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by

Peter W. Glynn and John E. Glynn

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26 November 1992

Abstract

A diffusion approximation for a network of continuous time reservoirs with power law release rules is examined. Under a mild assumption on the inflow processes, we show that for physically reasonable values of the power law constants, the system of processes converges to a multi-dimensional Gaussian diffusion process. We also illustrate how the limiting Gaussian process may be used to compute approximations to the original system of processes. In addition, we study the quality of our approximations by comparing them to results obtained by simulations of the original watershed model. The simulations offer support for the use of the approximation developed here.

1. Introduction

The linear reservoir plays a central role in hydrology (Nash, 1957, 1959). Although this model has been successfully used for many applications, a more general system of reservoirs with a power law storage-runoff relation can more accurately represent the physical characteristics of a natural storage system. The nonlinear reservoir has been examined by Laurensen (1964), Mein et al (1974), Klemeš (1978), Hughes and Murrel (1986), and Glynn (1989)) among others. The natural extension of this model to a nonlinear cascade of reservoirs has also been investigated (Klemeš et al. (1975,1985)). Unfortunately closed form formulae for the statistical properties of these reservoirs are only available for special cases (e.g. the linear reservoir, the fixed release reservoir, the power law release with special inflow distribution and exponent less than one).

Two approaches have generally been used to model the continuous time reservoir. In the first case, the inflows are directly modeled as diffusion processes and the problem requires solving nonlinear stochastic differential equations (Unny (1984), Unny and Karmeshu (1984)). Such models have also been used to study optimal control problems associated with reservoirs (Harrison and Taylor (1978), Pliska (1975)). The second approach assumes that inflows follow a compound Poisson process. The linear cascade with general additive homogeneous inflows has been investigated by Moran (1967). Exact solutions for the single reservoir with a power law release rule are available for the case of compound Poisson inflows where the power is less than one (Harrison and Resnick (1976)). Approximate solutions for general inflows have been obtained by Smith and Yeo (1981) using methods based on Hermite polynomial expansions.

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In this paper, we adopt a different approach to the analysis of complex reservoir models. Based on a careful mathematical analysis of the underlying non-linear reservoir model, we show that our watershed model can be approximated by a certain diffusion process. Such diffusion approximations have been successfully applied to biology, physics, computer science, and many other areas (see, for example, Karlin and Taylor (1981)). In particular, such approximations have been used by Yamada (1983, 1984) to study a single continuous time non-linear finite capacity reservoir with Poisson inflows. In a similar spirit, Harrison and Shepp (1984) develop a diffusion approximation for a cascade of two discrete time reservoirs.

Our diffusion approximation starts from a watershed network of continuous time nonlinear reservoirs with general (possibly correlated) inflows and power law release rules. This work differs from that of Yamada, Harrison, and Shepp both because we study an entire watershed of reservoirs (rather than a single reservoir) and because the power law release rule used here does not fit into their mathematical framework. (In particular, their diffusion limits apply to reservoirs in which the mean inflow rate is close to that of the maximal outflow rate, which is assumed to be finite.) Our approach leads to a diffusion limit that is especially tractable from a computational viewpoint, primarily because the approximating diffusion process turns out to be Gaussian. In addition, the mathematical conditions under which our approximation is valid appears to coincide with a physically reasonable subset of the parameter space that defines the family of power law release rules. Consequently, we believe that this approach offers a comprehensive means of developing approximations to large-scale watersheds.

This paper is organized as follows. In Section 2, we describe the basic watershed model, followed by a description of the diffusion approximation in Section 3. The diffusion approximation requires that the user supply some mean and covariance information for the exogenous inflow processes to the reservoirs in the watershed. Section 4 provides some background on how to calculate these model parameters. In Section 5, we discuss the numerical computation of the transient and steady-state distributions of the approximating diffusion process. Because the process is Gaussian, this reduces to calculating the mean and covariance of the corresponding Gaussian random variables. We show how these quantities can be calculated by solving a certain deterministic system of linear differential equations, the dimension of which equals the number, d, of reservoirs being studied. Section 6 illustrates a different type of computation, namely that associated with developing an approximation for the expected time required for a reservoir to exceed a given level. Here, one needs to solve a (deterministic) linear partial differential equation in at most $d + 1$ variables. In Section 7, we specialize our discussion to that of a watershed consisting of a single reservoir. In this simpler setting, closed-form formulae are available for several quantities of interest. Section 8 offers some numerical evidence which support the quality of our approximations, and Section 9 describes our conclusions.

2. Description of the Model

The drainage network model will consist of d reservoirs. In addition, the following

notational conventions will be used:

$P(j)$ - the collection of reservoirs flowing directly into reservoir j

$R(j)$ - the reservoirs that receive inflow water from reservoir j $(1 \le j \le d)$ (It is assumed throughout this paper that $|R(j)| = 1$),

$I_j(t)$ - the cumulative exogenous inflow into reservoir j $(1 \le j \le d)$, and

$S_j(t)$ - the storage content of reservoir j at time t.

We will assume that the storage levels follow "power law" release rules. The continuity equation then states that

$$S_j(t) = s_j + I_j(t) + \sum_{k \in P(j)} \int_0^t a_k S_k(u)^{b_k} du - \int_0^t a_j S_j(u)^{b_j} du, \quad 1 \le j \le d, \tag{1}$$

where $s_1, a_1, b_1, s_2, a_2, b_2, \ldots, s_d, a_d, b_d$, are positive (deterministic) constants..

