CMS Technical Summary Report #93-12

APPROXIMATION ORDERS OF AND APPROXIMATION MAPS FROM LOCAL PRINCIPAL SHIFT-INVARIANT SPACES

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May 1993

(Received May 17, 1993)

Approved for public release
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Sponsored by

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
1800 G. Street
Washington, DC 20550

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Abstract: Approximation orders of shift-invariant subspaces of $L_p(\mathbb{R}^d)$, $2 \le p \le \infty$, generated by the shifts of one compactly supported function are considered. In that course, explicit approximation maps are constructed. The approach avoids quasi-interpolation and applies to stationary and non-stationary refinements. The general results are specialized to box spline spaces, to obtain new results on their approximation orders.

AMS (MOS) Subject Classifications: 41A25, 41A63; 41A30, 41A15, 42B99, 46E30

Key Words and phrases: approximation order, Strang-Fix conditions, shift-invariant spaces, box splines, exponential box splines, principal shift-invariant spaces.

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This research was partially supported by the United States Army (under Contract DAAL03-G-90-0090) and by the National Science Foundation (grant DMS-9102857).

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1. Introduction

Let S be a function space consists of complex (or real) valued functions defined on $\mathbb{R}^d$. We say that S is **shift-invariant** (SI, for short) if S is invariant under all integer translations (referred to hereafter as **shifts**), i.e.,

$$\forall \alpha \in \mathbb{Z}^d \quad (f \in S \iff f(\cdot - \alpha) \in S). \tag{1.1}$$

In this paper we consider SI spaces which are subspaces of

$$L_p := L_p(\mathbb{R}^d),$$

for some $2 \leq p \leq \infty$. The simplest type of shift-invariant spaces is the **PSI** space ("P" for "principal") which is the case when S is closed (usually in the underlying p-norm, but sometimes in a weaker topology) and the shifts of a single function ϕ (=the **generator**) are fundamental in S. Approximation from PSI and other shift-invariant spaces is pertinent to the theory and applications of several subareas of analysis, and in particular to *Multivariate Splines*, *Radial Basis Approximation*, *Wavelets* and *Sampling Theory*.

In many actual approximations, the SI space S is *refined* to yield another approximating space S_h with, presumably, better approximation properties. The standard (known as **stationary**) refinement is by scaling, that is, S_h is obtained by dilating the functions in S:

$$S_h = \sigma_h S := \{\sigma_h f := f(\cdot/h) : \ f \in S\}.$$

Sometimes (cf. [DR]) it is necessary to refine S by means other than dilation.

The basic way for measuring the approximation "power" of S is via the tool of *approximation orders*. Roughly speaking, the collection of spaces $\{S_h\}_{h>0}$ is said to **provide approximation order** $k > 0$, if, for all sufficiently smooth f,

$$\operatorname{dist}(f, S_h) = O(h^k).$$

Here, dist is measured by the relevant p-norm or one of its relatives (a Sobolev norm, a local p-norm, etc.). For some time, the analysis of approximation orders of PSI spaces was largely dominated by the *Strang-Fix conditions*, [SF]. These conditions assert that, if S is generated by a *compactly supported* ϕ, if $\widehat{\phi}(0) \neq 0$, and if the scale $\{S_h\}_h$ is *stationary*, then the approximation orders provided by $\{S_h\}_h$ are determined by the order of the zero $\widehat{\phi}$ has at each of $2\pi\mathbb{Z}^d\backslash 0$. The standard method for converting the information about these zeros into approximation order results is the **polynomial reproduction / quasi-interpolation argument**; (cf. the book [C], the survey [B], and the references therein). However, several important PSI spaces that were introduced and studied in recent years do not satisfy the requirements imposed above on the PSI space. One difficulty arises in the area of radial basis functions, since there the typical generator ϕ is not compactly supported. A totally different difficulty arises in the area of box splines: while the box spline is compactly supported, its corresponding $\{S_h\}_h$ is not a stationary one (unless the generator ϕ is a *polynomial* box spline). The attempts to cover those cases by generalized quasi-interpolation arguments led to some remarkable achievements, but did not solve the problem in its entirety. In retrospect, it seems that the quasi-interpolation approach fails to realize the approximation order of general PSI spaces, a fortiori of general SI spaces.

New approaches that circumvent quasi-interpolation were recently developed in [BR2] ($p = \infty$) and [BDR1] ($p = 2$). While the two methods differ in the approximation scheme they employ as well as in their error analysis, they both perform the entire error analysis on the Fourier domain, hence, in turn, avoid the imposition of decay rates on the generator ϕ. Also, importantly, both characterizations do not require $\{S_h\}_h$ to be stationary, and, furthermore, [BDR1] even dispense with the regularity assumption $\widehat{\phi}(0) \neq 0$. Extensive discussions of the various applications of the results of [BR2] and [BDR1] to radial function approximation can be found in §3 of [BR2] and [R3] respectively. Further discussion of the literature can be found in §3.2.

In the present paper, we revisit PSI spaces generated by compactly supported functions (such spaces were coined **local** in [BDR2]). Primarily, we aim at establishing results on the approximation order of *box spline* spaces, and providing explicit approximation schemes that realize that order. The main findings here with respect to box spline spaces are as follows:

(a): We determine exactly (in Theorem 3.4) the approximation order of box spline spaces in the L_2-norm. In no norm has such result been known before, although, for $p = \infty$, [BR2] came very close to the mark. We also show that the saturation class is trivial (cf. Theorem 3.7).

(b): We present (in §3.3) an explicit approximation scheme in the form

$$f \approx \sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha)(Jf)(\alpha),$$

for a suitably chosen convolution operator J. We prove that the scheme realizes the L_2-approximation order, and, as a matter of fact, maintain the same approximation order in L_p-norms, $2 < p < \infty$ (Theorem 3.13). The results also apply to the L_∞-case, if a mild smoothness condition is imposed on ϕ ($\phi \in A$, with A the Wiener algebra).

(c): We develop a new error analysis method which differs from the two of [BR2] and [BDR1]. With the aid of that approach, we show that for $2 \leq p \leq \infty$ (with some additional smoothness conditions required of the box spline ϕ, in the $p = \infty$ case) the approximation scheme used provides simultaneous approximation to functions and their derivatives (Theorem 3.17).

Most of the analysis that is developed to deal with box spline spaces was found to apply also to general local PSI spaces. Due to that reason, we first present, in §2, results that concern approximation orders of local PSI spaces, and only then apply these results, in §3, to the box spline case. Most of the proofs are collected in §4.

Soon after completing the present paper, I received a preprint of Kyriazis' paper [K]. The paper [K] deals with similar problems to those considered in the present paper, and to a limited extent employs similiar analysis (primarily, both papers invoke the approximation scheme of [BR2], and also both paper apply the Hausdorff-Young theorem). However, the foci of the two papers seem to be quite different, hence they seem to be complementary one to the other. [K] focuses on *stationary* refinements, and attempts to impose as mild as possible decay condition of the generator ϕ. Therefore, a substantial effort is devoted there to proving that various constructed approximants are indeed taken from the underlying PSI space. The paper [K] suceeds in providing satisfactory results also to L_p norms where $p < 2$, its class of smooth functions is less restrictive than here, and it established Strang-Fix-like conditions similar to those derived in [BDR1]. The present paper focuses on box spline space, hence avoids on the one hand questions of decay rates of the generator, but needs on the other hand to deal with non-stationary refinements. Also, we present here results on simultaneous approximation to functions and derivatives, a topic which was not considered in [K], and, finally, our results apply to $p = \infty$, as well ([K] relies on the Michlin multiplier theorem, hence deals with $p < \infty$).

Notations: we have already used the notation σ_h for the dilation operator

$$\sigma_h : f \mapsto f(\cdot/h).$$

Norms of vectors $x \in \mathbb{R}^d$ are denoted by $|x|_p$, namely,

$$|x|_p := (\sum_{j=1}^{d} |x_j|^p)^{1/p},$$

with the default notation $|x| := |x|_2$. The function

$$x \mapsto |x|,$$

which is used extensively in the paper, is denoted by (the essentially self-understood notation)

$$|\cdot|$$

For $\theta \in \mathbb{C}^d$, the notation e_θ stands for the exponential function

$$e_\theta : \omega \mapsto e^{i\theta\cdot\omega}.$$

Unless otherwise stated, all domains of functions in this paper are taken to be $\mathbb{R}^d$. Thus, $L_p = L_p(\mathbb{R}^d)$, $S' = S'(\mathbb{R}^d)$ (the space of all d-dimensional complex-valued tempered distributions), $W_p^k = W_p^k(\mathbb{R}^d)$ (the Sobolev space of all functions whose derivatives up to order k are in L_p), etc. We also abbreviate

$$\|f\|_p := \|f\|_{L_p}.$$

2. Approximation from local PSI spaces

Our model is as follows. We are given an indexed set $\Phi := \{\phi_h\}_h \subset L_p$. The *locality* assumption usually means that each ϕ_h is supported in some bounded, h-independent domain Ω, but, while such an assumption holds indeed in the box spline case, we do not need it here. We only assume that each ϕ_h is compactly supported. Regardless of the value of p, we define, for any compactly supported ϕ, the PSI space $S(\phi)$ to be the *infinite* span of the shifts of ϕ:

$$S(\phi) := \{ \sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha)c(\alpha)\}.$$

The convergence of the infinite sums can be taken pointwise, since the sum is actually finite on compact domains. No a-priori growth condition is imposed on the coefficients $\{c(\alpha)\}_{\alpha\in\mathbb{Z}^d}$. Although this definition slightly deviates from the one given in the introduction (our space is not a subspace of L_p, nor $S(\phi)\cap L_p$ need be closed), that difference would not matter in subsequent discussions.

