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A. H. Nuttall

NAVAL UNDERWATER SYSTEMS CENTER

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Foreword

This collection of technical reports and technical memoranda deals with the following topics: accurate efficient evaluation of cumulative or exceedance probability distributions directly from characteristic functions; determination of the performance of general second-order processors with nonstationary Gaussian inputs; the exact operating characteristics of a sum of an envelope-detected narrowband Gaussian process and sine wave; the resolution of the right-left ambiguity of a randomly moving line array; the operating characteristics of a cross-correlator with sample mean removal; statistical characterization of the under-ice profile; and the evaluation of densities and distributions from knowledge of high-order moments.

Some of the material presented here is based heavily on earlier work by the author, which can be found in the following volumes in addition to the referenced technical reports:

Performance of Detection and Communication Systems, NUSC Scientific and Engineering Studies, 1974;

Spectral Estimation, NUSC Scientific and Engineering Studies, 1977;

Coherence Estimation, NUSC Scientific and Engineering Studies, 1979;

Receiver Performance Evaluation and Spectral Analysis, NUSC Scientific and Engineering Studies, 1981; and

Signal Processing Studies, NUSC Scientific and Engineering Studies, 1983.

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Table of Contents

Technical Report 7023
1 October 1983

Accurate Efficient Evaluation Of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions

A. H. Nuttall

ABSTRACT

An accurate and efficient method of evaluating the entire cumulative or exceedance probability distribution, via one fast Fourier transform of the sampled characteristic function, is presented. The sampling rate applied to the characteristic function results in aliasing of the probability density function, while the limited extent of the sampling gives rise to a systematic disturbance in the calculated probability distribution. Both types of errors are easily recognizable and can be controlled by a trial and error procedure whereby the calculated distributions are plotted for observation and modification.

The size of the fast Fourier transform determines the number of distribution values available, but has no effect on the accuracy of the result. Regardless of the number of characteristic function evaluations required for accurate results, the storage required is just that corresponding to the size of the fast Fourier transform.

A program for the procedure is presented and the inputs required of the user are indicated. Several representative examples and plots illustrate the utility of the approach.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

x	random variable of primary interest
$f_x(\xi)$	characteristic function of random variable x
$p_x(v)$	probability density function of random variable x
b	bias or shift added to x
y	random variable $y=x+b$
$f_y(\xi)$	characteristic function of random variable y
μ_x	mean of random variable x
$p_y(v)$	probability density function of random variable y
$P_x(v)$	cumulative distribution function of random variable x
$P_y(v)$	cumulative distribution function of random variable y
$g(\xi,v)$	auxiliary function (6)
Im	imaginary part
μ_y	mean of random variable y
$C(v)$	right-hand side of (8)
Δ	sampling increment in argument ξ of characteristic function
$\tilde{p}_y(v)$	aliased version of $p_y(v)$; (12)
$\delta_\Delta(\xi)$	impulse train (13)
M	size of FFT employed
z_n	sequence of characteristic function samples; (20)
$\hat{z}_n$	collapsed sequence; (21)
L	limit on integral of characteristic function; (28)
N	number of nonzero z_n; $N=L/\Delta$
overbar	ensemble average

ACCURATE EFFICIENT EVALUATION OF CUMULATIVE OR EXCEEDANCE PROBABILITY DISTRIBUTIONS DIRECTLY FROM CHARACTERISTIC FUNCTIONS

INTRODUCTION

The performance of a signal processor can often be evaluated in terms of the characteristic function of the decision variable, either numerically or in closed form; see for example, refs. 1 and 2. However, a closed form for the corresponding probability density function or cumulative distribution function is seldom available, and numerical procedures must be employed. Several such procedures have been published in the literature, refs. 3-8. However they have limited accuracy or they require extensive storage or analytical manipulations and calculations.

We present a technique which is limited in accuracy only by the round-off noise of the computer or by the errors of the special functions required in the characteristic function calculation. The amount of storage depends only on the number of cumulative or exceedance distribution function values requested and does not influence the accuracy of the final probability values. The entire cumulative and exceedance distribution function values result as the output of one fast Fourier transform (FFT). The size of the FFT dictates the storage required and the spacing of the calculated probability values, but not their accuracy.

The addition and subtraction of integrand functions given in ref. 7 can be entirely circumvented and yet enable use of an FFT, through proper manipulation of the origin contribution of the characteristic function. Specific connections with past results will be noted at appropriate points in the derivations.

DERIVATION OF PROCEDURE

Shifted Random Variable

The primary random variable of interest is the real quantity x with given characteristic function $f_x(\xi)$ which is related to the probability density function p_x of random variable x via Fourier transform *

$$f_x(\xi) = \int dv \exp(i\xi v)\ p_x(v). \tag{1}$$

We define secondary random variable y as

$$y = x+b, \tag{2}$$

where bias (shift) b is a constant, chosen such that random variable y has insignificant probability of being less than zero. However, we also pick b as small as possible, so that the characteristic function of y,

$$f_y(\xi) = f_x(\xi) \exp(ib\xi), \tag{3}$$

will vary slowly with ξ. In fact, b can be negative, as for example if x were limited to values larger than some positive threshold. The approach here is not limited to positive random variables x, as were some of the results in ref. 7, but is applicable to any random variable distribution.

By way of example, for an exponential probability density function for random variable x, we choose b=0; while for a Gaussian random variable, $b=-\mu_x+8\sigma_x$ yields a probability less than 1E-15 of y being negative. The probability density function of random variable y therefore appears as depicted in figure 1.

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* Integrals and sums without limits are over $(-\infty, +\infty)$.

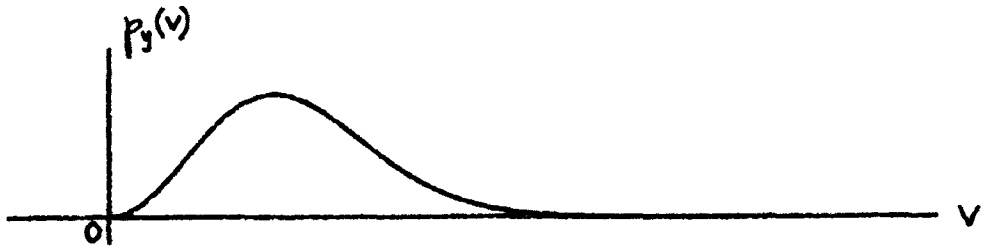

Figure 1. Probability Density Function of Secondary Random Variable y

The cumulative distribution functions of random variables y and x are related according to

$$\int_{-\infty}^{v} dt\, p_y(t) = P_y(v) = P_x(v-b); \quad P_x(v) = P_y(v+b). \tag{4}$$

Thus we can inspect $P_x(v)$ in the neighborhood of v=-b (the lower edge of interest of x) by looking at cumulative distribution function $P_y(v)$ in the neighborhood of v=0. More precisely, we will investigate $P_y(v)$ for values of v greater than zero, since this is the region of significant variation of $P_y(v)$; this is called the positive neighborhood of v=0.

Approximation to Cumulative Distribution Function

From ref. 4, eq. 7, we have the cumulative distribution function of random variable y in terms of the characteristic function according to

$$P_y(v) = \frac{1}{2} - \int_{0+}^{+\infty} d\xi\, g(\xi, v), \tag{5}$$

where we have defined auxiliary function

$$g(\xi, v) = \mathrm{Im}\left\{\exp(-i\xi v)\, \frac{f_y(\xi)}{\pi \xi}\right\}. \tag{6}$$

Observe for later use that

$$g(0+, v) = \lim_{\xi \to 0+} \mathrm{Im}\left\{(1-i\xi v)\, \frac{1+i\xi\mu_y}{\pi\xi}\right\} = \frac{\mu_y - v}{\pi}, \tag{7}$$

where μ_y is the mean of random variable y.

For v in the neighborhood of zero, $\exp(-i\xi v)$ in (6) varies slowly with ξ, and we have the approximation, via the Trapezoidal rule, to (5) as

$$P_y(v) \cong \frac{1}{2} - \frac{\Delta}{2} g(0+,v) - \sum_{n=1}^{+\infty} \Delta\, g(n\Delta,v) \equiv C(v), \tag{8}$$

where the right-hand side of (8) has been defined as C(v). Here, Δ is the sampling interval in ξ, and is small enough to track changes in $\exp(-i\xi v) * f_y(\xi)/\xi$. We choose the Trapezoidal rule in (8) over other integration rules, such as Simpson's rule, because it results in minimum aliasing for Fourier transforms relative to all other rules; see appendix A for elaboration and proof.

Observe from (8) that

$$P_y(0) \cong C(0) \quad \text{means} \quad C(0) \cong 0, \tag{9}$$

since $P_y(0)$ is insignificant by the choice of b in (2); this relation will be used later.

Relationship of Approximation

Although we want to evaluate the exact cumulative distribution function $P_y(v)$, we have instead arrived at an approximation C(v) via (8). How are these two related? To determine the relationship, we manipulate (6)-(8) as follows:

$$C(v) = \frac{1}{2} + \frac{\Delta}{2}\frac{v-\mu_y}{\pi} - \sum_{n=1}^{+\infty} \mathrm{Im}\left\{\exp(-in\Delta v)\,\frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{\Delta}{2}\frac{v-\mu_y}{\pi} - \mathrm{Im}\left\{\sum_{n=1}^{+\infty} \exp(-in\Delta v)\,\frac{f_y(n\Delta)}{\pi n}\right\} = \tag{10}$$

$$= \frac{1}{2} + \frac{\Delta}{2}\frac{v-\mu_y}{\pi} - \frac{1}{i2\pi}\sum_{n\neq 0} \exp(-in\Delta v)\,\frac{f_y(n\Delta)}{n}. \tag{11}$$

The removal of the imaginary operation from within the summation in (10) is a crucial step; it does not create a problem in divergence since $n > 0$. This is in contrast with the integral of (5) and (6), where removal of the imaginary operation would create a divergent integral. This postponement of the removal of the imaginary operation, until after the approximation to the integral was developed in (8), is the major difference with the results in ref. 7.

Taking a derivative of (11), we obtain

$$C'(v) = \frac{\Delta}{2\pi} + \frac{\Delta}{2\pi} \sum_{n\neq 0} \exp(-in\Delta v)\ f_y(n\Delta) =$$

$$= \frac{\Delta}{2\pi} \sum_{n} \exp(-in\Delta v)\ f_y(n\Delta) =$$

$$= \frac{1}{2\pi} \int d\xi\ \exp(-i\xi v)\ f_y(\xi)\ \Delta\delta_\Delta(\xi) =$$

$$= p_y(v) \circledast \delta_{\frac{2\pi}{\Delta}}(v) = \sum_n p_y\left(v - n\,\frac{2\pi}{\Delta}\right) \equiv \tilde{p}_y(v), \qquad (12)$$

where infinite impulse train

$$\delta_\Delta(\xi) = \sum_n \delta(\xi - n\Delta), \qquad (13)$$

where $\circledast$ denotes convolution, and where we have used the relation

$$\frac{1}{2\pi} \int dt\ \exp(-i\omega t)\ \Delta\delta_\Delta(t) = \delta_{\frac{2\pi}{\Delta}}(\omega). \qquad (14)$$

This last result follows from ref. 9, p. 28, rule 11, with $u(t) = \delta(t)$, $T=\Delta$, $F=1/T$, and $\omega=2\pi f$. Relation (12) indicates that $C'(v)$ is an infinitely aliased version of the probability density function $p_y(v)$, with resultant period $2\pi/\Delta$ in v. For small enough sampling increment Δ in (8), there will be very little overlap of the displaced versions of p_y in (12), thereby yielding the good approximation

$$\tilde{p}_y(v) = p_y(v) \quad \text{for} \quad 0 \leq v \leq 2\pi/\Delta. \tag{15}$$

The situation for relation (12) is depicted in figure 2.

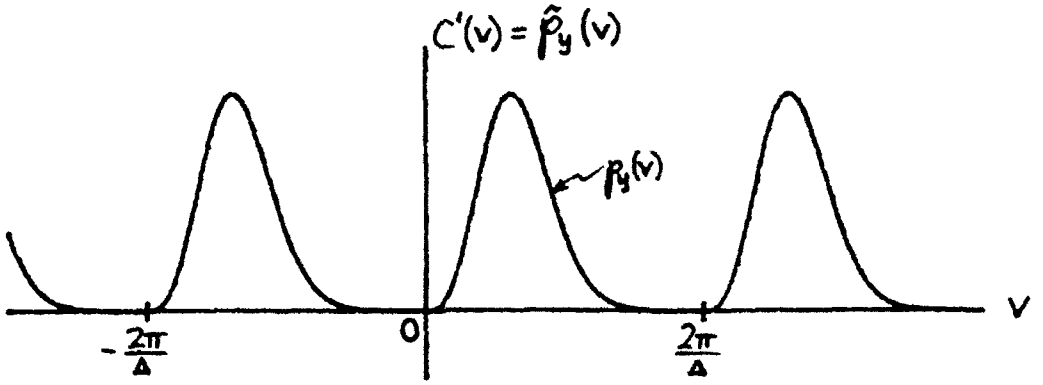

Figure 2. Infinitely Aliased Probability Density Function $\tilde{p}_y(v)$

There now follows from (12),

$$C(v) = C(0) + \int_0^v du\, \tilde{p}_y(u) \equiv C(0) + \tilde{P}_y(v), \tag{16}$$

where C(0) is given by (10) as

$$C(0) = \frac{1}{2} - \frac{\Delta\mu_y}{2\pi} - \mathrm{Im}\left\{\sum_{n=1}^{+\infty} \frac{f_y(n\Delta)}{\pi n}\right\}. \tag{17}$$

Relation (16) is an exact relation, showing that C(v) is the integral of the infinitely aliased version of $p_y(v)$, starting at v=0, plus an additive constant which is substantially zero; see (9).

So for v in the positive neighborhood of zero, (4), (8), (16), and (9) yield

$$P_x(v-b) = P_y(v) \cong C(v) = C(0) + \tilde{P}_y(v) \cong \tilde{P}_y(v). \tag{18}$$

Thus the quantity we want, the left-most term in (18), is well-approximated by calculated quantity C(v), which itself is approximately the integral of the infinitely aliased version of $p_y(v)$.

Calculation of C(v)

Let $v = \frac{2\pi k}{M\Delta}$ in (10), where M and k are arbitrary integers. Then

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{\Delta\mu_y}{2\pi} - \mathrm{Im}\left\{\sum_{n=1}^{+\infty} \exp(-i2\pi nk/M)\frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{+\infty} \exp(-i2\pi nk/M)\; z_n\right\}, \qquad (19)$$

where we define complex sequence

$$z_n = \begin{cases} i\,\frac{1}{2}\Delta\mu_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for} \quad n\geq 1 \end{cases}. \qquad (20)$$

Now define collapsed sequence (ref. 7, pp. 13-16) as

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1. \qquad (21)$$

Then since z_n receives the same weight as z_{n+Mj} in (19), regardless of the value of k, (19) can be expressed as

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\;\hat{z}_n\right\}. \qquad (22)$$

Relation (22) is exact and valid for all k. Since we are only interested in the positive neighbornood of v=0 in (18), we confine attention in (22) to $0 \leq k \leq M-1$.* Relation (22) can then be accomplished by an M-point FFT if M is chosen to be a power of 2. Notice that storage only for the M complex numbers $\{\hat{z}_n\}$ in (21) is required, even though the $\{z_n\}$ sequence in (20) is of infinite length.

---

* Values for other k are available from (22) when we observe that $C\left(\frac{2\pi(M+k)}{M\Delta}\right) = 1 + C\left(\frac{2\pi k}{M\Delta}\right)$ for all k.

Observe that the size of M in no way affects the error of the calculation of $C(\frac{2\pi k}{M\Delta})$ or estimation of $P_y(v)$. Rather, M specifies the spacing at which $C(\frac{2\pi k}{M\Delta})$ is calculated, and can be coarse if desired. The accuracy of the estimate of $P_y(v)$ is governed thus far by Δ, through the aliasing depicted in figure 2.

Reference to (18) now yields

$$P_x(\frac{2\pi k}{M\Delta} - b) \cong C(\frac{2\pi k}{M\Delta}) \quad \text{for} \quad 0 \leq k \leq M-1, \tag{23}$$

where the latter quantity is given by (22). Thus the M-point FFT sweeps out the argument range $(-b,-b+2\pi/\Delta)$ for the cumulative distribution function P_x.

If we want the exceedance distribution function of y instead of the cumulative distribution function, we use (18) and (22) to get

$$1 - C(\frac{2\pi k}{M\Delta}) = \frac{1}{2} - \frac{k}{M} + \frac{1}{\pi} \operatorname{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\ \hat{z}_n\right\} \text{ for } 0 \leq k \leq M-1. \tag{24}$$

(By the footnote to (22), we have $1-C(2\pi/\Delta) = -C(0)$.)

Since μ_y must be known in (20) in order to use this approach, we need the mean μ_x of random variable x, since from (2)

$$\mu_y = \mu_x + b. \tag{25}$$

The quantity μ_x can be found analytically from characteristic function $f_x(\xi)$ according to

$$f_x'(0) = i\mu_x; \tag{26}$$

see (1).

In addition to the error caused by aliasing associated with nonzero sampling increment Δ, an additional error occurs because we cannot calculate all the coefficients $\{z_n\}$ in (20) and (21) out to $n=+\infty$. Rather, we terminate the calculation at integer $n=N$, such that $|z_n|$ is sufficiently small as to be negligible for $n \geq N$. Letting

$$L = N\Delta, \tag{27}$$

this is equivalent to ignoring the contribution to (5) of the tail error

$$-\int_{L}^{+\infty} d\xi\ g(\xi,v) = -\mathrm{Im} \int_{L}^{+\infty} d\xi\ \exp(-i\xi v)\ \frac{f_y(\xi)}{\pi\xi}. \tag{28}$$

If the asymptotic behavior of $f_y(\xi)$ for large ξ is known, this error can sometimes be evaluated in closed form and used to ascertain an adequate value of L. Instead, we have observed that tail error (28) causes a characteristic low-level sinusoidal variation in the calculated cumulative distribution function for small v near 0, and in the calculated exceedance distribution function for large v near $2\pi/\Delta$. When this sinusoidal variation is deemed excessive, L can be increased until the effect disappears or decreases to acceptable levels. This trial and error approach avoids the necessity of analytically upper-bounding the magnitude of error (28), which is often very tedious and generally pessimistic.

So there are two errors to be concerned with: aliasing due to nonzero sampling interval Δ and tail error due to non infinite limit L. Later examples will demonstrate how these errors manifest themselves in the cumulative and exceedance distribution functions and how they can be controlled by a trial and error approach.

Relation to Requicha's Method, ref. 5

From ref. 5, eqs. 7, 9, 10, the cumulative distribution function is given by an expression that can be manipulated into the form (using current notation)

$$F_k = \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{M/2} \exp(-i2\pi kn/M)\,\frac{f_y(n\Delta)}{n}\right\} + \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{M/2}\frac{f_y(n\Delta)}{n}\right\}. \qquad (29)$$

Although this is similar to the upper line of (19) here, it differs in several important respects:

1. F_k does not use mean μ_y at all; it is therefore not using a direct approximation to the specified integral in (5) and (6).

2. From (29), there follows $F_0 = 0$, $F_M = 1$; however, these results are not strictly true for the actual cumulative distribution function at these end points, thereby leading to poor estimates in the neighborhoods of these points. This is due to the arbitrary origin established in ref. 5, eq. 6.

3. The sums in (29) utilize characteristic function samples $f_y(n\Delta)$ only for $n \leq M/2$, where M is the size of the FFT. This is a very severe and unnecessary restriction; in fact, the sum on n in (29) ought to be conducted to the point where the tail contribution, (28), is negligible, regardless of the value of M.

4. In ref. 5, if eq. 4 is substituted into eq. 1, and the summation limits are extended to $\pm\infty$, we get exactly the second line of (12) here. When the probability density function is integrated to get the cumulative distribution function in ref. 5, eq. 6, the resultant cumulative distribution function is arbitrarily set to zero at v=0. We instead have from (9) and (17),

$$P_y(0) \cong C(0) = \frac{1}{2} - \frac{\Delta\mu_y}{2\pi} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{\pm\infty}\frac{f_y(n\Delta)}{n}\right\}, \qquad (30)$$

which is small, but not necessarily zero. This consideration is very important on the tails of the cumulative and exceedance distribution functions.

Summary of Procedure

The cumulative distribution function of y is given by

$$P_y\left(\frac{2\pi k}{M\Delta}\right) = C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\,\hat{z}_n\right\}$$

$$\text{for} \quad 0 \le k \le M-1, \tag{31}$$

where M is the size of the FFT and storage employed. Also

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \le n \le M-1, \tag{32}$$

where

$$z_n = \begin{cases} i\frac{1}{2}\Delta\mu_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for } 1 \le n \le N \\ 0 & \text{for } n > N \end{cases}. \tag{33}$$

(The value for n=N should be scaled by 1/2 for the Trapezoidal rule). The zero values for z_n, when $n > N$, serve to terminate the collapsed sum in (32) at a finite upper limit. The value of N is given by the integer part of L/Δ, where Δ and L must be chosen so as to minimize aliasing and tail error, respectively. The characteristic function of random variable y needed in (33) is given by

$$f_y(\xi) = f_x(\xi)\,\exp(ib\xi), \tag{34}$$

in terms of the characteristic function of the primary random variable x, where shift b must be chosen such that $y = b+x$ is positive with probabilty virtually 1. The mean $\mu_y = b+\mu_x$ can be determined analytically from knowledge of characteristic function $f_x(\xi)$. Finally, the exceedance distribution function for random variable y is obtained by subtracting (31) from 1.

EXAMPLES

Programs for the following five examples are listed in appendix B.

1. Chi-Square

A chi-square variate of 2K degrees of freedom has probability density function (ref. 10)

$$p_x(v) = \frac{v^{K-1} \exp(-v/2)}{2^K (K-1)!} \quad \text{for } v > 0 \tag{35}$$

and characteristic function

$$f_x(\zeta) = (1-i2\zeta)^{-K}. \tag{36}$$

Since random variable x is obviously nonnegative by (35), we can choose shift b=0; i.e. y=x. A plot of the cumulative and exceedance distribution functions of random variable y obtained from characteristic function (36) with K=4 is given in figure 3 for $0 \leq v \leq 2\pi/\Delta$. The values of Δ and L have been chosen such that aliasing and tail error are insignificant.

The ordinate scale for figure 3 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region 1E-11, where round-off noise is encountered. The exceedance distribution function values continue to decrease with v until, finally, negative values (due to round-off noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the 1E-12 level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of Δ and L.

For this example, $N=L/\Delta=2666$, while M=256. Thus collapsing, according to (21) or (32), by over a factor of 10 has been employed and a small size FFT has been utilized. Nevertheless the error realized for the cumulative and exceedance distribution functions is in the 1E-12 range, the limit of accuracy of the Hewlett Packard 9845B Desk Calculator used here. Finer spacing in the distribution outputs is achievable by merely increasing M.

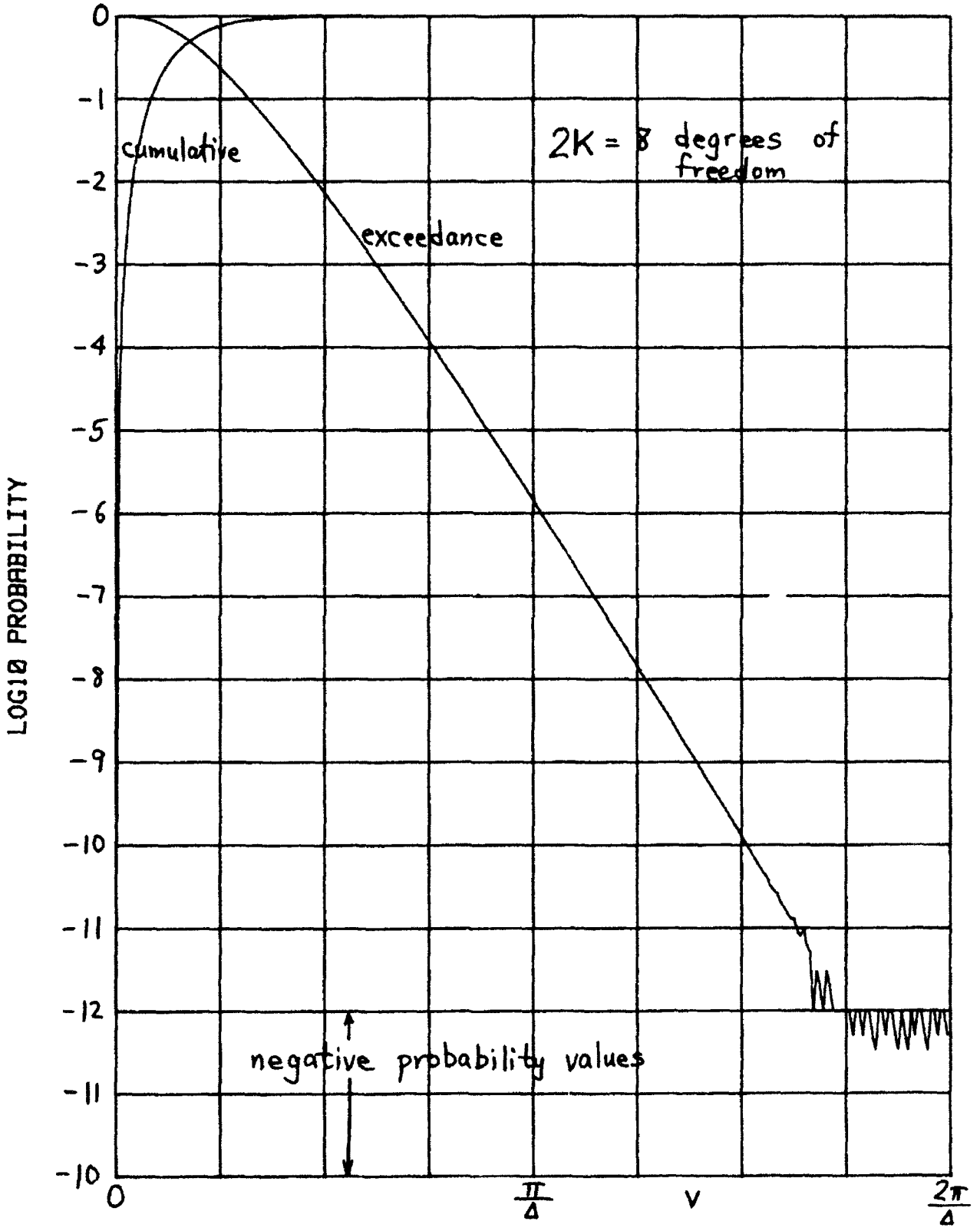

Figure 3. Chi-Square; L=200, Δ=.075, b=0, M=256

2. Gaussian

The characteristic function for a zero-mean unit-variance random variable is

$$f_x(\zeta) = \exp(-\zeta^2/2), \tag{37}$$

and the probability density function and cumulative distribution function are (ref. 11, eq. 10.5.3)

$$p_x(v) = (2\pi)^{-1/2} \exp(-v^2/2), \; P_x(v) = \Phi(v). \tag{38}$$

For $b = 5\pi/2$, using (4),

$$P_y(0) = P_x(-b) = \Phi(-b) = 2E\text{-}15. \tag{39}$$

which is negligible, as desired.

Plots of the cumulative and exceedance distribution functions for random variable y are given in figure 4 for L=7, Δ=.3. The logarithmic ordinate gives rise to the characteristic parabolic shape on the tails of the distributions. Once again, the probabilities decrease to the level of the round-off noise and fluctuate around 1E-12 near the edges of the fundamental aliased interval $(0,2\pi/\Delta)$. The fact that the cumulative distribution function of y starts in the round-off noise at v=0 indicates that $b=5\pi/2$ was large enough to guarantee $y > 0$ with probability virtually 1. Also indicated on the figure is the origin for random variable x. We have, from (4),

$$P_x(u) = P_y(u+b); \tag{40}$$

thus for example

$$\text{Prob}(x < 0) = P_x(0) = P_y(b) = .5. \tag{41}$$

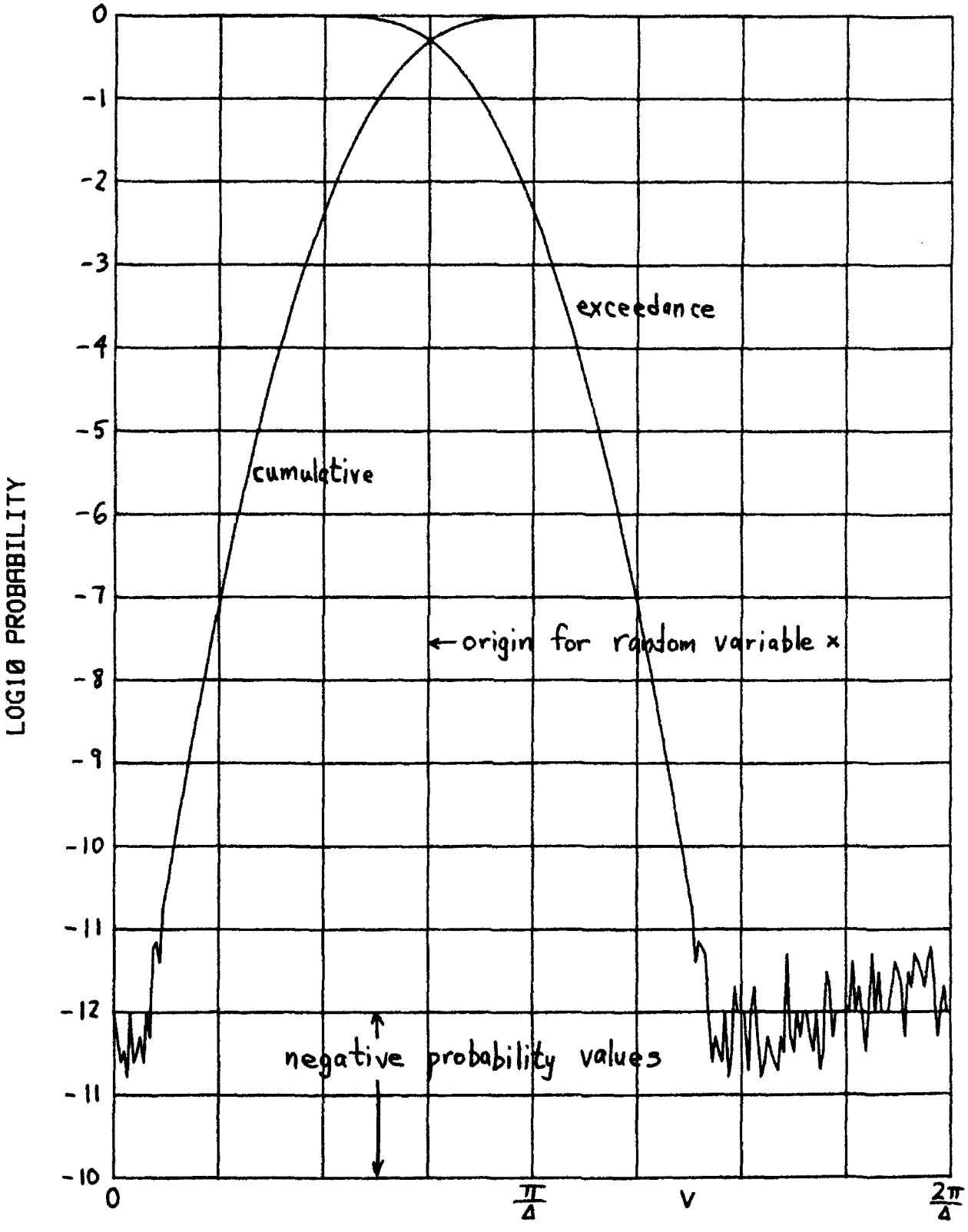

Figure 4. Gaussian; L=7, Δ=.3, b=2.5π, M=256

In figure 5, the only change is to decrease limit L from 7 to 6. The tail error mentioned in (28) et seq. then dominates the round-off noise and has a sinusoidal variation. Aliasing is not a problem, as witnessed by the fact that the cumulative and exceedance distribution functions of random variable y have decayed below 1E-12 well before the edges of the interval are reached.

When limit L is restored to 7, and sampling increment Δ is increased to .5, aliasing becomes significant, as shown in figure 6. The exceedance distribution function has not yet decayed to the round-off noise level at $v=2\pi/\Delta$, and the cumulative distribution function shows a large negative probability region near v=0. Shift b has been maintained at the value $5\pi/2$, corresponding to (39).

When L and Δ are restored to their values 7 and .3 as for figure 4, but b is decreased to $5\pi/3$, the probability of y becoming negative is, from (4) and (38), $\Phi(-5\pi/3) = .82E-7$. This is reflected in the cumulative distribution function for y in figure 7 at v=0, where the probability value is well above the round-off noise level. Also, the exceedance distribution function develops significantly negative values near $v = 2\pi/\Delta$.

Accurate evaluation of the cumulative and exceedance distribution functions can only be achieved when L, Δ, and b are properly chosen. Probably the optimum combination for the Gaussian variate is displayed in figure 8, where Δ has been increased to .4, the distributions are centered on the fundamental aliased interval $(0, 2\pi/\Delta)$ by choice of b, and L is taken at 7 to avoid tail error.

3. Smirnov

The limiting characteristic function of a measure of goodness of fit based on the sample distribution function was derived by Smirnov and is given by (ref. 12, eq. 30.104)

$$f_x(\xi) = \left(\frac{s}{\sin(s)}\right)^{1/2} \text{ where } s = (1+i)\sqrt{\xi} \text{ for } \xi \geq 0. \qquad (42)$$

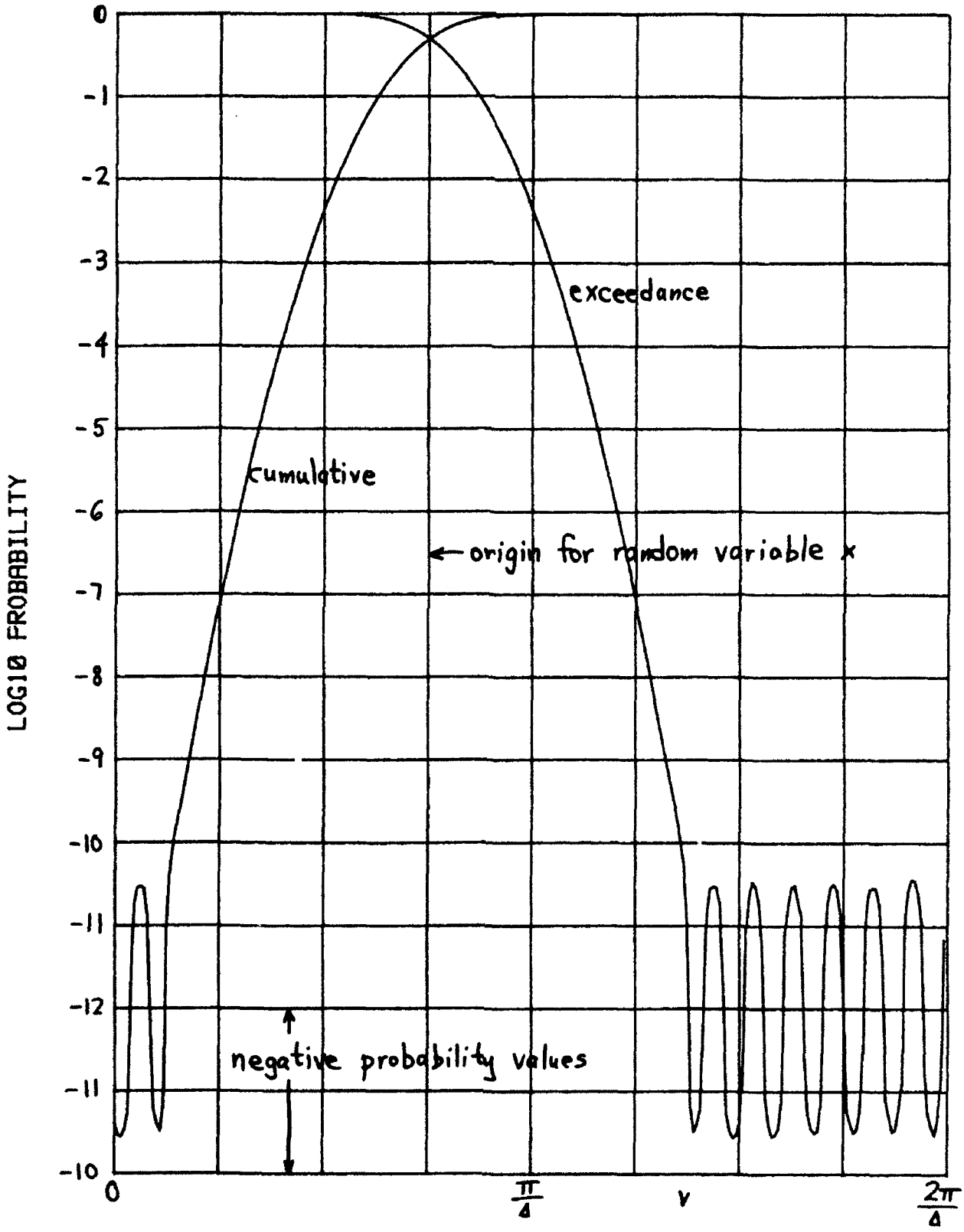

Figure 5. Gaussian; L=6, Δ=.3, b=2.5π, M=256

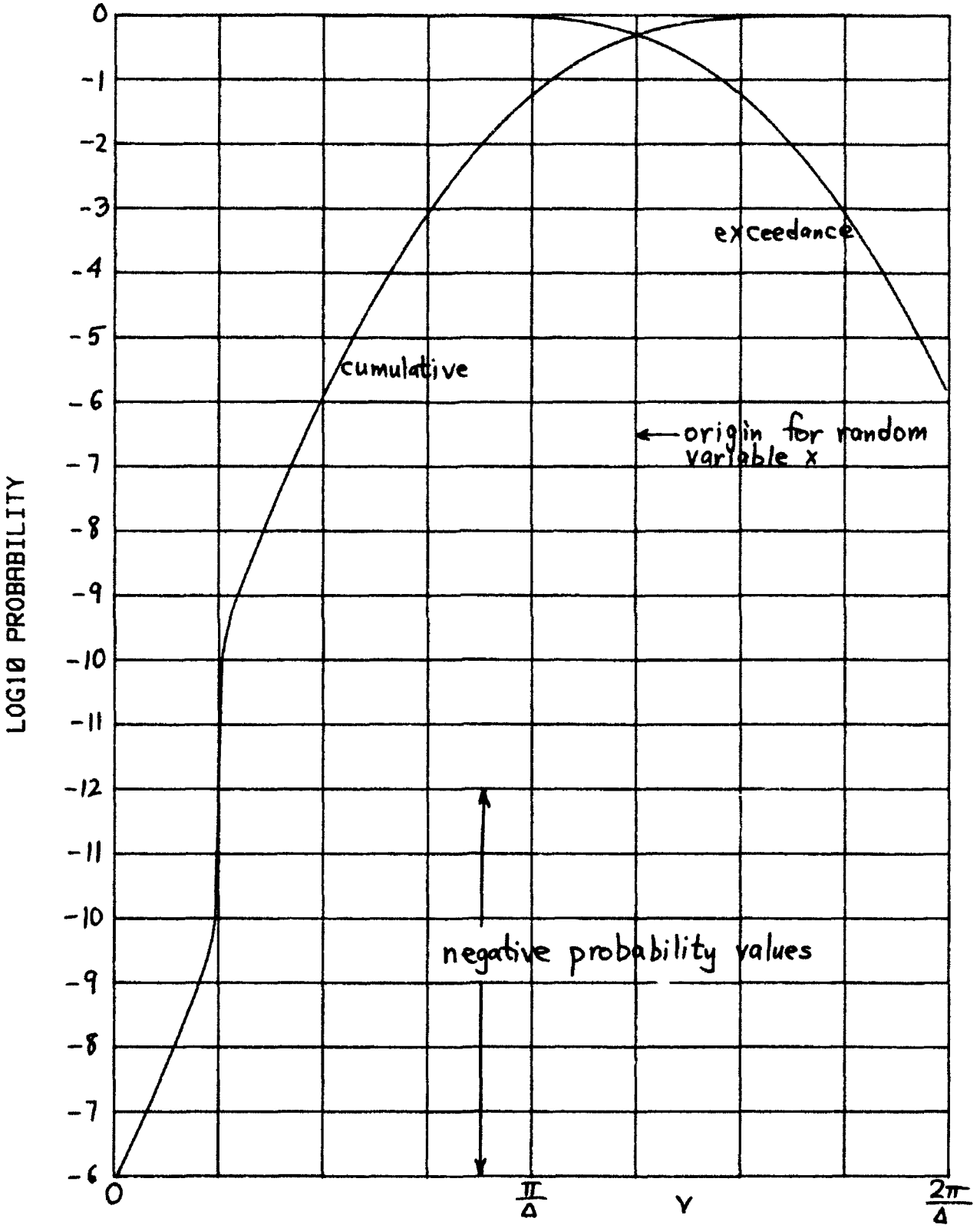

Figure 6. Gaussian; L=7, Δ=.5, b=2.5π, M=256

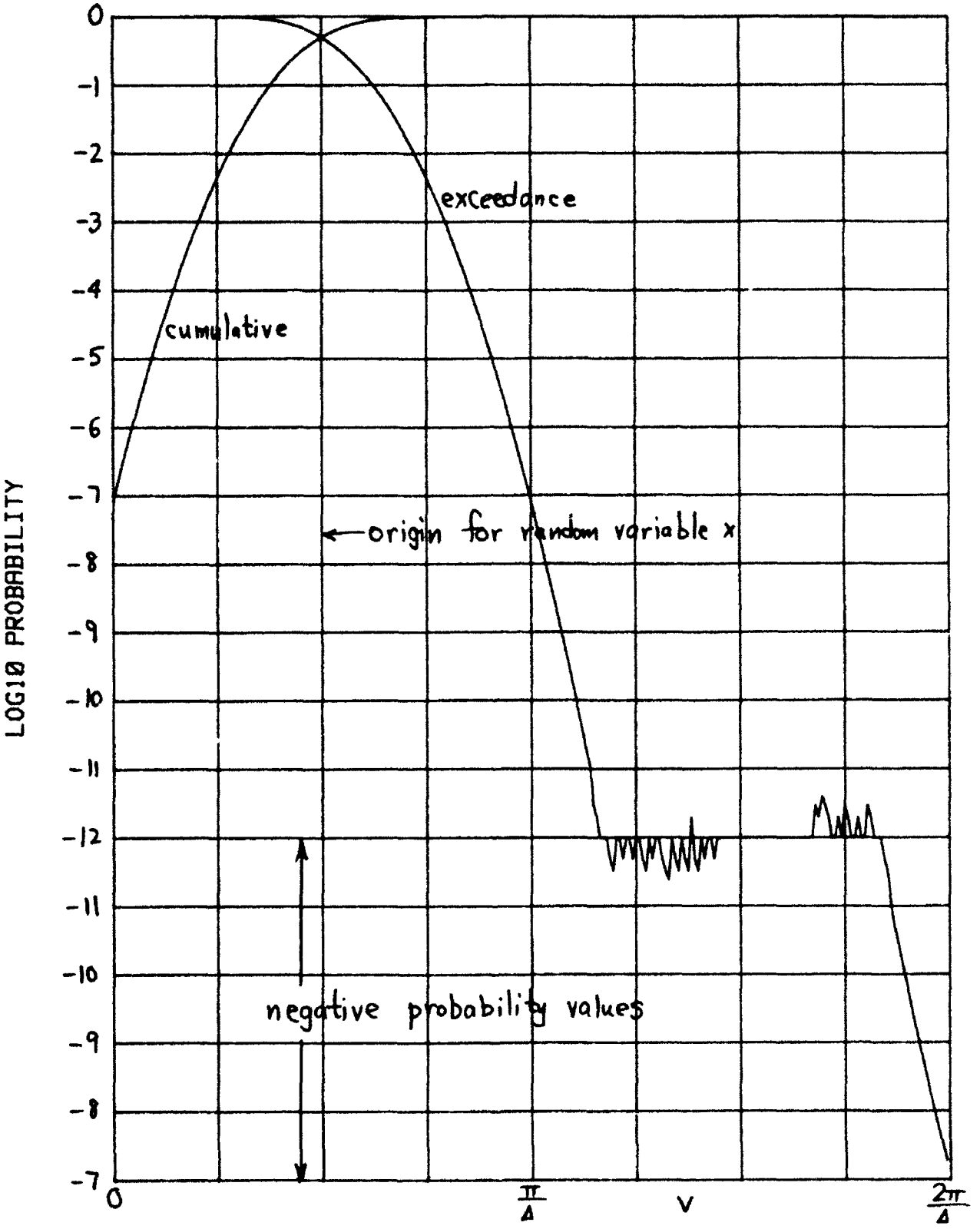

Figure 7. Gaussian; L=7, Δ=.3, b=5π/3, M=256

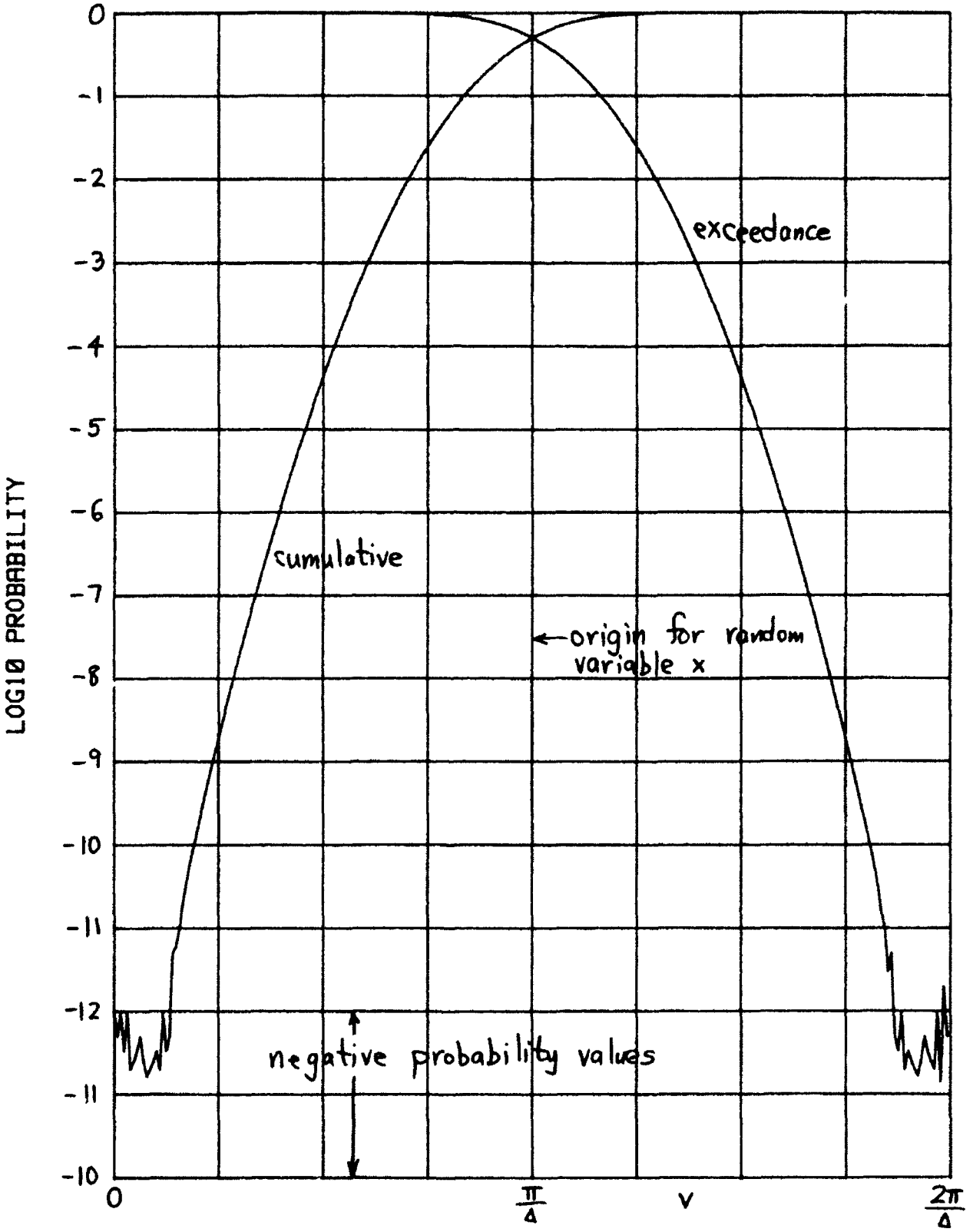

Figure 8. Gausstan; L=7, Δ=.4, b=2.5π, M=256

An expansion about $\xi=0$ yields

$$f_x(\xi) = 1+i\frac{1}{6}\xi - \frac{1}{40}\xi^2; \quad \text{i.e., } \mu_x = 1/6,\ \sigma_x^2 = 1/45. \tag{43}$$

And since the goodness of fit is always positive, random variable x is positive and we can choose

$$b=0. \tag{44}$$

Since

$$\sin((1+i)\sqrt{\xi}) \sim i\frac{1}{2}\exp(\sqrt{\xi}(1-i)) \quad \text{as } \xi\to+\infty, \tag{45}$$

it follows that

$$f_x(\xi) \sim 2^{3/4}\xi^{1/4}\exp(-\frac{1}{2}\sqrt{\xi} + i(\frac{1}{2}\sqrt{\xi} - \frac{\pi}{8})) \text{ as } \xi\to+\infty. \tag{46}$$

The phase of this term rotates according to $\sqrt{\xi}/2$; if we were to choose $b\neq 0$, $f_y(\xi)$ would rotate faster than $f_x(\xi)$ (linear with ξ rather than $\sqrt{\xi}$). This could necessitate a faster sampling rate, which is undesirable.

The cumulative and exceedance distribution functions are plotted in figure 9. L and Δ have been chosen so as to avoid tail error and aliasing. The exceedance distribution function is seen to decay exponentially until it reaches approximately 2E-11; the bump in the curve at this point is a manifestation of the limited accuracy of the trigonometric functions built into the calculator employed. Larger values of v lead to round-off noise around the 1E-12 level.

A comparison of results for this characteristic function, with Requicha's method described in (29) et seq., is given in figure 10 for FFT size M=1024. The plot labeled with N=L=512 is precisely Requicha's method. Aliasing is known to be insignificant for $\Delta=1$, as seen by reference to figure 9 and observing that extrapolation of the straight line section of the exceedance distribution function would result in probability values near 1E-13 at $v=2\pi/\Delta$. The dashed portion of the N=L=512 curve in figure 10 in fact

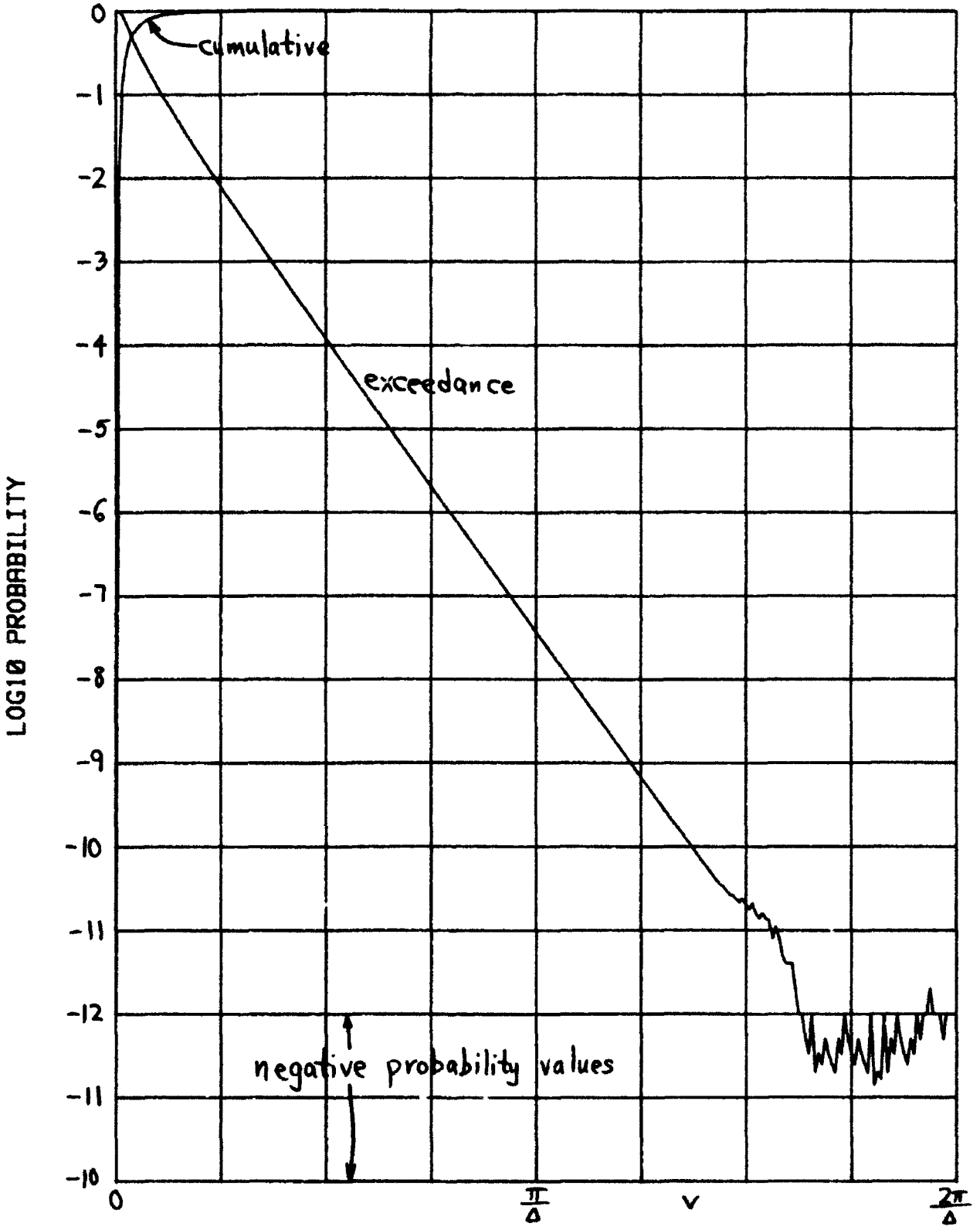

Figure 9. Smirnov; L=3000, Δ=1, b=0, M=256

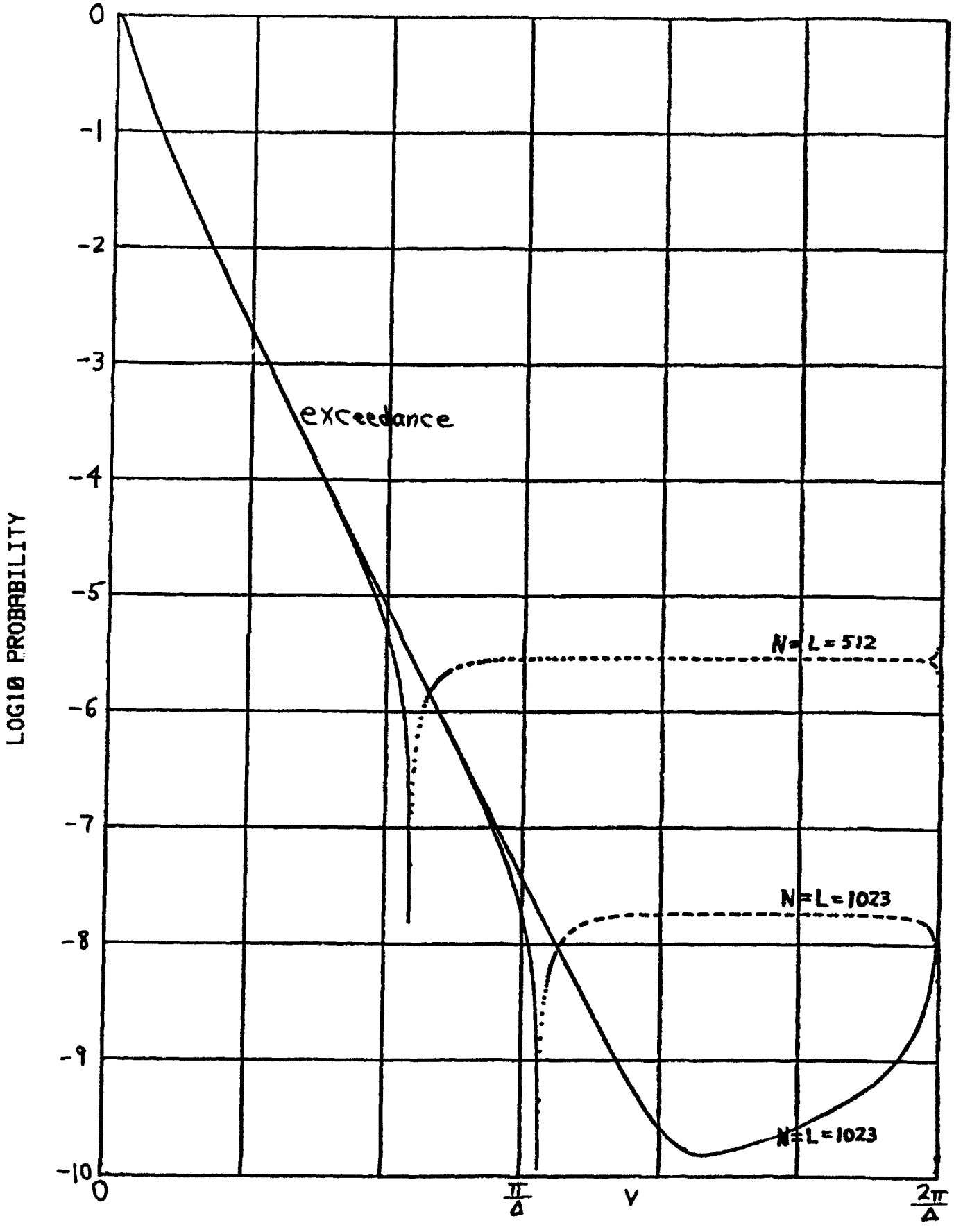

Figure 10. Comparison with Requicha's Method;
Δ=1, b=0, M=1024

corresponds to negative probability estimates; these grossly inaccurate results are due to an inadequate value of limit L, leading to large tail error.

When N is simply increased to 1023, the middle curve in figure 10 results from Requicha's method. Again, negative estimates are indicated by the dashed portion of this curve, although two orders of magnitude smaller than above. The reasons for these errors have been delineated in (29) et seq.

The bottom-most curve in figure 10 (solid curve) is that obtained by the method proposed in this report for L = 1023. Exceedance distribution function estimates in the 1E-10 range are obtained, but the error returns to the 1E-8 range at $v=2\pi/\Delta$. No negative probability values occur. Also, by simply increasing limit L, while keeping FFT size M fixed, the error can be reduced significantly further, as already witnessed by figure 9.

4. Noncentral Chi-Square

Here the random variable x is given by

$$x = \sum_{k=1}^{K} (g_k + d_k)^2 \tag{47}$$

where $\{d_k\}$ are constants, and $\{g_k\}$ are independent Gaussian random variables with zero-mean and unit variance. The characteristic function of x is

$$f_x(\xi) = (1-i2\xi)^{-K/2} \exp\left(\frac{id^2\xi}{1-i2\xi}\right), \tag{48}$$

where deflection d is defined according to

$$d^2 = \sum_{k=1}^{K} d_k^2. \tag{49}$$

We actually consider a more general characteristic function than (48), namely

$$f_X(\zeta) = (1-i2\zeta)^{-\nu} \exp\left(\frac{id^2\zeta}{1-i2\zeta}\right) = \exp\left(\frac{id^2\zeta}{1-i2\zeta} - \nu \ln(1-i2\zeta)\right), \qquad (50)$$

where ν is an arbitrary positive real constant. Suppose that we use the principal value logarithm for $\ln(z)$, where the branch cut lies along the negative real axis of the complex z plane (ref. 13, sect. 4.1.1). Then since the argument of the logarithm in (50) never crosses the branch cut, form (50) gives the correct characteristic function values automatically for all real ζ, and any ν.

The probability density function and exceedance distribution function corresponding to (50) are (ref. 14, 6.631 4)

$$p_X(v) = \frac{1}{2} \exp\left(-\frac{d^2+v}{2}\right)\left(\frac{\sqrt{v}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{v}) \quad \text{for} \quad v > 0,$$

$$1 - P_X(v) = \int_{\sqrt{v}}^{+\infty} dt\ t \exp\left(-\frac{d^2+t^2}{2}\right)\left(\frac{t}{d}\right)^{\nu-1} I_{\nu-1}(dt) \equiv Q_\nu(d,\sqrt{v}) \quad \text{for} \quad v > 0. \qquad (51)$$

Since the probability density function in (51) is never negative (ref. 13, sect. 9.6.1), (50) is a legal characteristic function. Also because random variable x is always positive according to (51), we choose shift b=0. Plots of the exceedance distribution function, as determined from characteristic function (50) are displayed for various values of d in figure 11. The values of L were chosen for each d value so as to control the tail error below the 1E-10 level plotted. Direct calculation of the exceedance distribution function directly from (51) would be a formidable task for arbitrary ν values.

5. Product of Correlated Gaussian Variates

Let

$$x = st \qquad (52)$$

where s and t are zero-mean unit-variance Gaussian random variables with correlation coefficient ρ. The joint probability density function of s and t is

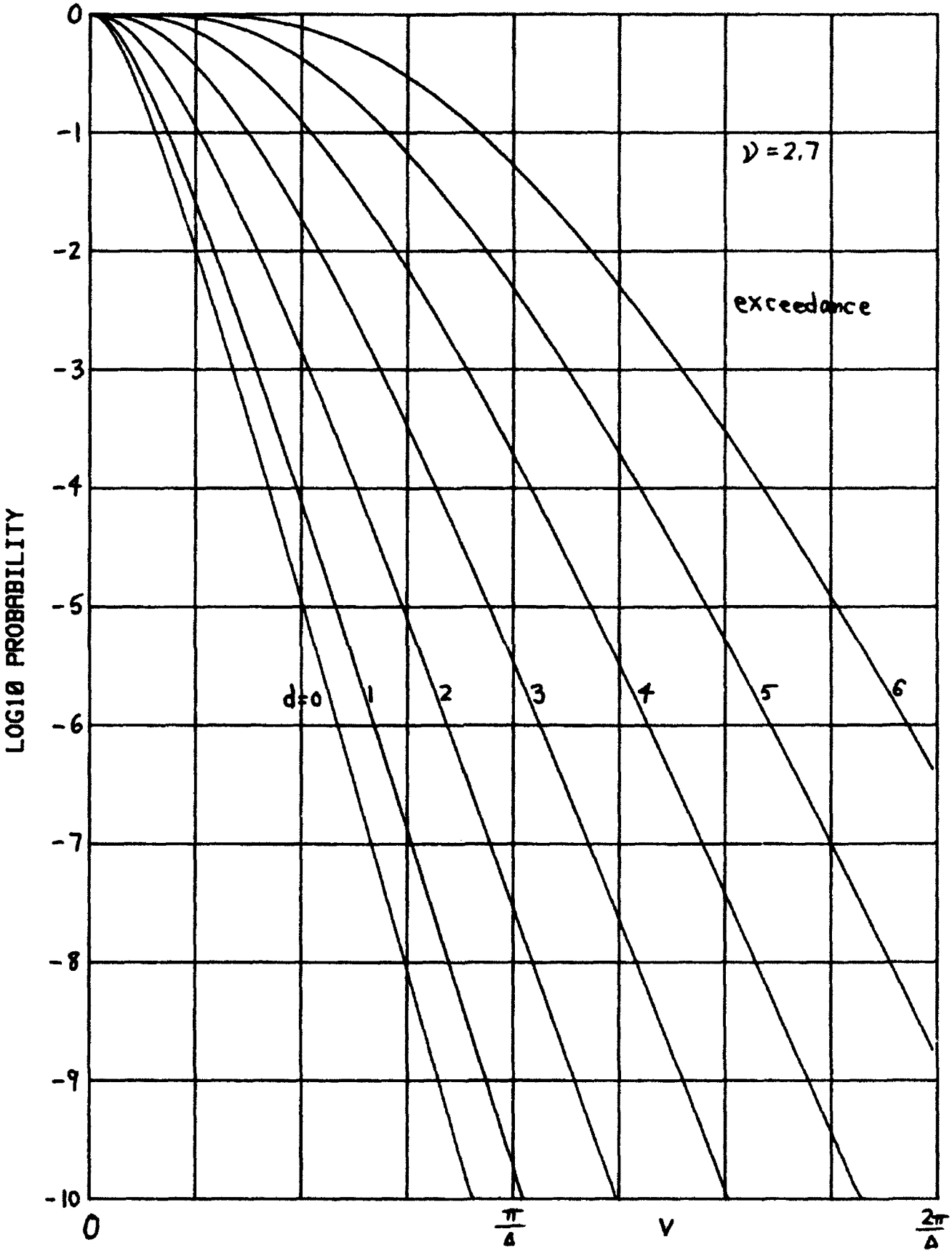

Figure 11. Non-Central Chi-Square; Δ=.05, b=0, M=256

$$p_{st}(u,v) = \left(2\pi\sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right]. \qquad (53)$$

The characteristic function of x is then

$$f_x(\xi) = \overline{\exp(i\xi st)} = \iint du\, dv \exp(i\xi uv)\, p_{st}(u,v) =$$

$$= \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-1/2} = \left[1-i(1+\rho)\xi\right]^{-1/2} \left[1+i(1-\rho)\xi\right]^{-1/2}, \qquad (54)$$

via repeated use of ref. 14, eq. 3.323 2. The corresponding probability density function of x is

$$p_x(v) = \int \frac{dy}{|y|}\, p_{st}(y,\tfrac{v}{y}) = \frac{1}{\pi\sqrt{1-\rho^2}} \exp\left(\frac{\rho v}{1-\rho^2}\right) K_0\left(\frac{|v|}{1-\rho^2}\right) \text{ for all } v, \qquad (55)$$

via ref. 14, eq. 3.478 4.

(If we transform this probability density function according to (1) and use ref. 14, eq. 6.611 9 and ref. 13, eq. 4.4.15, we get precisely (54). Alternatively, if we transform (54) and modify the contour to wrap around the branch line along the imaginary axis and then use ref. 14, eq. 3.388 2, we get (55). Or we can use ref. 14, eq. 3.754 2.)

We actually consider a more general characteristic function than (54), namely

$$f_x(\xi) = \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-\nu} = \exp\left(-\nu \ln\left[1-i2\rho\xi + (1-\rho^2)\xi^2\right]\right) =$$

$$= \left[\frac{1}{1-\rho^2} + \left\{\sqrt{1-\rho^2}\,\xi - i\,\frac{\rho}{\sqrt{1-\rho^2}}\right\}^2\right]^{-\nu}. \qquad (56)$$

The mean of this random variable x is given by

$$\mu_x = 2\nu\rho. \qquad (57)$$

The probability density function corresponding to (56) is

$$p_X(v) = \frac{1}{2\pi} \int d\xi \exp(-i\xi v)\, f_X(\xi) =$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty - i\rho/\sqrt{1-\rho^2}}^{+\infty - i\rho/\sqrt{1-\rho^2}} dy \exp\left(\frac{\rho v}{1-\rho^2} - i\frac{yv}{\sqrt{1-\rho^2}}\right)\left(\frac{1}{1-\rho^2} + y^2\right)^{-\nu}, \quad (58)$$

where we let

$$y = \sqrt{1-\rho^2}\,\xi - i\frac{\rho}{\sqrt{1-\rho^2}}. \quad (59)$$

We can move the contour in (58) to the real y-axis, because the branch points of the integrand are at $y = \pm i/\sqrt{1-\rho^2}$ which are outside the path of integration, since $|\rho| < 1$. Then using ref. 14, eq. 3.771 2 and ref. 13, eq. 6.1.17, we obtain

$$p_X(v) = \left(\sqrt{\pi}\,\Gamma(\nu)\sqrt{1-\rho^2}\right)^{-1}\left(\frac{|v|}{2}\right)^{\nu-\frac{1}{2}} \exp\left(\frac{\rho v}{1-\rho^2}\right) K_{\nu-\frac{1}{2}}\left(\frac{|v|}{1-\rho^2}\right) \quad \text{for all } v. \quad (60)$$

Since this probability density function is never negative (ref. 13, sect. 9.6.1), (56) is a legal characteristic function. If we Fourier transform (60) via ref. 14, 6.699 12. we get (56) directly.

There is no simple relation for the cumulative distribution function of this random variable. Nevertheless, it is a simple matter to evaluate directly from characteristic function (56). The $\ln$ in (56) causes no problems since its argument never crosses the branch cut. A plot for $\nu=7.7$ and $\rho=-.3$ is displayed in figure 12. The rate of decay of the distribution is different for each tail. The round-off noise is clearly visible at both ends of the range of v values.

