, A. (1954) Kraftfreie Magnetfelder, *Z. Astrophysik*, **34**: 263-282.

4. Chandrasekhar, S., and Kendall, P.C. (1957) On Force-free Magnetic Fields, *Astrophys. J.*, **126**. 457-460.

5. A DIALOG search on "Force-Free (Magnetic) Fields" conducted in July 1990 turned up 622 references during the period 1969-1990 of which 295 (47%) dealt in a significant way with solar magnetic fields. The partition of all papers into the three categories (theory, observation, experiment) by percent was (89, 9, 2).

6. Burlaga, L.F., and Lepping, R.P. (1990) *Global Configuration of a Magnetic Cloud*, Geophysical Monograph 58, AGU, pp. 373-377.

7. Klein, L.W., and Burlaga, L.F. (1982) Interplanetary Magnetic Clouds at 1 AU, *J. Geophys. Res.*, **87**: 613-624.

8. Lepping, R.P., Jones, J.A., and Burlaga, L.F. (1990) Magnetic Field Structure of Interplanetary Magnetic Clouds at 1 AU, *J. Geophys. Res.*, **95**: 11,957-11,965.

9. Yoshida, Z., and Giga, Y. (1990) Remarks on Spectra of Operator Rot, *Math. Z.*, **204**: 235-245.

10. Roberts, P.H. (1967) *An Introduction to Magnetohydrodynamics*, American Elsevier, New York.

11. Grad, H., and Rubin, H. (1959) Hydromagnetic Equilibria and Force-Free Fields, *Proc. Second U.N. Int'l Conf. on Peacefu Uses of Atomic Energy*, **31**: 190-197.

12. Sakurai, T. (1989) Computational Modeling of Magnetic Fields in Solar Active Regions, *Space Science Reviews*, **51**: 11-48.

13. Saks, R.S. (1971) On the Boundary Value Problems for the System rot $u + l\,u = h$, *Soviet*

Math. Dokl., **12**: 1240-1244.

14. Bostrom, R. (1973) Models of Force-Free Magnetic Fields in Resistive Media, *Astrophys Space Sci.*, **22**: 353-380.

15. Semel, M. (1988) Extrapolation functions for constant-α force-free fields. Green's method for the oblique boundary value, *Astron. Astrophys.*, **198**: 293-299.

16. Aly, J.J. (1989) On the Reconstruction of the Nonlinear Force-Free Coronal Magnetic Field from Boundary Data, *Solar Phys.*, **120**: 19-48.

17. Yan, Y., Yu, Q., and Kang, F. (1991) A Solar Magnetic Field Model and its 3-D Boundary Element Method Solution, *Solar Phys.*, **136**: 195-198.

18. Kress, R. (1977) A Remark on a Boundary Value Problem for Force-free Fields, *J. Appl. Math Phys.*, **28**: 715-722.

19. Kress, R. (1978) The treatment of a Neumann boundary value problem for force-free fields by an integral equation method, *Proc. Roy. Soc.* (Edinburgh), **82A**: 71-86.

20. Kress, R. (1986) On constant-alpha force-free fields in a torus, *J. Eng. Math.*, **20**: 323-344.

21. Chandrasekhar, S. (1956) On Force-Free Magnetic Fields, *Proc. National Acad. Sci. (USA)*, **42**: 1-5.

22. Ehrenpreis, L. (1970) *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, New York.

23. Palamodov, V.P. (1970) *Linear Differential Operators with Constant Coefficients*, V. 168 of Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, New York.

24. Sneddon, I.N. (1972) *The Use of Integral Transforms*, McGraw-Hill, New York.

25. Hansen, W.W. (1935) A New Type of Expansion in Radiation Problems, *Phys. Rev.*, **47**: 139-143.

26. Elsasser, W.M. (1946) Induction Effects in Terrestrial Magnetism Part I. Theory, *Phys. Rev.*, **69**: 106-116.

27. Miller, W. (1977) *Symmetry and Separation of Variables*, Addison-Wesley, Reading, MA.

28. Newton, R.G. (1992) A representation of solutions to the Helmholtz equation with application to crystal Green's functions, *J. Math. Phys.*, **33**: 44-55

29. Yemets', Yu. P., and Kovbasenko, Yu. P. (1983) Group Properties and Invariant Solutions of the Equations of Force-Free Magnetic Fields, *Dopovidi Akad. Nauk Ukr. SSR* No. 4, 67-70 (Ukrainian).