The scale of spaces $\{S_h\}_h$ is obtained by dilating the PSI spaces $S(\phi_h)$:

$$S_h := \{\sigma_h f = f(\cdot/h) : \ f \in S(\phi_h)\},$$

and the **approximation orders provided by** $\{\phi_h\}_h$ is concerned with the rate of decay of

$$\text{(2.1)} \qquad \operatorname{dist}_p(f, S_h) := \inf\{\|f - s\|_p : \ s \in S_h\}$$

as $h \to 0$. More precisely, we say that $\{\phi_h\}_h$ provides approximation order k, if for every f in some smoothness space $V_{p,k}$ and small enough h,

$$\operatorname{dist}_p(f, S_h) \le \text{const}\, h^k \|f\|_{p,k},$$

with $\|f\|_{p,k}$ some norm of f. The scale $\{S_h\}_h$ is **stationary** if $\phi_h = \phi_{h'}$ for all h, h'. In such a case, $\{S_h\}_h$ are all dilates of one basic PSI space $S(\phi)$.

The space $V_{p,k}$ of "smooth enough" test functions is defined as follows. For two conjugate exponents $1 \le q \le p \le \infty$, and $k \ge 0$,

$$V_{p,k} := \{f : \|f\|_{p,k} := \|(1+|\cdot|)^k \widehat{f}\|_{L_q(\mathbb{R}^d)} < \infty\}. \tag{2.2}$$

Note that, for an integer k, the Hausdorff-Young Theorem implies that $V_{p,k}$ is continuously embedded into the Sobolev space W_p^k.

Given $f \in V_{p,k}$, we seek an approximant for f from S_h. Since S_h is the h-dilate of $S(\phi_h)$, we can define the approximant for f in terms of an element $A_h(f) \in S(\phi_h)$, i.e., approximate f by $\sigma_h(A_h(f))$. $A_h(f)$ is, necessarily, a (possibly infinite) linear combination of the shifts of ϕ_h. We obtain the coefficients in this combination as the restriction to $\mathbb{Z}^d$ of a continuous (in fact, entire) function $J_h(f)$. In summary, we approximate f by $\sigma_h A_h(f)$, where

$$A_h(f) = \sum_{\alpha \in \mathbb{Z}^d} \phi_h(\cdot - \alpha) J_h(f)(\alpha). \tag{2.3}$$

Thus, the particular details of our approximation scheme rely on the choice of the maps

$$f \mapsto J_h(f).$$

As it turns out, the results below on approximation orders require four conditions of the maps $\{J_h\}_h$ and any collection that satisfies these four properties will do here. Three of these conditions are independent of the specific approximation order we are after, and are listed now.

(2.4) Conditions required from the maps $\{J_h\}_h$:

(a) Each J_h is a dilation followed by convolution, that is

$$J_h(f) = (T_h \widehat{\sigma_{1/h} f})^\vee. \tag{2.5}$$

(b) Each T_h is a function supported in some h-independent origin-neighborhood $B \subset [-\pi \,..\, \pi]^d$.

(c) For some $h_0 > 0$, $\{T_h\}_{h<h_0}$ are uniformly bounded on B (hence on $\mathbb{R}^d$).

Note that we do not impose smoothness conditions on T_h, and therefore J_h need not to map L_p into itself. However, $\{\sigma_h J_h\}_h$ are uniformly bounded endomorphisms on each $V_{p,k}$.

Theorem 2.6. *Let $2 \le p \le \infty$ be given and let q be its conjugate exponent. For $k > 0$, let $V_{p,k}$ be as in (2.2). Let $\{\phi_h\}_h$ be a family of compactly supported functions, and B an origin-neighborhood. Assume that the collection of sequences*

$$m_{k,h} : (2\pi\mathbb{Z}^d \backslash 0) \ni \beta \mapsto \|(h+|\cdot|)^{-k} \widehat{\phi_h}(\cdot + \beta)\|_{L_\infty(B)}, \quad h < h_0,$$

lies in $\ell_q(2\pi\mathbb{Z}^d \backslash 0)$ and is bounded there. Suppose that $\{J_h\}_{h<h_0}$ satisfy conditions (a), (b) and (c) of (2.4) (with respect to the present B), and, in addition,

$$\sup_{h<h_0} \|(h+|\cdot|)^{-k}(1 - \widehat{\phi_h} T_h)\|_{L_\infty(B)} < \infty. \tag{d}$$

Let A_h be defined by (2.3). Then:

$$\|f - \sigma_h(A_h(f))\|_{L_p(\mathbb{R}^d)} \le \text{const}\, h^k \|f\|_{p,k}, \quad f \in V_{p,k}.$$

Proof. cf. §4.1. ♠

We remark that the proof of Theorem 2.6 provides the following bound on the error:

$$\|f - \sigma_h A_h(f)\|_{L_p(\mathbb{R}^d)} \leq$$

$$\text{(2.7)} \qquad \text{const}\, h^k \|f\|_{p,k} \left(\|T_h\|_{L_\infty(B)} \|m_{k,h}\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)} + \|(h + |\cdot|)^{-k}(1 - \widehat{\phi}_h T_h)\|_{L_\infty(B)} + o(1)\right),$$

with the $o(1)$ expression always bounded by 1, decays to 0 with h, and otherwise depends only on f, p, k and B, and with const depending only on p. Therefore, assuming that B is fixed, one might try to choose J_h such that the sum $\|T_h\|_{L_\infty(B)} \|m_{k,h}\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)} + \|(h + |\cdot|)^{-k}(1 - \widehat{\phi}_h T_h)\|_{L_\infty(B)}$ is minimized.

A natural choice for J_h is given by $T_h := \chi / \widehat{\phi}_h$, with χ a characteristic function of some 0-neighborhood B. In this case $(h + |\cdot|)^{-k}(1 - \widehat{\phi}_h T_h)$ vanishes on B, hence condition (d) in the theorem trivially holds. The only condition that needs to be checked then is condition (c), viz., the uniform boundedness of $\{T_h\}$, which amounts to the uniform boundedness away of zero of $1/\widehat{\phi}_h$. Hence we have:

Corollary 2.8. *Assume that the Fourier transforms of the family* $\Phi := \{\phi_h\}_{h<h_0}$ *of compactly supported functions are uniformly bounded away from 0 on some origin-neighborhood* B. *Then, for every* $2 \leq p \leq \infty$, *every* $k > 0$ *and* $h < h_0$,

$$\text{dist}_p(f, S_h) \leq \text{const}\, h^k \|f\|_{p,k} (\|m_{k,h}\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)} + o(1)). \quad \forall f \in W_{p,k},$$

with $\{m_{k,h}\}$ *defined as in Theorem 2.6, with* const *independent of* k *and* f, *the* $o(1)$ *expression bounded by* 1, *and with* q *the conjugate of* p. *Hence,* $\{\phi_h\}_h$ *provides approximation order no smaller than* k *whenever* $\{\|m_{k,h}\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)}\}_h$ *are uniformly bounded for sufficiently small* h.

In order for this corollary to be useful in the derivation of approximation orders we need to find conditions which guarantee the boundedness of the sequences $\{m_{k,h}\}_h$. In the next section, we will see how this is done in the case of box splines. At present, we note that the essential part in the boundedness assumption of $\{m_{k,h}\}_h$ is the pointwise boundedness, that is, for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ the function

$$h \mapsto \|(h + |\cdot|)^{-k} \widehat{\phi}_h(\cdot + \beta)\|_{L_\infty(B)}, \quad h < h_0$$

should be bounded, and the bound should be uniform in β. The fact that we assume more than that pointwise boundedness in the theorem, is due to the technical details of the proof, and, in most practical examples is translated to smoothness conditions on $\{\phi_h\}_h$. Note also that in the stationary case, ϕ_h does not change with h, and the pointwise boundedness condition thus can be easily seen to be equivalent to $\widehat{\phi}$ having a k-order zero at each of $\beta \in 2\pi\mathbb{Z}^d\backslash 0$.

Under additional smoothness conditions on $\{\phi_h\}_h$, the approximants $\{A_h(f)\}_h$ can be shown to approximate f in Sobolev norms as well. We mention that such results (concerning simultaneous approximation from SI spaces) are a rarity, especially since there is no standard way to derive them from quasi-interpolation arguments. The most notable exception is [SF] that states such results in the L_2- and L_∞-norm (for the stationary case), and proves the L_2-statement.