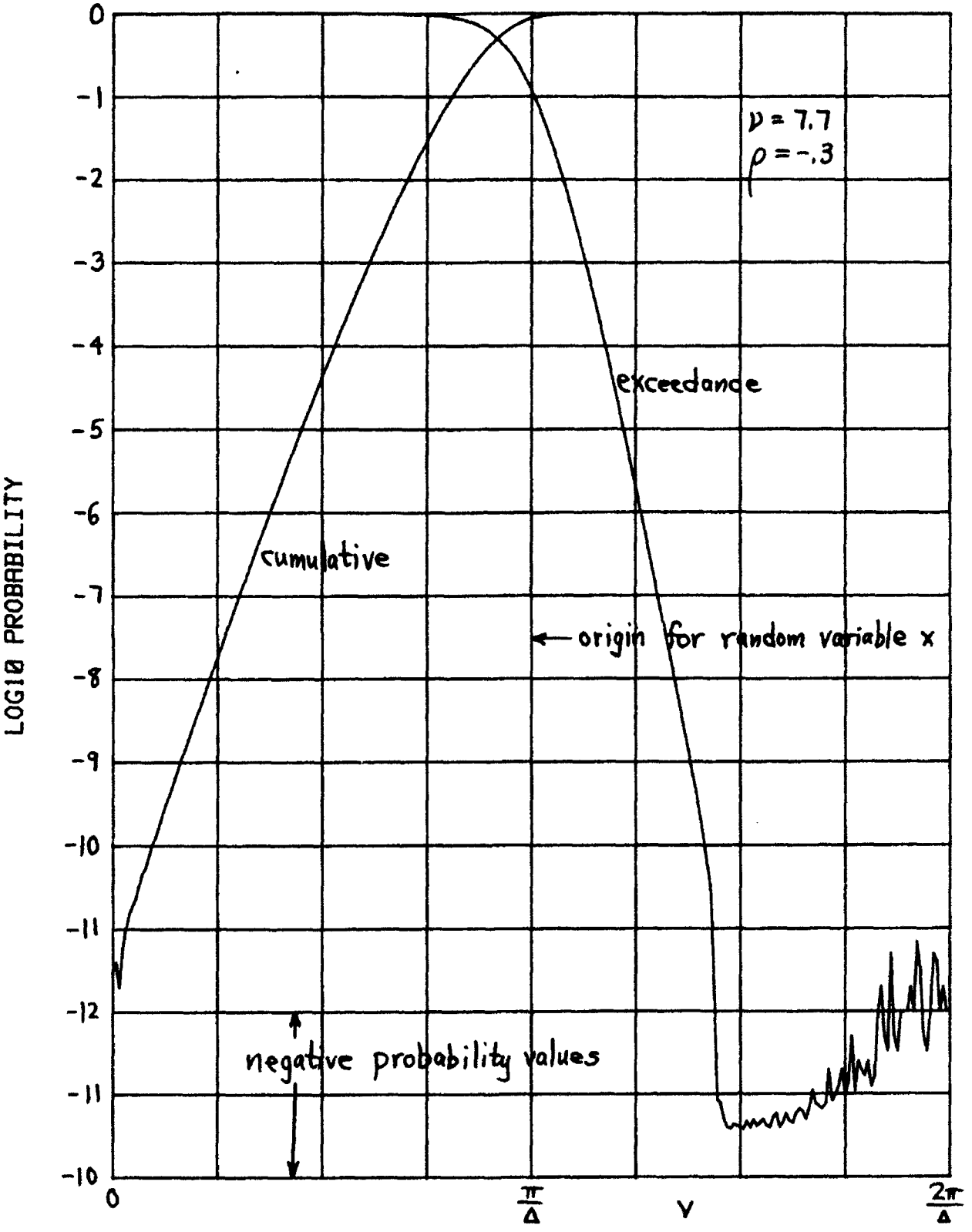

Figure 12. Gaussian Product; L=5, Δ=.06, b=50π/3, M=256

APPLICATIONS

We now have the capability to handle the following type of statistical problem in a fairly easy fashion. Consider random variable

$$x = \sum_{k=1}^{K} r_k^{\nu_k} , \tag{61}$$

where $\{r_k\}$ are arbitrary random variables, statistically independent of each other, and with different distributions. Power ν_k is arbitrary (except that ν_k must be a positive integer for those r_k that can become negative). Let the probability density function of random variable r_k be $p_k(v)$. Then the characteristic function of $r_k^{\nu_k}$ is

$$g_k(\xi) = \overline{\exp\left(i\xi r_k^{\nu_k}\right)} = \int dv \, \exp\left(i\xi v^{\nu_k}\right) p_k(v) =$$

$$= \frac{1}{\nu_k} \int \frac{dt}{t} \exp(i\xi t) \, t^{1/\nu_k} \quad p_k\left(t^{1/\nu_k}\right) . \tag{62}$$

If (62) is not integrable in closed form, it can be evaluated by means of an FFT (one for each k if the probability density functions or ν_k are all different). Then the characteristic function of random variable x in (61) is given by

$$f_x(\xi) = \prod_{k=1}^{K} \left\{g_k(\xi)\right\} . \tag{63}$$

Now the techniques of this report are directly applicable to (63).

An additional example is afforded by

$$x = \sum_{k=1}^{K} \alpha_k v_k^2 + \left(\sum_{k=1}^{K} \beta_k v_k \right)^2 + \sum_{k=1}^{K} \gamma_k v_k, \tag{64}$$

where $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ are constants, and $\{v_k\}$ are independent random variables with arbitrary probability density functions. The characteristic function of x is

$$f_x(\xi) = \overline{\exp(i\xi x)} = \int dV\ p_v(V)\ \exp\left(i\xi \left[\sum \alpha_k v_k^2 + \left(\sum \beta_k v_k \right)^2 + \sum \gamma_k v_k \right] \right), \tag{65}$$

where $V = (v_1, v_2, \ldots, v_K)$. Now since

$$\left(\frac{ia}{\pi} \right)^{1/2} \int dy\ \exp(-iay^2 + iby) = \exp\left(\frac{ib^2}{4a} \right) \text{ for } a \neq 0, \tag{66}$$

we identify $a = \xi/4$, $b = \xi \sum \beta_k v_k$, eliminate the square in the exponent, and express (65) as

$$f_x(\xi) = \int dV\ p_v(V)\ \exp\left(i\xi \sum \alpha_k v_k^2 + i\xi \sum \gamma_k v_k \right) *$$

$$* \left(\frac{i\xi}{4\pi} \right)^{1/2} \int dy\ \exp\left(-\frac{i\xi y^2}{4} + i\xi y \sum \beta_k v_k \right) =$$

$$= \left(\frac{i\xi}{4\pi} \right)^{1/2} \int dy\ \exp\left(-\frac{i\xi y^2}{4} \right) \prod_{k=1}^{K} \left\{ \int dv_k\ p_k(v_k)\ \exp(i\xi(\alpha_k v_k^2 + \gamma_k v_k + y\beta_k v_k)) \right\}, \tag{67}$$

where

$$P_v(V) = \prod_{k=1}^{K} \left\{ p_k(v_k) \right\} . \tag{68}$$

The inner integrals in (67) can either be done analytically or numerically. Then the remaining single integral on y must be numerically evaluated to find characteristic function $f_x(\xi)$. As an example, if v_k is exponentially distributed

$$p_k(v) = a_k \exp(-a_k v) \text{ for } v > 0, \tag{69}$$

then the inner integrals in (67) are w-functions; see ref. 13, ch. 7. A simpler method of handling general quadratic expressions like (64) with Gaussian $\bar{V}$ is presented in ref. 15.

SUMMARY

An accurate method for efficient evaluation of the cumulative and exceedance distribution functions has been derived and applied to several examples to illustrate its utility. Choice of the sampling increment Δ applied to the characteristic function controls the aliasing problem, and selection of the limit L minimizes the tail error; the effects of both of these parameters can be observed from sample plots of the distributions and can be modified if needed. Additionally, shift b must be chosen so as to yield a positive random variable with probability virtually 1. The number of distribution values yielded depends on the size of the FFT employed and can be independently selected to yield the desired spacing in distribution values.

APPENDIX A. SAMPLING FOR A FOURIER TRANSFORM

Suppose we are interested in evaluating Fourier transform

$$G(f) = \int dt \, \exp(-i2\pi ft) \, g(t). \tag{A-1}$$

If we sample at interval Δ in t in (A-1), and use integration weighting w(t), we have the approximation to G(f),

$$\begin{aligned} \tilde{G}(f) &\equiv \int dt \, \exp(-i2\pi ft) \, g(t) \, \delta_\Delta(t) \, w(t) \\ &= G(f) \circledast \frac{1}{\Delta} \delta_{\frac{1}{\Delta}}(f) \circledast W(f) \\ &= \frac{1}{\Delta} \sum_n G(f - \tfrac{n}{\Delta}) \circledast W(f), \end{aligned} \tag{A-2}$$

where infinite impulse train (sampling function)

$$\delta_\Delta(t) = \sum_n \delta(t - n\Delta), \tag{A-3}$$

and $\circledast$ denotes convolution.

The term

$$\frac{1}{\Delta} \sum_n G(f - \tfrac{n}{\Delta}) \tag{A-4}$$

in (A-2) is an infinitely aliased version of desired function G(f); this aliasing is an unavoidable effect due to sampling at increment Δ. However, to minimize any further aliasing in (A-2), we would like $W(f) = \delta(f)$, which requires $w(t) = 1$ for all t; strictly, all we need is

$$w(n\Delta) = 1 \quad \text{for all } n. \tag{A-5}$$

That is, the best weighting in (A-2) is uniform.

As an example, for Simpson's rule, we have weighting

$$w(n\Delta) = \ldots, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \ldots = 1 + \frac{1}{3}(-1)^n \text{ or } 1 - \frac{1}{3}(-1)^n, \tag{A-6}$$

which can be represented as samples of time function

$$w(t) = 1 + \frac{1}{3}\exp(i\pi t/\Delta) \text{ or } 1 - \frac{1}{3}\exp(i\pi t/\Delta). \tag{A-7}$$

The corresponding transform is

$$W(f) = \int dt \exp(-i2\pi ft)\, w(t) =$$

$$= \delta(f) + \frac{1}{3}\delta(f - \frac{1}{2\Delta}) \quad \text{or} \quad \delta(f) - \frac{1}{3}\delta(f - \frac{1}{2\Delta}) \ . \tag{A-8}$$

But this window function substituted in (A-2) results in an extra aliasing lobe in $\tilde{G}(f)$, halfway between the unavoidable major lobes of (A-4) at multiples of $1/\Delta$, of magnitude 1/3 as large. This effect very adversely affects the quality of $\tilde{G}(f)$ insofar as its approximation to the desired $G(f)$ is concerned. Thus the best sampling plan in (A-2) is the equal weight structure of (A-5) when one wants to approximate the Fourier transform of (A-1). For a bounded region, this is modified to the Trapezoidal rule, i.e., half-size weights at the boundaries.

APPENDIX B. LISTINGS OF PROGRAMS FOR FIVE EXAMPLES

The following listings are programs in BASIC for the Hewlett Packard 9845B Desktop Calculator. The FFT utilized is one with the capability of a zero subscript and is listed at the end of the appendix. Mathematically, the FFT programmed is

$$Z_m = \sum_{k=0}^{M-1} \exp(-i2\pi mk/M)\, z_k \quad \text{for } 0 \le m \le M-1,$$

where the arrays $\{z_k\}_0^{M-1}$ and $\{Z_m\}_0^{M-1}$ are handled directly, including the zero-subscript terms z_0 and Z_0.

A detailed explanation of the first program below for Chi-Squared random variables is as follows: line 20 specifies the parameter K, where 2K is the number of squared-Gaussian random variables summed to yield random variable x. Lines 30-60 require inputs L, Δ, b, M respectively, on the part of the user. Line 110 is the input of mean μ_x of random variable x, as evaluated analytically from characteristic function $f_x(\xi)$. Lines 180-210 specifically evaluate the characteristic function $f_y(\xi)$ at general point ξ. All of these lines mentioned thus far require inputs on the part of the user and are so noted in the listing by the presence of a single ! on each line; the comments after a double !! are for information purposes only and need not be modified. This convention is also adopted in the remaining listings.

Lines 220-240 accomplish the collapsing operation of (32)-(33). The cumulative and exceedance distribution functions are finally evaluated and stored in arrays X(*) and Y(*) in lines 400-410.

Some further elaboration is necessary for the listing of the Smirnov characteristic function as given by (42). Since a characteristic function is a continuous function of real ξ, the square root in (42) is <u>not</u> a principal value square root, but in fact must yield a continuous function in ξ. In

order to achieve this, the argument of the square root is traced continuously from ξ=0 (line 110). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 250) and corrects the final values of characteristic function $f_y(\xi)$ (lines 260-270). No problems are encountered with complex sin(z) since it is analytic for all z.

```
10 ! CHI-SQUARE CHARACTERISTIC FUNCTION 1/(1-i 2 xi)^4
20    K=4                      !  2K=8 degrees of freedom
30    L=200                    !  Limit on integral of char. function
40    Delta=.075               !  Sampling increment on char. function
50    Bs=0                     !  Shift b
60    M=2^8                    !  Size of FFT
70    PRINTER IS 0
80    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
90    REDIM X(0:M-1),Y(0:M-1)
100   DIM X(0:1023),Y(0:1023)
110   Mux=2*K                                !  Mean of random variable x
120   Muy=Mux+Bs
130   X(0)=0
140   Y(0)=.5*Delta*Muy
150   N=INT(L/Delta)
160   FOR Ns=1 TO N
170   Xi=Delta*Ns                          !!  Argument xi of char. fn.
180   C=Xi+Xi                               !  Calculation of
190   CALL Mul(1,-C,1,-C,A,B)               !  characteristic
200   CALL Mul(A,B,A,B,C,D)                 !  function fy(xi)
210   CALL Div(1,0,C,D,Fyr,Fyi)             !  for K=4
220   Ms=Ns MOD M                          !!  Collapsing
230   X(Ms)=X(Ms)+Fyr/Ns
240   Y(Ms)=Y(Ms)+Fyi/Ns
250   NEXT Ns
260   CALL Fft10z(M,X(*),Y(*))             !!  0 subscript FFT
```

```
270   PLOTTER IS "GRAPHICS"
280   GRAPHICS
290   SCALE 0,M,-14,0
300   LINE TYPE 3
310   GRID M/8,1
320   PENUP
330   LINE TYPE 1
340   B=Bs*M*Delta/(2*PI)                !!  Origin for random variable x
350   MOVE B,0
360   DRAW B,-14
370   PENUP
380   FOR Ks=0 TO M-1
390   T=Y(Ks)/PI-Ks/M
400   X(Ks)=.5-T                         !!  Cumulative probability in X(*)
410   Y(Ks)=Pr=.5+T                      !!  Exceedance probability in Y(*)
420   IF Pr>=1E-12 THEN Y=LGT(Pr)
430   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
440   IF ABS(Pr)<1E-12 THEN Y=-12
450   PLOT Ks,Y
460   NEXT Ks
470   PENUP
480   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
490   FOR Ks=0 TO M-1
500   Pr=X(Ks)
510   IF Pr>=1E-12 THEN Y=LGT(Pr)
520   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
530   IF ABS(Pr)<1E-12 THEN Y=-12
540   PLOT Ks,Y
550   NEXT Ks
560   PENUP
570   PAUSE
580   DUMP GRAPHICS
590   PRINT LIN(5)
600   PRINTER IS 16
610   END
620   !
630   SUB Mul(X1,Y1,X2,Y2,A,B)           !  Z1*Z2
640   A=X1*X2-Y1*Y2
650   B=X1*Y2+X2*Y1
660   SUBEND
670   !
680   SUB Div(X1,Y1,X2,Y2,A,B)           !  Z1/Z2
690   T=X2*X2+Y2*Y2
700   A=(X1*X2+Y1*Y2)/T
710   B=(Y1*X2-X1*Y2)/T
720   SUBEND
730   !
740   SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10 ! GAUSSIAN CHARACTERISTIC FUNCTION exp(-.5 xi^2)
20    L=7                      !  Limit on integral of char. function
30    Delta=.3                 !  Sampling increment on char. function
40    Bs=.375*(2*PI/Delta)     !  Shift b, as fraction of alias interval
50    M=2^8                    !  Size of FFT
60    PRINT IS 0
70    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
80    REDIM X(0:M-1),Y(0:M-1)
90    DIM X(0:1023),Y(0:1023)
100   Mux=0                                !  Mean of random variable x
110   Muy=Mux+Bs
120   X(0)=0
130   Y(0)=.5*Delta*Muy
140   N=INT(L/Delta)
150   FOR Ns=1 TO N
160   Xi=Delta*Ns                         !!  Argument xi of char. fn.
170   A=EXP(-.5*Xi*Xi)                     !  Calculation of
180   B=Bs*Xi                              !  characteristic
190   Fyr=A*COS(B)                         !  function
200   Fyi=A*SIN(B)                         !  fy(xi)
210   Ms=Ns MOD M                         !!  Collapsing
220   X(Ms)=X(Ms)+Fyr/Ns
230   Y(Ms)=Y(Ms)+Fyi/Ns
240   NEXT Ns
250   CALL Fft10z(M,X(*),Y(*))            !!  0 subscript FFT
260   PLOTTER IS "GRAPHICS"
270   GRAPHICS
280   SCALE 0,M,-14,0
290   LINE TYPE 3
300   GRID M/8,1
310   PENUP
320   LINE TYPE 1
330   B=Bs*M*Delta/(2*PI)                 !!  Origin for random variable x
340   MOVE B,0
350   DRAW B,-14
360   PENUP
370   FOR Ks=0 TO M-1
380   T=Y(Ks)/PI-Ks/M
390   X(Ks)=.5-T                          !!  Cumulative probability in X(*)
400   Y(Ks)=Pr=.5+T                       !!  Exceedance probability in Y(*)
410   IF Pr>=1E-12 THEN Y=LGT(Pr)
420   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
430   IF ABS(Pr)<1E-12 THEN Y=-12
440   PLOT Ks,Y
450   NEXT Ks
460   PENUP
470   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
480   FOR Ks=0 TO M-1
490   Pr=X(Ks)
500   IF Pr>=1E-12 THEN Y=LGT(Pr)
510   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
520   IF ABS(Pr)<1E-12 THEN Y=-12
530   PLOT Ks,Y
540   NEXT Ks
550   PENUP
560   PAUSE
570   DUMP GRAPHICS
580   PRINT LIN(5)
590   PRINTER IS 16
600   END
610   !
620   SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscri
```

```
10 ! SMIRNOV CHARACTERISTIC FUNCTION [s/sin(s)]^1/2 where s=(1+i)sqr(xi)
20    L=3000                  !  Limit on integral of char. function
30    Delta=1                 !  Sampling increment on char. function
40    Bs=0                    !  Shift b
50    M=2^8                   !  Size of FFT
60    PRINTER IS 0
70    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
80    REDIM X(0:M-1),Y(0:M-1)
90    DIM X(0:1023),Y(0:1023)
100   Mux=1/6                                !  Mean of random variable x
110   R=0                                    !  Argument of square root
120   P=1                                    !  Polarity indicator
130   Muy=Mux+Bs
140   X(0)=0
150   Y(0)=.5*Delta*Muy
160   N=INT(L/Delta)
170   FOR Ns=1 TO N
180   Xi=Delta*Ns                           !!  Argument xi of char. fn.
190   A=SQR(Xi)                              !  Calculation
200   CALL Sin(A,A,B,C)                      !  of
210   CALL Div(A,A,B,C,D,E)                  !  characteristic
220   CALL Sqr(D,E,A,B)                      !  function
230   Ro=R                                   !  fy(xi)
240   R=ATN(B/A)                             !
250   IF ABS(R-Ro)>1.6 THEN P=-P             !
260   Fyr=A*P                                !
270   Fyi=B*P                                !
280   Ms=Ns MOD M                           !!  Collapsing
290   X(Ms)=X(Ms)+Fyr/Ns
300   Y(Ms)=Y(Ms)+Fyi/Ns
310   NEXT Ns
320   CALL Fft10z(M,X(*),Y(*))              !!  0 subscript FFT
330   PLOTTER IS "GRAPHICS"
340   GRAPHICS
350   SCALE 0,M,-14,0
360   LINE TYPE 3
370   GRID M/8,1
380   PENUP
390   LINE TYPE 1
400   B=Bs*M*Delta/(2*PI)                   !!  Origin for random variable x
410   MOVE B,0
420   DRAW B,-14
430   PENUP
440   FOR Ks=0 TO M-1
450   T=Y(Ks)/PI-Ks/M
460   X(Ks)=.5-T                            !!  Cumulative probability in X(*)
470   Y(Ks)=Pr=.5+T                         !!  Exceedance probability in Y(*)
480   IF Pr>=1E-12 THEN Y=LGT(Pr)
490   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
500   IF ABS(Pr)<1E-12 THEN Y=-12
510   PLOT Ks,Y
520   NEXT Ks
```

```
530   PENUP
540   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
550   FOR Ks=0 TO M-1
560   Pr=X(Ks)
570   IF Pr>=1E-12 THEN Y=LGT(Pr)
580   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
590   IF ABS(Pr)<1E-12 THEN Y=-12
600   PLOT Ks,Y
610   NEXT Ks
620   PENUP
630   PAUSE
640   DUMP GRAPHICS
650   PRINT LIN(5)
660   PRINTER IS 16
670   END
680   !
690   SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
700   T=X2*X2+Y2*Y2
710   A=(X1*X2+Y1*Y2)/T
720   B=(Y1*X2-X1*Y2)/T
730   SUBEND
740   !
750   SUB Sqr(X,Y,A,B)                  !  PRINCIPAL SQR(Z)
760   IF X<>0 THEN 800
770   A=B=SQR(.5*ABS(Y))
780   IF Y<0 THEN B=-B
790   GOTO 910
800   F=SQR(SQR(X*X+Y*Y))
810   T=.5*ATN(Y/X)
820   A=F*COS(T)
830   B=F*SIN(T)
840   IF X>0 THEN 910
850   T=A
860   A=-B
870   B=T
880   IF Y>=0 THEN 910
890   A=-A
900   B=-B
910   SUBEND
920   !
930   SUB Sin(X,Y,A,B)                  !  SIN(Z)
940   E=EXP(Y)
950   A=.5*SIN(X)*(E+1/E)
960   IF ABS(Y)<.1 THEN 990
970   S=.5*(E-1/E)
980   GOTO 1010
990   S=Y*Y
1000  S=Y*(120+S*(20+S))/120
1010  B=COS(X)*S
1020  SUBEND
1030  !
1040  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscri
```

```
10 ! NON-CENTRAL CHI-SQUARE CHARACTERISTIC FUNCTION
20 ! exp(i d^2 xi / s) / s^nu   where s = 1-i 2 xi
30    Nu=2.7                  !  Power law nu
40    Ds=3                    !  Deflection d
50    L=500                   !  Limit on integral of char. function
60    Delta=.05               !  Sampling increment on char. function
70    Bs=0                    !  Shift b
80    M=2^8                   !  Size of FFT
90    PRINTER IS 0
100   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
110   REDIM X(0:M-1),Y(0:M-1)
120   DIM X(0:1023),Y(0:1023)
130   D2=Ds*Ds                              !  Calculate parameter
140   Mux=2*Nu+D2                           !  Mean of random variable x
150   Muy=Mux+Bs
160   X(0)=0
170   Y(0)=.5*Delta*Muy
180   N=INT(L/Delta)
190   FOR Ns=1 TO N
200   Xi=Delta*Ns                          !!  Argument xi of char. fn.
210   T=Xi+Xi                               !  Calculation of
220   CALL Div(0,D2*Xi,1,-T,A,B)            !  characteristic
230   CALL Log(1,-T,C,D)                    !  function
240   CALL Exp(A-Nu*C,B-Nu*D+Bs*Xi,Fyr,Fyi)    ! fy(xi)
250   Ms=Ns MOD M                          !!  Collapsing
260   X(Ms)=X(Ms)+Fyr/Ns
270   Y(Ms)=Y(Ms)+Fyi/Ns
280   NEXT Ns
290   CALL Fft10z(M,X(*),Y(*))             !!  0 subscript FFT
300   PLOTTER IS "GRAPHICS"
310   GRAPHICS
320   SCALE 0,M,-14,0
330   LINE TYPE 3
340   GRID M/8,1
350   PENUP
360   LINE TYPE 1
370   B=Bs*M*Delta/(2*PI)                  !!  Origin for random variable x
380   MOVE B,0
390   DRAW B,-14
400   PENUP
410   FOR Ks=0 TO M-1
420   T=Y(Ks)/PI-Ks/M
430   X(Ks)=.5-T                           !!  Cumulative probability in X(*)
440   Y(Ks)=Pr=.5+T                        !!  Exceedance probability in Y(*)
450   IF Pr>=1E-12 THEN Y=LGT(Pr)
460   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
470   IF ABS(Pr)<1E-12 THEN Y=-12
480   PLOT Ks,Y
490   NEXT Ks
500   PENUP
```

```
510   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
520   FOR Ks=0 TO M-1
530   Pr=X(Ks)
540   IF Pr>=1E-12 THEN Y=LGT(Pr)
550   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
560   IF ABS(Pr)<1E-12 THEN Y=-12
570   PLOT Ks,Y
580   NEXT Ks
590   PENUP
600   PAUSE
610   DUMP GRAPHICS
620   PRINT LIN(5)
630   PRINTER IS 16
640   END
650   !
660   SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
670   T=X2*X2+Y2*Y2
680   A=(X1*X2+Y1*Y2)/T
690   B=(Y1*X2-X1*Y2)/T
700   SUBEND
710   !
720   SUB Exp(X,Y,A,B)                  !  EXP(Z)
730   T=EXP(X)
740   A=T*COS(Y)
750   B=T*SIN(Y)
760   SUBEND
770   !
780   SUB Log(X,Y,A,B)                  !  PRINCIPAL LOG(Z)
790   A=.5*LOG(X*X+Y*Y)
800   IF X<>0 THEN 830
810   B=.5*PI*SGN(Y)
820   GOTO 850
830   B=ATN(Y/X)
840   IF X<0 THEN B=B+PI*(1-2*(Y(0))
850   SUBEND
860   !
870   SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscrip
```

```
10 ! GAUSSIAN PRODUCT CHARACTERISTIC FUNCTION (56)
20    Nu=7.7                    !  Power Nu
30    Rho=-.3                   !  Correlation coefficient
40    L=5                       !  Limit on integral of char. function
50    Delta=.06                 !  Sampling increment on char. function
60    Bs=.5*(2*PI/Delta)        !  Shift b, as fraction of alias interval
70    M=2^8                     !  Size of FFT
80    PRINTER IS 0
90    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
100   REDIM X(0:M-1),Y(0:M-1)
110   DIM X(0:1023),Y(0:1023)
120   T1=1-Rho*Rho                          !  Calculate
130   T2=2*Rho                              !  parameters
140   Mux=2*Nu*Rho                          !  Mean of random variable x
150   Muy=Mux+Bs
160   X(0)=0
170   Y(0)=.5*Delta*Muy
180   N=INT(L/Delta)
190   FOR Ns=1 TO N
200   Xi=Delta*Ns                        !!  Argument xi of char. fn.
210   CALL Log(1+T1*Xi*Xi,-T2*Xi,A,B)    !  Calculation of
220   CALL Exp(-Nu*A,Bs*Xi-Nu*B,Fyr,Fyi)!  characteristic function fy(xi)
230   Ms=Ns MOD M                        !!  Collapsing
240   X(Ms)=X(Ms)+Fyr/Ns
250   Y(Ms)=Y(Ms)+Fyi/Ns
260   NEXT Ns
270   CALL Fft10z(M,X(*),Y(*))           !!  0 subscript FFT
280   PLOTTER IS "GRAPHICS"
290   GRAPHICS
300   SCALE 0,M,-14,0
310   LINE TYPE 3
320   GRID M/8,1
330   PENUP
340   LINE TYPE 1
350   B=Bs*M*Delta/(2*PI)                !!  Origin for random variable x
360   MOVE B,0
370   DRAW B,-14
380   PENUP
```

```
390   FOR Ks=0 TO M-1
400   T=Y(Ks)/PI-Ks/M
410   X(Ks)=.5-T                          !!  Cumulative probability in X(*)
420   Y(Ks)=Pr=.5+T                       !!  Exceedance probability in Y(*)
430   IF Pr>=1E-12 THEN Y=LGT(Pr)
440   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
450   IF ABS(Pr)<1E-12 THEN Y=-12
460   PLOT Ks,Y
470   NEXT Ks
480   PENUP
490   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
500   FOR Ks=0 TO M-1
510   Pr=X(Ks)
520   IF Pr>=1E-12 THEN Y=LGT(Pr)
530   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
540   IF ABS(Pr)<1E-12 THEN Y=-12
550   PLOT Ks,Y
560   NEXT Ks
570   PENUP
580   PAUSE
590   DUMP GRAPHICS
600   PRINT LIN(5)
610   PRINTER IS 16
620   END
630   !
640   SUB Exp(X,Y,A,B)                    !  EXP(Z)
650   T=EXP(X)
660   A=T*COS(Y)
670   B=T*SIN(Y)
680   SUBEND
690   !
700   SUB Log(X,Y,A,B)                    !  PRINCIPAL LOG(Z)
710   A=.5*LOG(X*X+Y*Y)
720   IF X<>0 THEN 750
730   B=.5*PI*SGN(Y)
740   GOTO 770
750   B=ATN(Y/X)
760   IF X<0 THEN B=B+PI*(1-2*(Y<0))
770   SUBEND
780   !
790   SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10     SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
20     DIM C(0:256)
30     INTEGER I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,J,K
40     DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998795456205,.998475580573,.998118112900
50     DATA .997723066644,.997290456679,.996820299291,.996312612183,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
60     DATA .991709753669,.990902635428,.990058210262,.989176509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
70     DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
80     DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
90     DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.939459223602,.937339011913,.935183509939,.932992798835
100    DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
110    DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
120    DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
130    DATA .848120344803,.844853565250,.841554977437,.838224705555,.8348€2874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
140    DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
150    DATA .776888465673,.773010453363,.769103337646,.765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
160    DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
170    DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
180    DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763,.615231590581,.610382806276,.605511041404
190    DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745887,.565731810784,.560661576197,.555570233020
200    DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
210    DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
220    DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
230    DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717312,.336889853392
240    DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
250    DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058108281,.225083911360,.219101240157
260    DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068664150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
270    DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.980171403296E-1
280    DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
290    DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
300    READ C(*)
310    K=1024/N
320    FOR J=0 TO N/4
330    C(J)=C(K*J)
340    NEXT J
350    N1=N/4
360    N2=N1+1
370    N3=N2+1
```

```
380   N4=N1+N3
390   Log2n=INT(1.4427*LOG(N)+.5)
400   FOR I1=1 TO Log2n
410   I2=2^(Log2n-I1)
420   I3=2*I2
430   I4=N/I3
440   FOR I5=1 TO I2
450   I6=(I5-1)*I4+1
460   IF I6<=N2 THEN 500
470   N6=-C(N4-I6-1)
480   N7=-C(I6-N1-1)
490   GOTO 520
500   N6=C(I6-1)
510   N7=-C(N3-I6-1)
520   FOR I7=0 TO N-I3 STEP I3
530   I8=I7+I5
540   I9=I8+I2
550   N8=X(I8-1)-X(I9-1)
560   N9=Y(I8-1)-Y(I9-1)
570   X(I8-1)=X(I8-1)+X(I9-1)
580   Y(I8-1)=Y(I8-1)+Y(I9-1)
590   X(I9-1)=N6*N8-N7*N9
600   Y(I9-1)=N6*N9+N7*N8
610   NEXT I7
620   NEXT I5
630   NEXT I1
640   I1=Log2n+1
650   FOR I2=1 TO 10                    ! 2^10=1024
660   C(I2-1)=1
670   IF I2>Log2n THEN 690
680   C(I2-1)=2^(I1-I2)
690   NEXT I2
700   K=1
710   FOR I1=1 TO C(9)
720   FOR I2=I1 TO C(8) STEP C(9)
730   FOR I3=I2 TO C(7) STEP C(8)
740   FOR I4=I3 TO C(6) STEP C(7)
750   FOR I5=I4 TO C(5) STEP C(6)
760   FOR I6=I5 TO C(4) STEP C(5)
770   FOR I7=I6 TO C(3) STEP C(4)
780   FOR I8=I7 TO C(2) STEP C(3)
790   FOR I9=I8 TO C(1) STEP C(2)
800   FOR I10=I9 TO C(0) STEP C(1)
810   J=I10
820   IF K>J THEN 890
830   A=X(K-1)
840   X(K-1)=X(J-1)
850   X(J-1)=A
860   A=Y(K-1)
870   Y(K-1)=Y(J-1)
880   Y(J-1)=A
890   K=K+1
900   NEXT I10
910   NEXT I9
920   NEXT I8
930   NEXT I7
940   NEXT I6
950   NEXT I5
960   NEXT I4
970   NEXT I3
980   NEXT I2
990   NEXT I1
1000  SUBEND
```

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15 October 1983

Exact Performance of General Second-Order Processors for Gaussian Inputs

A. H. Nuttall

ABSTRACT

The characteristic function of general second-order processors with nonstationary nonzero mean Gaussian inputs is derived in closed form. Three classes of processors are considered; in the first, the decision variable is the sum of K independent terms, each of second-order form involving two statistically dependent Gaussian random variables; the second class of processor is a narrowband crosscorrelator of arbitrary Gaussian processes, with accumulation of K independent lowpass filter output samples; in the third class, the decision variable is a general quadratic-plus-linear form of M random variables, all statistically dependent on each other. Specializations to various forms of weighted energy detectors and correlators are made. Also, the characteristic function for the first class of processor subject to fading is evaluated.

Programs for evaluating the cumulative and exceedance distribution functions of all three classes of processors are given and have been used to plot representative examples of performance. A comparison with a simulation result corroborates the analysis and program of the first class of processor.

TABLE OF CONTENTS

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

s,t	jointly-Gaussian random variables
a,b,c,d,e	constant weights
x	output random variable of second-order processor
$f_x(\xi)$	characteristic function of random variable x
overbar	statistical average
D_1, D_2	denominator constants in characteristic function
N_0, N_1, N_2	numerator constants in characteristic function
υ_x	mean of random variable x
σ_x^2	variance of random variable x
K	number of independent terms in random variables x and v
M_{ij}	mean parameters, (12)
r	power scale factor, (18)-(19)
$s_1(t)$, $s_2(t)$	input signals to narrowband cross-correlator
$n_1(t)$, $n_2(t)$	input noises to narrowband cross-correlator
z(t)	lowpass filter output
v	narrowband cross-correlator system output random variable
$a_j(t)$, $b_j(t)$	in-phase and quadrature components of $s_j(t)$; j = 1,2
$A_j(t)$, $P_j(t)$	envelope and phase modulation of $s_j(t)$
$x_j(t)$, $y_j(t)$	in-phase and quadrature components of $n_j(t)$
σ_j^2	variance of $x_j(t)$ and $y_j(t)$, (27)-(28)
γ	complex correlation coefficient of noises, (27)
ρ,λ	real and imaginary parts of γ, (27)
w(k)	weight in narrowband cross-correlator
$\upsilon_z(s+n)$	mean of random variable z with signal and noise present
R_z	signal-to-noise ratio of random variable z
M	size of Hermitian and quadratic forms
X	random complex vector, Mx1
A	constant complex linear weight matrix
B	constant complex Hermitian weight matrix

LIST OF SYMBOLS (Con'd)

a_m	elements of A
b_{mn}	elements of B
q	Hermitian and quadratic form outputs
X^H	transpose and conjugate of X
E	mean of random vector X, (46)
$\tilde{X}$	ac component of X, (46)
K	covariance matrix of X, (46)
W	linearly transformed random variables, (54)
C	modal matrix of BK, (58)
Λ	eigenvalue matrix of BK, (58)
λ_m	elements of Λ
D	auxiliary vector, (59)
d_m	elements of D, (68)
ν_m	elements of $\overline{W}$, (68)
$\varkappa_q(n)$	n^{th} cumulant of random variable q
U,V	components of X, (71)
N	number of elements in U and V; M=2N
B^{ij}	partitioned matrices of B
$A^{(j)}$	partitioned vectors of A
u_n, v_n	components of U, V respectively
K_{uv}	partitioned matrices of K, (73)
K_o	common covariance matrix in special case, (89)
Q	modal matrix of K_o, (92)
Γ	eigenvalue matrix of K_o, (92)
γ_n	elements of Γ

EXACT PERFORMANCE OF GENERAL SECOND-ORDER PROCESSORS FOR GAUSSIAN INPUTS

INTRODUCTION

The performance of weighted energy detectors and correlators for processing deterministic and/or random signals in the presence of nonstationary noise is a topic of frequent interest. Most often, a second-moment approach is adopted, whereby the means and variances of the decision variable under the various hypotheses are evaluated and employed in a central limit assumption to get approximate false alarm and/or detection probabilities. This approach is suspect for small false alarm probabilities or for cases where the decision variable is not the sum of a large number of independent random variables all of comparable variance.

A recent technical report [1] has presented an accurate and efficient method for evaluating cumulative and exceedance distribution functions directly from characteristic functions. This approach is very fruitful for determining the performance of general time-varying second-order processors with nonstationary nonzero mean Gaussian inputs, since the characteristic function of the decision variable can be evaluated in closed form in these cases.

We will consider three classes of processors and derive the characteristic functions for all three decision variables in closed form. The first two classes are special cases of the third, but are of interest in their own right, since they include and immediately reduce to many practical processors in current use. Also there is no need to solve for the eigenvalues and eigenvectors of a general symmetric matrix that is encountered in the third more-general class of processors. Rather, the characteristic functions are given directly in terms of specified processor weights and input statistics.

There has been considerable effort on this problem in the past; for example, see [2,3] and the references listed therein. Most of the lengthy analytical derivations and results have been aimed at getting workable

expressions for the probability density function and/or cumulative distribution function. Here, when we consider our three classes of processors, we encounter characteristic functions which are more general than that given in the recent paper for a filtered analog processor [3, eq. 5]; thus specialization of our results will yield those of [3] and the references listed therein. The technique employed here to proceed directly to the cumulative and exceedance distribution functions is a numerical one, as given in [1], and does not require any series expansions or analytical manipulations at all. The asymptotic behaviors of the cumulative and exceedance distribution functions on both tails are easily observed and will be found to corroborate the comment made in [3, p. 673] that these tails are generally exponential rather than Gaussian; however, there can be a considerable transition region.

The programs listed in the appendices require the user merely to input his processor weights, signal constants, and noise statistical parameters in a series of data statements at the top of the program, and to select values for

L, limit on integral of characteristic function,
Δ, sampling increment on characteristic function,
b, additive constant, to guarantee a positive random variable, and
M_f, size of FFT and storage required.

Selection of L and Δ is largely a matter of trial and error and is amply documented in the examples in [1].

A PARTICULAR SECOND-ORDER PROCESSOR

Before we embark on the analysis of the particular second-order processor of interest in this section, we solve the following simpler statistical problem. Let s and t be real jointly-Gaussian random variables with means m_s, m_t, standard deviations σ_s, σ_t, and correlation coefficient ρ; thus s and t are statistically dependent. Consider the random variable

$$x = as^2 + bt^2 + cst + ds + et, \tag{1}$$

where weightings a, b, c, d, e are arbitrary real constants. The characteristic function of random variable x is defined by*

$$f_x(\xi) = \overline{\exp(i\xi x)} = \overline{\exp(i\xi(as^2 + bt^2 + cst + ds + et))} =$$

$$= \iint du\, dv\, \exp(i\xi(au^2 + bv^2 + cuv + du + ev))\, p_{st}(u,v), \tag{2}$$

where the joint probability density function of s and t is

$$p_{st}(u,v) = \left(2\pi\, \sigma_s\sigma_t\sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{\left(\frac{u-m_s}{\sigma_s}\right)^2 + \left(\frac{v-m_t}{\sigma_t}\right)^2 - 2\rho\left(\frac{u-m_s}{\sigma_s}\right)\left(\frac{v-m_t}{\sigma_t}\right)}{2(1-\rho^2)}\right]. \tag{3}$$

Substitution of (3) in (2) and use of the double integral

$$\iint dx\, dy\, \exp[-\alpha x^2 - \beta y^2 + 2\gamma xy + ux + vy] = \frac{\pi}{\sqrt{\alpha\beta-\gamma^2}} \exp\left[\frac{\beta u^2 + \alpha v^2 + 2\gamma uv}{4(\alpha\beta-\gamma^2)}\right]$$

$$\text{for} \quad \alpha_r > 0, \quad \beta_r > 0, \quad \alpha_r\beta_r > \gamma_r^2, \tag{4}$$

---

* Integrals without limits are over $(-\infty, +\infty)$.

(where sub r denotes the real part of complex constants α, β, γ, u, v) yields, after an extensive amount of manipulations, the characteristic function of random variable x as the compact closed form expression

$$f_x(\xi) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-1/2} \exp\left[i\xi \frac{N_0 - i\xi N_1 - \xi^2 N_2}{1 - i\xi D_1 - \xi^2 D_2}\right] . \qquad (5)$$

The required real constants in (5) are given directly in terms of the processor weights and statistical parameters as

$$D_1 = 2(a\sigma_s^2 + b\sigma_t^2 + c\rho\sigma_s\sigma_t) ,$$

$$D_2 = (4ab - c^2)(1 - \rho^2)\sigma_s^2\sigma_t^2 ,$$

$$N_0 = am_s^2 + bm_t^2 + cm_s m_t + dm_s + em_t ,$$

$$N_1 = (4ab - c^2)\left(\tfrac{1}{2} m_s^2\sigma_t^2 + \tfrac{1}{2} m_t^2\sigma_s^2 - \rho m_s m_t\sigma_s\sigma_t\right) +$$

$$+ (2ae - cd)\sigma_s(m_t\sigma_s - \rho m_s\sigma_t) +$$

$$+ (2bd - ce)\sigma_t(m_s\sigma_t - \rho m_t\sigma_s) -$$

$$- \left(\tfrac{1}{2} d^2\sigma_s^2 + \tfrac{1}{2} e^2\sigma_t^2 + de\rho\sigma_s\sigma_t\right) ,$$

$$N_2 = -(ae^2 + bd^2 - cde)(1-\rho^2)\sigma_s^2\sigma_t^2 . \qquad (6)$$

For later reference, the mean and variance of x follow from (5), upon expansion of $\ln f_x(\xi)$ in a power series in ξ, as

$$u_x = N_0 + \tfrac{1}{2} D_1,$$

$$\sigma_x^2 = \tfrac{1}{2} D_1^2 + 2N_0 D_1 - D_2 - 2N_1. \qquad (7)$$

(When $D_1 = 0$ in (6), it can be shown that $D_2 < 0$; thus characteristic function (5) never possesses any singularities along the real ξ axis.)

Second-Order Processor

Now let x be the sum of K independent terms of the form of (1):

$$x = \sum_{k=1}^{K} (a_k s_k^2 + b_k t_k^2 + c_k s_k t_k + d_k s_k + e_k t_k) \quad , \tag{8}$$

where real constants a_k, b_k, c_k, d_k, e_k can depend arbitrarily on k, and where means* m_{sk}, m_{tk}, standard deviations σ_{sk}, σ_{tk}, and correlation coefficients ρ_k are unrestricted (except that $\sigma_{sk} \geq 0$, $\sigma_{tk} \geq 0$, $|\rho_k| \leq 1$). The pair of random variables s_k, t_k is statistically independent of the pair s_n, t_n for all $k \neq n$. Thus random variable x is composed of a sum of K groups of random variables, where each group is statistically independent of every other group, but each group itself contains two statistically dependent random variables.

This processor in (8) is the general form of interest in this section. It can be time-varying when the weights $\{a_k, b_k, c_k, d_k, e_k\}$ vary with k, and nonstationary when the statistical parameters $\{m_{sk}, m_{tk}, \sigma_{sk}, \sigma_{tk}, \rho_k\}$ vary with k.

The characteristic function of (8) follows from (5) as

$$f_x(\xi) = \left[\prod_{k=1}^{K} \{1 - i\xi D_1(k) - \xi^2 D_2(k)\}\right]^{-1/2} *$$

$$* \exp\left[i\xi \sum_{k=1}^{K} \frac{N_0(k) - i\xi N_1(k) - \xi^2 N_2(k)}{1 - i\xi D_1(k) - \xi^2 D_2(k)}\right] \quad , \tag{9}$$

* These means can be interpreted as the deterministic signal components of the channels s and t, if desired.

where the identification of $D_1(k)$, etc., is obvious from (6). Only one (continuous) square root and one exponential per ζ value is required in (9), regardless of the number of terms added, K. The mean and variance of random variable x in (8) follows from (9) as

$$\upsilon_x = \sum_{k=1}^{K} \left[N_0(k) + \frac{1}{2} D_1(k) \right],$$

$$\sigma_x^2 = \sum_{k=1}^{K} \left[\frac{1}{2} D_1^2(k) + 2N_0(k)\, D_1(k) - D_2(k) - 2N_1(k) \right]. \tag{10}$$

Any analytical attempt at determining the probability density function or cumulative distribution function corresponding to characteristic function (9) would be a formidable task indeed. However, it is a very simple task via the method of [1] to get accurate numerical values for the cumulative and exceedance distribution functions. The program listing in appendix A accomplishes this task, based upon characteristic function (9) and the constants listed in (6). All the weights $\{a_k, b_k, c_k, d_k, e_k\}_1^K$ and statistical parameters $\{m_{sk}, m_{tk}, \sigma_{sk}, \sigma_{tk}, \rho_k\}_1^K$ are arbitrary. Observe that (9) is far more general than the characteristic function considered in [3, eq. 5], which itself required a very lengthy analytic treatment to get the probability density function and cumulative distribution function. In fact, there is little hope of getting any tractable analytic results for (9) when K is greater than 2.

Special Case 1

Suppose weightings a, b, c, d, e in (8) are independent of k and that statistics σ_s, σ_t, ρ are also independent of k. The decision variable x in (8) then simplifies to

$$x = \sum_{k=1}^{K} (as_k^2 + bt_k^2 + cs_k t_k + ds_k + et_k) \quad . \tag{11}$$

Then D_1, D_2, N_2 are independent of k. If we define mean parameters

$$M_{20} = \sum_{k=1}^{K} m_{sk}^2 , \quad M_{02} = \sum_{k=1}^{K} m_{tk}^2 , \quad M_{11} = \sum_{k=1}^{K} m_{sk}\, m_{tk} ,$$

$$M_{10} = \sum_{k=1}^{K} m_{sk} , \quad M_{01} = \sum_{k=1}^{K} m_{tk} , \tag{12}$$

the characteristic function of x in (9) then takes the simpler form

$$f_x(\xi) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-K/2} \exp\left[i\xi \frac{N_0' - i\xi N_1' - \xi^2 N_2'}{1 - i\xi D_1 - \xi^2 D_2} \right] , \tag{13}$$

where D_1 and D_2 are still given by (6), and

$$N_0' = aM_{20} + bM_{02} + cM_{11} + dM_{10} + eM_{01} ,$$

$$N_1' = (4ab - c^2)(\tfrac{1}{2}\sigma_t^2 M_{20} + \tfrac{1}{2}\sigma_s^2 M_{02} - \rho\sigma_s\sigma_t M_{11}) +$$

$$+ (2ae - cd)\sigma_s(\sigma_s M_{01} - \rho\sigma_t M_{10}) +$$

$$+ (2bd - ce)\sigma_t(\sigma_t M_{10} - \rho\sigma_s M_{01}) -$$

$$- K\,(\tfrac{1}{2} d^2\sigma_s^2 + \tfrac{1}{2} e^2\sigma_t^2 + de\rho\sigma_s\sigma_t) ,$$

$$N_2' = -K(ae^2 + bd^2 - cde)(1 - \rho^2)\sigma_s^2\sigma_t^2 . \tag{14}$$

(The choice of $K = 2$ and $N_2' = 0$ in (13) corresponds to the form given in [3, eq. 5].) Observe that the characteristic function in (13) (and therefore the performance) of the processor in (11) depends on the means $\{m_{sk}\}$ and $\{m_{tk}\}$ only through the parameters $\{M_{ij}\}$ defined in (12). The mean and variance

of random variable x in (11) follow from characteristic function (13) as

$$u_x = N_0' + \frac{1}{2} KD_1,$$

$$\sigma_x^2 = \frac{1}{2} KD_1^2 + 2N_0'D_1 - KD_2 - 2N_1' . \tag{15}$$

Special Case 2

Let us also assume d = 0, e = 0 in (11) above; then the pertinent decision variable is given by

$$x = \sum_{k=1}^{K} (as_k^2 + bt_k^2 + cs_k t_k) \quad . \tag{16}$$

D_1 and D_2 are still given by (6), and there follows from (14),

$$N_0' = aM_{20} + bM_{02} + cM_{11} \quad ,$$

$$N_1' = (4ab - c^2)(\frac{1}{2} \sigma_t^2 M_{20} + \frac{1}{2} \sigma_s^2 M_{02} - \rho\sigma_s\sigma_t M_{11}) \quad ,$$

$$N_2' = 0 \quad . \tag{17}$$

The characteristic function of x is given by (13), with $N_2' = 0$. The mean and variance of x in (16) are given by (15).

Fading for Special Case 2

Let the mean parameters $\{M_{ij}\}$ in (12) be subject to slow fading; i.e., replace

$$M_{20} \text{ by } rM_{20},\ M_{02} \text{ by } rM_{02},\ M_{11} \text{ by } rM_{11}, \tag{18}$$

where power scale factor r has probability density function

$$p_r(u) = \frac{\nu^\nu}{\Gamma(\nu)}\, u^{\nu-1}\, e^{-\nu u} \quad \text{for } u > 0,\ \nu > 0;$$

$$\bar{r} = 1,\ \sigma_r^2 = \frac{1}{\nu},\ \chi_r(n) = \frac{(n-1)!}{\nu^{n-1}} \quad \text{for } n \geq 1. \tag{19}$$

This form of fading is encountered in diversity combination receivers; see, for example, [4, eq. 9 et seq.] and [5, eq. 24 et seq.]. Then (13), (17), and (18) yield the conditional characteristic function, for a specified r, as

$$f_x(\xi|r) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-K/2} \exp\left[i\xi r\, \frac{N_0' - i\xi N_1'}{1 - i\xi D_1 - \xi^2 D_2}\right] . \tag{20}$$

Weighting (20) according to the probability density function in (19), and performing the integral, there follows, for the characteristic function of the decision variable x in (16), the result

$$f_x(\xi) = \frac{\left(1 - i\xi D_1 - \xi^2 D_2\right)^{\nu - \frac{K}{2}}}{\left(1 - i\xi(D_1 + N_0'/\nu) - \xi^2(D_2 + N_1'/\nu)\right)^{\nu}} . \tag{21}$$

(The limit of (21) as $\nu \to +\infty$ is again (13) with $N_2' = 0$, as in (17); this agrees with the fact that the corresponding limit of the probability density function in (19) is $p_r(u) = \delta(u-1)$.) The mean and variance of x in (16) follow from characteristic function (21) as

$$\mu_x = N_0' + \frac{1}{2} KD_1 ,$$

$$\sigma_x^2 = \frac{1}{2} KD_1^2 + 2N_0'D_1 - KD_2 - 2N_1' + N_0'^2/\nu . \tag{22}$$

Observe that mean μ_x is independent of ν, the power law in fading (19). A program for the cumulative and exceedance distribution functions corresponding to characteristic function (21) is given in appendix B.

SPECIAL FORMS OF SECOND-ORDER PROCESSOR (8)

Before embarking on the analysis of the other two classes of processors, we will explicitly detail some of the special forms that processor (8) reduces to, under particular selections of the weightings and statistical parameters. A rather broad collection of typical processors will be seen to be included. In the following, any unspecified weights $\{a_k, b_k, c_k, d_k, e_k\}$ are zero, and any unspecified statistical parameters that do not appear in the final characteristic function are irrelevant.

I. Gaussian

$$d_k = 1$$

$$x = \sum_{k=1}^{K} s_k$$

$$f_x(\xi) = \exp\left[i\xi \sum_{k=1}^{K} m_{sk} - \frac{1}{2}\xi^2 \sum_{k=1}^{K} \sigma_{sk}^2 \right]$$

II. Chi-square of K Degrees of Freedom

$$a_k = 1, \quad m_{sk} = 0, \quad \sigma_{sk} = 1$$

$$x = \sum_{k=1}^{K} s_k^2$$

$$f_x(\xi) = \left(1 - i\xi 2\right)^{-K/2}$$

III. Non-Central Chi-Square (Q_M Distribution if K = 2M)

$$a_k = 1, \quad \sigma_{sk} = \sigma_s$$

$$x = \sum_{k=1}^{K} s_k^2$$

$$f_x(\xi) = \left(1 - i\xi 2\sigma_s^2\right)^{-K/2} \exp\left[\frac{i\xi \sum_{k=1}^{K} m_{sk}^2}{1 - i\xi 2\sigma_s^2}\right]$$

IV. Weighted Energy Detector

$$a_k \neq 0, \quad d_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + d_k s_k)$$

$$f_x(\xi) = \left[\prod_{k=1}^{K} \left\{1 - i\xi 2a_k\sigma_{sk}^2\right\}\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{a_k m_{sk}^2 + d_k m_{sk} + i\xi \frac{1}{2} d_k^2 \sigma_{sk}^2}{1 - i\xi 2a_k\sigma_{sk}^2}\right]$$

V. Weighted Cross-Correlator

$$c_k \neq 0$$

$$x = \sum_{k=1}^{K} c_k s_k t_k$$

$$D_1(k) = 2c_k\rho_k\sigma_{sk}\sigma_{tk},$$

$$D_2(k) = -c_k^2(1 - \rho_k^2)\sigma_{sk}^2\sigma_{tk}^2,$$

$$N_0(k) = c_k m_{sk} m_{tk},$$

$$N_1(k) = -c_k^2\left(\frac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \frac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}\right),$$

$$N_2(k) = 0 \quad .$$

Characteristic function $f_x(\xi)$ is given by (9).

VI. Two-Channel Energy Detector

$$a_k \neq 0, \quad b_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + b_k t_k^2)$$

$$D_1(k) = 2(a_k \sigma_{sk}^2 + b_k \sigma_{tk}^2) \ ,$$

$$D_2(k) = 4a_k b_k (1 - \rho_k^2) \sigma_{sk}^2 \sigma_{tk}^2 \ ,$$

$$N_0(k) = a_k m_{sk}^2 + b_k m_{tk}^2,$$

$$N_1(k) = 4a_k b_k\left(\frac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \frac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}\right),$$

$$N_2(k) = 0.$$

Characteristic function $f_x(\xi)$ is given by (9). A simple application of this particular processor was encountered in [6, eqs. 25-26].

VII. Two-Channel Energy Detector and Cross-Correlator

$$a_k \neq 0, \quad b_k \neq 0, \quad c_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + b_k t_k^2 + c_k s_k t_k)$$

$$D_1(k) = 2(a_k \sigma_{sk}^2 + b_k \sigma_{tk}^2 + c_k \rho_k \sigma_{sk} \sigma_{tk}) \quad ,$$

$$D_2(k) = (4a_k b_k - c_k^2)(1 - \rho_k^2)\sigma_{sk}^2 \sigma_{tk}^2 \quad ,$$

$$N_0(k) = a_k m_{sk}^2 + b_k m_{tk}^2 + c_k m_{sk} m_{tk} \quad ,$$

$$N_1(k) = (4a_k b_k - c_k^2)(\tfrac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \tfrac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}) \quad ,$$

$$N_2(k) = 0 \quad .$$

Characteristic function $f_x(\mathfrak{F})$ is given by (9).

NARROWBAND CROSS-CORRELATOR

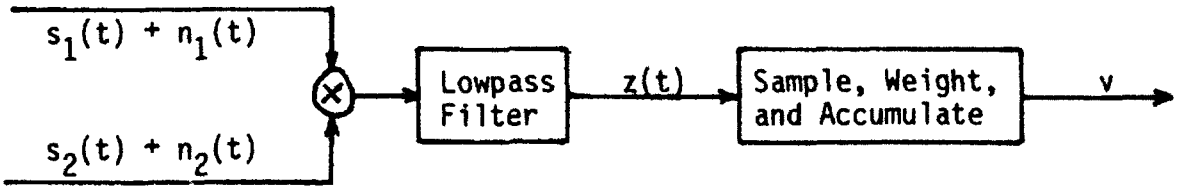

Figure 1. Narrowband Cross-Correlator

The processor of interest in this section is depicted in figure 1. Input signals $s_1(t)$ and $s_2(t)$ are arbitrary deterministic narrowband real waveforms:

$$s_j(t) = \mathrm{Re}\{\underline{s}_j(t)\exp(i2\pi f_o t)\} = A_j(t)\cos(2\pi f_o t + P_j(t)) =$$

$$= a_j(t)\cos(2\pi f_o t) - b_j(t)\sin(2\pi f_o t) \quad \text{for } j = 1,2, \tag{23}$$

where input signal complex envelope

$$\underline{s}_j(t) = A_j(t)\exp(iP_j(t)) = a_j(t) + ib_j(t) \tag{24}$$

in terms of polar or rectangular low-frequency components, respectively.

Input noises $n_1(t)$ and $n_2(t)$ are zero-mean correlated narrowband jointly-Gaussian processes which may be nonstationary:

$$n_j(t) = \mathrm{Re}\{\underline{n}_j(t)\exp(i2\pi f_o t)\} = x_j(t)\cos(2\pi f_o t) - y_j(t)\sin(2\pi f_o t), \tag{25}$$

where noise complex envelope

$$\underline{n}_j(t) = x_j(t) + iy_j(t) \quad \text{for } j = 1,2. \tag{26}$$

The statistics of the input noise complex envelopes are arbitrary:

$$\overline{|\underline{n}_1(t)|^2} = 2\sigma_1^2 ,$$

$$\overline{|\underline{n}_2(t)|^2} = 2\sigma_2^2 ,$$

$$\overline{\underline{n}_1(t)\,\underline{n}_2^*(t)} = 2\sigma_1\sigma_2\gamma, \quad \text{where } \gamma = \rho + i\lambda = |\gamma|\exp(i\phi),$$

$$\overline{\underline{n}_j(t)\,\underline{n}_m(t)} = 0 \text{ for all } j, m. \tag{27}$$

The quantities σ_1, σ_2, γ can all vary with time t, for nonstationary noise processes. There follows, for the statistics of the in-phase and quadrature components defined in (25),

$$\overline{x_1^2} = \overline{y_1^2} = \sigma_1^2, \quad \overline{x_1 y_1} = 0 ,$$

$$\overline{x_2^2} = \overline{y_2^2} = \sigma_2^2, \quad \overline{x_2 y_2} = 0 ,$$

$$\overline{x_1 x_2} = \overline{y_1 y_2} = \sigma_1\sigma_2\rho ,$$

$$\overline{x_2 y_1} = -\overline{x_1 y_2} = \sigma_1\sigma_2\lambda . \tag{28}$$

The reason for breaking out this narrowband cross-correlator as a separate problem is now apparent from (28). Namely, at each time instant, a group of four random variables are statistically dependent on each other. This case does not fall into the framework of (8) above, since only two random variables were dependent there.

Using the narrowband character of all the waveforms in (24) and (26), the lowpass filter output in figure 1 may be expressed as

$$z(t) = \tfrac{1}{2}[x_1(t) + a_1(t)][x_2(t) + a_2(t)] + \tfrac{1}{2}[y_1(t) + b_1(t)][y_2(t) + b_2(t)]. \tag{29}$$

The final system output in figure 1 is the weighted sum of K terms,

$$v = \sum_{k=1}^{K} w(k)\, z(t_k) \ , \tag{30}$$

where it is assumed that the time separations between samples at instants $\{t_k\}$ lead to statistically independent random variables $\{z(t_k)\}$. The weights and statistics can change with sample time t_k, in an arbitrary fashion.

Based upon the method in [7], we find the characteristic function of z(t) in (29) to be given by

$$f_z(\xi,t) = \frac{1}{1-i\xi D_1 + \xi^2 D_2} \exp\left[i\xi \frac{N_0 + i\xi N_1}{1-i\xi D_1 + \xi^2 D_2} \right], \tag{31}$$

where the constants (in their most compact form) are given by

$$D_1 = \sigma_1\sigma_2\rho \ ,$$

$$D_2 = \frac{1}{4}\sigma_1^2\sigma_2^2(1-\rho^2-\lambda^2) \ ,$$

$$N_0 = \frac{1}{2}(a_1a_2 + b_1b_2) \ ,$$

$$N_1 = \frac{1}{8}\left[\sigma_2^2(a_1^2 + b_1^2) + \sigma_1^2(a_2^2 + b_2^2) - 2\sigma_1\sigma_2\rho(a_1a_2 + b_1b_2) - 2\sigma_1\sigma_2\lambda(a_2b_1 - a_1b_2)\right]. \tag{32}$$

(The characteristic function and constants in (31) and (32) are not to be interchanged or confused with any earlier results in previous sections. In fact, observe there is no square root involved in (31).) All of the parameters in (32) can vary with time t.

In terms of the signal polar definitions in (24) and the complex noise correlation coefficient γ in (27), alternative expressions to (32) (where we have emphasized the t-dependence) are

$$D_1 = \sigma_1(t)\,\sigma_2(t)\,\mathrm{Re}\{\gamma(t)\} = \sigma_1(t)\,\sigma_2(t)\,|\gamma(t)|\,\cos\phi(t),$$

$$D_2 = \tfrac{1}{4}\,\sigma_1^2(t)\,\sigma_2^2(t)\,\left(1-|\gamma(t)|^2\right),$$

$$N_0 = \tfrac{1}{2}\,\mathrm{Re}\{\underline{s}_1^*(t)\,\underline{s}_2(t)\} = \tfrac{1}{2}\,A_1(t)\,A_2(t)\cos[P_1(t)-P_2(t)],$$

$$N_1 = \tfrac{1}{8}\left[\sigma_2^2(t)\,|\underline{s}_1(t)|^2 + \sigma_1^2(t)\,|\underline{s}_2(t)|^2 - 2\,\sigma_1(t)\,\sigma_2(t)\,\mathrm{Re}\{\underline{s}_1^*(t)\,\underline{s}_2(t)\,\gamma(t)\}\right] =$$

$$= \tfrac{1}{8}\left[\sigma_2^2(t)\,A_1^2(t) + \sigma_1^2(t)\,A_2^2(t) - 2\,\sigma_1(t)\,\sigma_2(t)\,A_1(t)\,A_2(t)\,|\gamma(t)|\cos[P_1(t)-P_2(t)-\phi(t)]\right]. \tag{33}$$

The mean and variance of z(t) in (29) follow from (31) as

$$u_z = D_1 + N_0,$$

$$\sigma_z^2 = D_1^2 + 2D_2 + 2D_1N_0 + 2N_1. \tag{34}$$

Finally, the characteristic function of the narrowband cross-correlator output v in (30) follows from (31) as

$$f_v(\xi) = \prod_{k=1}^{K} f_z(\xi w(k), t_k) =$$

$$= \left[\prod_{k=1}^{K}\left\{1-i\xi w(k)\,D_1(k) + \xi^2 w^2(k)\,D_2(k)\right\}\right]^{-1} \exp\left[i\xi\sum_{k=1}^{K}\frac{w(k)\,N_0(k) + i\xi w^2(k)\,N_1(k)}{1-i\xi w(k)\,D_1(k) + \xi^2 w^2(k)\,D_2(k)}\right], \tag{35}$$

where we have allowed all the parameters in (32) and (33) to vary with time t_k. The mean and variance of output v follow from (35) as

$$u_v = \sum_{k=1}^{K} w(k)\,[D_1(k) + N_0(k)],$$

$$\sigma_v^2 = \sum_{k=1}^{K} w^2(k)\left[D_1^2(k) + 2D_2(k) + 2D_1(k)N_0(k) + 2N_1(k)\right]. \tag{36}$$

A program for the evaluation of the cumulative and exceedance distribution functions via (35) is given in appendix C.

In comparison with earlier results in [3] and [7], we have obtained the following extensions here:

1. The input signals are arbitrary narrowband waveforms; they are not limited to two sine waves at the same frequency;
2. The Gaussian input noises can be nonstationary;
3. The number of terms summed to yield the narrowband cross-correlator output can be greater than 1;
4. The characteristic function is in its most compact form, and the constants are expressed directly in terms of given quantities, having eliminated all auxiliary variables.

Output Signal-to-Noise Ratio

It is sometimes desirable to have simple expressions for the output signal-to-noise ratio of the narrowband cross-correlator in figure 1. In terms of the lowpass filter output z(t), we observe first from (32)-(34) that

$$v_z(s) = v_z(s+n) - v_z(n) = N_0 = \frac{1}{2} A_1 A_2 \cos(P_1-P_2) . \tag{37}$$

We then have two alternative definitions of the signal-to-noise ratio at the lowpass filter output:

$$R_z(n) = \frac{v_z^2(s)}{\sigma_z^2(n)} = \frac{A_1^2 A_2^2 \cos^2(P_1-P_2)}{2 \sigma_1^2 \sigma_2^2(1+\rho^2-\lambda^2)} ,$$

$$R_z(s+n) = \frac{v_z^2(s)}{\sigma_z^2(s+n)} = \frac{A_1^2 A_2^2 \cos^2(P_1-P_2)}{4(D_1^2 + 2D_2 + 2D_1N_0 + 2N_1)} . \tag{38}$$

These closed form expressions allow for arbitrary noise correlations and are considerably simpler than [7, eqs. 41-43]. The signal-to-noise ratios of system output v in figure 1 are K times greater than either form in (38).

Specialization to Narrowband Energy Detector

If the signal and noise parameters in (24) and (27) are chosen as

$$a_1(t) = a_2(t) = a(t) ,$$

$$b_1(t) = b_2(t) = b(t) ,$$

$$\sigma_1(t) = \sigma_2(t) = \sigma(t) ,$$

$$\rho(t) = 1, \ \lambda(t) = 0 , \tag{39}$$

then figure 1 reduces to identical input channels, that is, a narrowband energy detector. There follows from (32),

$$D_1 = \sigma^2(t), \quad D_2 = 0 ,$$

$$N_0 = \frac{1}{2}\left(a^2(t) + b^2(t)\right) = \frac{1}{2} A^2(t), \quad N_1 = 0 , \tag{40}$$

and (31) becomes

$$f_z(\xi, t) = \frac{1}{1 - i\xi\sigma^2(t)} \exp\left[i\xi \frac{A^2(t)/2}{1 - i\xi\sigma^2(t)}\right]. \tag{41}$$

Corresponding results for the system output v are easily obtained from this.

$$P_z(u,t) = \left[1 - Q\left(\frac{A(t)}{\sigma(t)}, \frac{\sqrt{2u}}{\sigma(t)}\right)\right] U(u).$$

REDUCTION OF HERMITIAN AND LINEAR FORM

The most general case of interest in this section is as follows: random complex matrix

$$X = [x_1 \; x_2 \; \cdots \; x_M]^T \tag{42}$$

is Mx1; constant complex matrix

$$A = [a_1 \; a_2 \; \cdots \; a_M]^T \tag{43}$$

is Mx1; and constant complex matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1M} \\ & & \\ b_{M1} & \cdots & b_{MM} \end{bmatrix} \tag{44}$$

is MxM and Hermitian. The Hermitian and linear form we consider is

$$q = X^H B X + \frac{1}{2}(X^H A + A^H X) =$$

$$= \sum_{m,n=1}^{M} x_m^* b_{mn} x_n + \frac{1}{2} \sum_{m=1}^{M} (x_m^* a_m + a_m^* x_m), \tag{45}$$

which is real. Random variable q is a weighted sum of all possible products of $\{x_m^*\}$ and $\{x_n\}$, plus linear combinations.* A and B are called the weighting matrices.

---

* For M=2 or 4, and real variables and weights, (45) reduces to the earlier forms given in (1) and (29).

We will concentrate in this section on reducing form (45) to a weighted sum of squares of uncorrelated random variables. This stepping stone does not require any Gaussian assumptions on X and is therefore useful as a separate item.

The relevant statistics pertaining to random vector X are

$$\overline{X} = E \quad \text{(mean matrix)},$$

$$\tilde{X} = X - \overline{X} = X - E,$$

$$\text{Cov}\{X\} = \overline{\tilde{X}\tilde{X}^H} = K \quad \text{(covariance matrix)}, \tag{46}$$

where statistics matrices E and K are given. MxM matrix K is always Hermitian and non-negative definite. We assume K is positive definite; otherwise eliminate the linearly dependent components of $\tilde{X}$. We allow x_m and x_n to be correlated with each other for any m and n; this situation is much more general than the investigations above.

Let C be a constant MxM matrix and form the linearly transformed variables

$$W = C^H X = [w_1 \; w_2 \; \ldots \; w_M]^T. \tag{47}$$

Then the statistics of W are given by

$$\overline{W} = C^H E,$$

$$\tilde{W} = W - \overline{W} = C^H \tilde{X},$$

$$\text{Cov}\{W\} = \overline{\tilde{W}\tilde{W}^H} = C^H \overline{\tilde{X}\tilde{X}^H} C = C^H K C. \tag{48}$$

Also, from (47), since

$$X = C^{-H} W, \tag{49}$$

then we can express (45) as

$$q = W^H C^{-1} B C^{-H} W + \frac{1}{2}(W^H D + D^H W), \tag{50}$$

where we define constant Mx1 matrix

$$D = C^{-1}A = [d_1 \; d_2 \; \ldots \; d_M]^T. \tag{51}$$

We want to have, from (48) and (50),

$$C^H K C = I \tag{52}$$

and

$$C^{-1} B C^{-H} = \Lambda = \text{diag}(\lambda_1 \; \lambda_2 \; \ldots \; \lambda_M); \quad \text{i.e. } C^H B^{-1} C = \Lambda^{-1}, \tag{53}$$

for then, in addition to the relation between the means,

$$\bar{W} = C^H E, \tag{54}$$

we have the desirable properties

$$\text{Cov}\{W\} = I, \tag{55}$$

and

$$- = W^H \Lambda W + \frac{1}{2}(W^H D + D^H W) = \sum_{m=1}^{M} \lambda_m |w_m|^2 + \text{Re} \sum_{m=1}^{M} d_m^* w_m \quad . \tag{56}$$

That is, the random vector W given by (47) is composed of uncorrelated unit-variance components, and q is a weighted sum of magnitude-squares of these components, in addition to a linear sum.

We now have to address the problem of determining the MxM matrices C and Λ in (52) and (53). From [8, p. 106, Theorem 2], we identify

$$M \rightarrow K, \quad K \rightarrow B^{-1}, \; \Lambda \rightarrow \Lambda^{-1}; \tag{57}$$

then according to [8, p. 107, eq. 29], we must solve for C and Λ in the equation

$$B^{-1}C = KC\Lambda^{-1}, \quad \text{i.e.} \quad BKC = C\Lambda. \tag{58}$$

So the only matrix that need be considered is the MxM product BK. C is the modal matrix, and Λ the eigenvalue matrix, of BK. Also, from (51),

$$D = C^H A, \text{ since } C^{-1} = C^H. \tag{59}$$

Letting $C = [C^{(1)} \; \ldots \; C^{(M)}]$, where eigenvector $C^{(m)}$ is a Mx1 matrix, (58) can be expressed as

$$BKC^{(m)} = \lambda_m C^{(m)} \quad \text{for} \quad 1 \leq m \leq M. \tag{60}$$

Several important properties hold for Λ and C:

The $\{\lambda_m\}_1^M$ are all real, but can be positive, zero, or negative.
If K and B are real, then C is real.
If B is positive definite, then $\lambda_m > 0$ for $1 \leq m \leq M$.
If A = 0 and E = 0, there is no need to solve (58) for C, because D = 0 and $\overline{W} = 0$. (61)

QUADRATIC AND LINEAR FORM

If random vector X is real Gaussian, if A is real, and if B is real symmetric, then mean E and covariance K are real, and it follows that modal matrix C is also real. Also from (47) and (59), W and D are real. Equation (45) reduces to

$$q = X^T B X + X^T A = \sum_{m,n=1}^{M} x_m b_{mn} x_n + \sum_{m=1}^{M} a_m x_m \,, \tag{62}$$

which is a quadratic form and linear form.