We further assume that the input processes are well-behaved, in the sense that the following limits exist and are finite-valued:

$$\begin{aligned} \mu_j &= \lim_{t \to \infty} \frac{1}{t} E I_j(t) \qquad (1 \le j \le d), \text{ and} \\ C_{jk} &= \lim_{t \to \infty} \frac{1}{t} \mathrm{cov}(I_j(t), I_k(t)) \quad (1 \le j, k \le d) \ . \end{aligned} \tag{2}$$

Any input process with asymptotically stationary increments would typically satisfy (2).

An example of such a network of reservoirs is given in Figure 1. In this case the system of equations given by Equation (1) takes the following form:

$$\begin{aligned} S_1(t) &= s_1 + I_1(t) - \int_0^t a_1 S_1(u)^{b_1} du \ , \\ S_2(t) &= s_2 + I_2(t) - \int_0^t a_2 S_2(u)^{b_2} du \ , \text{ and} \\ S_3(t) &= s_3 + I_3(t) + \int_0^t a_1 S_1(u)^{b_1} du + \int_0^t a_2 S_2(u)^{b_2} du - \int_0^t a_3 S_3(u)^{b_3} du. \end{aligned} \tag{3}$$

3. The Diffusion Approximation

The application of the diffusion approximation is quite straightforward and follows a

sequence of steps which will be described in this section. A basic assumption of the approximation is that the a_i $(1 \le i \le d)$ appearing in (1) are small in a certain sense. To calculate the approximations then requires applying the following steps:

A-1. Calculate ε_* where

$$\varepsilon_* \stackrel{\Delta}{=} \max_{1 \le k \le d} a_k^{1/b_k}$$

Our approximation requires that ε_* should be "small". Storage-runoff models assuming a small value of a (and hence small ε_*) have been cited in the literature (see, for example Laurenson, (1964, p. 145) where the release rule, $r(x)$, is given by $r(x) = 0.0445\,x^{1.37}$ and $0.128\,x^{1.37}$) indicating that this model may be a reasonable approximation in some realistic situations. In any case, the storage continuity equations (1) can then be written in the form

$$S_j(t) = s_j + I_j(t) + \sum_{k \in P(j)} \int_0^t \alpha_k \varepsilon_*^{b_k} S_k(u)^{b_k} du - \int_0^t \alpha_j \varepsilon_*^{b_j} S_j(u)^{b_j} du, \tag{4}$$

for $1 \le j \le d$, where $\alpha_k \stackrel{\Delta}{=} a_k \varepsilon_*^{-b_k}$, $(\alpha_k \le 1,\ 1 \le k \le d)$.

A-2. Let $x_j \stackrel{\Delta}{=} \varepsilon_* s_j$, $1 \le j \le d$. Solve the following system of ordinary differential equations:

$$\begin{aligned} &\dot{x}_j(t) = \mu_j + \sum_{k \in P(j)} \alpha_k x_k(t)^{b_k} - \alpha_j x_j(t)^{b_j}, \quad 1 \le j \le d, \\ &\text{and such that } x_j(0) = x_j, \quad 1 \le j \le d. \end{aligned} \tag{5}$$

A-3. Solve the following system of stochastic differential equations:

$$dZ_j(t) = \sum_{k \in P(j)} \alpha_k b_k x_k(t)^{b_k - 1} Z_k(t) dt - \alpha_j b_j x_j(t)^{b_j - 1} Z_j(t) dt + dB_j(t), \qquad (1 \le j \le d) \tag{6}$$

(where $(B_1(t), B_2(t), \ldots, B_d(t))$ is a d- dimensional Brownian motion process with zero drift and covariance matrix $C = (C_{ij} : 1 \le i, j \le d)$).

We note that since (6) comprises a linear system of stochastic differential equations, it follows that the solution $(Z_1(t), \ldots, Z_d(t))$ is a Gaussian process. Such processes possess the convenient property that they are completely characterised by their (time-dependent) mean and covariance functions.

A-4. The approximation is then given by the formula

$$S_j(t) \overset{D}{\approx} \frac{1}{\varepsilon_*} x_j(\varepsilon_* t) + \frac{1}{\sqrt{\varepsilon_*}} Z_j(\varepsilon_* t) \tag{7}$$

($\overset{D}{\approx}$ denotes " has approximately the same distribution as the process")

It is clear that the right hand side of (7) inherits the Gaussian structure of the $Z_i(t)$'s. Furthermore the approximating process appearing in (7) must necessarily be a Markovian diffusion process, since the solution to the stochastic differential equation (6) is Markovian.

4. Calculating the Mean and Covariance Structure of the Exogenous Inflow Process

We now develop two theorectical models in which the mean vector $\mu = (\mu_j : 1 \le j \le d)$ and covariance matrix $C = (C_{ij} : 1 \le i, j \le d)$ can be calculated directly in terms of the underlying "building block" data. We shall need to develop some notation in order to precisely specify our models.

Let T_{kj} be the time at which the k'th rainfall event occurs in the j'th reservoir. (These events correspond to rainfall occurring directly into the j'th reservoir and exclude rainfall in upstream reservoirs.) We then set $T_{0j} = 0$ and $\tau_{kj} = T_{kj} - T_{k-1,j}$. If we let $N_j(t)$ be the number of events to occur by time t in the j'th reservoir, it follows that $N_j(t) = \max \{n \ge 0 : T_{nj} \le t\}$. Let X_{kj} be the amount of rainfall deposited into the j'th reservoir that propogates into the reservoir; and we assume for simplicity that propogation is immediate. Then the cumulative exogenous inflow into reservoir j up to time t is given by $I_j(t) = \sum_{k=1}^{N_j(t)} X_{kj}$.

