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## A Conservative Formulation for Plasticity

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In this paper we propose a fully conservative form for the continuum equations governing rate-dependent and rate-independent plastic flow in metals. The conservation laws are valid for discontinuous as well as smooth solutions. In the rate-dependent case, the evolution equations are in divergence form, with the plastic strain being passively convected and augmented by source terms. In the rate-independent case, the conservation laws involve a Lagrange multiplier that is determined by a set of constraints; we show that Riemann problems for this system admit scale-invariant solutions. © 1992 Academic Press, Inc.

### 1. INTRODUCTION

In a previous paper [24], we formulated the equations for elasticity in the Eulerian picture as conservation laws. The motivation for this work was that the Eulerian framework is useful in numerical computations of large-deformation flows. Furthermore, the most effective numerical meth-

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ods, such as second-order Godunov schemes and the front tracking method, rely on an understanding of the structure of nonlinear wave solutions. Such an understanding requires that the equations be in conservation form. Subsequent studies [5, 1] have shown that the use of a conservative formulation does indeed lead to substantially improved numerical calculations.

In the present paper, we show that a fully conservative formulation can be given for the equations governing an important class of rate-dependent and rate-independent plastic materials. Unlike the case of elastic materials, for which Lagrangian conservation laws were already available, we must appeal directly to experiment to decide which of the dynamical equations for plastic flow are conservation laws. For this reason, we focus on the plastic flow of metals induced by strong shock waves. We expect, however, that the conservative form applies to a broader class of plastic materials.

The formulation of finite-strain plasticity that we adopt has much in common with that of Simo and coworkers [30, 28, 29]. There are two novel aspects to the present work. For rate-dependent materials, we cast the governing equations in divergence form and argue, on the basis of experimental results for metal plasticity, that these equations represent conservation principles. For rate-independent materials, we show that the loading/unloading conditions, written as an optimality condition involving a Lagrange multiplier, hold for weak solutions; the resulting constrained system of conservation laws admits scale-invariant solutions.

Section 2 of this paper contains a brief summary of elasticity in the Lagrangian and Eulerian pictures. The various parts of Section 3 present the equations of viscoplasticity in the Lagrangian and Eulerian pictures and include a discussion of the kinematics of plastic deformation, constitutive relations and models, and plastic flow rules. In Section 4 it is argued, on the basis of experiment, that the plastic flow rule and hardening law are conservation laws. In Section 5, the conservative form for rate-independent plasticity is obtained from the rate-dependent theory in the zero relaxation-time limit. Finally, in Section 6, we examine scale-invariant solutions of the rate-independent theory.

## 2. ELASTICITY

In this section we summarize the continuum description of an elastically deformable medium, first in the Lagrangian (material) picture and then in the Eulerian (spatial) picture. We refer the reader to Ref. [24] and to the general literature (e.g., Refs. [15, 35, 21, 9, 2]) for further detail.

### 2a. Kinematics and Conservation Laws

The motion of a deformable body  $\mathcal{B}$  is represented mathematically by a time-dependent map  $\phi$ , the *motion*, embedding  $\mathcal{B}$  into an ambient space  $\mathcal{S}$ . Let  $X^\alpha$ ,  $\alpha = 1, 2, 3$ , denote *material coordinates* on  $\mathcal{B}$ , and let  $x^i$ ,  $i = 1, 2, 3$ , denote *spatial coordinates* on  $\mathcal{S}$ ; then  $x^i = \phi^i(X, t)$ . The motion  $\phi$  must respect the conservation of momentum and energy. These principles are expressed as partial differential equations involving the temporal and spatial derivatives of  $\phi$ , namely the *Lagrangian velocity*  $V^i := \partial\phi^i/\partial t$  and the *Lagrangian deformation gradient*  $F^i_\alpha := \partial\phi^i/\partial X^\alpha$ .

Let  $\rho_{\text{ref}}$  denote the *mass density* of the undeformed body,  $\mathcal{E}$  the *specific energy* (energy per unit mass),  $S^{\alpha\beta}$  the (*second*, or *symmetric*) *Piola-Kirchhoff stress tensor*, and  $Q^\alpha$  the *Lagrangian heat flux*. Then the conservation laws are

$$\rho_{\text{ref}} \dot{V}^i = (F^i_\alpha S^{\alpha\beta})_{;\beta}, \quad (2.1)$$

$$\rho_{\text{ref}} \left( \frac{1}{2} V_i V^i + \mathcal{E} \right) = (V_i F^i_\alpha S^{\alpha\beta})_{;\beta} - Q^\gamma_{;\gamma}. \quad (2.2)$$

Here the dot and the semicolon denote covariant differentiation with respect to time and space, respectively. Equations (2.1) and (2.2) are in divergence form; indeed, they derive from the integral form of the conservation laws. Therefore they hold in the sense of distributions.

Because the motion  $\phi$  enters the conservation laws only through its derivatives, the second-order equations can be regarded as first-order equations in which the velocity and the deformation gradient are fundamental dynamical variables. If  $\phi$  is eliminated in favor of the  $F^i_\alpha$ , however, the system of equations must be expanded. The additional equations are also conservation laws:

$$\dot{F}^i_\alpha = V^i_{;\alpha}, \quad (2.3)$$

as follows by equating mixed partial derivatives of  $\phi$ . Equations (2.1)–(2.3) express the governing equations for a deformable medium as first-order partial differential equations in conservative form.

### 2b. Constitutive Relations

The conservation laws involve the energy, stress, and heat flux; therefore they are incomplete unless supplemented by constitutive relations for these quantities. These relations characterize the material properties of the deformable medium.

The thermodynamic state of an elastic material is specified by the deformation gradient  $F^i_\alpha$  and the *specific entropy*  $S$ . Because of covariance with respect to rigidbody transformations of the spatial coordinates (the

principle of frame indifference), the state depends on  $F^i_{,a}$  only through the (right) Cauchy–Green tensor

$$C_{\alpha\beta} = F^k_{,\alpha} F^k_{,\beta} \tag{2.4}$$

or, equivalently, the Lagrangian strain tensor

$$E_{\alpha\beta} = \frac{1}{2} [F^k_{,\alpha} F^k_{,\beta} - \delta_{\alpha\beta}]. \tag{2.5}$$

(In general,  $\delta_{\alpha\beta}$  should be interpreted as the material metric tensor.) We postulate that the thermodynamic state determines the specific energy  $\mathcal{E}$  through a hyperelastic equation of state

$$\mathcal{E} = \hat{\mathcal{E}}(E_{\alpha\beta}, S). \tag{2.6}$$

(For simplicity, we assume that the material is uniform, i.e., neither  $\rho_{\text{ref}}$  nor  $\hat{\mathcal{E}}$  has explicit dependence on the material point  $X$ .) The heat flux is specified by a separate constitutive relation, such as

$$Q^\alpha = -\mathcal{K} T^{;\alpha}, \tag{2.7}$$

where  $T$  is the temperature and  $\mathcal{K}$  is the thermal conductivity.

Following the standard argument of Coleman [3], the second law of thermodynamics, as embodied in the Clausius–Duhem inequality

$$\rho_{\text{ref}} \dot{S} + (Q^\alpha / T)_{;\alpha} \geq 0, \tag{2.8}$$

leads to the identifications

$$S^{\alpha\beta} = \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}}, \tag{2.9}$$

$$T = \frac{\partial \hat{\mathcal{E}}}{\partial S}, \tag{2.10}$$

together with the constraint

$$-Q^\gamma T_{;\gamma} \geq 0. \tag{2.11}$$

For the constitutive relation (2.7), this constraint requires that  $\mathcal{K} \geq 0$ . It follows from the governing equations and the constitutive relations that the entropy satisfies

$$\rho_{\text{ref}} T \dot{S} = -Q^\gamma_{;\gamma}. \tag{2.12}$$

Given the constitutive relations (2.6), (2.9), (2.10), and (2.7), Eqs. (2.1)–(2.3) comprise a complete system of evolution equations in conservative form that describes elastic materials in the Lagrangian picture.

### 2c. Eulerian Formulation

To translate the governing equations into conservation laws in the Eulerian picture, we introduce notation for several Eulerian quantities, which are defined to be functions of  $x = \phi(X, t)$  and  $t$ :

$$f^i_\alpha := F^i_\alpha, \quad (2.13)$$

$$g^\alpha_i := (F^{-1})^\alpha_i, \quad (2.14)$$

$$\rho := J^{-1} \rho_{\text{ref}}, \quad (2.15)$$

$$v^i := V^i, \quad (2.16)$$

$$\varepsilon := \mathcal{E}, \quad (2.17)$$

$$\eta := S, \quad (2.18)$$

$$\theta := T, \quad (2.19)$$

$$\sigma^{ij} := J^{-1} F^i_\alpha S^{\alpha\beta} F^j_\beta, \quad (2.20)$$

$$q^i := J^{-1} F^i_\alpha Q^\alpha. \quad (2.21)$$

Here  $J$  is the *Jacobian determinant* of the map  $\phi$ , i.e., the determinant of the linear transformation  $F^i_\alpha$ , and  $\sigma^{ij}$  is the *Cauchy stress tensor*.

In the standard fashion, the transport theorem (see, e.g., Ref. [9]) shows that the Lagrangian conservation laws (2.1) and (2.2) for momentum and energy are equivalent to

$$\frac{\partial}{\partial t}(\rho v^i) + (\rho v^i v^j)_{;j} = \sigma^{ij}_{;j}, \quad (2.22)$$

$$\frac{\partial}{\partial t} \left[ \rho \left( \frac{1}{2} v_i v^i + \varepsilon \right) \right] + \left[ \rho \left( \frac{1}{2} v_i v^i + \varepsilon \right) v^j \right]_{;j} = (v_i \sigma^{ij})_{;j} - q^k_{;k}. \quad (2.23)$$

The conservation law (2.3) governing the deformation gradient translates into the Eulerian picture in a similar manner [24]. With the aid of the Piola identity,  $\{J(F^{-1})^\alpha_i\}_{;\alpha} = 0$ , Eq. (2.3) becomes

$$\frac{\partial}{\partial t}(\rho f^i_\alpha) + (\rho f^i_\alpha v^j)_{;j} = (\rho v^j f^i_\alpha)_{;j}. \quad (2.24)$$

We supplement this continuity equation with the conservation of mass

equation,

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0, \tag{2.25}$$

which derives from the Lagrangian equation  $\dot{\rho}_{\text{ref}} = 0$ .