Letting mean $\overline{W}$ in (54) be expressed as

$$\overline{W} = [\nu_1 \ \nu_2 \ \cdots \ \nu_M]^T, \tag{63}$$

the Gaussian character of X and the linear transformation (47) allow us to write the probability density function of W as a product:

$$p(W) = \prod_{m=1}^{M} \left\{(2\pi)^{-1/2} \exp\left(-\frac{1}{2}(w_m - \nu_m)^2\right)\right\}. \tag{64}$$

Here we used property (55). Since we now have, from (56),

$$q = \sum_{m=1}^{M} (\lambda_m w_m^2 + d_m w_m), \tag{65}$$

the characteristic function of q is

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp\left(i\xi \sum_{m=1}^{M} \left(\lambda_m w_m^2 + d_m w_m\right)\right)} =$$

$$= \left[\prod_{m=1}^{M} \left\{1 - i2\lambda_m \xi\right\}\right]^{-1/2} \exp\left[i\xi \sum_{m=1}^{M} \frac{\lambda_m \nu_m^2 + d_m \nu_m + i\xi d_m^2/2}{1 - i2\lambda_m \xi}\right], \tag{66}$$

where the square root must be a continuous function of ξ, not a principal value square root.* Notice that only one square root and one exponential is required per ξ value. Observe that the characteristic function depends on the separate values $\{\nu_m\}_1^M$ and $\{d_m\}_1^M$, not merely on their sums. If A = E = 0, the exponential is unity, by virtue of (54) and (59). And if M=2, (66) reduces to (5), while M=4 leads to form (31).

To summarize, the characteristic function $f_q(\xi)$ in (66) for random variable q in (62) requires the constants $\{\lambda_m\}$, $\{d_m\}$, and $\{\nu_m\}$ for $1 \leq m \leq M$. The initially given quantities are weighting matrices A, B and statistics matrices E, K. We first solve the equation (58),

$$BKC = C\Lambda, \tag{67}$$

for eigenvalue matrix Λ and modal matrix C corresponding to BK. Then

$$\Lambda = \text{diag}(\lambda_1 \ \lambda_2 \ \cdots \ \lambda_M) \ ;$$

$$D = C^T A = [d_1 \ d_2 \ \cdots \ d_M]^T \ ,$$

$$\overline{W} = C^T r = [\nu_1 \ \nu_2 \ \cdots \ \nu_M]^T \ . \tag{68}$$

If the mean of input X is zero, E = 0, and if the linear weighting form is zero, A = 0, then there is no need to solve for modal matrix C of BK in (67). Then $D = \overline{W} = 0$ and the exponential term in (66) is unity. One only need compute eigenvalue matrix Λ of BK in this case.

A program for the evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (66) is listed in appendix D. The inputs to the program are considered to be M, $\{\lambda_m\}$,

* That is, the square root is the analytic continuation of the function defined as 1 at ξ =0.

$\{d_m\}$, $\{\nu_m\}$; that is, it is presumed that (67) and (68) have already been solved prior to use of the program.

The cumulants of q are obtained from (66) as

$$\chi_q(n) = \begin{cases} \sum_{m=1}^{M} (\lambda_m + \lambda_m \nu_m^2 + d_m \nu_m) = v_q & \text{for } n = 1 \\ 2^{n-1}(n-1)! \sum_{m=1}^{M} \lambda_m^{n-2} \left[\lambda_m^2 + n\left(\lambda_m \nu_m + \frac{1}{2} d_m\right)^2 \right] & \text{for } n \geq 2 \end{cases} . \tag{69}$$

In particular, the variance of q is

$$\chi_q(2) = 2\sum_{m=1}^{M} \left[\lambda_m^2 + 2\left(\lambda_m \nu_m + \frac{1}{2} d_m\right)^2 \right] = \sigma_q^2 \quad . \tag{70}$$

If another random variable is formed by the sum of several independent random variables q_j with the form (62), but with different sizes M_j, the new characteristic function is the product of terms like (66).

Breakdown of X into Two Components

It is useful to investigate a particular version of the general results above, because the resultant forms correspond to some often-realized practical energy detectors and correlators. We let M = 2N, and

$$X = \begin{bmatrix} U \\ V \end{bmatrix} \quad , \quad B = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \quad , \quad A = \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} \quad , \tag{71}$$

where U, V, $A^{(1)}$, $A^{(2)}$ are Nx1 real matrices, and $\{B^{ij}\}$ are NxN real matrices. Also B^{11} and B^{22} are symmetric, while $B^{21} = B^{12^T}$. Then (62) can be expressed as

$$q = X^T B X + X^T A = \begin{bmatrix} U^T & V^T \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} U^T & V^T \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} =$$

$$= U^T B^{11} U + U^T B^{12} V + V^T B^{21} U + V^T B^{22} V + U^T A^{(1)} + V^T A^{(2)} =$$

$$= U^T B^{11} U + 2U^T B^{12} V + V^T B^{22} V + U^T A^{(1)} + V^T A^{(2)} =$$

$$= \sum_{m,n=1}^{N} \left(u_m \, b_{mn}^{11} \, u_n + 2u_m \, b_{mn}^{12} \, v_n + v_m \, b_{mn}^{22} \, v_n \right) + \sum_{n=1}^{N} \left(u_n a_n^{(1)} + v_n a_n^{(2)} \right) =$$

= all possible auto and cross combinations of random

variables $\{u_n\}_1^N$ and $\{v_n\}_1^N$, plus linear combinations. (72)

We also have, from (47) and (68),

$$W = C^T X = C^T \begin{bmatrix} U \\ V \end{bmatrix}, \quad \overline{W} = C^T \begin{bmatrix} \overline{U} \\ \overline{V} \end{bmatrix} = C^T \begin{bmatrix} E_u \\ E_v \end{bmatrix}, \quad D = C^T \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix},$$

$$K = \mathrm{Cov}\{X\} = \overline{\tilde{X}\tilde{X}^T} = \overline{\begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} \begin{bmatrix} \tilde{U}^T & \tilde{V}^T \end{bmatrix}} = \begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix}. \tag{73}$$

Then the fundamental matrix required in (67) is expressable as

$$BK = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix}, \tag{74}$$

which is a 2Nx2N matrix. Also random variable (65) is now

$$q = \sum_{m=1}^{2N} \left(\lambda_m w_m^2 + d_m w_m \right), \tag{75}$$

which has 2N terms. The characteristic function of q, in general, follows from (66) and (68) as

$$f_q(\xi) = \left[\prod_{m=1}^{2N} \{1-i2\lambda_m\xi\}\right]^{-1/2} \exp\left[i\xi \sum_{m=1}^{2N} \frac{\lambda_m \nu_m^2 + d_m \nu_m + i\xi d_m^2/2}{1-i2\lambda_m\xi}\right], \quad (76)$$

where

$$\bar{W} = [\nu_1 \cdots \nu_{2N}]^T = C^T \begin{bmatrix} E_u \\ E_v \end{bmatrix}, \quad D = [d_1\ d_2 \cdots d_{2N}]^T = C^T \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix}. \quad (77)$$

(If $A^{(1)} = A^{(2)} = E_u = E_v = 0$, then $D = 0$ and $\bar{W} = 0$, and there is no need to solve for modal matrix C; the exponential in $f_q(\xi)$ in (76) is then unity.)

As a special case, if $A = 0$, $B^{11} = 0$, $B^{22} = 0$, then (71) and (73) yield

$$B = \begin{bmatrix} 0 & B^{12} \\ B^{21} & 0 \end{bmatrix}, \quad D = 0\ , \quad (78)$$

and (72) gives

$$q = 2U^T B^{12} V = 2 \sum_{m,n=1}^{N} u_m\, b_{mn}^{12}\, v_n =$$

$$= \text{all possible cross combinations of } \{u_n\}_1^N \text{ and } \{v_n\}_1^N. \quad (79)$$

Then (74) specializes to

$$BK = \begin{bmatrix} 0 & B^{12} \\ B^{21} & 0 \end{bmatrix} \begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix} = \begin{bmatrix} B^{12}K_{vu} & B^{12}K_{vv} \\ B^{21}K_{uu} & B^{21}K_{uv} \end{bmatrix} \quad (80)$$

and (75) reduces to

$$q = \sum_{m=1}^{2N} \lambda_m w_m^2 \quad , \tag{81}$$

with characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{2N} \{1 - i2\lambda_m \xi\}\right]^{-1/2} \exp\left[i\xi \sum_{m=1}^{2N} \frac{\lambda_m \nu_m^2}{1 - i2\lambda_m \xi}\right] \tag{82}$$

following directly from (76) and (78).

For the particular example of

$$B^{12} = \frac{1}{2} \operatorname{diag}(\ell_1\ \ell_2 \cdots \ell_N) + \frac{1}{2} [g_1\ g_2 \cdots g_N]^T [h_1\ h_2 \cdots h_N], \tag{83}$$

then

$$q = \sum_{n=1}^{N} \ell_n u_n v_n + \left(\sum_{n=1}^{N} g_n u_n\right)\left(\sum_{n=1}^{N} h_n v_n\right), \tag{84}$$

with the same characteristic function (82).

As a still more-special case, let $B^{12} = \frac{1}{2} I$; then (79) and (81) give the simple cross-correlator (but with correlated inputs for all time separations)

$$q = \sum_{n=1}^{N} u_n v_n = \sum_{m=1}^{2N} \lambda_m w_m^2 \quad , \tag{85}$$

and (80) and (77) become

$$BK = \frac{1}{2} \begin{bmatrix} K_{vu} & K_{vv} \\ K_{uu} & K_{uv} \end{bmatrix}, \quad \overline{W} = C^T \begin{bmatrix} E_u \\ E_v \end{bmatrix} = [\nu_1 \cdots \nu_{2N}]^T . \tag{86}$$

The important equations that must be solved are always

$$BKC = C\Lambda \tag{87}$$

or

$$BKC^{(m)} = \lambda_m C^{(m)} \quad \text{for} \quad 1 \leq m \leq M = 2N, \tag{88}$$

where all matrices B, K, C, Λ are 2Nx2N. The characteristic function of (85) is again (82).

Special Case of Correlator (85)

Here we let components U and V have the same covariance and a scaled cross-correlation; that is, let

$$K_{uu} = K_o, \quad K_{vv} = K_o, \quad K_{uv} = \rho K_o \ , \tag{89}$$

where ρ is a scale factor. This case corresponds, for example, to a common signal in two independent components:

$$\begin{aligned} u(t) &= s(t) + n_1(t), \\ v(t) &= s(t) + n_2(t), \end{aligned} \tag{90}$$

where $s(t)$, $n_1(t)$, $n_2(t)$ are all independent and have a common covariance. Then (8ᵕ) becomes

$$BK = \frac{1}{2}\begin{bmatrix} \rho K_o & K_o \\ K_o & \rho K_o \end{bmatrix} . \tag{91}$$

Now suppose that we can determine the NxN eigenvalue matrix Γ and modal matrix Q of K_o, that is

$$K_o Q = Q\Gamma\,; \quad \Gamma = \mathrm{diag}(\gamma_1, \gamma_2 \cdots \gamma_N). \tag{92}$$

Then we have the standard relations [8]

$$K_o = Q\Gamma Q^T \quad \text{where} \quad QQ^T = I. \tag{93}$$

We can now express the 2Nx2N matrix in (91) as

$$BK = \frac{1}{2}\begin{bmatrix} Q\rho\Gamma Q^T & Q\Gamma Q^T \\ Q\Gamma Q^T & Q\rho\Gamma Q^T \end{bmatrix} =$$

$$= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \frac{1}{2}\rho\Gamma & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \frac{1}{2}\rho\Gamma \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}, \tag{94}$$

and 2Nx2N identity matrix

$$I_{2N} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \tag{95}$$

There follows

$$BK - \lambda I_{2N} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \frac{1}{2}\rho\Gamma - \lambda I & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \frac{1}{2}\rho\Gamma - \lambda I \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \tag{96}$$

But the middle matrix in (96) can be developed in detail in the partitioned form

$$\left[\begin{array}{ccc:ccc} \frac{1}{2}\rho\gamma_1 - \lambda & & & \frac{1}{2}\gamma_1 & & \\ & \ddots & & & \ddots & \\ & & \frac{1}{2}\rho\gamma_N - \lambda & & & \frac{1}{2}\gamma_N \\ \hdashline \frac{1}{2}\gamma_1 & & & \frac{1}{2}\rho\gamma_1 - \lambda & & \\ & \ddots & & & \ddots & \\ & & \frac{1}{2}\gamma_N & & & \frac{1}{2}\rho\gamma_N - \lambda \end{array}\right] . \qquad (97)$$

This matrix is singular when the kth row is equal to, or the negative of, the k+N th row. This leads to the eigenvalues $\{\lambda_n\}_1^{2N}$ of matrix BK:

$$\lambda_1 = \frac{1}{2}(\rho+1)\gamma_1, \ \ldots, \ \lambda_N = \frac{1}{2}(\rho+1)\gamma_N ,$$

$$\lambda_{N+1} = \frac{1}{2}(\rho-1)\gamma_1, \ \ldots, \ \lambda_{2N} = \frac{1}{2}(\rho-1)\gamma_N . \qquad (98)$$

Thus we need only solve for the N eigenvalues $\{\gamma_n\}_1^N$ of matrik K_o, and then use them as above to determine all 2N eigenvalues of BK; this is a significant shortcut.

If also $E_u = E_v = 0$, then $\overline{W} = 0$ from (86), and the characteristic function of q in (85) follows from (82) and (98) as

$$f_q(\xi) = \left[\prod_{m=1}^{N} \left\{(1-i(\rho+1)\gamma_m\xi)(1-i(\rho-1)\gamma_m\xi)\right\}\right]^{-1/2} =$$

$$= \left[\prod_{m=1}^{N} \left\{1-i2\rho\gamma_m\xi + (1-\rho^2)\gamma_m^2\xi^2\right\}\right]^{-1/2} . \qquad (99)$$

This is a generalization of [1, eq. 54], which held for a single pair of Gaussian random variables.

EXAMPLES

The program listed in appendix A for the second-order processor (8) and attendant characteristic function (9) has been employed to yield the result in figure 2. The particular values for the number of terms K, the weights, and the input statistics are listed in lines 20-120. There is no physical significance attached to this particular example; rather it has been run simply to illustrate the extreme generality that the technique is capable of. Some negative values for the weights, means, and correlation coefficients have been employed to emphasize this generality. This simple example (and others to follow) can be used as a check case on any user-written program to evaluate cumulative and exceedance distribution functions.

The selection of parameters L, Δ, b in lines 130-150 is discussed in detail in [1]; the reader is referred there for the deleterious effects that can occur for improper choices of L, Δ, b. The selection of M_f, the FFT size in line 160, is rather arbitrary; it controls the spacing at which the probability distributions are computed, but has no effect upon the accuracy of the results (except for round-off noise). Additional computational details on the particular program for characteristic function (9) are given in appendix A.

The ordinate scale for figure 2 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region 1E-11, where roundoff noise is encountered. The exceedance distribution function values continue to decrease with x until, finally, negative values (due to roundoff noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the 1E-12 level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of L and Δ.

The rates of decay of the cumulative and exceedance distribution functions in figure 2 are markedly different for this particular example. Additionally, since the decays are both linear on this logarithmic ordinate, it means that both tail distributions are exponential, not Gaussian. These attributes of the cumulative and exceedance distribution functions are easily

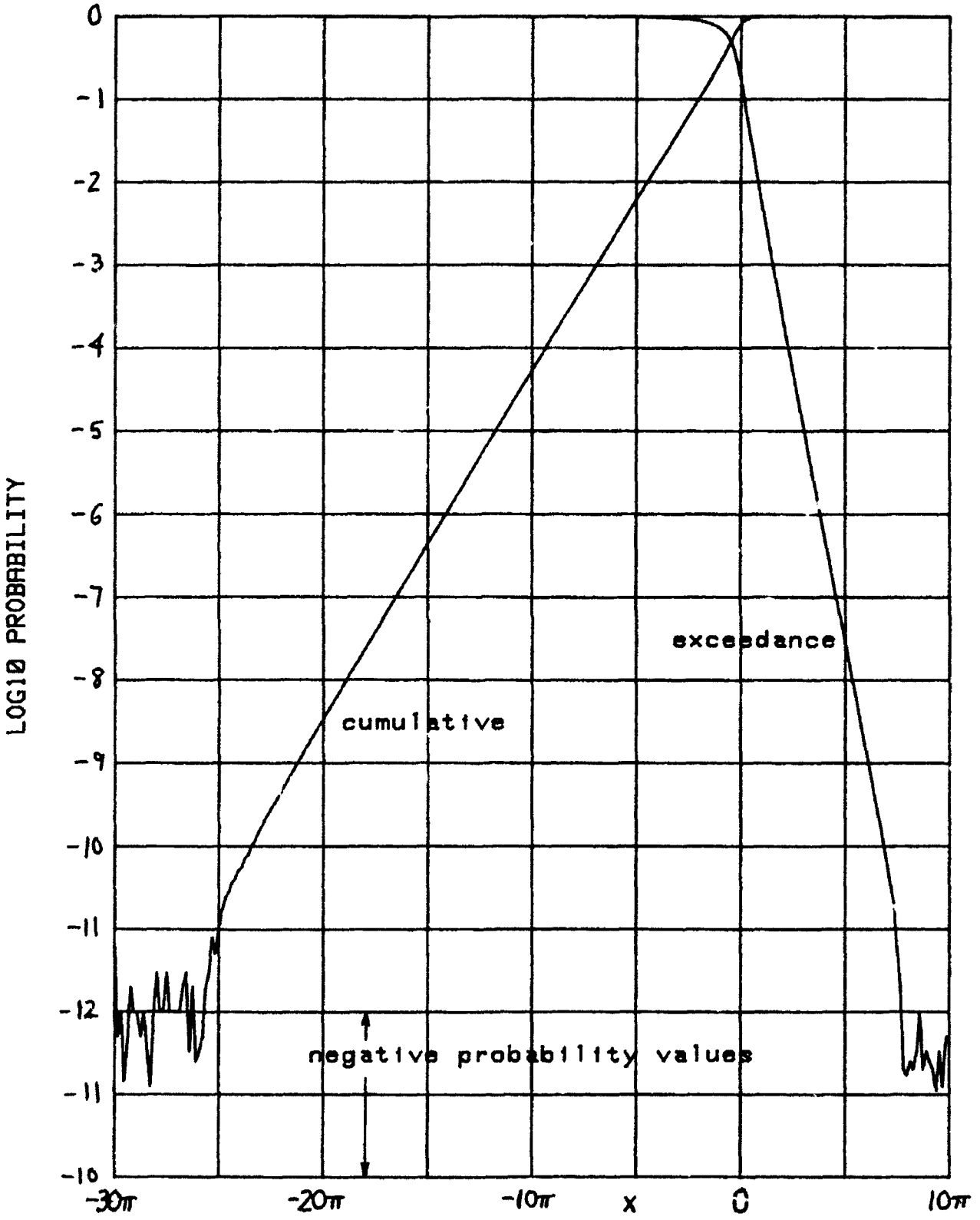

Figure 2. Distributions for Second-Order Processor

and quickly discernible by use of the numerical technique in [1], for a limitless variety of weights and input statistics, with a minimum of effort on the part of the user.

As a check on the program in appendix A, the second-order processor in (8) was simulated, and 10,000 independent trials were used to determine its performance for the exact same parameters as used for figure 2 above. The program is listed in appendix E and the results are given in figure 3. The corroboration is excellent, even near the 1E-4 probability level.

As the number of terms, K, in the second-order processor (8) is increased, and if the statistics are identical, the random variable x should approach Gaussian, at least near its mean. The example in figure 4 was run for K = 10, and all weights and statistics independent of k; the particular choices were

$$\begin{aligned} &a = .6,\ b = -.6,\ c = .3,\ d = -.2,\ e = .2, \\ &m_s = .5,\ m_t = -.5,\ \sigma_s = 1,\ \sigma_t = 1,\ \rho = .4, \\ &L = 4,\ \Delta = .05,\ b = 20\pi,\ M_f = 256. \end{aligned} \tag{100}$$

The cumulative and exceedance distribution functions in figure 4 both display a parabolic shape near the mean of x, which signifies Gaussian behavior of the random variable, as expected. However, on the tails, the distributions are tending to linear, which means an exponential decay there. This observation for this example confirms the comments of [7, p. 673].

The cumulative and exceedance distribution functions for an example of the second-order processor with fading are displayed in figure 5, as determined from characteristic function (21) and the corresponding program in appendix B. The power law, ν, for the fading probability density function (19) is 2.7 for this example, but can be easily changed. The particular constants employed are listed in lines 20–110 in appendix B.

An example of the distributions for the narrowband cross-correlator of figure 1 is presented in figure 6, as evaluated from characteristic function

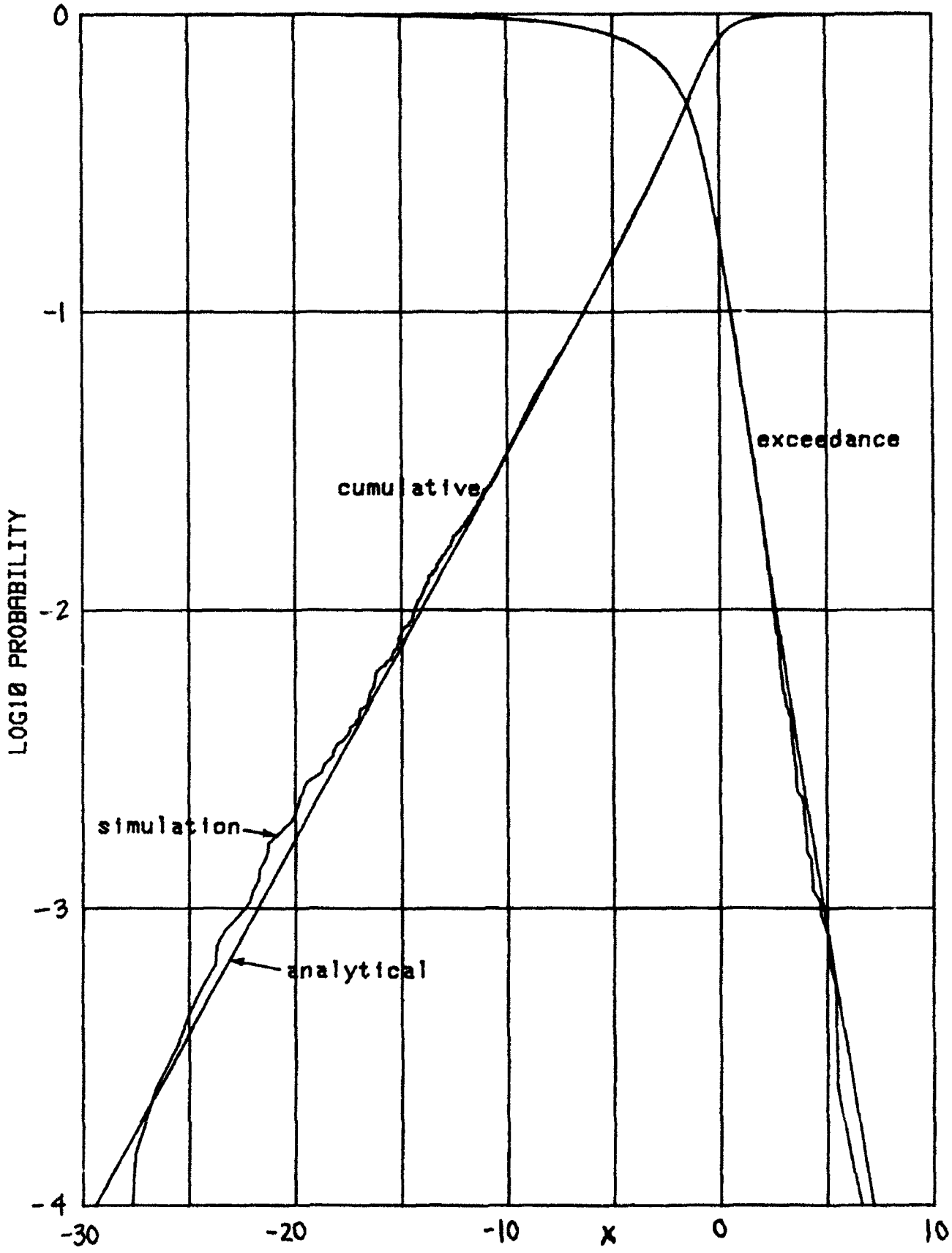

Figure 3. Simulation Comparison of Second-Order Processor

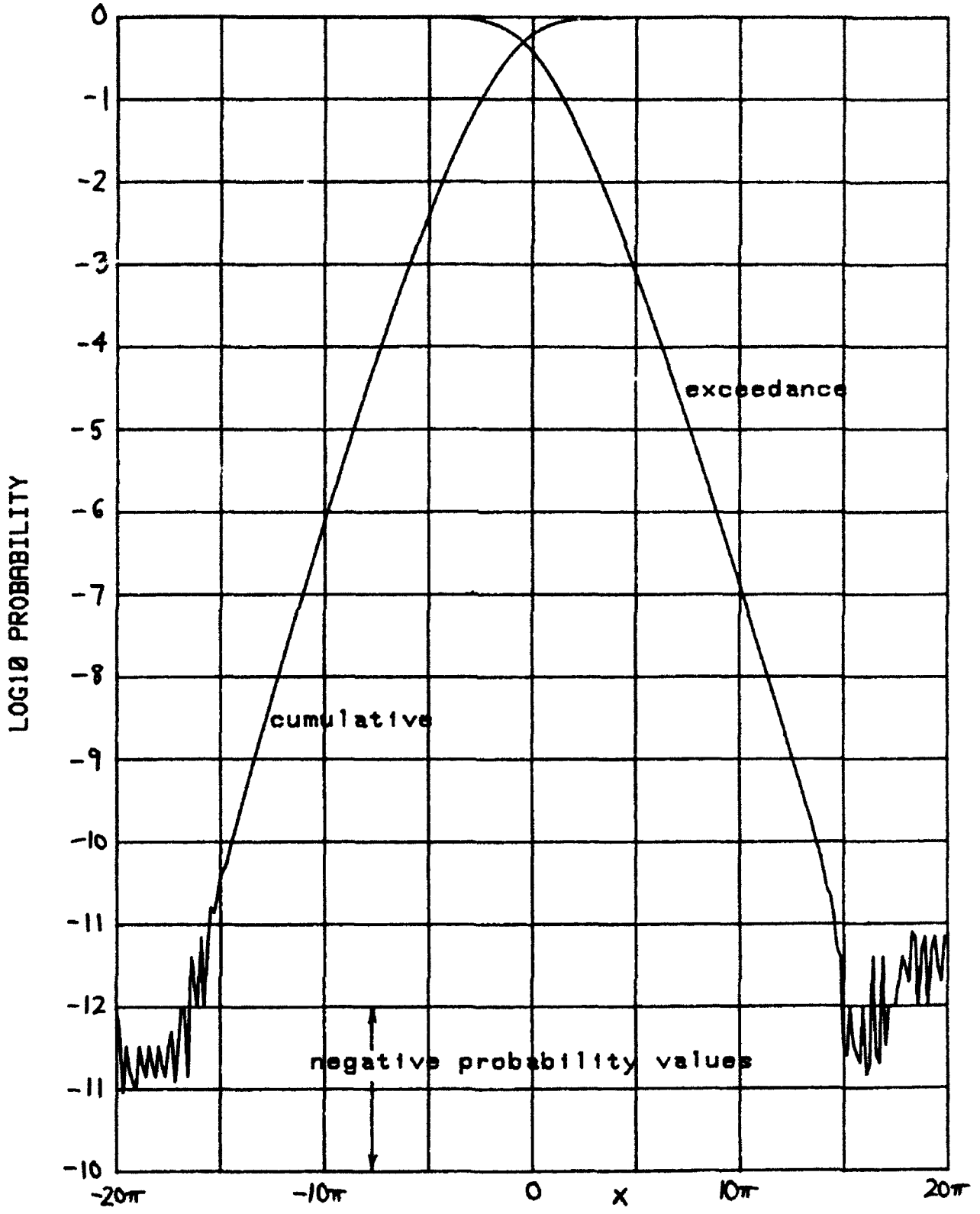

Figure 4. Second-Order Processor, K=10, Identical Statistics

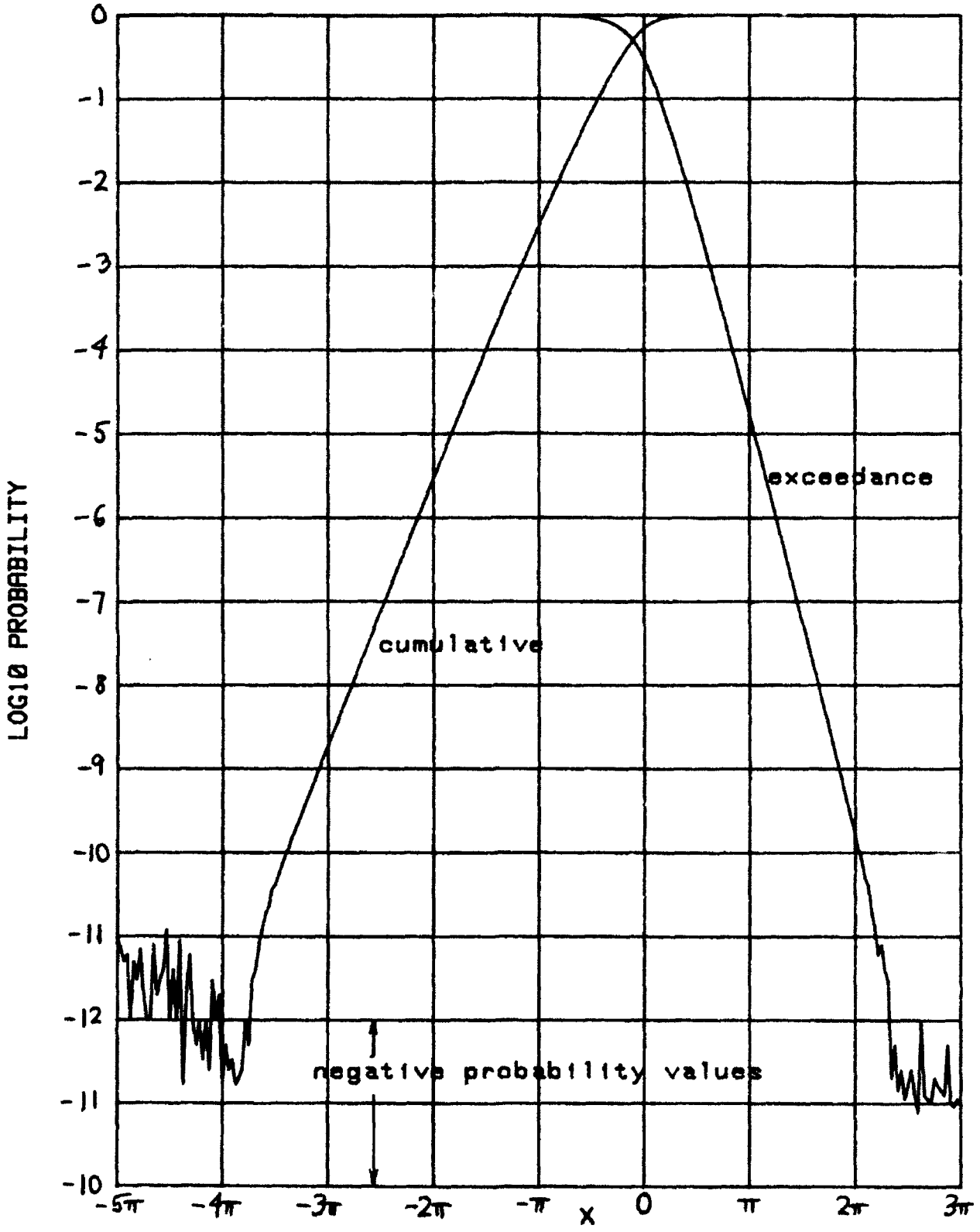

Figure 5. Second-Order Processor with Fading

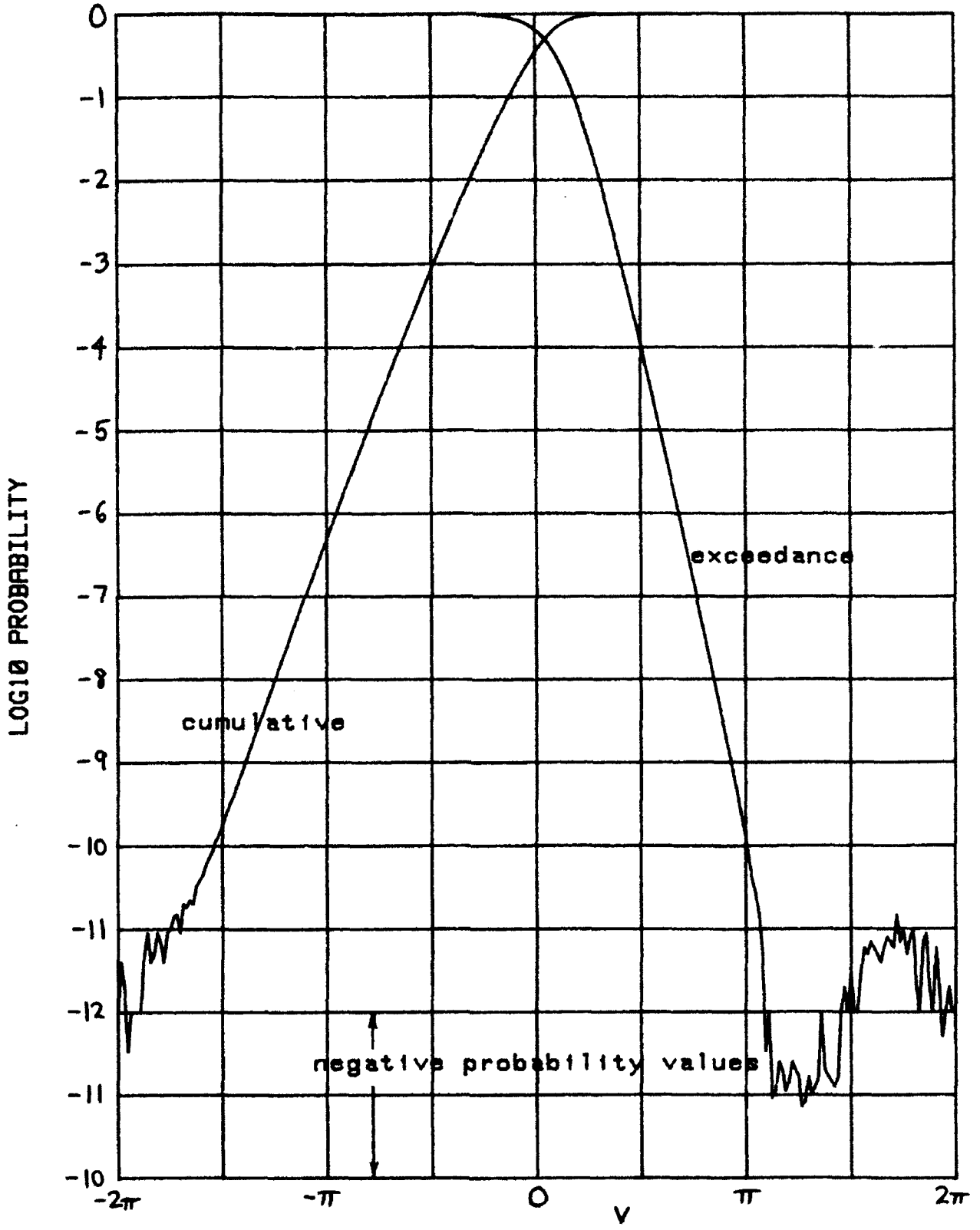

Figure 6. Distributions for Narrowband Cross-Correlator

(35) and the program in appendix C. The weightings, signal components, and noise statistics have no special values or interrelationships; the particular values used here are listed in lines 20-110.

The distributions for the reduced quadratic and linear form (65) and accompanying characteristic function (66) are presented in figure 7 for the numerical example employed in the program listing in appendix D. If the given form is instead that of (62), then (67)-(68) must first be solved before the program in appendix D can be employed; that is, one must augment these results with the capability for extracting the eigenvalues (and eigenvectors in some cases) of the MxM matrix BK. The size of the FFT, M_f, has been increased to 1024 in figure 7; this results in finer spacing of the distribution values and additional spikes in the round-off noise region centered about 1E-12.

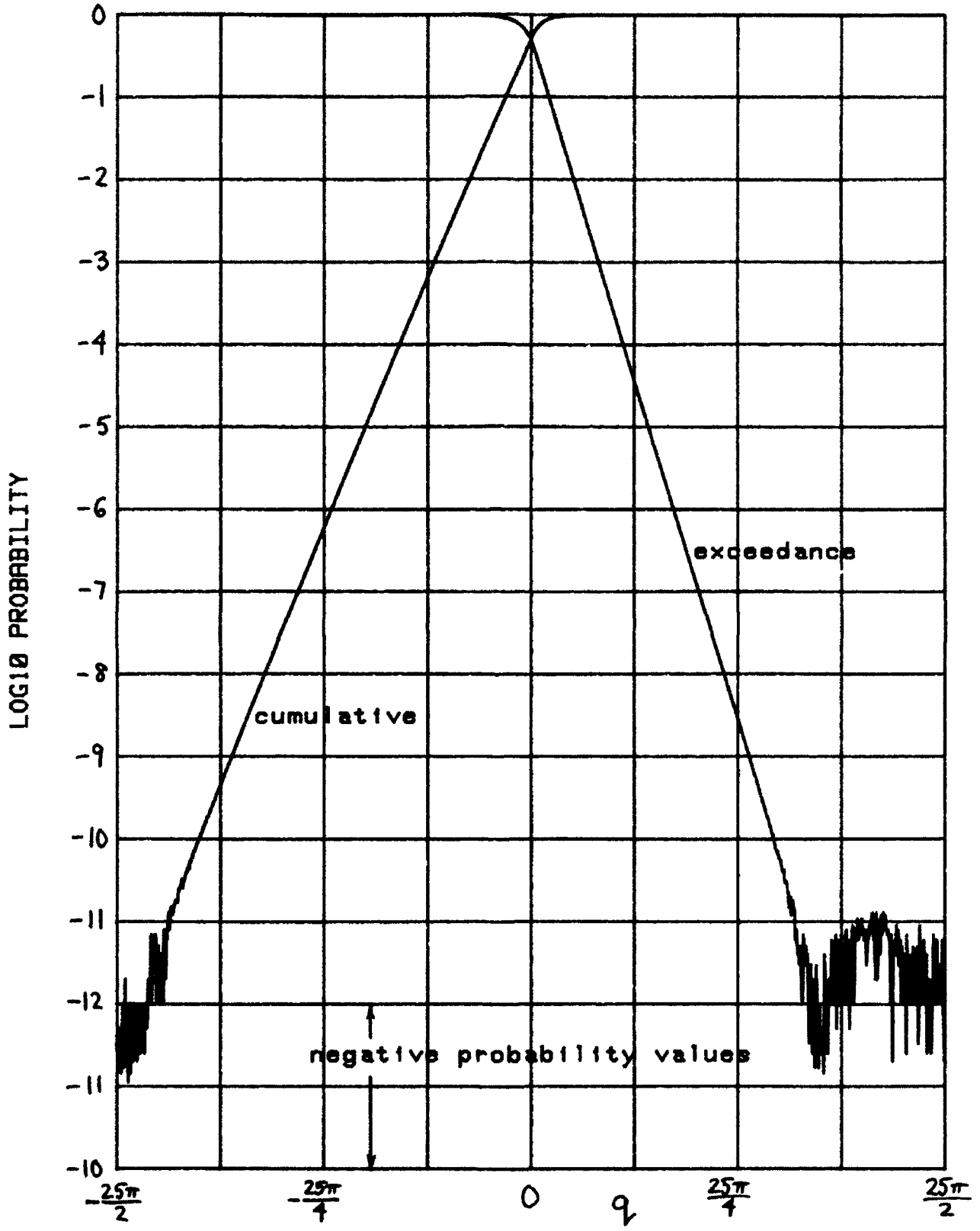

Figure 7. Distributions for Quadratic and Linear Form

SUMMARY AND DISCUSSION

Closed form expressions for the characteristic functions of the decision variables of three classes of second-order processors have been derived. The input noise to the processors must be Gaussian, but it can be nonstationary with arbitrary statistics. Programs for the direct evaluation of the exact cumulative and exceedance distribution functions have been generated and then exercised for completely general values of the weights, signal parameters, and noise statistics. There is no assumption needed about a large number of statistically independent contributors, nor need any signal-to-noise ratio be either small or large. The first two classes of processors are restricted in form, but include many of the practical devices often encountered in detection and estimation problems. The third class covers the most general second-order processor; it requires the solution for the eigenvalue and modal matrices of an MxM matrix (where M is the size of the general quadratic form) in addition to the program furnished here. The approach utilized here allows a user to quickly and easily obtain accurate quantitative information about the performance of a particular processor, and to investigate the effects of making changes in any of the input constants or parameters.

Approximations to the performance of continuous quadratic processors are possible by use of the above procedures. For example,

$$\iint dt_1\, dt_2\, x(t_1)\, \beta(t_1, t_2)\, x(t_2) \cong \Delta_1\, \Delta_2 \sum_{m,n} x(m\Delta_1)\, \beta(m\Delta_1, n\Delta_2)\, x(n\Delta_2), \qquad (101)$$

which is of the form $X^T B X$ encountered in (62). Also

$$\iint dt_1\, dt_2\, u(t_1)\, \beta(t_1, t_2)\, v(t_2) \cong \Delta_1\, \Delta_2 \sum_{m,n} u(m\Delta_1)\, \beta(m\Delta_1, n\Delta_2)\, v(n\Delta_2), \qquad (102)$$

which is of the form $U^T B^{12} V$ encountered in (79).

Receiver operating characteristics, that is, detection probability vs false alarm probability, can be easily determined from the above results. First store the exceedance distribution for zero signal strength in an array. Then plot the exceedance distributions for nonzero signal strengths vs this stored array of numbers, each point for a common threshold. The common thresholds are most easily realized by keeping sampling increment Δ and FFT size M_f the same throughout all the computations.

APPENDIX A. SECOND-ORDER PROCESSOR

This program computes the cumulative and exceedance distribution functions of random variable (8) via characteristic function (9). The required inputs are listed in lines 20-120 and are annotated consistently with (8). The parameters D_1, D_2, N_0, N_1, N_2 required in characteristic function (9) are pre-computed once in loop 290-510 for the sake of execution time. The mean of x is entered in line 520. When we enter loop 590-830 for the actual calculation of the characteristic function (9), the number of computations are minimized. For example, only one complex exponential and square root are required per ξ value, in lines 740-750. The square root in (9) is <u>not</u> a principal value square root, but in fact must yield a continuous function in ξ. In order to achieve this, the argument of the square root is traced continuously from $\xi = 0$ (line 530). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 780) and corrects the final values of characteristic function $f_y(\xi)$ (line 790). More detail on the selection of L, Δ, b in lines 130-150 is available in [1].

```
10 ! SECOND-ORDER PROCESSOR
20    K=5                          !  Number of terms summed
30    DATA .6,-.5,.4,-.3,.2        !  a(k) weightings
40    DATA .9,.8,.7,-.6,-.5        !  b(k) weightings
50    DATA -.6,-.8,1,1.2,1.4       !  c(k) weightings
60    DATA .1,-.2,-.3,.4,.5        !  d(k) weightings
70    DATA -.7,.6,.5,.4,-.3        !  e(k) weightings
80    DATA .2,.3,.4,-.5,-.6        !  Means of random variables s(k)
90    DATA .8,-.7,-.6,.5,.4        !  Means of random variables t(k)
100   DATA .1,.3,.5,.7,.9          !  Standard deviations of s(k)
110   DATA .2,.4,.6,.8,1           !  Standard deviations of t(k)
120   DATA .4,-.5,.6,.7,-.8        !  Correlation coeffs. of s(k) and t(k)
130   L=25                         !  Limit on integral of char. function
140   Delta=.05                    !  Sampling increment on char. function
150   Bs=.75*(2*PI/Delta)          !  Shift b, as fraction of alias interval
160   Mf=2^8                       !  Size of FFT
170   PRINTER IS 0
180   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
190   REDIM A(1:K),B(1:K),C(1:K),D(1:K),E(1:K)
200   REDIM Ms(1:K),Mt(1:K),Ss(1:K),St(1:K),Rho(1:K)
210   REDIM D1(1:K),D2(1:K),N0(1:K),N1(1:K),N2(1:K)
220   REDIM X(0:Mf-1),Y(0:Mf-1)
230   DIM A(1:10),B(1:10),C(1:10),D(1:10),E(1:10)
240   DIM Ms(1:10),Mt(1:10),Ss(1:10),St(1:10),Rho(1:10)
250   DIM D1(1:10),D2(1:10),N0(1:10),N1(1:10),N2(1:10)
260   DIM X(0:1023),Y(0:1023)
270   READ A(*),B(*),C(*),D(*),E(*)      !  Enter
280   READ Ms(*),Mt(*),Ss(*),St(*),Rho(*)  ! constants
```

```
290   FOR J=1 TO K                          !  Calculation
300   T1=Ms(J)^2                            !  of
310   T2=Mt(J)^2                            !  parameters
320   T3=Ss(J)^2
330   T4=St(J)^2
340   T5=Ms(J)*Mt(J)
350   T6=Rho(J)*Ss(J)*St(J)
360   T7=4*A(J)*B(J)-C(J)^2
370   T8=(1-Rho(J)^2)*T3*T4
380   T9=Mt(J)*Ss(J)
390   T10=Ms(J)*St(J)
400   T11=D(J)^2
410   T12=E(J)^2
420   T13=D(J)*E(J)
430   D1(J)=2*(A(J)*T3+B(J)*T4+C(J)*T6)
440   D2(J)=T7*T8
450   N0(J)=A(J)*T1+B(J)*T2+C(J)*T5+D(J)*Ms(J)+E(J)*Mt(J)
460   T=T7*(.5*(T1*T4+T2*T3)-T5*T6)
470   T=T+(2*A(J)*E(J)-C(J)*D(J))*Ss(J)*(T9-Rho(J)*T10)
480   T=T+(2*B(J)*D(J)-C(J)*E(J))*St(J)*(T10-Rho(J)*T9)
490   N1(J)=T-.5*(T11*T3+T12*T4)-T13*T6
500   N2(J)=-(A(J)*T12+B(J)*T11-C(J)*T13)*T8
510   NEXT J
520   Mux=SUM(N0)+.5*SUM(D1)                !  Mean of random variable x
530   R=0                                   !  Argument of square root
540   P=1                                   !  Polarity indicator
550   Muy=Mux+Bs
560   X(0)=0
570   Y(0)=.5*Delta*Muy
580   N=INT(L/Delta)
590   FOR Ns=1 TO N
600   Xi=Delta*Ns                           !  Argument xi of char. fn.
610   X2=Xi*Xi                              !  Calculation
620   Pr=1                                  !  of
630   Pi=Sr=Si=0                            !  characteristic
640   FOR J=1 TO K                          !  function
650   Dr=1-X2*D2(J)                         !  fy(xi)
660   Di=-Xi*D1(J)
670   CALL Mul(Pr,Pi,Dr,Di,A,B)
680   Pr=A
690   Pi=B
700   CALL Div(N0(J)-X2*N2(J),-Xi*N1(J),Dr,Di,A,B)
710   Sr=Sr+A
720   Si=Si+B
730   NEXT J
740   CALL Exp(-Xi*Si,Xi*(Sr+Bs),A,B)
750   CALL Sqr(Pr,Pi,C,D)
760   Ro=R
770   R=ATN(D/C)
780   IF ABS(R-Ro)>1.6 THEN P=-P
790   CALL Div(A,B,C*P,D*P,Fyr,Fyi)
800   Ms=Ns MOD Mf                          !  Collapsing
810   X(Ms)=X(Ms)+Fyr/Ns
820   Y(Ms)=Y(Ms)+Fyi/Ns
830   NEXT Ns
840   CALL Fft10z(Mf,X(*),Y(*))             !  0 subscript FFT
```

```
850   PLOTTER IS "GRAPHICS"
860   GRAPHICS
870   SCALE 0,Mf,-14,0
880   LINE TYPE 3
890   GRID Mf/8,1
900   PENUP
910   LINE TYPE 1
920   B=Bs*Mf*Delta/(2*PI)                    !  Origin for random variable x
930   MOVE B,0
940   DRAW B,-14
950   PENUP
960   FOR Ks=0 TO Mf-1
970   T=Y(Ks)/PI-Ks/Mf
980   X(Ks)=.5-T                              !  Cumulative probability in X(*)
990   Y(Ks)=Pr=.5+T                           !  Exceedance probability in Y(*)
1000  IF Pr>=1E-12 THEN Y=LGT(Pr)
1010  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1020  IF ABS(Pr)<1E-12 THEN Y=-12
1030  PLOT Ks,Y
1040  NEXT Ks
1050  PENUP
1060  PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
1070  FOR Ks=0 TO Mf-1
1080  Pr=X(Ks)
1090  IF Pr>=1E-12 THEN Y=LGT(Pr)
1100  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1110  IF ABS(Pr)<1E-12 THEN Y=-12
1120  PLOT Ks,Y
1130  NEXT Ks
1140  PENUP
1150  PAUSE
1160  DUMP GRAPHICS
1170  PRINT LIN(5)
1180  PRINTER IS 16
1190  END
1200  !
1210  SUB Mul(X1,Y1,X2,Y2,A,B)                !  Z1*Z2
1220  A=X1*X2-Y1*Y2
1230  B=X1*Y2+X2*Y1
1240  SUBEND
1250  !
1260  SUB Div(X1,Y1,X2,Y2,A,B)                !  Z1/Z2
1270  T=X2*X2+Y2*Y2
1280  A=(X1*X2+Y1*Y2)/T
1290  B=(Y1*X2-X1*Y2)/T
1300  SUBEND
1310  !
1320  SUB Exp(X,Y,A,B)                        !  EXP(Z)
1330  T=EXP(X)
1340  A=T*COS(Y)
1350  B=T*SIN(Y)
1360  SUBEND
1370  !
```

```
1380  SUB Sqr(X,Y,A,B)                    !  PRINCIPAL SQR(Z)
1390  IF X<>0 THEN 1430
1400  A=B=SQR(.5*ABS(Y))
1410  IF Y<0 THEN B=-B
1420  GOTO 1540
1430  F=SQR(SQR(X*X+Y*Y))
1440  T=.5*ATN(Y/X)
1450  A=F*COS(T)
1460  B=F*SIN(T)
1470  IF X>0 THEN 1540
1480  T=A
1490  A=-B
1500  B=T
1510  IF Y>=0 THEN 1540
1520  A=-A
1530  B=-B
1540  SUBEND
1550  !
1560  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
1570  DIM C(0:256)
1580  INTEGER I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,J,K
1590  DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998795456205,.998475580573,.998118112900
1600  DATA .997723066644,.997290456679,.996820299291,.996312612183,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
1610  DATA .991709753669,.990902635428,.990058210262,.989176509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
1620  DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
1630  DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
1640  DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.939459223602,.937339011913,.935183509939,.932992798835
1650  DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
1660  DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
1670  DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
1680  DATA .848120344803,.844853565250,.841554977437,.838224705555,.834862874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
1690  DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
1700  DATA .776888465673,.773010453363,.769103337646,.765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
1710  DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
1720  DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
1730  DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763,.615231590581,.610382806276,.605511041404
1740  DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745887,.565731810784,.560661576197,.555570233020
1750  DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
```

```
1760  DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
1770  DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
1780  DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717312,.336889853392
1790  DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
1800  DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058108281,.225083911360,.219101240157
1810  DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068624150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
1820  DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.980171403296E-1
1830  DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
1840  DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
1850  READ C(*)
1860  K=1024/N
1870  FOR J=0 TO N/4
1880  C(J)=C(K*J)
1890  NEXT J
1900  N1=N/4
1910  N2=N1+1
1920  N3=N2+1
1930  N4=N1+N3
1940  Log2n=INT(1.4427*LOG(N)+.5)
1950  FOR I1=1 TO Log2n
1960  I2=2^(Log2n-I1)
1970  I3=2*I2
1980  I4=N/I3
1990  FOR I5=1 TO I2
2000  I6=(I5-1)*I4+1
2010  IF I6<=N2 THEN 2050
2020  N6=-C(N4-I6-1)
2030  N7=-C(I6-N1-1)
2040  GOTO 2070
2050  N6=C(I6-1)
2060  N7=-C(N3-I6-1)
2070  FOR I7=0 TO N-I3 STEP I3
2080  I8=I7+I5
2090  I9=I8+I2
2100  N8=X(I8-1)-X(I9-1)
2110  N9=Y(I8-1)-Y(I9-1)
2120  X(I8-1)=X(I8-1)+X(I9-1)
2130  Y(I8-1)=Y(I8-1)+Y(I9-1)
2140  X(I9-1)=N6*N8-N7*N9
2150  Y(I9-1)=N6*N9+N7*N8
2160  NEXT I7
2170  NEXT I5
2180  NEXT I1
2190  I1=Log2n+1
2200  FOR I2=1 TO 10                    ! 2^10=1024
2210  C(I2-1)=1
2220  IF I2>Log2n THEN 2240
2230  C(I2-1)=2^(I1-I2)
2240  NEXT I2
```

```
2250  K=1
2260  FOR I1=1 TO C(9)
2270  FOR I2=I1 TO C(8) STEP C(9)
2280  FOR I3=I2 TO C(7) STEP C(8)
2290  FOR I4=I3 TO C(6) STEP C(7)
2300  FOR I5=I4 TO C(5) STEP C(6)
2310  FOR I6=I5 TO C(4) STEP C(5)
2320  FOR I7=I6 TO C(3) STEP C(4)
2330  FOR I8=I7 TO C(2) STEP C(3)
2340  FOR I9=I8 TO C(1) STEP C(2)
2350  FOR I10=I9 TO C(0) STEP C(1)
2360  J=I10
2370  IF K>J THEN 2440
2380  A=X(K-1)
2390  X(K-1)=X(J-1)
2400  X(J-1)=A
2410  A=Y(K-1)
2420  Y(K-1)=Y(J-1)
2430  Y(J-1)=A
2440  K=K+1
2450  NEXT I10
2460  NEXT I9
2470  NEXT I8
2480  NEXT I7
2490  NEXT I6
2500  NEXT I5
2510  NEXT I4
2520  NEXT I3
2530  NEXT I2
2540  NEXT I1
2550  SUBEND
```

APPENDIX B. FADING FOR SECOND-ORDER PROCESSOR

This program computes the cumulative and exceedance distribution functions for characteristic function (21), when the power fading factor r in (18) has probability density function (19). The parameters D_1, D_2, N_0', N_1' are pre-computed once in lines 210-310. The logarithms in lines 430 and 440 have arguments that never cross the branch line along the negative real axis for the principal value logarithm; hence the calculated characteristic function is automatically continuous for all .

```
10 ! FADING FOR SECOND-ORDER PROCESSOR
20    Nu=2.7                    !  Power law for fading
30    K=5                       !  Number of terms summed
40    Ak=.7                     !  a(k) weighting
50    Bk=-.9                    !  b(k) weighting
60    Ck=-.6                    !  c(k) weighting
70    DATA .2,.3,.4,-.5,-.6     !  Means of random variables s(k)
80    DATA .8,-.7,-.6,.5,.4     !  Means of random variables t(k)
90    Ss=.3                     !  Standard deviation of s(k)
100   St=.2                     !  Standard deviation of t(k)
110   Rho=-.4                   !  Correlation coeff. of s(k) and t(k)
120   L=150                     !  Limit on integral of char. function
130   Delta=.25                 !  Sampling increment on char. function
140   Bs=.625*(2*PI/Delta)      !  Shift b, as fraction of alias interval
150   Mf=2^8                    !  Size of FFT
160   PRINTER IS 0
170   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
180   REDIM Ms(1:K),Mt(1:K),X(0:Mf-1),Y(0:Mf-1)
190   DIM Ms(1:10),Mt(1:10),X(0:1023),Y(0:1023)
200   READ Ms(*),Mt(*)                    !  Enter constants
210   M20=DOT(Ms,Ms)                      !  Calculation
220   M02=DOT(Mt,Mt)                      !  of
230   M11=DOT(Ms,Mt)                      !  parameters
240   T1=Ss*Ss
250   T2=St*St
260   T3=Rho*Ss*St
270   T4=4*Ak*Bk-Ck*Ck
280   N0p=Ak*M20+Bk*M02+Ck*M11
290   N1p=T4*(.5*(T2*M20+T1*M02)-T3*M11)
300   D1=2*(Ak*T1+Bk*T2+Ck*T3)
310   D2=T4*(1-Rho*Rho)*T1*T2
320   D1p=D1+N0p/Nu
330   D2p=D2+N1p/Nu
340   Mux=N0p+.5*K*D1                     !  Mean of random variable x
350   Muy=Mux+Bs
360   T=Nu-.5*K
```

```
370   X(0)=0
380   Y(0)=.5*Delta*Muy
390   N=INT(L/Delta)
400   FOR Ns=1 TO N
410   Xi=Delta*Ns                          !  Argument xi of char. fn.
420   X2=Xi*Xi                            !  Calculation
430   CALL Log(1-X2*D2,-Xi*D1,A,B)        !  of
440   CALL Log(1-X2*D2p,-Xi*D1p,C,D)      !  characteristic
450   T1=T*A-Nu*C                         !  function
460   T2=T*B-Nu*D+Bs*Xi                   !  fy(xi)
470   CALL Exp(T1,T2,Fyr,Fyi)
480   Ms=Ns MOD Mf                        !  Collapsing
490   X(Ms)=X(Ms)+Fyr/Ns
500   Y(Ms)=Y(Ms)+Fyi/Ns
510   NEXT Ns
520   CALL Fft10z(Mf,X(*),Y(*))           !  0 subscript FFT
530   PLOTTER IS "GRAPHICS"
540   GRAPHICS
550   SCALE 0,Mf,-14,0
560   LINE TYPE 3
570   GRID Mf/8,1
580   PENUP
590   LINE TYPE 1
600   B=Bs*Mf*Delta/(2*PI)                !  Origin for random variable x
610   MOVE B,0
620   DRAW B,-14
630   PENUP
640   FOR Ks=0 TO Mf-1
650   T=Y(Ks)/PI-Ks/Mf
660   X(Ks)=.5-T                          !  Cumulative probability in X(*)
670   Y(Ks)=Pr=.5+T                       !  Exceedance probability in Y(*)
680   IF Pr>=1E-12 THEN Y=LGT(Pr)
690   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
700   IF ABS(Pr)<1E-12 THEN Y=-12
710   PLOT Ks,Y
720   NEXT Ks
730   PENUP
740   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
750   FOR Ks=0 TO Mf-1
760   Pr=X(Ks)
770   IF Pr>=1E-12 THEN Y=LGT(Pr)
780   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
790   IF ABS(Pr)<1E-12 THEN Y=-12
800   PLOT Ks,Y
810   NEXT Ks
820   PENUP
830   PAUSE
840   DUMP GRAPHICS
850   PRINT LIN(5)
860   PRINTER IS 16
870   END
880   !
```

```
890   SUB Exp(X,Y,A,B)                    !  EXP(Z)
900   T=EXP(X)
910   A=T*COS(Y)
920   B=T*SIN(Y)
930   SUBEND
940   !
950   SUB Log(X,Y,A,B)                    !  PRINCIPAL LOG(Z)
960   A=.5*LOG(X*X+Y*Y)
970   IF X<>0 THEN 1000
980   B=.5*PI*SGN(Y)
990   GOTO 1020
1000  B=ATN(Y/X)
1010  IF X<0 THEN B=B+PI*(1-2*(Y<0))
1020  SUBEND
1030  !
1040  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

APPENDIX C. NARROWBAND CROSS-CORRELATOR

This program computes the cumulative and exceedance distribution functions of random variable (30) via characteristic function (35). The parameters D_1, D_2, N_0, N_1 are pre-computed in lines 280-390 and weighted according to (35)-(36) in lines 400-440. All the functions employed are analytic.

```
10  ! NARROWBAND CROSS-CORRELATOR
20    K=5                          !  Number of terms summed
30    DATA .6,-.5,.4,-.3,.2        !  w(k)      weightings
40    DATA .9,.8,.7,-.6,-.5        !  a1(k)     signal 1 in-phase components
50    DATA -.6,-.8,1,1.2,1.4       !  b1(k)     signal 1 quadrature components
60    DATA .1,-.2,-.3,.4,.5        !  a2(k)     signal 2 in-phase components
70    DATA -.7,.6,.5,.4,-.3        !  b2(k)     signal 2 quadrature components
80    DATA .1,.3,.5,.7,.9          !  sigma1(k)  noise 1 standard deviations
90    DATA .2,.4,.6,.8,1           !  sigma2(k)  noise 2 standard deviations
100   DATA .4,-.5,.6,.7,-.8        !  rho(k)    noise in-phase corr. coeffs.
110   DATA .9,-.7,-.5,.3,-.1       !  lambda    noise quadrature corr. coeffs.
120   L=50                         !  Limit on integral of char. function
130   Delta=.5                     !  Sampling increment on char. function
140   Bs=.5*(2*PI/Delta)           !  Shift b, as fraction of alias interval
150   Mf=2^8                       !  Size of FFT
160   PRINTER IS 0
170   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
180   REDIM W(1:K),A1(1:K),B1(1:K),A2(1:K),B2(1:K)
190   REDIM S1(1:K),S2(1:K),Rho(1:K),Lambda(1:K)
200   REDIM D1(1:K),D2(1:K),N0(1:K),N1(1:K)
210   REDIM X(0:Mf-1),Y(0:Mf-1),W2(1:K)
220   DIM W(1:10),A1(1:10),B1(1:10),A2(1:10),B2(1:10)
230   DIM S1(1:10),S2(1:10),Rho(1:10),Lambda(1:10)
240   DIM D1(1:10),D2(1:10),N0(1:10),N1(1:10)
250   DIM X(0:1023),Y(0:1023),W2(1:10)
260   READ W(*),A1(*),B1(*),A2(*),B2(*)  !  Enter
270   READ S1(*),S2(*),Rho(*),Lambda(*)  !  constants
280   FOR J=1 TO K                       !  Calculation
290   S1s=S1(J)^2                        !  of
300   S2s=S2(J)^2                        !  parameters
310   T1=S1(J)*S2(J)
320   D1(J)=T2=T1*Rho(J)
330   D2(J)=.25*S1s*S2s*(1-Rho(J)^2-Lambda(J)^2)
340   T3=A1(J)*A2(J)+B1(J)*B2(J)
350   N0(J)=.5*T3
360   T4=A2(J)*B1(J)-A1(J)*B2(J)
370   T5=S2s*(A1(J)^2+B1(J)^2)+S1s*(A2(J)^2+B2(J)^2)
380   N1(J)=.125*(T5-2*T2*T3-2*T1*Lambda(J)*T4)
390   NEXT J
400   MAT W2=W.W
410   MAT D1=W.D1
420   MAT D2=W2.D2
430   MAT N0=W.N0
440   MAT N1=W2.N1
450   Mux=SUM(N0)+SUM(D1)                !  Mean of random variable v
460   Muy=Mux+Bs
```

```
470   X(0)=0
480   Y(0)=.5*Delta*Muy
490   N=INT(L/Delta)
500   FOR Ns=1 TO N
510   Xi=Delta*Ns                          !  Argument xi of char. fn.
520   X2=Xi*Xi                             !  Calculation
530   Pr=1                                 !  of
540   Pi=Sr=Si=0                           !  characteristic
550   FOR J=1 TO K                         !  function
560   Dr=1+X2*D2(J)                        !  fy(xi)
570   Di=-Xi*D1(J)
580   CALL Mul(Pr,Pi,Dr,Di,A,B)
590   Pr=A
600   Pi=B
610   CALL Div(N0(J),Xi*N1(J),Dr,Di,A,B)
620   Sr=Sr+A
630   Si=Si+B
640   NEXT J
650   CALL Exp(-Xi*Si,Xi*(Sr+Bs),A,B)
660   CALL Div(A,B,Pr,Pi,Fyr,Fyi)
670   Ms=Ns MOD Mf                         !  Collapsing
680   X(Ms)=X(Ms)+Fyr/Ns
690   Y(Ms)=Y(Ms)+Fyi/Ns
700   NEXT Ns
710   CALL Fft10z(Mf,X(*),Y(*))            !  0 subscript FFT
720   PLOTTER IS "GRAPHICS"
730   GRAPHICS
740   SCALE 0,Mf,-14,0
750   LINE TYPE 3
760   GRID Mf/8,1
770   PENUP
780   LINE TYPE 1
790   B=Bs*Mf*Delta/(2*PI)                 !  Origin for random variable v
800   MOVE B,0
810   DRAW B,-14
820   PENUP
830   FOR Ks=0 TO Mf-1
840   T=Y(Ks)/PI-Ks/Mf
850   X(Ks)=.5-T                           !  Cumulative probability in X(*)
860   Y(Ks)=Pr=.5+T                        !  Exceedance probability in Y(*)
870   IF Pr>=1E-12 THEN Y=LGT(Pr)
880   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
890   IF ABS(Pr)<1E-12 THEN Y=-12
900   PLOT Ks,Y
910   NEXT Ks
920   PENUP
930   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
940   FOR Ks=0 TO Mf-1
950   Pr=X(Ks)
960   IF Pr>=1E-12 THEN Y=LGT(Pr)
970   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
980   IF ABS(Pr)<1E-12 THEN Y=-12
990   PLOT Ks,Y
1000  NEXT Ks
1010  PENUP
1020  PAUSE
```

```
1030  DUMP GRAPHICS
1040  PRINT LIN(5)
1050  PRINTER IS 16
1060  END
1070  !
1080  SUB Mul(X1,Y1,X2,Y2,A,B)          !  Z1*Z2
1090  A=X1*X2-Y1*Y2
1100  B=X1*Y2+X2*Y1
1110  SUBEND
1120  !
1130  SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
1140  T=X2*X2+Y2*Y2
1150  A=(X1*X2+Y1*Y2)/T
1160  B=(Y1*X2-X1*Y2)/T
1170  SUBEND
1180  !
1190  SUB Exp(X,Y,A,B)                  !  EXP(Z)
1200  T=EXP(X)
1210  A=T*COS(Y)
1220  B=T*SIN(Y)
1230  SUBEND
1240  !
1250  SUB Fft10z(N,X(*),Y(*))     ! N  = 2^10 = 1024, N=2^INTEGER     0 subscript
```

APPENDIX D. REDUCED QUADRATIC AND LINEAR FORM

This program computes the cumulative and exceedance distribution functions of random variable (65) via characteristic function (66). The required inputs to the program are M and the $\{\lambda_m\}$, $\{d_m\}$, $\{\nu_m\}$ of (68). The square root in (66) must again be continuous and is handled exactly as in appendix A. The parameters required in the exponential of (66) are pre-computed in lines 170-210, and the mean of q is entered in line 220.