Theorem 2.9. *Adopting the notations and assumptions of Theorem 2.6, assume, in addition, that for some positive integer* $r < k$, *the sequences*

$$m_{k,h}^r : (2\pi\mathbb{Z}^d\backslash 0) \ni \beta \mapsto |\beta|^r \|(h + |\cdot|)^{-k} \widehat{\phi_h}(\cdot + \beta)\|_{L_\infty(B)}, \quad h < h_0$$

lie in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$ and are uniformly bounded there. Then,

$$\|f - \sigma_h(A_h(f))\|_{W_p^r} \le \text{const}_{p,r}\|f\|_{p,k}h^{k-r}.$$

Proof. cf. §4.1. ♠

3. Approximation from box spline spaces

3.1. Statment of the problem and its L_2-solution

To define a box spline, we let Ξ be a *rational* matrix of d rows which is also considered as the multiset of its columns $\{\xi\}_{\xi\in\Xi}$, with each column $\xi \in \Xi$ (referred to sometimes as "a direction") is assumed to be a non-zero vector. The matrix Ξ is augmented by a row vector $\lambda = \lambda_\Xi \in \mathbb{C}^\Xi$, and the resulted matrix, denoted by (Ξ, λ), is used to define the **box spline** $M := M_{\Xi,\lambda}$ whose Fourier transform is

$$\widehat{M}(\omega) = \prod_{\xi\in\Xi} \frac{e^{\lambda_\xi - i\xi\cdot\omega} - 1}{\lambda_\xi - i\xi\cdot\omega}, \quad \omega \in \mathbb{R}^d. \tag{3.1}$$

In general M is a compactly supported measure defined on $\mathbb{R}^d$, but upon assuming that

$$\operatorname{rank}\Xi = d \tag{3.2}$$

(as we always do hereafter), the box spline is a bounded compactly supported piecewise-exponential-polynomial function supported in the *zonotope*

$$Z_\Xi := \{\sum_{\xi\in\Xi} t_\xi \xi : \ t_\xi \in [0\,..\,1]\}. \tag{3.3}$$

Polynomial box splines correspond to the choice $\lambda = 0$. Exponential B-splines are obtained when $d = 1$ and $\xi = 1$, all ξ. Tensor splines are obtained whenever all the directions are standard unit vectors. The box spline is positive in the interior of Z_Ξ whenever λ is real-valued.

We now define the scale $\{S_h\}_h$ of box spline spaces. For this, we fix M (i.e., fix Ξ and λ), and define, for a given refinement parameter $h > 0$, the box spline M_h as

$$M_h := M_{\Xi,h\lambda}.$$

The rest of the definition is as in the introduction, i.e., $S_h := \sigma_h S(M_h)$. Our space of "test functions" remains the space $V_{p,k}$ defined in (2.2).

Since the ladder of spaces $\{S_h\}_h$ is determined as soon as M is chosen (in affect, as soon as Ξ and λ are chosen), we refer to the relevant approximation orders as *provided by* M, rather than "provided by $\{M_h\}$". Note that each S_h is "spanned" by the $h\mathbb{Z}^d$-shifts of the dilated function $\sigma_h M_h$. Furthermore, in case $\lambda = 0$, $M_h = M$, all h, and the scale $\{S_h\}_h$ becomes stationary. The motivation behind the particular definition of S_h in the non-stationary case, is that, while S_h becomes invariant under finer and finer shifts as $h \to 0$, the functions in S_h are always piecewise in some finite-dimensional h-independent space $\mathcal{H}$ (of exponential-polynomials).

We have seen in the last section that approximation orders from PSI spaces can be understood in terms of the behaviour of the various generators around $2\pi\mathbb{Z}^d\backslash 0$. In the box spline case, however, such results cannot be considered as satisfactory: the immediately available information on the box spline is the matrix (Ξ, λ), and therefore we wish to characterize the approximation order of box spline spaces in those terms: that is, given $k > 0$, we need to find all (Ξ, λ) whose corresponding box spline $M_{\Xi,\lambda}$ provides an approximation order k.

For the L_2-norm, we provide in this paper the following complete answer to the above problem.

Theorem 3.4. *The L_2-approximation order provided by the box spline $M_{\Xi,\lambda}$ is the number*

$$k'(\Xi) := \min\{\#K_\beta : \beta \in 2\pi\mathbb{Z}^d\backslash 0\}, \tag{3.5}$$

with

$$K_\beta := K_\beta(\Xi) := \{\xi \in \Xi : \xi \cdot \beta \in 2\pi\mathbb{Z}\backslash 0\}. \tag{3.6}$$

In particular, the approximation order is independent of λ.

Proof. cf. §4.3. ♠

Note that for $p = 2$ and integer k, $V_{2,k} = W_2^k$, and hence the above stated approximation orders apply to the entire Sobolev space.

We will also show that the saturation class associated with the above problem is trivial. Precisely, we have:

Theorem 3.7. *Let $\{S_h\}_h$ be the box spline space scale associated with a box spline $M_{\Xi,\lambda}$. Let $k := k'(\Xi)$, and let $f \in W_2^k\backslash\{0\}$. Then, for every sequence $h_j \underset{j\to\infty}{\longrightarrow} 0$,*

$$\mathrm{dist}_2(f, S_{h_j}) \neq o(h_j^k).$$

Proof. cf. §4.3. ♠

The definition of $k'(\Xi)$ is entirely in terms of the matrix Ξ (i.e., does not require any information on the underlying box spline $M_{\Xi,\lambda}$), and, moreover $k'(\Xi)$ can be computed by a finite algorithm.

In the important special case when Ξ is an *integer matrix* Theorem 3.4 implies the following (known) result:

Corollary 3.8. *Assume that Ξ is an integer matrix. Then the L_2-approximation order provided by the box spline $M_{\Xi,\lambda}$ is*

$$k(\Xi) := \min\{\#X : X \subset \Xi,\ \mathrm{rank}(\Xi\backslash X) < d\}. \tag{3.9}$$

Here and hereafter, $X \subset \Xi$ means that X is obtained from Ξ by the deletion of some columns, and $\#X$ is the number of columns in X.

Proof. In view of Theorem 3.4, it suffices to show that, for an integer matrix Ξ, $k'(\Xi) = k(\Xi)$. Let $\xi \in \Xi$. Since ξ is integer, $\xi \cdot \beta \in 2\pi\mathbb{Z}$ for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, hence K_β of (3.6) can equivalently be defined here as

$$K_\beta = \{\xi \in \Xi : \xi \cdot \beta \neq 0\}. \tag{3.10}$$

Thus, $(\Xi\backslash K_\beta)^T\beta = 0$ and hence $\mathrm{rank}(\Xi\backslash K_\beta) < d$. This show that $k(\Xi) \le \#K_\beta$, and consequently $k(\Xi) \le k'(\Xi)$. The reverse inequality does not require the integrality of Ξ, but only its rationality: assume that, for some $X \subset \Xi$, $\mathrm{rank}(\Xi\backslash X) < d$. Since Ξ is a rational matrix, $(\Xi\backslash X)$ is rank-deficient if and only if there exists a non-zero *integer* vector α perpendicular to all $\xi \in (\Xi\backslash X)$. In view of (3.6), we have $K_{2\pi\alpha} \subset X$, and hence $k'(\Xi) \le \#K_{2\pi\alpha} \le \#X$. It follows that $k'(\Xi) \le k(\Xi)$. ♠

3.2. Further literature discussion

Now that the two numbers $k(\Xi)$ and $k'(\Xi)$ are introduced and their connection to approximation orders is revealed, we are able to discuss the history of the problem in further detail. In this regard, it seems instructive to separate the discussion of the polynomial box spline case ($\lambda = 0$) from the general exponential case. As mentioned before, the problem of the former case is ***stationary***, i.e., the spaces $\{S_h\}$ are all obtained from the original space $S(M)$ by dilation, and there is a variety of papers (including, but not restricted to, [SF], [DM1,2], [BJ], and [JL]) which treat such setting for a general compactly supported ϕ, and links restrictive (hence stronger) notions of approximation order (known as "controlled" and "local") to the *polynomials* in $S(M)$. Further, more recently, it was shown in [R2] (L_∞-norm) and [BDR1] (L_2-norm) that whenever ϕ is compactly supported and $\widehat{\phi}(0) \neq 0$ (which is certainly the case for a polynomial box spline $\phi = M$) the polynomials in $S(M)$ characterize the *unconstrained* approximation order (i.e., the one defined and analyzed in the present paper). Thus, at least in essence, the characterization of the approximation order provided by a polynomial box spline amounts to the identification of the polynomials in $S(M)$. These polynomials were characterized by de Boor and DeVore in [BD] for the *three-directional* polynomial box spline ($d = 2$, $\lambda = 0$, $\xi^T \in \{(1,0),(0,1),(1,1)\}$, $\forall \xi \in \Xi$) which were also introduced there. Polynomial box splines associated with a general integer Ξ were introduced and studied by de Boor and Höllig in [BH], with the identification of the underlying polynomial space being among the highlights of that paper. The abstract argument provided in [BH] for the conversion of the knowledge on the polynomials into lower bounds on the approximation order has become a standard tool since then. Another proof of that result is included in the subsequent work of Dahmen and Micchelli, [DM1]. The characterization of the approximation order of a polynomial box spline associated with a general Ξ was only recently established in [RS], where, again, the main result is concerned with the identification of the polynomials in $S(M)$.