The Eulerian system of Eqs. (2.22)–(2.25) is complete in the following sense (cf. [24]): Consider a solution to the Eulerian system that is either smooth or piecewise constant; suppose that there is a deformation  $\phi_0$  such that at some initial time  $t = t_0$ ,  $f'_{\alpha}$  and  $J$  are the gradient and Jacobian of  $\phi_0$ , respectively. Then there is a motion  $\phi$ , coinciding with  $\phi_0$  at  $t = t_0$ , that satisfies the corresponding Lagrangian system of Eqs. (2.1)–(2.3) and relates to the Eulerian solution as in the definitions above.

An alternative to Eqs. (2.24) and (2.25) has been suggested by Trangenstein and Colella [34], viz., the equality of mixed partial derivatives for  $\phi^{-1}$ :

$$\frac{\partial}{\partial t} g^{\alpha}_i + [g^{\alpha}_k v^k]_{,i} = 0. \tag{2.26}$$

An equivalence, in the sense above, between the Lagrangian system and the system comprising Eqs. (2.22), (2.23), and (2.26) is straightforward to establish in the manner of Ref. [24]. Moreover, Wagner [37] has used the methods of Ref. [36] to extend the equivalence to general weak solutions.

The conservation laws must be supplemented by an equation of state, which expresses the energy as a function of the *Finger tensor*

$$b^{ij} := f^{i\gamma} f^j_{\gamma}, \tag{2.27}$$

or, equivalently, the *Almansi strain tensor*

$$e_{ij} := \frac{1}{2} [\delta_{ij} - (b^{-1})_{ij}]. \tag{2.28}$$

Note that  $e_{ij} := (F^{-1})^{\alpha}_i E_{\alpha\beta} (F^{-1})^{\beta}_j$ . The equation of state can be written [30] as

$$\varepsilon = \hat{\varepsilon}(e_{ij}, f^k_{\gamma}, \eta), \tag{2.29}$$

where  $\hat{\varepsilon}$  relates to  $\hat{\mathcal{E}}$  through

$$\hat{\varepsilon}(e_{kl}, f^k_{\gamma}, \eta) := \hat{\mathcal{E}}(f^m_{\gamma} e_{mn} f^n_{\delta}, \eta). \tag{2.30}$$

Differentiating this definition yields the formulae

$$\sigma^{ij} = \rho \frac{\partial \hat{\varepsilon}}{\partial e_{ij}}, \quad (2.31)$$

$$\theta = \frac{\partial \hat{\varepsilon}}{\partial \eta}. \quad (2.32)$$

In calculating the derivative in Eq. (2.31),  $e_{ij}$  and  $f^k_\gamma$  are regarded as independent variables, and their relationship is invoked only after the differentiation has been performed.

*Remark.* The dependence of  $\hat{\varepsilon}$  on  $f^k_\gamma$  is essential. In fact,

$$\rho f^i_\alpha \frac{\partial \hat{\varepsilon}}{\partial f^j_\alpha} = 2\sigma^{ik} e_{kj}, \quad (2.33)$$

so that if  $\hat{\varepsilon}$  were independent of  $f^k_\gamma$ , then the stress  $\sigma^{ij}$  would vanish. In other words, one cannot assume that the energy depends solely on  $e_{ij}$  and  $\eta$  (unless the material response to deformation is trivial). Note, however, that  $\partial \hat{\varepsilon} / \partial f^k_\gamma$  vanishes at the undeformed state  $e_{ij} = 0$ , which means that the effects of  $f^k_\gamma$  under small deformations are of higher order. Similar conclusions hold in the case of plasticity.

The constitutive relation (2.7) for the heat flux becomes

$$q^i = -k\theta^{;i}, \quad (2.34)$$

with  $k := J^{-1}\mathcal{K}$ .

Assuming the constitutive relations (2.29), (2.31), (2.32), and (2.34), Eqs. (2.22) and (2.23), together with either Eqs. (2.24) and (2.25) or Eq. (2.26), comprise a complete set of conservation laws in the Eulerian picture.

### 3. PLASTICITY

The plastic response of real materials is extremely diverse. For instance, the microphysical pictures of metal plasticity and of the flow of granular materials are quite different. Furthermore, passage from the microphysical to the macroscopic description requires scientific modeling across a hierarchy of length scales. Nevertheless, there are two concepts that apply to a broad class of macroscopic models: plastic deformation and plastic flow rule.



### 3a. Plastic Deformation

One principal feature of plastic behavior is irrecoverable deformation: whereas an elastic body will return to its undeformed state when surface forces are relaxed, a plastic material might not. The permanent deformation suffered depends on the manner in which the body was worked, i.e., on the history of deformation of the material. In a continuum theory of this phenomenon, there must be "internal" state variables that record this deformation history. Because the deformation history varies from point to point in the body, a plastic material is inhomogeneous.

To understand these ideas on a physical level, consider a small neighborhood of a material point  $X$  in a body that has undergone plastic deformation. If the neighborhood is cut out of the material and the forces on its surface are relaxed, then it deforms, recovering a stress-free configuration. Because of the plastic working, this configuration differs from the original configuration of the neighborhood before the body was deformed. Thus at every point there is a local irrecoverable deformation taking the neighborhood from its virgin state to a stress-free configuration. In general, however, there is no global deformation that relaxes the internal stresses simultaneously throughout the body. The local irrecoverable deformations at neighboring points may be incompatible, in the sense that the relaxed neighborhoods do not fit together to form a continuous body without gaps or tears.

These ideas can be expressed mathematically, using the theory of inhomogeneous materials developed by Noll [35]. For a neighborhood that is sufficiently small on the macroscopic scale, the irrecoverable deformation is approximated well by its gradient, which is a linear map from the tangent space at  $X$  to three-dimensional Euclidean space  $\mathbf{R}^3$ . Mathematically, such a linear map defines a frame at  $X$ , for its inverse carries the standard basis for  $\mathbf{R}^3$  to a basis for the tangent space at  $X$ . We denote the components of this *local reference frame* at  $X$  by  $(F_{\text{ref}}(X))^\alpha_a$ , where  $a = 1, 2, 3$ .

Let a local reference frame  $(F_{\text{ref}}(X))^\alpha_a$  be chosen at each material point  $X$ . As discussed above, there need not exist a deformation of the entire body whose gradient is  $(F_{\text{ref}})^\alpha_a$ : there is no "intermediate" stress-free configuration of an inhomogeneous body. Indeed, the body is considered to be *homogeneous* if and only if the frame field  $F_{\text{ref}}$  is a gradient. Nevertheless, the gradient  $F^\alpha_a$  of an arbitrary deformation can be referred, at each material point, to the local reference frame. Thus we define  $\bar{F}^i_a$  by decomposing the deformation gradient  $F^\alpha_a$  as a product involving the frame  $(F_{\text{ref}})^\alpha_a$ :

$$F^\alpha_a = \bar{F}^i_a (F_{\text{ref}})^\alpha_a. \quad (3.1)$$

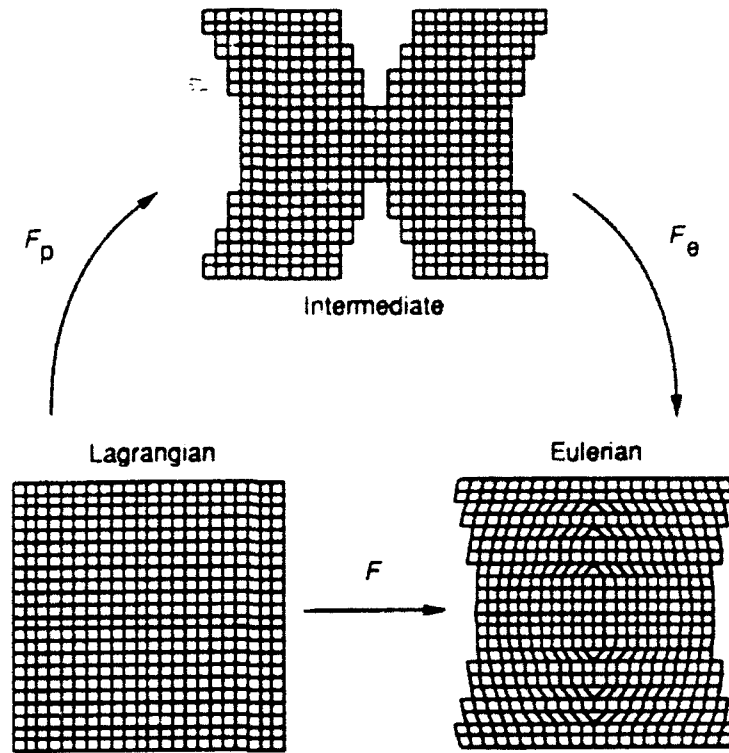


FIG. 1. The multiplicative decomposition of the deformation gradient:  $F'_\alpha = (F'_e)_\alpha (F_p)_\alpha$ . The plastic deformation  $(F_p)_\alpha$ , which represents the irrecoverable deformation, maps neighborhoods in the body to local stress-free configurations. Here this deformation involves the slip of horizontal crystal planes. As indicated, the relaxed neighborhoods do not fit together continuously. The elastic deformation  $(F'_e)_\alpha$  distorts the relaxed neighborhoods into the final configuration of the body.

(See Fig. 1.) For a homogeneous body, the transformation to local reference frames amounts to a change of material coordinates.

For example, the frame field can be given a concrete interpretation in terms of the theory of dislocations in crystals (see, e.g., Refs. [35, 6, 13, 12, 10]). Each of the three basis vectors defined by the local reference frame at  $X$  is regarded as a displacement by a fixed number of lattice spacings along a crystal axis. Thus the local reference frames reflect, at a continuum level, the crystallographic structure. The tensor field

$$\rho^{a\alpha} := -\varepsilon^{\alpha\beta\gamma} (F_{\text{ref}})_\beta{}^\gamma{}^a, \quad (3.2)$$

which is a measure of the nonintegrability of the frame field, can be interpreted as the *dislocation density*.