Model 1: Assume that each of the d inter-event time sequences $(\tau_{kj} : k \ge 1)$ and d rainfall magnitude sequences $(X_{kj} : k \ge 1)$ are independent and identically distributed (i.i.d.), and further assume that the $2d$ seqeunces are independent of one another. One implication of this assumption is that the rainfall patterns in each of the d subwatersheds are statistically independent of one another. This is clearly only reasonable for a reservoir system covering a large geographic area and, even in that case, only in a very approximate sense. Let

$$\lambda_j = 1/E\tau_{kj}$$
$$\sigma_j^2 = \mathrm{var}(\tau_{kj})$$
$$\nu_j = EX_{kj}$$
$$\eta_j^2 = \mathrm{var}(X_{kj}) \quad .$$

Note that $E[I_j(t) \mid N_j(t)] = N_j(t)\nu_j$. Since $N_j(t)$ is a so-called renewal process, it follows that $N_j(t) \sim \lambda_j t$ (See Karlin and Taylor (1975)) and hence $EI_j(t)/t \to \lambda_j\nu_j$ as $t \to \infty$. For the covariances, note that the reservoir independence discussed earlier implies that $C_{jk} = 0$ for $j \neq k$. As for C_{jj}, we use the variance identity

$$\mathrm{var}(I_j(t)) = E[\mathrm{var}(I_j(t)|N_j(t))] + \mathrm{var}(E(I_j(t)|N_j(t))).$$

(see, for example, Bratley, Fox, and Schrage (1987)). Since $\mathrm{var}(I_j(t)|N_j(t)) = N_j(t)\eta_j^2$, it is evident that $\mathrm{var}(I_j(t)) = \eta_j^2 EN_j(t) + \nu_j^2 \mathrm{var}(N_j(t))$. Again, standard renewal theory implies that $\mathrm{var}(N_j(t)) \sim \lambda_j^3\sigma_j^2 t$ as $t \to \infty$ (see p. 208, Karlin and Taylor (1975)), and consequently,

$$C_{jj} = \lambda_j[\eta_j^2 + \lambda_j^2\nu_j^2\sigma_j^2]. \tag{8}$$

Model 2. In this model we will permit correlation between the exogenous inflows into each of the d reservoirs. In fact, we will assume that a common sequence of rainfall events affects all d reservoirs. More precisely, we will suppose that $T_{kj} = T_{k1}$ for all $1 \le j \le d$, $k \ge 1$. Of course, in this setting, it is no longer reasonable to assume that the rainfall amounts X_{kj} $(1 \le k \le d)$ are independent of one another. Let $\tau_k = \tau_{k1}$.

We shall require that $(\tau_k : k \ge 1)$ is an i.i.d. sequence independent of $\{(X_{kj} : 1 \le k \le d), j \ge 1\}$, which is itself assumed to be an i.i.d. sequence (of d-vectors). Let

$$\lambda = 1/E\tau_k$$
$$\sigma^2 = \mathrm{var}(\tau_k)$$
$$\nu_k = EX_{kj}$$
$$\eta_{jk} = \mathrm{cov}(X_{ji}, X_{ki}) \quad .$$

Then the mean and the covariance for the Gaussian inflow process are given by

$$\mu_j = \lambda\nu_j, \quad \text{and}$$
$$C_{jk} = \lambda[\eta_{jk} + \lambda\nu_j\nu_k\sigma^2] \quad . \tag{9}$$

We conclude this section with a brief discussion of how the mean vector μ and the covariance matrix C may be statistically estimated directly from actual measured exogenous inflow data. The statistical discussion assumes no model structure such as that associated with Models 1 and 2. (If we want to exploit such a model structure a more efficient statistical estimator for C could be developed.) Suppose that $(I_j(t) : 0 \le t \le T,\ 1 \le j \le d)$ is observed. Then, for $1 \le j \le d$, $\hat{\mu}_j = I_j(T)/T$ is an appropriate estimator for μ_j. To estimate the covariance matrix C, let $\dot{I}_j(t)$ be the measured rate at which the exogenous inflow into the j'th reservoir occurs at time t. It is often reasonable to assume that the d-dimensional process $((\dot{I}_1, \ldots, \dot{I}_d(t)) : t \ge 0)$ is stationary. In this case,

$$C_{jk} = \int_0^\infty \mathrm{cov}(\dot{I}_j(0),\dot{I}_k(t))dt + \int_0^\infty \mathrm{cov}(\dot{I}_k(0),\dot{I}_j(t))dt.$$

We note that C_{jk} is related to the cross-spectral density of the two processes $(\dot{I}_j(t) : t \ge 0)$ and $(\dot{I}_k(t) : t \ge 0)$ evaluated at the frequency 0. Consequently, estimators from the theory of spectral density estimation can be used to calculate $\hat{C}_{jk}$ for $1 \le j, k \le d$. (See Brillinger (1981) for details.) One class of such estimators takes the form

$$\hat{C}_{jk} = \int_0^T w_T(t)\, \hat{\mathrm{cov}}(\dot{I}_j(0),\dot{I}_k(t))dt + \int_0^T w_T(t)\, \hat{\mathrm{cov}}(\dot{I}_k(0),\dot{I}_j(t))dt, \tag{10}$$

where $w_T(t)$ is an appropriately chosen weighting function having the properties that $w_T(t) \to 0$ as $t \to \infty$ and $w_T(t) \to 1$ as $T \to \infty$, and where

$$\hat{\mathrm{cov}}(\dot{I}_j(0),\dot{I}_k(t)) = \frac{1}{T-t}\int_0^{T-t}(\dot{I}_j(s) - \hat{\mu}_j)(\dot{I}_k(s+t) - \hat{\mu}_k)ds.$$

5. Distributional Characterization of the Approximation

As noted in Section 3, the diffusion approximation for the vector storage process $(S(t) : t \ge 0)$ is a Gaussian process, and so it follows that

$$S(t) \overset{D}{\sim} N(m(t),K(t))$$

where $m(t)$ and $K(t)$ are (respectively) the mean and the covariance of the d-dimensional normal random variable that approximates $S(t)$. We will now describe, in a step-by-step fashion, how to calculate the functions $m(t)$ and $K(t)$, thereby completing the characterization of the approximation. (For additional detail, we refer the reader to Chapter 8 of Arnold (1974).)