```
10 ! REDUCED QUADRATIC AND LINEAR FORM
20    M=5                          !  Number of terms summed
30    DATA .2,-.3,.4,.5,-.6        !  Lambda values
40    DATA -.1,.3,.5,.7,-.9        !  d values
50    DATA .6,.5,-.4,-.3,.2        !  Nu values
60    L=800                        !  Limit on integral of char. function
70    Delta=.08                    !  Sampling increment on char. function
80    Bs=.5*(2*PI/Delta)           !  Shift b, as fraction of alias interval
90    Mf=2^10                      !  Size of FFT
100   PRINTER IS 0
110   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
120   REDIM Lambda(1:M),D(1:M),Nu(1:M),A(1:M),B(1:M),C(1:M)
130   REDIM X(0:Mf-1),Y(0:Mf-1)
140   DIM Lambda(1:10),D(1:10),Nu(1:10),A(1:10),B(1:10),C(1:10)
150   DIM X(0:1023),Y(0:1023)
160   READ Lambda(*),D(*),Nu(*)          !  Enter constants
170   FOR Ms=1 TO M                      !  Calculation
180   A(Ms)=2*Lambda(Ms)                 !  of parameters
190   B(Ms)=(Lambda(Ms)*Nu(Ms)+D(Ms))*Nu(Ms)
200   C(Ms)=.5*D(Ms)^2
210   NEXT Ms
220   Muq=SUM(Lambda)+SUM(B)             !  Mean of random variable q
230   R=0                                !  Argument of square root
240   P=1                                !  Polarity indicator
250   Muy=Muq+Bs
260   X(0)=0
270   Y(0)=.5*Delta*Muy
280   N=INT(L/Delta)
290   FOR Ns=1 TO N
300   Xi=Delta*Ns                        !  Argument xi of char. fn.
310   Pr=1                               !  Calculation
320   Pi=Sr=Si=0                         !  of
330   FOR Ms=1 TO M                      !  characteristic
340   T=-A(Ms)*Xi                        !  function
350   CALL Mul(Pr,Pi,1,T,A,B)            !  fy(xi)
360   Pr=A
370   Pi=B
380   CALL Div(B(Ms),C(Ms)*Xi,1,T,A,B)
390   Sr=Sr+A
400   Si=Si+B
410   NEXT Ms
```

```
420   CALL Exp(-Si*Xi,(Sr+Bs)*Xi,A,B)
430   CALL Sqr(Pr,Pi,C,D)
440   Ro=R
450   R=ATN(D/C)
460   IF ABS(R-Ro)>1.6 THEN P=-P
470   CALL Div(A,B,C*P,D*P,Fyr,Fyi)
480   Ms=Ns MOD Mf                          !  Collapsing
490   X(Ms)=X(Ms)+Fyr/Ns
500   Y(Ms)=Y(Ms)+Fyi/Ns
510   NEXT Ns
520   CALL Fft10z(Mf,X(*),Y(*))             !  0 subscript FFT
530   PLOTTER IS "GRAPHICS"
540   GRAPHICS
550   SCALE 0,Mf,-14,0
560   LINE TYPE 3
570   GRID Mf/8,1
580   PENUP
590   LINE TYPE 1
600   B=Bs*Mf*Delta/(2*PI)                  !  Origin for random variable q
610   MOVE B,0
620   DRAW B,-14
630   PENUP
640   FOR Ks=0 TO Mf-1
650   T=Y(Ks)/PI-Ks/Mf
660   X(Ks)=.5-T                            !  Cumulative probability in X(*)
670   Y(Ks)=Pr=.5+T                         !  Exceedance probability in Y(*)
680   IF Pr>=1E-12 THEN Y=LGT(Pr)
690   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
700   IF ABS(Pr)<1E-12 THEN Y=-12
710   PLOT Ks,Y
720   NEXT Ks
730   PENUP
740   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
750   FOR Ks=0 TO Mf-1
760   Pr=X(Ks)
770   IF Pr>=1E-12 THEN Y=LGT(Pr)
780   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
790   IF ABS(Pr)<1E-12 THEN Y=-12
800   PLOT Ks,Y
810   NEXT Ks
820   PENUP
830   PAUSE
840   DUMP GRAPHICS
850   PRINT LIN(5)
860   PRINTER IS 16
870   END
880   !
```

```
890   SUB Mul(X1,Y1,X2,Y2,A,B)            !  Z1*Z2
900   A=X1*X2-Y1*Y2
910   B=X1*Y2+X2*Y1
920   SUBEND
930   !
940   SUB Div(X1,Y1,X2,Y2,A,B)            !  Z1/Z2
950   T=X2*X2+Y2*Y2
960   A=(X1*X2+Y1*Y2)/T
970   B=(Y1*X2-X1*Y2)/T
980   SUBEND
990   !
1000  SUB Exp(X,Y,A,B)                    !  EXP(Z)
1010  T=EXP(X)
1020  A=T*COS(Y)
1030  B=T*SIN(Y)
1040  SUBEND
1050  !
1060  SUB Sqr(X,Y,A,B)                    !  PRINCIPAL SQR(Z)
1070  IF X<>0 THEN 1110
1080  A=B=SQR(.5*ABS(Y))
1090  IF Y<0 THEN B=-B
1100  GOTO 1220
1110  F=SQR(SQR(X*X+Y*Y))
1120  T=.5*ATN(Y/X)
1130  A=F*COS(T)
1140  B=F*SIN(T)
1150  IF X>0 THEN 1220
1160  T=A
1170  A=-B
1180  B=T
1190  IF Y>=0 THEN 1220
1200  A=-A
1210  B=-B
1220  SUBEND
1230  !
1240  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

APPENDIX E. SIMULATION OF SECOND-ORDER PROCESSOR

This program simulates random variable x in second-order processor (8) directly. The weights and statistics are entered in lines 30-130. A pair of independent zero-mean unit-variance Gaussian random variables are generated in lines 310-380. The sample cumulative and exceedance distribution functions are computed in lines 510-590.

```
10 ! SIMULATION OF SECOND-ORDER PROCESSOR
20    Tt=10000                    !  Number of trials
30    K=5                         !  Number of terms summed
40    DATA .6,-.5,.4,-.3,.2       !  a(k) weightings
50    DATA .9,.8,.7,-.6,-.5       !  b(k) weightings
60    DATA -.6,-.8,1,1.2,1.4      !  c(k) weightings
70    DATA .1,-.2,-.3,.4,.5       !  d(k) weightings
80    DATA -.7,.6,.5,.4,-.3       !  e(k) weightings
90    DATA .2,.3,.4,-.5,-.6       !  Means of random variables s(k)
100   DATA .8,-.7,-.6,.5,.4       !  Means of random variables t(k)
110   DATA .1,.3,.5,.7,.9         !  Standard deviations of s(k)
120   DATA .2,.4,.6,.8,1          !  Standard deviations of t(k)
130   DATA .4,-.5,.6,.7,-.8       !  Correlation coeffs. of s(k) and t(k)
140   REDIM A(1:K),B(1:K),C(1:K),D(1:K),E(1:K)
150   REDIM Ms(1:K),Mt(1:K),Ss(1:K),St(1:K),Rho(1:K)
160   REDIM A1(1:K),Be(1:K),X(1:Tt)
170   DIM A(1:10),B(1:10),C(1:10),D(1:10),E(1:10)
180   DIM Ms(1:10),Mt(1:10),Ss(1:10),St(1:10),Rho(1:10)
190   DIM A1(1:10),Be(1:10),X(1:10000)
200   READ A(*),B(*),C(*),D(*),E(*)
210   READ Ms(*),Mt(*),Ss(*),St(*),Rho(*)
220   FOR J=1 TO K
230   A1(J)=Ss(J)*Rho(J)
240   Be(J)=Ss(J)*SQR(1-Rho(J)^2)
250   NEXT J
260   RANDOMIZE SQR(.6)
270   L=LOG(.25)
```

```
280   FOR I=1 TO Tt
290   X=0
300   FOR J=1 TO K
310   V1=RND-.5                          ! GENERATE TWO
320   V2=RND-.5                          ! INDEPENDENT
330   S=V1*V1+V2*V2                      ! GAUSSIAN
340   IF S>.25 THEN 310                  ! RANDOM
350   Q=(L-LOG(S))/S                     ! VARIABLES VIA
360   Q=SQR(Q+Q)                         ! ACCEPTANCE
370   G1=V1*Q                            ! AND
380   G2=V2*Q                            ! REJECTION
390   S=Ms(J)+A1(J)*G1+Be(J)*G2
400   T=Mt(J)+St(J)*G1
410   X=X+A(J)*S*S+B(J)*T*T+C(J)*S*T+D(J)*S+E(J)*T
420   NEXT J
430   X(I)=X
440   NEXT I
450   MAT SORT X
460   PLOTTER IS "GRAPHICS"
470   GRAPHICS
480   SCALE -30,10,-4,0
490   GRID 5,1
500   PENUP
510   FOR I=1 TO Tt
520   Y=LGT((I-.5)/Tt)
530   PLOT X(I),Y
540   NEXT I
550   PENUP
560   FOR I=1 TO Tt
570   Y=LGT(1-(I-.5)/Tt)
580   PLOT X(I),Y
590   NEXT I
600   PENUP
610   END
```

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Exact Operating Characteristics for Linear Sum of Envelopes of Narrowband Gaussian Process and Sinewave

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ABSTRACT

The characteristic function of a linear sum of M independent Rice variates is derived and evaluated exactly and then used in a numerical procedure to determine the exceedance distribution function, as a function of the threshold, the input signal-to-noise ratio (SNR), and M. Plots of the detection probability and false alarm probability for a wide range of SNR's are given, for values of M up to 8192. In addition, the required threshold values and input SNR's are tabulated and plotted for specific values of M, false alarm probability, and detection probability. A program and explanation are included for those users interested in extending results to their particular application.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

M	Number of independent envelope samples summed
e_m	m-th envelope sample
x	Decision variable; sum of M envelope samples
P_D	Detection probability
P_{FA}	False alarm probability
A	Sinewave amplitude
σ	Noise standard deviation
α	A/σ, voltage measure of input signal-to-noise ratio
S/N	$\alpha^2/2$, power measure of input signal-to-noise ratio
$p_e(u)$	Probability density function of envelope random variable e_m
$f_e(\zeta)$	Characteristic function of envelope random variable e_m
I_o	Modified Bessel function of order zero
${}_1F_1$	Confluent hypergeometric function
μ_x	Mean of random variable x
$f_x(\zeta)$	Characteristic function of decision variable x
$Q_x(u)$	Exceedance distribution function of random variable x; (8)
Im, Re	Imaginary part, Real part
Δ	Sampling increment in ζ; (12)
N	Size of FFT; (13)-(14)
L	Limit employed on integral on ζ in (9)
b	Bias employed to shift random variable x
Φ	Cumulative distribution function of normalized Gaussian random variable; (19)
$\tilde{\Phi}$	Inverse function to Φ defined in (19)
σ_x	Standard deviation of random variable x
dB	Required input signal-to-noise ratio per-sample in decibels
β	Parameter incorporating specified P_D and P_{FA}; (28)

EXACT OPERATING CHARACTERISTICS FOR LINEAR SUM OF ENVELOPES OF NARROWBAND GAUSSIAN PROCESS AND SINEWAVE

INTRODUCTION

The operating characteristics for a linear envelope-detector of a sinewave in narrowband Gaussian noise, followed by summation of M independent envelope samples, were presented in [1] and [2, sect. 8.3]. That approach was based upon evaluation of the first 31 moments of the envelope variate and their use in a type A Gram-Charlier series approximation, or in modified approximations involving averages over different numbers of terms in the series [1, pp. 758-9]. However, there are possible pitfalls to the above approach. First, evaluation of very low exceedance probabilities, like 10^{-10}, may be inaccurate; see [1, Fig. 1]. Second, the effect of a systematic error would be hard to detect, if present, since the method yields only an approximation to the exceedance distribution function, and not its exact value.

We will use an exact approach here, based upon evaluation of the characteristic function of the envelope detector output, from which the exceedance distribution function can be precisely evaluated numerically [3,4]. In this fashion, we avoid moment evaluations altogether; we can evaluate false alarm probabilities in the 10^{-10} range easily (with double precision computer arithmetic); and we can control truncation and aliasing errors to any desired degree; see [3] for details. The results of [4] can not be applied here because each independent envelope sample is the result of a nonlinear operation, namely a square root, applied to a sum of two squares of Gaussian random variables with non-zero means.

In the plots of detection probability vs. false alarm probability to be presented herein, both abscissa and ordinate use the same normal probability scales, regardless of the number of envelope samples M considered. This allows for easier interpolation, and is in distinction to [1], where a different false alarm probability abscissa was used for each M [1, pp. 759-62]. Also, the parameter employed here for indexing the curves is α, a voltage signal-to-noise ratio which is equal to the ratio of the sinewave amplitude to the

rms noise level, rather than the dB parameter employed in [1]. This leads to curves that are more nearly equally spaced, and therefore to easier and finer interpolation capability.

Finally, we present five figures for the required input signal-to-noise ratio per sample required to realize specified false alarm and detection probabilities, as a function of M, the number of envelope samples added. The five figures correspond to detection probability P_D=.5, .9, .95, .99, and .999 respectively, and each figure contains false alarm probabilities $P_{FA}=10^{-n}$ for n=1(1)8. This total of 40 curves greatly augments the 2 cases presented in [1, Fig. 16] and [2, Fig. 8.18].

A program for the evaluation of the input signal-to-noise ratio required for a specified set of values of M, P_{FA}, and P_D is furnished, along with an explanation of its use. In this fashion, values of M, P_{FA}, and P_D intermediate to those considered here can be easily investigated.

METHOD OF EVALUATION

Characteristic Function Details

In [3,4], a method of calculating the cumulative and exceedance distribution functions directly from a given characteristic function was presented. To utilize those results here, we need the characteristic function of summation random variable

$$x = \sum_{m=1}^{M} e_m \quad , \tag{1}$$

where e_m is the envelope of a narrowband filter output with a sinewave signal of amplitude A and Gaussian noise of power σ^2. Through proper normalization, the probability density function of envelope e_m takes the familiar Rice form

$$p_e(u) = u \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad \text{for } u \geq 0 \quad , \tag{2}$$

where the single parameter

$$\alpha = \frac{A}{\sigma} \tag{3}$$

is a voltage measure of signal-to-noise ratio per envelope sample. The power measure of signal-to-noise ratio per sample is

$$\frac{S}{N} = \frac{A^2/2}{\sigma^2} = \frac{\alpha^2}{2} \quad . \tag{4}$$

The quantities in (3) and (4) will be referred to as input signal-to-noise ratios, since they are per-sample measures, prior to the summation in (1) which yields the output or decision variable x.

The characteristic function corresponding to random variable e in (2) is given by Fourier transform

$$f_e(\ ) = \int_{-\infty}^{+\infty} du\ \exp(i\xi u)\ p_e(u) = \int_{0}^{+\infty} du\ u\ \exp\left(i\xi u - \frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad , \qquad (5)$$

and will be called the Rice characteristic function. A series expansion for (5) is developed in appendix A, and has been programmed in double precision for numerical use here. As a particular special case, for $\alpha=0$, no signal, we have the Rayleigh probability density function and characteristic function:

$$p_e^{(o)}(u) = u\ \exp(-u^2/2) \quad \text{for } u \geq 0 \quad ,$$

$$f_e^{(o)}(\xi) = \exp(-\xi^2/2)\left[{}_1F_1\left(-\frac{1}{2};\ \frac{1}{2};\ \frac{\xi^2}{2}\right) + i\left(\frac{\pi}{2}\right)^{1/2}\xi\right] . \qquad (6)$$

The latter follows by use of [5, 3.896 3&4] and via manipulation of the hypergeometric function series along with Kummer's transformation [5, 9.212 1]. Formula (6) is particularly attractive numerically, since the series expansion of ${}_1F_1$ contains all negative terms except for one. It should be observed that the imaginary part of Rayleigh characteristic function $f_e^{(o)}(\xi)$ in (6) decays very rapidly with ξ; this useful feature will also be shared by the Rice characteristic function, $f_e(\xi)$, and is due to the fact that the odd part of the Rice probability density function in (2) is smooth for all u, and is in fact entire in u, for any α. By contrast, the even part of the Rice probability density function in (2) has a discontinuous derivative for real u, thereby leading to slow decay of the real part of $f_e(\xi)$.

The characteristic function of output variable x in (1), for statistically independent envelope samples $\{e_m\}$, is given by

$$f_x(\xi) = [f_e(\xi)]^M \quad , \qquad (7)$$

in terms of the Rice characteristic function (5). This relation could be used directly to find the exceedance distribution function of x according to [3, (5)-(6)]

$$Q_x(u) = \int_{u}^{+\infty} dt\ p_x(t) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi\ \mathrm{Im}\left\{\exp(-iu\xi)\ \frac{f_x(\xi)}{\pi\xi}\right\} \quad . \qquad (8)$$

However, the slow decay of $\mathrm{Re}\{f_x(\xi)\}$ prompts us to use a modified version given in [6, (15)]:

$$Q_x(u) = \frac{2}{\pi} \int_{0+}^{+\infty} \frac{d\xi}{\xi} \cos(u\xi)\ \mathrm{Im}\{f_x(\xi)\} \quad \text{for} \quad u > 0 \quad . \tag{9}$$

This form is applicable to positive random variables, of which x, as given by (1) and (2), is certainly a member.

To see why form (9) is preferred over (8), we develop (7) as

$$f_x(\xi) = [f_r(\xi) + i f_i(\xi)]^M = \sum_{m=0}^{M} \binom{M}{m} i^m [f_i(\xi)]^m [f_r(\xi)]^{M-m} \quad , \tag{10}$$

where $f_r(\xi)$ and $f_i(\xi)$ are the real and imaginary parts of Rice characteristic function $f_e(\xi)$. Then

$$\mathrm{Im}\{f_x(\xi)\} = \sum_{\substack{m=1 \\ m\ \mathrm{odd}}}^{M} (-1)^{\frac{m-1}{2}} \binom{M}{m} [f_i(\xi)]^m [f_r(\xi)]^{M-m} \tag{11}$$

contains $f_i(\xi)$ to at least the first power in all terms, thereby yielding a rapid decay with ξ.

Development (11) has been used to show why $\mathrm{Im}\{f_x(\xi)\}$ decays rapidly with ξ. However, when we employ (9) in a numerical evaluation, we simply take the imaginary part of the power in (7), and do not use (11) at all; (11) is an alternating series of large terms for large M.

Actual numerical evaluation of (9) proceeds as follows [3]: for the Trapezoidal rule with sampling increment Δ in ξ,

$$Q_x(u) \cong \frac{2}{\pi}\left[\frac{1}{2}\mu_x\Delta + \sum_{n=1}^{\infty} \frac{1}{n} \cos(un\Delta)\ \mathrm{Im}\{f_x(n\Delta)\}\right] \quad , \tag{12}$$

where we used $f_x(\xi) \sim 1 + i\mu_x\xi$ as $\xi \to 0$. Then, restricting the u values to a particular selection,

$$Q_x\left(\frac{2\pi m}{N\Delta}\right) = \frac{2}{\pi}\left[\frac{1}{2}\mu_x\Delta + \sum_{n=1}^{\infty}\frac{1}{n}\cos(2\pi mn/N)\ \mathrm{Im}\{f_x(n\Delta)\}\right] =$$

$$= \frac{2}{\pi}\,\mathrm{Re}\sum_{n=0}^{N-1} z_n \exp(-i2\pi mn/N) \quad , \tag{13}$$

where collapsed sequence $\{z_n\}_0^{N-1}$ is defined as

$$z_0 = \frac{1}{2}\mu_x\Delta + \sum_{j=1}^{\infty}\frac{1}{jN}\,\mathrm{Im}\{f_x(jN\Delta)\} \quad ,$$

$$z_n = \sum_{j=0}^{\infty}\frac{1}{n+jN}\,\mathrm{Im}\{f_x((n+jN)\Delta)\} \quad \text{for} \quad 1 \le n \le N-1 \quad . \tag{14}$$

Form (13) is particularly attractive since it can be accomplished via an N-point FFT. It can be shown that only the values for $0 \le m \le N/2$ are useful in (13); the remainder are heavily aliased and must be discarded. Thus there is a trade-off: use of only the imaginary part of $f_x(\xi)$ results in aliasing twice as coarse. However, the rapid decay of the imaginary part far outweighs the aliasing.

The summations in (12) and (14) cannot be conducted to infinity. Rather the integral on ξ in (9) is terminated at limit L, where the truncation error is guaranteed to be sufficiently small. A trial and error procedure [3] yielded the following rules which control the truncation and aliasing errors:

$$\begin{aligned} L &= \min\,(9,\ 17/\sqrt{M}), \\ \Delta &= .12/\sqrt{M}, \\ b &= \min\,(0,\ -M\sqrt{\pi/2} + \sqrt{M}\,6). \end{aligned} \tag{15}$$

The inverse $\sqrt{M}$ dependence of L and Δ for large M can be anticipated by observing that the characteristic function of random variable x in (1) then

approaches a Gaussian function with argument proportional to $M\delta^2$. The bias (or shift) b is added to random variable x in order to yield a new random variable that remains just positive, even for large M; this allows us to take maximum advantage of the fundamental aliasing interval $(0, \pi/\Delta)$ in u in (12) and (13). The linear term (in M) of b in (15) is due to the mean of the Rayleigh variate (for $\alpha=0$) which is $\sqrt{\pi/2}$; the algebraic term in $\sqrt{M}$ is due to the fact that the standard deviation of random variable x in (1) increases according to $\sqrt{M}$.

In order to use this characteristic function approach, we also need the mean of random variable x in (1). Using (2), this is given by [5, 6.631 1]

$$\mu_x = M\mu_e = M \int_0^\infty du\, u^2 \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) =$$

$$= M \left(\frac{\pi}{2}\right)^{1/2} \exp\left(-\frac{\alpha^2}{2}\right) {}_1F_1\left(\frac{3}{2};\, 1;\, \frac{\alpha^2}{2}\right) \quad . \tag{16}$$

This non-alternating series yields accurate values for the mean.

Special Cases

For general M, the characteristic function approach described above must be used. However, for M = 1 and 2, closed form expressions for the false alarm and detection probabilities are possible. Specifically, from (1) and (2), for $u \geq 0$,

$$\left.\begin{aligned} P_{FA} &= \int_u^\infty dt\, p_e(t) = \int_u^\infty dt\, t \exp(-t^2/2) = \exp(-u^2/2) \\ P_D &= \int_u^\infty dt\, t \exp\left(-\frac{t^2+\alpha^2}{2}\right) I_0(\alpha t) = Q(\alpha,u) \end{aligned}\right\} \text{for } M = 1. \tag{17}$$

And for M = 2, the false alarm probability can be determined by convolving two Rayleigh probability density functions of the form of (6), to give, for $u \geq 0$,

$$P_{FA} = \exp(-u^2/2) + \sqrt{\pi}\, u \exp(-u^2/4)\left[\Phi\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\right] \quad \text{for } M = 2. \tag{18}$$

Here, Φ is the cumulative distribution function of a normalized Gaussian random variable:

$$\Phi(u) = \int_{-\infty}^{u} dt\ (2\pi)^{-1/2} \exp(-t^2/2) \quad . \tag{19}$$

The detection probability of random variable x in (1) is not available in closed form for M > 1.

Asymptotic Performance for Large M

For large M, decision variable x in (1) is approximately Gaussian. The mean of x was given in (16); a similar approach for the mean square of x yields the variance as

$$\sigma_x^2 = M\sigma_e^2 = M(2+\alpha^2-\mu_e^2) \quad . \tag{20}$$

The probability density function of x is then approximately

$$p_x(u) \cong \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{(u-\mu_x)^2}{2\sigma_x^2}\right], \tag{21}$$

with exceedance distribution function

$$Q_x(u) \cong \Phi\left(\frac{\mu_x - u}{\sigma_x}\right) = \Phi\left(\frac{M\mu_e - u}{\sqrt{M}\,\sigma_e}\right) \quad . \tag{22}$$

For input signal-to-noise ratio S/N=0, we have α=0 from (4), and (22), (16), and (20) specialize to

$$P_{FA} \cong \Phi\left(\frac{M\sqrt{\pi/2} - u}{\sqrt{M}\sqrt{2-\frac{\pi}{2}}}\right) . \tag{23}$$

On the other hand, for S/N>0, (22) yields the detection probability P_D. We now use the inverse function $\tilde{\Phi}$ to definition (19) and solve (23) and (22) according to

$$\frac{M\sqrt{\pi/2}-u}{\sqrt{M}\sqrt{2-\frac{\pi}{2}}} = \tilde{\Phi}(P_{FA}) \quad , \quad \frac{M\mu_e-u}{\sqrt{M}\,\sigma_e} = \tilde{\Phi}(P_D) \quad . \tag{24}$$

Eliminating threshold u in (24), we have

$$\sigma_e\tilde{\Phi}(P_D) = \sqrt{M}\left(\mu_e-\sqrt{\frac{\pi}{2}}\right) + \sqrt{2-\frac{\pi}{2}}\,\tilde{\Phi}(P_{FA}) \quad . \tag{25}$$

But also, for large M, the required per-sample input signal-to-noise ratio α will be small, giving

$$\mu_e = \sqrt{\frac{\pi}{2}}\,{}_1F_1\left(-\frac{1}{2};\ 1;\ -\frac{\alpha^2}{2}\right) \cong \sqrt{\frac{\pi}{2}}\left(1+\frac{\alpha^2}{4}\right) \quad ,$$

$$\sigma_e^2 = 2 + \alpha^2 - \mu_e^2 \cong 2 - \frac{\pi}{2} \quad . \tag{26}$$

Substituting these results in (25) and solving for α, we have the required per-sample input signal-to-noise ratio measures for large M in the alternative forms

$$\alpha \cong 2\left(\frac{4-\pi}{\pi}\right)^{1/4}\frac{\beta^{1/2}}{M^{1/4}} = 1.446\,\frac{\beta^{1/2}}{M^{1/4}} \quad ,$$

$$\frac{S}{N} = \frac{\alpha^2}{2} \cong 2\left(\frac{4-\pi}{\pi}\right)^{1/2}\frac{\beta}{M^{1/2}} = 1.045\,\frac{\beta}{M^{1/2}} \quad ,$$

$$dB = 10\log\frac{S}{N} \cong 10\log\left(2\sqrt{\frac{4-\pi}{\pi}}\right)+10\log(\beta)-5\log(M) = .193+10\log(\beta)-5\log(M), \tag{27}$$

where the single parameter

$$\beta = \tilde{\Phi}(P_D) - \tilde{\Phi}(P_{FA}) \tag{28}$$

incorporates the specified false alarm and detection probabilities. (27) displays the familiar 5 log M decibel decay for large M associated with the incoherent addition in (1); see also [2, p. 279, Ex. 8.8].

RESULTS

For a given value of M, the output variable in (1),

$$x = \sum_{m=1}^{M} e_m \quad , \tag{29}$$

will exceed threshold u with false alarm probability P_{FA} when signal-to-noise ratio α is zero. That is

$$P_{FA} = \text{Prob}(x>u \mid \alpha=0;\ M). \tag{30}$$

For specified values of M and P_{FA}, this relation can be solved numerically for u; the values of normalized threshold u/M are listed in table 1 for $M=2^n$, n=0(1)13 and for $P_{FA}=10^{-n}$, n=1(1)8.

The detection probability depends on threshold u, M, and signal-to-noise ratio $\alpha(>0)$:

$$P_D = \text{Prob}(x>u \mid \alpha;\ M). \tag{31}$$

For specified values of M, P_D, and u, this relation can be solved numerically for the required input signal-to-noise ratio α. When the threshold results in Table 1 are employed, the results yield the required input signal-to-noise ratio for specified false alarm probability and detection probability at a particular M. These are plotted in figures 1-5 for

$$P_D = .5,\ .9,\ .95,\ .99,\ .999, \tag{32}$$

respectively. The abscissa is $\log_2 M$, and the ordinate is in decibels, as defined in (27). The fit of (27) is very good for large M, especially for the larger P_{FA} values. These results in figures 1-5 greatly extend the one in [1, Fig. 16] and [2, Fig. 8.18].

Table 1. Normalized Thresholds Required for Specified M and P_{FA}

M \ P_{FA}	1E-1	1E-2	1E-3	1E-4
1	2.14596603	3.03485426	3.71692219	4.29193205
2	1.87154046	2.46578168	2.92459903	3.31372579
4	1.68491649	2.08494224	2.39281962	2.65432267
8	1.55592564	1.82779134	2.03544098	2.21134522
16	1.46605729	1.65246898	1.79362769	1.91266565
32	1.40314416	1.53192213	1.62866385	1.70984877
64	1.35896377	1.44846093	1.51524477	1.57104117
128	1.32787317	1.39035933	1.43673968	1.47534630
256	1.30596258	1.34974198	1.38210498	1.40896493
512	1.29050601	1.32125803	1.34392160	1.36268942
1024	1.27959472	1.30123656	1.31715039	1.33030674
2048	1.27188832	1.28713956	1.29833595	1.30758095
4096	1.26644357	1.27720181	1.28509047	1.29159844
8192	1.26259580	1.27018998	1.27575385	1.28034098

M \ P_{FA}	1E-5	1E-6	1E-7	1E-8
1	4.79852591	5.25652177	5.67769243	6.06970852
2	3.65817649	3.97074674	4.25904998	4.52806135
4	2.88639585	3.09755766	3.29282208	3.47544423
8	2.36734857	2.50933650	2.64073862	2.76376208
16	2.01795589	2.11363367	2.20209577	2.28487698
32	1.78141625	1.84629005	1.90615996	1.96210527
64	1.62006566	1.66438962	1.70520835	1.74328423
128	1.50917003	1.53967893	1.567 1937	1.59383081
256	1.43244246	1.45357776	1.47297026	1.49100181
512	1.37906412	1.39378248	1.40726893	1.41979378
1024	1.34176981	1.35206123	1.36148146	1.37022180
2048	1.31562790	1.32284604	1.32944798	1.33556910
4096	1.29725887	1.30233301	1.30697131	1.31126956
8192	1.28432858	1.28790149	1.29116614	1.29419029

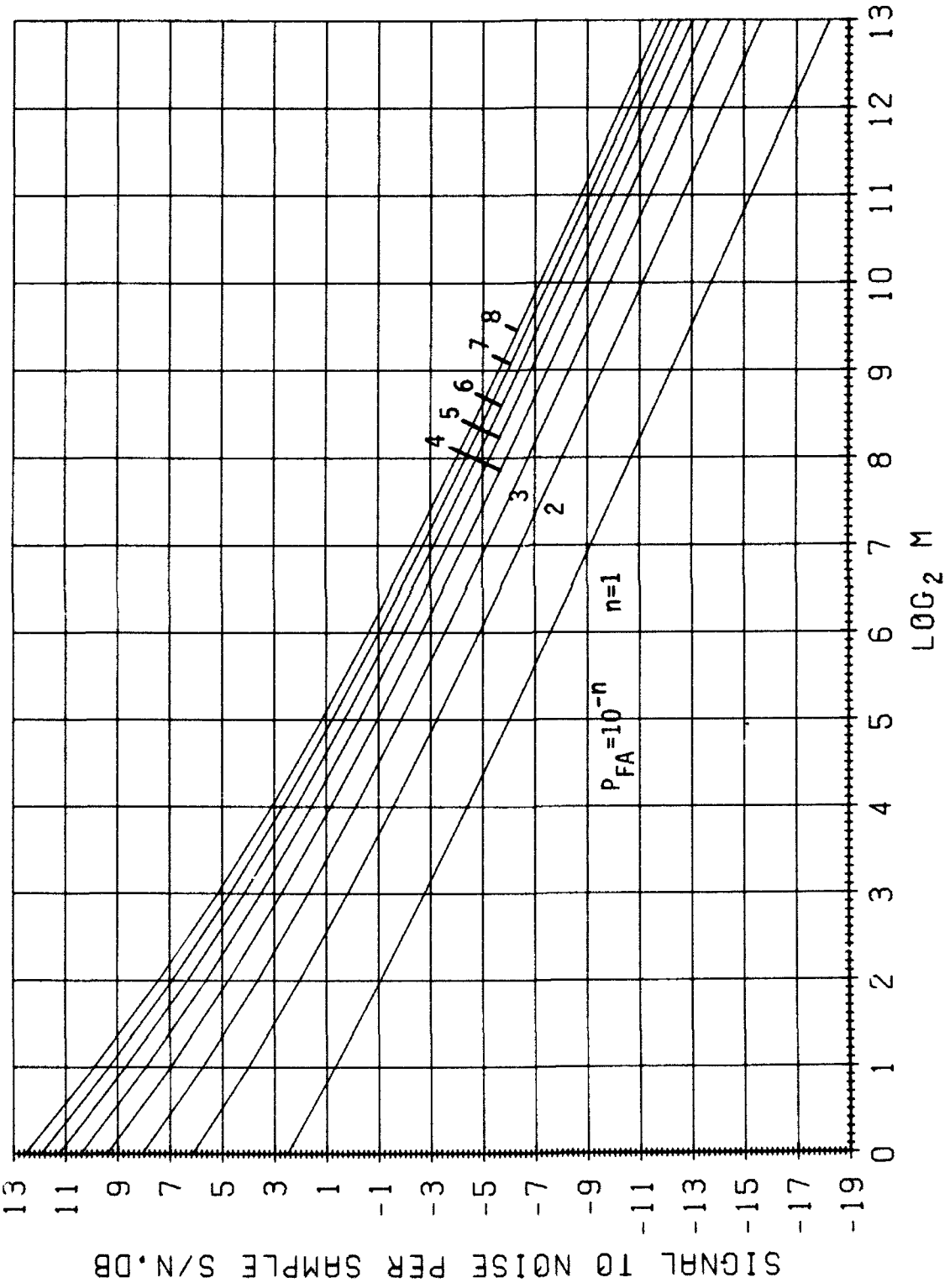

Figure 1. Required Input S/N for P_n=.5

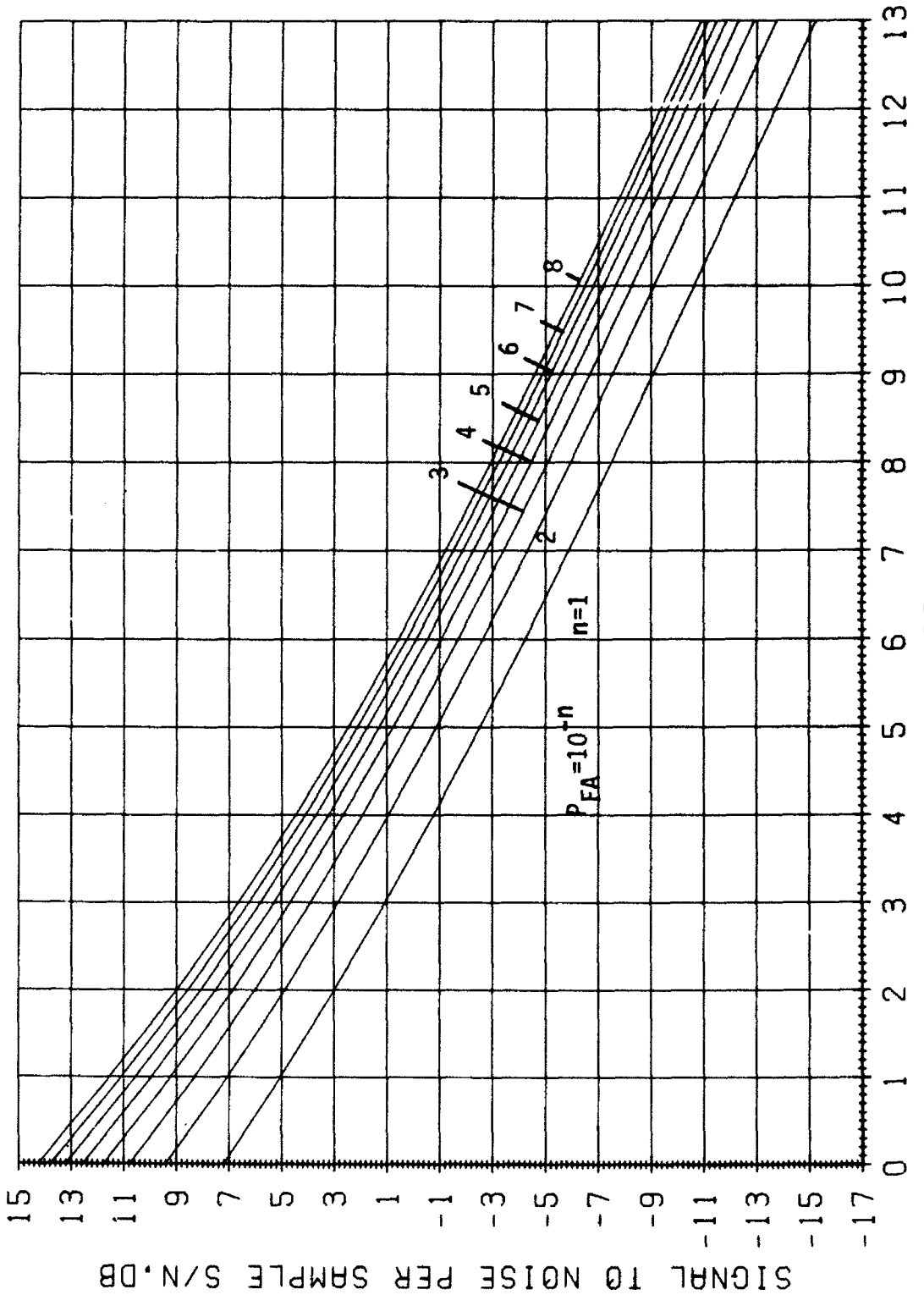

Figure 2. Required Input S/N for $P_D=.9$

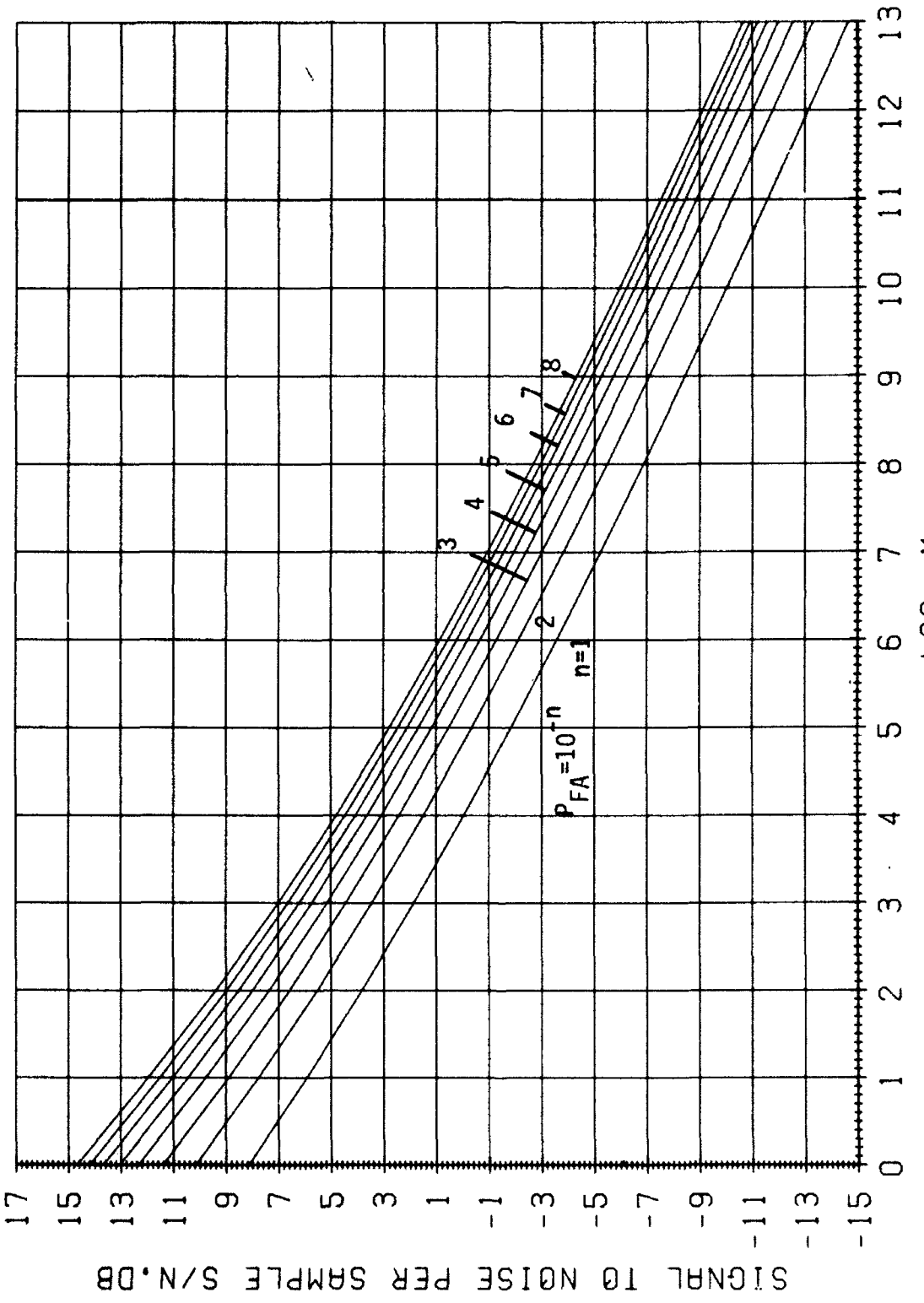

Figure 3. Required Input S/N for P_D=.95

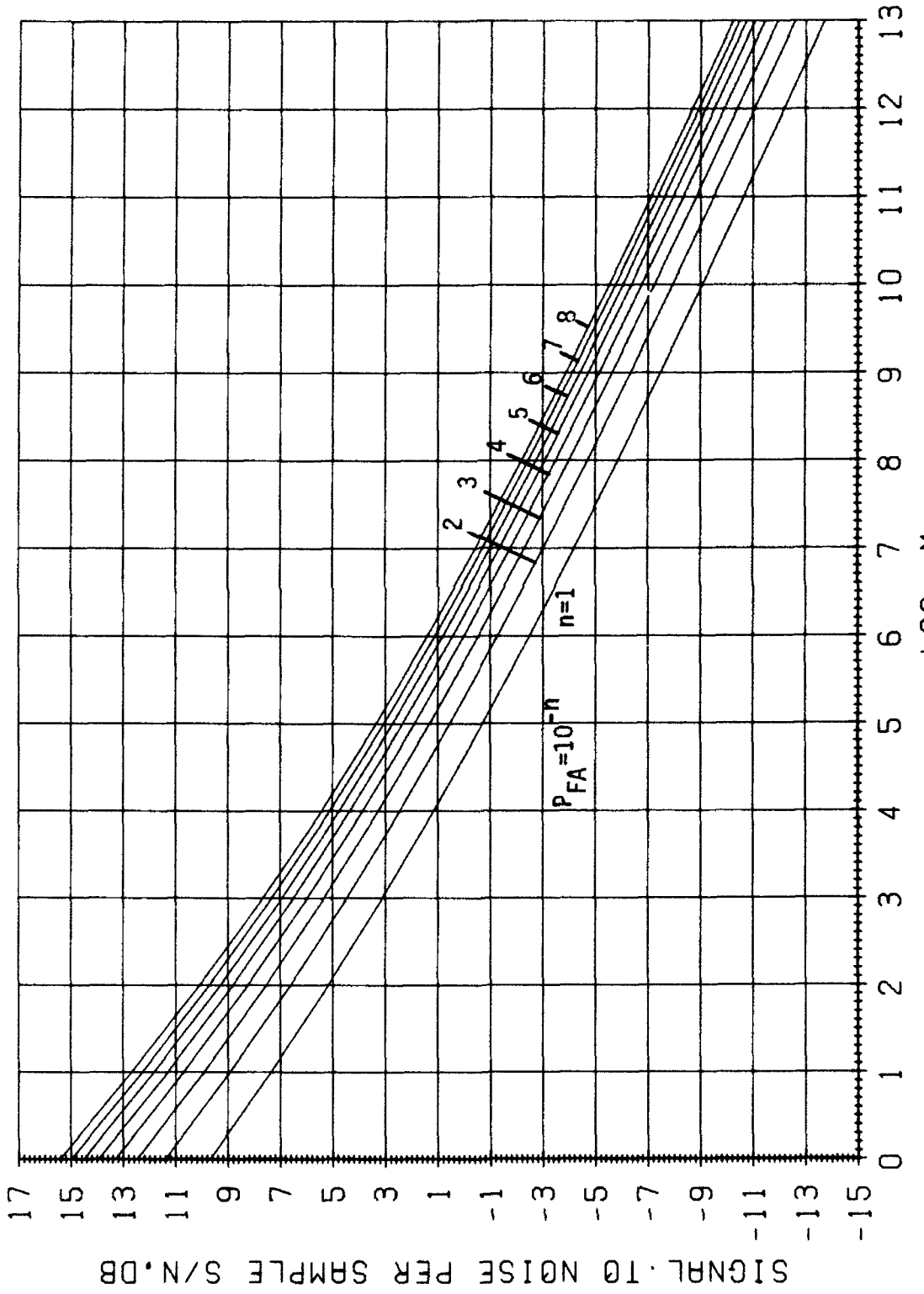

Figure 4. Required Input S/N for P_D=.99

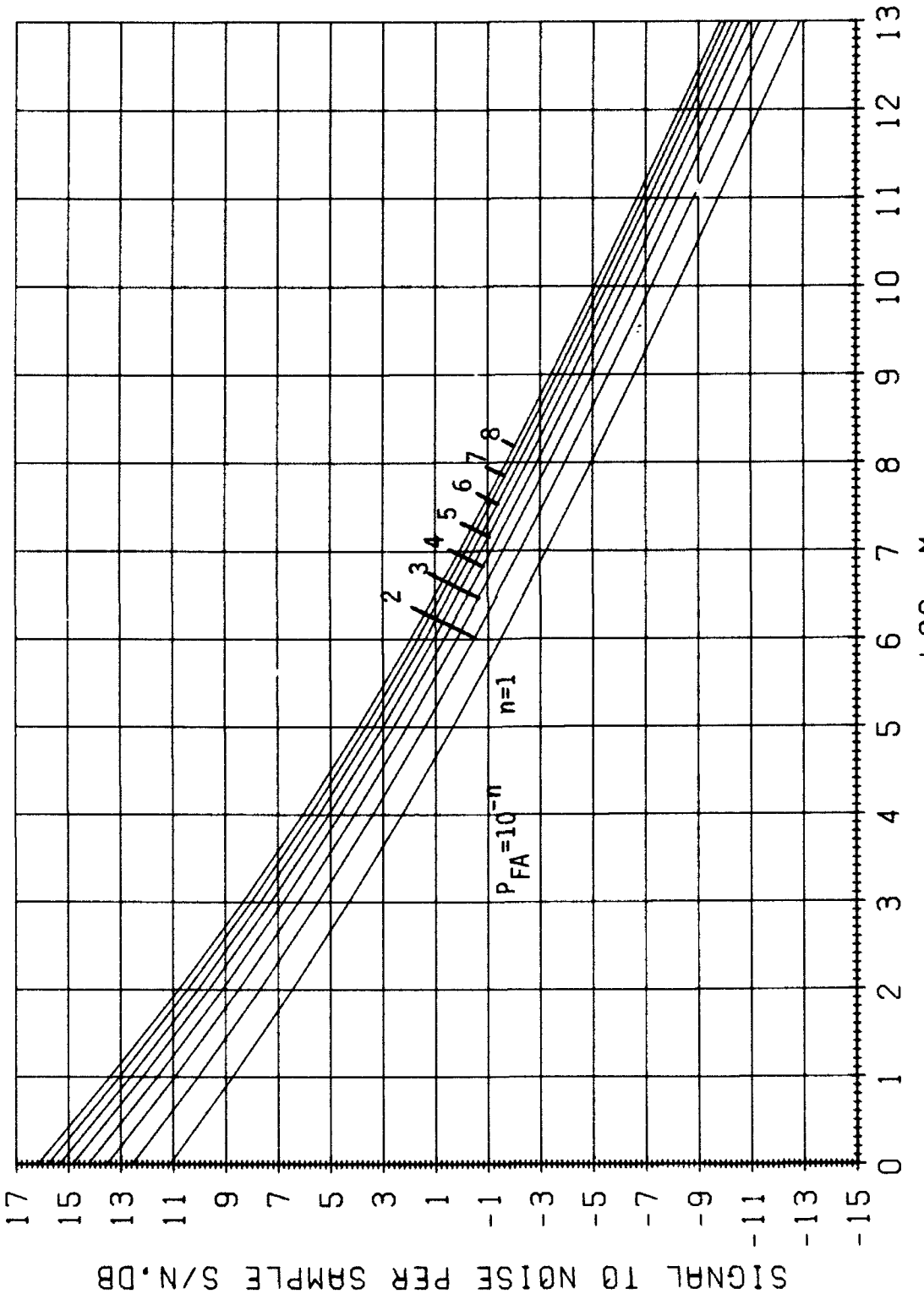

Figure 5. Required Input S/N for P_D=.999

The results in figures 1 through 5 only cover a selected set of detection and false alarm probability values. A more complete description is afforded by the receiver operating characteristics, namely detection probability vs. false alarm probability, with signal-to-noise ratio as a parameter. In figures 6 through 19 are given these operating characteristics for

$$M = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, \quad (33)$$

respectively. The false alarm probability covers the range 10^{-10} to .5, while the detection probability covers 10^{-10} to .999. Both abscissa and ordinate in these figures employ the inverse function to the Gaussian cumulative distribution function Φ defined in (19); thus, a truly Gaussian random variable would plot as a series of equally spaced parallel straight lines (with parameter α). Observe that the curves are nearly equally spaced with parameter α, except for very small α, where the nonlinear envelope operation causes small signal suppression and a crowding together of the curves.

If the decision variable x is presumed Gaussian, and the operating characteristics overlayed on the exact results in figures 6-19, it is found that the two sets of curves for M=8192 are virtually identical in the range of P_{FA} and P_D plotted. However, for M=16, the Gaussian approximation is somewhat optimistic; for example, the exact curve for α=2.75 is well-approximated by the Gaussian approach for α=2.62. For small M, the Gaussian approximation is overly optimistic for small P_{FA}; however, the two sets cross near P_{FA}=.5, which is not a practical range of interest anyway.