With  $(F_{\text{ref}})_\alpha$  chosen, the material response depends on  $\bar{F}^i_\alpha$ , the entropy, and the material position  $X$ . If there exists a choice of local

reference frames such that this response does not depend explicitly on  $X$ , then the body is said to be *materially uniform*. Note that there is a possible ambiguity in the decomposition (3.1), depending on the symmetry of the material. Indeed, a *material isomorphism* at a point is a change of local reference frame that preserves the response of the material to deformations [35, 10]. These isomorphisms form the local symmetry group at the point; a materially uniform body has the same local symmetry group at each point. One is free to transform the local reference frame at each point of the body, provided that the transformation belongs to the symmetry group and depends continuously on position. For a crystal, the symmetry group is finite, so that the transformation must be the same at each point. Thus  $(F_{\text{ref}})'_a$  is essentially unique, being defined modulo a finite group of transformations. By contrast, the symmetry group is the rotation group for a material such as a metal. Consequently, the constitutive equation for the energy depends on  $(F_{\text{ref}})'_a$  only through the strain

$$E'_{\alpha\beta} := \frac{1}{2} \left[ (F_{\text{ref}})'_{\alpha\alpha} (F_{\text{ref}})'_{\beta\beta} - \delta_{\alpha\beta} \right]. \quad (3.3)$$

The theory of inhomogeneous materials can be adapted to model plasticity by allowing the local reference frames of Noll to be dynamic [6]. In this approach, the frame field  $(F_{\text{ref}})'_a$  is regarded as the *plastic deformation*  $(F_p)'_a$ , while  $\bar{F}'_a$  is the *elastic deformation*  $(F_e)'_a$ . Thus we are led to the fundamental kinematic decomposition

$$F'_a = (F_e)'_a (F_p)'_a. \quad (3.4)$$

Such a multiplicative decomposition has been advocated by several workers in plasticity theory [4, 17, 6, 16, 41, 20, 19, 25, 14].

The plastic deformation  $(F_p)'_a$  is the principal, but not necessarily the only, internal variable in a model of a plastic material. Other variables, such as a *hardening parameter*, characterize the effects of plastic deformation on the material response. We will assume that there is a single scalar hardening parameter  $K$ ; the theory is easily extended to account for several tensorial internal variables.

### 3b. Constitutive Relations

Internal variables  $(F_p)'_a$  and  $K$  enter the equations of motion in two places: in the dynamical equations that govern their evolution; and in the constitutive equation for the energy, which depends on the local reference frame and hardening parameter as well as the deformation gradient and the entropy. Choosing a plasticity model amounts to specifying the evolution and constitutive equations.

A general thermodynamical framework for plasticity, which encompasses a variety of specific models, has been given by Green and Naghdi [7]. This framework allows for various choices of the measure of plastic strain. A model based on the multiplicative decomposition (3.4) can be accommodated in two ways, depending on the material symmetry [6]. For crystals, one chooses  $(F_p)^\alpha_\alpha$  as the plastic strain measure; for metals, one defines the *plastic strain*  $E_{\alpha\beta}^p$  by Eq. (3.3):

$$E_{\alpha\beta}^p := \frac{1}{2} \left[ (F_p)_{\alpha\alpha} (F_p)^\alpha_\beta - \delta_{\alpha\beta} \right]. \quad (3.5)$$

In the following we will focus on metals, although the development could be adapted to treat crystals.

The constitutive equation for the energy expresses  $\mathcal{E}$  as a function of the deformation gradient  $F^\alpha_\alpha$ , the plastic deformation  $(F_p)^\alpha_\alpha$ , the hardening parameter  $K$ , and the entropy  $S$ . By the principle of frame indifference,  $\mathcal{E}$  depends on the deformation gradient only through the (total) strain  $E_{\alpha\beta}$ ; and for a metal, the plastic deformation appears only through the plastic strain  $E_{\alpha\beta}^p$ . Therefore we assume the hyperelastic equation of state

$$\mathcal{E} = \hat{\mathcal{E}}(E_{\gamma\delta}, E_{\gamma\delta}^p, K, S). \quad (3.6)$$

This assumption, together with the Clausius–Duhem inequality, implies formulae for the stress and temperature [3, 7].

To derive these formulae, we introduce the notation

$$D_{\alpha\beta} := \frac{1}{2} (F_{k\alpha} V^k_{;\beta} + V_{k;\alpha} F^k_\beta), \quad (3.7)$$

$$\Omega_{\alpha\beta} := \frac{1}{2} (F_{k\alpha} V^k_{;\beta} - V_{k;\alpha} F^k_\beta) \quad (3.8)$$

for the *Lagrangian strain rate* and *vorticity tensors*. These tensors are the symmetric and antisymmetric parts of  $F_{k\alpha} \dot{F}^k_\beta$ , respectively; in particular,

$$\dot{E}_{\alpha\beta} = D_{\alpha\beta}. \quad (3.9)$$

Note that the energy conservation equation (2.2), combined with the momentum conservation equation (2.1), implies that

$$\rho_{\text{ref}} \dot{\mathcal{E}} = S^{\alpha\beta} D_{\alpha\beta} - Q^\gamma_{;\gamma}. \quad (3.10)$$

Furthermore, differentiating the equation of state (3.6) shows that

$$\dot{\mathcal{E}} = \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}} \dot{E}_{\alpha\beta} + \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \dot{E}_{\alpha\beta}^p + \frac{\partial \hat{\mathcal{E}}}{\partial K} \dot{K} + \frac{\partial \hat{\mathcal{E}}}{\partial S} \dot{S}. \quad (3.11)$$

Combining these two equations with inequality (2.8) yields

$$\begin{aligned} & \left( S^{\alpha\beta} - \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}} \right) \dot{E}_{\alpha\beta} - \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \dot{E}_{\alpha\beta}^p - \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial K} \dot{K} \\ & - \rho_{\text{ref}} \left( T - \frac{\partial \hat{\mathcal{E}}}{\partial S} \right) \dot{S} - \frac{Q^\gamma T_{;\gamma}}{T} \geq 0. \end{aligned} \quad (3.12)$$

Assuming, as is conventional, that  $\dot{E}_{\alpha\beta}$  and  $\dot{S}$  can be chosen arbitrarily even if there is neither a temperature gradient nor plastic flow (i.e.,  $T_{;\gamma} = 0$ ,  $\dot{E}_{\alpha\beta}^p = 0$ , and  $\dot{K} = 0$ ), inequality (3.12) leads to the identifications

$$S^{\alpha\beta} = \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}} \quad (3.13)$$

and

$$T = \frac{\partial \hat{\mathcal{E}}}{\partial S} \quad (3.14)$$

along with the constraints

$$\mathcal{D} \geq 0, \quad (3.15)$$

$$- \frac{Q^\gamma T_{;\gamma}}{T} + \mathcal{D} \geq 0. \quad (3.16)$$

Here

$$\mathcal{D} := -\rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \dot{E}_{\alpha\beta}^p - \rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial K} \dot{K} \quad (3.17)$$

is the *plastic dissipation*, since entropy is generated according to

$$\rho_{\text{ref}} T \dot{S} = -Q^\gamma_{;\gamma} + \mathcal{D}. \quad (3.18)$$

Thus the plastic dissipation is  $\mathcal{D} = S_p^{\alpha\beta} \dot{E}_{\alpha\beta}^p + \Omega \dot{K}$ , where

$$S_p^{\alpha\beta} := -\rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p}, \quad (3.19)$$

$$\Omega := -\rho_{\text{ref}} \frac{\partial \hat{\mathcal{E}}}{\partial K} \quad (3.20)$$

are the dissipative forces conjugate to  $E_{\alpha\beta}^p$  and  $K$ , respectively. In general,

the plastic stress  $S_p^{ab}$  and the stress  $S^{ab}$  differ, although they are approximately equal in the small-strain limit. Inequalities (3.15) and (3.16) place thermodynamic restrictions on the dependence of  $\hat{\mathcal{E}}$  on the internal variables and on the evolution equations that govern them.

### 3c. Constitutive Models

Specifying the constitutive function  $\hat{\mathcal{E}}$  involves modeling assumptions. For instance, a natural assumption in plasticity is that the energy is determined primarily by the elastic part of the strain: the history of plastic deformation has only mild effects on the material response, accounted for by the hardening parameter. Furthermore, the anisotropic component of the elastic strain is typically small because of plastic yielding, so that a small-anisotropy approximation is useful for a material that is isotropic in the local stress-free configurations.

Two alternative measures of the elastic strain are used in the literature. Green and Naghdi define the *elastic strain*  $E_{\alpha\beta}^c$ , as measured with respect to the Lagrangian frame, through the additive decomposition

$$E_{\alpha\beta} = E_{\alpha\beta}^c + E_{\alpha\beta}^p \quad (3.21)$$

of the Lagrangian strain. Equation (3.21) is analogous to the additive decomposition of infinitesimal strain in classical plasticity. Alternatively, Lee and Liu [17, 16] measure the elastic strain with respect to the local reference frames using

$$\bar{E}_{ab}^c := \frac{1}{2} \left[ (F_c)_{ka} (F_c)^k_b - \delta_{ab} \right], \quad (3.22)$$

instead of  $E_{\alpha\beta}^c$ . These strain measures are related by

$$E_{\alpha\beta}^c = (F_p)_{\alpha}^a \bar{E}_{ab}^c (F_p)^b_{\beta}. \quad (3.23)$$

Either measure of elastic strain can be used in constructing the equation of state (3.6). This is evident for the additive decomposition (3.21). In terms of the strain measure  $\bar{E}_{ab}^c$ , the general constitutive equation (3.6) can be written as

$$\mathcal{E} = \bar{\mathcal{E}}(\bar{E}_{ab}^c, (F_p)_{\alpha}^a, K, S), \quad (3.24)$$

where

$$\begin{aligned} \bar{\mathcal{E}}(\bar{E}_{ab}^c, (F_p)_{\alpha}^a, K, S) := & \hat{\mathcal{E}}\left((F_p)_{\alpha}^a \bar{E}_{ab}^c (F_p)^b_{\beta} + \frac{1}{2} \left[ (F_p)_{a\alpha} (F_p)^a_{\beta} - \delta_{\alpha\beta} \right], \right. \\ & \left. \frac{1}{2} \left[ (F_p)_{a\alpha} (F_p)^a_{\beta} - \delta_{\alpha\beta} \right], K, S\right). \end{aligned} \quad (3.25)$$

Conversely, for Eq. (3.24) to represent an equation of state, Eq. (3.25) must hold for some  $\mathcal{E}$ . Indeed, when the local symmetry group is the rotation group,  $\bar{E}_{ab}^c$  and  $(F_p)^a_\alpha$  enter  $\mathcal{E}$  only through the variables  $E_{\alpha\beta}$  and  $E_{\alpha\beta}^p$ . In this sense, the alternative formulations are equivalent [8, 27]; for a particularly lucid explanation, see Ref. [30].