Let $A(t) = (A_{ij}(t) : 1 \le i, j \le d)$ be the $d \times d$ matrix with elements given by

$$A_{kj}(t) = \begin{cases} \alpha_k b_k x_k(t)^{b_k - 1}, & \text{if } k \in P(j), \\ -\alpha_j b_j x_j(t)^{b_j - 1}, & \text{if } k = j, \text{ and} \\ 0, & \text{else,} \end{cases}$$

where $(x_1(t), \ldots, x_d(t))'$ is the solution of the system (5) of ordinary differential equations.

The rows of the matrix A are numbered in such a way that the further upstream the reservoir, the lower its number; in other words we require that $k \in P(j) \Rightarrow k < j$ for $1 \le j \le d$. This numbering convention will be used throughout the remainder of the paper.

Our procedure is to:

B-1. Solve the following deterministic matrix-valued differential equation for $\Phi(t)$:

$$\dot{\Phi}(t) = A(t)\,\Phi(t)$$
$$\text{such that } \Phi(0) = I.$$

B-2. Calculate

$$\Gamma(t) = \Phi(t) \int_0^t \Phi(u)^{-1} C (\Phi(u)^{-1})' du\, \Phi(t)'$$

where A' denotes the transpose of A.

The matrix $\Gamma(t)$ can alternatively be calculated by solving the matrix-valued differential equation

$$\dot{\Gamma}(t) = A(t)\Gamma(t) + \Gamma(t)A(t)' + C \qquad (11)$$
$$\text{such that } \Gamma(0) = 0.$$

B-3. Then,

$$m(t) = \frac{1}{\varepsilon_*} x(\varepsilon_* t)$$
$$K(t) = \frac{1}{\varepsilon_*} \Gamma(\varepsilon_* t). \qquad (12)$$

Of particular interest is the steady-state distribution of the storage. It is easily seen that $m(t) \to m$ as $t \to \infty$, where $m = (m_j : 1 \le j \le d)$ is the stable solution of the differential equation (5) and is given by

$$m_j = \left(\frac{\mu_j + \sum_{k \in P(j)} \alpha_k m_k^{b_k}}{\alpha_j} \right)^{1/b_j}. \tag{13}$$

We note that the above system can be solved recursively in closed-form by solving first for the furthest upstream reservoirs and then working one's way downstream.

It is clear that $A(t) \to A$ as $t \to \infty$, where

$$A_{kj} = \begin{cases} \alpha_k b_k m_k^{b_k - 1}, & \text{if } k \in P(j), \\ -\alpha_j b_j m_j^{b_j - 1}, & \text{if } k = j, \text{ and} \\ 0, & \text{else} \end{cases}.$$

It is then reasonable (see (11)) to expect that $\Gamma(t) \to \Gamma$ as $t \to \infty$, where Γ solves

$$0 = A\Gamma + \Gamma A' + C. \tag{14}$$

Since A is lower triangular (due to the numbering system adopted at the beginning of this section), and has negative diagonal elements, it therefore has strictly negative eigenvalues and is hence stable. Consequently a unique solution to the above matrix equation is guaranteed (see Arnold (1974)).

If $S(\infty)$ is the d-dimensional random variable representing the steady-state vector storage of the system, the above discussion then suggests using the approximation

$$S(\infty) \overset{D}{\approx} N(m/\varepsilon, \Gamma/\varepsilon). \tag{15}$$

6. An Approximation for the Expected Time for a Reservoir to Exceed a Given Level

In this section, we exploit the fact that our approximating process is a diffusion. (This is in contrast to Section 5, where our analysis relied on the Gaussian character of the approximation.) The Markov structure of a diffusion process permits one to develop (deterministic) partial differential equations (P.D.E.'s) that describe a variety of different quantitative characteristics of the process. We shall illustrate the power of this property by

applying this idea to a specific calculation, namely the development of an approximation for the expected time for a reservoir to exceed a given level. Specifically, we shall be interested in calculating the expected time required for reservoir d's content to exceed level β. (We note that any other reservoir can be viewed as the furthest downstream reservoir to the reservoirs that feed into it. Consequently, if one is interested in reservoir j ($j \neq d$), one can replace the original system by the appropriate smaller system of reservoirs. Thus, our assumption is without loss of generality.)