30. Meyer, R.X. (1983) Spatially Periodic, Force-Free Magnetic Fields with Resistive Decay, *Quart. Appl. Math.*, **20**: 377-384.

31. Dombre, T., Frisch, U., Greene, J.M., Hénon, M., Mehr, A., and Soward, A.M. (1986) Chaotic streamlines in the ABC flows, *J. Fluid Mech.*, **167**: 353-391.

32. McLaughlin, D., and Pironneau, O. (1991) Some notes on periodic Beltrami fields in Cartesian geometry, *J. Math Phys.*, **32**: 797-804.

33. Constantin, P., and Majda, A. (1988) The Beltrami Spectrum for Incompressible Fluid Flows, *Comm. Math. Phys.*, **115**: 435-456.

34. Barton, G. (1989) *Elements of Green's Functions and Propagation; Potentials, Diffusion and Waves*, Clarendon Press, Oxford.

35. Chandrasekhar, S., and Woltjer, L. (1958) On Force-Free Magnetic Fields, *Proc. National Acad.Sci. (USA)*, **44**: 285-289.

36. Woltjer, L (1958) A Theorem on Force-Free Magnetic Fields, *Proc. National Acad. Sci. (USA)*, **44**: 489-491.

37. Woltjer, L (1958) The Stability of Force-Free Magnetic Fields, *Astrophys. J.*, **128**: 384-391.

38. Woltjer, L (1959) Hydromagnetic Equilibrium II. Stability in the Variational Formulation, *Proc National Acad. Sci. (USA)*, **45**: 769-771.

39. Laurence, P., and Avellaneda, M. (1991) On Woltjer's variational principle for force-free fields: *J. Math. Phys.*, **32**: 1240-1253.

40. Moffatt, H.K. (1978) *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, Cambridge.

41. Rañada, A.F. (1992) On the magnetic helicity, *Eur. J. Phys.*, **13**: 70-76.

42. Berger, M.A., and Field, G.B. (1984) The topological properties of magnetic helicity, *J Fluid Mech.* **147** : 133-148.

43. Ericksen, J.L. (1960) Tensor Fields, in *Principles of Classical Mechanics and Field Theory, Encyclopedia of Physics III/1*, ed. S. Flügge, Springer-Verlag, Berlin.

44. Truesdell, C. (1954) *The Kinematics of Vorticity*, Indiana University Press, Bloomington.

45. Beltrami, E. (1889) Considerazioni idrodinamiche, *Rend. Reale Istituto Lombardo* (2), **22**: 121-130.

46. Trkal, V. (1919) Poznámka k hydrodynamice vazkych tekutin, *Casopis Pest. Mat. Fis.*, **48**: 302-311 (Czech).

47. Bjørgum, O., (1951) On Beltrami Vector Fields and Flows ($\nabla\times\mathbf{V} = \Omega\mathbf{V}$) Part I A Comparative Study of Some Basic Types of Vector Fields, *Universitet i Bergen, Årbok. Naturv. rekke*, Nr. 1.

48. Bjørgum, O and Godal, T. (1952) On Beltrami Vector Fields and Flows ($\nabla\times\mathbf{V} = \Omega\mathbf{V}$) Part II The case when W is constant in space, *Universitet i Bergen, Årbok, Naturv. rekke*, Nr. 13.

49. Godal, T. (1957) On Beltrami Vector Fields and Flows ($\nabla\times\mathbf{V} = \Omega\mathbf{V}$) Part III Some considerations on the general case, *Universitet i Bergen, Årbok 1957, Naturv. rekke*, Nr. 12.

50. Moffatt, H.K. (1985) Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations, *J. Fluid Mech.*, **166**: 359-378.

51. Moffatt, H.K. (1986) On the existence of localized rotational disturbances which propagate without change of structure in an inviscid fluid, *J. Fluid Mech.*, **173**: 289-302.

52. Hide, R. (1989) Superhelicity, Helicity and Potential Vorticity, *Geophys. Astrophys. Fluid Dynamics*, **48**: 69-79.