Exponential box splines were introduced in [R1], and that paper also contained the first result on their approximation order (showing that for $\lambda \subset \mathbb{R}$ and an integer Ξ, the approximation order in the L_∞-norm is at least 1). The first comprehensive discussion of approximation orders for exponential box splines is found in [DR], where, for general λ but integer Ξ, $k(\Xi)$ was proved to be a lower bound on the L_∞-approximation order. (It was further shown there that the exponential reproduction argument cannot provide better bounds). The extension of these results to $p < \infty$ was done by Lei and Jia in [LJ], where, in addition, the local structure of the spline space was used to provide matching upper bounds (Thus, Corollary 3.8 is [LJ]'s).

As already alluded to before, all the aforementioned results employed the quasi-interpolation argument: first, the space $\mathbf{H}(M)$ of all exponential-polynomials (polynomials, if $\lambda = 0$) in $\cap_h S_h$ is computed (either explicitly or as the kernel of explicit differential operators) and then the approximation power of $\mathbf{H}(M)$ around the origin is studied. This local approximation order of $\mathbf{H}(M)$ is then converted to lower bounds on the approximation orders via the quasi-interpolation argument (for this argument, in the exponential case, see [DR], [LJ], [R2], [BR1] and [CW]). However, in contrast with the stationary case, there is no general theory that can be applied to the exponential box spline to show that the lower bounds obtained by quasi-interpolation are the best approximation orders. Indeed, we draw (in §3.3) an example of a box spline M (necessarily with a non-integer Ξ and a non-zero λ) such that $k'(\Xi) = 1$ while the corresponding $\mathbf{H}(M)$ is trivial. We stress, however, that examples of this type are the exception rather than the rule.

The only reference that we are aware of which treats the approximation order provided by $M_{\Xi,\lambda}$ for a rational Ξ and general λ is [BR2], where approximation in ∞-norm is considered. The approach in that paper is based on the theory for L_∞-approximation orders developed there, a theory which indeed circumvents quasi-interpolation. Theorem 3.13 is similar to its L_∞-counterpart from [BR2], with one important difference: while the results of [BR2] require some minimal smoothness conditions of the basis function under consideration (here the box spline M), hence exclude box splines of low smoothness, no such exclusion exists in L_2-analogous results from [BDR1]. We will elaborate on this point in the next subsection, since Theorem 3.13 and its proof provide a better understanding of the nature of the smoothness restriction on M which was required in [BR2].

3.3. Approximation orders of box spline spaces in L_p, $p \geq 2$

In order to derive approximation orders for box spline spaces we invoke Theorems 2.6 and 2.9. For that task we need first to find, for the given box spline scale $\{\phi_h = M_h\}_h$, corresponding maps $\{J_h\}_h$ that satisfy the four requirements specified in Theorem 2.6. Upon completing that part, we will turn our attention to the main problem: identifying the largest integer k for which the uniform boundedness of the sequences $\{m_{k,h}\}_h$ is satisfied.

Lemma 3.11. *Let $\{M_h\}_h$ be any box spline scale, and let χ be the characteristic function of a 0-neighborhood B. Then, for sufficiently small B, the operators*

$$J_h : f \mapsto (\chi\widehat{\sigma_{1/h}f}/\widehat{M_h})^\vee$$

satisfy the four requirements of Theorem 2.6.

Proof. J_h certainly has the form required in (2.5), with $T_h = \chi/\widehat{M_h}$. Condition (b) there can be satisfied by ensuring $\operatorname{supp}\chi \subset [-\pi \,..\, \pi]^d$. Condition (d) (listed in Theorem 2.6) holds, since on $B := \operatorname{supp}\chi$, $1 - \widehat{M_h}T_h = 0$. It remains to deal with condition (c) of (2.4). For that, we first observe that since

$$\widehat{M_h}(\omega) = \prod_{\xi\in\Xi}\int_0^1 e^{(h\lambda_\xi - i\xi\cdot\omega)t}\,dt,$$

$\{\widehat{M_h}\}_h$ converges uniformly on $\operatorname{supp}\chi$ to

$$\prod_{\xi\in\Xi}\int_0^1 e^{(-i\xi\cdot\omega)t}\,dt,$$

and this latter expression is bounded away of zero on, say, $[-\pi..\pi]^d$. Thus, for sufficiently h_0, $\{T_h = \chi/\widehat{M_h}\}_{h<h_0}$ are uniformly bounded, as required. ♠

In order to deal with the essential requirement of Theorem 2.6, that is the uniform boundedness in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$ of the sequences $\{m_{k,h}\}_h$, we will prove the following:

Lemma 3.12. *Let $\{M_h\}_h$ be a box spline scale generated by $M = M_{\Xi,\lambda}$, and let M_0 be the corresponding polynomial box spline, i.e., $M_0 = M_{\Xi,0}$. Given an origin-neighborhood $B \subset [-\pi..\pi]^d$, define, as in Theorem 2.6,*

$$m_{k,h} : (2\pi\mathbb{Z}^d\backslash 0) \ni \beta \mapsto \|(h + |\cdot|)^k\widehat{M_h}(\cdot+\beta)\|_{L_\infty(B)}.$$

If, for $1 \leq q \leq \infty$, $\widehat{M_0} \in L_q(\mathbb{R}^d)$, then, for small enough h_0, the sequences $\{m_{k,h}\}_{h<h_0}$ are uniformly bounded in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$, for $k := k'(\Xi)$.

Proof. cf. §4.2. ♠

With the aid of the two last lemmas, we establish the following theorem:

Theorem 3.13. *Let $\{M_h\}_h$ be the box spline scale associated with the box spline $M := M_{\Xi,\lambda}$. Consider the approximation maps*

$$f \approx \sigma_h A_h(f),$$

with

$$A_h(f) = \sum_{\alpha\in\mathbb{Z}^d} M_h(\cdot - \alpha) J_h(f)(\alpha),$$

where

$$J_h(f) := (\chi \widehat{f}/\widehat{M_h})^\vee,$$

and χ is the support function of some origin-neighborhood $B \subset [-\pi \, .. \, \pi]^d$. Then, for $k = k'(\Xi)$, and $2 \le p < \infty$,

$$\|f - \sigma_h A_h(f)\|_p \le \mathrm{const}_p \|f\|_{p,k} h^k,$$

for every $f \in V_{p,k}$ and every small enough h. The result is valid for $p = \infty$, as well, provided that $\widehat{M_0} \in L_1$, with $M_0 = M_{\Xi,0}$ the associated polynomial box spline.

Proof. The claim of the present theorem with respect to $p = \infty$ follows by an application of Lemma 3.12 and Lemma 3.11 to Theorem 2.6. The same is true also for $2 \le p < \infty$, as soon as we show that for such p, $\widehat{M_0}$ necessarily lies in L_q. Since we assume that the rank condition

$$\operatorname{rank}\Xi = d$$

holds, we can find a $d \times d$ invertible submatrix $X \subset \Xi$. Then

$$M_0 = M_1 * M_2,$$

with $M_1 := M_{X,0}$ and $M_2 = M_{(\Xi\backslash X),0}$. The trivial bound

$$\left|\int_0^1 e^{-i\xi\cdot\omega t}\, dt\right| \le 1, \quad \xi,\omega \in \mathbb{R}^d,$$

proves that $\widehat{\phi} \in L_\infty$ for any polynomial box spline ϕ, and therefore $\widehat{M_2} \in L_\infty$. Consequently, it suffices to prove that $\widehat{M_1} \in L_q$. By applying a linear change of variables, we may assume without loss that X is the identity matrix, and thus $\widehat{M_1}$ becomes the tensor product of the univariate function

$$\omega \mapsto \int_0^1 e^{-i\omega t}\, dt = \frac{1 - e^{-i\omega}}{i\omega},$$

which lies in $L_q(\mathbb{R})$ for $q > 1$. It follows that $\widehat{M_1}$, hence also $\widehat{M_0}$ are in $L_q(\mathbb{R}^d)$. ♠

As mentioned in the introduction, results similar to Theorem 3.4 were derived in [BR2], but with respect to ∞-norm. It is shown there that the L_∞-approximation order provided by M is always bounded above by $k'(\Xi)$, and this bound is proved there to be the exact approximation order under the additional assumption

$$\sum_{\beta\in\mathbb{Z}^d\backslash 0}\ \prod_{\xi\in\Xi,\xi\cdot\beta\ne 0} |\xi\cdot\beta|^{-1} < \infty. \tag{3.14}$$

In comparison, Theorem 3.13 requires the more verifiable condition $\widehat{M_0} \in L_1$ (which is shown to imply (3.14)). For example, the latter condition is satisfied whenever Ξ can be partitioned into two matrices $\Xi = X \cup Y$ both of rank d. Indeed, the full rank assumption on X and Y implies that $\widehat{M_{X,0}}, \widehat{M_{Y,0}} \in L_2$, and hence their convolution product $M_{\Xi,0}$ is in the algebra $A := \{\widehat{f} : \ f \in L_1\}$.