If T is the time required for reservoir d's level to exceed β, then T can be defined as $T = \inf\{t \geq 0 : S_d(t) \geq \beta\}$.. Using the approximation (7), we conclude that

$$\begin{aligned} T &\overset{D}{\approx} \inf\{t \geq 0 : x_d(\varepsilon_\bullet t)/\varepsilon_\bullet + Z_d(\varepsilon_\bullet t)/\sqrt{\varepsilon_\bullet} \geq \beta\} \\ &= \frac{1}{\varepsilon_\bullet}\inf\{u \geq 0 : x_d(u)/\varepsilon_\bullet + Z_d(u)/\sqrt{\varepsilon_\bullet} \geq \beta\} \\ &= \frac{1}{\varepsilon_\bullet}\inf\{u \geq 0 : W_d(u) \geq \beta\} \end{aligned}$$

where $W(u) = (W_1(u), \ldots, W_d(u)) : u \geq 0)$ is the diffusion process with i'th component given by

$$W_i(u) = \frac{1}{\varepsilon_\bullet}x_i(u) + \frac{1}{\sqrt{\varepsilon_\bullet}}Z_i(u).$$

Let $T_t = \inf\{u \geq 0 : W_d(u + t) \geq \beta\}$ be the elapsed time for reservoir d's level to exceed β, taken relative to a re-defined time origin at t. Then, we can set $u(w,t) = E[T_t \mid W(t) = w]$ for $t \geq 0$, $w \in \mathbb{R}^d$. The diffusion $W(\cdot)$ has infinitesimal generator L given by the linear partial differential operator

$$L = \sum_{i=1}^{d}\left(\frac{\dot{x}_i(t)}{\varepsilon_\bullet} + \sum_{k\in P(i)}\frac{\alpha_k b_k x_k(t)^{b_k-1}w_k}{\sqrt{\varepsilon_\bullet}} - \frac{\alpha_i b_i x_i(t)^{b_i-1}w_i}{\sqrt{\varepsilon_\bullet}}\right)\frac{\partial u}{\partial w_i} + \frac{1}{2}\sum_{i,j=1}^{d}\frac{C_{ij}}{\varepsilon_\bullet}\frac{\partial^2 u}{\partial w_i \partial w_j}.$$

The theory of diffusion processes (see Karlin and Taylor (1981)) establishes that u satisfies the P.D.E. given by

$$\frac{\partial}{\partial t}u + Lu = -1 \qquad (16)$$

subject to $u(w,t) = 0$ for $w_d \geq \beta$. Hence, we can calculate an approximation to the expected time $E(T)$ required for reservoir d's level to exceed β by following the procedure:

C-1. Solve the (deterministic) P.D.E.

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \left(\frac{\dot{x}_i(t)}{\varepsilon_*} + \sum_{k \in P(i)} \frac{\alpha_k b_k x_k(t)^{b_k - 1} w_k}{\sqrt{\varepsilon_*}} - \frac{\alpha_j b_j x_j(t)^{b_j - 1} w_j}{\sqrt{\varepsilon_*}} \right) \frac{\partial u}{\partial w_j} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{C_{ij}}{\varepsilon_*} \frac{\partial^2 u}{\partial w_i \partial w_j} = -1$$

for $u(w,t)$ in the domain $w \in \mathbf{R}^d$, $t \geq 0$, subject to $u(w,t) = 0$ for $w_d \geq \beta$.

C-2. Then $E(T) \approx \frac{1}{\varepsilon_*} u(s_1, \ldots, s_d, 0)$. (17)

7. The Single Reservoir Case.

The mass balance equation for a single reservoir is given by the following relation

$$S(t) = s_0 + I(t) - \int_0^t a S(u)^b \, du$$

where s_0, a, $b > 0$, and $I(t)$ is a non-decreasing process that represents the cumulative input to time t. As required by assumption (2), we assume that the limits $\mu = \lim_{t \to \infty} EI(t)/t$ and $C = \lim_{t \to \infty} \operatorname{var} I(t)/t$ exist.

Following the procedure outlined in Section 3, it is first necessary to solve the ordinary differential equation given by

$$\dot{x}(t) = \mu - \alpha x(t)^b, \quad \text{with}$$
$$x(0) = x_0$$

where $x_0 = a^{1/b} s_0$ and $\alpha = 1$. The solution to the differential equation is then given by

$$x(t) = \varphi^{-1}(t), \quad \text{where}$$

$$\varphi(t) = \int_{x_0}^{x} \frac{du}{\mu - u^b} .$$

In particular, if $b = 1$, then

$$\varphi(x) = \log\left(\frac{\mu - x_0}{\mu - x}\right)$$

and so the solution of the differential equation becomes $x(t) = \mu - e^{-t}(\mu - x_0)$. The next step in the procedure is to solve the stochastic differential equation for the diffusion approximation which is given by $dZ(t) = -bx(t)^{b-1} Z(t)\,dt + dB(t)$ where $B(\cdot)$ is a Brownian motion having a variance parameter C. Then, in the linear case, we can apply (11) to obtain the result $\Gamma(t) = C(1 - \exp(-2t))/2$. Since $\varepsilon_\infty = a$, the transient approximation (7) becomes

$$S(t) \overset{D}{\approx} N((\mu - e^{-at}(\mu - s_0 a))/a,\ C(1 - e^{-2at})/2a) \quad . \tag{18}$$

In the case where b is not equal to one, we can easily supply a closed-form solution for the steady-state storage $S(\infty)$. In particular, (14) takes the form $\Gamma = C\mu^{(1-b)/b}/2b$. Then (15) yields

$$S(\infty) \overset{D}{\approx} N\left(\left(\frac{\mu}{a}\right)^{1/b},\ C\left(\frac{\mu}{a}\right)^{1/b} \frac{1}{2b\mu}\right) \tag{19}$$

8. Numerical Examples

We now consider two numerical examples. In the first case we will compare the diffusion approximation to exact known results from Harrison and Resnick (1976), and in the second case we will compare the diffusion approximation to simulations for a cascade of two reservoirs.