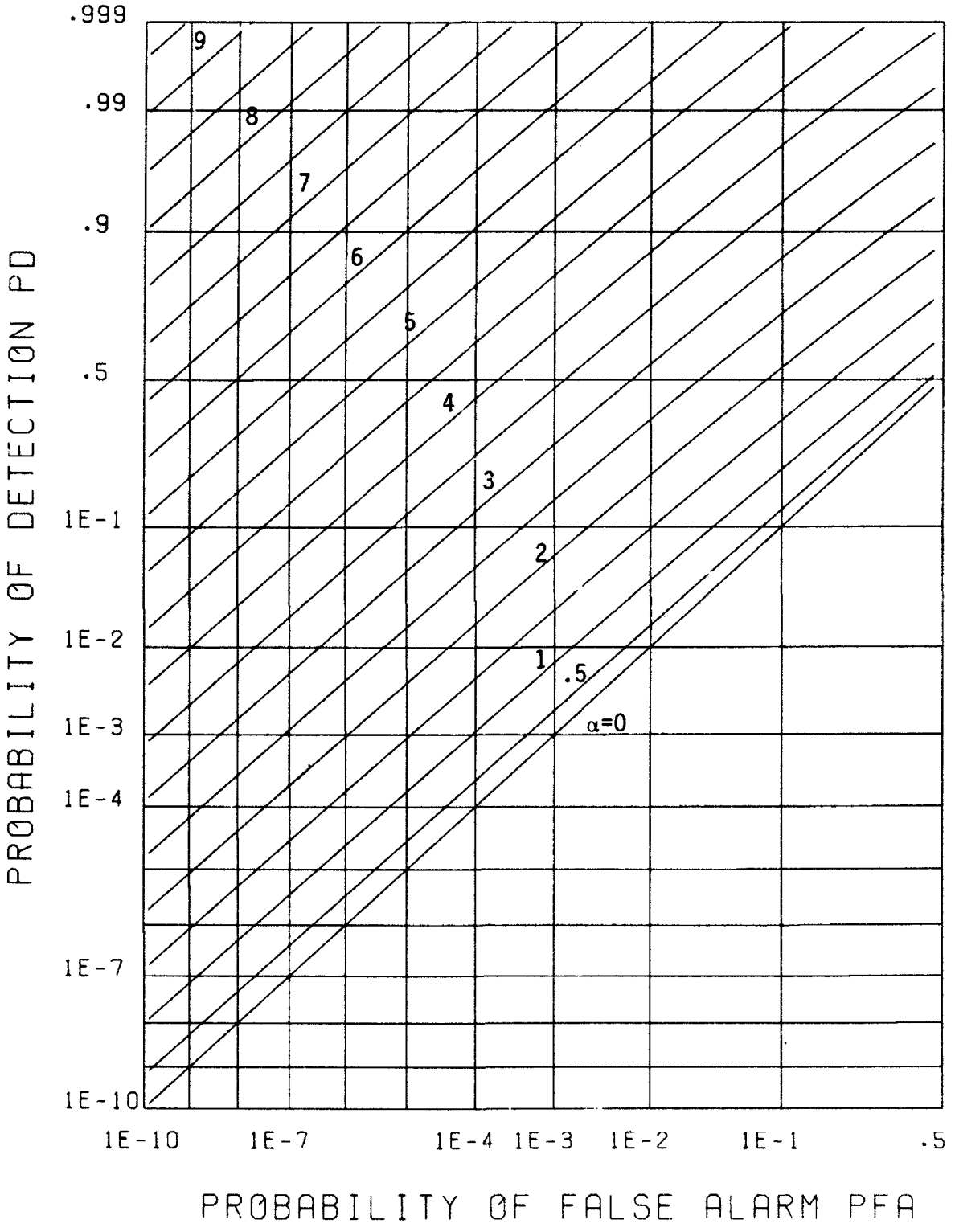

Figure 6. Receiver Operating Characteristics for M=1

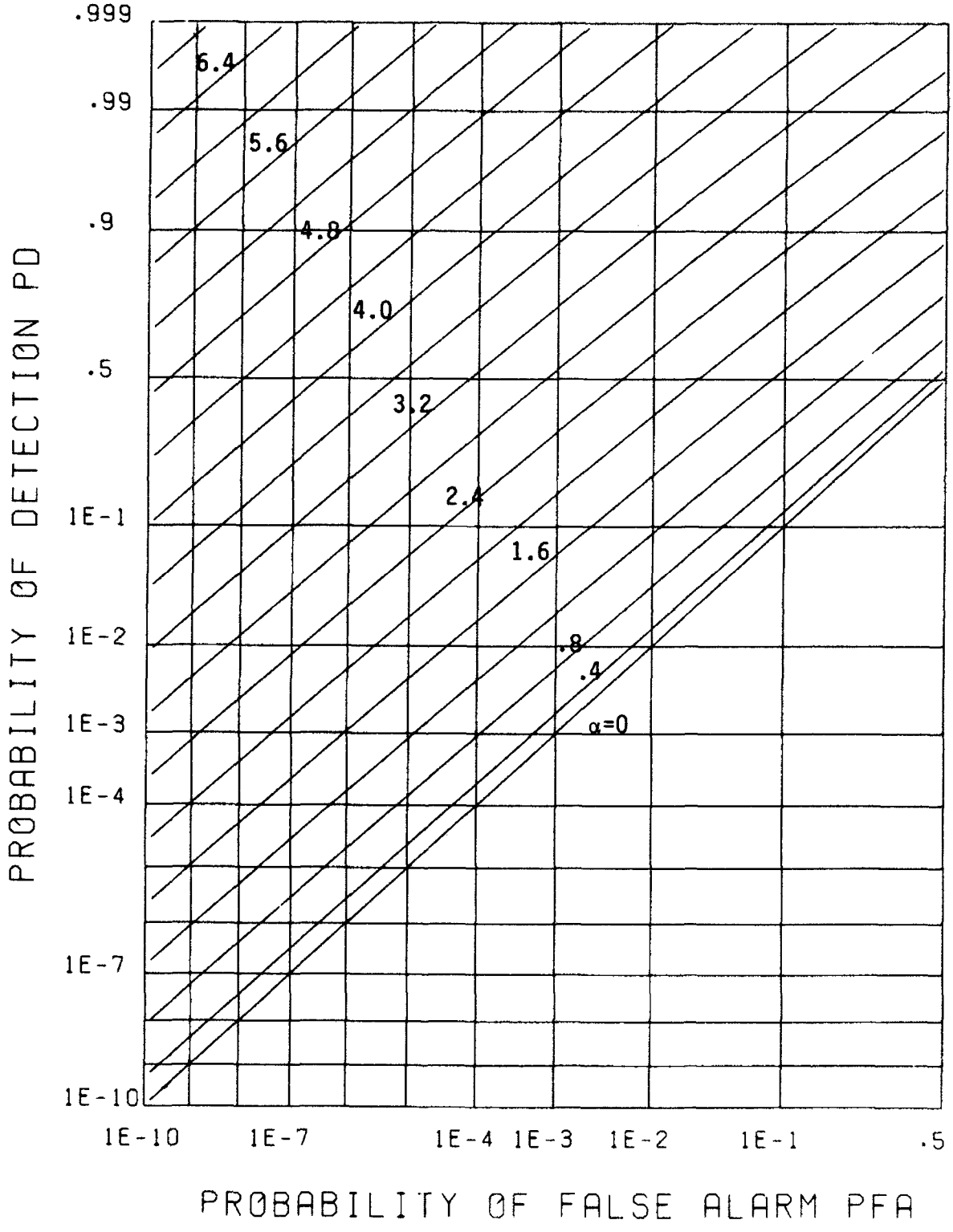

Figure 7. Receiver Operating Characteristics for M=2

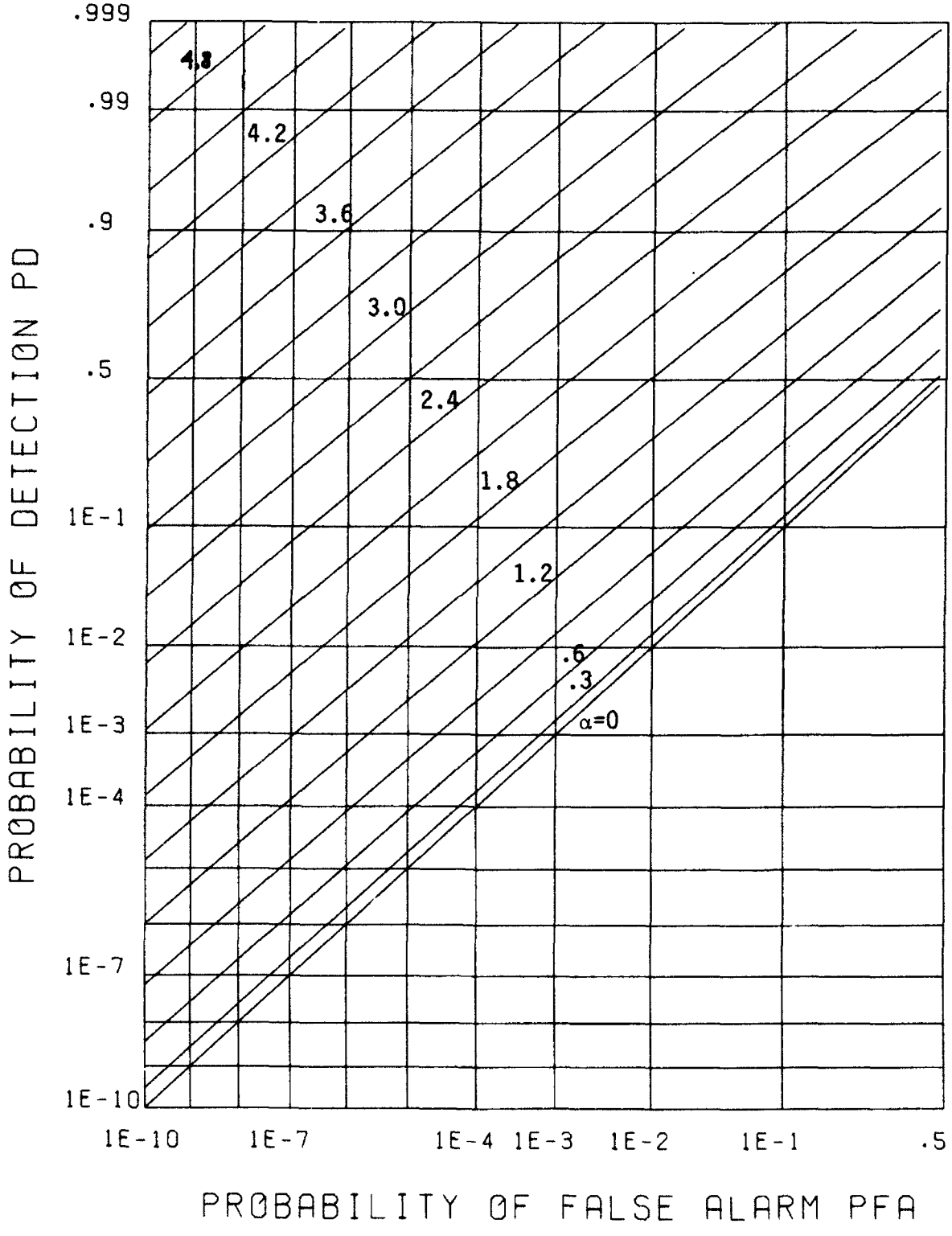

Figure 8. Receiver Operating Characteristics for M=4

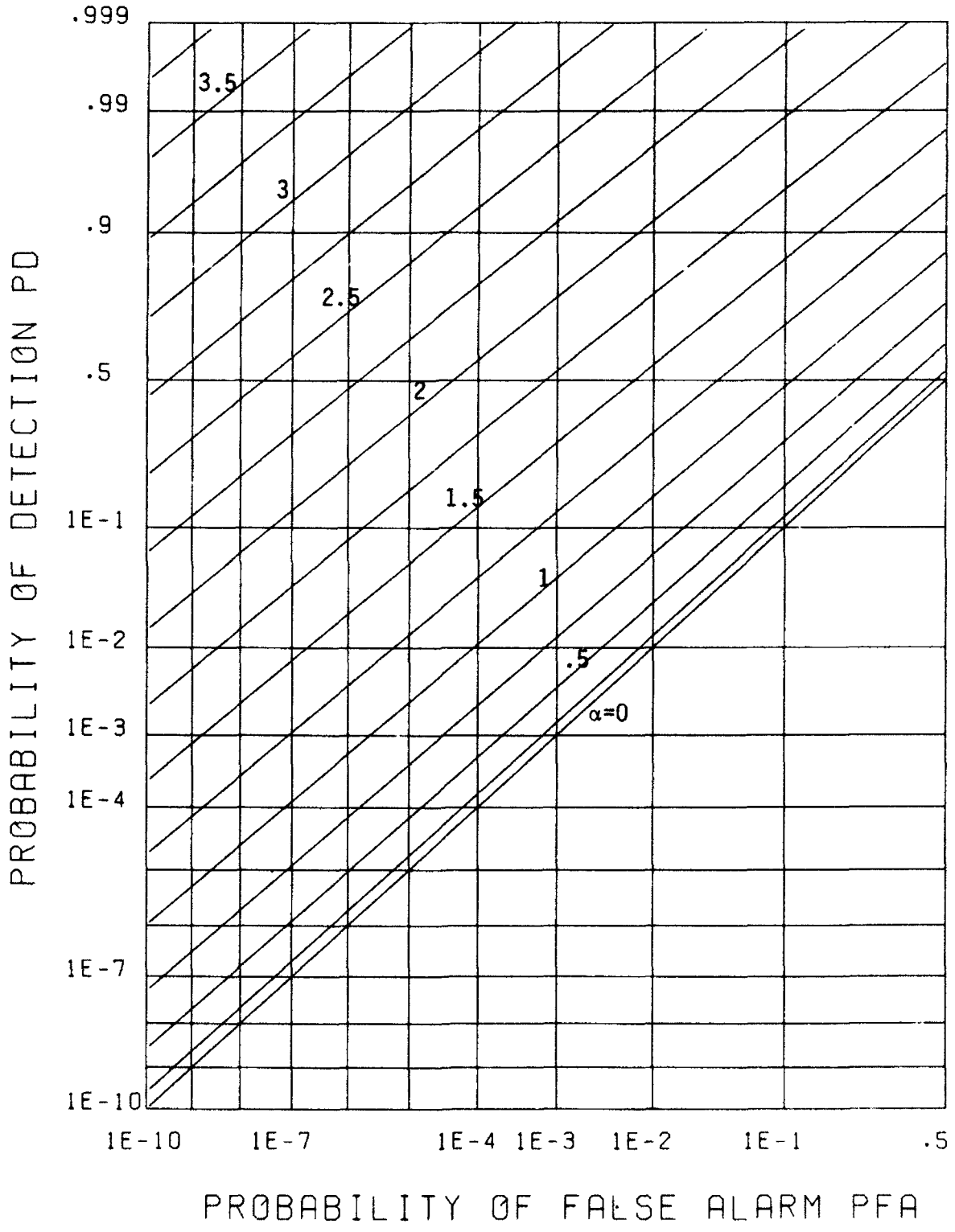

Figure 9. Receiver Operating Characteristics for M=8

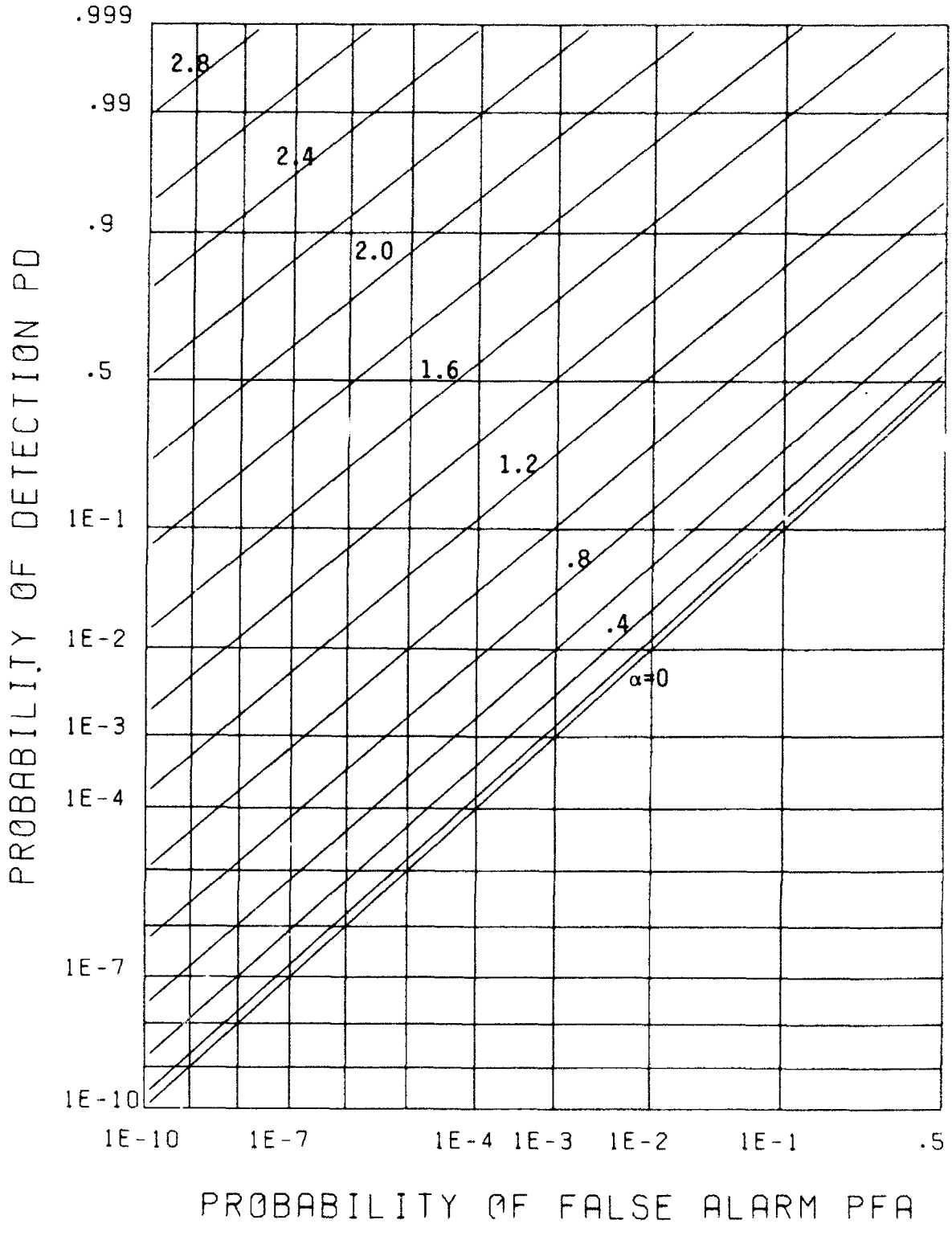

Figure 10. Receiver Operating Characteristics for M=16

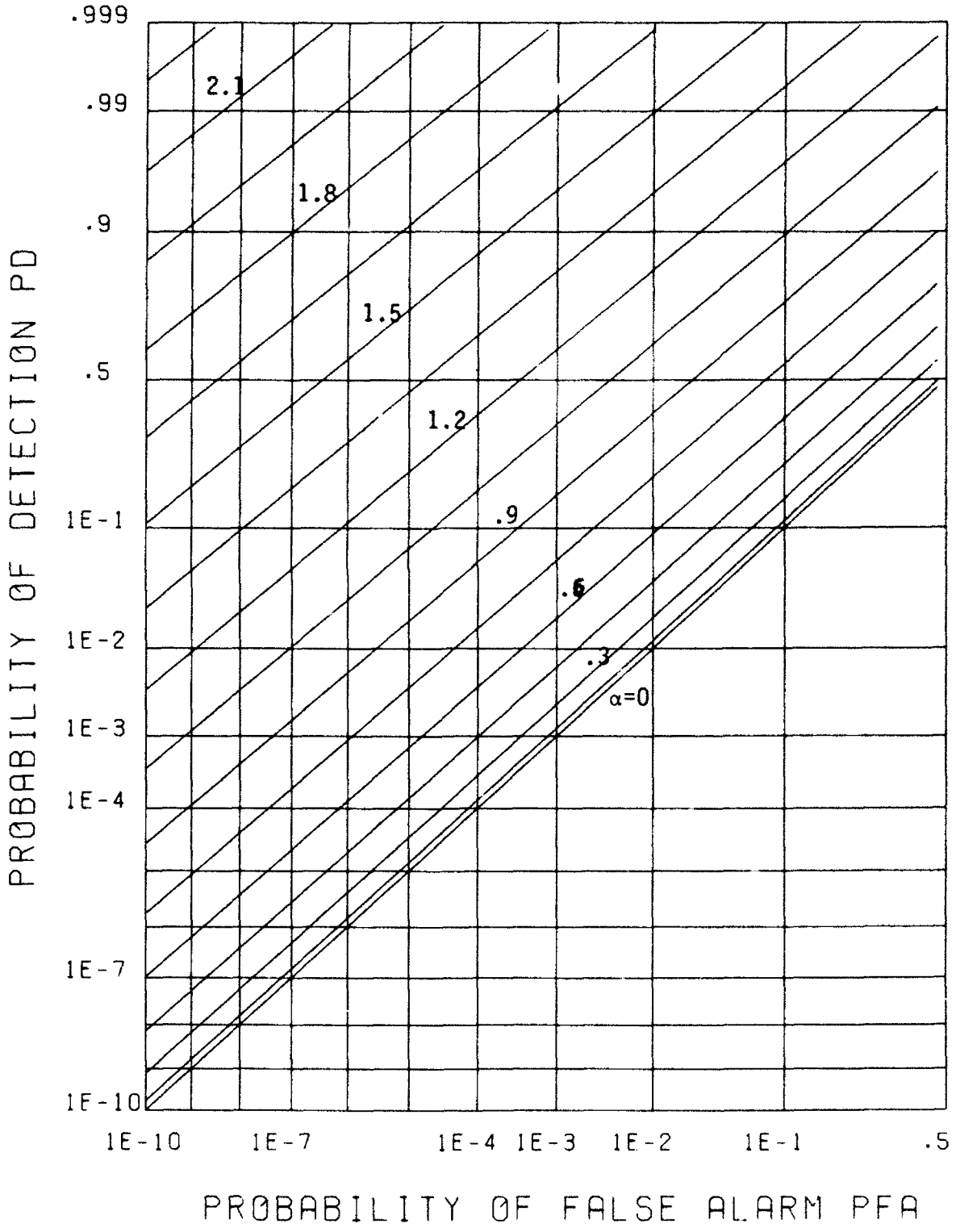

Figure 11. Receiver Operating Characteristics for M=32

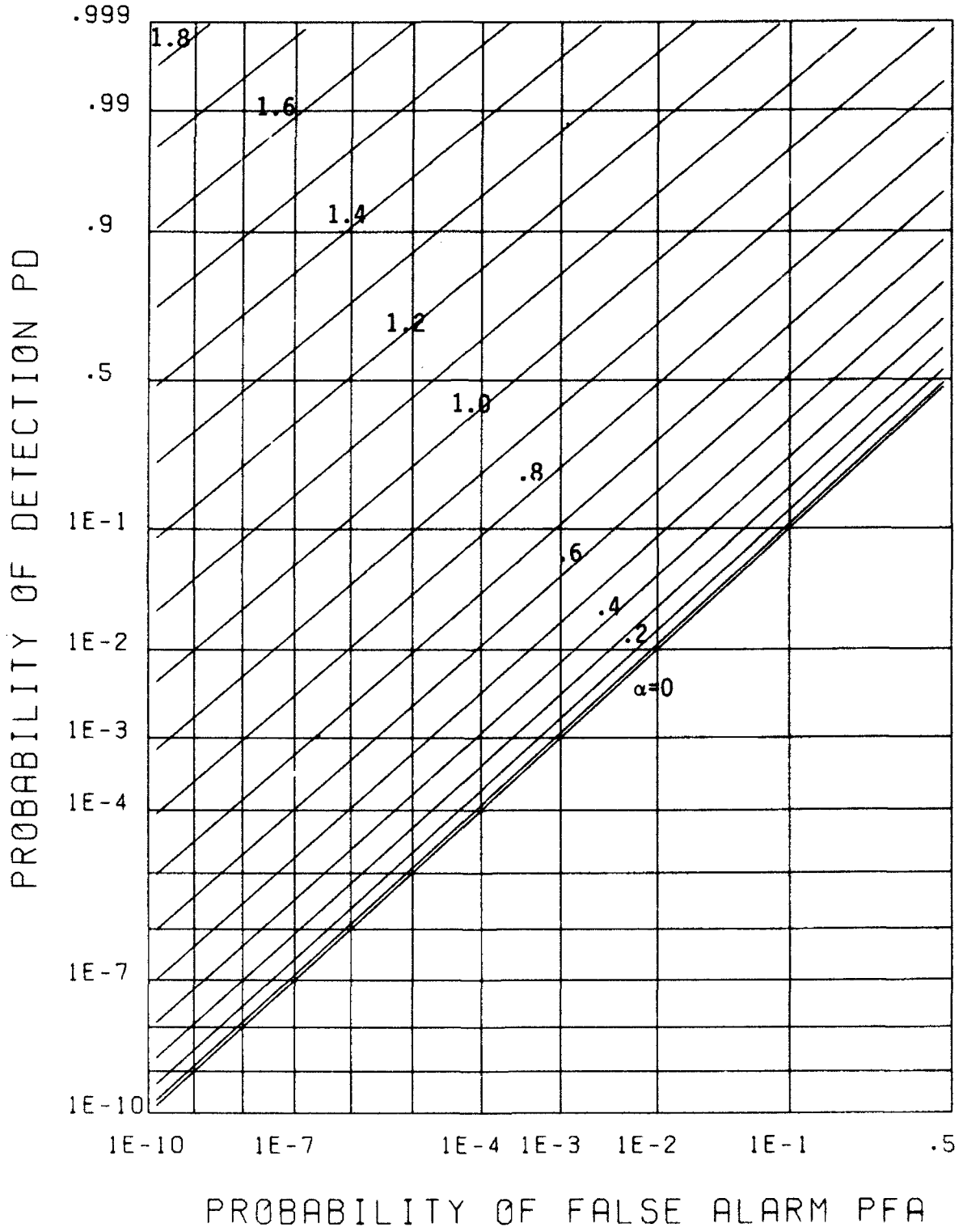

Figure 12. Receiver Operating Characteristics for M=64

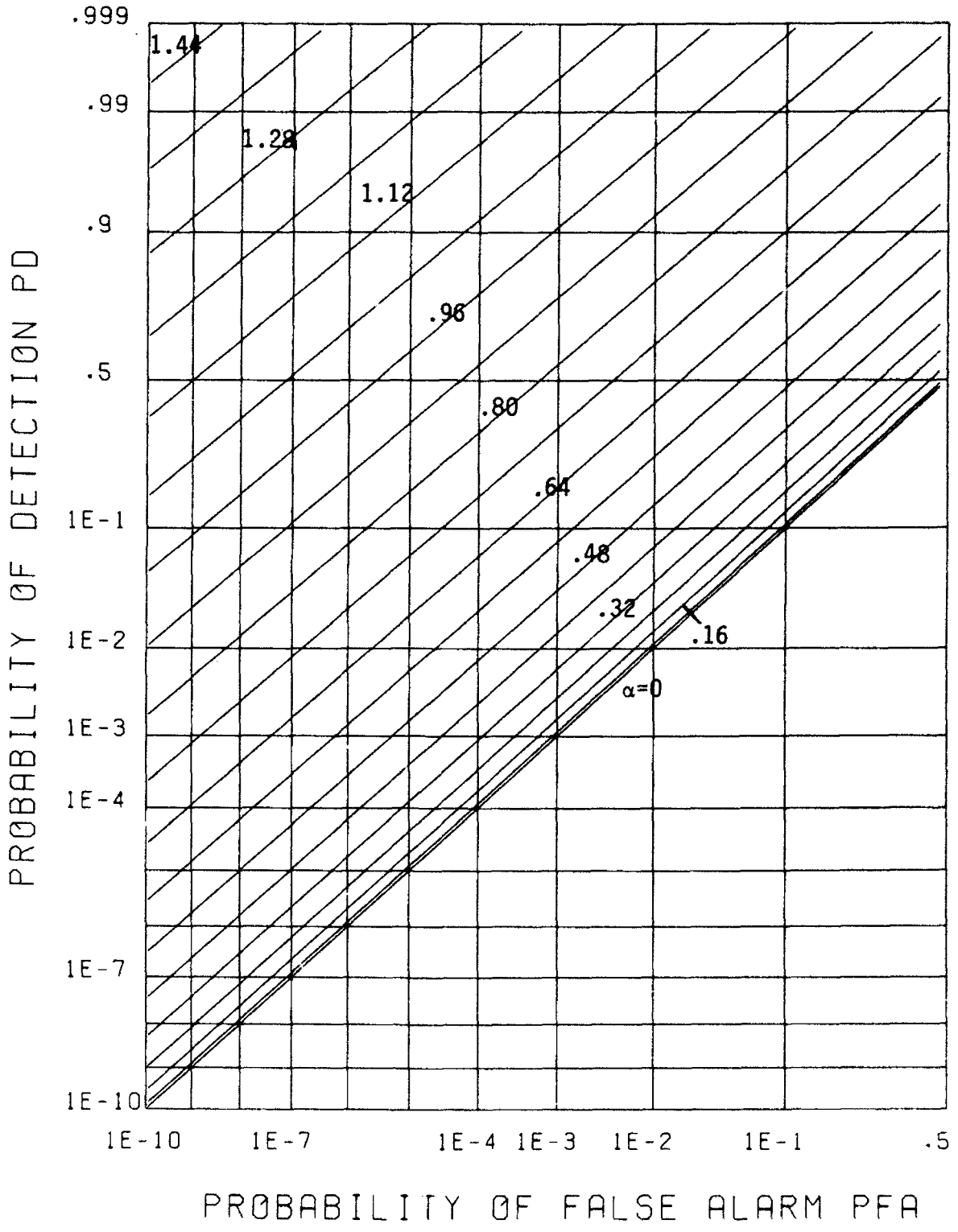

Figure 13. Receiver Operating Characteristics for M=128

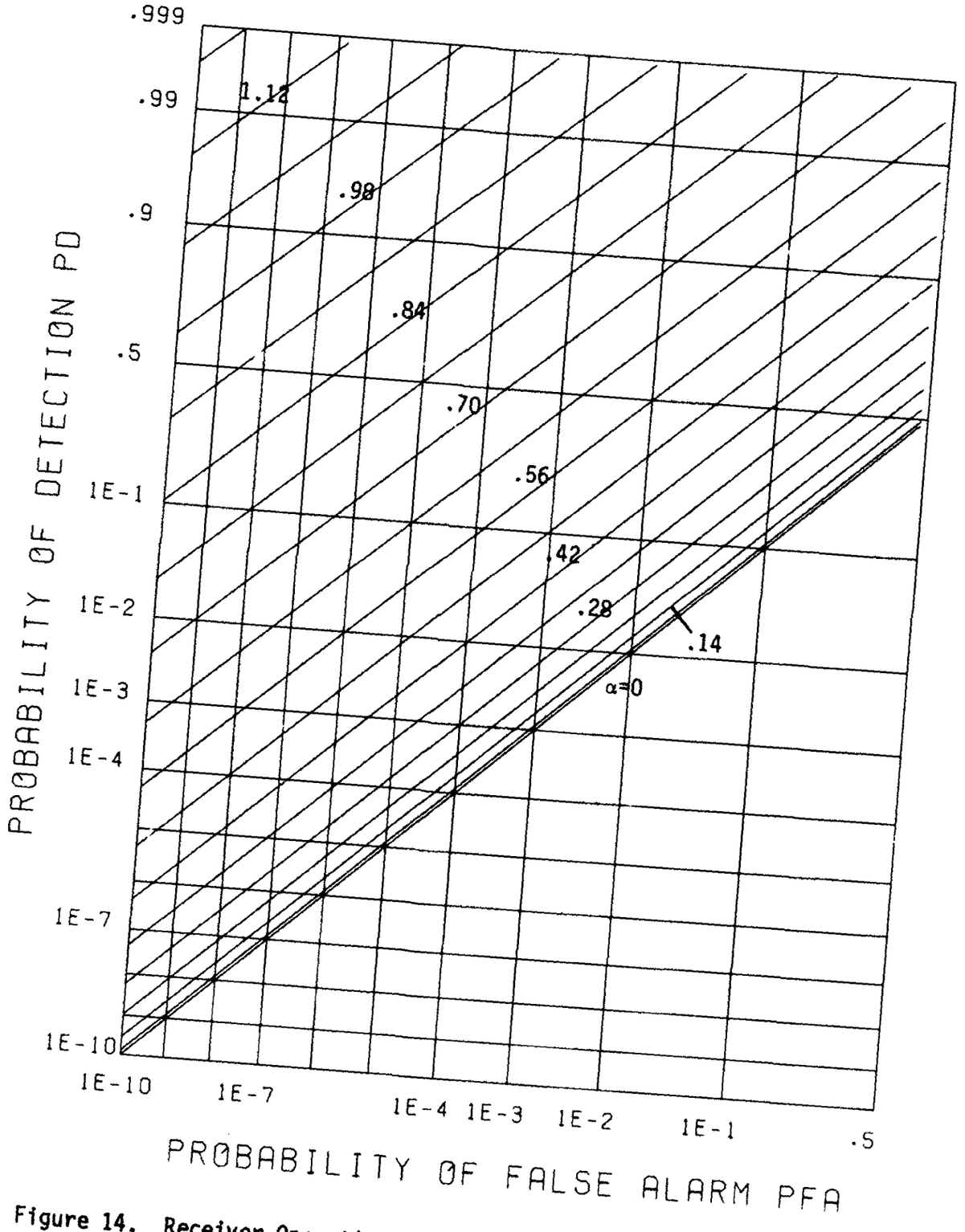

Figure 14. Receiver Operating Characteristics for M=256

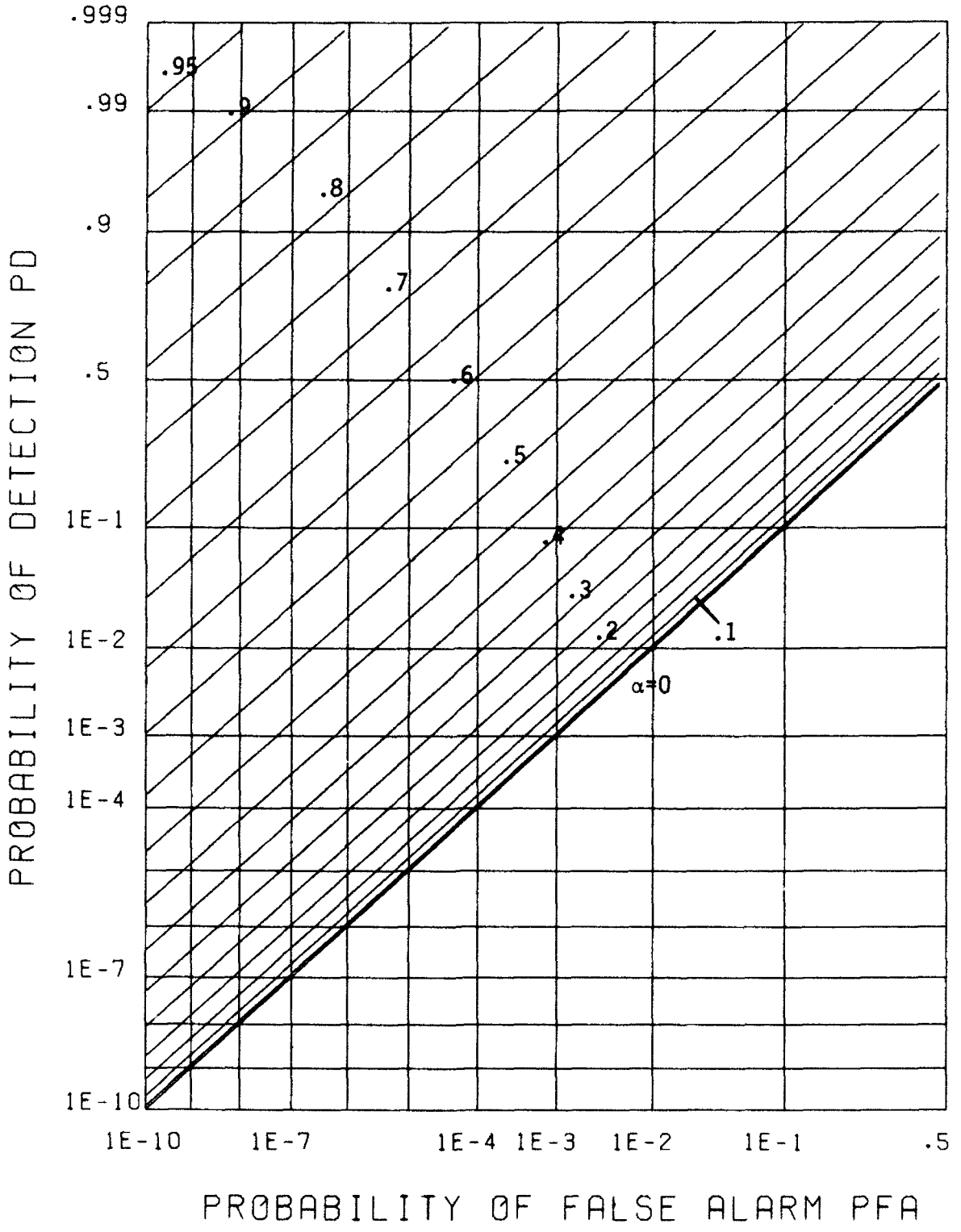

Figure 15. Receiver Operating Characteristics for M=512

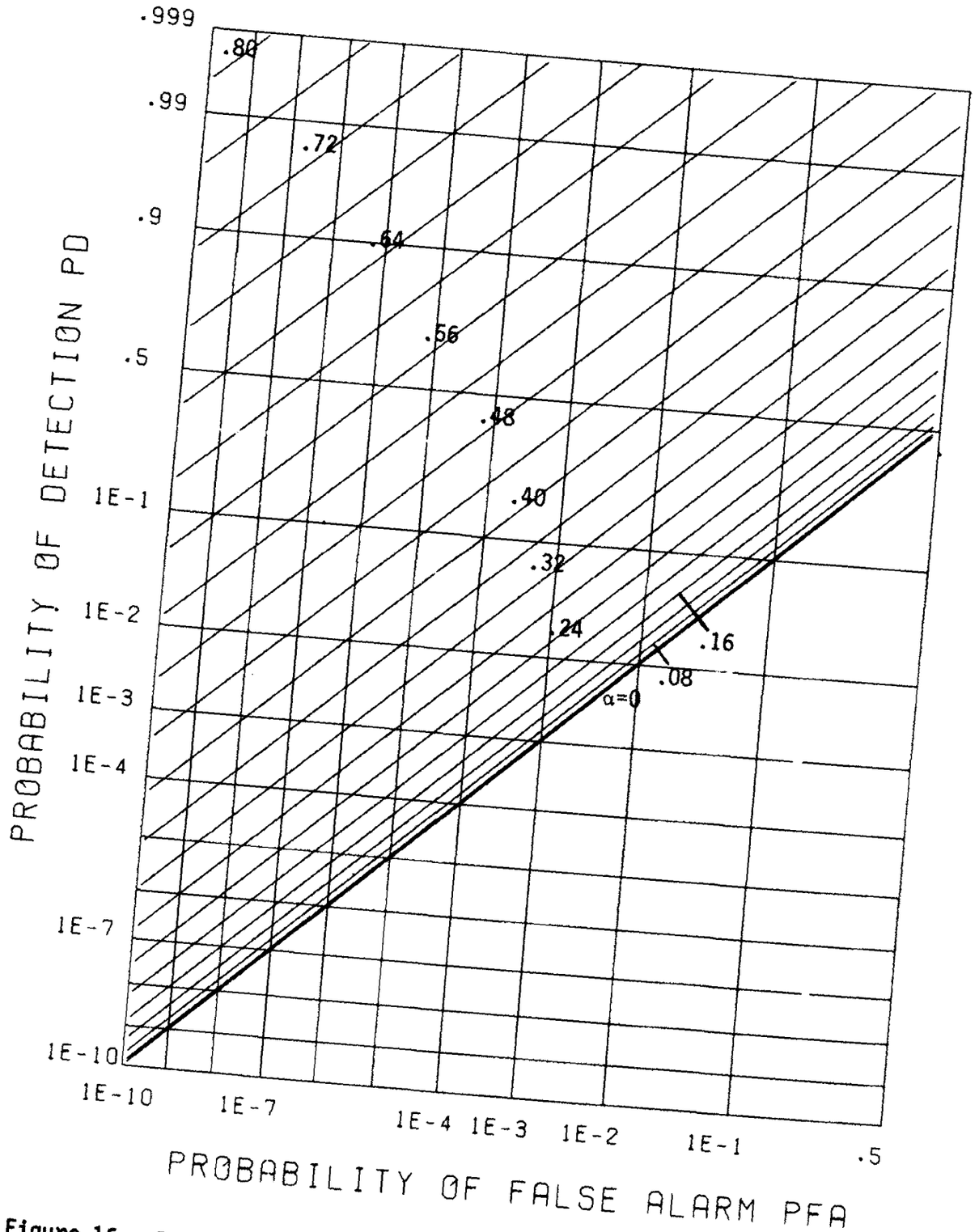

Figure 16. Receiver Operating Characteristics for M=1024

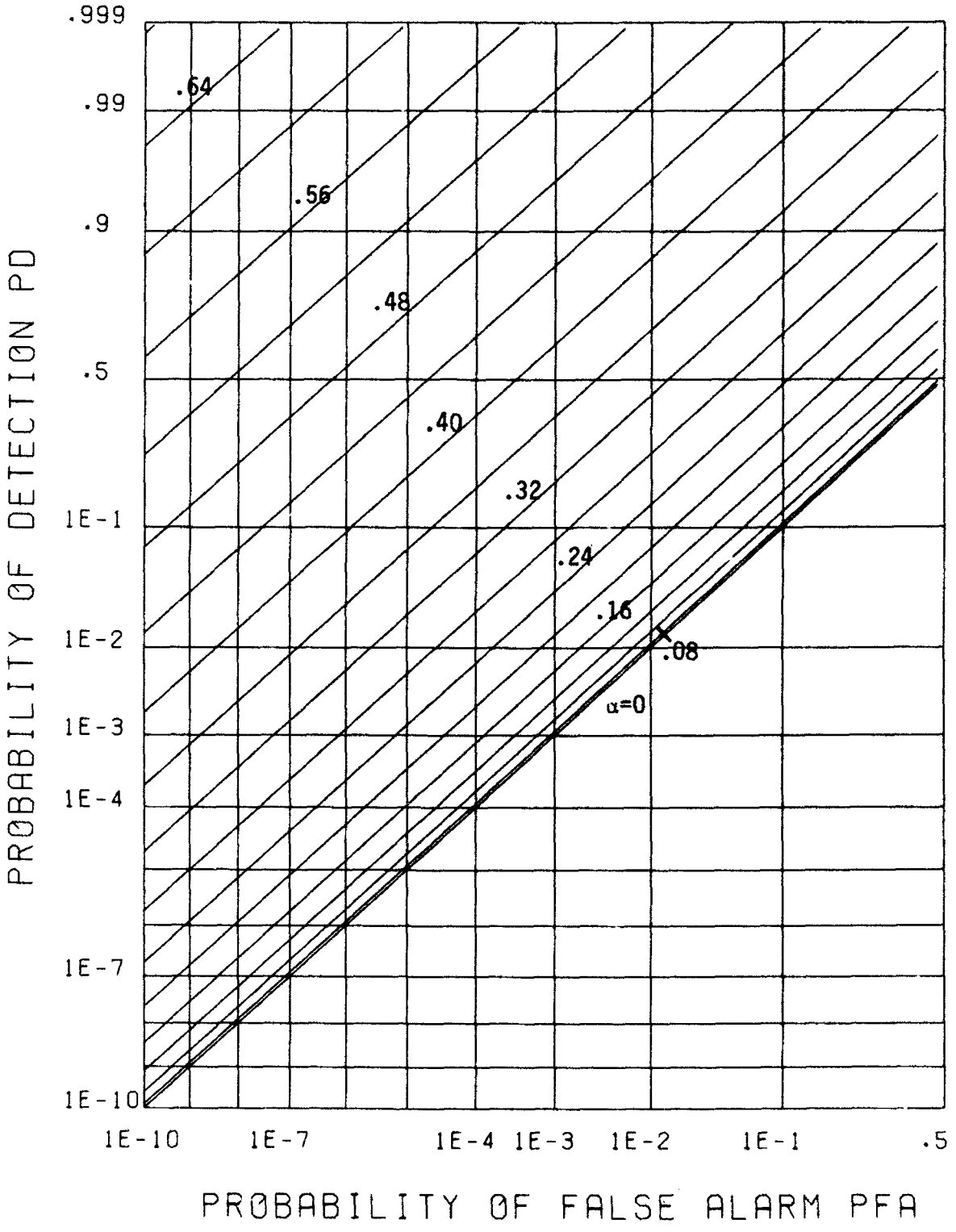

Figure 17. Receiver Operating Characteristics for M=2048

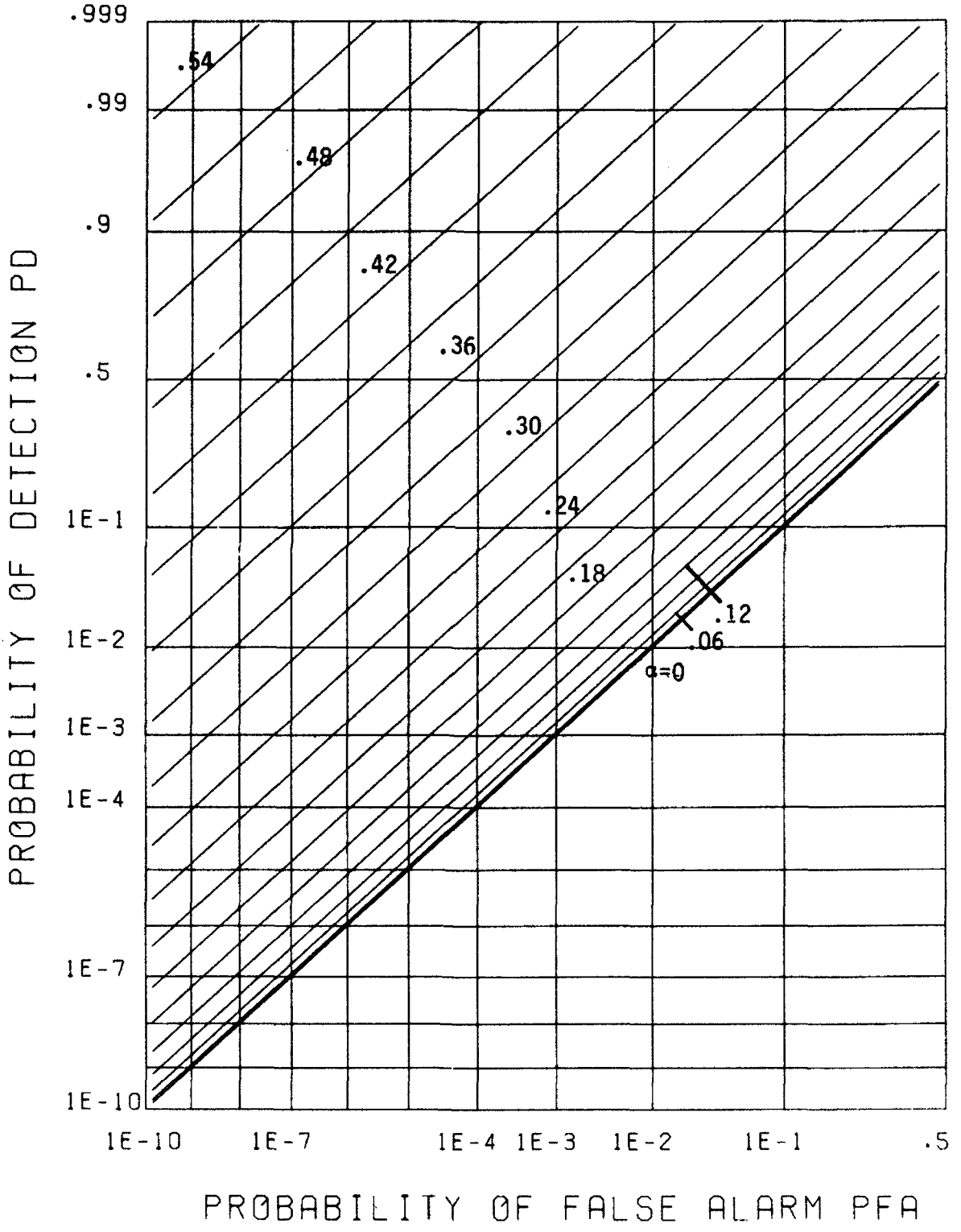

Figure 18. Receiver Operating Characteristics for M=4096

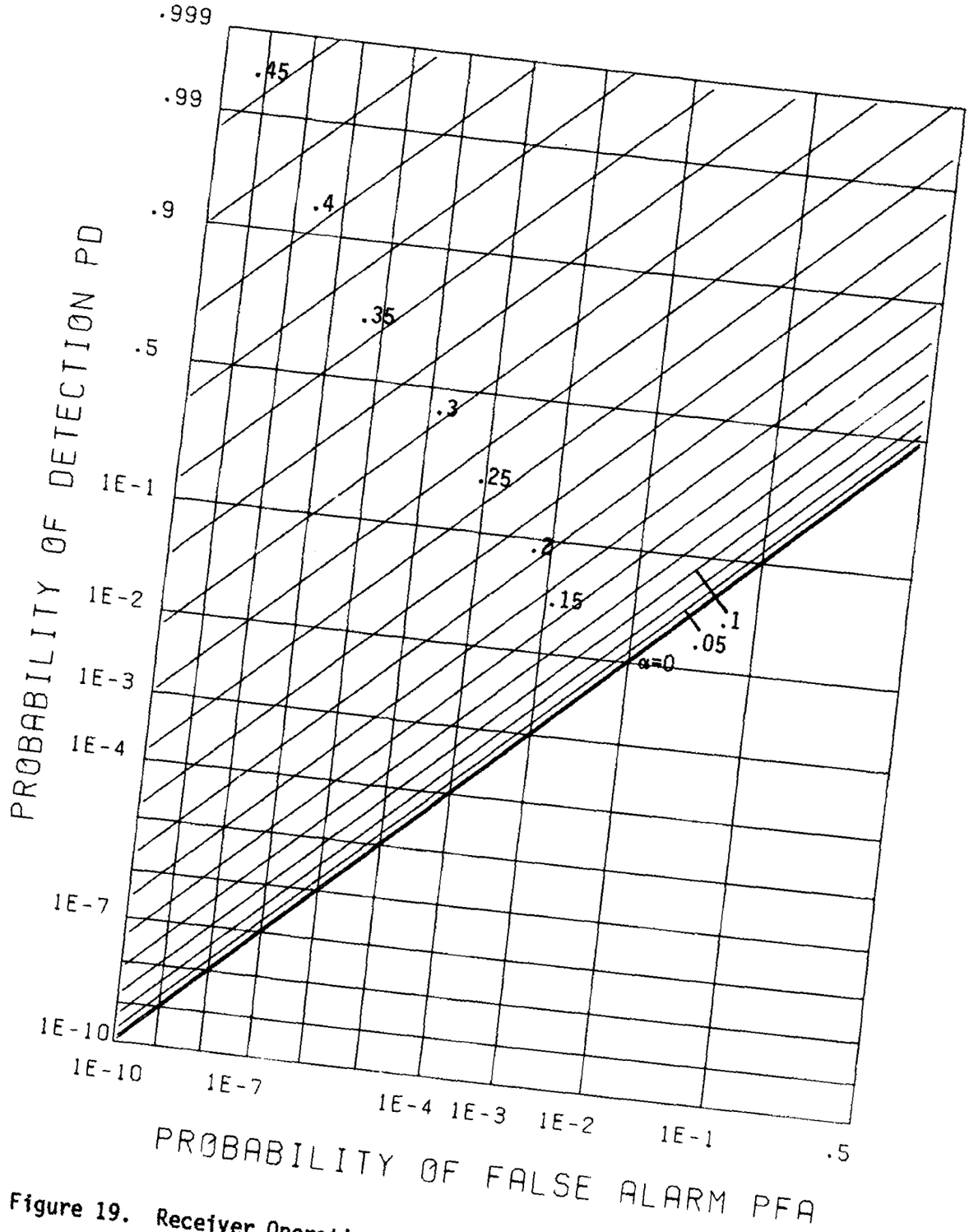

Figure 19. Receiver Operating Characteristics for M=8192

SUMMARY

A method for exact evaluation of the exceedance distribution function, of a linear sum of M envelopes of a narrowband Gaussian process and sinewave, has been utilized to determine the receiver operating characteristics for a wide range of values of M and signal-to-noise ratio. Also, the required input signal-to-noise ratio vs. M has been determined for a selected set of false alarm and detection probabilities. Programs are also supplied by which other values of the various parameters can be investigated by the user.

Agreement between the current results and those in [1,2] is very good over the range of common values plotted. For M larger than 8192, the approximation given in (27) and (28) is recommended, since the summation variable is then well represented by a Gaussian random variable.

APPENDIX A. DERIVATION OF RICE CHARACTERISTIC FUNCTION

The normalized probability density function of a Rice random variable was given in (2) as

$$p_e(u) = u \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad \text{for} \quad u \geq 0 \quad . \tag{A-1}$$

The corresponding characteristic function is

$$f_e(\xi) = \int_{-\infty}^{+\infty} du \exp(i\xi u)\, p_e(u) = \int_0^{+\infty} du\, u \exp\left(i\xi u - \frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) =$$

$$= \exp(-r) \sum_{n=0}^{\infty} \frac{(r/2)^n}{(n!)^2} \int_0^{\infty} du\, u^{2n+1} \exp(i\xi u - u^2/2) \quad , \tag{A-2}$$

where we have expanded I_0 in a power series [5, 8.447 1] and defined power signal-to-noise ratio

$$r = \alpha^2/2 \quad . \tag{A-3}$$

(If desired, a power series in ξ could be developed by expanding $\exp(i\xi u)$ in a power series instead of I_0.)

We define

$$C_n(\xi) = \frac{1}{2^n (n!)^2} \int_0^{\infty} du\, u^{2n+1} \exp(i\xi u - u^2/2) \quad \text{for } n \geq 0 \quad , \tag{A-4}$$

and get the characteristic function series

$$f_e(\xi) = \exp(-r) \sum_{n=0}^{\infty} r^n\, C_n(\xi) \quad . \tag{A-5}$$

In order to get a recurrence on $C_n(\xi)$, we also define

$$B_k(\xi) = \int_0^{\infty} dw\, w^k \exp(i\xi w - w^2/2) \quad \text{for } k \geq 0 \quad , \tag{A-6}$$

for then

$$C_n(\xi) = \frac{B_{2n+1}(\xi)}{2^n(n!)^2} \quad . \tag{A-7}$$

By integrating by parts on (A-6), there follows

$$B_k = i\xi B_{k-1} + (k-1)B_{k-2} \quad \text{for } k \geq 1 \quad . \tag{A-8}$$

This recurrence can be started with [5, 3.896 3&4]

$$B_o = \exp(-\xi^2/2)\left[\sqrt{\frac{\pi}{2}} + i\xi\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{\xi^2}{2}\right)\right] \quad . \tag{A-9}$$

By looking at three adjacent terms of recurrence (A-8), we can generate the alternative recurrence

$$B_k = (2k-3-\xi^2)B_{k-2} - (k-2)(k-3)B_{k-4} \quad . \tag{A-10}$$

By means of (A-7), this translates into

$$C_n = \frac{1}{n^2}\left[\left(2n - \frac{1+\xi^2}{2}\right)C_{n-1} - \frac{n-\frac{1}{2}}{n-1}C_{n-2}\right] \quad \text{for } n \geq 2 \quad . \tag{A-11}$$

Starting values are (via manipulation of hypergeometric series and Kummer's transformation) expressable as

$$C_0 = \exp(-\xi^2/2)\left[{}_1F_1\left(-\frac{1}{2}; \frac{1}{2}; \frac{\xi^2}{2}\right) + i\sqrt{\frac{\pi}{2}}\,\xi\right],$$

$$C_1 = \exp(-\xi^2/2)\left[{}_1F_1\left(-\frac{3}{2}; \frac{1}{2}; \frac{\xi^2}{2}\right) + i\sqrt{\frac{\pi}{2}}\,(3-\xi^2)\frac{\xi}{2}\right]. \tag{A-12}$$

Each of the series for ${}_1F_1$ consists of terms of the same polarity, except for one term, and are therefore useful for obtaining very accurate initial values. C_0 is the characteristic function of the Rayleigh probability

density function. Relations (A-11)-(A-12) constitute recurrences on both the real and imaginary parts of C_n.

It was found that the terms $\exp(-r)\ r^n$ in (A-5) became very large for large n, while the $C_n(\xi)$ terms became very small. In order to avoid overflow and underflow, we defined the total term

$$A_n = \exp(-r)\ r^n\ C_n(\xi) \quad . \tag{A-13}$$

Reference to (A-11) readily yields the recurrence on A_n, and (A-12) furnishes corresponding obvious starting values for A_0 and A_1.

APPENDIX B. DESCRIPTION OF PROGRAMS AND LISTINGS

Overview

Information obtained via evaluation of the Rice characteristic function may be displayed in three formats.

FORMAT 1: Display PD vs. PFA

The user defines the number of samples M and the range of values for alpha, a voltage signal-to-noise ratio measure. An algorithm then utilizes the Rice characteristic function for alpha=0 and for the alphas specified by the user. This results in the production of a threshold vs. PFA and M (alpha=0) and threshold vs. PD and M (alpha>0) tables. These two tables are stored on an output file. For each user-defined M, a plot routine displays PD vs. PFA for the set of user-defined alphas.

FORMAT 2: Display SNR vs M

The user supplies the input which specifies a PD. The algorithm then solves for the threshold values corresponding to PFA=10**(-IPFA), (IPFA=1,..,8) and M=2**IM, (IM=0,...,13) and alpha=0. A root finding technique is then employed to solve for the SNR defined by a threshold value and user-defined PD. An SNR is found for each threshold value. The results are stored in an output file. A plot routine displays the required SNR vs. M for PFA=10**(-IPFA), (IPFA=1,2,...,8).

FORMAT 3: Print SNR

The user specifies a value for PD, PFA, M. The program solves for the threshold corresponding to PFA and M. A root finding technique is then employed to determine the SNR corresponding to this threshold and user-defined PD and M. The results are printed.

Description of Input

Inputs to the program consist of cards which either specify values (PARAMETER CARDS), activate the reading of tabularized values (TABLES), assign files (FILE NAME CARDS), process data (COMMAND CARDS), or specify a plot device (PLOT DEVICE CARDS). The basic format of a card is

CARD NAME = value units

where CARD NAME is an alphanumeric expression from Tables 2-6. The alphanumeric must begin in column 1, value is a floating point or integer number, and units is an alphanumeric.

Parameter cards, file names, and tables constitute the data upon which commands operate. If two cards with the same name specify different data, then the last entry overrides the other.

For the programmers convenience, FORTRAN variable names associated with file names or parameters may be located in the Tables 2 through 6. Since input and values stored represent the same physical quantity, it is convenient to refer to both in this paper by the same variable name. The convention adopted is to express the variable by the lower case letters and reserve upper case letters for constants.

Parameter Cards

Parameter cards are used to specify an axis length or assign a range of values to a parameter. These cards are shown in Table 2. For example,

```
NUMBER OF SAMPLES MINIMUM = 1.
NUMBER OF SAMPLES MAXIMUM = 8192.
NUMBER OF SAMPLES FACTOR  = 2.
```

implies that the program will process data for M=1,2,4,8,16,....,4096,8192.

Table Cards

A table card contains the values that are to be assigned to a variable. The last card that must appear in a table is an EOF card. This card terminates the reading of the table. Table cards exist for PD and PFA only. A list of the table cards appears in Table 4. For example,

```
PROBABILITY OF DETECTION TABLE
.5
.7
.99
EOF
```

This table assigns values of .5, .7, .99 to PD.

Files Cards

A file card allows for dynamic assignment of all mass storage files. This is accomplished by linking internal FORTRAN unit numbers to files during execution. The file card is shown in Table 4. Two of the three algorithms use files. They are

```
Display PD vs PFA : A file is used to store output.
Display SNR vs M  : A file is used to store output.
```

For example,

```
OUTPUT FILE = PDFILE
```

directs the output of a program to a file called PDFILE.

Command Cards

Command cards are used to compute, plot, or terminate a run stream. Command cards are given in Table 5.

Plot Device Cards

Plot device cards direct the plot output to either a TEKTRONIX, FR80, or a CALCOMP plotter. The cards necessary for that operation are shown in Table 6.

Examples of Output

Example 1: Display PD vs PFA

The input deck for the first example appears in Table 7. This deck designates that PD vs. PFA data will be computed for M=1 and alpha=.5,1.0,1.5,...,9.5. The output is stored on a file called FILE1. The plot corresponding to the data is shown in figure 6. The second half of the run stream computes PD vs. PFA data for M=2 and alpha=0.,.4,.8,...,7.2. The output is stored in FILE2. The plot of the data appears in figure 7.

Example 2: Display SNR vs. M

The input deck for the second example appears in Table 8. The first half of the input deck designates that the SNR vs. M plots will be computed for a value PD=.5. The output is displayed in figure 1. The parameter cards specify that the axis will be scaled as follows: -19 DB (minimum), 13 DB (maximum), 2 DB (increment), and 5 inches long for the SNR axis and 6.86 inches long for the number of samples axis. It should be noted that the limits for the number of samples axis are predefined by the program to be 1 (minimum), 8192 (maximum), 2 (factor). The output is stored in a file called PDFIL1. The second half of the run stream computes SNR vs. M for a value PD=.9. The axis limits for SNR were changed to -17 DB (minimum), 15 DB (maximum), 2 DB (increment). Alpha curves were computed for alpha=0.,.4,.8,...,7.2. This output is stored in file PDFIL2. A plot of this data appears in figure 2.

Example 3: Print SNR

The input deck for the third example appears in Table 9. The output appears in Table 10.

TABLE 2. PARAMETER CARDS

INPUT CARDS	UNITS
SNR AXIS LENGTH = snraxs	IN
SAMPLE AXIS LENGTH = smpaxs	IN
PD AXIS LENGTH = pdaxs	IN
PFA AXIS LENGTH = pfaaxs	IN
SNR MINIMUM = snrmin	DB
SNR MAXIMUM = snrmax	DB
SNR INCREMENT = snrinc	DB
ALPHA MINIMUM = alpmin	
ALPHA MAXIMUM = alpmax	
ALPHA INCREMENT = alpinc	
NUMBER OF SAMPLES MINIMUM = smpmin	
NUMBER OF SAMPLES MAXIMUM = smpmax	
NUMBER OF SAMPLES FACTOR = smpfct	

TABLE 3. TABLE CARDS

INPUT CARDS	VARIABLE
PROBABILITY OF DETECTION TABLE	PD
PROBABILITY OF FALSE ALARM TABLE	PFA

TABLE 4. FILE CARDS

INPUT CARDS
OUTPUT FILE = name

TABLE 5. COMMAND CARDS

INPUT CARDS
RUN MAIN
COMPUTE PD VS PFA
COMPUTE SNR VS M
PLOT PD VS PFA
PLOT SNR VS M
END

TABLE 6. PLOT DEVICE CARDS

INPUT CARDS	OPTIONS
BAUD RATE = 960.	
PLOT DEVICE = device	FR80,TEKTRO,CALCOMP
RESET PLOT DEVICE	

TABLE 7. SAMPLE INPUT DECK FOR PD VS PFA

```
RUN MAIN
BAUD RATE = 960.
PLOT DEVICE = TEKTRO
RESET PLOT DEVICE
PD AXIS LENGTH = 6.86 IN
PFA AXIS LENGTH = 5. IN
OUTPUT FILE = FILE1
NUMBER OF SAMPLES MINIMUM = 1
ALPHA MINIMUM = .5
ALPHA MAXIMUM = 9.5
ALPHA INCREMENT = .5
COMPUTE PD VS PFA
PLOT PD VS PFA
OUTPUT FILE = FILE2
NUMBER OF SAMPLES MINIMUM = 2
ALPHA MINIMUM = 0.
ALPHA MAXIMUM = 7.2
ALPHA INCREMENT = .4
COMPUTE PD VS PFA
PLOT PD VS PFA
END
```

TABLE 8. SAMPLE INPUT DECK FOR SNR vs M

```
RUN MAIN
BAUD RATE = 960.
TEMPORARY FILE = FALSE
PLOT DEVICE = TEKTRO
RESET PLOT DEVICE
OUTPUT FILE = PDFIL1
SNR MINIMUM = -19. DB
SNR MAXIMUM = 13. DB
SNR INCREMENT = 2. DB
SNR AXIS LENGTH = 5. IN
SAMPLE AXIS LENGTH = 6.86 IN
PROBABILITY OF DETECTION TABLE
.5
EOF
COMPUTE SNR VS M
PLOT SNR VS M
OUTPUT FILE = PDFIL2
SNR MINIMUM = -17. DB
SNR MAXIMUM = 15. DB
SNR INCREMENT = 2. DB
PROBABILITY OF DETECTION TABLE
.9
EOF
COMPUTE SNR VS M
PLOT SNR VS M
END
```

TABLE 9. SAMPLE INPUT DECK FOR PRINTING SNR

```
RUN MAIN
PROBABILITY OF DETECTION TABLE
.5
.9
EOF
PROBABILITY OF FALSE ALARM TABLE
.1
.001
EOF
NUMBER OF SAMPLES MINIMUM = 1.
NUMBER OF SAMPLES MAXIMUM = 2048.
NUMBER OF SAMPLES FACTOR = 2.
PRINT SNR
END
```

TABLE 10. PRINT OUT OF SNR VS M

PD = 0.500	PFA = 0.100D+00	PD = 0.900	PFA = 0.100D+00
M = 1	SNR = 2.50	M = 1	SNR = 7.18
M = 2	SNR = 0.68	M = 2	SNR = 5.03
M = 4	SNR = -1.08	M = 4	SNR = 2.99
M = 8	SNR = -2.77	M = 8	SNR = 1.05
M = 16	SNR = -4.41	M = 16	SNR = -0.79
M = 32	SNR = -6.02	M = 32	SNR = -2.55
M = 64	SNR = -7.59	M = 64	SNR = -4.24
M = 128	SNR = -9.15	M = 128	SNR = -5.89
M = 256	SNR = -10.69	M = 256	SNR = -7.50
M = 512	SNR = -12.22	M = 512	SNR = -9.08
M = 1024	SNR = -13.74	M = 1024	SNR = -10.64
M = 2048	SNR = -15.26	M = 2048	SNR = -12.18

PD = 0.500	PFA = 0.100D-02	PD = 0.900	PFA = 0.100D-02
M = 1	SNR = 8.06	M = 1	SNR = 10.76
M = 2	SNR = 5.74	M = 2	SNR = 8.29
M = 4	SNR = 3.63	M = 4	SNR = 6.01
M = 8	SNR = 1.69	M = 8	SNR = 3.89
M = 16	SNR = -0.14	M = 16	SNR = 1.92
M = 32	SNR = -1.87	M = 32	SNR = 0.06
M = 64	SNR = -3.54	M = 64	SNR = -1.72
M = 128	SNR = -5.16	M = 128	SNR = -3.42
M = 256	SNR = -6.75	M = 256	SNR = -5.07
M = 512	SNR = -8.31	M = 512	SNR = -6.68
M = 1024	SNR = -9.86	M = 1024	SNR = -8.26
M = 2048	SNR = -11.39	M = 2048	SNR = -9.82

Listing of Program

This section contains a listing of three master programs and associated subroutines. Subroutines which read input and plot the output have been omitted. Table 11 contains a list of the subroutine names and a brief description of the pertinent subroutines.

TABLE 11. DESCRIPTION OF SUBROUTINES

NAME	DESCRIPTION
CMPDVA	MASTER PROGRAM FOR COMPUTING PD VS PFA
CMPSVS	MASTER PROGRAM FOR COMPUTING SNR VS M
PRTSNR	MASTER PROGRAM FOR COMPUTING AND PRINTING SNR
FFT	COMPUTES THE FAST FOURIER TRANSFORM OF A FUNCTION
RDC	COMPUTES AN APPROXIMATE S/N FOR A GIVEN PD, PFA, M (SEE REF 7)
FNPD	COMPUTES THE PROBABILITY OF DETECTION FOR A GIVEN M, S/N, AND THRESHOLD
FNPF	COMPUTES THE PROBABILITY OF FALSE ALARM FOR A GIVEN M AND THRESHOLD
FNF11	COMPUTES THE CONFLUENT HYPERGEOMETRIC FUNCTION
RICE	COMPUTES THE CHARACTERISTIC FUNCTION OF A RICE VARIATE
FNIPHI	COMPUTES THE INVERSE OF THE CUMULATIVE GAUSSIAN DISTRIBUTION
DIST	COMPUTES THE EXCEEDANCE DISTRIBUTION FUNCTION FOR A GIVEN M AND S/N

```
      SUBROUTINE FFT(N,X,Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION C(0:256),X(0:1023),Y(0:1023),L(0:9)
      DATA PI/3.14159265358979324D0/

      T=2.D0*PI/N
      J1=N/4
      DO 100 J=0,J1
      C(J)=DCOS(T*DFLOTJ(J))
100   CONTINUE

      N1=N/4
      N2=N1+1
      N3=N2+1
      N4=N3+N1
      L2=JIDINT(1.4427D0*DLOG(DFLOTJ(N))+.5D0)
      DO 600 I1=1,L2
      I2=2**(L2-I1)
      I3=2D0*I2
      I4=N/I3

      DO 500 I5=1,I2
      I6=I4*(I5-1)+1
      IF( I6.LE.N2 ) GO TO 350
      V6=-C(N4-I6-1)
      V7=-C(I6-N1-1)
      GO TO 375
350   V6=C(I6-1)
      V7=-C(N3-I6-1)
375   L3=N-I3

      DO 400 I7=0,L3,I3
      I8=I7+I5
      I9=I8+I2
      V8=X(I8-1)-X(I9-1)
      V9=Y(I8-1)-Y(I9-1)
      X(I8-1)=X(I8-1)+X(I9-1)
      Y(I8-1)=Y(I8-1)+Y(I9-1)
      X(I9-1)=V6*V8-V7*V9
      Y(I9-1)=V6*V9+V7*V8
400   CONTINUE
500   CONTINUE
600   CONTINUE


      I1=L2+1
      DO 700 I2=1,10
      L(I2-1)=1.D0
      IF( I2.GT.L2 ) GO TO 700
      L(I2-1)=2**(I1-I2)
700   CONTINUE
```

```
      IC0=L(0)
      IC1=L(1)
      IC2=L(2)
      IC3=L(3)
      IC4=L(4)
      IC5=L(5)
      IC6=L(6)
      IC7=L(7)
      IC8=L(8)
      IC9=L(9)

      K=1
      DO 1900 I1=1,IC9
      DO 1800 I2=I1,IC8,IC9
      DO 1700 I3=I2,IC7,IC8
      DO 1600 I4=I3,IC6,IC7
      DO 1500 I5=I4,IC5,IC6
      DO 1400 I6=I5,IC4,IC5
      DO 1300 I7=I6,IC3,IC4
      DO 1200 I8=I7,IC2,IC3
      DO 1100 I9=I8,IC1,IC2
      DO 1000 I10=I9,IC0,IC1
      J=I10
      IF( K.GT.J ) GO TO 900
      A=X(K-1)
      X(K-1)=X(J-1)
      X(J-1)=A
      A=Y(K-1)
      Y(K-1)=Y(J-1)
      Y(J-1)=A
  900 K=K+1
 1000 CONTINUE
 1100 CONTINUE
 1200 CONTINUE
 1300 CONTINUE
 1400 CONTINUE
 1500 CONTINUE
 1600 CONTINUE
 1700 CONTINUE
 1800 CONTINUE
 1900 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE FNPD(ALPHA,V,AM,AL,AD,ABS,PD)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/

      SNR=.5D0*ALPHA*ALPHA
      CALL FNF11(1.5D0,1.D0,SNR,F11)
      FAC=DSQRT(.5D0*PI)*EXP(-SNR)*F11
      AMUY=AM*FAC+ABS
      AM2=AM/2.D0
      VD=V*AD
      EXC=.5*AD*AMUY
      NS1=JIDINT(AL/AD)
      DO 100 NS=1,NS1
      XI=AD*NS
      CALL RICE(XI,SNR,FR,FI)
      A=DATAN2(FI,FR)
      FYI=DSIN(AM*A+ABS*XI)*(FR*FR+FI*FI)**AM2
      ADD=FYI*DCOS(VD*DFLOTJ(NS))/DFLOTJ(NS)
      EXC=EXC+ADD
100   CONTINUE
      PD=2.D0*EXC/PI

      RETURN
      END


      SUBROUTINE FNPF(V,AM,AL,AD,ABS,PF)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/

      FAC=DSQRT(.5D0*PI)
      AMUY=AM*FAC+ABS
      AM2=AM/2.D0
      VD=V*AD
      EXC=.5*AD*AMUY
      NS1=JIDINT(AL/AD)
      DO 100 NS=1,NS1
      XI=AD*NS
      X2=.5D0*XI*XI
      E=EXP(-X2)
      CALL FNF11(-.5D0,.5D0,X2,F11)
      FR=E*F11
      FI=E*FAC*XI
      A=DATAN2(FI,FR)
      FYI=DSIN(AM*A+ABS*XI)*(FR*FR+FI*FI)**AM2
      ADD=FYI*DCOS(VD*DFLOTJ(NS))/DFLOTJ(NS)
      EXC=EXC+ADD
100   CONTINUE
      PF=2.D0*EXC/PI

      RETURN
      END
```

```
      SUBROUTINE RDC(AM,PF,PD,ALPHA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

      A=DLOG(.62D0/PF)
      B=DLOG(PD/(1.D0-PD))
      FACT=6.2D0 + 4.54D0/DSQRT(AM+.44D0)
      SNRDB=-5.D0*DLOG10(AM) + DLOG10(A+.12D0*A*B+1.7D0*B)*FACT
      ALPHA=DSQRT(2.D0*10.D0**(.1D0*SNRDB))
      RETURN
      END


      SUBROUTINE FNF11(A,B,X,F11)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

      F11=1.D0
      T=1.D0
      DO 100 K=1,300
      U=K-1
      T=T*(A+U)*X/((B+U)*K)
      F11=F11+T
      IF( DABS(T).LE.DABS(F11)*1.D-18 ) GO TO 200
  100 CONTINUE
      PRINT 101
  101 FORMAT(2X,'300 TERMS IN FNF11')
  200 CONTINUE

      RETURN
      END


      SUBROUTINE FNIPHI(X,PHI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

  100 Y=DMAX1(X,1.D-12)
      Y=DMIN1(Y,1.D0-1.D-12)
      D=X-.5D0
      IF( DABS(D).GT. .01D0 ) GO TO 250
      PHI=2.506628274631D0*D*(1.D0+D*D*1.047197551200)
      GO TO 300
  250 PHI=Y
      IF( Y.GT. .5D0 ) PHI=.5D0-(Y-.5D0)
      PHI=DSQRT(-2.D0*DLOG(PHI))
      T=1.D0+PHI*(1.432788D0+PHI*(.189269D0+PHI*.001308D0))
      PHI=PHI-(2.515517D0+PHI*(.802853D0+PHI*.010328D0))/T
      IF( Y.LT. .5D0 ) PHI=-PHI
  300 RETURN
      END
```

```
      SUBROUTINE DIST(AM,ALPHA,MF,X,Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION X(0:1023), Y(0:1023)
      COMMON /PDVPF/AL,AD,ABS
      DATA PI/3.14159265358979324D0/


      SNR=.5D0*ALPHA*ALPHA
      CALL FNF11(1.5D0,1.D0,SNR,F11)
      AMU=DSQRT(.5D0*PI)*DEXP(-SNR)*F11
      AMUS=AM*AMU+ABS
      AM2=AM/2.D0

      DO 100 I=0,1023
      X(I)=0.D0
      Y(I)=0.D0
  100 CONTINUE

      X(0)=.5D0*AMUS*AD
      NS1=JIDINT(AL/AD)
      DO 1000 NS=1,NS1
      XI=AD*NS
      CALL RICE(XI,SNR,U,V)
      T=DATAN2(V,U)
      FI=DSIN(AM*T+ABS*XI)*(U*U+V*V)**AM2
      MS=JMOD(NS,MF)
      X(MS)=X(MS)+FI/NS
 1000 CONTINUE

      CALL FFT(MF,X,Y)

      FAC=2.D0/PI
      KS1=MF/2.D0
      DO 2000 KS=0,KS1
      T=X(KS)*FAC
      X(KS)=1.D0-T
      Y(KS)=T
 2000 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE RICE(X,SNR,FR,FI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/


      X2=.5D0*X*X
      E=DEXP(-X2-SNR)
      CALL FNF11(-.5D0,.5D0,X2,F11)
      AOR=E*F11
      AOI=E*DSQRT(.5D0*PI)*X
      CALL FNF11(-1.5D0,.5D0,X2,F11)
      ANR=E*SNR*F11
      ANI=SNR*(1.5D0-X2)*AOI
      FR=AOR+ANR
      FI=AOI+ANI
      BR=DMAX1(DABS(AOR),DABS(FR))
      BI=DMAX1(DABS(AOI),DABS(FI))
      T=.5D0+X2

      SNR2=SNR**2
      DO 100 N=2,200
      FO=N**2
      F1=SNR*(N+N-T)/FO
      F2=SNR2*(N-.5D0)/((N-1)*FO)
      R=F1*ANR-F2*AOR
      V=F1*ANI-F2*AOI
      AOR=ANR
      AOI=ANI
      ANR=R
      ANI=V
      FR=FR+R
      FI=FI+V
      BR=DMAX1(BR,DABS(FR))
      BI=DMAX1(BI,DABS(FI))
      IF( DABS(V).LE.5.D-19*DABS(FI) .AND. DABS(R).LE.5.D-19*DABS(FR))
     1 GO TO 200
  100 CONTINUE
       PRINT 101
  101 FORMAT(2X,'200 TERMS IN RICE')
  200 DR=18.-DLOG10(DABS(BR/FR))
      DI=18.-DLOG10(DABS(BI/FI))
      RETURN
      END
```

```
      SUBROUTINE CMPDVA
      PARAMETER MF=2**10
      PARAMETER PRDNUM=18
      DOUBLE PRECISION AL,AD,ABS,RSA,AM,ALPHA,ALFA,X(0:1023),Y(0:1023)
      PARAMETER  (NUMFIL=30)
      CHARACTER*6  FILES(NUMFIL)
      COMMON  /FILEC/FILES
      CHARACTER*6  PRDNAM
      EQUIVALENCE
     1  (PRDNAM,FILES(18))
      PARAMETER  (NUMPAR=200)
      COMMON  /PARAMC/PARAMS(NUMPAR)
      EQUIVALENCE
     1  (SMPMIN,PARAMS(187)),
     1  (SNMIN,PARAMS(184)), (SNMAX,PARAMS(185)), (SNDEL,PARAMS(186))
      COMMON/PDVPF/AL,AD,ABS
      DOUBLE PRECISION PI
      DATA PI/3.14159265358979324D0/

C
C     OPEN THE FILE
      CALL OPNFIL(PRDNUM,PRDNAM)
C
C     COMPUTE THE NUMBER OF SNR CURVES
C
      NSN=(SNMAX-SNMIN)/SNDEL + 1

C
C     STORE HEADER INFO
      WRITE(PRDNUM) SMPMIN,SNMIN,SNMAX,SNDEL,NSN

      AM = SMPMIN
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      RSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,RSA)
C
C     COMPUTE SNR VS PFA
      ALFA=0.D0
      CALL DIST(AM,ALFA,MF,X,Y)
C
C     STORE THE SNR VS PD
      WRITE(PRDNUM) (Y(I),I=0,512)

      DO 1000 ISN=1,NSN
      SNR= SNMIN + SNDEL*(ISN-1)

      ALPHA = SNR
      CALL DIST(AM,ALPHA,MF,X,Y)
C
C     STORE THE SNR VS PD
      WRITE(PRDNUM) (Y(I),I=0,512)

 1000 CONTINUE

 2000 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE PRTSNR
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION PFA(10),PD(10),V(14,8),SNR(14,8)
      DATA PI/3.14159265358979324D0/
      REAL SMPMIN,SMPMAX,SMPFCT,PARAMS
      PARAMETER NUMPAR=200
      COMMON/PARAMC/PARAMS(NUMPAR)
      EQUIVALENCE
     1  (SMPMIN,PARAMS(187)),(SMPMAX,PARAMS(188)), (SMPFCT,PARAMS
      COMMON/PDPF/NPD,NPFA,PD,PFA


      MMAX=ALOG10(SMPMAX/SMPMIN)/ALOG10(SMPFCT) + 1



      F1=DSQRT(.5D0*PI)
      F2=DSQRT(2.D0-.5D0*PI)
      DO 1000 IM=1,MMAX
      AM=SMPMIN*SMPFCT**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      AMU=F1*AM
      SIG=F2*DSQRT(AM)
      DO 900 IPF=1,NPFA
      PF=PFA(IPF)
      IF( AM.GT. 1.D0 ) GO TO 250
      VN=DSQRT(-2.*DLOG(PF))
      GO TO 750
  250 CALL FNIPHI(PF,YF)
      V1=AMU-SIG*YF+ABS
      IF( IPF.GT.1 ) V1=DMAX1(V1,VN)
      V2=V1+.5D0
      IF( V1.NE.VN ) GO TO 300
      P1=PN
      GO TO 325
  300 CALL FNPF(V1,AM,AL,AD,ABS,P1)
  325 CALL FNPF(V2,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PF).LT.DABS(P2-PF) ) GO TO 350
      V0=V1
      P0=P1
      VN=V2
      PN=P2
      GO TO 400
  350 V0=V2
      P0=P2
      VN=V1
      PN=P1
  400 CALL FNIPHI(P0,Y0)
      GO TO 550
  500 CALL FNPF(VN,AM,AL,AD,ABS,PN)
  550 CALL FNIPHI(PN,YN)
      IF( DABS(PN-PF).LE.1D-9*PF ) GO TO 750
      T=(V0*(YN-YF)+VN*(YF-Y0))/(YN-Y0)
```

```
      V0=VN
      Y0=YN
      VN=T
      GO TO 500
  750 V(IM,IPF)=VN
  900 CONTINUE
 1000 CONTINUE


      DO 4000 IPD=1,NPD


      CALL FNIPHI(PD(IPD),YD)
      DO 3000 IM=1,MMAX
      AM=SMPMIN*SMPFCT**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      DO 2900 IPF=1,NPFA
      PF=PFA(IPF)
      CALL RDC(AM,PF,PD(IPD),A1)
      A2=A1*1.01D0
      VV=V(IM,IPF)
      CALL FNPD(A1,VV,AM,AL,AD,ABS,P1)
      CALL FNPD(A2,VV,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PD(IPD)).LT.DABS(P2-PD(IPD)) ) GO TO 2350
      A0=A1
      P0=P1
      AN=A2
      PN=P2
      GO TO 2400
 2350 A0=A2
      P0=P2
      AN=A1
      PN=P1
 2400 CALL FNIPHI(P0,Y0)
      GO TO 2550
 2500 CALL FNPD(AN,VV,AM,AL,AD,ABS,PN)
 2550 CALL FNIPHI(PN,YN)
      IF( DABS(PN-PD(IPD)).LE.1D-6*PD(IPD) ) GO TO 2750
      T=(A0*(YN-YD)+AN*(YD-Y0))/(YN-Y0)
      A0=AN
      Y0=YN
      AN=T
      GO TO 2500
 2750 SNR(IM,IPF)=10.*DLOG10(.5D0*AN*AN)
 2900 CONTINUE
 3000 CONTINUE
```

```
      DO 3200 IPF=1,NPFA
      PRINT 3001
3001  FORMAT(2(/))
      PRINT 3011, PD(IPD),PFA(IPF)
3011  FORMAT(2X,'PD =',F10.3,5X,'PFA =',D10.3)
      DO 3100 IM=1,MMAX
      M=SMPMIN*SMPFCT**(IM-1)
      PRINT 3021, M,SNR(IM,IPF)
3021  FORMAT(2X,'M =',I5,5X,'SNR =',F7.2)
3100  CONTINUE
3200  CONTINUE

4000  CONTINUE


      RETURN
      END




      SUBROUTINE CMPSVS
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER MMAX=14
      PARAMETER NUMFIL=30, PRDNUM=18
      CHARACTER*6 FILES(NUMFIL)
      COMMON/FILEC/FILES
      CHARACTER*6 PRDNAM
      EQUIVALENCE (PRDNAM,FILES(18))
      DIMENSION PFA(10),PD(10),V(14,8),ALPHA(14,8)
      DIMENSION THRS(14,8)
      DATA PI/3.14159265358979324D0/
      COMMON/PDPF/NPD,NPFA,PD,PFA


      CALL OPNFIL(PRDNUM,PRDNAM)
```

```
      F1=DSQRT(.5D0*PI)
      F2=DSQRT(2.D0-.5D0*PI)
      DO 1000 IM=1,MMAX
      AM=2.**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      AMU=F1*AM
      SIG=F2*DSQRT(AM)
      DO 900 IPF=1,8
      PF=10.**(-DFLOTJ(IPF))
      IF( AM.GT. 1.D0 ) GO TO 250
      VN=DSQRT(-2.*DLOG(PF))
      GO TO 750
  250 CALL FNIPHI(PF,YF)
      V1=AMU-SIG*YF+ABS
      IF( IPF.GT.1 ) V1=DMAX1(V1,VN)
      V2=V1+.5D0
      IF( V1.NE.VN ) GO TO 300
      P1=PN
      GO TO 325
  300 CALL FNPF(V1,AM,AL,AD,ABS,P1)
  325 CALL FNPF(V2,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PF).LT.DABS(P2-PF) ) GO TO 350
      V0=V1
      P0=P1
      VN=V2
      PN=P2
      GO TO 400
  350 V0=V2
      P0=P2
      VN=V1
      PN=P1
  400 CALL FNIPHI(P0,Y0)
      GO TO 550
  500 CALL FNPF(VN,AM,AL,AD,ABS,PN)
  550 CALL FNIPHI(PN,YN)
      IF( DABS(PN-PF).LE.1D-9*PF ) GO TO 750
      T=(V0*(YN-YF)+VN*(YF-Y0))/(YN-Y0)
      V0=VN
      Y0=YN
      VN=T
      GO TO 500
  750 V(IM,IPF)=VN
      THRS(IM,IPF)=(VN-ABS)/AM
  900 CONTINUE
 1000 CONTINUE
```

```
      WRITE(PBDNUM)   NPD,(PD(I),I=1,10)
      DO 4000 IPD=1,NPD


      CALL FNIPHI(PD(IPD),YD)
      DO 3000 IM=1,MMAX
      AM=2.D0**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      DO 2900 IPF=1,8
      PF=10.D0**(-DFLOTJ(IPF))
      CALL RDC(AM,PF,PD(IPD),A1)
      A2=A1*1.01D0
      VV=V(IM,IPF)
      CALL FNPD(A1,VV,AM,AL,AD,ABS,P1)
      CALL FNPD(A2,VV,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PD(IPD)).LT.DABS(P2-PD(IPD)) ) GO TO 2350
      A0=A1
      P0=P1
      AN=A2
      PN=P2
      GO TO 2400
2350  A0=A2
      P0=P2
      AN=A1
      PN=P1
2400  CALL FNIPHI(P0,Y0)
      GO TO 2550
2500  CALL FNPD(AN,VV,AM,AL,AD,ABS,PN)
2550  CALL FNIPHI(PN,YN)
      IF( DABS(PN-PD(IPD)).LE.1D-6*PD(IPD) ) GO TO 2750
      T=(A0*(YN-YD)+AN*(YD-Y0))/(YN-Y0)
      A0=AN
      Y0=YN
      AN=T
      GO TO 2500
2750  ALPHA(IM,IPF)=AN
2900  CONTINUE
3000  CONTINUE


      WRITE(PBDNUM) ((ALPHA(IM,IPF),IPF=1,8),IM=1,MMAX)

4000  CONTINUE


      RETURN
      END
$
```

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Resolution of Ambiguity For Randomly Moving Line Array

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ABSTRACT

The generalized likelihood ratio detector, for deciding between the right-left ambiguity of a line array attempting to estimate the angle of arrival of a plane wave, is derived. Two scenarios are considered, the first with noisy measured antenna angle, the second with noiseless antenna angle measurements. The detector for both cases is a cross-correlator of the sample ac components of the measured antenna and source angle waveforms.

INTRODUCTION

A line array inherently has a cone of ambiguity in its response. When the array lies in the horizontal plane, and a source is located in that same plane, the ambiguity reduces to a right-left uncertainty, which cannot be resolved without some manuevering on the part of the source or array. If the line array is moving randomly, unintentionally or uncontrollably, this movement can serve as a means of making a high quality decision about the source direction, if the array angle, as well as the source angle relative to the line array, are measured.

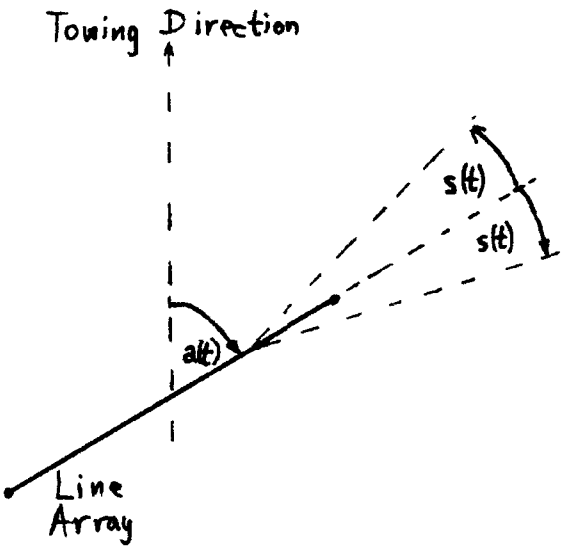

Figure 1. Geometry of Line Array and Source

The situation of interest here is described in figure 1. The line array is being towed due north; however, it is undergoing rigid bar rotation about this direction in a random manner, as described by random process a(t), which is the actual antenna angle relative to the towing direction.

The actual source angle, relative to the line array end-fire direction, is s(t). Furthermore, the actual source angle, relative to the towing direction, is θ, an unknown constant; it is presumed that θ is constant

throughout the observation interval. Reference to figure 1 reveals that these various quantities are interrelated according to the alternatives

$$\theta = \begin{cases} a(t) + s(t) \text{ for hypothesis 1, } H_1 \\ a(t) - s(t) \text{ for hypothesis 2, } H_2 \end{cases} . \tag{1}$$

However, it is unknown which hypothesis is correct; nevertheless, it is desired to make a reliable decision, so that an accurate estimate of the source direction can be made. From (1), observe that we can express

$$s(t) = \begin{cases} \theta - a(t) \text{ for } H_1 \\ -\theta + a(t) \text{ for } H_2 \end{cases} , \tag{2}$$

which will be needed in later developments.

NOISY MEASURED ANTENNA ANGLE

In this section, the measured antenna angle is not a(t) as desired, but rather is

$$x(t) = a(t) + m(t) \ , \tag{3}$$

where m(t) is an unavoidable additive noise process. Also the measured source angle is not s(t), but instead is

$$y(t) = s(t) + n(t) \ , \tag{4}$$

where n(t) is likewise an undesirable additive perturbation, due to limited observation time, array length, ambient noise, etc. The three random processes a(t), m(t), n(t) are presumed to be zero-mean Gaussian processes, independent of each other.

Combining (2)-(4), the situation is as follows: the available measurements upon which a decision must be reached are the two waveforms

$$\left.\begin{aligned} x(t) &= a(t) + m(t) \\ y(t) &= \theta - a(t) + n(t) \end{aligned}\right\} \text{for } H_1 \ , \tag{5A}$$

or

$$\left.\begin{aligned} x(t) &= a(t) + m(t) \\ y(t) &= -\theta + a(t) + n(t) \end{aligned}\right\} \text{for } H_2 \ . \tag{5B}$$

On the basis of waveforms x(t) and y(t), what is the best decision and what is the corresponding estimate of θ?

Derivation of Generalized Likelihood Ratio

Let Δ be the time sampling increment applied to measurement waveforms x(t) and y(t); assume that the samples of the three processes are statistically independent at this rate. Denote

$$x_k = x(k\Delta) \text{ for } 1 \leq k \leq K ,$$
$$y_k = y(k\Delta) \text{ for } 1 \leq k \leq K , \tag{6}$$

where $K\Delta$ is the total observation time, and let the collections of samples be denoted by

$$X = x_1, x_2, \ldots, x_K, \quad Y = y_1, y_2, \ldots, y_K, \quad A = a_1, a_2, \ldots, a_K . \tag{7}$$

Then for a fixed A and a hypothesized value θ_h for the source angle, the conditional probability density function under H_1, of the total set of measurements X, Y, is

$$p_1(X, Y|\theta_h, A) = \prod_{k=1}^{K} \left\{ \frac{1}{\sigma_m} \phi\left(\frac{x_k - a_k}{\sigma_m}\right) \frac{1}{\sigma_n} \phi\left(\frac{y_k - \theta_h + a_k}{\sigma_n}\right) \right\} , \tag{8}$$

where the normalized Gaussian probability density function is

$$\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2) . \tag{9}$$

Here we used (5A) and the Gaussian character of processes m(t) and n(t) with standard deviations σ_m and σ_n respectively.