Example 3.15. Let $d = 2$. Then one easily checks that Ξ can always be partitioned into two matrices of rank 2 unless $\Xi = X \cup \{\xi\}$ with X is a rank-1 matrix, or, in terms of $k(\Xi)$ (cf. (3.9)) if and only if $k(\Xi) = 1$. As observed in the proof of Corollary 3.8, $k'(\Xi) \leq k(\Xi)$, hence in such a case $k'(\Xi) \in \{0, 1\}$. Now, *if $k'(\Xi) = 0$, then, since the L_∞-approximation order is proved in [BR2] to* be bounded above by $k'(\Xi)$, we conclude that M provides approximation order 0. Therefore, the only bivariate box splines whose ∞-approximation order cannot be decided directly by the results here are those associated with a matrix Ξ that satisfies $k(\Xi) = k'(\Xi) = 1$.

Next, we want to show, with the aid of an example, that the approximation order $k'(\Xi)$ can exceed in times the local approximation order of the space $\mathbf{H}(M)$ (cf. third paragraph of §3.2):

Example 3.16. Let $d = 2$,

$$\Xi = \begin{pmatrix} 1/2 & 0 & 1/2 & 1 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 1/2 & 1 & -1/2 \end{pmatrix},$$

and λ yet to be determined. It is easily verified that $k'(\Xi) = 1 < k(\Xi) = 5$, and hence, by Theorem 3.1 and in view of the previous example, the L_p-approximation order provided by $M_{\Xi,\lambda}$ is 1, for every $2 \leq p \leq \infty$ (and regardless of the choice of λ). The choice $\lambda = 0$ leads to a stationary situation, and the approximation order 1 must then imply that the shifts of $M_{\Xi,0}$ partition the unity, as one can verify, with the aid of Poisson's summation formula, from the fact that

$$\widehat{M}_{\Xi,0}(2\pi\beta) = \delta_{\beta,0}, \quad \beta \in \mathbb{Z}^d.$$

On the other hand, for a generic choice of λ, $\mathbf{H}(M) = \{0\}$. *This can be proved as follows:* if $\mathbf{H}(M)$ is non-trivial, then by Lemma 3.1 of [BAR], it contains an exponential $x \mapsto e^{i\theta \cdot x}$. The frequency $i\theta$ of that exponential must satisfy the following two conditions:

(a) $\lambda_X + iX^T\theta = 0$, for some 2×2 $X \subset \Xi$ of rank 2 ; and

(b) $\widehat{M}_{\Xi,\lambda}(\theta + 2\pi\beta) = 0$, $\beta \in \mathbb{Z}^2\backslash 0$. (cf. [R1;§4] for (a) and [BR1;§2] for (b).)

To see that the above (a) and (b) can hold only in exceptional circumstances, we proceed as follows: we fix the above Ξ, X and λ_X. This determines a unique θ (see (a) above). We now try to define λ_ξ, $\xi \in \Xi$, so that (b) above is valid. For that, one verifies first (directly) that, regardless of the specific choice of the 2×2 $X \subset \Xi$, $\widehat{M_{X,\lambda_X}}$ can vanish only on a subset of some proper sublattice L_X of $2\pi\mathbb{Z}^2$. Selecting $\beta \in 2\pi\mathbb{Z}^2\backslash\{0\}$ in the complement of this sublattice, (b) above implies the existence of $\xi \in \Xi\backslash X$ such that $\int_0^1 e^{(\lambda_\xi - i\xi\cdot(\theta+\beta))t}\,dt = 0$, or, equivalently,

$$\lambda_\xi - i\xi \cdot (\theta + \beta) \in 2\pi\mathbb{Z}.$$

This shows that one of λ_ξ, $\xi \in \Xi$ must be chosen from the countable set $-i\xi \cdot (\theta + 2\pi\mathbb{Z}^2)$, and thererfore, enerically, $\mathbf{H}(M) = \{0\}$, as claimed. ♠

Finally, we state our result concerning simultaneous approximation.

Theorem 3.17. *In the notations of Theorem 3.13, and for $2 \leq p < \infty$,*

$$\|f - \sigma_h A_h(f)\|_{W_p^r} \leq \text{const}_{p,r}\, h^{k-r} \|f\|_{p,k},$$

for every $f \in V_{p,k}$, every small enough h, and every integer $r < k$. The same holds for $p = \infty$, provided that M_0 and all its derivatives up to order r inclusive lie in the Wiener algebra A (or, equivalently, $|\cdot|^r \widehat{M}_0 \in L_1$.)

Proof. cf. §4.4. ♠

4. Proofs

4.1. Proofs of Theorems 2.6 and 2.9

The approximation maps $\{A_h\}_h$ that we employ are intimately related to those used in [BR2]. In fact, the latter, albeit a special case of the present schemes, seem to be their most natural choice. In contrast, the error analysis of [BR2] cannot be adopted here: that analysis makes use of the optimal approximation to the exponential functions

$$e_\theta : x \mapsto e^{i\theta \cdot x}, \quad \theta \in \mathbb{R}^d$$

of [R2], and synthesize those optimal approximations on the Fourier domain to yield optimal approximation to other smooth functions. However, when $p < \infty$ the above exponentials are not in L_p any more, hence the [BR2] approach cannot go through. Instead, we use here the following identity

$$\sum_{\alpha \in \mathbb{Z}^d} \psi(\cdot - \alpha) g(\alpha) = \sum_{\beta \in 2\pi\mathbb{Z}^d} \psi * (e_\beta g), \tag{4.1}$$

which is valid for any compactly supported distribution ψ provided that g is sufficiently smooth, say, $g \in C^\infty(\mathbb{R}^d)$ (cf. Theorem 2.6 of [RS]). The convergence of the right hand side of (4.1) is valid in the topology of tempered distributions (and in most circumstances, in much stronger topologies), provided that g and all its derivatives grow no faster than polynomially at ∞.

Proof of Theorem 2.6. Fix $2 \le p \le \infty$, and let $f \in V_{p,k}$. We try to estimate the error

$$\|f - \sigma_h(A_h(f))\|_p = h^{d/p} \|\sigma_{1/h} f - A_h(f)\|_p, \tag{4.2}$$

with the right hand side in (4.2) obtained from the left hand side by scaling. Invoking (4.1) with respect to $A_h(f)$ (i.e. with $g := J_h(f)$ and $\psi := \phi_h$), we obtain that

$$\begin{aligned} \|\sigma_{1/h} f - A_h(f)\|_p &= \|\sigma_{1/h} f - \phi_h * J_h(f) - \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \phi_h * (e_\beta J_h(f))\|_p \\ &\le \|\sigma_{1/h} f - \phi_h * J_h(f)\|_p + \|\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \phi_h * (e_\beta J_h(f))\|_p. \end{aligned} \tag{4.3}$$

We estimate each of the two terms in the last line of (4.3) with the aid of the Hausdorff-Young inequality:

$$\|f\|_p \le \mathrm{const}_p \|\widehat{f}\|_q, \quad 1/p + 1/q = 1, \tag{4.4}$$

valid for $2 \le p \le \infty$, provided that $\widehat{f} \in L_q(\mathbb{R}^d)$.

We first estimate in the proposition below the second term in the second line of (4.3). For later use, we derive that estimate in a slightly more general setup than needed here.

Proposition 4.5. *Let w be some sequence defined on $2\pi\mathbb{Z}^d \backslash 0$ and having (at most) polynomial growth. Then, in the notations of Theorem 2.6, and under the assumptions there,*

$$h^{d/p} \|\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} w(\beta) \phi_h * (e_\beta J_h(f))\|_p \le \mathrm{const}\, h^k \|f\|_{p,k} \|m_{k,h} w\|_{\ell_q(2\pi\mathbb{Z}^d \backslash 0)}.$$

Proof. There is nothing to prove in case $\|m_{k,h}w\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)} = \infty$. Otherwise, by (4.4) it suffices to bound

$$h^{d/p}\|(\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w(\beta)\phi_h * (e_\beta J_h(f)))^\wedge\|_q = h^{d/p}\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w(-\beta)\widehat{\phi}_h\widehat{J_h(f)}(\cdot+\beta)\|_q. \tag{4.6}$$

(The justification for the term-by-term application of the Fourier transform is given in the sequel). Substituting $h^{-d}\widehat{f}(\cdot/h)T_h$ for $\widehat{J_h(f)}$ (cf. (2.5)) we obtain from (4.6) the equivalent expression

$$h^{-d/q}\|\widehat{\phi}_h\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w(-\beta)\widehat{f}((\cdot+\beta)/h)T_h(\cdot+\beta)\|_q. \tag{4.7}$$

Here, the infinite sum in the above expression trivially converges (since $\operatorname{supp} T_h\subset B\subset[-\pi\,..\,\pi]^d$), and the limit is supported in $(2\pi\mathbb{Z}^d\backslash 0)+B$. Also, since the weights $\{w(\beta)\}$ are of polynomial growth, the convergence holds in S'. In retrospect, this justifies the term-by-term application of the Fourier transform in (4.6), as well as the changing of the order of summation and multiplication by ϕ_h is the display afterwards.