Example 1. The behaviour of the system for a single reservoir with Poisson inflows and exponentially distributed jump sizes will first be examined. The statistics of the inflow process can then be calculated as in Section (4). This case has been solved in closed form by Harrison and Resnick in the special case where the storage system empties in finite time i.e. for the case where $b < 1$. In this case define two functions $\rho(x)$ and $K(x)$ by the formulae

$$\rho(x) = \lambda/(\nu\alpha x^b), \quad \text{and}$$
$$K(x) = \nu\,\rho(x) \exp(-\nu \int_0^x (1 - \rho(t))\,dt) \quad ,$$

where λ and ν are defined as in Model 1 of Section 4.

The steady-state density, $f_s(x)$, of the storage process is then given by the formula

$$f_s(x) = K(x)/(1 + \int_0^\infty K(t)dt) \quad . \tag{20}$$

Similarly the density of the runoff process $f_r(x)$ is given by the formula $f_r(x) = abx^{b-1}f_s(ax^b)$. It is clear from these formulae and from the fact that the storage is depleted in finite time that the distributions of the storage and runoff processes have atoms at 0. A comparison of the densities of storage for the diffusion process and for the exact solution of the storage process for various values of a is shown in Figure 2.

Example 2. Consider a cascade of two storage systems fed by a single exogenous inflow process $I_1(\cdot)$ into the first storage system. In this case the matrix A (see Section 5) takes the form

$$A = \begin{pmatrix} -\alpha_1 b_1 m_1^{b_1-1} & 0 \\ \alpha_1 b_1 m_1^{b_1-1} & -\alpha_2 b_2 m_2^{b_2-1} \end{pmatrix} \quad . \tag{21}$$

We can calculate the steady-state variance of the storage process by solving Equation (14) for Γ, which results in the following formula

$$\Gamma = \frac{-C_{11}}{2} \begin{pmatrix} \dfrac{1}{A_{11}} & \dfrac{1}{A_{11}+A_{22}} \\ \dfrac{1}{A_{11}+A_{22}} & \dfrac{A_{11}}{A_{22}(A_{11}+A_{22})} \end{pmatrix} \tag{22}$$

where $C_{11} = \lim_{t\to\infty} \operatorname{var} I_1(t)/t$ and the A_{ij}'s are the corresponding elements of the matrix A defined in (21). The steady-state distribution of this non-linear cascade can then be approximated as in (15).

The diffusion approximation for the cascade was compared with simulation runs for the process. The simulation model was based on approximating the continuous time model given by equation (1) with the following approximate discrete time model

$$\begin{aligned} S_1(n+1) - S_1(n) &= I_n(\Delta t) - a_1 \Delta t S_1(n+1)^{b_1} \\ S_2(n+1) - S_2(n) &= a_1 \Delta t S_1(n+1)^{b_1} - a_2 \Delta t S_2(n+1)^{b_2} \quad . \end{aligned} \tag{23}$$

where $\{S_1(n) : n \geq 0\}$ and $\{S_2(n) : n \geq 0\}$ are the storages of the first and second system

respectively. Such a discretization is necessary in order to solve the system of integral equations (1) numerically. The inflow process $\{I_n(\Delta t) : n \geq 0\}$ was simulated using the formula

$$I_n(\Delta t) = \sum_{j=1}^{N_n(\Delta t)} G_{nj} \tag{24}$$

where the G_{nj} are independent and identically distributed Gamma random variables and the $N_n(\Delta t)$ are independent and identically distributed Poisson random variables with mean $\lambda \Delta t$.

In our cascade example, the parameter λ for the Poisson process $\{N_j(\Delta t) : j \geq 0\}$ was assumed to be one, as was the mean, ν, and the standard deviation, η, of the gamma deviates, N_{nj}. In this case, C_{11} is equal to two, and C_{12}, C_{21}, and C_{22} are all zero. The time parameter Δt was put equal to 0.1 and the storage process simulation was run for 10000 time steps (to $t = 1000$). Examples of the resulting sample paths for runoff from the two storage systems are shown in Figures 3 and 4, and the results of simulations are provided in Figure 5. Confidence intervals for the estimated parameters were calculated using an assumed t distribution based on ten replications of each simulation. The justification for this confidence interval methodology is given in Law and Kelton (1981).

9. Conclusions

In this paper, we have developed a diffusion approximation for a watershed consisting of d reservoirs with non-linear power law release rules. Our simulation experiments suggest that it works reasonably well for some values of the release rule parameters which are physically plausible. Of course, the assumptions guaranteeing the validity of our approximation are not universally applicable, and further research is therefore necessary to fully delineate its domain of applicability.

Appendix - Justification for the Diffusion Approximation

We wish to show that when the parameter ε is small, then the diffusion approximation is reasonable. To formulate this problem mathematically, let

$$S_j(\varepsilon, t) = S_j(\varepsilon, 0) + I_j(t) + \sum_{k \in P(j)} \int_0^t \alpha_k \varepsilon^{b_k} S_k(\varepsilon, u)^{b_k} du - \int_0^t \alpha_j \varepsilon^{b_j} S_j(\varepsilon, u)^{b_j} du, \quad 1 \le j \le d, \tag{25}$$

be the storage contents processes associated with the parameter ε. We need to show that $S_j(\varepsilon, t)$ can be approximated as in (A-4) when $\varepsilon \downarrow 0$.