We now must weight (8) by the Gaussian probability density function for process a(t) and integrate over A, to determine the unconditional probability density function of X, Y, for hypothesized value θ_h. Carrying out the integrals and simplifying the result, we obtain

$$p_1(X, Y|\theta_h) = \left(2\pi\sigma_a\sigma_m\sigma_n R_3^{1/2}\right)^{-K} \quad *$$

$$* \exp\left[-\frac{R_{an}}{2R_3\sigma_m^2}\sum_k x_k^2 - \frac{R_{am}}{2R_3\sigma_n^2}\sum_k (y_k-\theta_h)^2 - \frac{1}{R_3\sigma_m^2\sigma_n^2}\sum_k x_k(y_k-\theta_h)\right], \tag{10}$$

where we define

$$R_{am} = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_m^2}\ , \ R_{an} = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_n^2}\ , \ R_3 = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\ , \tag{11}$$

and use the shorthand notation

$$\sum_k = \sum_{k=1}^{K}\ . \tag{12}$$

In a similar fashion, the unconditional probability density function of X, Y under H_2, for hypothesized value θ_h, is given by

$$p_2(X, Y|\theta_h) = \left(2\pi\sigma_a\sigma_m\sigma_n R_3^{1/2}\right)^{-K} \quad *$$

$$* \exp\left[-\frac{R_{an}}{2R_3\sigma_m^2}\sum_k x_k^2 - \frac{R_{am}}{2R_3\sigma_n^2}\sum_k (y_k+\theta_h)^2 + \frac{1}{R_3\sigma_m^2\sigma_n^2}\sum_k x_k(y_k+\theta_h)\right]. \tag{13}$$

Now if θ_h were known, we could evaluate the likelihood ratio by taking the ratio of (10) and (13). However, we must resort instead to a generalized likelihood ratio, by computing the two values of θ_h that maximize (10) and (13) respectively, and then taking the ratio of the two maxima [1, p. 92]. This procedure is not optimum in any sense; however, it often leads to physical processors that perform well.

The values of θ_h that maximize (10) and (13) are given respectively by

$$\theta_1 = \frac{1}{K}\sum_k y_k + \frac{\sigma_a^2}{\sigma_a^2 + \sigma_m^2}\frac{1}{K}\sum_k x_k \quad \text{for } H_1 , \tag{14A}$$

$$\theta_2 = -\frac{1}{K}\sum_k y_k + \frac{\sigma_a^2}{\sigma_a^2 + \sigma_m^2}\frac{1}{K}\sum_k x_k \quad \text{for } H_2 , \tag{14B}$$

These results have a reasonable physical interpretation: From (5A), the sum $x(t) + y(t)$ would eliminate the random process $a(t)$, and the sample mean of the sum would give an estimate of θ under H_1. However, $m(t)$ contaminates the $a(t)$ contribution according to (3); thus the scale factor $\sigma_a^2/(\sigma_a^2 + \sigma_m^2)$ in (14A) indicates how trustworthy the sample mean of $x(t)$ is. The noise $n(t)$ in $y(t)$ is unavoidable but is partially suppressed by the inherent averaging of the sample mean of $y(t)$. A similar argument holds for $x(t) - y(t)$ under H_2.

The logarithm of the generalized likelihood ratio is (proportional to)

$$T = \frac{1}{2} R_3\, \sigma_m^2\, \sigma_n^2\, \ell n\, \frac{p_2(X, Y|\theta_2)}{p_1(X, Y|\theta_1)} =$$

$$= -\frac{1}{4} R_{am}\, \sigma_m^2 \sum_k (y_k + \theta_2)^2 + \frac{1}{2}\sum_k x_k (y_k + \theta_2) +$$

$$+ \frac{1}{4} R_{am}\, \sigma_m^2 \sum_k (y_k - \theta_1)^2 + \frac{1}{2}\sum_k x_k (y_k - \theta_1) , \tag{15}$$

where we used (10) and (13) with θ_1 and θ_2 substituted for θ_h. Now let

$$S_x = \sum_k x_k, \quad S_y = \sum_k y_k , \quad P = \sum_k x_k y_k . \tag{16}$$

Then (15) becomes, upon use of (14) and simplification and cancellation of various terms,

$$T = P - \frac{1}{K} S_x S_y =$$

$$= \sum_k x_k y_k - \frac{1}{K} \sum_m x_m \sum_n y_n = \tag{17A}$$

$$= \sum_k \tilde{x}_k \tilde{y}_k \ , \tag{17B}$$

where

$$\tilde{x}_k = x_k - \frac{1}{K} \sum_m x_m \quad \text{for } 1 \leq k \leq K \ ,$$

$$\tilde{y}_k = y_k - \frac{1}{K} \sum_n y_n \quad \text{for } 1 \leq k \leq K \ , \tag{18}$$

are defined as the sample ac components of measurements X and Y; that is, the sample means are subtracted from the measurements.

The generalized likelihood ratio test in (17) says to cross-correlate the sample ac components of both measured waveforms and to compare with zero (assuming H_1 and H_2 are equally likely apriori). That is, the test is

$$T = \sum_{k=1}^{K} \tilde{x}_k \tilde{y}_k \underset{H_1}{\overset{H_2}{\gtrless}} 0 \ . \tag{19}$$

Observe that this decision rule makes no use of the variances of any of the processes a(t), m(t), n(t), although this information was presumed known in the above derivation. Of course, the source angle estimates in (14) do require knowledge of the signal-to-noise ratio σ_a^2/σ_m^2 in the x(t) measurement of the antenna angle a(t). Once the decision of H_1 vs H_2 is made via (19), the corresponding estimate of the actual source angle θ is taken from (14).

On the Performance of Test (19)

From (5) and (6), we find that

$$\overline{x_k y_k} = \begin{cases} -\sigma_a^2 & \text{for } H_1 \\ \sigma_a^2 & \text{for } H_2 \end{cases},$$

$$\overline{x_m y_n} = 0 \quad \text{for } m \neq n \ . \tag{20}$$

Then (17A) yields the mean value of the generalized likelihood ratio test statistic as

$$\overline{T} = \begin{cases} -(K-1)\ \sigma_a^2 & \text{for } H_1 \\ (K-1)\ \sigma_a^2 & \text{for } H_2 \end{cases} . \tag{21}$$

Since the statistics of T are desired different under the two hypotheses, (21) indicates that large K and σ_a^2 are desired. That is, a large observation time and a widely-moving antenna give better performance of the test; both of these conclusions are physically plausible.

Although generalized likelihood ratio test (19) does not require knowledge of any variances, the performance (in terms of the error probability) does depend on all the variances. However, the performance does not depend on the actual value θ of the source angle. To see this, we employ (5) in (18) to obtain

$$\tilde{y}_k = \begin{cases} -a_k + n_k - (-S_a + S_n)/K & \text{for } H_1 \\ a_k + n_k - (S_a + S_n)/K & \text{for } H_2 \end{cases}, \tag{22}$$

where

$$S_a = \sum_k a_k, \quad S_n = \sum_k n_k \ . \tag{23}$$

Thus θ is absent from (22), and since θ is not involved in $x(t)$ or x_k, test statistic T is independent of θ.

If we develop (19) in more detail and make use of (18) and (5), we find we can express

$$T_1 = -\left[\sum_k (a_k + m_k)(a_k - n_k) - \frac{1}{K}\sum_k (a_k + m_k)\sum_k (a_k - n_k)\right] \text{ for } H_1 \ ,$$

$$T_2 = \left[\sum_k (a_k + m_k)(a_k + n_k) - \frac{1}{K}\sum_k (a_k + m_k)\sum_k (a_k + n_k)\right] \text{ for } H_2 \ . \tag{24}$$

Thus the statistics of T_1 are identical to those of $-T_2$.

Based upon the results in [2,3], an exact analysis of the cumulative and exceedance distribution functions of test statistic (17A) is possible and will be documented in a NUSC technical report shortly; in fact, a more general processor, where the sample means term is scaled prior to subtraction, will be analyzed.

NOISELESS MEASURED ANTENNA ANGLE

In this section, the noise m(t) in the antenna angle measurement is zero; thus, from (3),

$$x(t) = a(t) \tag{25}$$

under H_1 and H_2. We also remove the Gaussian assumption on the statistics of antenna movement a(t), and allow a(t) to be completely general. Furthermore, we allow any statistical dependence among the samples A of a(t) in (7). However, we retain the Gaussian assumption on the additive noise process n(t) in (4) and (5), and keep the statistical independence of its samples $\{n(k\Delta)\}_1^K$.

Derivation of Generalized Likelihood Ratio

Based upon these premises, the conditional probability density function under H_1 of measurements X, Y, for a fixed A and hypothesized θ_h, is

$$p_1(X, Y|\theta_h, A) = \prod_{k=1}^{K} \left\{ \delta(x_k - a_k) \frac{1}{\sigma_n} \phi\left(\frac{y_k - \theta_h + a_k}{\sigma_n} \right) \right\} . \tag{26}$$

(This is also the limit of (8) as $\sigma_m \to 0+$.) Then letting joint probability density function $p_a(A)$ represent the arbitrary statistical dependence of samples A, the unconditional probability density function under H_1 of X, Y is

$$p_1(X, Y|\theta_h) = \int dA\, p_a(A)\, p_1(X, Y|\theta_h, A) =$$

$$= (\sqrt{2\pi}\,\sigma_n)^{-K} \exp\left[-\frac{1}{2\sigma_n^2} \sum_k (y_k - \theta_h + x_k)^2 \right] p_a(X) , \tag{27}$$

using (26), (9), and the sifting property of delta functions. (If measurement noise m(t) in (3) were non-zero, this simplification of the probability density function in (27) would not be possible.)

The value of θ_h that maximizes probability density function (27), regardless of the form of the probability density function p_a, is

$$\theta_1 = \frac{1}{K}\sum_k (x_k + y_k) \ , \tag{28}$$

which is simply the sample mean of the sum waveform x(t) + y(t). This result is consistent with the earlier one in (14A), for $\sigma_m = 0$, and the ensuing discussion.

In a similar fashion, the probability density function of X, Y under H_2 is

$$p_2(X, Y|\theta_h) = (\sqrt{2\pi}\,\sigma_n)^{-K} \exp\left[-\frac{1}{2\sigma_n^2}\sum_k (y_k + \theta_h - x_k)^2\right] p_a(X) \ , \tag{29}$$

and the maximizing choice of θ_h is

$$\theta_2 = \frac{1}{K}\sum_k (x_k - y_k) \ , \tag{30}$$

which is the sample mean of difference waveform x(t) - y(t).

There follows, from (27)-(30), the logarithm of the generalized likelihood ratio as

$$2\sigma_n^2 \ln \frac{p_2(X, Y|\theta_2)}{p_1(X, Y|\theta_1)} = -\sum_k (y_k + \theta_2 - x_k)^2 + \sum_k (y_k - \theta_1 + x_k)^2 \ . \tag{31}$$

The generalized likelihood ratio test for the two hypotheses in (1) is therefore

$$\sum_k (x_k + y_k - \theta_1)^2 \underset{H_1}{\overset{H_2}{\gtrless}} \sum_k (x_k - y_k - \theta_2)^2 , \tag{32}$$

or upon substitution of (28) and (30), and use of (18), simply

$$\sum_k \tilde{x}_k \tilde{y}_k \underset{H_1}{\overset{H_2}{\gtrless}} 0 \quad . \tag{33}$$

As in the previous section, the cross-correlation of the sample ac components of the measurements X, Y should be compared with zero. This decision rule holds for any statistics of antenna movement a(t).

The alternative form in (32) has an interesting interpretation: Reference to (27)-(28) reveals that θ_1 is the best constant fit to $\{x_k + y_k\}_1^K$ in a least squares sense. Thus the left side of (32) is the actual value of the least squares error of a constant fit to the sum waveform. Similarly, the right side of (32) is the least squares error of a constant fit to the difference waveform. Whichever error is smaller, that hypothesis is selected. This decision rule, (32), is consistent with the observation from (5) and (25) that

$$x(t) + y(t) = \theta + n(t) \quad \text{under } H_1 ,$$

$$x(t) - y(t) = \theta - n(t) \quad \text{under } H_2 . \tag{34}$$

That is, except for zero-mean measurement noise n(t), the sum waveform is constant under H_1, whereas the difference waveform is constant under H_2.

An alternative form for (32) and (33) is

$$T = \sum_k x_k y_k - \frac{1}{K} \sum_m x_m \sum_n y_n \underset{H_1}{\overset{H_2}{\gtrless}} 0 , \qquad (35)$$

just as in (17A). All the ensuing discussion there through (24) is directly relevant for this case as well.

SUMMARY

The generalized likelihood ratio test statistic, for both the noisy as well as the noiseless antenna angle measurement, is a cross-correlator of the sample ac components of the measured antenna and source angle waveforms. Exact performance of this processor can be accomplished, since the test statistic is a quadratic form of correlated Gaussian random variables; in fact, the complete cumulative and exceedance distribution functions of the quantity

$$\sum_{k=1}^{K} x_k y_k - \frac{\gamma}{K} \sum_{m=1}^{K} x_m \sum_{n=1}^{K} y_n \ , \tag{36}$$

for any scaling γ, is capable of exact analysis and will be presented in a future NUSC technical report.

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Operating Characteristics of Crosscorrelator With or Without Sample Mean Removal

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ABSTRACT

The characteristic function of the output of a crosscorrelator, with the sample means removed from each channel, is derived in closed form. More generally, if scaled versions of the sample means are subtracted prior to multiplication of the channel inputs and summation, a closed form for the characteristic function of the correlator output is derived. These results are used to plot the exact operating characteristics of the crosscorrelator, as functions of the threshold, the general scaling factors applied to the sample means, the number of terms, N, summed to yield the output, the actual means at the inputs, and the signal-to-noise ratios of the random signal components at each of the system inputs. Programs for the various cases considered are documented and exercised. Comparisons are made with a Gaussian approximation, which can be used to extend the results to larger values of N than considered here, if needed. Asymptotic results for the exceedance distribution functions also have been derived, but they are not too useful for large N.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF SYMBOLS

N	Number of terms summed to yield crosscorrelator output
u_n, v_n	Two channel inputs at sample time n
overbar	Statistical average
μ_u, μ_v	Means in two input channels
σ_u, σ_v	Standard deviations in two input channels
ρ	Correlation coefficient between two input channels
$u_s(n)$, $v_s(n)$	Random signal components at inputs
$u_d(n)$, $v_d(n)$	Random noise disturbances at inputs
S_u, S_v	Powers of random signal components
ρ_s	Correlation coefficient between signal components
D_u, D_v	Powers of random noise disturbances
R_u, R_v	Signal-to-noise ratios of random components of two input channels, (8)
$\tilde{u}_n$, $\tilde{v}_n$	Sample ac components of inputs
q_n	Crosscorrelator output, (12)
α, β	Scale factors applied to sample means
γ	Scale factor utilized in crosscorrelator output, (12)
U, V	Column matrices of two channel inputs
Q	NxN matrix, (16)
1	Column matrix of ones, (17)
$p_q(u)$	Probability density function of random variable q and argument u
$f_q(\xi)$	Characteristic function of random variable q and argument ξ
det	Determinant
$E_1, E_2, F_1, F_2, G_1, G_2$	Parameters of characteristic function, (24)
$\chi_q(n)$	n-th cumulant of random variable q
μ_q, σ_q	Mean and standard deviation of q
$I_\nu(z)$	Modified Bessel function of first kind
$K_\nu(z)$	Modified Bessel function of second kind
$P_q(u)$	Cumulative distribution function of q
$1-P_q(u)$	Exceedance distribution function of q
h	Normalized crosscorrelator output, (49), (94)
R	Signal-to-noise ratio for identical signal components, (56A), (78)

LIST OF SYMBOLS (Cont'd)

C_+, C_-	Contours for determining cumulative and exceedance distribution functions, (63), (64)
P_F, P_D	False alarm and detection probabilities
ϕ	Normalized Gaussian probability density function, (84)
Φ	Normalized Gaussian cumulative distribution function, (84)
Φ_x	Inverse Φ function
m_k, σ_k	Mean and standard deviation under hypothesis k, (86), (128)
r_u, r_v	Normalized means in two channels, (97)
r	Common normalized mean, (99), (124)
ω	Auxiliary variable equal to 1+2R, (102)
$U(u)$	Unit step function, (109)
$Q(a,b)$	Q-function of Marcum
$Q_M(a,b)$	Q_M-function, ref. 8
$q_M(u), \tilde{q}_M(u)$	Auxiliary functions, (115)

OPERATING CHARACTERISTICS OF CROSSCORRELATOR WITH OR WITHOUT SAMPLE MEAN REMOVAL

INTRODUCTION

The detection of weak signals in two channels is often accomplished by crosscorrelating the two waveforms and comparing with a threshold. For the case where a large number of independent products are added to yield the correlator output, the central limit theorem is often employed, with questionable validity for low false alarm probabilities, i.e. large thresholds. Also, this approximation may not be valid for intermediate numbers of terms added.

Here we wish to get exact operating characteristics for the crosscorrelator, namely detection probability vs. false alarm probability, even for probabilities as low as 1E-10. In particular, we desire results for an arbitrary number of products summed, for any degree of correlation between corresponding individual samples of the two channel inputs, and for any input signal and noise power levels.

Furthermore, it sometimes happens that the two input channels contain dc components, which can be considered either desirable or otherwise, depending on the application. Here we will consider these dc components as nuisance terms and will subtract them out prior to crosscorrelation. More precisely, since the actual values of the dc components in each channel will generally be unknown, we will estimate them via the sample means (over the available record lengths) and subtract these estimates from the available data. This subtraction feature creates new random variables, all of which are statistically dependent on each other, and thereby significantly complicates the analysis. Nevertheless, this crosscorrelation of the sample ac components of the input channels is encountered in practical situations, and in one recent study [1], it was in fact the generalized likelihood ratio detector under two different realistic scenarios. Accordingly, it merits study and accurate quantitative evaluation of performance capability.

More generally, we consider subtraction of scaled versions of the sample means of each channel prior to multiplication and summation. Then as special cases, we can investigate the crosscorrelator with or without sample mean removal, or any intermediate case of interest.

The major analytical result here is a closed form for the characteristic function of the correlator output, in the most compact form involving only two rooting operations and one exponential. Although this processor could be analyzed by the general method given in [2], in terms of the eigenvalues and eigenvectors of a correlation matrix, it would be less accurate and considerably more time consuming, even with computer aid, especially for a large number of terms summed. The actual numerical procedure adopted here for proceeding from the characteristic function to the exceedance distribution functions (false alarm and detection probabilities) is that given in [3], and utilized to advantage in [2,3,4].

PROBLEM DEFINITION

INPUT STATISTICS

The two channel inputs to the crosscorrelator are synchronously sampled in time, yielding random variables $\{u_n\}_1^N$ and $\{v_n\}_1^N$, where N is the total number of data samples taken in each channel. These random variables are Gaussian with the following statistics:

$$\left.\begin{array}{ll} \text{means} & \overline{u_n} = \mu_u,\ \overline{v_n} = \mu_v, \\ \text{variances} & \overline{(u_n-\overline{u_n})^2} = \sigma_u^2,\ \overline{(v_n-\overline{v_n})^2} = \sigma_v^2, \\ \text{covariances} & \overline{(u_n-\overline{u_n})(v_n-\overline{v_n})} = \rho\sigma_u\sigma_v, \end{array}\right\} \text{ all independent of } n. \qquad (1)$$

(An overbar denotes a statistical average.) That is, the means and variances in each channel, although different, do not change with time, and the degree of correlation between channels is constant. Also

$$\begin{array}{l} u_m \text{ is statistically independent of } u_n \text{ if } m\neq n, \\ v_m \text{ is statistically independent of } v_n \text{ if } m\neq n, \\ u_m \text{ is statistically independent of } v_n \text{ if } m\neq n. \end{array} \qquad (2)$$

However, u_n and v_n are statistically dependent on each other, for all n, to the extent ρ indicated in (1).

A SIGNAL AND NOISE MODEL

To better fix the mathematical definitions above, consider in this subsection the following possible signal and noise model:

$$\left.\begin{array}{l} u_n = \mu_u + u_s(n) + u_d(n) \\ v_n = \mu_v + v_s(n) + v_d(n) \end{array}\right\} \quad \text{for } 1 \leq n \leq N, \qquad (3)$$

where random signal components $u_s(n)$, $v_s(n)$ are zero-mean and partially correlated with each other:

$$\left.\begin{aligned} \overline{u_s(n)} = 0,\ \overline{v_s(n)} = 0, \\ \overline{u_s^2(n)} = S_u,\ \overline{v_s^2(n)} = S_v,\ \overline{u_s(n)\ v_s(n)} = \rho_s(S_u S_v)^{1/2} \end{aligned}\right\} \text{for all } n. \quad (4)$$

Thus S_u, S_v are the powers of the random signal components in each channel.

Also, the random noise disturbances $u_d(n)$, $v_d(n)$ in (3) are zero-mean and independent of each other:

$$\left.\begin{aligned} \overline{u_d(n)} = 0,\ \overline{v_d(n)} = 0, \\ \overline{u_d^2(n)} = D_u,\ \overline{v_d^2(n)} = D_v,\ \overline{u_d(n)\ v_d(n)} = 0 \end{aligned}\right\} \text{for all } n. \quad (5)$$

Thus D_u, D_v are the powers of the random noise disturbances in each channel. Finally, except for the statistical dependencies indicated in (4) between $u_s(n)$ and $v_s(n)$, all the 4N random components in (3) are independent of each other.

For this particular signal and noise model in (3)-(5), the master parameters in (1) take the special form

$$\sigma_u^2 = S_u + D_u,\ \sigma_v^2 = S_v + D_v,\ \rho\sigma_u\sigma_v = \rho_s(S_u S_v)^{1/2}, \quad (6)$$

from which there follows

$$\rho = \rho_s\left(\frac{R_u}{1+R_u}\,\frac{R_v}{1+R_v}\right)^{1/2}, \quad (7)$$

where the signal-to-noise ratios (per sample) of the random components in (3) have been defined as

$$R_u = \frac{S_u}{D_u} = \frac{\overline{u_s^2(n)}}{\overline{u_d^2(n)}}, \quad R_v = \frac{S_v}{D_v} = \frac{\overline{v_s^2(n)}}{\overline{v_d^2(n)}} \quad \text{for all } n. \tag{8}$$

Thus the parameters σ_u, σ_v, ρ in (1) depend only on the statistics of the random components in model (3), and not on the dc components μ_u and μ_v. Observe that even if $\rho_s=1$ and $R_u=\infty$, ρ would still be less than 1; the one noisy channel prevents full correlation between inputs.

CROSSCORRELATOR OUTPUT

We define the sample ac components of each channel of the crosscorrelator as

$$\left.\begin{aligned} \tilde{u}_n &= u_n - \frac{1}{N}\sum_{m=1}^{N} u_m \\ \tilde{v}_n &= v_n - \frac{1}{N}\sum_{m=1}^{N} v_m \end{aligned}\right\} \text{for } 1 \le n \le N, \tag{9}$$

where we have subtracted the corresponding sample means from each and every data sample. Thus $\{\tilde{u}_n\}_1^N$ and $\{\tilde{v}_n\}_1^N$ have zero-means and have statistics completely independent of the unknown actual values of input means μ_u, μ_v. However, in trade, we now must deal with a new set of 2N random variables, all of which are statistically dependent on each other; this is the feature which complicates the ensuing analysis. The test statistic (decision variable) of interest is the crosscorrelator output after sample mean removal,

$$q = \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n = \sum_{n=1}^{N} u_n v_n - \frac{1}{N}\sum_{m=1}^{N} u_m \sum_{n=1}^{N} v_n , \tag{10}$$

which is independent of the actual unknown values of input means μ_u and μ_v. If we knew the input means, we could subtract them directly and not have to resort to sample means.

More generally, we consider the modified channel components

$$\left.\begin{aligned} \tilde{u}_n &= u_n - \frac{\alpha}{N}\sum_{m=1}^{N} u_m \\ \tilde{v}_n &= v_n - \frac{\beta}{N}\sum_{m=1}^{N} v_m \end{aligned}\right\} \text{ for } 1 \le n \le N \qquad (11)$$

and the crosscorrelator output

$$q = \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n = \sum_{n=1}^{N} u_n v_n - \frac{\gamma}{N}\sum_{m=1}^{N} u_m \sum_{n=1}^{N} v_n \,, \qquad (12)$$

instead of (9) and (10). Scale factors α and/or β in (11) may be unequal to 1; the final parameter γ in (12) is given by

$$\gamma = \alpha + \beta - \alpha\beta = 1 - (\alpha-1)(\beta-1). \qquad (13)$$

The case of $\gamma=0$ in (12) obviously corresponds to the case of no sample mean removal. On the other hand, if either* $\alpha=1$ or $\beta=1$, then $\gamma=1$, and we have removal of the sample mean; i.e., (12) reduces to (10). We shall be interested here in the analysis of the general case represented by (12), for arbitrary γ.

* It is demonstrated in appendix A that if scale factor $\alpha=1$ but $\beta\neq1$, correlator output q is completely irdependent of μ_u, μ_v, β.

CHARACTERISTIC FUNCTION OF CROSSCORRELATOR OUTPUT

DERIVATION

We express the collection of random variables in (1) and (2) in column matrix form according to

$$U = [u_1\ u_2\ \ldots\ u_N]^T, \quad V = [v_1\ v_2\ \ldots\ v_N]^T\ , \tag{14}$$

where superscript T denotes transpose. The crosscorrelator output q in (12) can then be written as quadratic form

$$q = U^T Q\ V\ , \tag{15}$$

where: NxN matrix

$$Q = I - \frac{\gamma}{N}\ 1\ 1^T\ , \tag{16}$$

I is the NxN identity matrix, and

$$1 = [1\ \ 1\ \ldots\ 1]^T \tag{17}$$

is a Nx1 column matrix of ones.

Since U and V are Gaussian, their joint probability density function is, in terms of the parameters in (1),

$$p(U,V) = \left[2\pi\sigma_u\sigma_v\left(1-\rho^2\right)^{1/2}\right]^{-N} \exp\left[-\frac{1}{2(1-\rho^2)} *\right.$$

$$\left. * \left\{\frac{1}{\sigma_u^2}(U-\mu_u 1)^T(U-\mu_u 1) + \frac{1}{\sigma_v^2}(V-\mu_v 1)^T(V-\mu_v 1) - \frac{2\rho}{\sigma_u\sigma_v}(U-\mu_u 1)^T(V-\mu_v 1)\right\}\right]. \tag{18}$$

The characteristic function of correlator output q in (15) is then given by the statistical average

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp(i\xi U^T Q V)} =$$

$$= \iint dU\, dV\, p(U,V)\, \exp(i\xi U^T Q V) =$$

$$= \left[2\pi\sigma_u\sigma_v\left(1-\rho^2\right)^{1/2}\right]^{-N} \iint dU\, dV \exp\left[i\xi U^T Q V - \frac{1}{2(1-\rho^2)} *\right.$$

$$\left. * \left\{ \frac{1}{\sigma_u^2}(U - \mu_u 1)^T(U - \mu_u 1) + \frac{1}{\sigma_v^2}(V - \mu_v 1)^T(V - \mu_v 1) - \frac{2\rho}{\sigma_u\sigma_v}(U - \mu_u 1)^T(V - \mu_v 1) \right\} \right]. \quad (19)$$

At this point, in order to evaluate this 2N-fold integral, we employ the general integral result (B-2) and (B-6) in appendix B, identifying the matrices there as

$$A = \frac{1}{\sigma_u^2(1-\rho^2)} I, \quad B = \frac{1}{\sigma_v^2(1-\rho^2)} I, \quad C = i\xi Q + \frac{\rho}{\sigma_u\sigma_v(1-\rho^2)} I,$$

$$D = \frac{\sigma_v\mu_u - \rho\sigma_u\mu_v}{\sigma_u^2\sigma_v(1-\rho^2)} 1, \quad E = \frac{\sigma_u\mu_v - \rho\sigma_v\mu_u}{\sigma_u\sigma_v^2(1-\rho^2)} 1. \quad (20)$$

We also need the following auxiliary results for special matrix forms; namely, for arbitrary scalars c_1, c_2, the matrix determinant

$$\det(c_1 I + c_2\, 1\, 1^T) = c_1^{N-1}(c_1 + Nc_2), \quad (21)$$

and the matrix inverse

$$(c_1 I + c_2\, 1\, 1^T)^{-1} = \frac{1}{c_1} I - \frac{c_2}{c_1(c_1 + Nc_2)} 1\, 1^T . \quad (22)$$

Employment of appendix B and (20)-(22) then yields, after a very considerable amount of effort, a closed form for the characteristic function in (19) (in its most compact form)

$$f_q(\xi) = \frac{\exp\left[i\xi \dfrac{G_1 + i\xi G_2}{1 - i\xi F_1 + \xi^2 F_2}\right]}{\left(1 - i\xi E_1 + \xi^2 E_2\right)^{\frac{N-1}{2}} \left(1 - i\xi F_1 + \xi^2 F_2\right)^{\frac{1}{2}}}, \tag{23}$$

where

$$E_1 = 2\rho\sigma_u\sigma_v, \quad E_2 = \sigma_u^2\sigma_v^2(1-\rho^2),$$

$$F_1 = E_1(1-\gamma), \quad F_2 = E_2(1-\gamma)^2,$$

$$G_1 = N(1-\gamma)\mu_u\mu_v, \quad G_2 = \tfrac{1}{2}N(1-\gamma)^2(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 - 2\rho\sigma_u\sigma_v\mu_u\mu_v). \tag{24}$$

The square roots in (23) are principal value, being +1 at ξ=0. This characteristic function has four branch points and two essential singularities which overlap two of the branch points; the complexity of this characteristic function of q precludes tractable analytical results for the probability density function or exceedance distribution function of the correlator output, except in very special cases. Nevertheless, since the characteristic function in (23) is easily numerically evaluated with computer aid, it readily lends itself to the procedure presented in [2,3]. A program for the evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (23)-(24) is given in appendix C for arbitrary values of

- N, number of terms summed
- γ, scale factor in sample mean removal
- μ_u, mean in u-channel
- μ_v, mean in v-channel
- σ_u, standard deviation in u-channel
- σ_v, standard deviation in v-channel
- ρ, correlation coefficient between channels.

A sample plot of the cumulative and exceedance distribution functions for a typical selection of numerical values for the above parameters is also presented in appendix C.

CUMULANTS OF CORRELATOR OUTPUT

By taking the natural logarithm of the characteristic function in (23) and expanding in a power series in ξ, the cumulants of random variable q can be extracted:

$$\begin{aligned}\chi_q(n) &= \tfrac{1}{2}(n-1)![N-1+(1-\gamma)^n]\,(\sigma_u\sigma_v)^n(S^n + D^n) + \\ &+ \tfrac{1}{2}n!N(1-\gamma)^n(\sigma_u\sigma_v)^{n-1}\,\mu_u\mu_v(S^{n-1} + D^{n-1}) + \\ &+ \tfrac{1}{4}n!N(1-\gamma)^n(\sigma_u\sigma_v)^{n-2}(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2)(S^{n-1} - D^{n-1}) ,\end{aligned} \tag{25}$$

where here

$$S = \rho + 1, \quad D = \rho - 1 . \tag{26}$$

In particular, the mean and variance of q are available by using n=1 and 2 respectively in (25):

$$\begin{aligned}\mu_q &= (N-\gamma)\rho\sigma_u\sigma_v + N(1-\gamma)\mu_u\mu_v, \\ \sigma_q^2 &= (N-2\gamma+\gamma^2)(1+\rho^2)\sigma_u^2\sigma_v^2 + N(1-\gamma)^2(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 + 2\rho\sigma_u\sigma_v\mu_u\mu_v).\end{aligned} \tag{27}$$

SPECIAL CASE OF γ=1, SAMPLE MEAN REMOVAL

For γ=1, the general characteristic function in (23) reduces to

$$f_q(\xi;\ \gamma=1) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-\frac{N-1}{2}} , \tag{28}$$

where E_1 and E_2 are still given by (24), and are independent of means μ_u and μ_v, as shown earlier. The cumulants in (25) reduce to

$$\chi_q(n) = \frac{1}{2}(n-1)!(N-1)(\sigma_u\sigma_v)^n[(\rho+1)^n + (\rho-1)^n] , \tag{29}$$

and in particular, the mean and variance of q are

$$\mu_q = (N-1)\rho\sigma_u\sigma_v,$$

$$\sigma_q^2 = (N-1)(1+\rho^2)\sigma_u^2\sigma_v^2 . \tag{30}$$

SPECIAL CASE OF $\gamma=0$, SAMPLE MEAN NOT REMOVED

For $\gamma=0$, the characteristic function in (23) reduces to

$$f_q(\xi;\gamma=0) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-N/2} \exp\left[i\xi \frac{G_1^{(o)} + i\xi G_2^{(o)}}{1-i\xi E_1 + \xi^2 E_2}\right], \tag{31}$$

where

$$E_1 = 2\rho\sigma_u\sigma_v, \quad E_2 = \sigma_u^2\sigma_v^2(1-\rho^2),$$

$$G_1^{(o)} = N\mu_u\mu_v, \quad G_2^{(o)} = \frac{1}{2}N(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 - 2\rho\sigma_u\sigma_v\mu_u\mu_v). \tag{32}$$

The cumulants are obtained by setting $\gamma=0$ in (25), and in particular, the mean and variance of correlator output q are

$$\mu_q = N(\rho\sigma_u\sigma_v + \mu_u\mu_v),$$

$$\sigma_q^2 = N\left[(1+\rho^2)\sigma_u^2\sigma_v^2 + \sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 + 2\rho\sigma_u\sigma_v\mu_u\mu_v\right]. \tag{33}$$

INTERRELATIONSHIP OF TWO SPECIAL CASES

Let the general characteristic function in (23) be denoted by $f_q(\xi;\ N,\ \gamma,\ \mu_u,\ \mu_v)$. We have already seen the expression for $f_q(\xi;\ N,\ 1,\ \mu_u,\ \mu_v)$ in (28). At the same time, from (23) and (24), there follows

$$f_q(\xi;\ N\text{-}1,\ 0,\ 0,\ 0,) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-\frac{N-1}{2}}, \tag{34}$$

which is identical to (28). That is,

$$f_q(\xi;\ N,\ 1,\ \mu_u,\ \mu_v) = f_q(\xi;\ N\text{-}1,\ 0,\ 0,\ 0). \tag{35}$$

Thus the characteristic functions of the two following random variables are identical:

(1) Sum of N terms with sample mean removal, and the true means arbitrary,

(2) Sum of N-1 terms without sample mean removal, but the true means zero. (36)

The removal of the sample means has eliminated the dependence of the correlator output on the unknown means but has reduced the number of degrees of freedom by 1.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For general scaling factor γ and arbitrary input means μ_u, μ_v, and for the model introduced earlier in (3)-(6), the general characteristic function of the correlator output is still given by (23), but with the parameters in (24) now specialized to the form

$$E_1 = 2\rho_s(S_uS_v)^{1/2}, \quad E_2 = D_uD_v+D_uS_v+D_vS_u+S_uS_v(1-\rho_s^2),$$

$$F_1 = E_1(1-\gamma), \qquad F_2 = E_2(1-\gamma)^2,$$

$$G_1 = N(1-\gamma)\mu_u\mu_v, \; G_2 = \tfrac{1}{2}N(1-\gamma)^2[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2-2\rho_s(S_uS_v)^{1/2}\mu_u\mu_v] \; . \qquad (37)$$

The general n-th cumulant is still given by (25); however, the use of (6) allows for determination in terms of the fundamental quantities of the signal and noise model, namely S_u, S_v, D_u, D_v, ρ_s defined in (4)-(5). In particular, the mean and variance of correlator output q are

$$\mu_q = (N-\gamma)\rho_s(S_uS_v)^{1/2} + N(1-\gamma)\mu_u\mu_v,$$

$$\sigma_q^2 = (N-2\gamma+\gamma^2)[D_uD_v+D_uS_v+D_vS_u+(1+\rho_s^2)S_uS_v]+$$

$$+ N(1-\gamma)^2[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2+2\rho_s(S_uS_v)^{1/2}\mu_u\mu_v]. \qquad (38)$$

ANALYTIC RESULTS FOR $\gamma=1$, SAMPLE MEAN REMOVAL

In this section and the next, we will confine attention solely to the case of scale factor $\gamma=1$. The characteristic function of the crosscorrelator output q follows from (28) and (24) as

$$f_q(\xi) = [1 - i\xi E_1 + \xi^2 E_2]^{-\frac{N-1}{2}} = [1 - i\xi 2\rho\sigma_u\sigma_v + \xi^2\sigma_u^2\sigma_v^2(1-\rho^2)]^{-\frac{N-1}{2}} =$$

$$= \left\{[1+i\xi\sigma_u\sigma_v(1-\rho)]\,[1-i\xi\sigma_u\sigma_v(1+\rho)]\right\}^{-\frac{N-1}{2}} \quad \text{for } \gamma=1\ , \tag{39}$$

where we must have $N\geq 2$. We observe, for later numerical use in appendix D, that since $|1 \pm i\xi b| = (1+\xi^2 b^2)^{1/2}$ is monotonically increasing for $\xi \geq 0$, then $|f_q(\xi)|$ is monotonically decreasing for all $\xi \geq 0$ and any N, σ_u, σ_v, ρ.

GENERAL PROBABILITY RESULTS

The cumulants of q have already been listed in (29) and (30). The probability density function corresponding to characteristic function (39) is given by [5, 6.699 12]

$$p_q(u) = \left[\Gamma\left(\frac{N-1}{2}\right)\pi^{1/2}(1-\rho^2)^{1/2}\sigma_u\sigma_v\right]^{-1}\left(\frac{|u|}{2\sigma_u\sigma_v}\right)^{\frac{N}{2}-1} *$$

$$* K_{\frac{N}{2}-1}\left(\frac{|u|}{\sigma_u\sigma_v(1-\rho^2)}\right)\exp\left(\frac{\rho u}{\sigma_u\sigma_v(1-\rho^2)}\right) \quad \text{for all } u,\ \gamma=1\ , \tag{40}$$

where $K_\nu(z)$ is a modified Bessel function of the second kind [6, section 9.6]. If the number of terms added, N, to yield correlator output q, is odd, simple relations for the probability density function in (40) can be obtained [6, 10.2.15 and 10.1.9, last equation]; letting $n = \frac{N-3}{2}$ for N odd, we find the exact result

$$p_q(u) = \frac{(1-\rho^2)^n}{2\sigma_u\sigma_v\, 4^n\, n!} \exp\left(\frac{\rho u - |u|}{\sigma_u\sigma_v(1-\rho^2)}\right) \sum_{m=0}^{n} \frac{(2n-m)!}{(n-m)!\; m!} \left(\frac{2|u|}{\sigma_u\sigma_v(1-\rho^2)}\right)^m$$

$$\text{for all } u;\ n = \frac{N-3}{2},\ N = 3, 5, 7, \ldots . \tag{41}$$

For example, for N=3, we have n=0, yielding

$$p_q(u) = \frac{1}{2\sigma_u\sigma_v} \exp\left(\frac{\rho u - |u|}{\sigma_u\sigma_v(1-\rho^2)}\right) \quad \text{for all } u. \tag{42}$$

The corresponding cumulative distribution function for N=3 is

$$P_q(u) = \int_{-\infty}^{u} dt\; p_q(t) = \frac{1-\rho}{2} \exp\left(\frac{u}{\sigma_u\sigma_v(1-\rho)}\right) \quad \text{for } u \le 0, \tag{43A}$$

while the exceedance distribution function is

$$1 - P_q(u) = \int_{u}^{+\infty} dt\; p_q(t) = \frac{1+\rho}{2} \exp\left(\frac{-u}{\sigma_u\sigma_v(1+\rho)}\right) \quad \text{for } u \ge 0. \tag{43B}$$

This dichotomy, of presenting the cumulative distribution function for negative arguments, and the exceedance distribution function for positive arguments, turns out to be notationally convenient and physically meaningful and will be adopted throughout this report.

Although closed form expressions for the exceedance distribution function corresponding to probability density function (40) are not available for general N, the use of [6, 9.7.2] on (40) leads to the dominant term in the asymptotic expansion of the exceedance distribution function:

$$1 - P_q(u) \sim \frac{1+\rho}{2\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{u}{2\sigma_u\sigma_v}\right)^{\frac{N-3}{2}} \exp\left(\frac{-u}{\sigma_u\sigma_v(1+\rho)}\right) \text{ as } u \to +\infty . \tag{44}$$

For N=3, this is precise; see (43B).

POSSIBLE NORMALIZATIONS OF q

If we define a normalized random variable

$$x = \frac{q}{E_2^{1/2}} = \frac{q}{\sigma_u \sigma_v (1-\rho^2)^{1/2}}, \tag{45}$$

then the characteristic function of x is given by (39) as

$$f_x(\xi) = f_q(\xi/E_2^{1/2}) = \left[1 - i\xi E_1/E_2^{1/2} + \xi^2\right]^{-\frac{N-1}{2}} =$$

$$= \left[1 - i\xi 2\rho(1-\rho^2)^{-1/2} + \xi^2\right]^{-\frac{N-1}{2}}, \tag{46}$$

which has only two fundamental parameters, namely, N and ρ.

A second possibility is the random variable defined by

$$y = \frac{q}{\sigma_u \sigma_v}, \tag{47}$$

for which characteristic function

$$f_y(\xi) = f_q\left(\frac{\xi}{\sigma_u \sigma_v}\right) = \left[1 - i\xi 2\rho + \xi^2(1-\rho^2)\right]^{-\frac{N-1}{2}} \tag{48}$$

also depends only on N and ρ. However, neither of the normalizations, (45) and (47), are of interest to us here; an alternative normalization and reasons for its selection are given below.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For the model presented earlier in (3)-(8), the original E_1, E_2 parameters in (24) take the form already given in the upper line of (37). Let a normalized random variable, relative to the additive random noise disturbances, be defined according to

$$h = \frac{q}{(D_u D_v)^{1/2}} ; \tag{49}$$

see (5). This normalization for the particular signal model (3) is different from both x and y in the general case above. The reason we employ h is that the normalization depends only on the power of the additive noise disturbances, and not on the signal strengths or correlation coefficients; this is consistent with a system which monitors the noise-only background and sets a threshold for a desired false alarm probability.

The characteristic function of the normalized random variable h in (49) is given by

$$f_h(\xi) = f_q\left(\xi/(D_u D_v)^{1/2}\right) = \left[1 - i\xi\frac{E_1}{(D_u D_v)^{1/2}} + \xi^2 \frac{E_2}{D_u D_v}\right]^{-\frac{N-1}{2}} =$$

$$= \left[1 - i\xi 2\alpha + \xi^2(\beta^2 - \alpha^2)\right]^{-\frac{N-1}{2}} , \tag{50}$$

where we define auxiliary parameters here as

$$\alpha = \rho_s (R_u R_v)^{1/2} , \quad \beta = [(1+R_u)(1+R_v)]^{1/2} . \tag{51}$$

Here we used (39), (37), and (8). This characteristic function in (50) depends on the four fundamental parameters N, ρ_s, R_u, R_v, where the latter two quantities are the signal-to-noise ratios per sample of the random components of model (3); see (8).

Reference to (40) reveals that the probability density function of h corresponding to characteristic function (50) is given by

$$p_h(u) = \left[\Gamma\left(\frac{N-1}{2}\right)\pi^{1/2}(\beta^2-\alpha^2)^{1/2}\right]^{-1}\left(\frac{|u|}{2\beta}\right)^{\frac{N}{2}-1} * \\ * K_{\frac{N}{2}-1}\left(\frac{\beta|u|}{\beta^2-\alpha^2}\right)\exp\left(\frac{\alpha u}{\beta^2-\alpha^2}\right) \quad \text{for all } u. \tag{52}$$

For N odd, alternative forms are available from (41), if desired. The asymptotic behavior of the exceedance distribution function of h follows in a manner similar to that used for (44):

$$1 - P_h(u) \sim \frac{1+\alpha/\beta}{2\Gamma\left(\frac{N-1}{2}\right)}\left(\frac{u}{2\beta}\right)^{\frac{N-3}{2}}\exp\left(\frac{-u}{\beta+\alpha}\right) \quad \text{as } u \to +\infty . \tag{53}$$

The cumulants of h follow from (29), (26), and (6)-(8):

$$\chi_h(n) = \tfrac{1}{2}(n-1)!(N-1)[(\alpha+\beta)^n + (\alpha-\beta)^n] , \tag{54}$$

and in particular, the mean and variance of h are

$$\mu_h = (N-1)\alpha = (N-1)\rho_s(R_u R_v)^{1/2} ,$$

$$\sigma_h^2 = (N-1)(\alpha^2+\beta^2) = (N-1)[1 + R_u + R_v + R_u R_v(1+\rho_s^2)] . \tag{55}$$

The two parameters, α and β, are given here by (51), in terms of the fundamental quantities R_u, R_v, ρ_s of the signal and noise model.

REDUCTION TO IDENTICAL SIGNAL COMPONENTS

At this point, we will further specialize the results for the signal and noise model in the above subsection. We presume that

$$R_u = R_v = R \quad \text{and} \quad \rho_s = 1 \ , \tag{56A}$$

giving, from (51),

$$\alpha = R, \quad \beta = 1+R; \tag{56B}$$

that is, the signal-to-noise ratios in the two channels are equal, and the two channel signals are fully correlated. This corresponds physically to a case where the random signal components in (3) are identical, $u_s(n) = v_s(n)$, and the independent random noise disturbances have the same power level. This situation will hold for the rest of this section and all of the next section where the graphical results are presented.

Equations (50) and (56B) then yield the characteristic function for normalized random variable h in (49) as

$$f_h(\xi) = [1 - i\xi 2R + \xi^2(1+2R)]^{-\frac{N-1}{2}} =$$

$$= [(1 + i\xi)(1 - i\xi(1+2R))]^{-\frac{N-1}{2}} \ . \tag{57}$$

The cumulants in (54) reduce to

$$\chi_h(n) = \tfrac{1}{2}(n-1)!(N-1)[(1+2R)^n+(-1)^n] \ , \tag{58}$$

and in particular, the mean and variance become

$$\mu_h = (N-1)R, \quad \sigma_h^2 = (N-1)(1+2R+2R^2) \ . \tag{59}$$

All the above statistical descriptions depend only on the two parameters R, the per-sample signal-to-noise ratio, and N, the number of terms added to yield the correlator output.

The probability density function for h follows from (52) as

$$p_h(u) = \left[\Gamma\left(\frac{N-1}{2}\right) \pi^{1/2} (1+2R)^{1/2}\right]^{-1} \left(\frac{|u|}{2(1+R)}\right)^{\frac{N}{2}-1} * \\ * K_{\frac{N}{2}-1}\left(\frac{1+R}{1+2R}|u|\right) \exp\left(\frac{Ru}{1+2R}\right) \quad \text{for all } u\,, \tag{60}$$

and the asymptotic exceedance distribution function from (53):

$$1 - P_h(u) \sim \frac{1+2R}{2(1+R)\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{u}{2(1+R)}\right)^{\frac{N-3}{2}} \exp\left(\frac{-u}{1+2R}\right) \quad \text{as } u \to +\infty\,. \tag{61}$$

An important word of caution must be mentioned at this point: when N is large, (61) is inadequate for evaluating small false alarm and detection probabilities, since the succeeding terms in the asymptotic expansion contribute significantly. For example, when R=0, the maximum value of the dominant term (61) occurs when u = (N-3)/2 which, for N=128, yields false alarm probability 3.86E-21, a value far below those of interest. Thus (61) has limited applicability, being best for small N; in fact, the first correction term to (61) yields the multiplicative factor

$$1 + \frac{1+2R}{1+R} \frac{(N-3)(N+3+4R)}{8u}\,. \tag{62}$$

It indicates that, for large N, u must be of the order of N^2 in order for the dominant term (61) to be fairly accurate.

Although for N odd, an alternative closed form to the probability density function (60) of h is available from (41), the exceedance distribution function will generate a double sum and be rather cumbersome for large N. On the other hand, the characteristic function in (57) decays rapidly with ξ when N is large and yields very nicely to the numerical approach given in [2,3]. The only difficult cases are in fact those for small N; accordingly, some analytic results for N = 2, 3, 4, 5 will now be presented, based on characteristic function (57).

GENERAL DISTRIBUTION INTEGRALS

Suppose a random variable y has characteristic function $f_y(\xi)$. The cumulative distribution function of y can be written as a contour integral [3, (5)& (6)]

$$P_y(u) = -\frac{1}{i2\pi} \int_{C_+} d\xi \, \frac{f_y(\xi)}{\xi} \exp(-iu\xi) \quad \text{for all } u \,, \tag{63}$$

where C_+ is a contour along the real axis of the complex ξ-plane, with an upward indentation at the origin $\xi=0$, to avoid the pole of the integrand there.

Similarly, the exceedance distribution function of random variable y can be expressed as

$$1 - P_y(u) = \frac{1}{i2\pi} \int_{C_-} d\xi \, \frac{f_y(\xi)}{\xi} \exp(-iu\xi) \quad \text{for all } u \,, \tag{64}$$

where C_- is a contour along the real ξ axis, with a downward indentation at $\xi=0$.

For $u<0$, both contours can be moved into the upper-half ξ-plane, since the exp term furnishes rapid decay there. Similarly, for $u>0$, both contours can be moved into the lower-half ξ-plane, to realize exponential decay on the circular arcs tending to infinity.

DISTRIBUTIONS FOR N=2

From (57), the characteristic function of normalized correlator output h is

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-1/2} \,, \tag{65}$$

and the probability density function follows from (60) as

$$p_h(u) = \frac{1}{\pi(1+2R)^{1/2}} \; K_0\left(\frac{1+R}{1+2R}|u|\right) \exp\left(\frac{Ru}{1+2R}\right) \quad \text{for all } u \; . \tag{66}$$

There is no closed form for the indefinite integral of a K_0 function; see [6, 11.1.8 and 11.1.9]. Instead, we use (65) in (63) and move the contour upwards until it wraps around the branch point at $\zeta=i$ and extends vertically upward from there; this is in fact the steepest descent direction for the exponential when $u<0$. The contributions of the small and large circular arcs tend to zero as the radii tend to zero and infinity, respectively. Under a change of variable, there follows the cumulative distribution function in the form

$$P_h(u) = \frac{2}{\pi} \int_0^{+\infty} \frac{dt \, \exp[u(1+t^2)]}{(1+t^2)[1+(1+2R)(1+t^2)]^{1/2}} \quad \text{for } u \le 0 \; . \tag{67}$$

This is a useful exact result for several reasons: the integrand decays rapidly, has no cusps, and involves only elementary functions which are easily computed; also the integral is a sum of positive quantities and retains significance even for large $|u|$.

In a similar fashion, if characteristic function (65) is substituted in (64) and the contour moved down and wrapped around the branch point at $\zeta = -i/(1+2R)$ and along the vertical steepest descent direction for the exponential when $u>0$, the exceedance distribution function becomes, upon a change of variable,

$$1 - P_h(u) = \frac{2}{\pi}(1+2R) \exp\left(\frac{-u}{1+2R}\right) \int_0^{+\infty} \frac{dt \, \exp(-ut^2)}{[1+(1+2R)t^2][1+(1+2R)(1+t^2)]^{1/2}} \quad \text{for } u \ge 0. \tag{68}$$

This is useful for the same reasons given above.

There is one closed form result possible; namely, for $u=0$, direct integration of probability density function (66) yields [5, 6.611 9]

$$P_h(0) = \frac{1}{\pi} \arccos\left(\frac{R}{1+R}\right) \; , \; 1-P_h(0) = \frac{1}{\pi} \arccos\left(\frac{-R}{1+R}\right) \; . \tag{69}$$

DISTRIBUTIONS FOR N=3

Use of [6, 10.2.17] on (60) immediately yields probability density function

$$p_h(u) = \begin{cases} \frac{1}{2(1+R)} \exp(u) & \text{for } u \leq 0 \\ \\ \frac{1}{2(1+R)} \exp\left(\frac{-u}{1+2R}\right) & \text{for } u \geq 0 \end{cases} . \tag{70}$$

The cumulative and exceedance distribution functions easily follow as

$$\begin{aligned} P_h(u) &= \frac{1}{2(1+R)} \exp(u) && \text{for } u \leq 0 , \\ 1 - P_h(u) &= \frac{1+2R}{2(1+R)} \exp\left(\frac{-u}{1+2R}\right) && \text{for } u \geq 0 . \end{aligned} \tag{71}$$

This latter result corroborates (61) and (62).

DISTRIBUTIONS FOR N=4

The only closed form result possible is obtained by direct integration of probability density function (60) to get origin value

$$P_h(0) = \frac{1}{\pi} \left[\arccos\left(\frac{R}{1+R}\right) - \frac{R(1+2R)^{1/2}}{(1+R)^2} \right] . \tag{72}$$

This follows by use of the integral

$$\int_0^{+\infty} dx\, e^{-\alpha x} \times K_1(\beta x) = \frac{\beta \arccos(\alpha/\beta)}{(\beta^2-\alpha^2)^{3/2}} - \frac{\alpha}{\beta(\beta^2-\alpha^2)} \quad \text{for } \beta > -\alpha , \tag{73}$$

which follows from [5, 6.611 9] by applying $\partial/\partial\beta$ to both sides.

DISTRIBUTIONS FOR N=5

Use of [6, 10.2.17] on (60) immediately yields probability density function

$$p_h(u) = \begin{cases} \dfrac{1+2R-(1+R)u}{4(1+R)^3}\exp(u) & \text{for } u \le 0 \\ \dfrac{1+2R+(1+R)u}{4(1+R)^3}\exp\left(\dfrac{-u}{1+2R}\right) & \text{for } u \ge 0 \end{cases} . \qquad (74)$$

The cumulative and exceedance distribution functions follow as

$$P_h(u) = \frac{2+3R-(1+R)u}{4(1+R)^3}\exp(u) \quad \text{for } u \le 0,$$

$$1 - P_h(u) = \frac{1+2R}{4(1+R)^3}\left[(1+2R)(2+R)+(1+R)u\right]\exp\left(\frac{-u}{1+2R}\right) \text{ for } u \ge 0 . \qquad (75)$$

This latter result corroborates (61) and (62). Also, this example was used as a check on the numerical procedure [3] applied directly to the characteristic function, which is used in the following section; the agreement was ten decimals for numerous values of R and u.

GRAPHICAL RESULTS FOR $\gamma=1$, SAMPLE MEAN REMOVAL

SUMMARY OF PARTICULAR CASE CONSIDERED

We first summarize here the particular case that will be considered quantitatively in this section. The input samples are

$$\left.\begin{aligned} u_n &= \mu_u + u_s(n) + u_d(n) \\ v_n &= \mu_v + v_s(n) + v_d(n) \end{aligned}\right\} \quad \text{for } 1 \leq n \leq N, \tag{76}$$

where these Gaussian random variables have statistics

$$\overline{u_s(n)} = \overline{v_s(n)} = \overline{u_d(n)} = \overline{v_d(n)} = 0 ,$$

$$\overline{u_s^2(n)} = S_u,\ \overline{v_s^2(n)} = S_v,\ \overline{u_s(n)v_s(n)} = (S_u S_v)^{1/2} ,$$

$$\overline{u_d^2(n)} = D_u,\ \overline{v_d^2(n)} = D_v,\ \overline{u_d(n)v_d(n)} = 0 . \tag{77}$$

We presume that the simultaneous signal components $u_s(n)$, $v_s(n)$ in the two channels are fully correlated, that all other random variables are independent, and that the two channel input signal-to-noise ratios

$$\frac{S_u}{D_u} = \frac{S_v}{D_v} = R \tag{78}$$

have a common value R. More general situations have been considered in earlier sections; however, only this special case will be numerically evaluated here.

The normalized crosscorrelator output, with sample mean removal (γ=1), is

$$h = \frac{1}{(D_u D_v)^{1/2}} \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n \ , \tag{79}$$

where the sample ac components

$$\tilde{u}_n = u_n - \frac{1}{N} \sum_{m=1}^{N} u_m \ , \qquad \tilde{v}_n = v_n - \frac{1}{N} \sum_{m=1}^{N} v_m \ . \tag{80}$$

The characteristic function of h is given by (57) as

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-\frac{N-1}{2}} \tag{81}$$

and depends only on signal-to-noise ratio R and number of terms N. We must have $N \geq 2$.

If R=0 and we evaluate the exceedance distribution function corresponding to (81), we then have the false alarm probability. But when R>0, the exceedance distribution function corresponding to (81) is the detection probability. In the following, we plot the detection probability vs. the false alarm probability, with signal-to-noise ratio R as a parameter; different values of N are handled in separate plots.

OPERATING CHARACTERISTICS FOR γ=1

A sample program for evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (81), and thereby the detection probability vs. false alarm probability operating characteristics of the crosscorrelator with sample mean removal, is given in appendix D. It is heavily based on the technique developed and explained in [3].

In figures* 1-14 are presented the operating characteristics for the crosscorrelator with sample mean removal, for values of

$$N = 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 128, 256, \tag{82}$$

respectively. The case of N=2 was accomplished by use of (67)-(69); results for N=3 relied on (71); and the remainder for $N \geq 4$ employed a numerical procedure [3] proceeding directly from characteristic function (81) to the exceedance distribution function. False alarm probabilities P_F in the range 1E-10 to .5 and detection probabilities P_D covering 1E-10 to .999 are presented. The abscissa and ordinate on these plots are according to a normal probability transformation, as explained below. Values of signal-to-noise ratio R are taken as $R=2^n$, where n assumes values appropriate for each plot in order to cover the full range of probabilities of interest.

GAUSSIAN APPROXIMATION

Suppose the decision variable of a processor is Gaussian with mean and standard deviation m_0, σ_0 respectively when the input signal is absent, and m_1, σ_1 when signal is present. Then for threshold λ, the false alarm probability and detection probability are

$$P_F = \int_{\lambda}^{+\infty} du\, \frac{1}{\sigma_0}\, \phi\left(\frac{u-m_0}{\sigma_0}\right) = \Phi\left(\frac{m_0-\lambda}{\sigma_0}\right),$$

$$P_D = \int_{\lambda}^{+\infty} du\, \frac{1}{\sigma_1}\, \phi\left(\frac{u-m_1}{\sigma_1}\right) = \Phi\left(\frac{m_1-\lambda}{\sigma_1}\right), \tag{83}$$

respectively, where ϕ and Φ are the normalized Gaussian probability density function and cumulative distribution function:

$$\phi(u) = (2\pi)^{-1/2} \exp(-u^2/2), \qquad \Phi(u) = \int_{-\infty}^{u} dt\, \phi(t)\ . \tag{84}$$

If we let Φ_I be the inverse function to Φ, and define

$$x = \Phi_I(P_F), \quad y = \Phi_I(P_D)\ , \tag{85}$$

*See page 51 et seq.

then threshold Λ can be eliminated from (83) to yield

$$y = \frac{m_1 - m_o + \sigma_o x}{\sigma_1} . \tag{86}$$

Equation (85) corresponds to the transformation to normal probability coordinates; thus a plot of P_D vs P_F on normal probability paper is the straight line (86) when the decision variable is Gaussian under both hypotheses of signal absent as well as present.

Reference to (59) reveals that, for our application,

$$m_o = 0, \quad m_1 = (N-1)R, \quad \sigma_o^2 = N-1, \quad \sigma_1^2 = (N-1)(1+2R+2R^2) , \tag{87}$$

since setting signal-to-noise ratio R=0 corresponds to hypothesis 0, signal absent. Substitution in (86) yields

$$y = \frac{(N-1)^{1/2} R + x}{(1+2R+2R^2)^{1/2}} ; \tag{88}$$

that is, if normalized crosscorrelator output h were Gaussian, the operating characteristics would be straight lines dictated by (88). These straight lines are superposed as dashed lines in figures 12-14 for N=96, 128, 256 respectively. Despite the large value of N=96 in figure 12, the Gaussian approximation is not that accurate, especially for small false alarm probabilities and large detection probabilities. The exact curve (solid) and the Gaussian approximation (dashed) cross each other, and are labelled at the crossing with the corresponding value of n in signal-to-noise ratio $R=2^n$. For N=256 in figure 14, agreement is better and the Gaussian approximation is probably adequate for larger N. If not, an additional term or two in an Edgeworth expansion could be investigated with the aid of the cumulants given in (58).

An obvious shortcoming of the Gaussian approximation (88) may be seen immediately:

$$\lim_{R\to\infty} y = \left(\frac{N-1}{2}\right)^{1/2} < 1 \quad \text{for any } x \, . \tag{89}$$

Reference to (85) then yields the interpretation

$$\lim_{R\to\infty} P_D = \Phi\left(\left(\frac{N-1}{2}\right)^{1/2}\right) < 1 \quad \text{for any } P_F \, . \tag{90}$$

That is, as input signal-to-noise ratio R tends to infinity, the approximate detection probability saturates at a value less than 1, regardless of the false alarm probability. Thus the Gaussian approximation must certainly deteriorate for large R; the exact discrepancy for probabilities of practical interest is displayed in figures 12-14.

ANALYTIC RESULTS FOR $\gamma=0$, SAMPLE MEAN NOT REMOVED

In this section and the next, attention will be confined solely to the case of scale factor $\gamma=0$. The characteristic function of the crosscorrelator output q is then given in (31) and (32), and the mean and variance of q are listed in (33). Due to the complexity of the exponential term in characteristic function (31), there are no general probability density function or cumulative distribution function results for arbitrary N, like those given earlier in (40), (41), and (44) for $\gamma=1$. Here, we can have $N \geq 1$.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For the model presented earlier in (3)-(8), the original parameters in (24) take the form already given in (37), but now with $\gamma=0$. Specifically, characteristic function (31) is

$$f_q(\xi) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-N/2} \exp\left[i\xi \frac{G_1^{(o)} + i\xi G_2^{(o)}}{1-i\xi E_1 + \xi^2 E_2}\right], \tag{91}$$

where

$$E_1 = 2\rho_s(S_u S_v)^{1/2}, \quad E_2 = D_u D_v + D_u S_v + D_v S_u + S_u S_v(1-\rho_s^2) ,$$

$$G_1^{(o)} = N\mu_u\mu_v, \quad G_2^{(o)} = \tfrac{1}{2}N[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2-2\rho_s(S_u S_v)^{1/2}\mu_u\mu_v]. \tag{92}$$

The mean and variance of crosscorrelator output q follow from (38) according to

$$\mu_q = N[\rho_s(S_u S_v)^{1/2} + \mu_u\mu_v] ,$$

$$\sigma_q^2 = N[D_u D_v + D_u S_v + D_v S_u + (1+\rho_s^2)S_u S_v +$$

$$+(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2+2\rho_s(S_u S_v)^{1/2}\mu_u\mu_v] . \tag{93}$$

NORMALIZED CROSSCORRELATOR OUTPUT

As in (49), and for the same reasons, we define a normalized crosscorrelator output, relative to the additive random noise disturbances $u_d(n)$ and $v_d(n)$ in (3) and (5), according to

$$h = \frac{q}{(D_u D_v)^{1/2}} . \tag{94}$$

The characteristic function of h is available from (91) and (92):

$$f_h(\xi) = f_q\left(\xi/(D_u D_v)^{1/2}\right) =$$

$$= [1-i\xi 2\alpha+\xi^2(\beta^2-\alpha^2)]^{-N/2} \exp\left[i\xi \frac{a+i\xi b}{1-i\xi 2\alpha+\xi^2(\beta^2-\alpha^2)}\right], \tag{95}$$

where

$$\alpha = \rho_s(R_u R_v)^{1/2} , \quad \beta = [(1+R_u)(1+R_v)]^{1/2} ,$$

$$a = N r_u r_v, \; b = \tfrac{1}{2}N[(1+R_u)r_v^2 + (1+R_v)r_u^2 - 2\rho_s(R_u R_v)^{1/2} r_u r_v] , \tag{96}$$

and where we have defined

$$R_u = \frac{S_u}{D_u} , \; R_v = \frac{S_v}{D_v} , \; r_u = \frac{\mu_u}{D_u^{1/2}} , \; r_v = \frac{\mu_v}{D_v^{1/2}} . \tag{97}$$

The characteristic function in (95) depends on six fundamental parameters, namely N, ρ_s, R_u, R_v, r_u, r_v. The mean and variance of h follow from (94), (93), and (97):

$$\mu_h = N[\rho_s(R_u R_v)^{1/2} + r_u r_v] ,$$

$$\sigma_h^2 = N[(1+R_u)(1+R_v)+\rho_s^2 R_u R_v+(1+R_u)r_v^2+(1+R_v)r_u^2+2\rho_s(R_u R_v)^{1/2} r_u r_v] . \quad (98)$$

r_u and r_v are referred to as normalized means.

REDUCTION TO IDENTICAL SIGNAL COMPONENTS

In order to prepare for numerical evaluation of the operating characteristics of the crosscorrelator with $\gamma=0$, we further specialize the signal and noise model to the case where

$$R_u = R_v = R, \quad \rho_s = 1, \quad r_u = r_v = r ; \quad (99)$$

see (56) et seq. This leads to

$$\alpha = R, \quad \beta = 1+R, \quad a = b = Nr^2 ,$$

via (96). The characteristic function in (95) then reduces to

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-N/2} \exp\left[\frac{i\xi Nr^2}{1-i\xi(1+2R)}\right], \quad (100)$$

and the mean and variance in (98) become

$$\mu_h = N(R+r^2), \quad \sigma_h^2 = N[1+2R+2R^2+2(1+2R)r^2] . \quad (101)$$

The characteristic function in (100) has a branch point at $\xi = i$, and another branch point at $\xi = -i/(1+2R)$ which overlaps an essential singularity; this complicates some of the analytical development to follow.

There are three fundamental parameters in (100), namely N, R, r. Since normalized mean r appears only through its square, we can presume $r \geq 0$ without loss of generality. Furthermore, if r=0, characteristic function (100)

reduces to (57) if N-1 there is replaced by N. Thus the curves for r=0 here can be obtained from the earlier curves for γ=1 in figures 1-14 by looking at a value for N there which is one greater; accordingly we can confine attention to r>0 in this and the next section.

When the random signal components $u_s(n)$, $v_s(n)$ in model (3) are absent, then R=0, and the exceedance distribution function corresponding to characteristic function (100) becomes the false alarm probability. However, (100) still depends on r, meaning that the false alarm probability must, likewise. Thus, a non-zero mean in (3), i.e. r>0, is not considered a signal attribute here, but rather is a nuisance quantity; it may, in fact, degrade the operating characteristics if not removed.

For notational convenience in the following, we define

$$\omega = 1+2R \ . \tag{102}$$

The magnitude of the exponential term in characteristic function (100) then can be expressed as

$$\exp\left[\frac{-\xi^2 \omega N r^2}{1+\xi^2\omega^2}\right], \tag{103}$$

which is monotonically decreasing for $\xi \geq 0$. Coupled with the observation immediately under (39), it is seen that $|f_h(\xi)|$ in (100) is monotonically decreasing for all $\xi \geq 0$ and any N,R,r. This property allows for a convenient termination procedure in the numerical transformation [3] of characteristic function (100). It should however be observed that (103) does not decrease to zero, but saturates at value $\exp(-Nr^2)$, regardless of how ξ increases to infinity; thus the eventual decay of the characteristic function (100) is furnished only by the leading factor.

ASYMPTOTIC BEHAVIOR OF CUMULATIVE AND EXCEEDANCE DISTRIBUTION FUNCTIONS

In appendix E, it is shown that if characteristic function (100) is substituted in (63) and (64), and the contours moved appropriately in the complex ξ-plane, then the following asymptotic behaviors obtain. The cumulative distribution function

$$P_h(u) \sim \left[\Gamma\left(\frac{N}{2}\right) 2^{N/2}(1+R)^{N/2}\right]^{-1} (-u)^{\frac{N}{2}-1} \exp\left[u - \frac{Nr^2}{2(1+R)}\right] *$$

$$* \left[1 - \frac{\frac{N}{2}-1}{u}\left(1 + \frac{N(1+2R)}{4(1+R)} + \frac{Nr^2}{4(1+R)^2}\right)\right] \quad \text{as } u \to -\infty. \tag{104}$$

We see again, in similar fashion to (61) and (62), that in order for the correction term in the second line of (104) not to be too significant, we must have $u < -N^2$. For reasons elucidated in (61) et seq., (104) is not useful for large N.

As checks on (104), we note that for r=0 and N=2, (104) reduces precisely to the upper line of (71); this latter result pertains to γ=1, N=3 and is consistent with the observation already made in the paragraph below (101). In addition, if we let r=0 and N=4 in (104), it reduces to the upper line of (75); this latter result holds for γ=1, N=5 and is likewise consistent.

Also given in appendix E are a variety of asymptotic expansions for the exceedance distribution function; the simplest one is

$$1 - P_h(u) \sim \left[2\pi^{1/2}\, 2^{N/2}\, (1+R)^{N/2}\, \left(Nr^2\right)^{\frac{N-1}{4}}\right]^{-1} (1+2R)^{\frac{N+1}{2}} *$$

$$* u^{\frac{N-3}{4}} \exp\left[-\frac{\left(u^{1/2} - N^{1/2}r\right)^2}{1+2R}\right] \quad \text{as } u \to +\infty; \quad r>0\ . \tag{105}$$

However, the same reservations as above, regarding u large relative to N^2, are again in order.

DISTRIBUTIONS FOR N=1

If characteristic function (100) with N=1 is substituted in (63), and if the contour is moved as indicated under (66), there follows the exact result for the cumulative distribution function

$$P_h(u) = \frac{2}{\pi}\int_0^{+\infty} \frac{dt}{(1+t^2)(1+\omega(1+t^2))^{1/2}} \exp\left[u(1+t^2) - \frac{r^2(1+t^2)}{1+\omega(1+t^2)}\right] \text{ for } u\leq 0. \quad (106)$$

(This reduces to (67) for r=0, as it must.) This integral form possesses all the desirable attributes listed under (67).

If characteristic function (100) with N=1 is substituted in (64) in an attempt to get the exceedance distribution function, the analysis becomes rather difficult, due to the overlapping essential singularity and branch point of the integrand at $\xi = -i/\omega$. This problem is treated in detail in appendix F, with the result that the exceedance distribution function can be found via the characteristic function approach in terms of two integrals; see (F-21)-(F-23). However, a better numerical procedure for the exceedance distribution function is the direct result derived in (F-33); this latter integral is the one actually used here to generate the operating characteristics for γ=0, N=1.