When the equation of state is formulated as in Eq. (3.24), it is useful to work with various tensors referred to the intermediate frame. For instance, with

$$\bar{S}^{ab} := (F_p)^a_\alpha S^{\alpha\beta} (F_p)^b_\beta, \tag{3.26}$$

Eqs. (3.13) and (3.25) imply that

$$\bar{S}^{ab} = \rho_{\text{rel}} \frac{\partial \bar{\mathcal{E}}}{\partial \bar{E}_{ab}^c}. \tag{3.27}$$

Furthermore, in formulating the dynamical equation governing the plastic strain (see Section 3d), the following definitions will be helpful:

$$\bar{E}_{ab}^p := (F_p^{-1})^a_\alpha E_{\alpha\beta}^p (F_p^{-1})^b_\beta, \tag{3.28}$$

$$(L^p \bar{E}^p)_{ab} := (F_p^{-1})^a_\alpha \dot{E}_{\alpha\beta}^p (F_p^{-1})^b_\beta. \tag{3.29}$$

Here  $(L^p \bar{E}^p)_{ab}$  is the *plastic Lie derivative* of  $\bar{E}_{ab}^p$  [30]; cf. Eq. (3.63) below.

Let us return to the example of a plastic material that is isotropic. The physical picture provided by the multiplicative decomposition of the deformation gradient (see Fig. 1) suggests that the isotropy is manifested in the stress-free intermediate configuration. This means that  $\mathcal{E}$  depends solely on the principal invariants of  $(\bar{E}^c)^a_b$ , or equivalently of  $(\bar{C}^c)^a_b := 2(\bar{E}^c)^a_b + \delta^a_b$ . (Recall that the principal invariants of a  $3 \times 3$  matrix  $A$  are  $\iota_1(A) = \text{tr } A$ ,  $\iota_2(A) = \frac{1}{2}[(\text{tr } A)^2 - \text{tr } A^2]$ , and  $\iota_3(A) = \det A$ ; see, e.g., Refs. [2, 9].) Thus Lee and Liu [17, 16] adopt the equation of state

$$\mathcal{E} = \bar{\mathcal{E}}(I_1, I_2, I_3, K, S), \tag{3.30}$$

where  $I_k := \iota_k(\bar{C}^c)$ ,  $k = 1, 2, 3$ . This equation of state conforms with the requirement embodied in Eq. (3.25) because the invariants  $I_1$ ,  $I_2$ , and  $I_3$  can be expressed in terms of  $E_{\alpha\beta}$  and  $E_{\alpha\beta}^p$ . To see this, we let  $C_{\alpha\beta}^p := 2E_{\alpha\beta}^p + \delta_{\alpha\beta}$  and note that

$$(C_p^{-1})^{\alpha\gamma} C_{\gamma\beta} = (F_p^{-1})^a_\alpha (\bar{C}^c)^a_b (F_p)^b_\beta; \tag{3.31}$$

as the invariants are unaffected by a similarity transformation,  $\iota_k(C_p^{-1}C) = \iota_k(\bar{C}^e)$ ,  $k = 1, 2, 3$ .

*Remark.* For an equation of state such as Eq. (3.30), the plastic stress  $S_p^{\alpha\beta}$  of Eq. (3.19) that enters the plastic dissipation  $\mathcal{D}$  can be related to the stress  $S^{\alpha\beta}$ :

$$S_p^{\alpha\beta} = (C_p^{-1})^{\alpha\gamma} C_{\gamma\delta} S^{\delta\beta} \quad (3.32)$$

(the matrix on the right is, in fact, symmetric). When the elastic strain  $\bar{E}_{\gamma\delta}^e$  is small,  $S_p^{\alpha\beta}$  is approximately equal to  $S^{\alpha\beta}$ .

### 3d. Plastic Flow Rule

The other fundamental feature of plastic behavior is the *plastic flow rule*, which specifies the time evolution of the plastic deformation. Such a flow field has the general form [23, 18]

$$\dot{E}_{\alpha\beta}^p = \Lambda_{\alpha\beta}(E_{\gamma\delta}, E_{\gamma\delta}^p, K, S) \quad (3.33)$$

for a rate-dependent plastic, i.e., *viscoplastic*, material. In addition, a *hardening law*

$$\dot{K} = H(E_{\gamma\delta}, E_{\gamma\delta}^p, K, S) \quad (3.34)$$

governs the evolution of the hardening parameter. The flow rule and hardening law are constrained by the requirement (3.15) that the plastic dissipation  $\mathcal{D}$  be nonnegative.

Characteristic of viscoplastic flow (as opposed to viscoelastic flow) is that plastic deformation occurs only if a threshold has been reached. According to the plastic flow rule (3.33), the material deforms plastically only in the region of state space where  $\Lambda_{\alpha\beta} \neq 0$ . Since the hardening parameter records the effects of plastic deformation on the material properties, hardening occurs only during plastic flow; therefore  $H$  is assumed to vanish when  $\Lambda_{\alpha\beta} = 0$ . The interior of the region where  $\Lambda_{\alpha\beta} = 0$ , in which the behavior is purely elastic, is called the *elastic range*. The boundary of the elastic range is called the (*static*) *yield surface*. The elastic range and yield surface are usually characterized by a yield function  $\Phi$ :  $\Phi < 0$  in the elastic range and  $\Phi = 0$  on the yield surface. The yield function  $\Phi$  depends on the variables  $E_{\gamma\delta}$ ,  $E_{\gamma\delta}^p$ ,  $K$ , and  $S$ ; frequently, however,  $\Phi$  depends on the strains and the entropy only through the stress and the temperature.

We illustrate these concepts with a model that has been used to describe high strain-rate plastic flow in metals [11, 31, 32]. In the case of finite deformations, this model is specified most simply in terms of tensors



referred to the intermediate frame (see Section 3c and the first remark below). First, the *von Mises yield criterion*

$$\Phi := \|\text{dev } \bar{S}\| - \sqrt{2/3} Y_0(\bar{P}, T, K) \tag{3.35}$$

is adopted. In this equation,  $Y_0$  is the *static yield strength* (as specified below),  $\bar{P} := \frac{1}{3} J^{-1} \bar{S}^c_c$  is the *mean pressure*,  $(\text{dev } \bar{S})^{ab}$  is the *deviatoric stress*,

$$(\text{dev } \bar{S})^{ab} := \bar{S}^{ab} - \frac{1}{3} \bar{S}^c_c \delta^{ab}, \tag{3.36}$$

and the norm  $\|\cdot\|$  is defined by  $\|A\|^2 := A^{ab} A_{ab}$ . The flow rule is taken to be the *Lévy-St. Venant flow rule*, given by

$$(L^p \bar{E}^p)_{ab} = \Lambda(\bar{P}, \|\text{dev } \bar{S}\|, T, K) \frac{(\text{dev } \bar{S})_{ab}}{\|\text{dev } \bar{S}\|}. \tag{3.37}$$

In particular, the model is complete once  $\|L^p \bar{E}^p\| = \Lambda(\bar{P}, \|\text{dev } \bar{S}\|, T, K)$  has been specified.

*Remarks.* (1) Plastic flow in metals is often assumed to be volume preserving [17, 16], so that  $\det F_p = 1$ . Since

$$(\det F_p)^{-1} (\det F_p)^c = (\dot{F}_p)^c (F_p^{-1})^c_c = (L^p \bar{E}^p)^c_c, \tag{3.38}$$

the plastic flow rule preserves volume if and only if  $(L^p \bar{E}^p)^a_b$  is trace-free; this has been emphasized to us by F. Wang [40]. For this reason, we have formulated Eqs. (3.35) and (3.37) in terms of  $(\text{dev } \bar{S})^{ab}$ , rather than  $(\text{dev } \sigma)^{ij}$ , as would be conventional (see, e.g., Ref. [28]). The corresponding Eulerian formulation is given in Eqs. (3.71) and (3.72) below.

(2) When  $Y_0$  is independent of  $\bar{P}$ , the Lévy-St. Venant flow rule is *associative*, in the sense that

$$(L^p \bar{E}^p)_{ab} = \Lambda(\bar{P}, \|\text{dev } \bar{S}\|, T, K) \frac{\partial \Phi}{\partial \bar{S}^{ab}}. \tag{3.39}$$

Thus the direction of  $\Lambda_{\alpha\beta}$  is determined by the normal to the yield surface in stress space.

(3) In terms of Lagrangian quantities, the flow rule (3.37) is

$$\dot{E}^p_{\alpha\beta} = \Lambda(\bar{P}, \|\text{dev}_{C^p} S\|_{C^p}, T, K) C^p_{\alpha\gamma} \frac{(\text{dev}_{C^p} S)^{\gamma\delta}}{\|\text{dev}_{C^p} S\|_{C^p}} C^p_{\delta\beta}, \tag{3.40}$$

where  $\bar{P} := \frac{1}{3}J^{-1}S^{\gamma\delta}C_{\gamma\delta}^p$ ,

$$(\text{dev}_{C^p} S)^{\alpha\beta} := S^{\alpha\beta} - \frac{1}{3}S^{\gamma\delta}C_{\gamma\delta}^p(C^p)^{\alpha\beta}, \quad (3.41)$$

and the norm  $\|\cdot\|_{C^p}$  is defined by

$$\|A\|_{C^p}^2 := A^{\alpha\beta}C_{\beta\gamma}^p A^{\gamma\delta}C_{\delta\alpha}^p. \quad (3.42)$$

Therefore this flow rule does, indeed, have the form of Eq. (3.33), provided that  $S^{\alpha\beta}$  and  $T$  are expressed in terms of  $E_{\gamma\delta}$ ,  $E_{\gamma\delta}^p$ ,  $K$ , and  $S$  through Eqs. (3.13) and (3.14).

The specification of  $\|L^p\bar{E}^p\|$  derives from a model for the *dynamic yield strength*  $Y = \hat{Y}(\bar{P}, T, \|L^p\bar{E}^p\|, K)$ . This quantity is the yield strength observed in plastic flow at nonzero strain rate; in other words,

$$\|\text{dev } \bar{S}\| = \sqrt{2/3} \hat{Y}(\bar{P}, T, \|L^p\bar{E}^p\|, K) \quad (3.43)$$

during plastic flow. The dynamic yield strength must exceed the static yield strength

$$Y_0(\bar{P}, T, K) := \hat{Y}(\bar{P}, T, 0, K) \quad (3.44)$$

and increase monotonically with strain rate.