To study the behaviour of $\{S_j(\varepsilon, t) : 1 \le j \le d, t \ge 0\}$ as $\varepsilon \downarrow 0$, we need to strengthen the hypothesis (2) on the inflow processes somewhat. (In particular, we need to say something about the distribution of the inflow processes.) In addition, since our approximation requires a re-scaling of the spatial co-ordinates, our initial condition for $S_j(\varepsilon, 0)$ ought to reflect this. Our precise mathematical requirement is:

$$\begin{aligned} &\varepsilon^{-1/2} [\varepsilon S_1(\varepsilon,0) - x_1, \ldots, \varepsilon S_d(\varepsilon,0) - x_d, \varepsilon I_1(t/\varepsilon) - \mu_1 t, \ldots, \varepsilon I_d(t/\varepsilon) - \mu_d t] \\ &\quad \Rightarrow [Z_1(0), \ldots, Z_d(0), B_1(t), \ldots, B_d(t)], \end{aligned} \tag{26}$$

as $\varepsilon \downarrow 0$, where $\Rightarrow$ denotes convergence in distribution and the process $(B_1(t), \cdots, B_d(t))$ is assumed to be a Brownian motion with zero mean and covariance matrix C. From a practical standpoint, the stengthening of (2) to the above distributional requirement is quite mild. Virtually all stochastic processes that exhibit the behaviour (2) also exhibit the Brownian behaviour assumed above (Ethier and Kurtz (1986)).

Under the assumption (26), we can now proceed to analyze $S_j(\varepsilon, \cdot)$ as $\varepsilon \downarrow 0$. Note that (25) can be re-written as

$$\varepsilon S_j(\varepsilon,t/\varepsilon) = \varepsilon S_j(\varepsilon,0) + \varepsilon I_j(t/\varepsilon) + \sum_{k \in P(j)} \int_0^t \alpha_k (\varepsilon S_k(\varepsilon,u/\varepsilon))^{b_k} du - \int_0^t \alpha_j (\varepsilon S_j(\varepsilon,u/\varepsilon))^{b_j} du \quad .$$

By assumption, we have $\varepsilon I_j(t/\varepsilon) \Rightarrow \mu_j t$ as $\varepsilon \downarrow 0$, and so assuming in addition that the limit process

$$x_j(t) = \lim_{\varepsilon \downarrow 0} \varepsilon S_j(\varepsilon, t/\varepsilon) \tag{27}$$

exists for $1 \le j \le d$, the limit must satisfy the integral equation

$$x_j(t) = x_j + \mu_j t + \sum_{k \in P(j)} \int_0^t \alpha_k x_k(u)^{b_k} du - \int_0^t \alpha_j x_j(u)^{b_j} du, \quad 1 \le j \le d.$$

Expressed as a differential equation, the above result becomes

$$\dot{x}_j(t) = \mu_j + \sum_{k \in P(j)} \alpha_k x_k(t)^{b_k} - \alpha_j x_j(t)^{b_j}, \quad 1 \le j \le d, \quad \text{and}$$
$$x_j(0) = x_j, \quad 1 \le j \le d.$$

In order to establish the existence of the limit, one needs to prove the relative compactness of the family of processes $\{\varepsilon S_j(\varepsilon, t/\varepsilon) : \varepsilon \ge 0\}$. This can be done by appealing to standard tools from, for example, Ethier and Kurtz (1986). We can view (27) as a law of large numbers for $S_j(\varepsilon, \cdot)$. To develop a "Central Limit Theorem" we re-write (25) as

$$\begin{aligned}\varepsilon^{-1/2}[\varepsilon S_j(\varepsilon, t/\varepsilon) - x_j(t)] &= \varepsilon^{-1/2}[\varepsilon S_j(\varepsilon,0) - x_j] + \varepsilon^{-1/2}[\varepsilon I_j(t/\varepsilon) - \mu_j t] \\ &+ \sum_{k \in P(j)} \int_0^t \alpha_k \varepsilon^{-1/2}[(\varepsilon S_k(\varepsilon,u/\varepsilon))^{b_k} - x_k(u)^{b_k}]du \\ &- \int_0^t \alpha_j \varepsilon^{-1/2}[(\varepsilon S_j(\varepsilon,u/\varepsilon))^{b_j} - x_j(u)^{b_j}]du.\end{aligned}$$

A Taylor series expansion to one term then establishes that

$$\varepsilon^{-1/2}[(\varepsilon S_k(\varepsilon,u/\varepsilon))^{b_k} - x_k(u)^{b_k}] = b_k \zeta_k(\varepsilon,u)^{b_k - 1}[\varepsilon^{-1/2}(\varepsilon S_k(\varepsilon,u,\varepsilon) - x_k(u))]$$

where $\zeta_k(\varepsilon,u)$ lies between $x_k(u)$ and $\varepsilon S_k(\varepsilon,u/\varepsilon)$ and hence tends to $x_k(u)$ as $\varepsilon \downarrow 0$.

Assume, for the moment, that we can establish the existence of the limit

$$\varepsilon^{-1/2}(\varepsilon S_1(\varepsilon,t/\varepsilon) - x_1(t), \cdots, \varepsilon S_d(\varepsilon,t/\varepsilon) - x_d(t)) \Rightarrow (Z_1(t), \cdots, Z_d(t)) \quad .$$

Since $\varepsilon^{-1/2}[\varepsilon S_1(\varepsilon,0) - x_1, \cdots, \varepsilon S_d(\varepsilon,0) - x_d] \Rightarrow [Z_1(0), \cdots, Z_d(0)]$, the limit process $(Z_1(t), \ldots, Z_d(t))$ must then satisfy the following stochastic differential equation

$$Z_j(t) = Z_j(0) + B_j(t) + \sum_{k \in P(j)} \int_0^t \alpha_k b_k x_k(u)^{b_k - 1} Z_k(u)du - \int_0^t \alpha_j b_j x_j(u)^{b_j - 1} Z_j(u)du \quad .$$

As before, the existence of the limit may be justified by using a relative compactness argument (Ethier and Kurtz (1986)).