We fix $\alpha\in 2\pi\mathbb{Z}^d\backslash 0$, and compute that

$$\begin{aligned}
&\|\widehat{\phi}_h\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w(-\beta)\widehat{f}((\cdot+\beta)/h)T_h(\cdot+\beta)\|^q_{L_q(\alpha+B)}\\
&=\|w(\alpha)\widehat{\phi}_h\widehat{f}((\cdot-\alpha)/h)T_h(\cdot-\alpha)\|^q_{L_q(\alpha+B)}\\
&\le w(\alpha)^q\|T_h\|^q_{L_\infty(\mathbb{R}^d)}\,\|(h+|\cdot|)^k\widehat{f}(\cdot/h)\|^q_{L_q(B)}\,\|(h+|\cdot|)^{-k}\widehat{\phi}_h(\cdot+\alpha)\|^q_{L_\infty(B)}\\
&=h^{d+kq}w(\alpha)^q\,\|T_h\|^q_{L_\infty(\mathbb{R}^d)}\,\|(1+|\cdot|)^k\widehat{f}\|^q_{L_q(B/h)}\,\|(h+|\cdot|)^{-k}\widehat{\phi}_h(\cdot+\alpha)\|^q_{L_\infty(B)}.
\end{aligned}$$

Summing over all $\alpha\in 2\pi\mathbb{Z}^d\backslash 0$ and using the uniform boundedness of $\{T_h\}$ we obtain the bound

$$h^{-d}\|\widehat{\phi}_h\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w(-\beta)\widehat{f}((\cdot+\beta)/h)T_h(\cdot+\beta)\|^q_{L_q(\mathbb{R}^d)}\le \text{const}\, h^{kq}\|f\|^q_{p,k}\|m_{k,h}w\|^q_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)},$$

and the required result follows. ♠

In view of (4.2), (4.3) and the claim of the last proposition for the choice $w=1$, the proof of Theorem 2.6 is reduced to the study of the first term on second line of (4.3). Here we have, for some positive const,

$$\text{const}\|\sigma_{1/h}f-\phi_h * J_h(f)\|_p\le$$

$$\|\widehat{\sigma_{1/h}f}-\widehat{\phi}_h\widehat{J_h(f)}\|_q\le\|\widehat{\sigma_{1/h}f}(1-\widehat{\phi}_hT_h)\|_{L_q(B)}+\|\widehat{\sigma_{1/h}f}\|_{L_q(\mathbb{R}^d\backslash B)}, \tag{4.8}$$

with the first inequality by (4.4) and the second equality because $\widehat{J_h(f)}=T_h\widehat{\sigma_{1/h}f}$ and T_h is supported in B. Changing variables, the term $\|\widehat{\sigma_{1/h}f}\|_{L_q(\mathbb{R}^d\backslash B)}$ can be bounded as follows:

$$\begin{aligned}
\|\widehat{\sigma_{1/h}f}\|_{L_q(\mathbb{R}^d\backslash B)} &= h^{-d}\|\sigma_h\widehat{f}\|_{L_q(\mathbb{R}^d\backslash B)}\\
&= h^{-d+d/q}\|\widehat{f}\|_{L_q(\mathbb{R}^d\backslash(B/h))}\\
&\le h^{-d/p}(1+c/h)^{-k}\|(1+|\cdot|)^k\widehat{f}\|_{L_q(\mathbb{R}^d\backslash(B/h))}=h^{-d/p}h^k\|f\|_{p,k}\,o(1),
\end{aligned}$$

with the $o(1)$ expression uniformly bounded in h and $f \in V_{p,k}$. As for the other term in (4.8),

$$\|\widehat{\sigma_{1/h} f}(1 - \widehat{\phi}_h T_h)\|_{L_q(B)} \le \|(h + |\cdot|)^k \widehat{\sigma_{1/h} f}\|_{L_q(B)} \|(h + |\cdot|)^{-k}(1 - \widehat{\phi}_h T_h)\|_{L_\infty(B)}.$$

By assumption (d), $\|(h + |\cdot|)^{-k}(1 - \widehat{\phi}_h T_h)\|_{L_\infty(B)}$ is bounded independently of h, while a change of variables yields

$$\|(h + |\cdot|)^k \widehat{\sigma_{1/h} f}\|_{L_q(B)} = h^{-d}\|(h + |\cdot|)^k \sigma_h \widehat{f}\|_{L_q(B)} = h^{k-d/p}\|(1 + |\cdot|)^k \widehat{f}\|_{L_q(B/h)} \le h^{k-d/p}\|f\|_{p,k}.$$

Thus, we have shown that each of the two terms in the last line of (4.3) is of order $O(h^{k-d/p})$, and the claim of Theorem 2.6 then follows. ♠

Proof of Theorem 2.9. The proof of Theorem 2.9 closely follows that of its special case, Theorem 2.6. We thus only outline the proof, emphasizing parts of the proof that deviate from their counterparts in Theorem 2.6.

We let $P \in \Pi_r$ be a homogeneous polynomial, and $P(D)$ being its associated constant-coefficient differential operator. We want to establish the bound

$$\|P(D)(f - \sigma_h A_h(f))\|_p \le \text{const}\, h^{k-\deg P}\|f\|_{p,k}\|m^r_{k,h}\|. \tag{4.9}$$

Such an estimate leads to the desired result, since we may range P over some fixed homogeneous basis for Π_r. Here, const should be independent of h and f, but may depend on P, p, and r.

For the proof of (4.9), we consider

$$P(D)(f - \sigma_h A_h(f)) = P(D)f - h^{-\deg P}\sigma_h P(D) A_h(f).$$

By (4.1) (with $\psi := \phi_h$ and $g := J_h(f)$),

$$P(D)A_h(f) = P(D)\sum_{\beta \in 2\pi \mathbb{Z}^d} \phi_h * (e_\beta J_h(f)) = \sum_{\beta \in 2\pi \mathbb{Z}^d} \phi_h * (P(D)(e_\beta J_h(f))), \tag{4.10}$$

with the changing of order of summation and differentiation justified by the S'-convergence of the sum. The term corresponding to $\beta = 0$ is $\phi_h * (P(D)J_h(f)) = h^{\deg P}\phi_h * J_h(P(D)f)$, and we first estimate

$$P(D)f - \sigma_h(\phi_h * J_h(P(D)f)).$$

Since $f \in V_{p,k}$, $P(D)f \in V_{p,k-\deg P}$, and hence the proof of Theorem 2.6 yields that for small enough h we have

$$\|P(D)f - \sigma_h(\phi_h * J_h(P(D)f))\|_p \le \text{const} h^{k-\deg P}\|P(D)f\|_{p,k-\deg P}.$$

It remains to bound the expression

$$\sum_{\beta \in 2\pi \mathbb{Z}^d \backslash 0} \phi_h * P(D)(e_\beta J_h(f)).$$

Here, we fix $\beta \ne 0$, and expand

$$P(D)(e_\beta J_h(f)) = \sum_{\alpha \ge 0} \frac{1}{\alpha!}(D^\alpha P)(D)(e_\beta)\, D^\alpha J_h(f) = e_\beta \sum_{\alpha \ge 0} \frac{1}{\alpha!}(D^\alpha P)(i\beta)\, D^\alpha J_h(f),$$

and use summation by parts (allowed, since the range of α is actually finite, and since, for any fixed α, the Fourier transform of the various β-summands have pairwise disjoint supports) followed by the triangle inequality to estimate the norm of that part as follows:

$$\begin{aligned}
&\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \phi_h * P(D)(e_\beta J_h(f))\|_p \\
&\leq \sum_{\alpha\geq 0} \frac{1}{\alpha!}\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} (D^\alpha P)(i\beta)\,\phi_h * (e_\beta(D^\alpha J_h(f)))\|_p \\
&= \sum_{\alpha\geq 0} \frac{h^{|\alpha|_1}}{\alpha!}\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} (D^\alpha P)(i\beta)\,\phi_h * (e_\beta J_h(D^\alpha f))\|_p
\end{aligned} \tag{4.11}$$

Invoking Proposition 4.5 with $w(\beta) := w_\alpha(\beta) := (D^\alpha P)(i\beta)$ (and with f, k replaced by $D^\alpha f$ and $k - |\alpha|_1$ respectively), we obtain that

$$\begin{aligned}
&h^{|\alpha|_1}\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} (D^\alpha P)(i\beta)\,\phi_h * (e_\beta J_h(D^\alpha f))\|_p = \\
&h^{|\alpha|_1}\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} w_\alpha(\beta)\phi_h * (e_\beta J_h(D^\alpha f))\|_p \leq \text{const} h^k \|D^\alpha f\|_{p,k-|\alpha|_1}\|m_{k,h} w_\alpha\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)}.
\end{aligned} \tag{4.12}$$