DISTRIBUTIONS FOR N=2

The characteristic function is available from (100):

$$f_h(\xi) = (1+i\xi)^{-1} * (1-i\xi\omega)^{-1} \exp\left(\frac{i\xi 2r^2}{1-i\xi\omega}\right) \equiv f_1(\xi) * f_2(\xi) , \quad (107)$$

where $\omega = 1+2R$. The probability density functions corresponding to these two characteristic functions are [5, 6.631 4]

$$p_1(u) = \exp(u)\, U(-u) ,$$

$$p_2(u) = \frac{1}{\omega} \exp\left(-\frac{u+2r^2}{\omega}\right) I_o\left(\frac{2r}{\omega}(2u)^{1/2}\right) U(u) , \qquad (108)$$

where U is the unit step function

$$U(u) = \left\{\begin{matrix} 0 & \text{for} & u<0 \\ 1 & \text{for} & u>0 \end{matrix}\right\} . \qquad (109)$$

The probability density function of h is given by convolution

$$p_h(u) = \int_{-\infty}^{+\infty} dt\; p_1(t)\; p_2(u-t) \quad \text{for all } u . \qquad (110)$$

Substitution of (108) in (110) yields

$$p_h(u) = \begin{cases} \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) & \text{for } u \le 0 \\ \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) Q\left(\frac{2r}{\omega^{1/2}(1+\omega)^{1/2}} , \left(\frac{2(1+\omega)u}{\omega}\right)^{1/2}\right) & \text{for } u \ge 0 \end{cases} \qquad (111)$$

where the Q-function is defined in [7] and the two integrals encountered have been evaluated by use of [5, 6.631 4] and [7, (9)], respectively. The cumulative and exceedance distribution functions of h readily follow from (111), the latter by means of [7, (42)]:

$$P_h(u) = \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) \quad \text{for } u \le 0 ,$$

$$1 - P_h(u) = Q\left(\frac{2r}{\omega^{1/2}}, \left(\frac{2u}{\omega}\right)^{1/2}\right) -$$

$$- \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) Q\left(\frac{2r}{\omega^{1/2}(1+\omega)^{1/2}}, \left(\frac{2(1+\omega)u}{\omega}\right)^{1/2}\right) \quad \text{for } u \ge 0. \qquad (112)$$

(As a check, for r=0, then (111) and (112) reduce to (70) and (71) respectively, as they must.)

DISTRIBUTIONS FOR N=4

The characteristic function is available from (100):

$$f_h(\xi) = (1+i\xi)^{-2} * (1-i\xi\omega)^{-2} \exp\left(\frac{i\xi 4r^2}{1-i\xi\omega}\right) \equiv f_1(\xi) * f_2(\xi) , \qquad (113)$$

where $\omega = 1+2R$. The probability density functions corresponding to these two characteristic functions are [5, 6.631 4]

$$p_1(u) = -u \exp(u)\, U(-u) ,$$

$$p_2(u) = \frac{1}{2\omega r} \exp\left(-\frac{u+4r^2}{\omega}\right) u^{1/2} I_1\left(4ru^{1/2}/\omega\right) U(u) . \qquad (114)$$

The probability density function of h is given by convolution (110). In preparation for that result, we use the shorthand notation

$$q_M(u) = Q_M\left(2r\left(\frac{2}{\omega(1+\omega)}\right)^{1/2}, \left(\frac{2u(1+\omega)}{\omega}\right)^{1/2}\right) ,$$

$$\tilde{q}_M(u) = Q_M\left(2r\left(\frac{2}{\omega}\right)^{1/2}, \left(\frac{2u}{\omega}\right)^{1/2}\right), \qquad (115)$$

where the Q_M-function is defined in [8]. We also present a new integral result that will be needed in the sequel,

$$\int_C^{+\infty} dx\ x\ Q_M(b,\ ax) = \frac{1}{2a^2}[b^2 Q_{M+1}(b,ac)+2M\ Q_M(b,ac)-a^2c^2Q_{M-1}(b,ac)], \quad (116)$$

which can be interpreted as the limit of [8, (31)] as $p \to 0+$. Substitution of (114) in (110) yields probability density function

$$p_h(u) = \begin{cases} \dfrac{1}{(1+\omega)^4} \exp\left(u - \dfrac{4r^2}{1+\omega}\right)\left[4r^2+2\omega(1+\omega)-(1+\omega)^2u\right] & \text{for } u \le 0 \\ \dfrac{1}{(1+\omega)^4} \exp\left(u - \dfrac{4r^2}{1+\omega}\right)\left[4r^2q_3(u)+2\omega(1+\omega)q_2(u)-(1+\omega)^2uq_1(u)\right] & \text{for } u \ge 0 \end{cases} \quad (117)$$

where (115) has been used; the upper line employed [5, 6.631 4], while the lower line used (116) and an integration by parts procedure to be elaborated below in the exceedance distribution function evaluation.

The cumulative distribution function for $u \le 0$ follows readily by integration of (117):

$$P_h(u) = \frac{1}{(1+\omega)^4} \exp\left(u - \frac{4r^2}{1+\omega}\right)\left[4r^2+(1+\omega)(1+3\omega)-(1+\omega)^2u\right] \quad \text{for } u \le 0. \quad (118)$$

For $u \ge 0$, a modified approach is required. Integration of (110) yields cumulative distribution function

$$P_h(u) = \int_{-\infty}^{+\infty} dt\ p_2(t)\ P_1(u-t)\ . \quad (119)$$

Probability density functions p_1 and p_2 are available in (114), and cumulative distribution function P_1 follows as

$$P_1(u) = \begin{cases} (1-u)\exp(u) & \text{for } u \leq 0 \\ \\ 1 & \text{for } u \geq 0 \end{cases} . \qquad (120)$$

Substitution of (114) and (120) in (119) yields, for $u \geq 0$,

$$P_h(u) = \int_0^u dt\, \frac{1}{2\omega r} \exp\left(-\frac{t+4r^2}{\omega}\right) t^{1/2}\, I_1\left(4rt^{1/2}/\omega\right) +$$

$$+ \int_u^{+\infty} dt\, \frac{1}{2\omega r} \exp\left(-\frac{t+4r^2}{\omega}\right) t^{1/2}\, I_1\left(4rt^{1/2}/\omega\right) (1-u+t) \exp(u-t) =$$

$$= 1-\tilde{q}_2(u) + \frac{1}{2\omega r} \exp\left(u-\frac{4r^2}{\omega}\right) \int_u^{+\infty} dt\, t^{1/2} \exp\left(-t\,\frac{1+\omega}{\omega}\right) I_1\left(4rt^{1/2}/\omega\right) (1-u+t), \qquad (121)$$

via [8, (22)]. We now integrate by parts, letting $U(t)=1-u+t$, and the remainder $dV(t)$. Then using [8, (22)] again, we find

$$V(t) = -\frac{2\omega r}{(1+\omega)^2} \exp\left(\frac{4r^2}{\omega(1+\omega)}\right) q_2(t) . \qquad (122)$$

Combining these results, (121) and (116) yield

$$1 - P_h(u) = \tilde{q}_2(u) - \frac{1}{(1+\omega)^2} \exp\left(u - \frac{4r^2}{1+\omega}\right) q_2(u) - \int_u^{+\infty} dt\, q_2(t) =$$

$$= \tilde{q}_2(u) - \frac{1}{(1+\omega)^4} \exp\left(u - \frac{4r^2}{1+\omega}\right)\left[4r^2 q_3(u)+(1+\omega)(1+3\omega)q_2(u)-(1+\omega)^2 u q_1(u)\right]$$

$$\text{for } u \geq 0. \qquad (123)$$

The final results for N=4 are given by (115), (118), and (123). This case was used as a numerical check on the computational approach [3] proceeding directly from characteristic function (100) to the exceedance distribution function, with excellent agreement for numerous values of R, r, and u.

GRAPHICAL RESULTS FOR $\gamma=0$, SAMPLE MEAN NOT REMOVED

SUMMARY OF PARTICULAR CASE CONSIDERED

The situation of interest has already been summarized in (76)-(78); in addition, we have a common value for the normalized means,

$$\frac{\mu_u}{D_u^{1/2}} = \frac{\mu_v}{D_v^{1/2}} = r \ , \tag{124}$$

and the normalized crosscorrelator output is not (79)-(80), but rather is, for $\gamma=0$,

$$h = \frac{1}{(D_u D_v)^{1/2}} \sum_{n=1}^{N} u_n v_n \ . \tag{125}$$

The characteristic function is given by (100):

$$f_h(\xi) = [(1+i\xi)(1-i\xi\omega)]^{-N/2} \exp\left(\frac{i\xi N r^2}{1-i\xi\omega}\right), \tag{126}$$

where $\omega = 1+2R$. When the signal is absent, then R=0; however the false alarm probability corresponding to this characteristic function still depends on r. Thus since the sample mean has not been removed, the operating characteristics will also depend on r. Since results for r=0 can be found from an earlier section, we only consider r>0 here.

OPERATING CHARACTERISTICS FOR $\gamma=0$

A sample program for evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (126), and thereby the detection probability vs. false alarm probability operating characteristics of the crosscorrelator without sample mean removal, is given in appendix G. In figures* 15-35 are presented the operating characteristics for values of

*See page 65 et seq.

$$N = 1, 2, 3, 4, 8, 16, 32, 64, 128, 256, \tag{127}$$

and for various values of r. The case of N=1 was accomplished by use of (106) and (F-33); results for N=2 employed (112); and the remainder for $N \geq 3$ employed a numerical procedure [3] proceeding directly from characteristic function (126) to the exceedance distribution function. False alarm probabilities P_F in the range 1E-10 to .5 and detection probabilities P_D covering 1E-10 to .999 are presented. The abscissa and ordinate on these plots employ a normal probability transformation, as explained earlier in (84)-(86).

Values of signal-to-noise ratio R are taken as $R=2^n$, where n assumes values appropriate for each plot in order to cover the full range of probabilities of interest. Values of normalized mean r in (124) have been taken as r=1 and 2, with the exception of figure 19 where one example for r=4 was added.

Without exception, increasing r from zero degrades the operating characteristics of the crosscorrelator. For example, figures 17, 18, 19 give a succession of operating characteristics for r = 1, 2, 4 respectively, and for common values of signal-to-noise ratio R. (In order to determine the operating characteristics for r=0 here, we can look at the earlier results in figures 1-14, but for a value of N which is one greater there.) Thus, not removing the sample mean from the crosscorrelator output requires a larger threshold setting for a specified false alarm probability and thereby lowers the detection probability and degrades performance.

GAUSSIAN APPROXIMATION

If the crosscorrelator output is Gaussian, for both signal absent as well as present, the earlier derivations in (83)-(86) pertain. Now reference to (101) yields statistics

$$m_0 = Nr^2, \quad m_1 = N(R+r^2) ,$$

$$\sigma_0^2 = N(1+2r^2), \quad \sigma_1^2 = N[1+2R+2R^2+2(1+2R)r^2], \tag{128}$$

since setting signal-to-noise ratio R=0 corresponds to hypothesis 0, signal absent. Substitution in (86) yields the normal probability approximation

$$y = \frac{N^{1/2}\, R+(1+2r^2)^{1/2}\, x}{[1+2R+2R^2+2(1+2R)r^2]^{1/2}} . \tag{129}$$

These straight lines are superposed as dashed lines in figures 32-35 for N=128 and 256. The Gaussian approximation is moderately good for large N such as 256, and in fact crosses the exact curves (solid) at a point which is labeled with the corresponding value of n in signal-to-noise ratio $R=2^n$.

An obvious shortcoming of the Gaussian approximation (129) is apparent:

$$\lim_{R\to\infty} y = \left(\frac{N}{2}\right)^{1/2} \quad \text{for any } x,\ r . \tag{130}$$

Reference to (85) then yields the interpretation

$$\lim_{R\to\infty} P_D = \bar{\Phi}\left(\left(\frac{N}{2}\right)^{1/2}\right) < 1 \quad \text{for any } P_F,\ r . \tag{131}$$

That is, as input signal-to-noise ratio R tends to infinity, the approximate detection probability saturates at a value less than 1, regardless of the false alarm probability and normalized mean r. Thus the Gaussian approximation must certainly be inaccurate for large R; the exact discrepancy for probabilities of practical interest is displayed in figures 32-35.

SUMMARY

A closed form expression for the characteristic function of the output of a crosscorrelator, with or without sample mean removal, has been derived in (23)-(24) for general values of: the number of terms added to yield the correlator output, the means and variances in each of the two input channels, the degree of correlation between the two channels, and the scale factor employed in the sample mean removal. A program for the evaluation of the cumulative and exceedance distribution functions of this general case has also been presented. These results can furnish the basis of a study of the error probabilities of a correlator required to decide between alternative hypotheses on the input statistics [1]; this problem will in fact be the subject of a future technical report by this author.

The general results were first specialized to a signal and noise model, and then to the two distinct cases of sample mean removal (γ=1) and no sample mean removal (γ=0). Plots of the operating characteristics for numerous values of N and signal-to-noise ratio R were then displayed for a wide range of detection probability vs false alarm probability. Some new analytic results for cumulative and exceedance distribution functions, especially for small N, were derived and used as checks on the general numerical procedure. Comparisons with a Gaussian approximation indicated quantitatively when that simplification is valid. Asymptotic results derived were useful for small N, but not for large N except in the region of probabilities too small to be of practical importance.