Given a model for the dynamic yield strength, the corresponding flow rule is derived as follows. When  $\|\text{dev } \bar{S}\|$  is below the static yield strength, there is no plastic flow, and  $\|L^p\bar{E}^p\| = 0$ ; otherwise,  $\|\text{dev } \bar{S}\| > \sqrt{2/3} Y_0(\bar{P}, T, K)$ , so that Eq. (3.43) can be solved for  $\|L^p\bar{E}^p\|$  in terms of the remaining quantities, obtaining

$$\|L^p\bar{E}^p\| = \bar{\Lambda}(\bar{P}, \|\text{dev } \bar{S}\|, T, K). \quad (3.45)$$

Therefore the plastic flow rule is

$$(L^p\bar{E}^p)_{ab} = \begin{cases} 0 & \text{if } \|\text{dev } \bar{S}\| \leq \sqrt{2/3} Y_0(\bar{P}, T, K), \\ \bar{\Lambda}(\bar{P}, \|\text{dev } \bar{S}\|, T, K) \frac{(\text{dev } \bar{S})_{ab}}{\|\text{dev } \bar{S}\|} & \text{otherwise.} \end{cases} \quad (3.46)$$

To be more concrete, we consider a model for the dynamic yield strength given by Steinberg and Lund [32]. In this model,  $Y$  is composed of

a thermally activated part  $Y_T$  and a strain-hardening part  $Y_A$ :

$$Y = \frac{G(\bar{P}, T)}{G_0} [Y_T + Y_A(K)], \quad (3.47)$$

where the thermally activated part  $Y_T$ ,  $0 \leq Y_T \leq Y_p$ , is the solution of the equation

$$\sqrt{2/3} \|L^p \bar{E}^p\| = Y_T / \left\{ C_1 - C_1^{-1} Y_T \exp \left[ \frac{2U_\zeta}{kT} \left( 1 - \frac{Y_T}{Y_p} \right)^2 \right] \right\}. \quad (3.48)$$

In these expressions, the functions  $G$  and  $Y_A$  and the parameters  $G_0$ ,  $C_1$ ,  $C_2$ ,  $Y_p$ , and  $U_\zeta/k$  are material-dependent quantities. The hardening law (3.34) is taken to be

$$K = \sqrt{2/3} \|L^p \bar{E}^p\|; \quad (3.49)$$

in other words,  $K$  is the *equivalent plastic strain*, and Eq. (3.49) models *isotropic strain-hardening*.

The function  $\bar{\Lambda}$  for this model is obtained as follows: According to Eq. (3.48),  $Y_T = 0$  when  $\|L^p \bar{E}^p\| = 0$ , so that

$$Y_0(\bar{P}, T, K) = \frac{G(\bar{P}, T)}{G_0} Y_A(K). \quad (3.50)$$

Since  $\|\text{dev } \bar{S}\| = \sqrt{2/3} Y$  during plastic flow, Eq. (3.47) implies that

$$Y_T = \frac{\|\text{dev } \bar{S}\| - \sqrt{2/3} Y_0(\bar{P}, T, K)}{\sqrt{2/3} G(\bar{P}, T)/G_0}; \quad (3.51)$$

thus  $Y_T$  coincides with a measure of the extent to which the static yield condition has been exceeded. Substituting this expression into the right-hand side of Eq. (3.48) gives  $\sqrt{2/3} \bar{\Lambda}$ .

### 3c. Eulerian Formulation

Corresponding to the internal variables  $E_{\gamma_0}^p$  and  $K$  characterizing a plastic material in the Lagrangian picture, we define the Eulerian quantities

$$(f_c)'_a := (F_c)'_a, \quad (3.52)$$

$$e_{ij}^p := (F^{-1})^\alpha{}_i E_{\alpha\beta}^p (F^{-1})^\beta{}_j, \quad (3.53)$$

$$\kappa := K, \quad (3.54)$$

which are regarded as functions of  $x = \phi(X, t)$  and  $t$ . In terms of the elastic Finger tensor

$$\bar{b}_e^{ij} := (f_e)^{ia} (f_e)^j_a, \quad (3.55)$$

$$e_{ij}^p := \frac{1}{2} [(b_e^{-1})_{ij} - (b^{-1})_{ij}]. \quad (3.56)$$

The equation of state is

$$\varepsilon = \hat{\varepsilon}(e_{kl}, e_{kl}^p, f^k_\gamma, \kappa, \eta), \quad (3.57)$$

where

$$\hat{\varepsilon}(e_{kl}, e_{kl}^p, f^k_\gamma, \kappa, \eta) := \hat{\mathcal{E}}(f^m_\gamma e_{mn} f^n_\delta, f^m_\gamma e_{mn}^p f^n_\delta, \kappa, \eta). \quad (3.58)$$

Just as for elastic materials, the Eulerian conservation laws of momentum, energy, and continuity are Eqs. (2.22), (2.23), and either Eqs. (2.24) and (2.25) or Eq. (2.26); the constitutive equations are Eqs. (3.57), (2.31), (2.32), and (2.34). The thermodynamic constraints (3.15) and (3.16) become

$$d \geq 0, \quad (3.59)$$

$$-\frac{q^k \theta_{;k}}{\theta} + d \geq 0, \quad (3.60)$$

where

$$d := -\rho \frac{\partial \hat{\varepsilon}}{\partial e_{ij}^p} (L_\nu e^p)_{ij} - \rho \frac{\partial \hat{\varepsilon}}{\partial \kappa} L_\nu \kappa \quad (3.61)$$

is the plastic dissipation  $J^{-1}\mathcal{D}$ , and entropy is generated according to the equation

$$\rho \theta L_\nu \eta = -q^k_{;k} + d. \quad (3.62)$$

In these equations we have introduced the Lie derivatives [21, 30] of  $e_{ij}^p$ ,  $\kappa$ , and  $\eta$ , as defined by

$$\begin{aligned} (L_\nu e^p)_{ij} &:= (f^{-1})^\alpha_i \dot{E}_{\alpha\beta}^p (f^{-1})^\beta_j \\ &= \frac{\partial}{\partial t} e_{ij}^p + v^k e_{ij;k}^p + v^k_{;i} e_{kj}^p + e_{ik}^p v^k_{;j}, \end{aligned} \quad (3.63)$$

$$L_\nu \kappa := \frac{\partial \kappa}{\partial t} + v^k \kappa_{;k}, \quad (3.64)$$

$$L_\nu \eta := \frac{\partial \eta}{\partial t} + v^k \eta_{;k}. \quad (3.65)$$

To write the plastic flow rule (3.33) in the Eulerian formulation, we multiply Eq. (3.33) by  $(f^{-1})^\alpha_i (f^{-1})^\beta_j$  to find that

$$(L, e^p)_{ii} = \lambda_{ii}(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta); \tag{3.66}$$

in addition, the hardening law (3.34) becomes

$$L, \kappa = h(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta). \tag{3.67}$$

Here we have made the identifications

$$\lambda_{ii}(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta) := (F^{-1})^\alpha_i \Lambda_{\alpha\beta}(E_{\gamma\delta}, E^p_{\gamma\delta}, K, S)(F^{-1})^\beta_j, \tag{3.68}$$

$$h(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta) := H(E_{\gamma\delta}, E^p_{\gamma\delta}, K, S). \tag{3.69}$$

Similarly, the Eulerian yield function  $\varphi$  is given by

$$\varphi(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta) := \Phi(E_{\gamma\delta}, E^p_{\gamma\delta}, K, S). \tag{3.70}$$

For example, the von Mises yield criterion (3.35) and the Lévy-St. Venant flow rule (3.37), as expressed in the Eulerian picture, are

$$\varphi := \|\text{dev}_{b_c^{-1}} \sigma\|_{b_c^{-1}} - \sqrt{2/3} y_0(p, \theta, \kappa). \tag{3.71}$$

$$L, e^p_{ij} = \lambda(p, \|\text{dev}_{b_c^{-1}} \sigma\|_{b_c^{-1}}, \theta, \kappa)(b_c^{-1})_{ik} \frac{(\text{dev}_{b_c^{-1}} \sigma)^{kl}}{\|\text{dev}_{b_c^{-1}} \sigma\|_{b_c^{-1}}} (b_c^{-1})_{li}, \tag{3.72}$$

where  $p := \frac{1}{3} \sigma^{kl} (b_c^{-1})_{kl}$  is the mean pressure  $\bar{P}$ ,

$$(\text{dev}_{b_c^{-1}} \sigma)^{ij} := \sigma^{ij} - \frac{1}{3} \sigma^{kl} (b_c^{-1})_{kl} b_c^{ij}, \tag{3.73}$$

and the norm  $\|\cdot\|_{b_c^{-1}}$  is defined by

$$\|A\|_{b_c^{-1}}^2 := A^{ij} (b_c^{-1})_{jk} A^{kl} (b_c^{-1})_{li}. \tag{3.74}$$

Also,  $\lambda$  and  $y_0$  are identified with  $\Lambda$  and  $Y_0$ , respectively. Therefore this flow rule has the form of Eq. (3.66) when  $\sigma^{ij}$  and  $\theta$  are expressed in terms of  $e_{kl}$ ,  $e^p_{kl}$ ,  $\kappa$ , and  $\eta$  through Eqs. (2.31) and (2.32).

#### 4. FLOW RULES AS CONSERVATION LAWS

In the previous section we have seen that the evolution of a plastic medium is governed by the conservation laws of momentum, energy, and

continuity together with flow rules for the internal variables. This discussion implicitly assumed that the flow was smooth. Here we consider the possibility of discontinuous solutions, as are expected to occur on the basis of experiment and the structure of the governing system of partial differential equations. Analysis of discontinuous solutions is best carried out when the equations are written as conservation laws. This poses a question: When are the dynamical equations for the internal variables conservation laws?

Answering the question of when an evolution equation is a conservation law has two aspects. First, the equation must be written in divergence form. Second, the equation must hold in the sense of distributions. To understand these aspects, we recall the example of gas dynamics.