Since (i): the actual range of α in (4.11) is finite ($|\alpha|_1 \leq \deg P$), (ii): $\|D^\alpha f\|_{p,k-|\alpha|_1} \leq \|f\|_{p,k}$, and (iii): $|(D^\alpha P)(i\beta)| \leq \text{const}|\beta|^r$, we derive from (4.11) and (4.12) the inequality

$$\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \phi_h * P(D)(e_\beta J_h(f))\|_p \leq \text{const} h^k \|f\|_{p,k}\|m^r_{k,h}\|_{\ell_q(2\pi\mathbb{Z}^d\backslash 0)},$$

from which (4.9) follows. ♠

4.2. Proof of Lemma 3.12

Given $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, our first goal is to estimate $|\widehat{M_h}(\omega+\beta)|$, for small ω. Initially, this is done *without* using the assumption $\widehat{M_0} \in L_q$. We consider, one by one, the factors

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\omega+\beta))t}\,dt| \tag{4.13}$$

that form $|\widehat{M_h}|$. For this, fixing $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, we partition Ξ according to the behaviour of their corresponding factors into three groups: the first of which is K_β (cf. (3.6)) and the other two are defined as follows

$$L_\beta := \{\xi \in \Xi :\ \xi\cdot\beta \notin \mathbb{Z}\},$$
$$O_\beta := \{\xi \in \Xi :\ \xi\cdot\beta = 0\}.$$

Case I: $\xi \in O_\beta$. In this case, for sufficiently small h, we have

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\omega+\beta))t}\,dt| < 2.$$

Case II: $\xi \in L_\beta$. Here we use the estimate

$$(4.14) \qquad \left|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\omega+\beta))t}\,dt\right| < \frac{3}{|h\lambda_\xi - i\xi\cdot(\omega+\beta)|},$$

valid for all ω and sufficiently small h. The rationality of Ξ implies the existence of $n \in \mathbb{Z}$ such that $n\Xi$ is integral, and thus we have $|\xi\cdot\beta| \geq 2\pi/n$, and (4.14) shows that for sufficiently small h and ω

$$\left|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\omega+\beta))t}\,dt\right| < \frac{c}{|\xi\cdot\beta|},$$

with c depending only on Ξ.

Case III: $\xi \in K_\beta$. In this final case, we write

$$(4.15) \qquad \left|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\omega+\beta))t}\,dt\right| = \frac{|e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1|}{|h\lambda_\xi - i\xi\cdot(\omega+\beta)|} = \frac{|e^{h\lambda_\xi - i\xi\cdot\omega} - 1|}{|h\lambda_\xi - i\xi\cdot(\omega+\beta)|}.$$

The denominator in the right hand side of (4.15) can be estimated as in the previous case, while the numerator, for sufficiently small h and $|\omega|$, can be bounded by $c(h+|\omega|)$, hence we obtain in this case the estimate

$$\left|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot\omega)t}\,dt\right| \leq c'\,\frac{(h+|\omega|)}{|\xi\cdot\beta|}.$$

Combining these various estimates we obtain that, for ω in some h-independent neighborhood of the origin, for h sufficiently small and for some (h,ω,β)-independent const,

$$(4.16) \quad |\widehat{M_h}(\omega+\beta)| \leq \mathrm{const}(h+|\omega|)^{\#K_\beta} \prod_{\xi\in K_\beta\cup L_\beta} |\xi\cdot\beta|^{-1} \leq \mathrm{const}(h+|\omega|)^{k'(\Xi)} \prod_{\xi\in K_\beta\cup L_\beta} |\xi\cdot\beta|^{-1}.$$

the second inequality since $k'(\Xi) \leq \#K_\beta$ for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$.

For later use, we record this intermediate estimate:

Proposition 4.17. *Let $\{M_h\}_h$ be a box spline scale associated with Ξ and λ. Then, for $k := k'(\Xi)$, for some origin-neighborhood B, for sufficiently small h, and for some (h,β)-independent const we have*

$$m_{k,h}(\beta) := \|(h+|\cdot|)^{-k}\widehat{M_h}(\cdot+\beta)\|_{L_\infty(B)} \leq \mathrm{const} \prod_{\xi\in K_\beta\cup L_\beta} |\xi\cdot\beta|^{-1}.$$

To complete the proof of Lemma 3.12, we need to show that the sequence

$$c_\beta = \prod_{\xi\in K_\beta\cup L_\beta} |\xi\cdot\beta|^{-1}, \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0$$

is in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$, whenever $\widehat{M_0} \in L_q$. A slightly more general assertion is proved in the following proposition.

Proposition 4.18. *Assume that, for the polynomial box spline M_0 and for some $r \geq 0$, $|\cdot|^r\widehat{M_0} \in L_q$. Then*

$$(4.19) \qquad \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\beta|^{rq} \prod_{\xi\in K_\beta\cup L_\beta} |\xi\cdot\beta|^{-q} < \infty.$$

Proof of the Proposition. Choose $\omega \in \mathbb{R}^d$ that satisfies the following conditions:
(a) For every $\xi \in \Xi$, $\xi \cdot \omega$ is not 2π-rational.
(b) The series

$$\sum_{\beta \in 2\pi \mathbb{Z}^d \backslash 0} |\beta|^{rq} |\widehat{M_0}(\omega + \beta)|^q \tag{4.20}$$

converges.
(c)

$$|\xi \cdot (\omega + \beta)| \leq 2|\xi \cdot \beta|, \quad \xi \in \Xi, \ \beta \in 2\pi\mathbb{Z}^d, \ \beta \cdot \xi \neq 0.$$

It is clear that the set of points $\omega \in \mathbb{R}^d$ that violate (a) is of measure 0. Also, because $|\cdot|^r \widehat{M_0} \in L_q(\mathbb{R}^d)$, the series

$$\sum_{\beta \in 2\pi \mathbb{Z}^d \backslash 0} |\omega + \beta|^{rq} |\widehat{M_0}(\omega + \beta)|^q$$

converges a.e. on $[-\pi \,.\,.\, \pi]^d$, implying thereby the a.e. convergence of (4.20). On the other hand, because $\xi \in \Xi$ is rational, $\inf\{|\xi \cdot \beta| : \ \beta \in 2\pi\mathbb{Z}^d\backslash 0, \ \xi \cdot \beta \neq 0\} > 0$, and hence condition (c) is satisfied by all small enough ω. This proves that there is ω that satisfies all the above conditions.

The rationality of Ξ implies that for any fixed $\xi \in \Xi$ the range of the map

$$(2\pi\mathbb{Z}^d \backslash 0) \ni \beta \mapsto e^{-i\xi \cdot (\omega + \beta)} - 1$$

is finite, and, because of condition (a), this range does not contain 0. Condition (a) also implies that $\int_0^1 e^{-i\xi \cdot \omega t}\, dt \neq 0$, for all $\xi \in \Xi$. Thus, we obtain from condition (c) the estimate

$$\begin{aligned} |\widehat{M_0}(\omega + \beta)| &= \prod_{\xi \in O_\beta} \Big| \int_0^1 e^{-i\xi \cdot \omega t}\, dt \Big| \prod_{\xi \in K_\beta \cup L_\beta} \Big| \frac{e^{-i\xi \cdot (\omega + \beta)} - 1}{\xi \cdot (\omega + \beta)} \Big| \\ &\geq \text{const} \prod_{\xi \in K_\beta \cup L_\beta} |\xi \cdot \beta|^{-1}. \end{aligned} \tag{4.21}$$

Condition (b) then implies the desired result. ♠

4.3. Proofs of Theorems 3.4 and 3.7

The *positive* statement in Theorem 3.4, i.e., that the approximation order provided by $M_{\Xi,\lambda}$ (to functions in $V_{2,k'(\Xi)} = W_2^{k'(\Xi)}$) is *at least* $k'(\Xi)$, follows from Theorem 2.6 when combined with Lemma 3.11 and Lemma 3.12. Indeed, we only need to verify that the requirement $\widehat{M_0} \in L_2$ (needed for the application of Lemma 3.12) holds. That was proved in Theorem 3.13, but, as matter of fact, also follows directly from the fact that $M_0 \in L_2$.

The *negative* statement of Theorem 3.4 will follow from Theorem 3.7. Indeed, Theorem 3.7 provides $k'(\Xi)$ as an upper bound on the approximation order (and in the strongest possible sense).

Therefore, only Theorem 3.7 requires a proof.

For that proof, we need to borrow some of the general tools and results developed in [BDR1]. We remark that the definition of the PSI space $S(\phi)$ in [BDR1] differs from the one given here: it is defined there as the L_2-closure of the algebraic span of the shifts of ϕ. However, as Theorem 2.13 of [BDR1] asserts, $S(\phi) \cap L_2$ of the present paper is dense in $S(\phi)$ of [BDR1], hence the two spaces share the same approximation orders, and to the same functions.