53. Silberstein, L. (1907) Elektromagnetische Grundgleichungen in bivektorieller Behandlung, *Ann Physik*, (4) **22**: 579-586.

54. Stratton, J.A. (1941) *Electromagnetic Theory*, McGraw-Hill, New York.

55. Lakhtakia, A. (1991) First Order Characterization of Electromagnetic Fields in Isotropic Chiral Media, *Arch. Elek. Übertragungstechnik*, **45**: 57-59.

56. Lakhtakia, A., Varadan, V.K., and Varadan, V.V. (1989) *Time-Harmonic Electromagnetic*

Fields in Chiral Media, Lecture Notes in Physics No. 335, Springer-Verlag, New York.

57. Weber, H. (1919) *Die partiellen Differential-Gleichungen der mathematischen Physik*. Vieweg & Sohn, Braunschweig.

58. Bateman, H. (1914) *The Mathematical Analysis of Electrical and Optical Wave-Motion of the Basis of Maxwell's Equations*, Cambridge University Press; reprinted by Dover Publications, New York, 1955.

59. Lüst, R., Schlüter, A., and Trefftz, E. (1953) Verallgemeinerte Multipolfelder, *J. Rational Mech. Anal.*, **2**: 485-517.

60. Arnol'd, V. (1965) Sur la topologie des écoulements stationnaires des fluides parfaits. *C. R. Acad. Sci. Paris*, **261**: 17-20.

61. Sweet, P.A. (1958) The Topology of Force-Free Magnetic Fields, *The Observatory*, **78**: 30-32.

62. Cary, J.R., and Littlejohn, R.G. (1983) Noncanonical Hamiltonian Mechanics and Its Application to Magnetic Field Line Flow, *Ann. Physics*, **151**: 1-34.

63. Zaslavsky, G.M., Sagdeev, R.Z., Usikov, D.A., and Chernikov, A.A. (1991) *Weak Chaos and Quasi-Regular Patterns*, Cambridge University Press, Cambridge.

64. Courant, R., and Hilbert, D. (1953) *Methods of Mathematical Physics*, Vol. I, Interscience, New York.

65. Kalnins, E.G, Miller, W., Jr., and Tratnik, M.V. (1991) Families of Orthogonal and Biorthogonal Polynomials on the N-Sphere, *SIAM J. Math. Anal.*, **22**: 272-294.

66. Mandelbrot, B.B. (1982) *The Fractal Geometry of Nature*, Freeman, San Francisco.

67. Edgar, G.A. (1990) *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York.

68. Finn, J.M., and Ott, E. (1990) The fast kinematic magnetic dynamo and the dissipationless limit, *Phys. Fluids B*, **2**: 916-926.

69. Moffatt, H.K (1981) Some developments in the theory of turbulence, *J. Fluid Mech.*, **106**: 27-47.

70. André, J.C., and Lesieur, M. (1977) Influence of helicity on the evolution of isotropic turbulence at high Reynolds number, *J. Fluid Mech.*, **81**: 187-207.

71. Kraichnan, R.H. (1973) Helical turbulence and absolute equilibrium, *J. Fluid Mech.*, **59**: 745-752.

72. Frisch, U., Pouquet, A., Léorat, J., and Mazure, A. (1975) Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence, *J. Fluid Mech.*, **68**: 769-778.

73. Steenbeck, M., Krause, F., and Rädler, K.-H. (1966) Berechnung der mittleren Lorentz-Feldstärke für ein elektrisch leitendes Medium in turbulenter, durch Coriolis-Kräfte beeinflußter Bewegung, *Z. Naturforschung*, **21a**: 369-376.

74. Krause, F., and Rädler, K.-H. (1981) *Mean Field Magnetohydrodynamics and Dynamo Theory*, Pergamon, New York.

75. Marris, A.W., and Wang, C.-C. (1970) Solenoidal Screw Fields of Constant Magnitude, *Arch. Rat. Mech. Anal.*, **39**: 227-244.

76. Marris, A.W. (1979) The Impossibility of Steady Screw Motions of a Navier-Stokes Fluid: *Arch. Rational Mech. Anal.*, **70**: 47-60.

77. Marris, A.W. (1983) On the Completeness of Trkalian Motions of Navier-Stokes Fluids, *Arch. Rational Mech. Anal.*, **82**: 117-164.