For smooth gas flows, the equations governing the time evolution of both energy  $\varepsilon$  and entropy  $\eta$  are in divergence form:

$$\frac{\partial}{\partial t} \left[ \rho \left( \frac{1}{2} v_i v^i + \varepsilon \right) \right] + \left[ \rho \left( \frac{1}{2} v_i v^i + \varepsilon \right) v^j \right]_{;j} = -(\rho v^j)_{;j}, \quad (4.1)$$

$$\frac{\partial}{\partial t} (\rho \eta) + (\rho \eta v^j)_{;j} = 0. \quad (4.2)$$

However, one cannot require both of these equations to hold for discontinuous flows. Indeed, the jump conditions enforcing conservation of energy entail an increase in entropy, whereas the jump conditions for conservation of entropy lead to a decrease in energy. Although both Eq. (4.1) and Eq. (4.2) hold up until shock formation, a choice between them must be made at that point. This choice is not dictated by the mathematical structure of the equations. Rather, it must be made on the basis of physics: energy is conserved and entropy increases. Similarly, in plasticity, we must appeal to physics to decide when a flow rule is expressing a conservation law.

For plasticity, we first ask whether the evolution equations for the internal variables have divergence form. Referring to Eqs. (3.33) and (3.34), we see that the plastic flow rule and hardening law have divergence form in the Lagrangian formulation: the time derivatives of the internal variables are given by source terms involving the state variables but not their derivatives. By contrast, the corresponding equations (3.66) and (3.67) in the Eulerian formulation have not been written in divergence form; this is because  $L_{;i} e^p$  and  $L_{;i} \kappa$  involve spatial gradients multiplied by functions of the state variables. We therefore replace these equations by ones in divergence form.

The Eulerian equations in divergence form that correspond to the Lagrangian flow rule and hardening law are obtained as in Ref. [24]:

(a) the equations are written in integral form; (b) a change of variable is made in the integral; and (c) the transport theorem is applied. The equations that result can also be obtained by a simple argument (which is valid only for smooth solutions). Thus we multiply Eq. (3.33) by  $\rho$  and add the equation of conservation of mass, multiplied by  $E_{\alpha\beta}^p$ ; this yields

$$\frac{\partial}{\partial t}(\rho E_{\alpha\beta}^p) + (\rho E_{\alpha\beta}^p v^k)_{;k} = \rho \Lambda_{\alpha\beta}. \quad (4.3)$$

Since  $E_{\alpha\beta}^p = f^i_{;\alpha} e_{ij}^p f^j_{;\beta}$ , we obtain

$$\frac{\partial}{\partial t}(\rho f^i_{;\alpha} e_{ij}^p f^j_{;\beta}) + (\rho f^i_{;\alpha} e_{ij}^p f^j_{;\beta} v^k)_{;k} = \rho f^i_{;\alpha} \Lambda_{ij} f^j_{;\beta}. \quad (4.4)$$

Similarly, the hardening law becomes

$$\frac{\partial}{\partial t}(\rho \kappa) + (\rho \kappa v^k)_{;k} = \rho h. \quad (4.5)$$

The question remains whether Eqs. (4.4) and (4.5) represent physical conservation laws. In this paper, we address this question within the context defined by experiments on shock wave propagation in metals [26, 38, 39, 31-33].

Distinct wave structures are observed in the three regimes of low, intermediate, and high pressure driving the metal. If the driving pressure is below the Hugoniot elastic limit (on the order of 10 kbar), a purely elastic shock wave occurs. For an intermediate pressure (less than about 100 kbar), a two-wave structure is observed. At still higher pressures, the two-wave structure collapses into a single strong shock wave.

In the intermediate pressure regime, the leading wave is relatively sharp, has an amplitude that is generally independent of the driving pressure, and propagates at a speed determined by the elastic properties of the metal. Therefore it is interpreted as an elastic precursor wave. The wave following the precursor has a broader profile and an amplitude that increases with driving pressure; its structure is sensitive to the strain rate. Furthermore, the relaxation time associated with the strain-rate dependence is long compared with the rise time for the elastic precursor. These observations suggest that the plastic deformation occurs predominantly after the passage of the elastic precursor.

On this basis, it is concluded that (a) the elastic wave can be modeled effectively as a mathematical discontinuity, involving no change in the plastic deformation and (b) the plastic flow occurs within a broad profile plastic wave following the elastic precursor. This picture is consistent with

the assumption that the flow rule and hardening law, Eqs. (3.33) and (3.34) or Eqs. (4.4) and (4.5), are genuine conservation laws.

To justify this statement, we consider a plane wave, propagating at (Lagrangian) speed  $S_N$  in the direction of the vector  $N_\alpha$ , across which a discontinuity occurs in the state variables. Using  $\Delta$  to denote the jump in a quantity across the wave, the jump conditions for Eqs. (2.1)–(2.3) and Eqs. (3.33) and (3.34) are

$$-\rho_{\text{ref}} S_N \Delta V^i = \Delta(F^i_\alpha S^{\alpha\beta}) N_\beta, \quad (4.6)$$

$$-\rho_{\text{ref}} S_N \Delta\left(\frac{1}{2} V_i V^i + \mathcal{E}\right) = \Delta(V_i F^i_\alpha S^{\alpha\beta}) N_\beta - \Delta Q^\gamma N_\gamma, \quad (4.7)$$

$$-S_N \Delta F^i_\alpha = \Delta V^i N_\alpha, \quad (4.8)$$

$$-S_N \Delta E^p_{\alpha\beta} = 0, \quad (4.9)$$

$$-S_N \Delta K = 0. \quad (4.10)$$

Therefore no change in the internal variables  $E^p$  and  $K$  are sustained unless  $S_N = 0$ ; i.e., the wave moves with the speed of the fluid particles.

Thus for the case of shock wave propagation in metals, we have established a conservative form for the governing equations. This conservative form might well apply to a broader class of rate-dependent plastic materials.

## 5. RATE-INDEPENDENT PLASTICITY

While plastic flow is fundamentally a rate-dependent process, a rate-independent approximation is useful in certain flow regimes. In this section, we shall see how the rate-independent equations emerge as a singular limit of rate-dependent plasticity. We also explain the sense in which the rate-independent equations are conservation laws.

We assume that the static yield surface associated with the flow rule  $\lambda_{ij}$  is given by Eq. (3.70). In particular,  $\|\lambda\| > 0$  when  $\varphi > 0$ , so that we can define the *relaxation-time parameter*  $\tau > 0$  to be  $\tau := \varphi / \|\lambda\|$ . (For instance, in the Steinberg–Lund model,  $\tau$  is proportional to  $C_2$ .) Noting that  $\lambda_{ij} = 0$  and  $h = 0$  when  $\varphi \leq 0$ , Eqs. (3.66) and (3.67) can be written

$$(L_v e^p)_{ij} = \frac{1}{\tau} \left[ \varphi(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta) \right] + \bar{\lambda}_{ij}(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta), \quad (5.1)$$

$$L_{i,\kappa} = \frac{1}{\tau} \left[ \varphi(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta) \right] + \bar{h}(e_{kl}, e^p_{kl}, f^k_\gamma, \kappa, \eta), \quad (5.2)$$



where  $\|\tilde{\lambda}\| = 1$ . Here  $[\cdot]_+$  denotes the positive part of its argument:  $[\varphi]_+$  is  $\varphi$  when  $\varphi \geq 0$  and is zero otherwise.

If the material is in the plastic region  $\varphi > 0$ , then Eq. (5.1) dictates that  $e_{ij}^p$  relaxes to the yield surface  $\varphi = 0$  on the time scale of  $\tau$ . In the limit  $\tau \rightarrow 0$ , the flow process is approximately rate-independent. This approximation is valid when the time scale for returning the plastic material to the yield surface is much smaller than the time scale for other flow processes. The limiting equations can be formulated as

$$(L_t e^p)_{,i} = \mu \tilde{\lambda}_{,i} (e_{kl}, e_{kl}^p, f^k_\gamma, \kappa, \eta) \tag{5.3}$$

$$L_t \kappa = \mu \tilde{h} (e_{kl}, e_{kl}^p, f^k_\gamma, \kappa, \eta), \tag{5.4}$$

where  $\mu$  is a *Lagrange multiplier* that is subject to the *loading/unloading conditions*

$$\mu \geq 0, \quad \varphi \leq 0, \quad \mu \varphi = 0, \tag{5.5}$$

written as a *Kuhn-Tucker complementarity condition*, and the *persistence condition* that

$$L_t \varphi = 0 \quad \text{if } \mu > 0. \tag{5.6}$$

*Remark.* This formulation has been extensively discussed by Simo and coworkers [30, 28, 29]. Conditions (5.5) can be derived as optimality conditions for a constrained optimization problem: maximize the rate of dissipation of energy during plastic flow, subject to the constraint  $\varphi \leq 0$ .

In the following we describe how the rate-independent equations arise from the rate-dependent equations as  $\tau \rightarrow 0$ , assuming that the flow is smooth. Comparing Eqs. (5.1) and (5.2) with Eqs. (5.3) and (5.4) suggests the identification

$$\mu = \lim_{\tau \rightarrow 0} \frac{1}{\tau} [\varphi]_+. \tag{5.7}$$

Therefore we calculate  $\lim_{\tau \rightarrow 0} \tau^{-1} [\varphi]_+$ .

As a preliminary, note that the yield function  $\varphi$  satisfies the equation

$$L_t \varphi = [L_t \varphi]_{\text{elastic}} - \frac{\alpha}{\tau} [\varphi]_+, \tag{5.8}$$

where

$$[L_t \varphi]_{\text{elastic}} := \frac{\partial \varphi}{\partial e_{ij}} (L_t e)_{,i} - \frac{1}{\rho \theta} \frac{\partial \varphi}{\partial \eta} q^k_{;k} \tag{5.9}$$

and

$$\alpha := - \left( \frac{\partial \varphi}{\partial e_{ij}^p} \varepsilon_{ij} - \frac{1}{\theta} \frac{\partial \varphi}{\partial \eta} \frac{\partial \hat{f}}{\partial e_{ij}^p} \right) \tilde{\lambda}_{ij} - \left( \frac{\partial \varphi}{\partial \kappa} - \frac{1}{\theta} \frac{\partial \varphi}{\partial \eta} \frac{\partial \hat{f}}{\partial \kappa} \right) \tilde{h}. \quad (5.10)$$

Equation (5.8) can be derived by first calculating  $\Phi$  in Lagrangian coordinates and then transforming to Eulerian coordinates.