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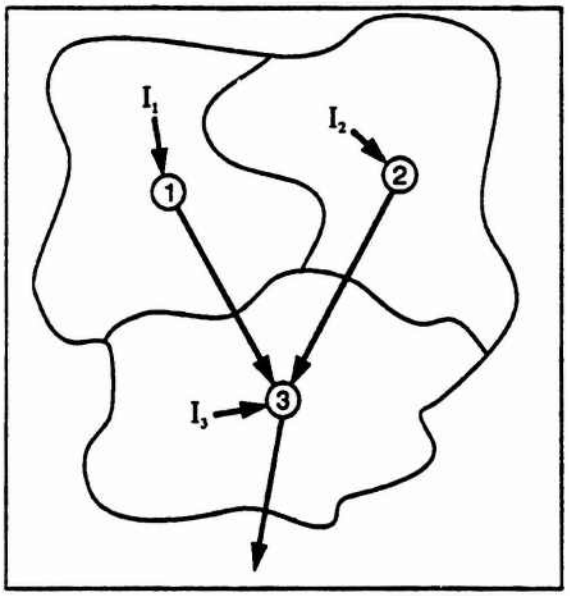

Figure 1. A storage system consisting of three reservoirs where reservoirs one and two feed into reservoir three.

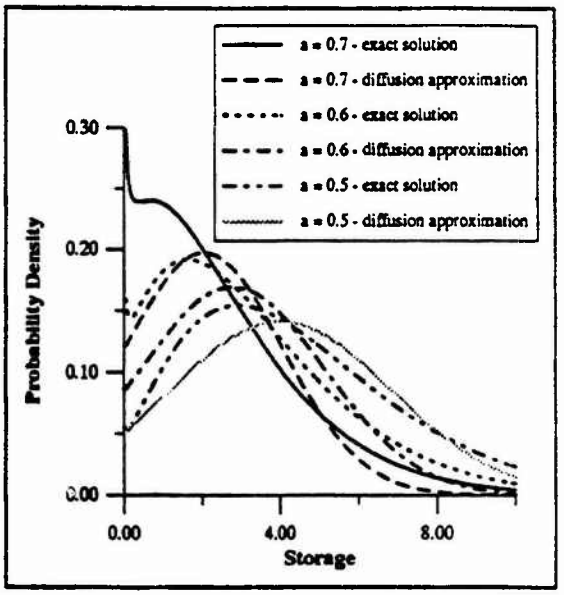

Figure 2. A Comparison of the exact solution and the diffusion approximation for a single reservoir (b = 0.5, λ = 1.0, ν = 1.0)

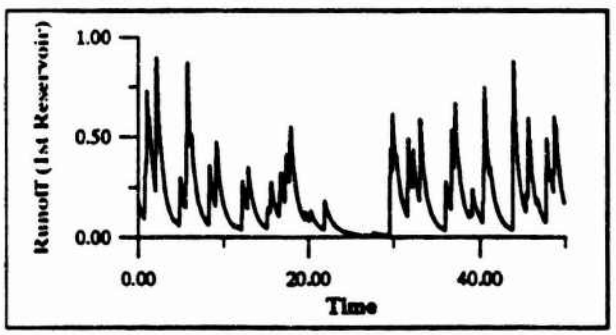

Figure 3. Sample path of first reservoir fed by compound Poisson inflows (rate 1.0) with gamma distributed jumps (mean jump size 2.0) and with Δt = 0.1, a_1 = 0.5, b_1 = 1.5.

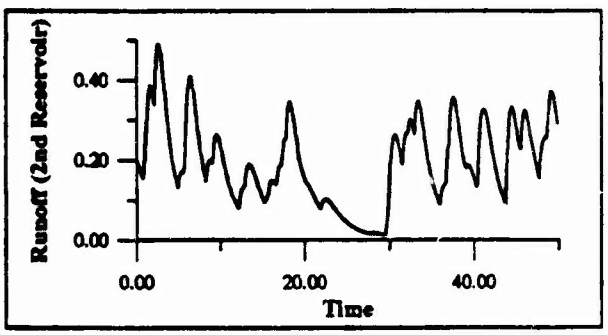

Figure 4. Sample path of second reservoir with same parameters as first reservoir (see Figure 3)

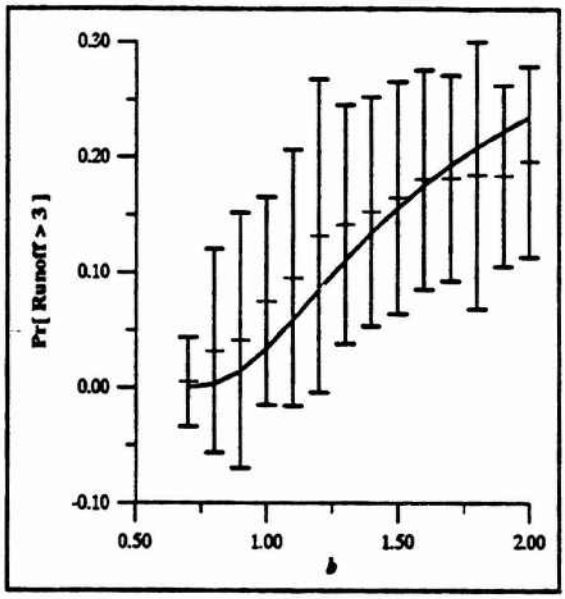

Figure 5. Exceedence probabilities for runoff from second reservoir with $a_1 = a_2 = 0.3$; Solid line represents diffusion and error bars represent 95% C.I. for simulation.