We first define for every $h \geq 0$

$$N_h := M_h * \overline{M_h(-\cdot)}.$$

Note that

$$\widehat{N_h} = |\widehat{M_h}|^2.$$

Because of the rank assumption (3.2), $M_h \in L_2$, and therefore N_h (which is clearly compactly supported) is continuous, as any convolution product of two L_2-functions is. We make a substantial use of the **symbol** $\widetilde{N_h}$ of N_h defined as

$$\widetilde{N_h} := \sum_{\alpha \in \mathbb{Z}^d} N_h(\alpha) e_{-\alpha}.$$

A standard application of Poisson's summation formula shows that

$$\widetilde{N_h} = \sum_{\beta \in 2\pi\mathbb{Z}^d} \widehat{N_h}(\cdot + \beta) = \sum_{\beta \in 2\pi\mathbb{Z}^d} |\widehat{M_h}(\cdot + \beta)|^2.$$

Thus, we see that $\widetilde{N_h}$ is a non-negative trigonometric polynomial, and, further, $\widetilde{N_h}(y) = 0$ only if $\widehat{N_h}(y) = 0$.

The L_2-approximation orders provided by M are determined [BDR1], by the behaviour around the origin of the functions

(4.22) $$\Lambda_h := (1 - \widehat{N_h}/\widetilde{N_h})^{1/2}, \quad h > 0$$

(here, 0/0 is defined as zero, but in any case, $\widetilde{N_h}$, as a trigonometric polynomial, vanishes only on a null-set). Note that Λ_h is non-negative and bounded by 1. The precise result that we need here follows from Theorem 2.20 and Corollary 3.10 of [BDR1], and reads as follows:

Result 4.23. *Let $f \in W_2^k$. Let $\{\phi_h\}_h$ be a subset of L_2. Then*

(4.24) $$\operatorname{dist}_2(f, \sigma_h(S(\phi_h))) = o(h^k)$$

only if

$$h^{-d/2} \|\Lambda_h \sigma_h \widehat{f}\|_{L_2(B)} = o(h^k),$$

on some origin-neighborhood B.

We next attempt to replace Λ_h in this result by simpler expressions. It is clear that we might replace Λ_h by $(\widetilde{N_h} - \widehat{N_h})^{1/2}$ in case the ratio

$$\frac{(\widetilde{N_h} - \widehat{N_h})^{1/2}}{\Lambda_h} = \widetilde{N_h}^{1/2}$$

is bounded around the origin by h-independent positive constants. For this we need the following lemma, in which we make use of the fact that every N_h, and in particular N_0, is supported in the symmetric region $Z_{\Xi \cup (-\Xi)} = Z_\Xi - Z_\Xi$ (which follows from the fact that $\overline{M(-\cdot)} = M_{(-\Xi),\overline{\lambda}}$; cf. (3.3) for the definition of Z_Ξ).

Lemma 4.25. *$N_h \underset{h\to 0}{\longrightarrow} N_0$ uniformly and hence $\widetilde{N_h} \to \widetilde{N_0}$ in any p-norm, $1 \le p \le \infty$.*

Proof. The first claim easily follows from the distributional definition of box splines (cf. e.g. Definition 2.1 in [R1]. More specifically, one can apply to $N_h - N_0$ the argument used in the proof of Lemma 5.1 in [DR]). The second claim follows from the first, since all N_h are supported in the same compact domain $Z_\Xi - Z_\Xi$. ♠

Thanks to Lemma 4.25, we know that $\{\widetilde{N_h}\}_h$ are uniformly bounded around the origin, hence may replace $\{\Lambda_h\}_h$ in Result 4.23 by

$$(\widetilde{N_h} - \widehat{N_h})^{1/2} = \Big(\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{M_h}(\cdot + \beta)|^2\Big)^{1/2}.$$

Thus, if, for some $f \in W_2^k$, $\mathrm{dist}_2(f, \sigma_h(S(\phi_h))) = o(h^k)$, then we must have, for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$,

$$h^{-d/2}\|\widehat{M_h}(\cdot + \beta)\sigma_h\widehat{f}\|_{L_2(B)} = o(h^k),$$

which implies by scaling that

$$\|\widehat{M_h}(h\cdot + \beta)\widehat{f}\|_{L_2(h^{-1}B)} = o(h^k). \tag{4.26}$$

Let $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ be chosen with $\#K_\beta = k$. Since $\widehat{f} \in L_2$, it is supported on a set of positive measure, and therefore, for sufficiently small ε, the set

$$A_\varepsilon := \{\omega \in \mathbb{R}^d : \ |\lambda_\xi - i\xi\cdot\omega| \ge \varepsilon, \ \forall \xi \in \Xi\}$$

has a positive measure intersection with $\mathrm{supp}\,\widehat{f}$. We fix such ε, and we let Ω be any bounded measurable subset of A_ε for which $\|\widehat{f}\|_{L_2(\Omega)} > 0$.

A straightforward computation shows that, for any $\theta \in \mathbb{R}^s$ and $\xi \in \Xi\backslash K_\beta(\Xi)$,

$$\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(h\theta+\beta))t}\,dt \underset{h\to 0}{\longrightarrow} \int_0^1 e^{-i\xi\cdot\beta t}\,dt \ne 0, \tag{4.27}$$

uniformly. Further, for $\xi \in K_\beta$, we get that

$$h^{-1}\Big|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(h\theta+\beta))t}\,dt\Big| \underset{h\to 0}{\longrightarrow} \Big|\frac{\lambda_\xi - i\xi\cdot\theta}{\xi\cdot\beta}\Big| \ge \mathrm{const} > 0, \quad \forall \theta \in A_\varepsilon,$$

with the convergence being uniform on compact sets, hence on Ω. Thus, for small enough h_0,

$$\inf\{h^{-1}\Big|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(h\theta+\beta))t}\,dt\Big| : \ \theta \in \Omega, \ h < h_0\} > 0, \quad \forall \xi \in K_\beta.$$

This, together with (4.27), implies, with

$$r_h := \inf\{|M_h(h\theta + \beta)| : \ \theta \in \Omega\},$$

that, for small enough h,

$$h^{-k} r_h = h^{-k}\inf\{\prod_{\xi\in\Xi}\Big|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(h\theta+\beta))t}\,dt\Big| : \ \theta \in \Omega\} > \mathrm{const} > 0.$$

Thus, by (4.26),

$$o(h^k) = \|M_h(h \cdot +\beta)\widehat{f}\|_{L_2(\Omega)} \geq \|\widehat{f}\|_{L_2(\Omega)} r_h > \mathrm{const}\|\widehat{f}\|_{L_2(\Omega)} h^k,$$

a contradiction to the fact that $\|\widehat{f}\|_{L_2(\Omega)} \neq 0$.

This completes the proof of Theorem 3.7, and thereby the proof of Theorem 3.4. ♠

4.4. Proof of Theorem 3.17

The proof invokes Theorem 2.9 for the choice $\phi_h := M_h$. Thus, we need to verify that for an integer $r < k$ (with $k := k'(\Xi)$), the sequences

$$m^r_{k,h} : (2\pi\mathbb{Z}^d\backslash 0) \ni \beta \mapsto |\beta|^r \|(h + |\cdot|)^{-k}\widehat{M_h}(\cdot + \beta)\|_{L_\infty(B)}, \quad h < h_0$$

lie in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$ and are bounded there.

For that, we first invoke Proposition 4.17 to conclude that

$$m^r_{k,h}(\beta) \leq \mathrm{const}|\beta|^r \prod_{\xi \in K_\beta \cup L_\beta} |\xi \cdot \beta|^{-1}.$$

Therefore, by Proposition 4.18, the uniform boundedness of $\{m^r_{k,h}\}_h$ in $\ell_q(2\pi\mathbb{Z}^d\backslash 0)$ is implied by the condition $|\cdot|^r\widehat{M_0} \in L_q$. For $p = \infty$ (i.e., $q = 1$) this latter condition is *assumed* in the present theorem. We prove here the validity of the condition for $q > 1$.

We first observe that the condition the condition $|\cdot|^r\widehat{M_0}$ is equivalent to the statement: "for each homogeneous polynomial P of degree r, $(P(D)M_0)^\wedge \in L_q$." Here, $r \leq k'(\Xi) - 1 \leq k(\Xi) - 1$ (cf. the proof of Corollary 3.8). On the other hand, it is known, [BH], that for any polynomial P of degree $< k(\Xi)$, $P(D)M_0$ can be written as a finite sum

$$\sum_X c_X M_X(\cdot - \alpha_X), \tag{4.28}$$

where each X in the above sum is a submatrix of Ξ of *full rank d*, where c_X are some coefficients, $\alpha_X \in \mathbb{R}^d$, and $M_X = M_{X,0}$ is the polynomial box spline defined by X. Since each X above satisfies $\mathrm{rank}\, X = d$, then, as established in the proof of Theorem 3.13, $\widehat{M_X} \in L_q$ for every $q > 1$. Therefore, the Fourier transform of the sum in (4.28) is in L_q, namely, $(P(D)M_0)^\wedge \in L_q$. Consequently, $|\cdot|^r\widehat{M_0} \in L_q$. ♠

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