78. Syrovatskii, S.I. (1978) On the Time Evolution of Force-Free Fields, *Solar Phys.*, **58**, 89-94.

79. Parker, E.N. (1990) Tangential Discontinuities and the Optical Analogy for Stationary Fields. VII. Effects of Resistivity, *Geophys. Astrophys. Dynamics*, **55**, 161-182.

80. Stenzel, R.L., and Urrutia, J.M. (1990) Force-Free Electromagnetic Pulses in a Laboratory Plasma, *Phys. Rev. Lett.* **65**: 2011-2014.

Appendix

Summary of Properties of the Moses Curl Eigenfunctions

Moses[1] introduced the transform space vectors $\mathbf{Q}_\lambda$ defined by

$$\mathbf{Q}_0(\mathbf{k}) \equiv -\frac{\mathbf{k}}{k},$$
$$\mathbf{Q}_\lambda(\mathbf{k}) \equiv -\frac{\lambda}{\sqrt{2}}\left[\frac{k_1(k_1 + i\lambda k_2)}{k(k+k_3)} - 1,\ \frac{k_2(k_1 + i\lambda k_2)}{k(k+k_3)} - i\lambda,\ \frac{k_1 + i\lambda k_2}{k}\right] \quad \text{(A.1)}$$

which have the following useful properties:

$$\mathbf{Q}_\lambda^*(\mathbf{k}) \cdot \mathbf{Q}_\mu(\mathbf{k}) = \delta_{\lambda\mu},$$
$$\sum_\lambda \mathbf{Q}_{\lambda i}^*(\mathbf{k}) \cdot \mathbf{Q}_{\lambda j}(\mathbf{k}) = \delta_{ij},$$
$$\mathbf{k} \cdot \mathbf{Q}_\lambda(\mathbf{k}) = 0 \quad \textit{for } \lambda = \pm 1, \quad \text{(A.2)}$$
$$\mathbf{k} \times \mathbf{Q}_\lambda(\mathbf{k}) = -i\lambda\,\mathbf{Q}_\lambda(\mathbf{k}).$$

Using these vectors he defined the vector functions $\boldsymbol{\chi}_\lambda$

$$\boldsymbol{\chi}_\lambda(\mathbf{x}|\mathbf{k}) \equiv (2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{Q}_\lambda(\mathbf{k}) \quad \text{(A.3)}$$

which he was able to show are eigenfunctions of the curl operator:

$$\nabla\times\boldsymbol{\chi}_\lambda(\mathbf{x}|\mathbf{k}) = k\lambda\,\boldsymbol{\chi}_\lambda(\mathbf{x}|\mathbf{k}),$$
$$\nabla\cdot\boldsymbol{\chi}_\lambda(\mathbf{x}|\mathbf{k}) = 0 \quad \textit{for } \lambda = \pm 1, \quad \text{(A.4)}$$
$$\nabla\cdot\boldsymbol{\chi}_0(\mathbf{x}|\mathbf{k}) = -ik(2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

and which satisfy the following orthogonality and completeness relations:

$$\int d\mathbf{x}\, \boldsymbol{\chi}_{\lambda}{}^{*}(\mathbf{x}|\mathbf{k}) \cdot \boldsymbol{\chi}_{\mu}(\mathbf{x}|\mathbf{k}') = \delta_{\lambda\mu}\, \delta(\mathbf{k} - \mathbf{k}') ,$$

$$\sum_{\lambda} \int d\mathbf{k}\, \boldsymbol{\chi}_{\lambda_i}{}^{*}(\mathbf{x}|\mathbf{k}) \cdot \boldsymbol{\chi}_{\lambda_j}(\mathbf{x}'|\mathbf{k}) = \delta_{ij}\, \delta(\mathbf{x} - \mathbf{x}') . \qquad (A.5)$$

The latter relations enabled him to show that any vector function **u** could be expanded as

$$\mathbf{u}(\mathbf{x}) = \sum_{\lambda} \int d\mathbf{k}\, \boldsymbol{\chi}_{\lambda}(\mathbf{x}|\mathbf{k})\, U_{\lambda}(\mathbf{k}) \qquad (A.6)$$

It is the latter relation that is central to the current report. Moses original paper contains several other significant results, including an extension to the Helmholtz vector function decomposition theorem.