*Remark.* The combinations of derivatives of  $\varphi$  that appear as coefficients in the definition of  $\alpha$  are the derivatives of  $\varphi$  with  $\varepsilon$ , rather than  $\eta$ , held fixed.

At this point we must assume that  $\alpha > 0$ ; this constitutive assumption is a *stability condition*, in that when the material is beyond the yield surface, i.e.,  $\varphi > 0$ , the second term in Eq. (5.8) drives the material toward the yield surface  $\varphi = 0$ .

We regard Eq. (5.8) as an ordinary differential equation for  $\varphi$  along a particle path, with  $[L_t \varphi]_{\text{elastic}}$  serving as a known forcing term. To simplify the analysis, we assume that  $t$  has been rescaled so that  $\alpha$  is constant. Furthermore, we suppose that the flow begins within the yield surface, i.e.,  $\varphi$  takes on the ( $\tau$ -independent) initial-value  $\varphi|_{t=0} \leq 0$  at time  $t = 0$ . In order to solve the initial-value problem, we consider three cases.

First, if  $\varphi|_{t=0} = 0$  and  $[L_t \varphi]_{\text{elastic}}|_{t=0} > 0$ , then  $\varphi \geq 0$  for small  $t \geq 0$  so that  $[\varphi]_{-} = \varphi$ . Therefore multiplying Eq. (5.8) by  $\exp(\alpha t/\tau)$  and integrating both sides along a particle path shows that  $\varphi$  is given by

$$\varphi = \int_0^t e^{-\alpha(t-s)/\tau} [L_s \varphi]_{\text{elastic}} ds. \quad (5.11)$$

Note that when  $t > 0$  is fixed,  $(\alpha/\tau) \exp[-\alpha(t-s)/\tau]$  approaches a delta-function at  $s = t$  as  $\tau \rightarrow 0$ . Consequently, Eq. (5.11) implies that

$$\lim_{t \rightarrow 0^+} \lim_{\tau \rightarrow 0} \frac{1}{\tau} [\varphi]_{-} = \frac{1}{\alpha} [L_t \varphi]_{\text{elastic}}. \quad (5.12)$$

(Obviously, taking the limits  $t \rightarrow 0^+$  and  $\tau \rightarrow 0$  in the opposite order yields zero; thus  $\mu$  is discontinuous at  $t = 0$ .)

Next, if either  $\varphi|_{t=0} < 0$  or  $\varphi|_{t=0} = 0$  and  $[L_t \varphi]_{\text{elastic}}|_{t=0} < 0$ , then  $\varphi \leq 0$  for small  $t \geq 0$ . Therefore

$$\lim_{t \rightarrow 0^+} \lim_{\tau \rightarrow 0} \frac{1}{\tau} [\varphi]_{-} = 0. \quad (5.13)$$

Finally, suppose that  $\varphi|_{t=0} = 0$  and  $[L_t \varphi]_{\text{elastic}}|_{t=0} = 0$ . Since  $[\varphi]_{-} \geq \varphi$ ,

the argument leading to Eq. (5.11) yields the inequality

$$[\dot{\varphi}] \geq \max \left\{ 0, \int_{\Omega} e^{-\mu t} \rho \theta^{-1} [L_t \varphi]_{\text{elastic}} \, ds \right\}. \quad (5.14)$$

Thus

$$0 \leq \lim_{t \rightarrow \infty} \limsup_{\tau \rightarrow \infty} \int_{\Omega} [\dot{\varphi}] \geq \max \{ 0, [L_t \varphi]_{\text{elastic}}(\infty) \} = 0. \quad (5.15)$$

As concerns the Lagrange multiplier  $\mu \geq 0$  of Eq. (5.7), we draw the following conclusions. Equation (5.13) says that  $\mu > 0$  if  $\dot{\varphi} > 0$ . Moreover, by Eqs. (5.12), (5.13), and (5.15),  $\mu > 0$  only if Eq. (5.12) holds, in which case Eq. (5.8) implies that  $L_t \varphi = 0$ . In other words, Eqs. (5.3) and (5.4) and conditions (5.5) and (5.6) are obtained in the rate-independent limit.

These conditions lead to the strain-space formulation of the loading criterion for a rate-independent material [22]. We define  $\mu_{\text{trial}}$  to be the right-hand side of Eq. (5.12), i.e.,

$$\mu_{\text{trial}} = \frac{\frac{\partial \varphi}{\partial e_{ij}} (L_t e)_{ij} - \frac{1}{\rho \theta} \frac{\partial \varphi}{\partial \eta} q^k \dot{z}_k}{-\left( \frac{\partial \varphi}{\partial e_{ij}^p} - \frac{1}{\theta} \frac{\partial \varphi}{\partial \eta} \frac{\partial \hat{e}}{\partial e_{ij}^p} \right) \dot{\lambda}_{ij} - \left( \frac{\partial \varphi}{\partial \kappa} - \frac{1}{\theta} \frac{\partial \varphi}{\partial \eta} \frac{\partial \hat{e}}{\partial \kappa} \right) \dot{h}}. \quad (5.16)$$

and say that the flow is undergoing *loading* when  $\dot{\varphi} = 0$  and  $\mu_{\text{trial}} > 0$ , *neutral loading* when  $\dot{\varphi} = 0$  and  $\mu_{\text{trial}} = 0$ , *unloading* when  $\dot{\varphi} = 0$  and  $\mu_{\text{trial}} < 0$ , and *elastic deformation* when  $\dot{\varphi} < 0$ . Then the Lagrange multiplier  $\mu$  that appears in Eqs. (5.3) and (5.4) is

$$\mu = \begin{cases} \mu_{\text{trial}} & \text{in the case of loading,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.17)$$

This classical formulation is implied by conditions (5.5) and (5.6), as follows [29]. First suppose that  $\dot{\varphi} = 0$  and  $\mu_{\text{trial}} > 0$ . Then in order for  $\dot{\varphi}$  to remain nonpositive, we must have that  $L_t \varphi \leq 0$ . Since  $L_t \varphi = [L_t \varphi]_{\text{elastic}} - \mu \alpha$ , this implies that  $\mu \geq \mu_{\text{trial}} > 0$ . By the persistency condition, we conclude that  $L_t \varphi = 0$ , i.e.,  $[L_t \varphi]_{\text{elastic}} - \mu \alpha = 0$ . By the persistency condition, we conclude that  $L_t \varphi = 0$ , i.e.,  $[L_t \varphi]_{\text{elastic}} - \mu \alpha = 0$ . Therefore  $\mu = \mu_{\text{trial}}$ . Next suppose that  $\dot{\varphi} < 0$ . Since  $\mu \dot{\varphi} = 0$ , this implies that  $\mu = 0$ . Finally, if  $\mu_{\text{trial}} \leq 0$ , then  $\mu = 0$  as well; otherwise  $\mu$  would be positive, implying that  $L_t \varphi = 0$ , which would entail that  $\mu = \mu_{\text{trial}}$ , a contradiction.

The classical flow rule is inadequate, however, when the flow is not smooth. In general we wish to allow flows containing discontinuities, i.e., solutions with bounded-variation that satisfy the governing equations in the weak sense (the sense of distributions). In such a case, Eq. (5.16) might involve multiplying a discontinuous function by a distribution with a delta-function singularity, a process that is ill-defined. In contrast, the Kuhn-Tucker conditions (5.5) are sensible for bounded-variation solutions: the conditions hold almost everywhere. Furthermore, if the persistency condition (5.6) is replaced by the equivalent condition

$$\frac{\partial}{\partial t}(\rho\varphi) + (\rho\varphi t^k)_{,k} = 0 \quad \text{in the interior of the support of } \mu, \quad (5.18)$$

then it also can be required of weak solutions.

Thus the weak formulation of the governing equations for rate-independent plasticity consists of the conservation laws (2.22), (2.23) and either (2.24) and (2.25) or (2.26), together with the flow rule

$$\frac{\partial}{\partial t}(\rho f^i_{,\alpha} e^p_{ij} f^j_{,\beta}) + (\rho f^i_{,\alpha} e^p_{ij} f^j_{,\beta} t^k)_{,k} = \rho \mu f^i_{,\alpha} \tilde{\lambda}_{ij} f^j_{,\beta}, \quad (5.19)$$

$$\frac{\partial}{\partial t}(\rho\kappa) + (\rho\kappa t^k)_{,k} = \rho\mu\tilde{h}, \quad (5.20)$$

where  $\mu$  is constrained by conditions (5.5) and (5.18). We call the resulting system a *constrained system of conservation laws*.

## 6. WAVE STRUCTURE

In this section we give a preliminary analysis of scale-invariant solutions for rate-independent plasticity. In the Lagrangian picture, the governing equations (with heat conduction neglected) are

$$\dot{F}^i_{,\alpha} = V^i_{,\alpha}, \quad (6.1)$$

$$\rho_{\text{ref}} \dot{V}^i = (F^i_{,\alpha} S^{\alpha\beta})_{,\beta}, \quad (6.2)$$

$$\rho_{\text{ref}} \left( \frac{1}{2} V_i V^i + \mathcal{E} \right)' = (V_i F^i_{,\alpha} S^{\alpha\beta})_{,\beta}, \quad (6.3)$$

$$\dot{E}^p_{\alpha\beta} = \mu \tilde{\lambda}_{\alpha\beta}, \quad (6.4)$$

$$\dot{K} = \mu \tilde{H}, \quad (6.5)$$

where  $\mu$  is subject to the constraints

$$\mu \geq 0, \quad \Phi \leq 0, \quad \mu\Phi = 0, \quad (6.6)$$

$$\Phi = 0 \quad \text{if } \mu > 0. \quad (6.7)$$

We seek scale-invariant solutions of these equations. By this we mean that the flow variables  $F^i_\alpha$ ,  $V^i$ ,  $S$ ,  $E^p_{\alpha\beta}$ , and  $K$  depend solely on  $\xi := N_\alpha X^\alpha / t$ , whereas  $\mu$  has the form  $\bar{\mu}(\xi)/t$ .

If such a solution has a jump discontinuity at speed  $S_N$ , then each (distributional) derivative appearing in Eqs. (6.1)–(6.5) is proportional to the delta-function  $\delta(N_\gamma X^\gamma - S_N t)$  supported on the plane  $\xi = S_N$ . For instance,  $E^p_{\alpha\beta} = -S_N \Delta E^p_{\alpha\beta} \delta$  and  $V^i_{;\alpha} = N_\alpha \Delta V^i \delta$ . The source terms on the right-hand sides of Eqs. (6.4) and (6.5), however, are bounded variation functions, with no delta-function component. Therefore the jump conditions (4.6)–(4.10) are satisfied (with  $\Delta Q^\gamma = 0$ ), subject to the constraint that  $\Phi \leq 0$  on both sides of the discontinuity. In other words, jump discontinuities are unaffected by the plastic source terms. There is no need to determine  $\mu$ ; in fact, since the plane  $\xi = S_N$  has measure zero, the value of the distribution  $\mu$  on this plane is indeterminate.

For a smooth solution, on the other hand, Eq. (6.2) becomes

$$\begin{aligned} \rho_{\text{ref}} \dot{V}^i &= \mathbf{A}^{i\beta j\delta} F_{j\delta;\beta} - \rho_{\text{ref}} T F^i_{;\alpha} \Gamma^{\alpha\beta} S_{;\beta} \\ &+ F^i_{;\alpha} \left[ \frac{\partial S^{\alpha\beta}}{\partial E^p_{\gamma\delta}} E^p_{\gamma\delta;\beta} + \frac{\partial S^{\alpha\beta}}{\partial K} K_{;\beta} \right]. \end{aligned} \quad (6.8)$$

Here  $\mathbf{A}^{i\beta k\delta}$  is the *adiabatic elasticity tensor* and the  $\Gamma^{\alpha\beta}$  are the Grüneisen coefficients:

$$\mathbf{A}^{i\beta k\delta} := \rho_{\text{ref}} \frac{\partial^2 \hat{\mathcal{E}}}{\partial F_{i\beta} \partial F_{k\delta}} \quad (6.9)$$

$$- T \Gamma^{\alpha\beta} := \frac{\partial^2 \hat{\mathcal{E}}}{\partial E_{\alpha\beta} \partial S}. \quad (6.10)$$

The elasticity tensor is related to the *adiabatic elastic moduli*

$$\mathbf{C}^{\alpha\beta\gamma\delta} := \rho_{\text{ref}} \frac{\partial^2 \hat{\mathcal{E}}}{\partial E_{\alpha\beta} \partial E_{\gamma\delta}} \quad (6.11)$$

through the equation  $\mathbf{A}^{i\beta k\delta} = F^i_{;\alpha} \mathbf{C}^{\alpha\beta\gamma\delta} F^k_{;\gamma} + \delta^{ik} S^{\beta\delta}$ .

Furthermore, Eq. (6.3) can be replaced by Eq. (3.18), i.e.,

$$TS^{\bar{\nu}} = -\mu \left[ \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \bar{\Lambda}_{\alpha\beta} + \frac{\partial \hat{\mathcal{E}}}{\partial K} \bar{H} \right]. \quad (6.12)$$

This implies that  $\dot{\Phi} = \dot{\Phi}|_{\text{elastic}} - \mu\alpha$  with

$$\dot{\Phi}|_{\text{elastic}} = \frac{\partial \Phi}{\partial E_{\gamma\delta}} \dot{E}_{\gamma\delta}, \quad (6.13)$$

and

$$\alpha = - \left( \frac{\partial \Phi}{\partial E_{\alpha\beta}^p} - \frac{1}{T} \frac{\partial \Phi}{\partial S} \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \right) \bar{\Lambda}_{\alpha\beta} - \left( \frac{\partial \Phi}{\partial K} - \frac{1}{T} \frac{\partial \Phi}{\partial S} \frac{\partial \hat{\mathcal{E}}}{\partial K} \right) \bar{H}. \quad (6.14)$$

Therefore, when a scale-invariant solution is differentiable, Eqs. (6.1)–(6.5) become

$$-\xi(F'_{\alpha})_{\xi} = N_{\alpha}(V')_{\xi}, \quad (6.15)$$

$$\begin{aligned} -\rho_{\text{rel}} \xi(V')_{\xi} &= N_{\beta} \mathbf{A}^{(\beta)\delta} (F_{j\delta})_{\xi} - \rho_{\text{rel}} T F'_{\alpha} N_{\beta} \Gamma^{\alpha\beta} S_{\xi} \\ &\quad + F'_{\alpha} N_{\beta} \left[ \frac{\partial S^{\alpha\beta}}{\partial E_{\gamma\delta}^p} (E_{\gamma\delta}^p)_{\xi} + \frac{\partial S^{\alpha\beta}}{\partial K} K_{\xi} \right], \end{aligned} \quad (6.16)$$

$$-\xi TS_{\xi} = -\bar{\mu} \left[ \frac{\partial \hat{\mathcal{E}}}{\partial E_{\alpha\beta}^p} \bar{\Lambda}_{\alpha\beta} + \frac{\partial \hat{\mathcal{E}}}{\partial K} \bar{H} \right], \quad (6.17)$$

$$-\xi(E_{\alpha\beta}^p)_{\xi} = \bar{\mu} \bar{\Lambda}_{\alpha\beta}, \quad (6.18)$$

$$-\xi K_{\xi} = \bar{\mu} \bar{H}. \quad (6.19)$$

If we let

$$\bar{\mu}_{\text{trial}} := \frac{1}{\alpha} \frac{\partial \Phi}{\partial E_{\gamma\delta}} F_{j\gamma} N_{\delta} (V')_{\xi}, \quad (6.20)$$

then according to conditions (6.6) and (6.7),  $\bar{\mu} = \bar{\mu}_{\text{trial}}$  when  $\Phi = 0$  and  $\bar{\mu}_{\text{trial}} > 0$ , whereas  $\bar{\mu} = 0$  otherwise.

When  $\xi = 0$ , Eq. (6.15) implies that  $(V')_{\xi} = 0$ , while Eqs. (6.17)–(6.19) are solved by taking  $\bar{\mu} = 0$ . Thus there is no velocity variation and no plastic flow at  $\xi = 0$ . Equation (6.16) represents three linear conditions connecting the remaining unknowns  $(F_{j\delta})_{\xi}$ ,  $S_{\xi}$ ,  $(E_{\gamma\delta}^p)_{\xi}$ , and  $K_{\xi}$ . These conditions are the infinitesimal version of the requirement that the normal

stress  $F^i_\alpha S^{\alpha\beta} N_\beta$  be continuous across a discontinuity propagating at speed  $S_N = 0$  (cf. Eq. (4.6)).

On the other hand, when  $\bar{\xi} \neq 0$ , Eq. (6.16) can be replaced by

$$\rho_{\text{ref}} \bar{\xi}^2 (V^i)_\xi = (\mathbf{A}_N)'_j (V^i)_\xi + \bar{\mu} F^i_\alpha N_\beta \left[ \left( \frac{\partial S^{\alpha\beta}}{\partial E^p_{\gamma\delta}} + \rho_{\text{ref}} \Gamma^{\alpha\beta} \frac{\partial \hat{\mathcal{E}}}{\partial E^p_{\gamma\delta}} \right) \bar{\lambda}_{\gamma\delta} + \left( \frac{\partial S^{\alpha\beta}}{\partial K} + \rho_{\text{ref}} \Gamma^{\alpha\beta} \frac{\partial \hat{\mathcal{E}}}{\partial K} \right) \bar{H} \right], \quad (6.21)$$

where  $\mathbf{A}'_N = N_\beta \mathbf{A}^{i\beta j\delta} N_\delta$  is the *acoustic tensor*. Since  $\bar{\mu}_{\text{trial}}$  is proportional to  $(V^i)_\xi$ , Eq. (6.21) represents an eigenvalue problem: if  $\Phi = 0$  and  $\bar{\mu}_{\text{trial}} > 0$ , then  $\rho_{\text{ref}} \bar{\xi}^2$  is an eigenvalue of the *plastic acoustic tensor*  $(\mathbf{A}^p_N)'_j$ , defined by

$$(\mathbf{A}^p_N)'_j = (\mathbf{A}_N)'_j + \frac{1}{\alpha} F^i_\alpha N_\beta \left[ \left( \frac{\partial S^{\alpha\beta}}{\partial E^p_{\gamma\delta}} + \rho_{\text{ref}} \Gamma^{\alpha\beta} \frac{\partial \hat{\mathcal{E}}}{\partial E^p_{\gamma\delta}} \right) \bar{\lambda}_{\gamma\delta} + \left( \frac{\partial S^{\alpha\beta}}{\partial K} + \rho_{\text{ref}} \Gamma^{\alpha\beta} \frac{\partial \hat{\mathcal{E}}}{\partial K} \right) \bar{H} \right] \frac{\partial \Phi}{\partial E_{\epsilon\eta}} F_{j\epsilon} N_\eta; \quad (6.22)$$

and otherwise  $\rho_{\text{ref}} \bar{\xi}^2$  is an eigenvalue of the (elastic) acoustic tensor  $(\mathbf{A}_N)'_j$ . The eigenvector in each case is  $(V^i)_\xi$ , and once it is known, the remaining unknown quantities  $(F^i_\alpha)_\xi$ ,  $S_\xi$ ,  $(E^p_{\alpha\beta})_\xi$ , and  $K_\xi$  are determined by Eqs. (6.15) and (6.17)–(6.19).

Assuming the eigenvalues of  $(\mathbf{A}_N)'_j$  and  $(\mathbf{A}^p_N)'_j$  to be positive, the wave modes occur in three pairs. For instance, in the elastic flow of an isotropic material, the fastest wave corresponds to longitudinal stress (i.e., pressure) modes, while the two slower waves correspond to radial shear stress (i.e., necking) and angular shear stress (i.e., torsion) modes. In plastic flow ( $\bar{\mu} > 0$ ), waves propagate at speeds corresponding to the eigenvalues of the matrix  $(\mathbf{A}^p_N)'_j$ , which is a rank-one perturbation of  $(\mathbf{A}_N)'_j$ . Typically the constitutive model leads to plastic wave speeds that are smaller than the corresponding elastic wave speeds.

## 7. CONCLUSIONS

In this paper we have formulated the equations governing rate-dependent and rate-independent plastic flow of metals in a conservative form, applicable to both smooth and discontinuous solutions. A preliminary analysis of the wave structures arising in the rate-independent case was

also given. This work provides the foundation for further studies of nonlinear waves in plastic flow, and for the use of this information in numerical computations.

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