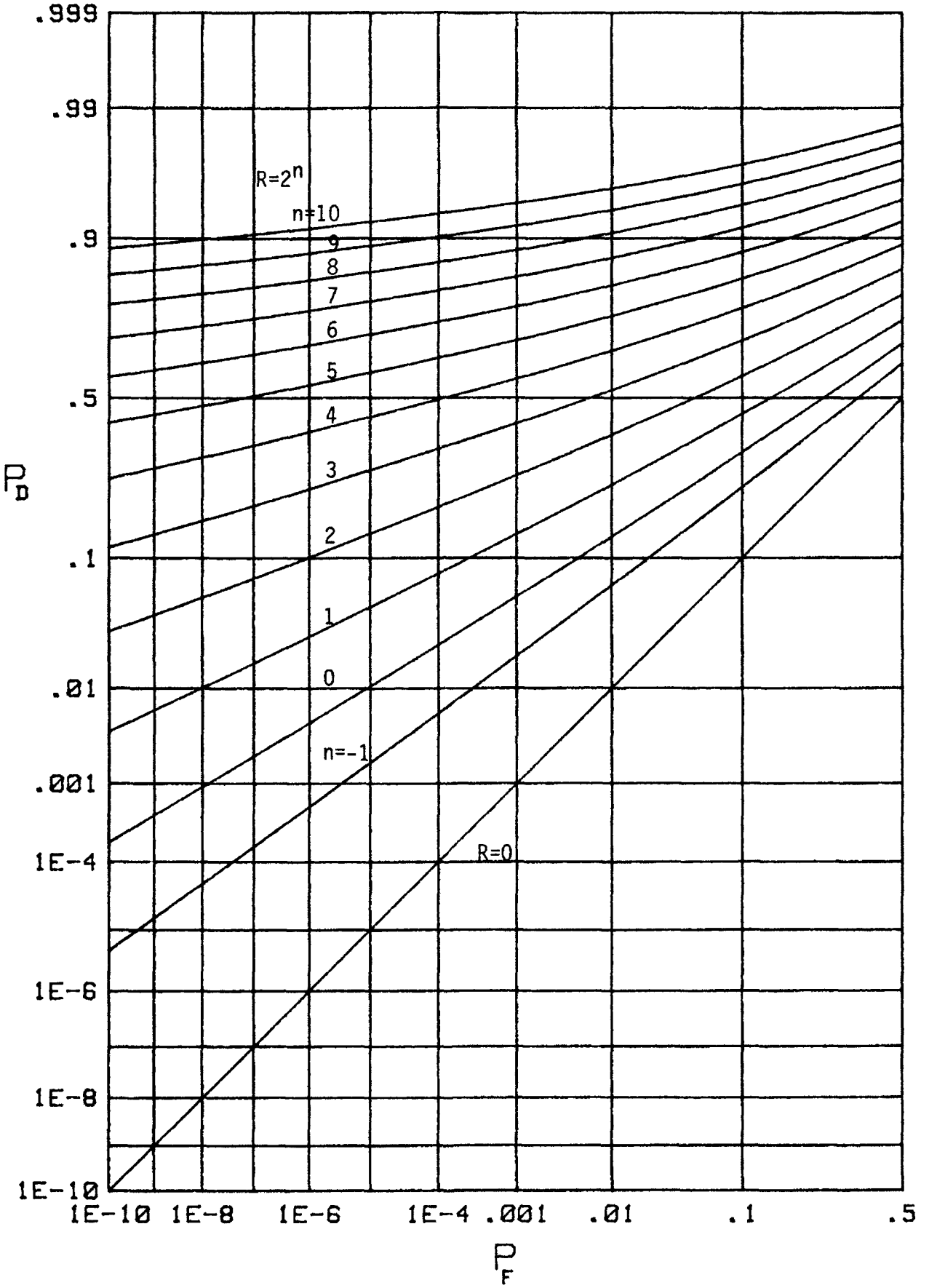

Figure 1. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 2

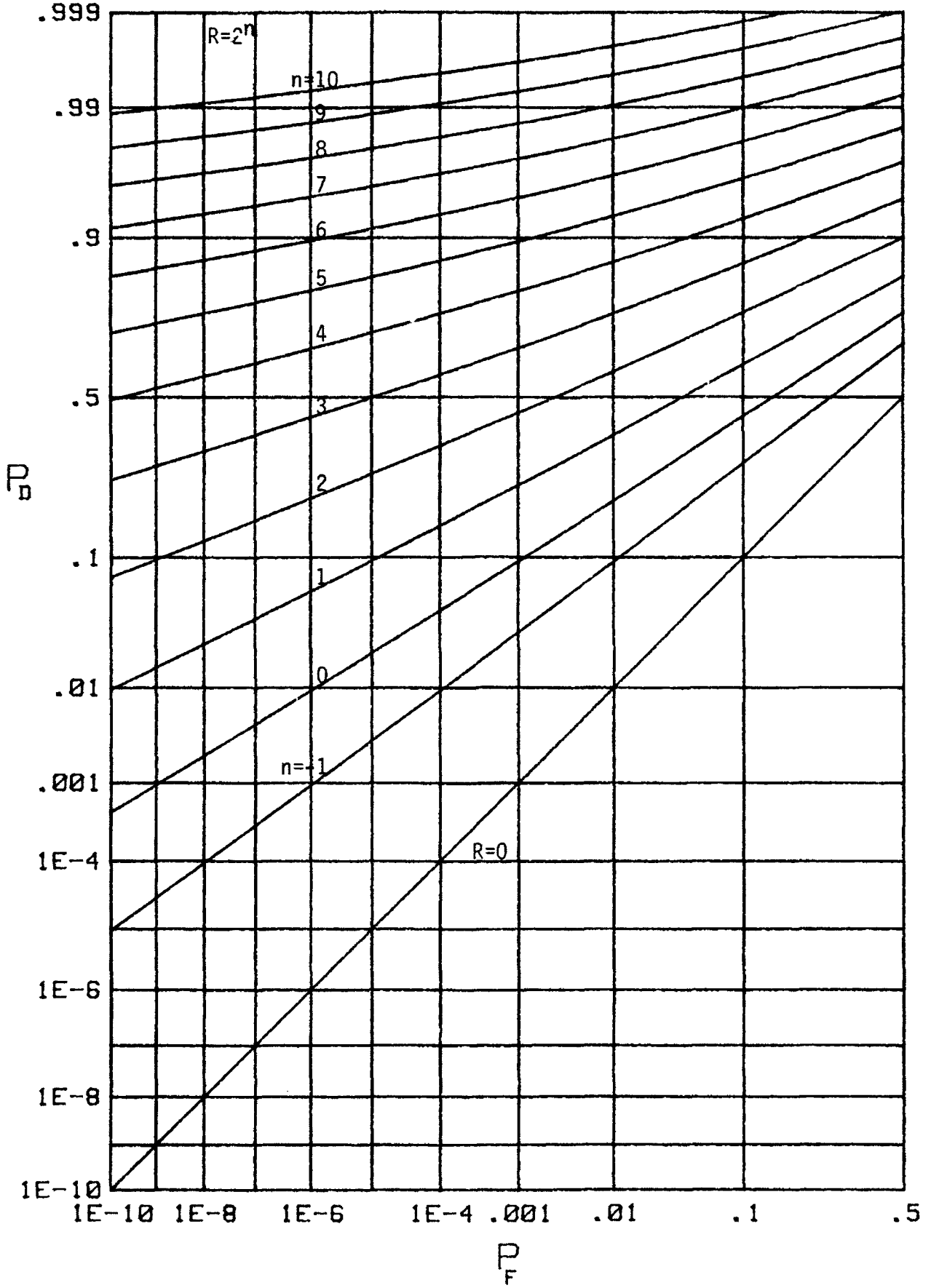

Figure 2. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 3

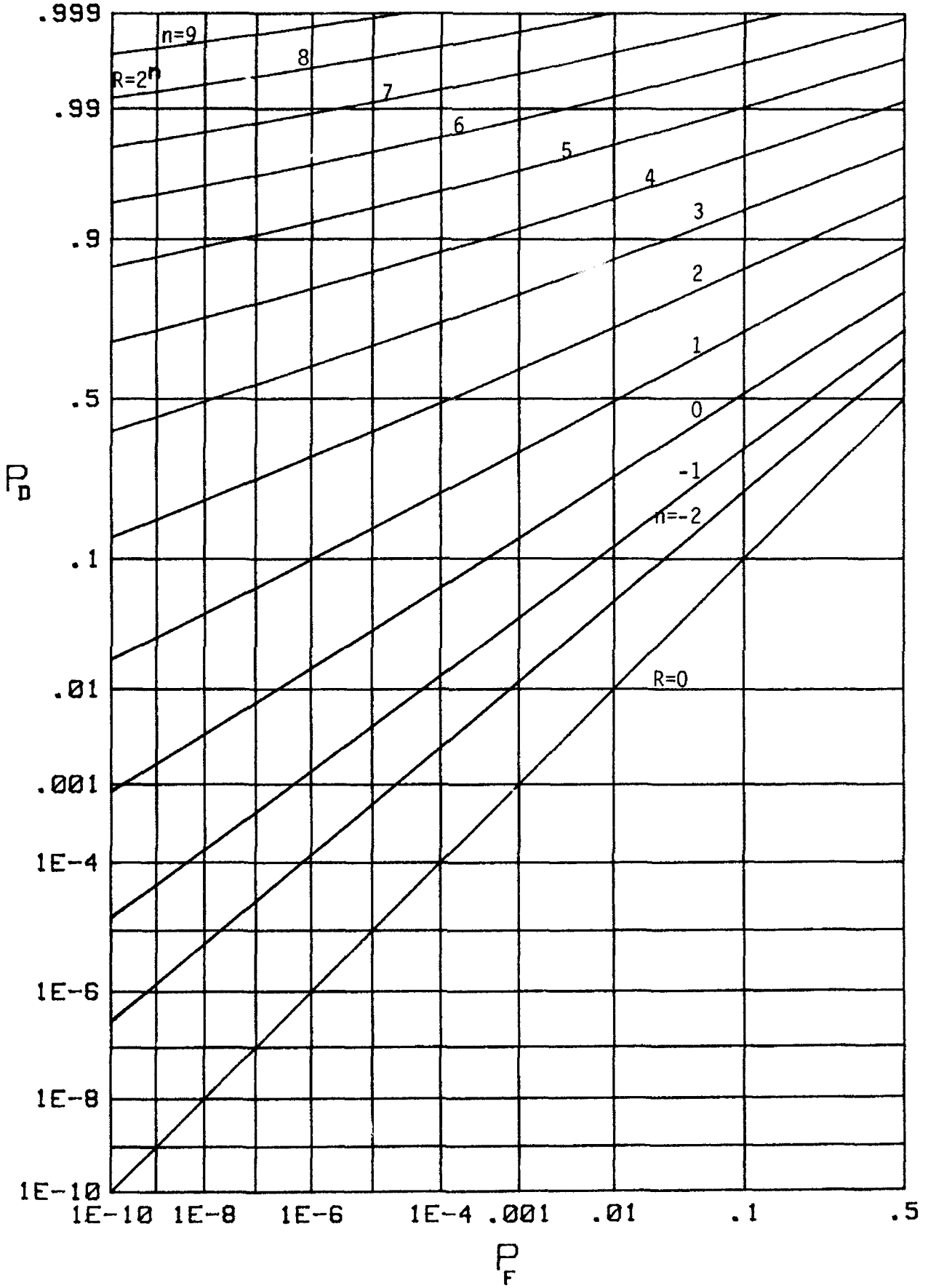

Figure 3. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 4

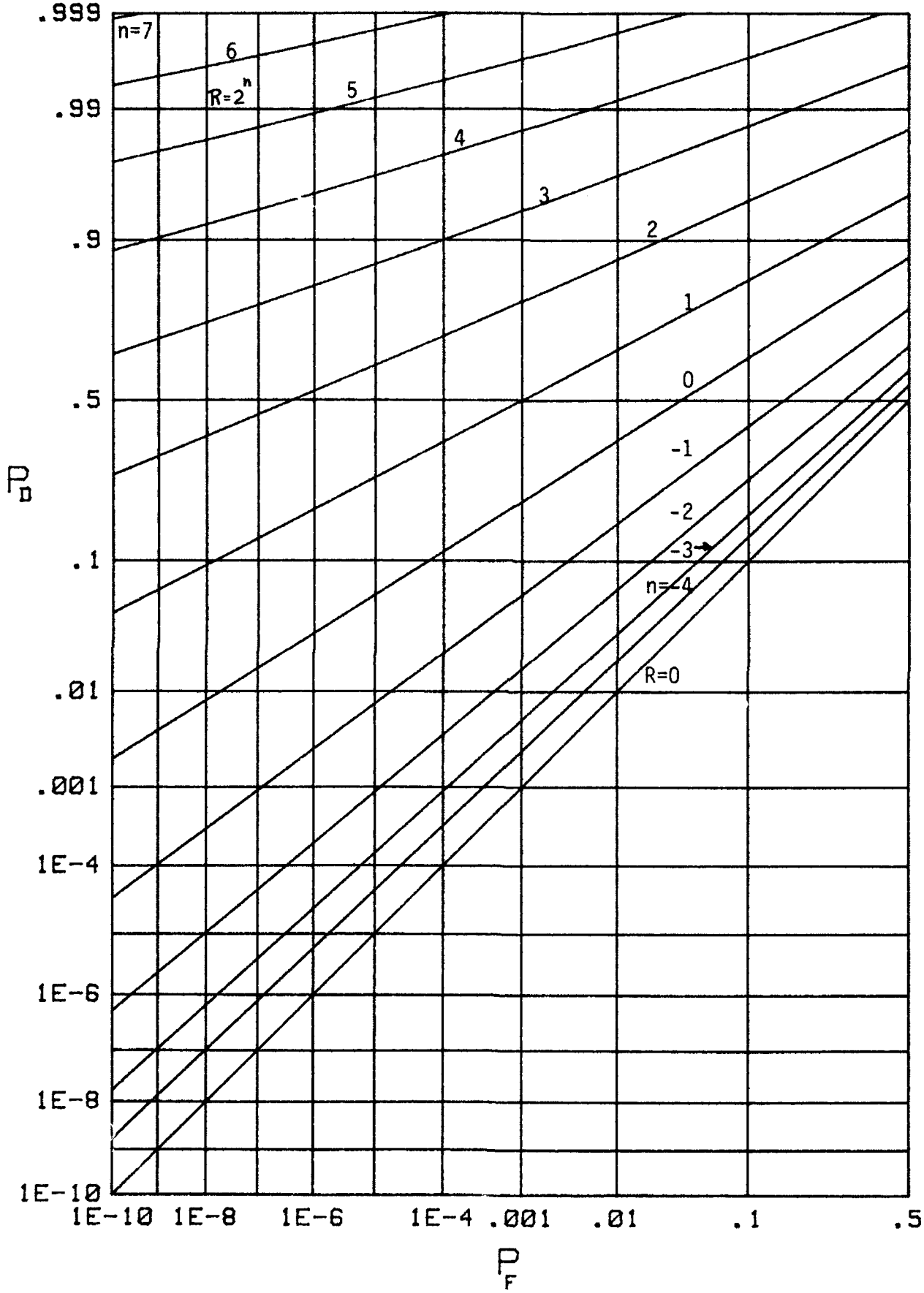

Figure 4. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 6

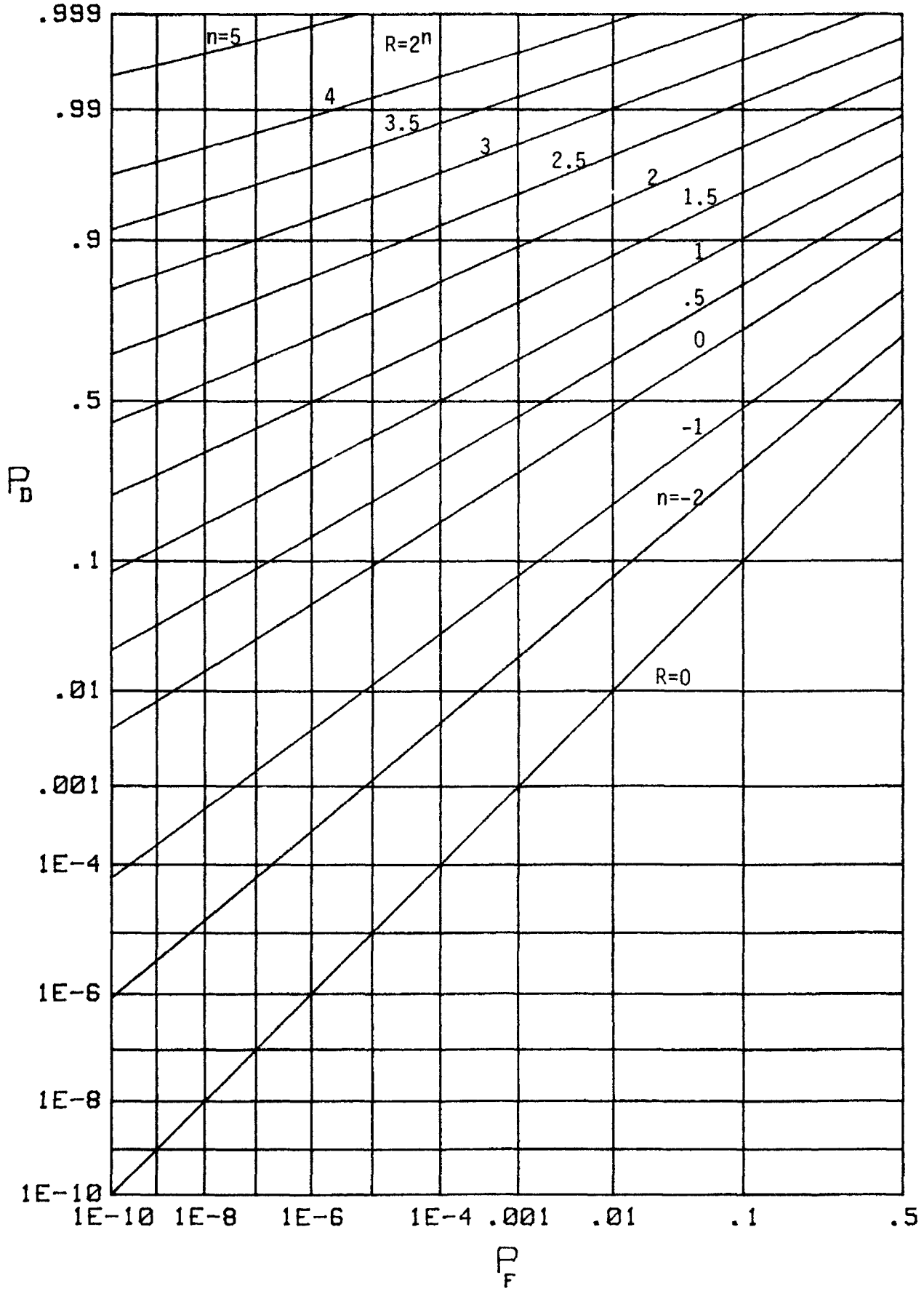

Figure 5. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 8

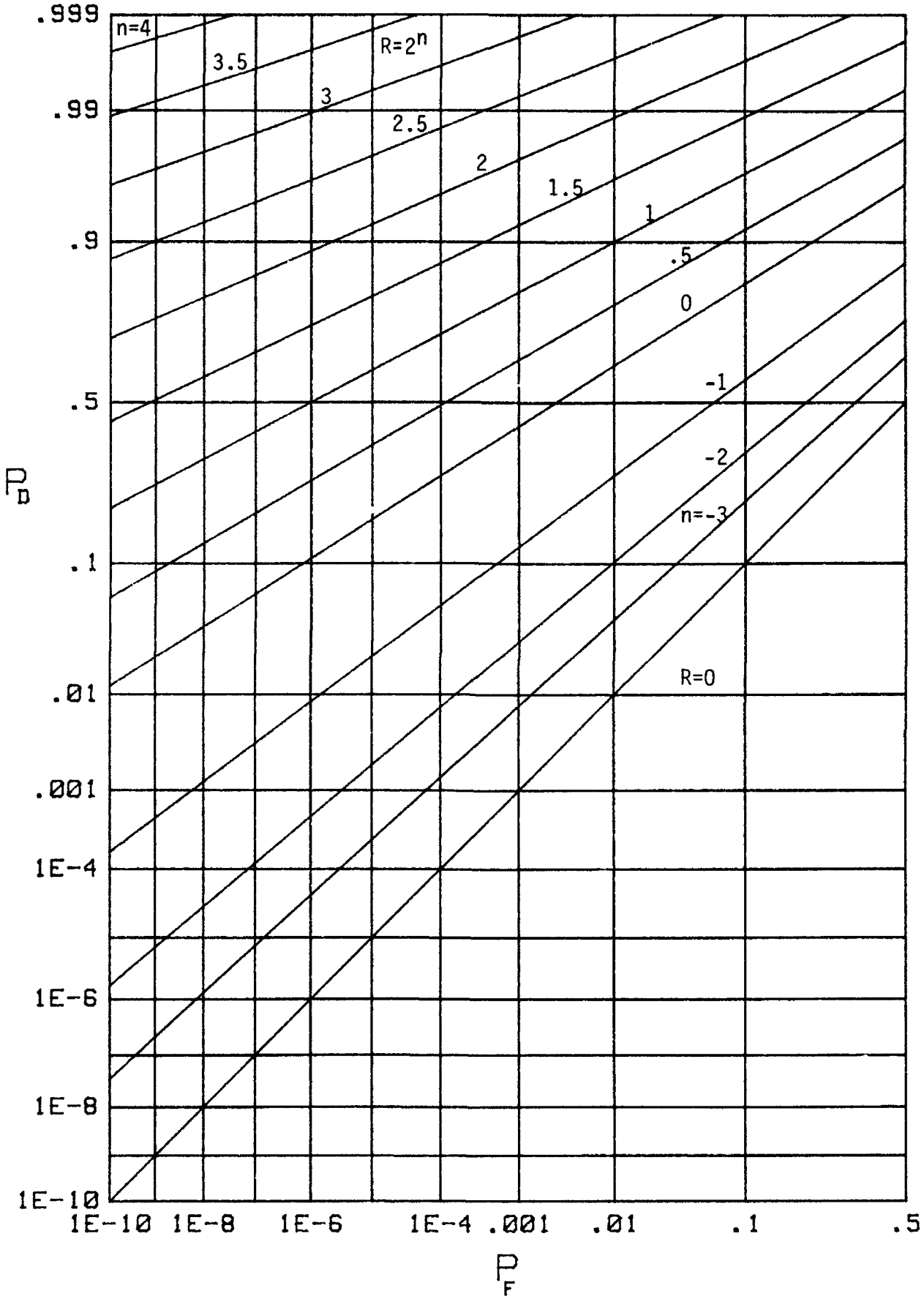

Figure 6. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 12

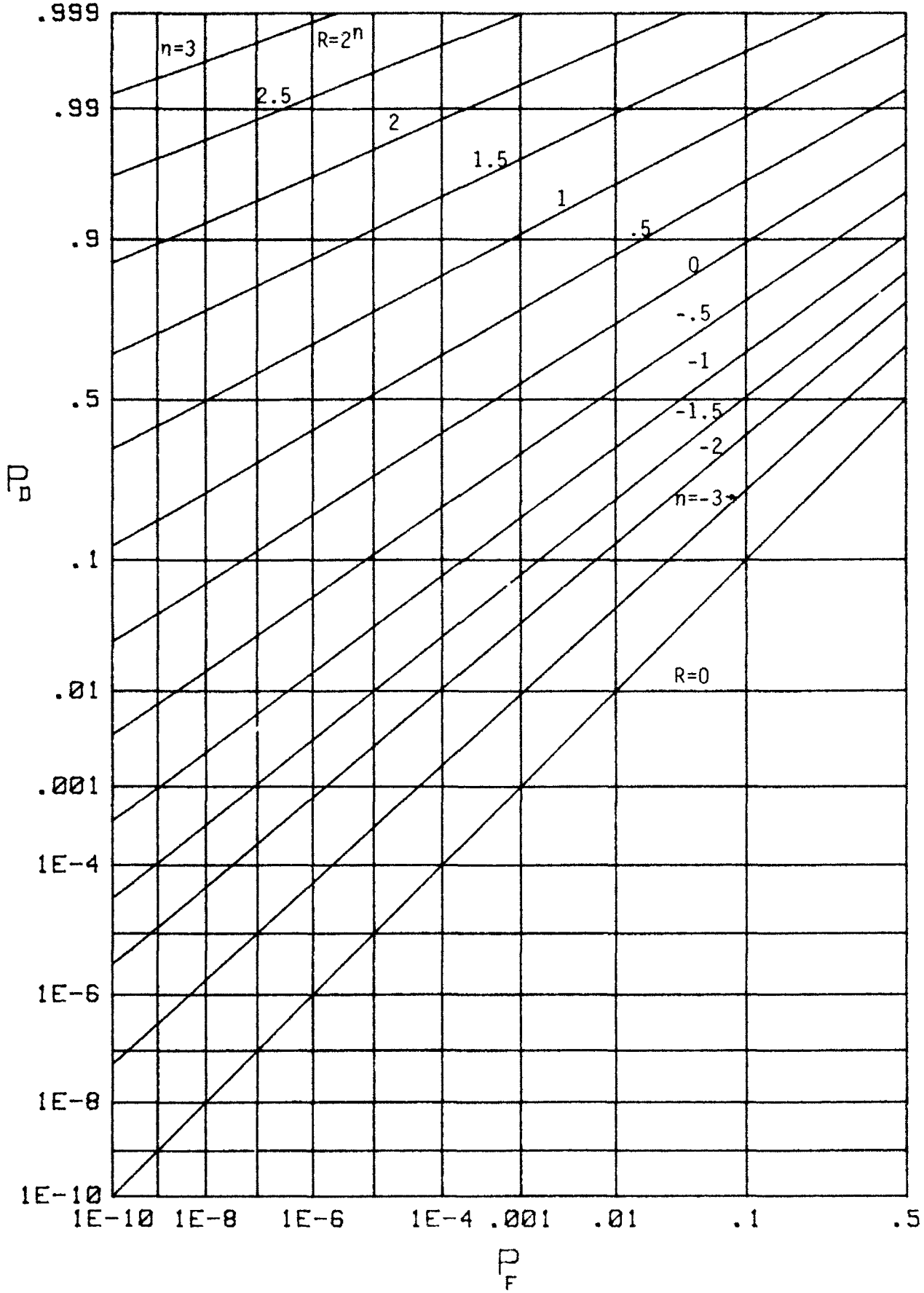

Figure 7. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 16

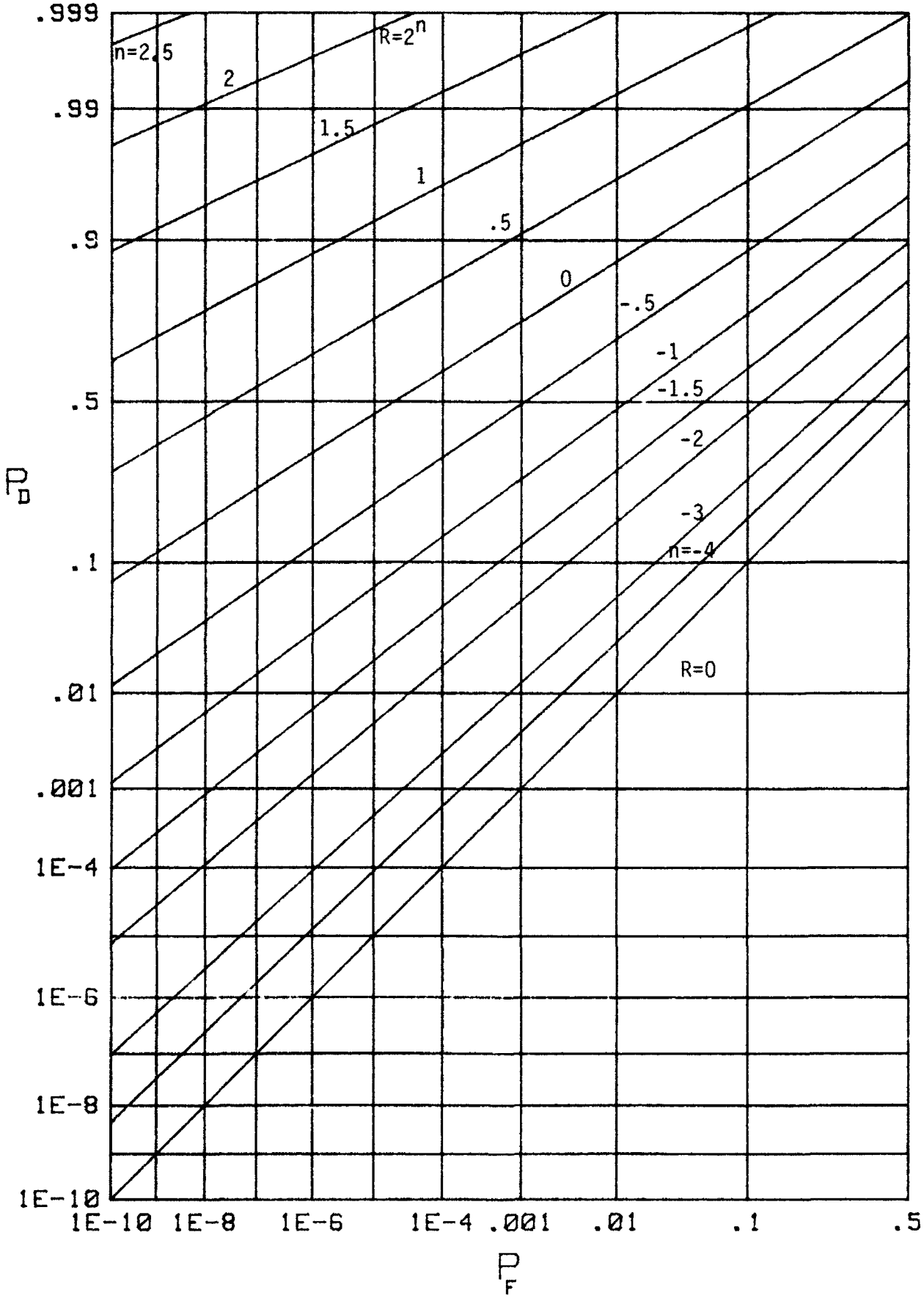

Figure 8. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 24

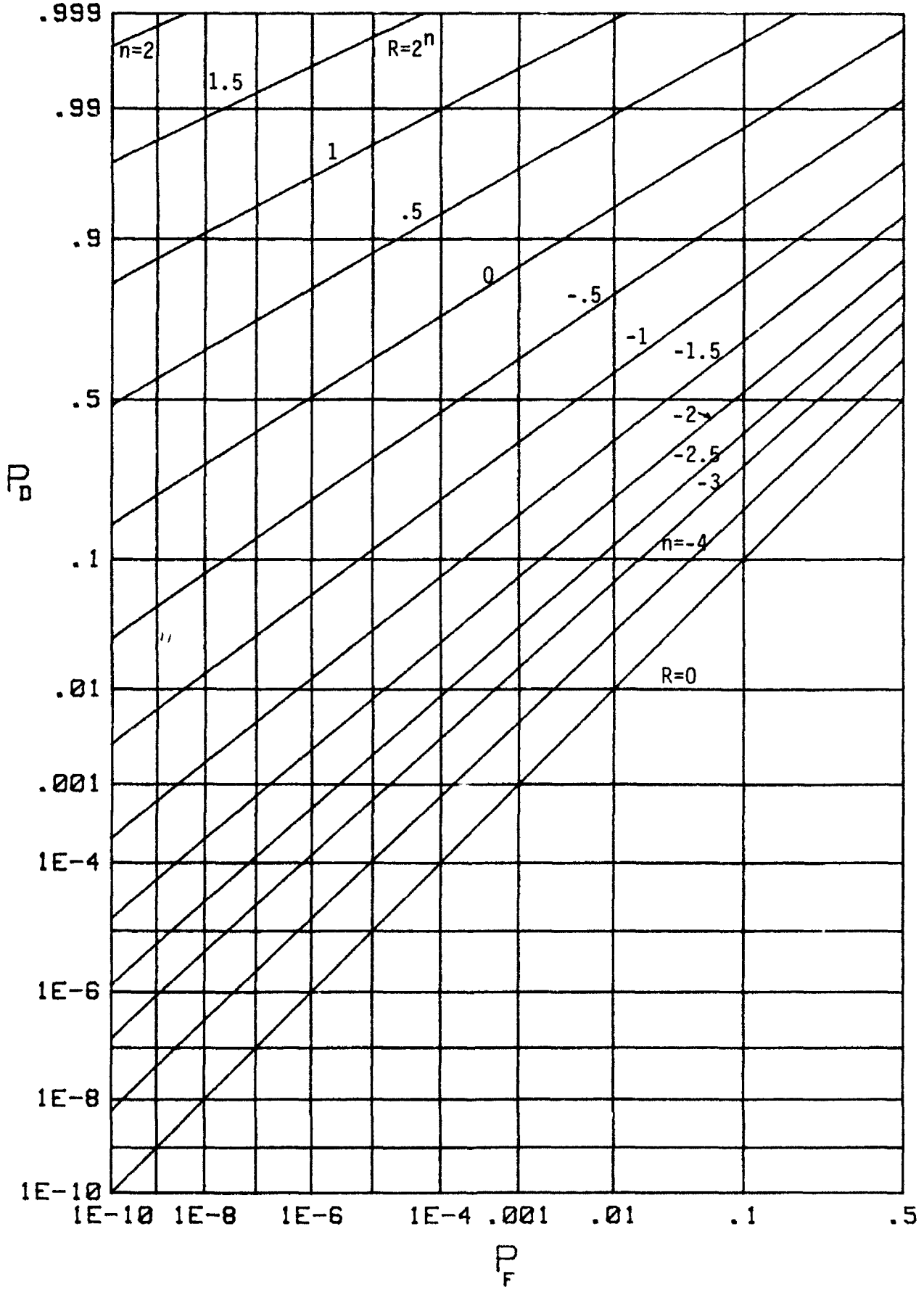

Figure 9. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 32

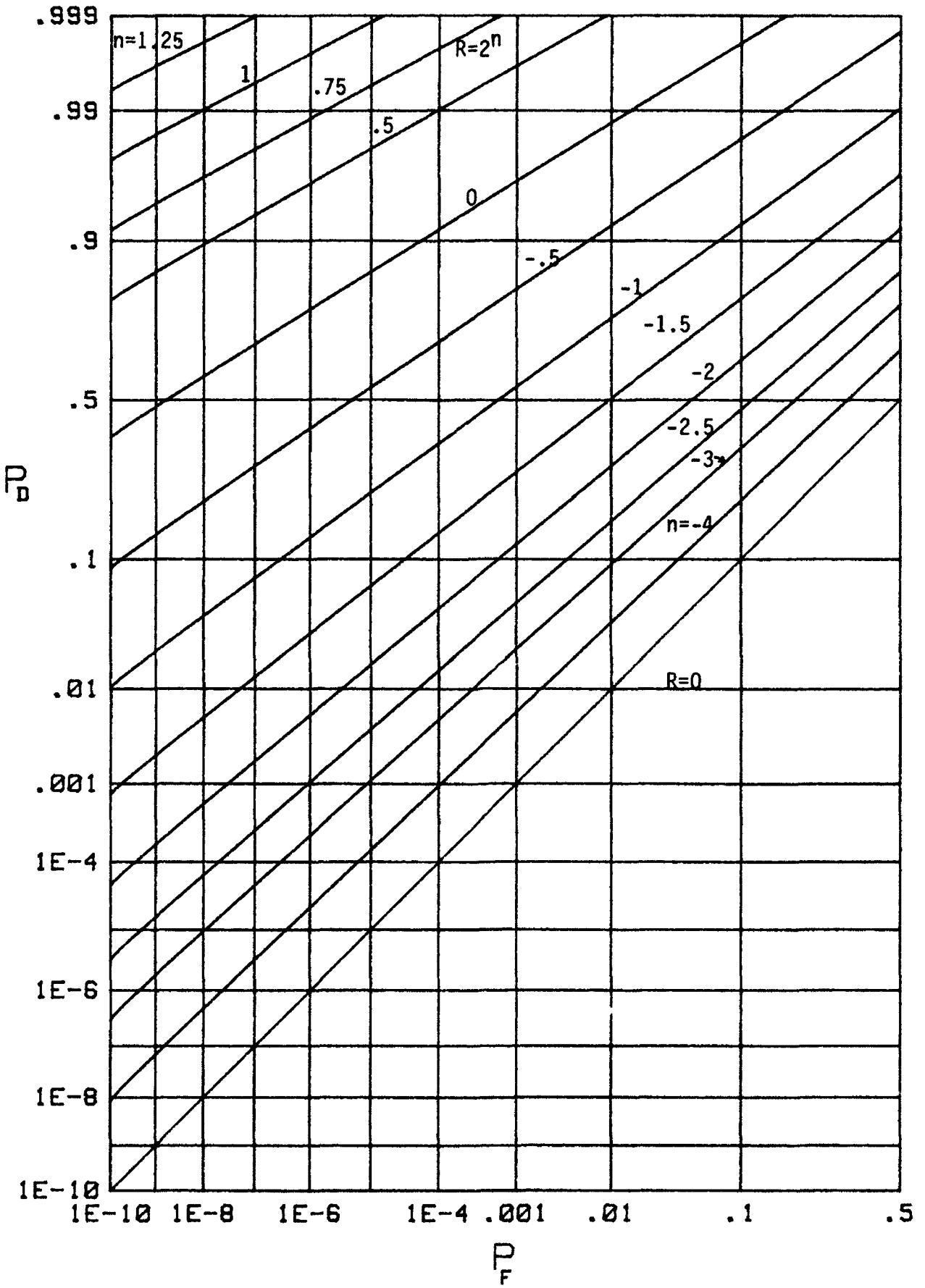

Figure 10. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 48

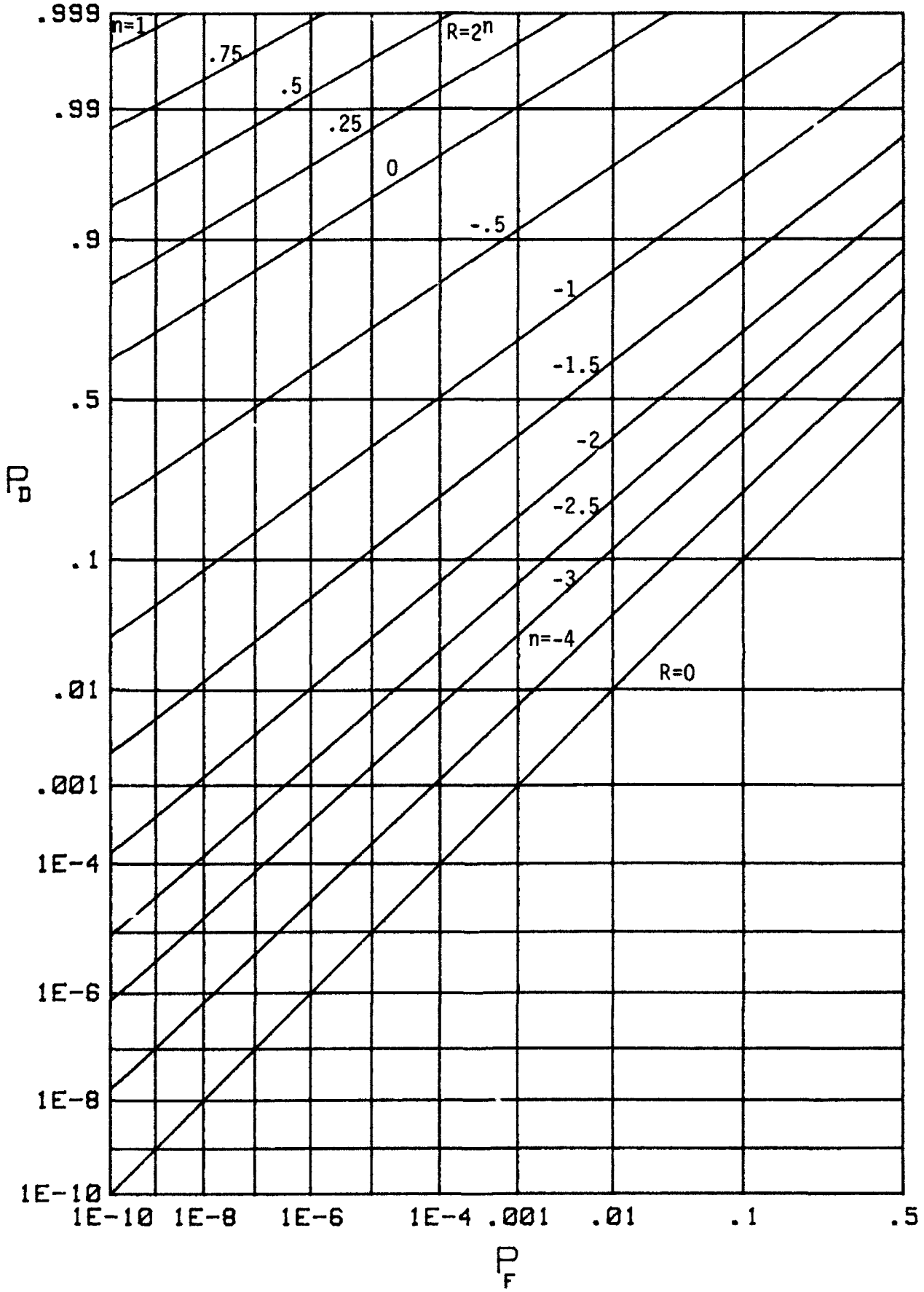

Figure 11. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 64

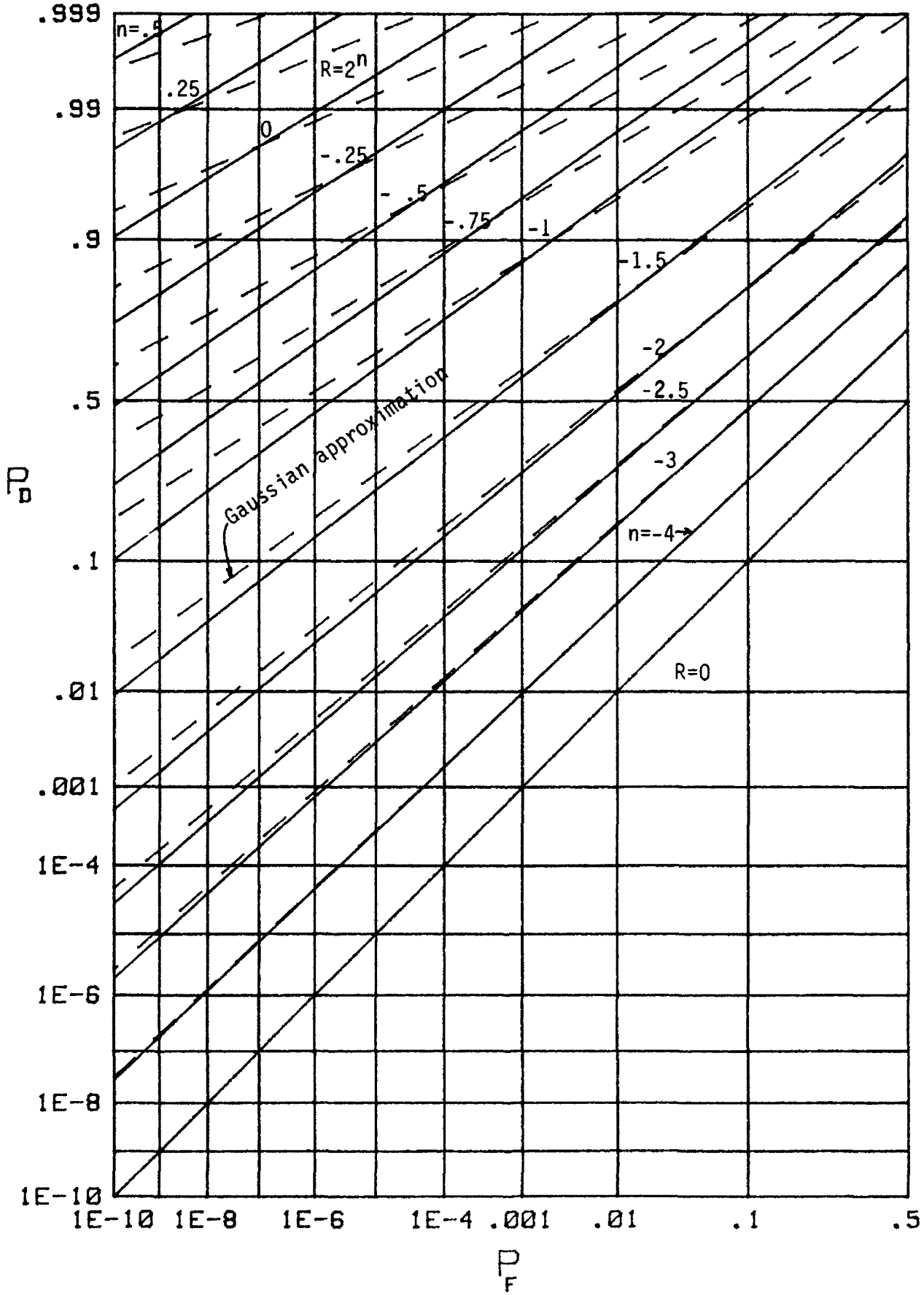

Figure 12. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 96

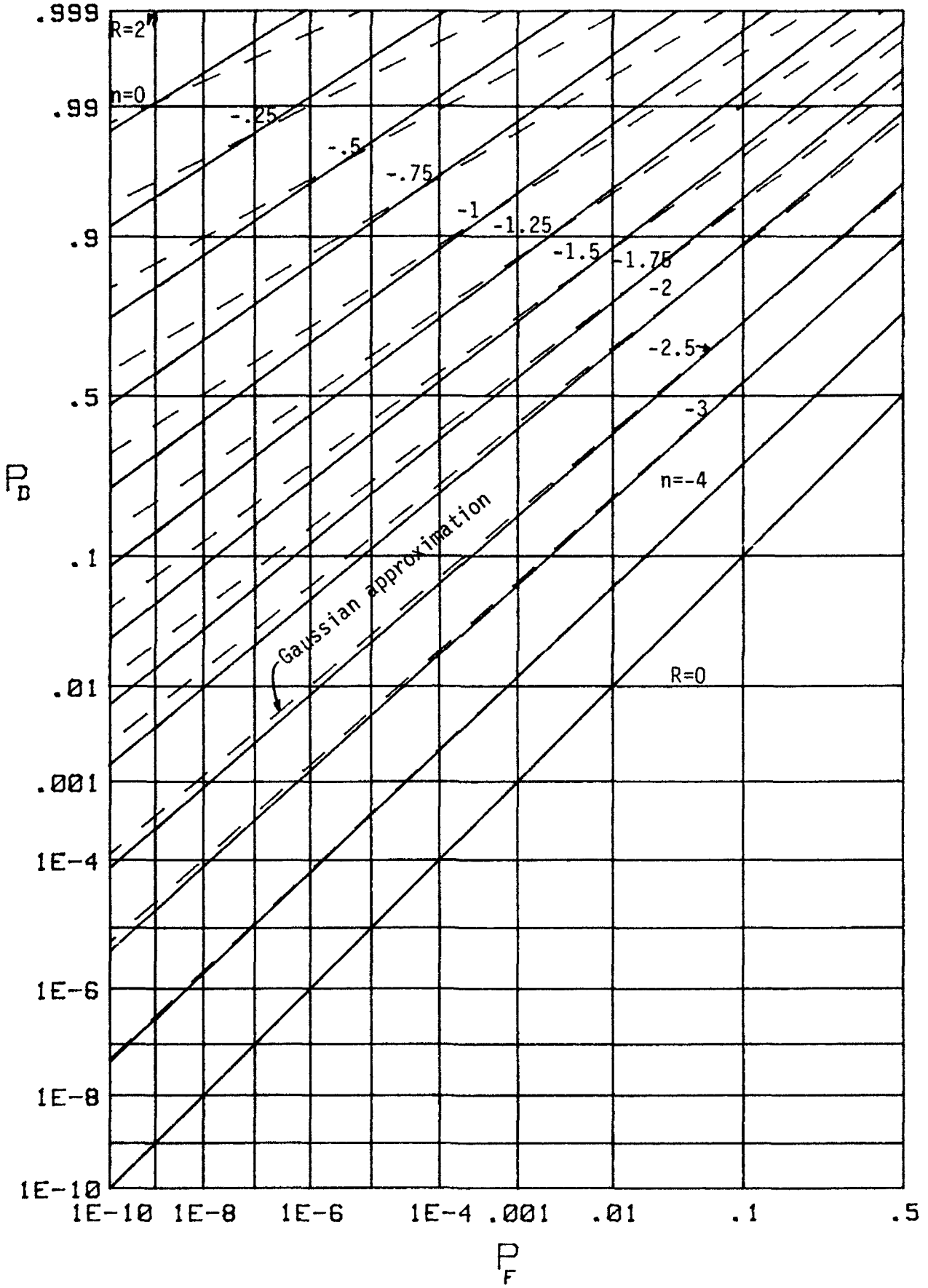

Figure 13. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 128

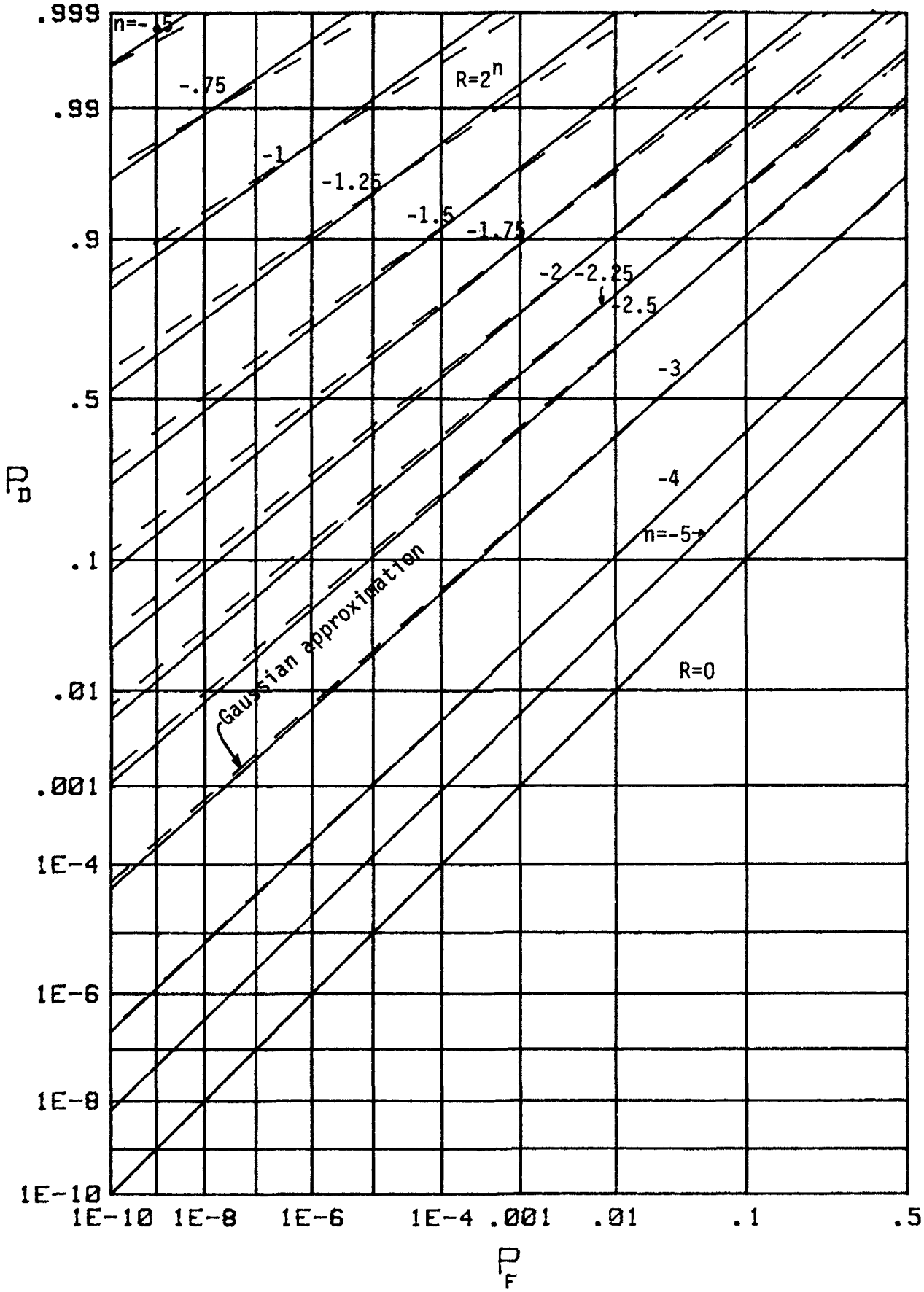

Figure 14. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 256

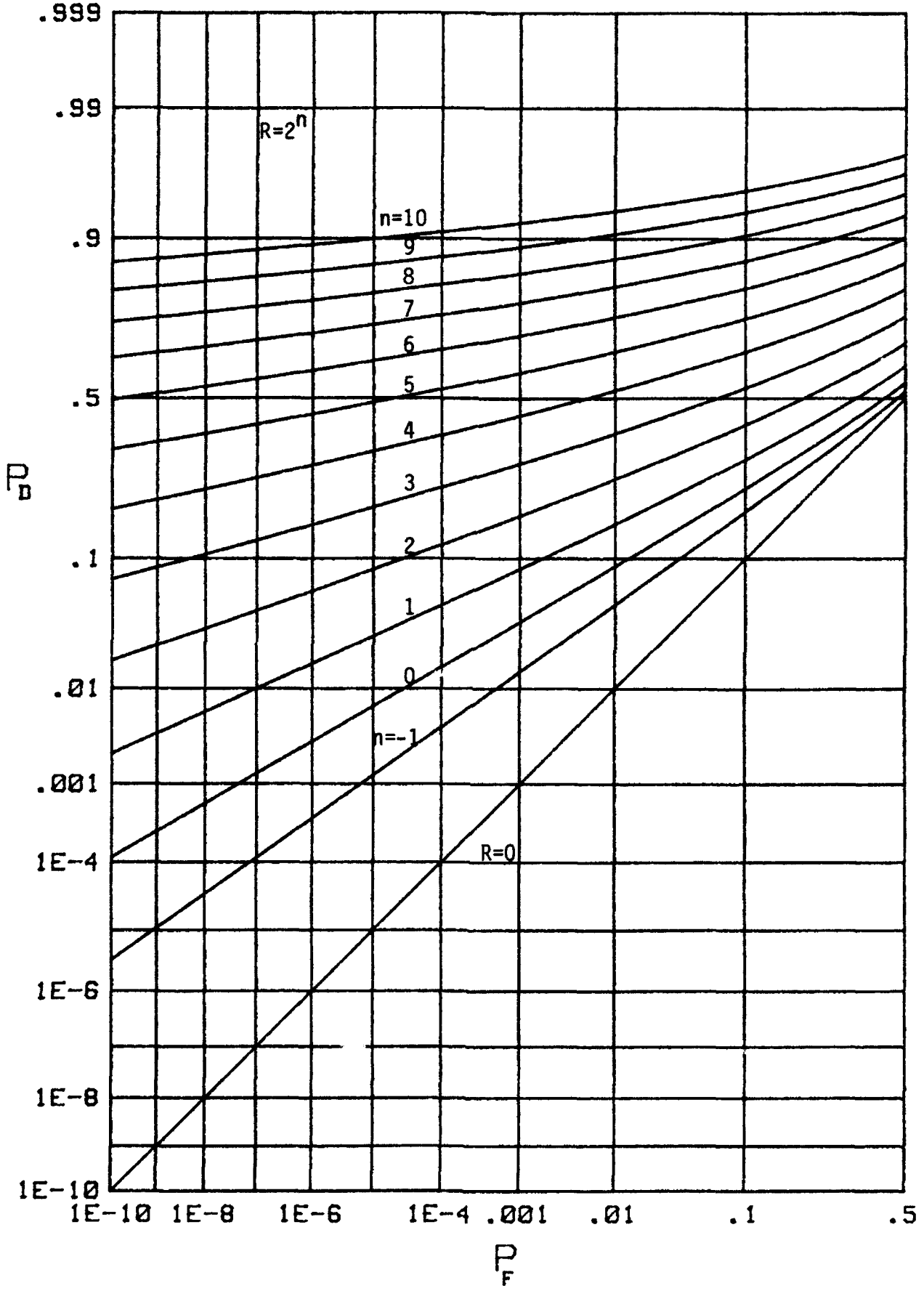

Figure 15. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=1, r=1

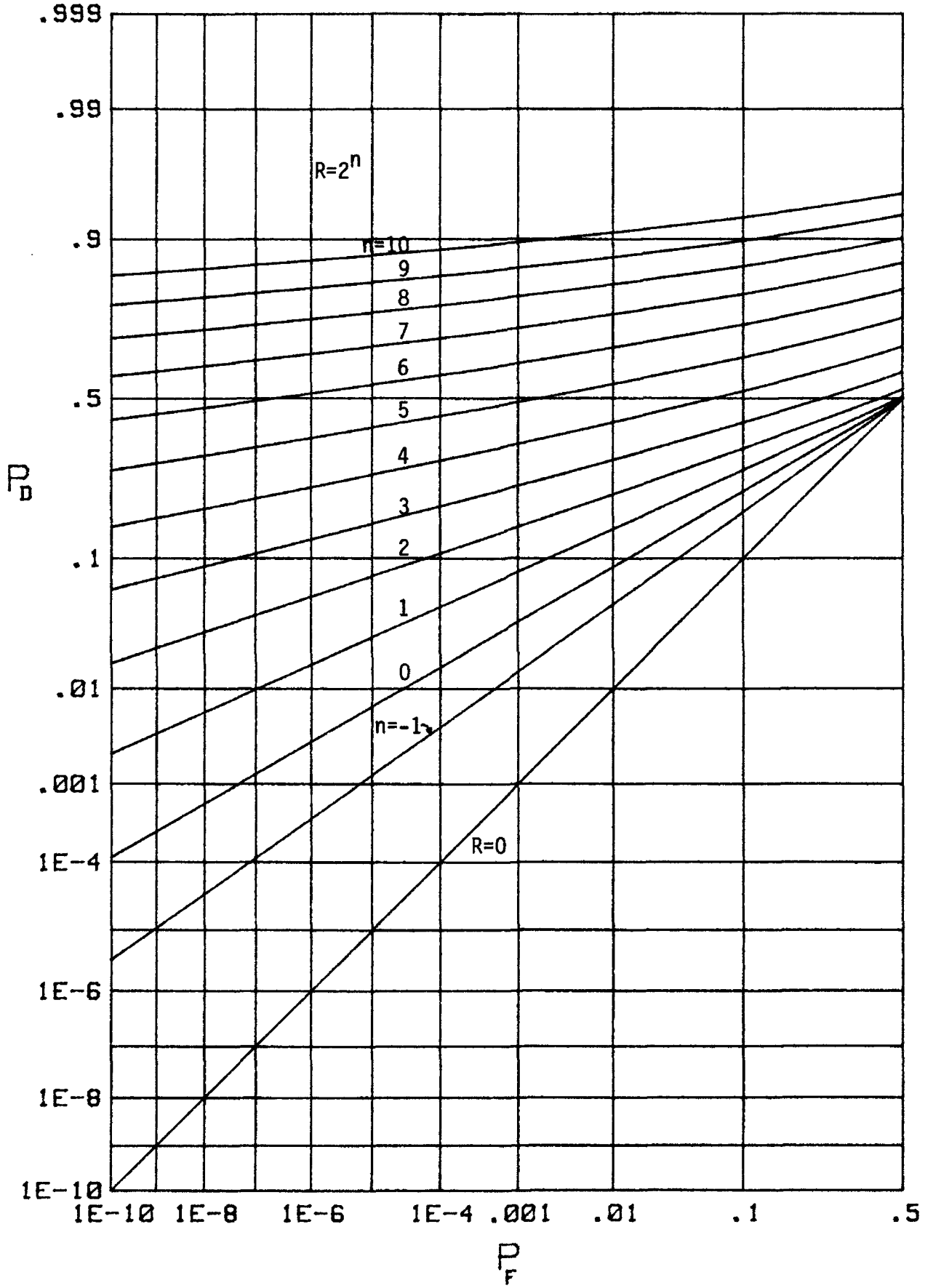

Figure 16. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=1, r=2

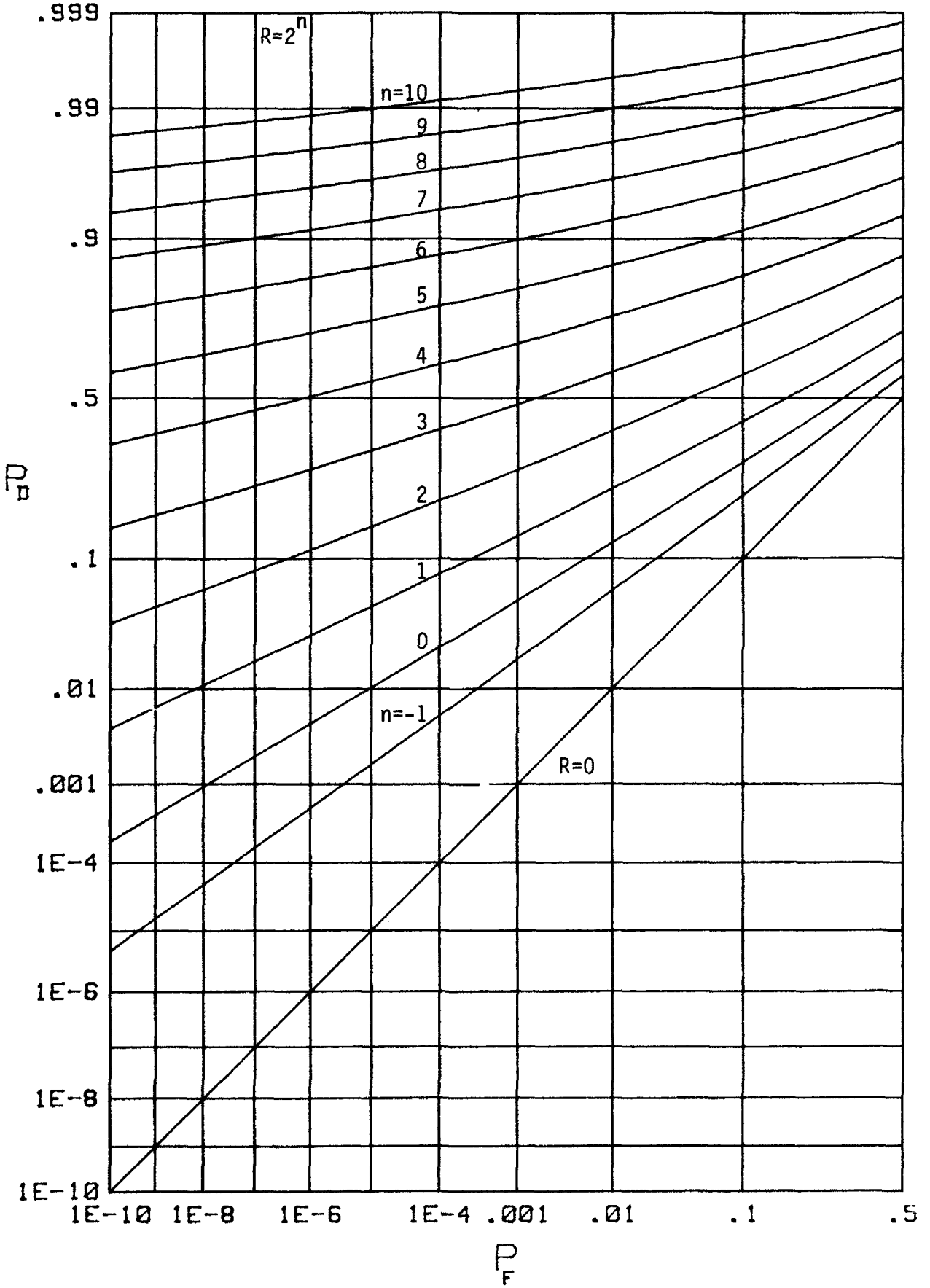

Figure 17. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=1

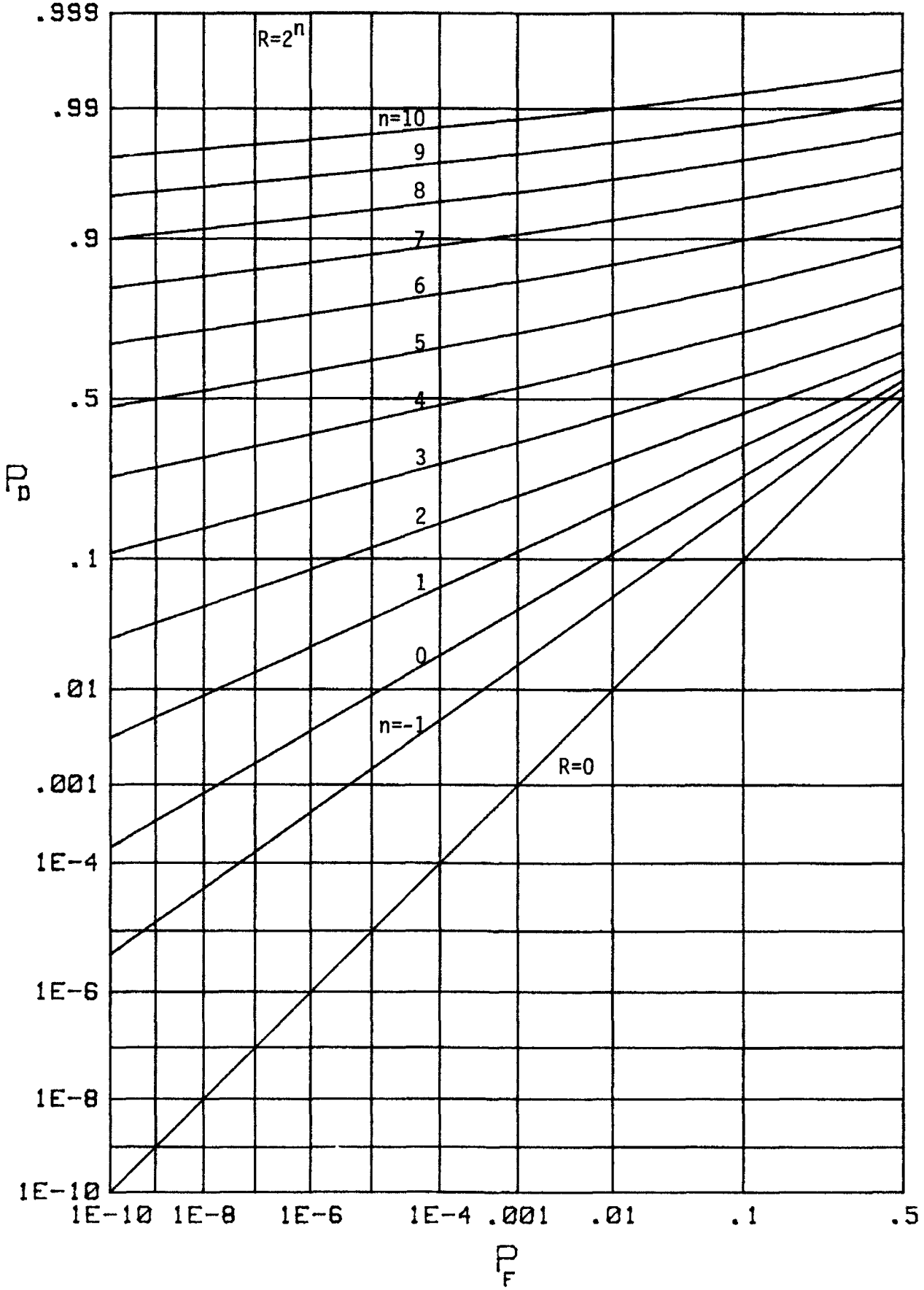

Figure 18. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=2

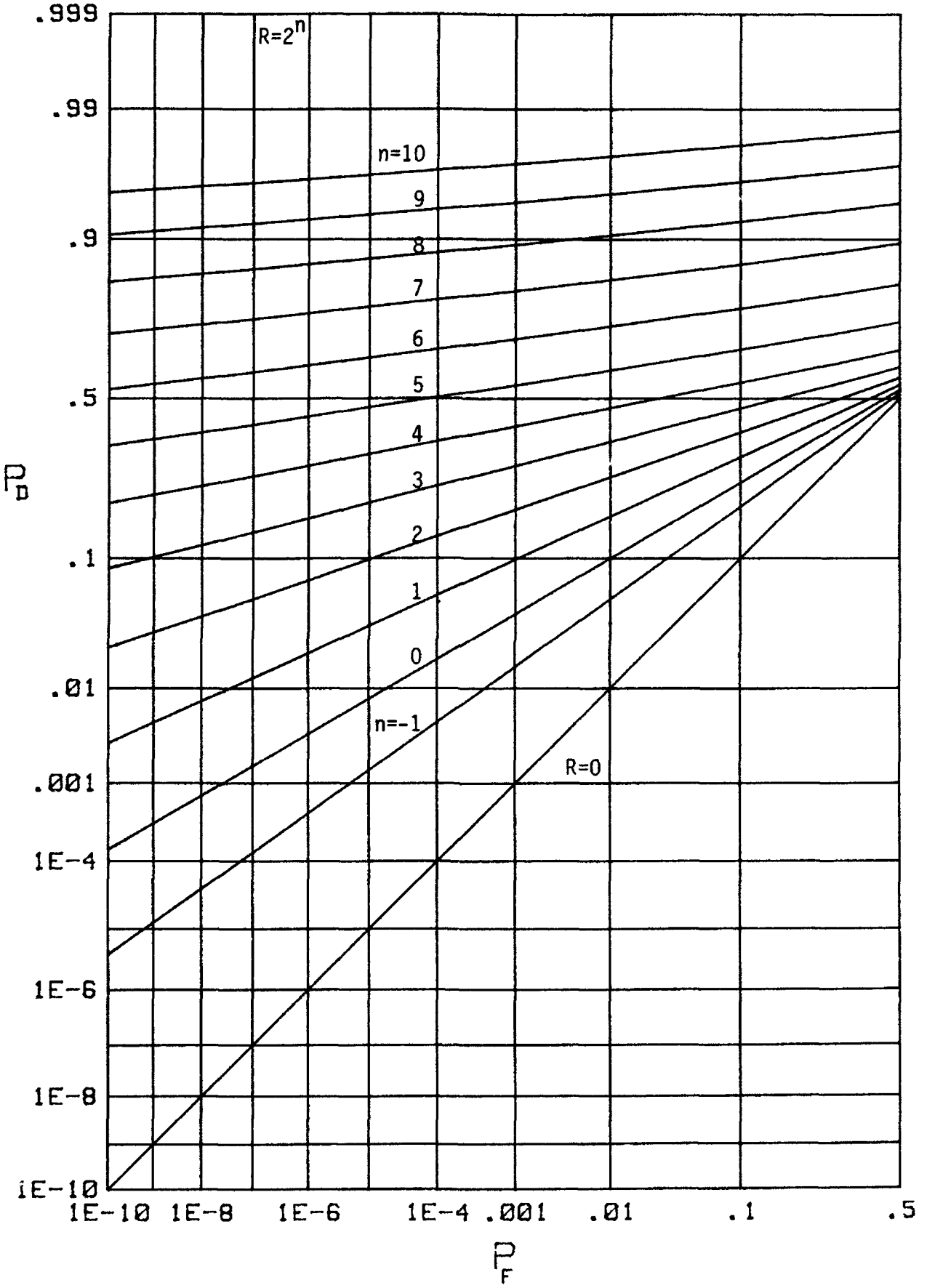

Figure 19. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=4

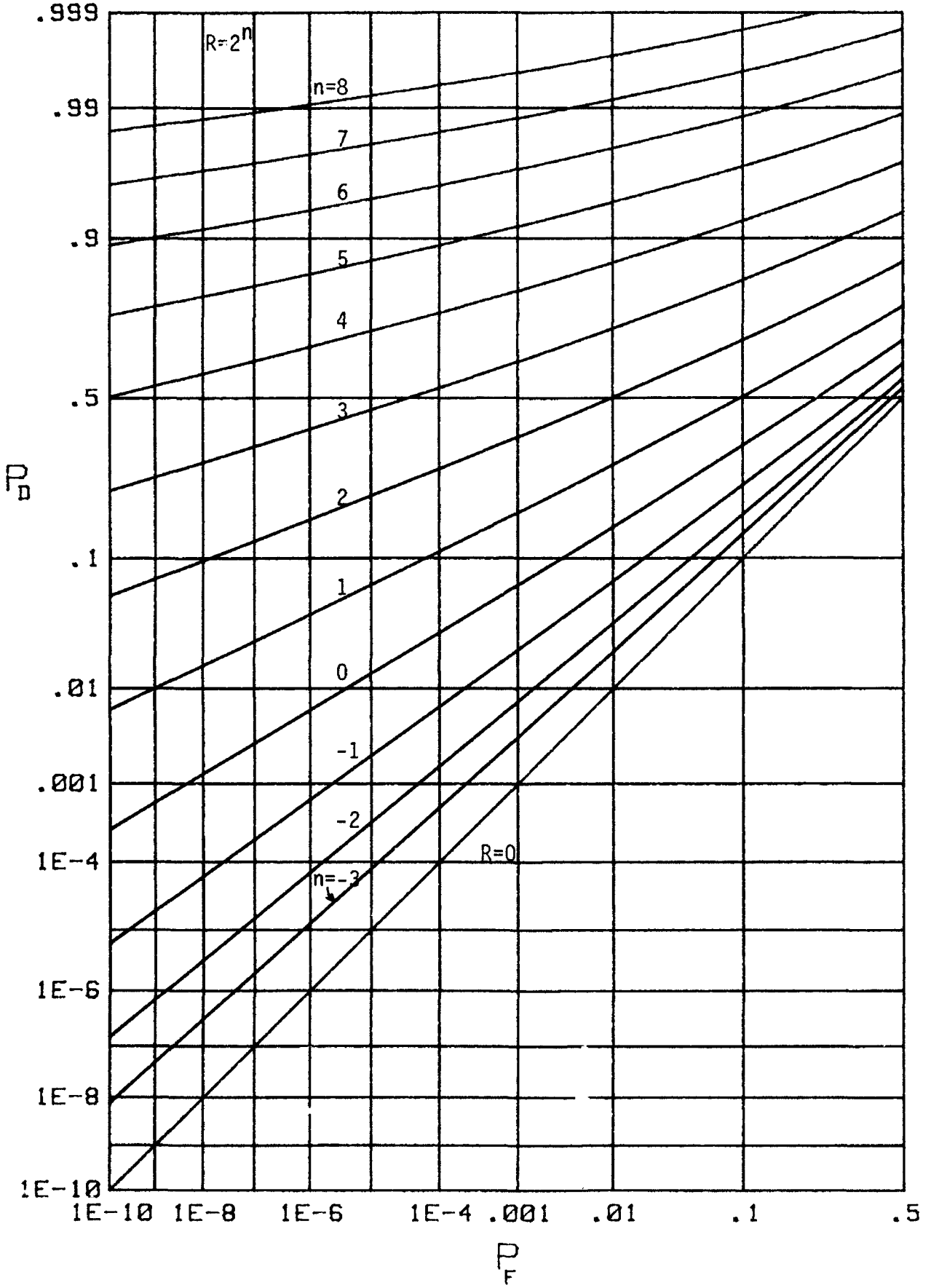

Figure 20. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=3, r=1

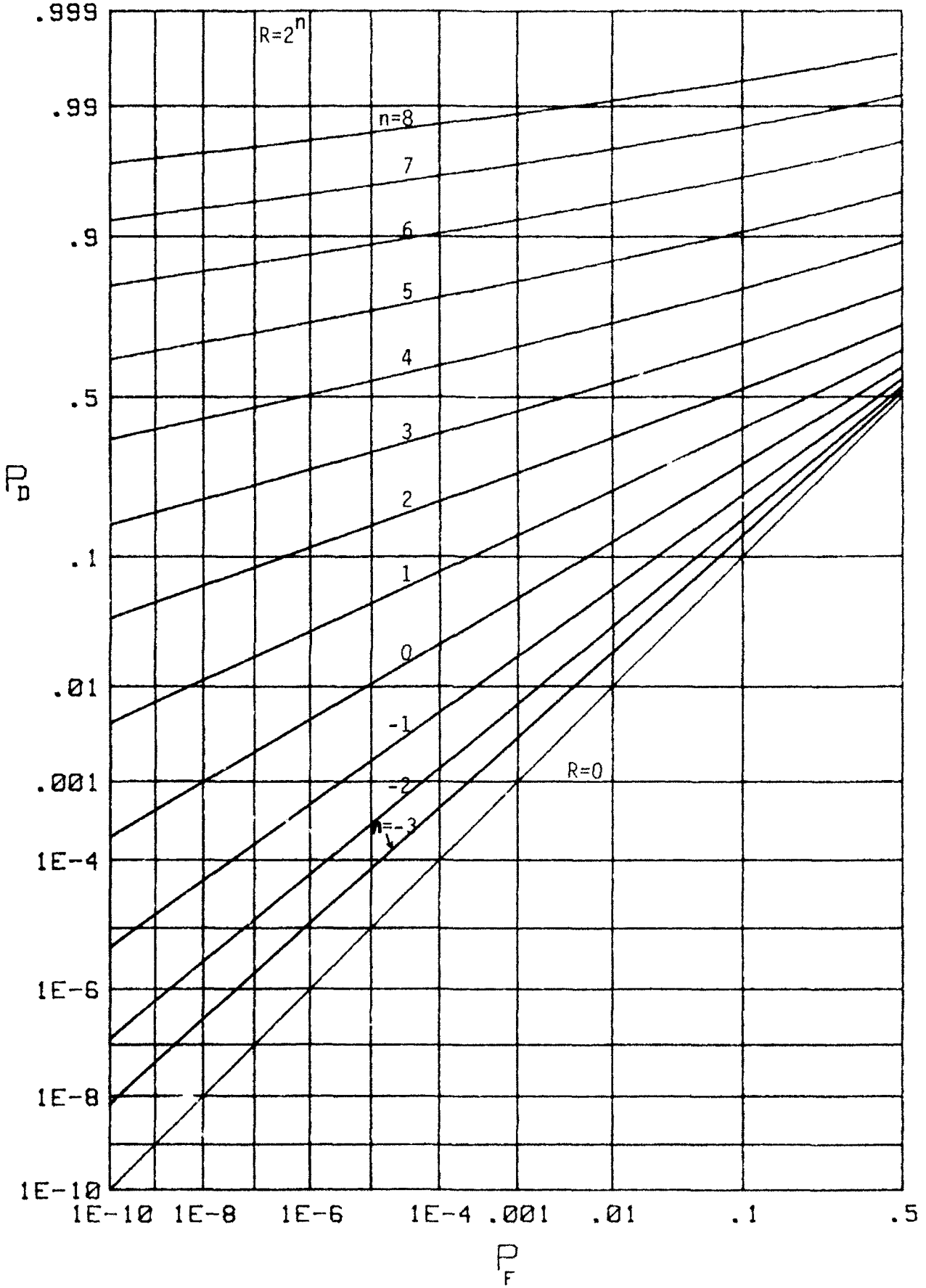

Figure 21. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=3, r=2

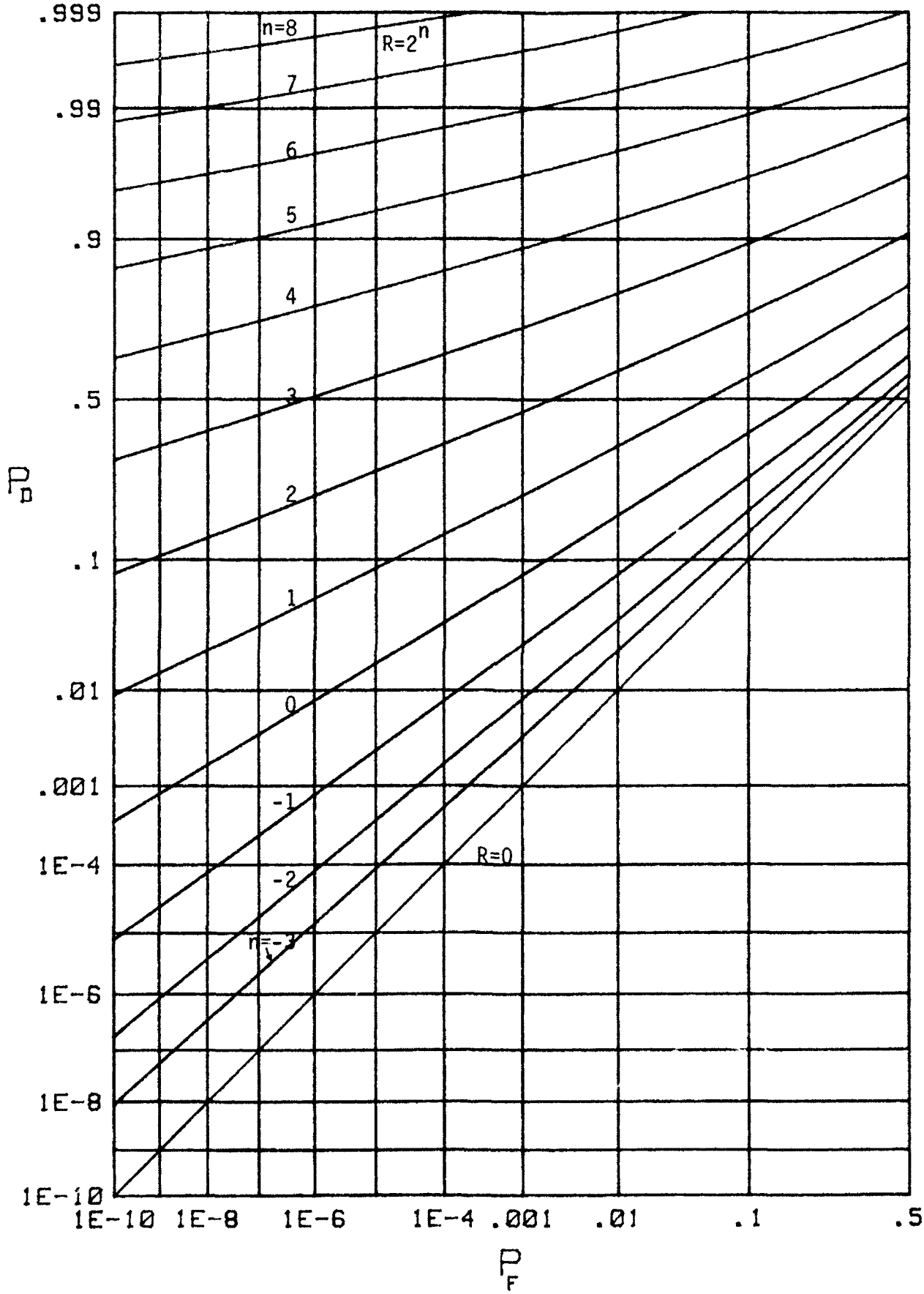

Figure 22. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=4, r=1

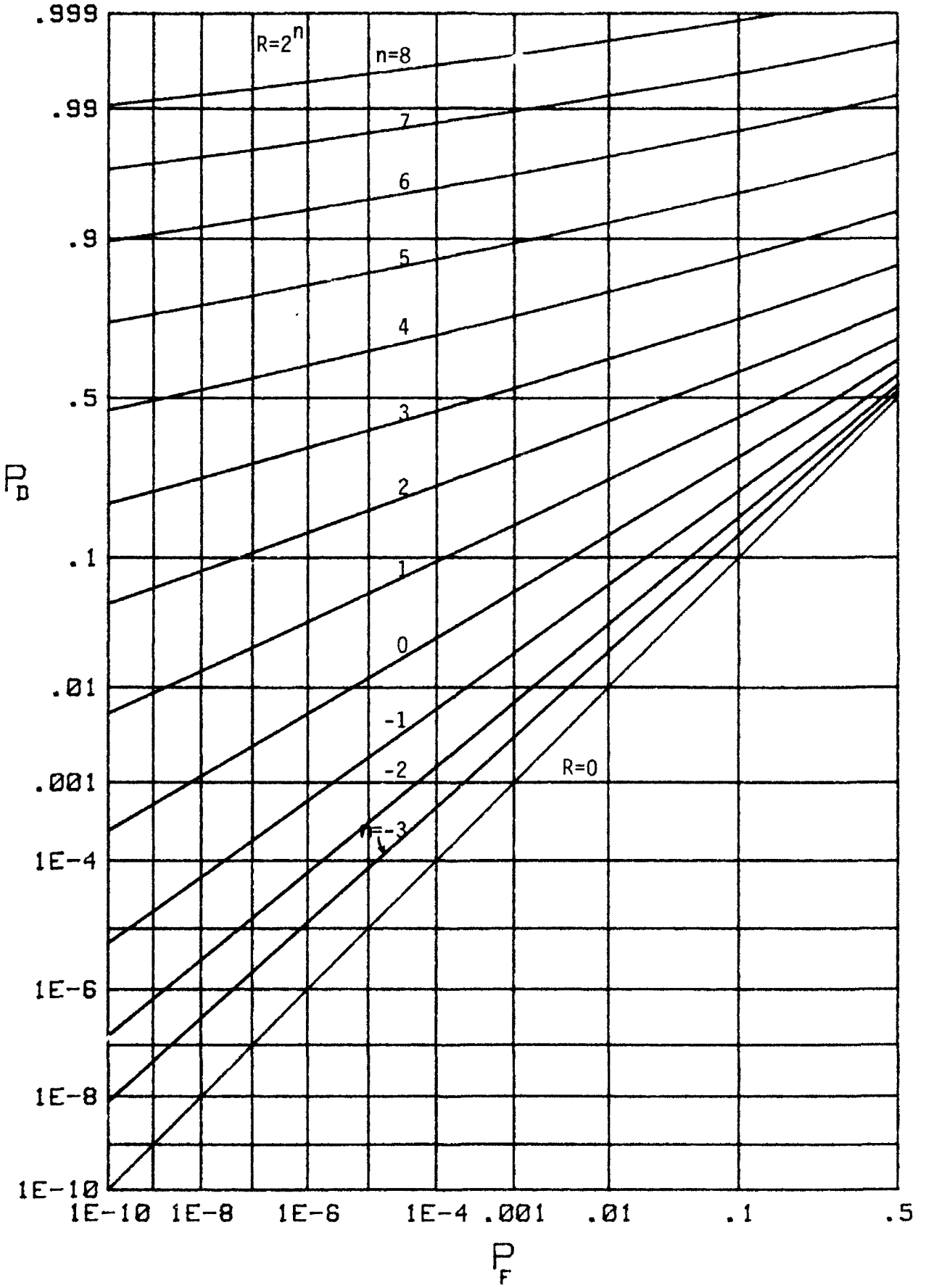

Figure 23. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=4, r=2

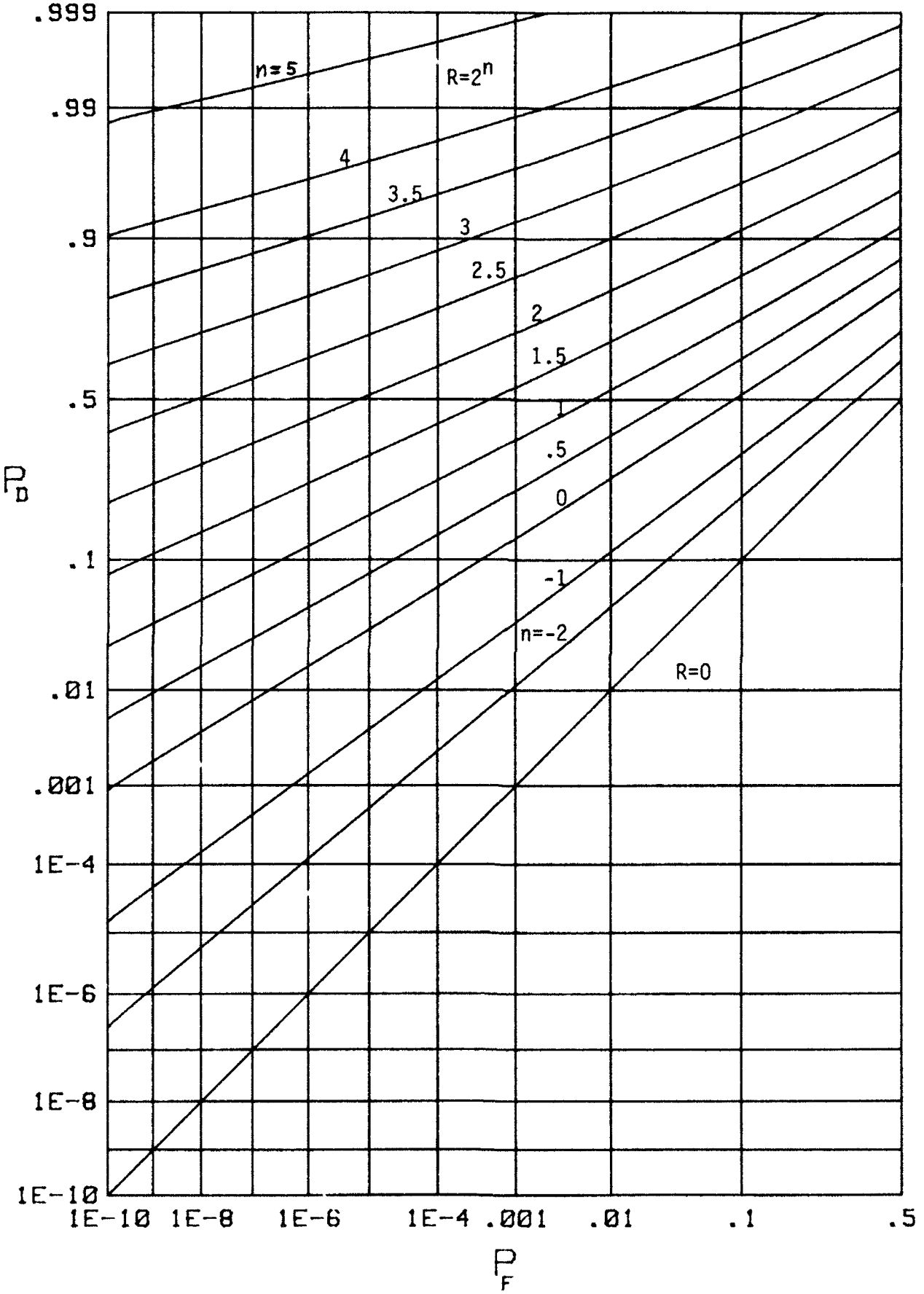

Figure 24. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=8, r=1

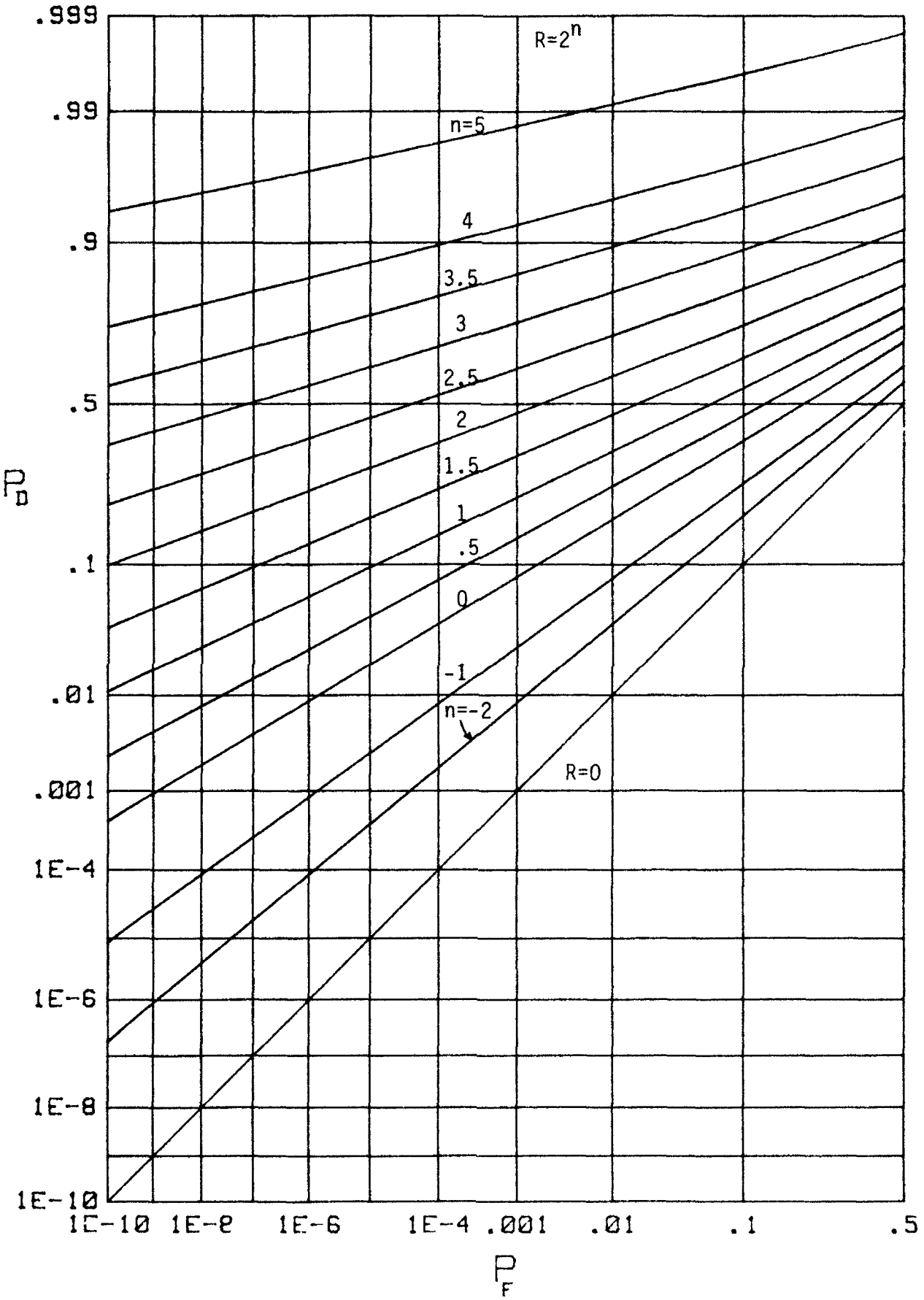

Figure 25. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=8, r=2

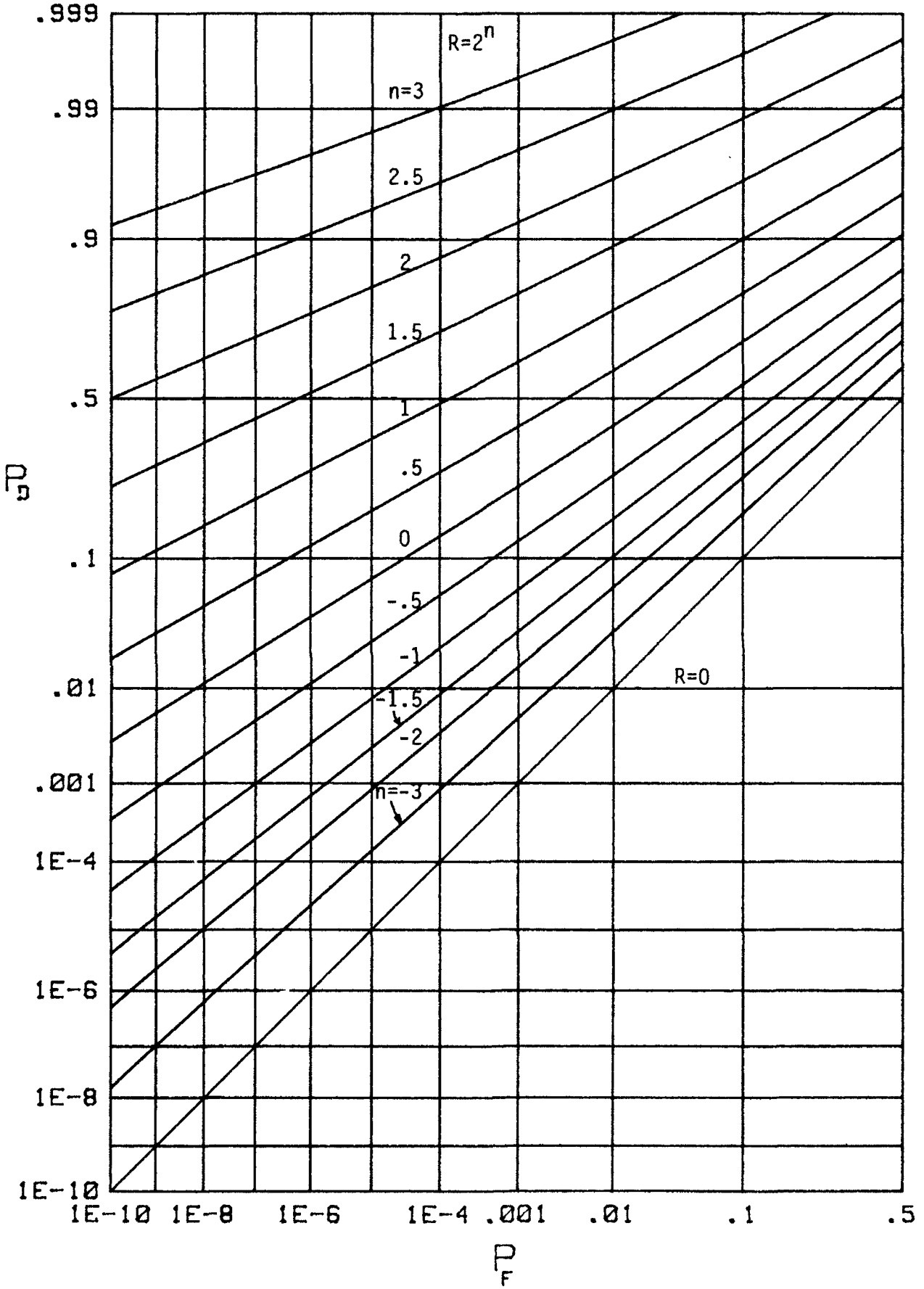

Figure 26. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=16, r=1

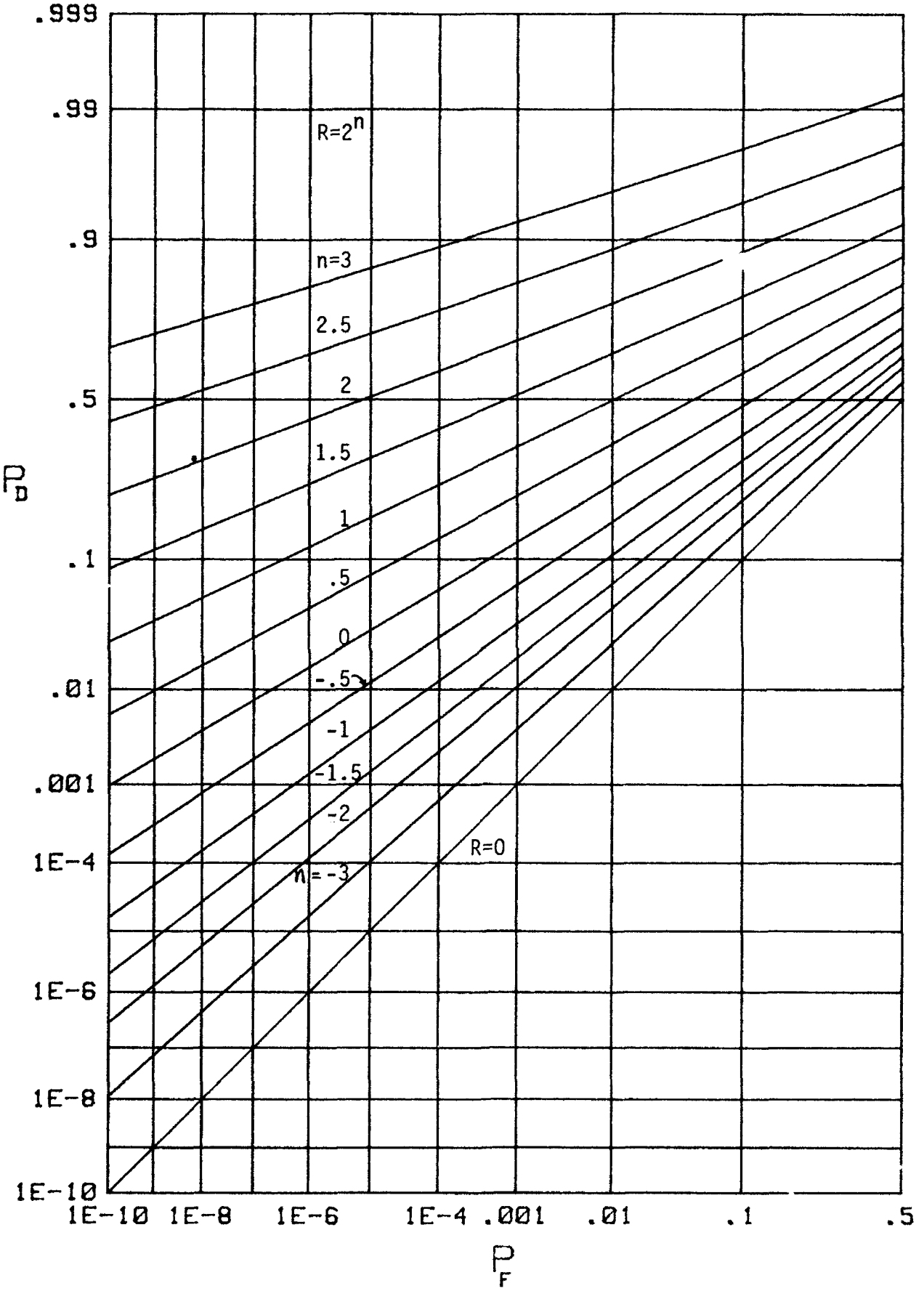

Figure 27. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=16, r=2

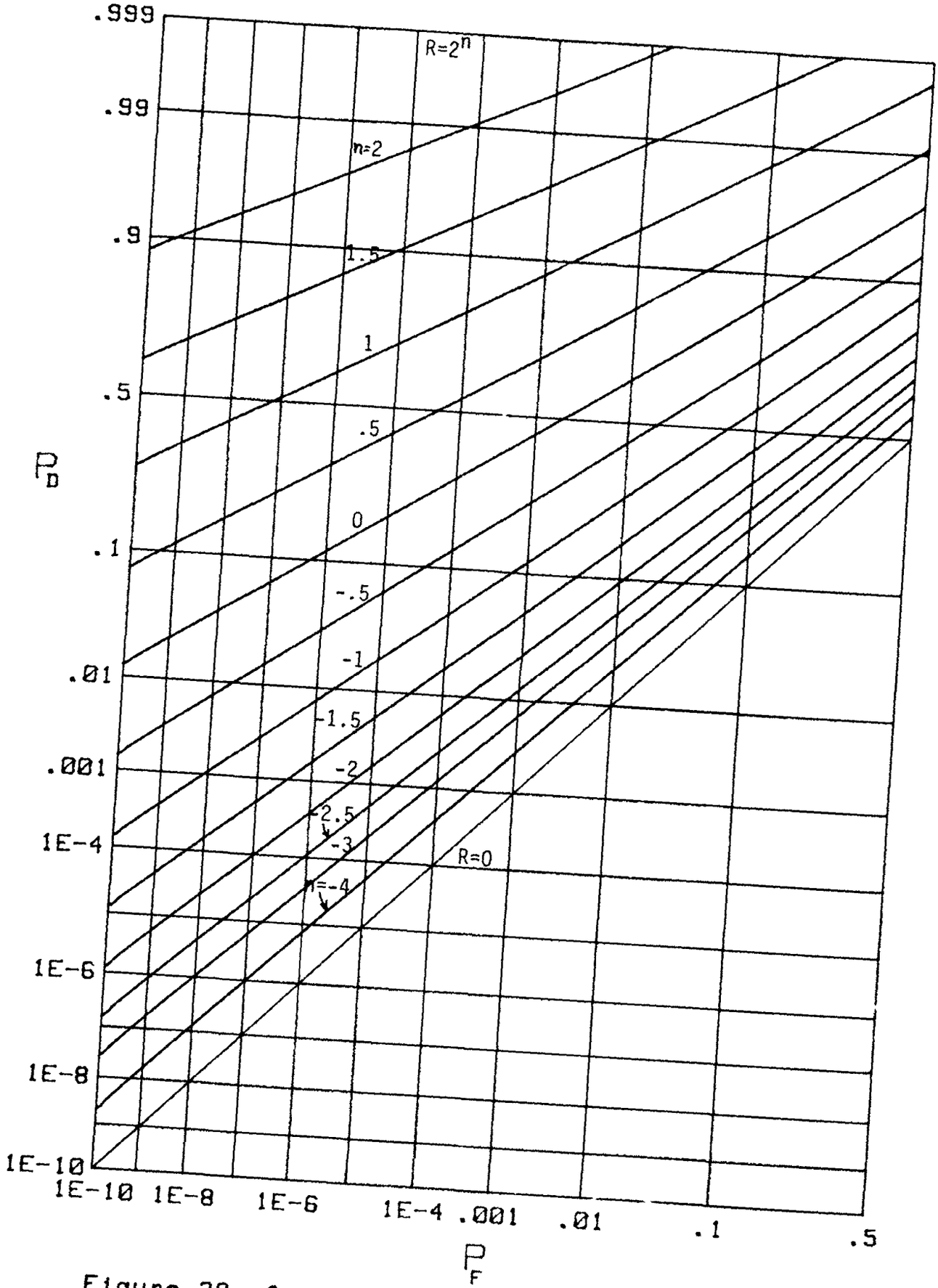

Figure 28. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=32, r=1

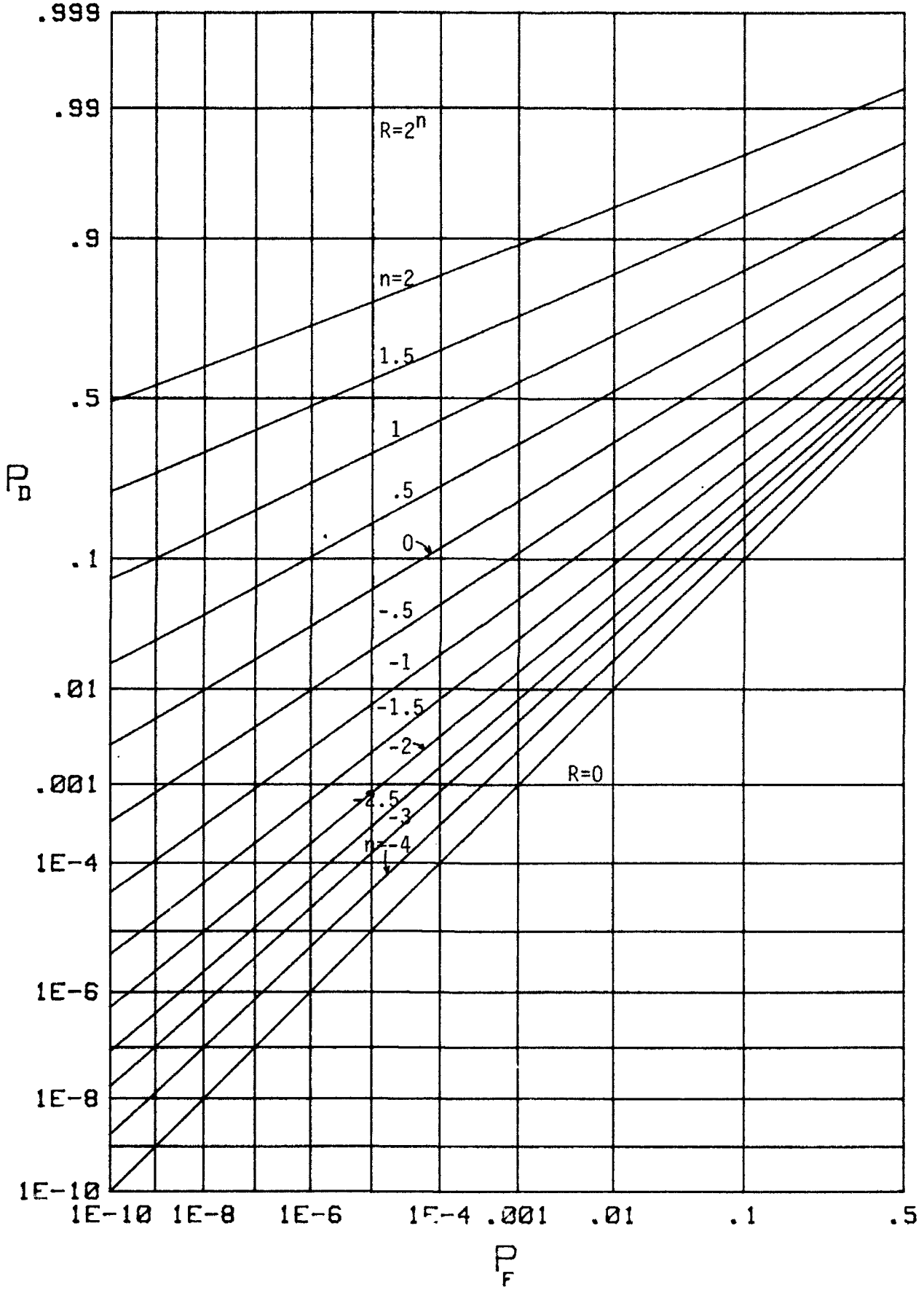

Figure 29. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=32, r=2

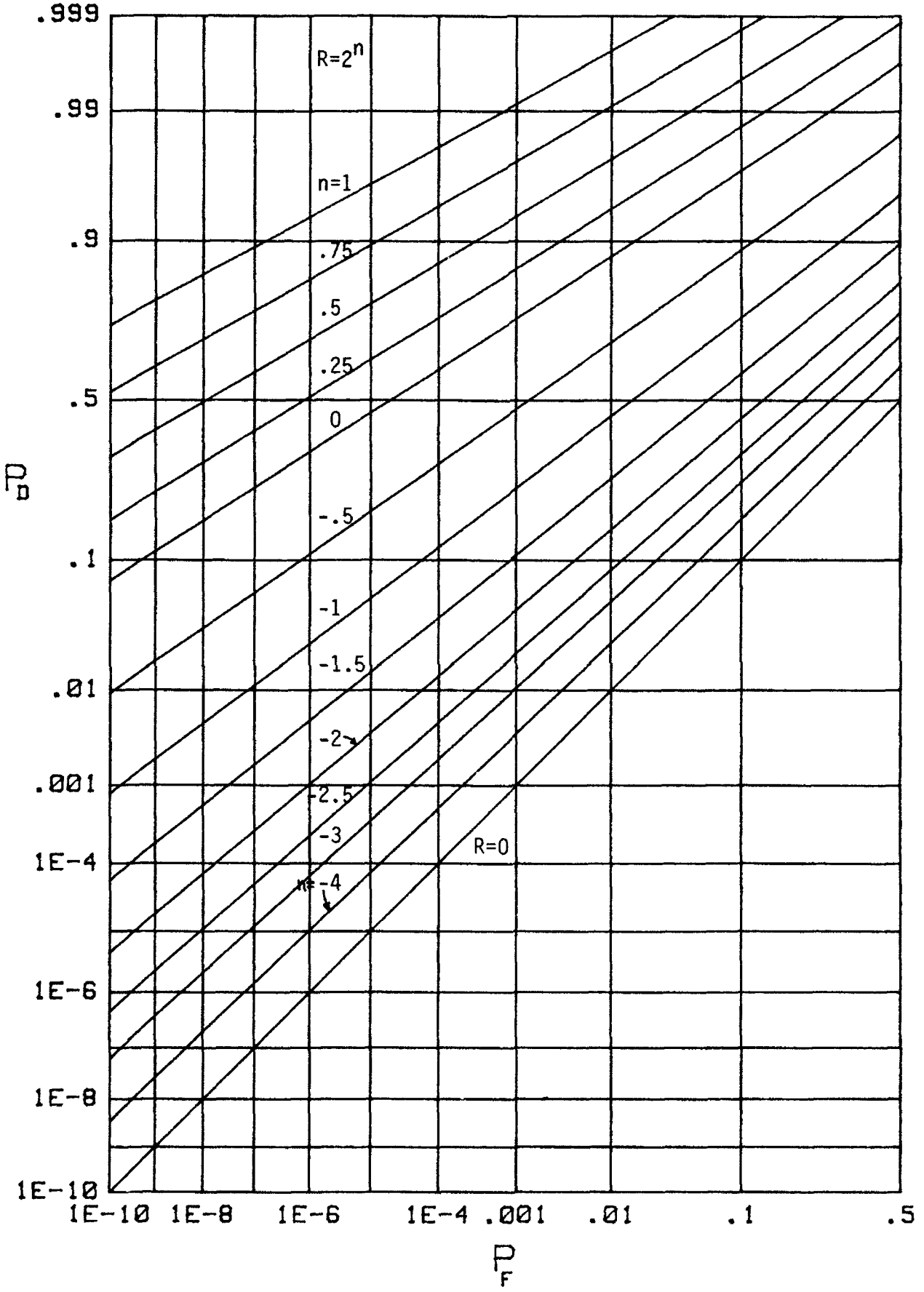

Figure 30. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=64, r=1

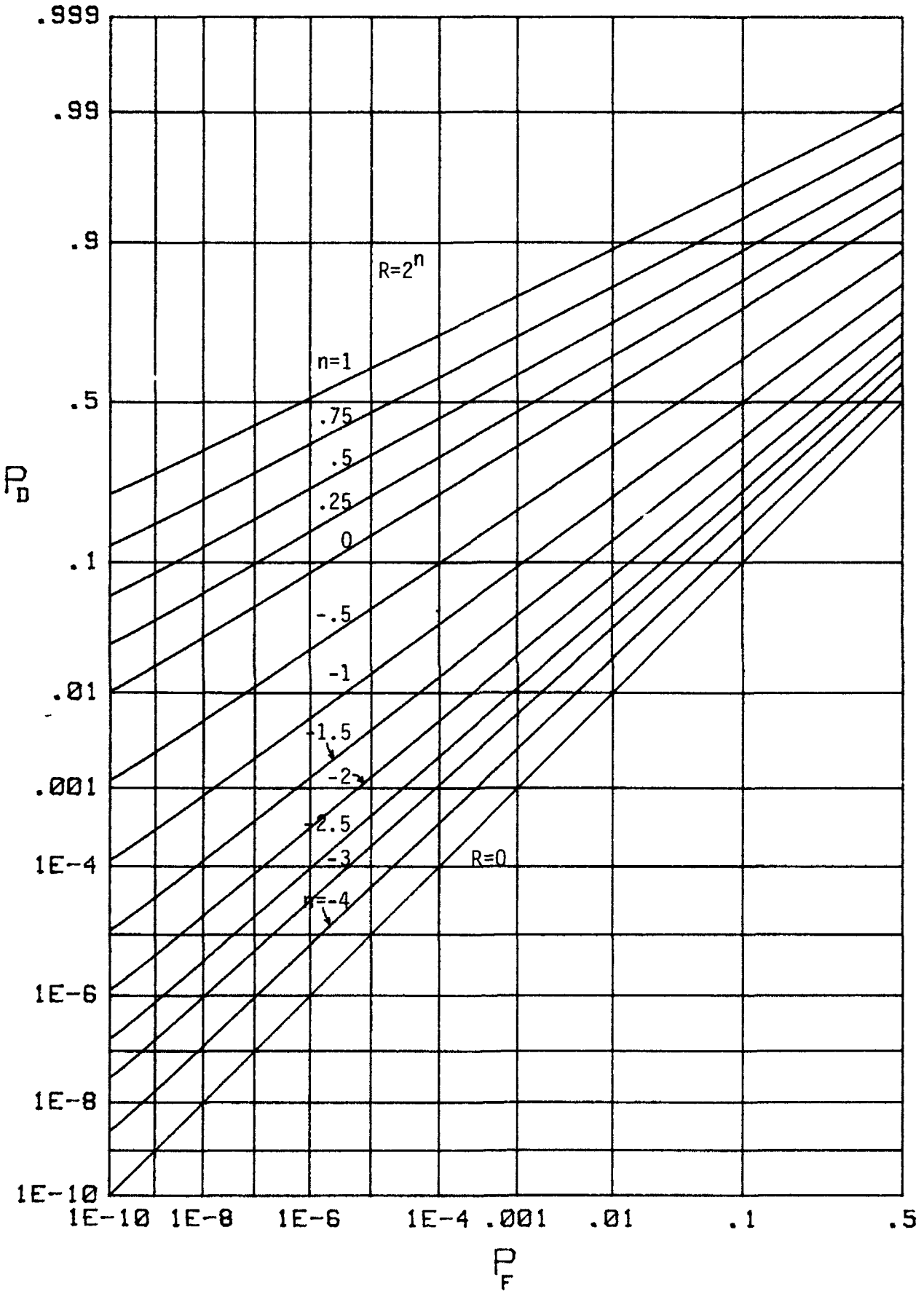

Figure 31. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=64, r=2

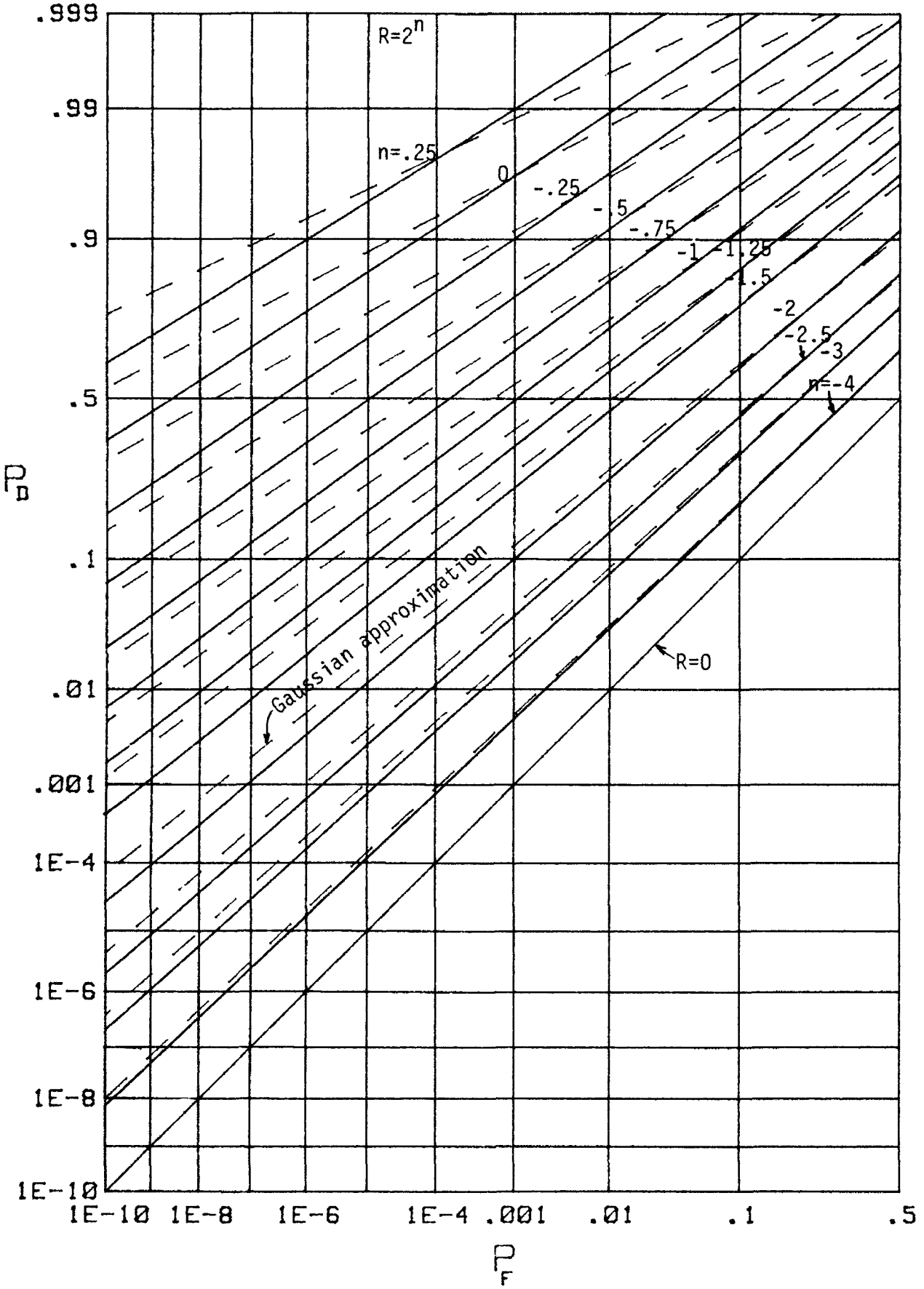

Figure 32. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=128, r=1

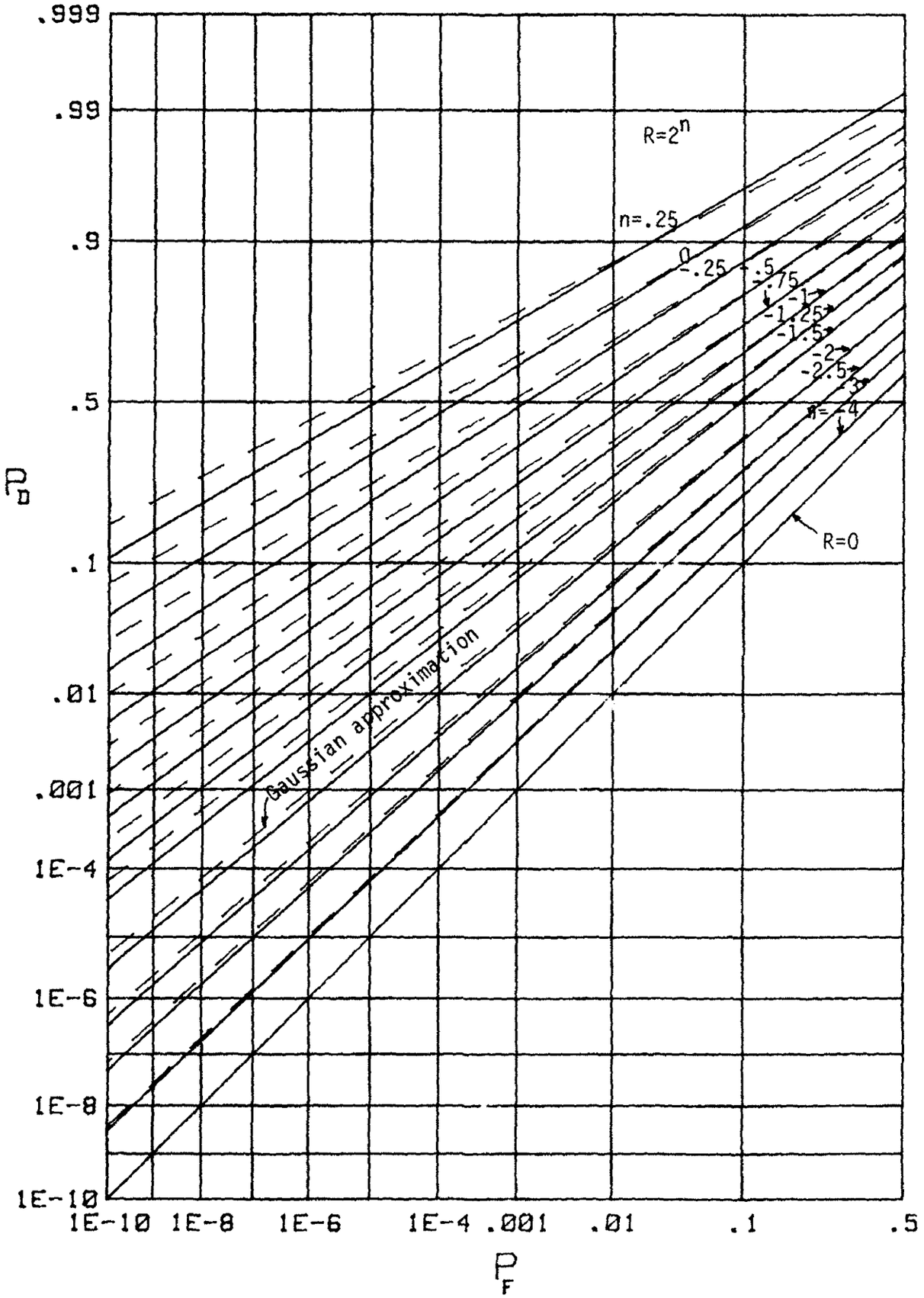

Figure 33. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=128, r=2

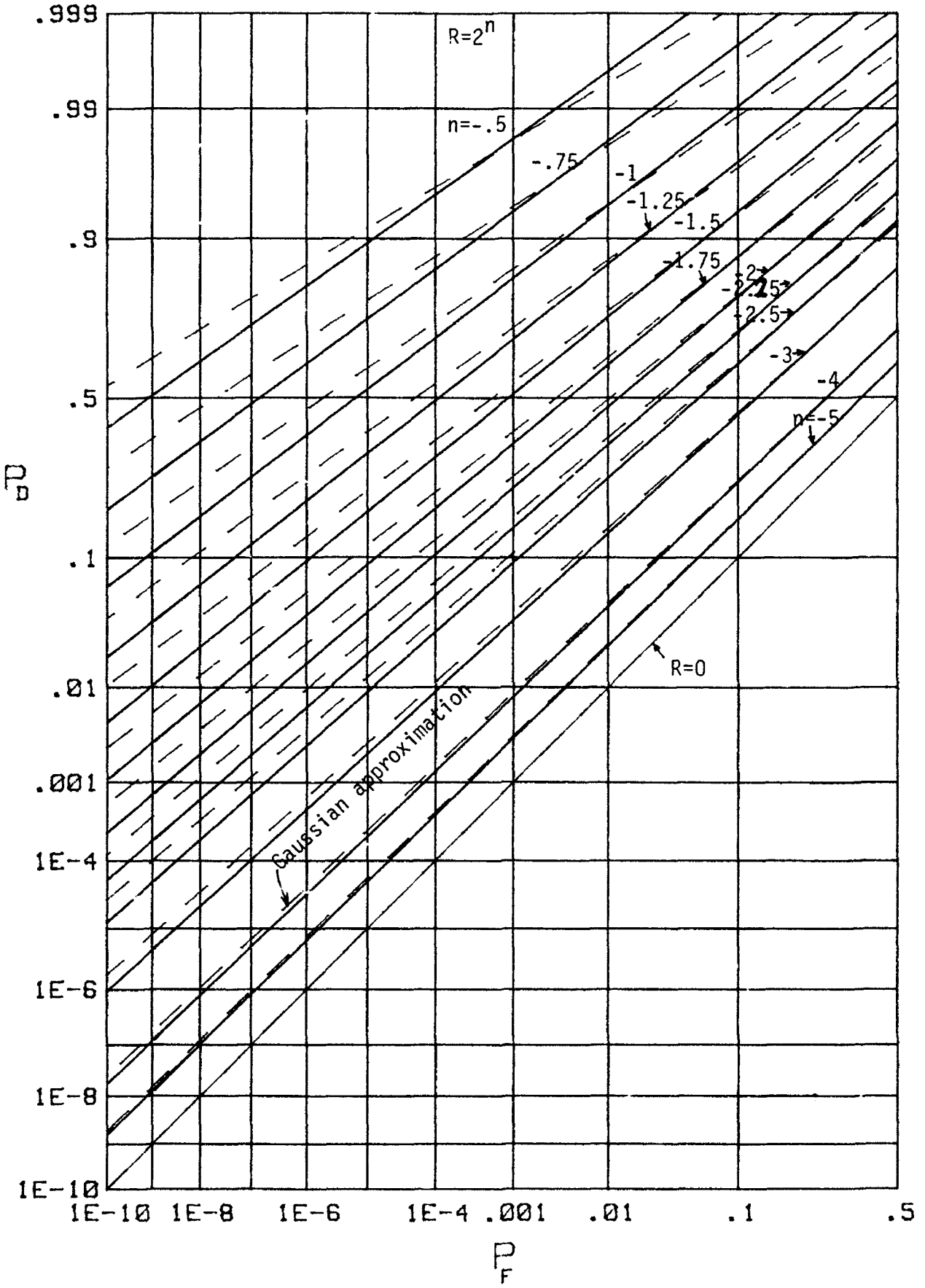

Figure 34. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=256, r=1

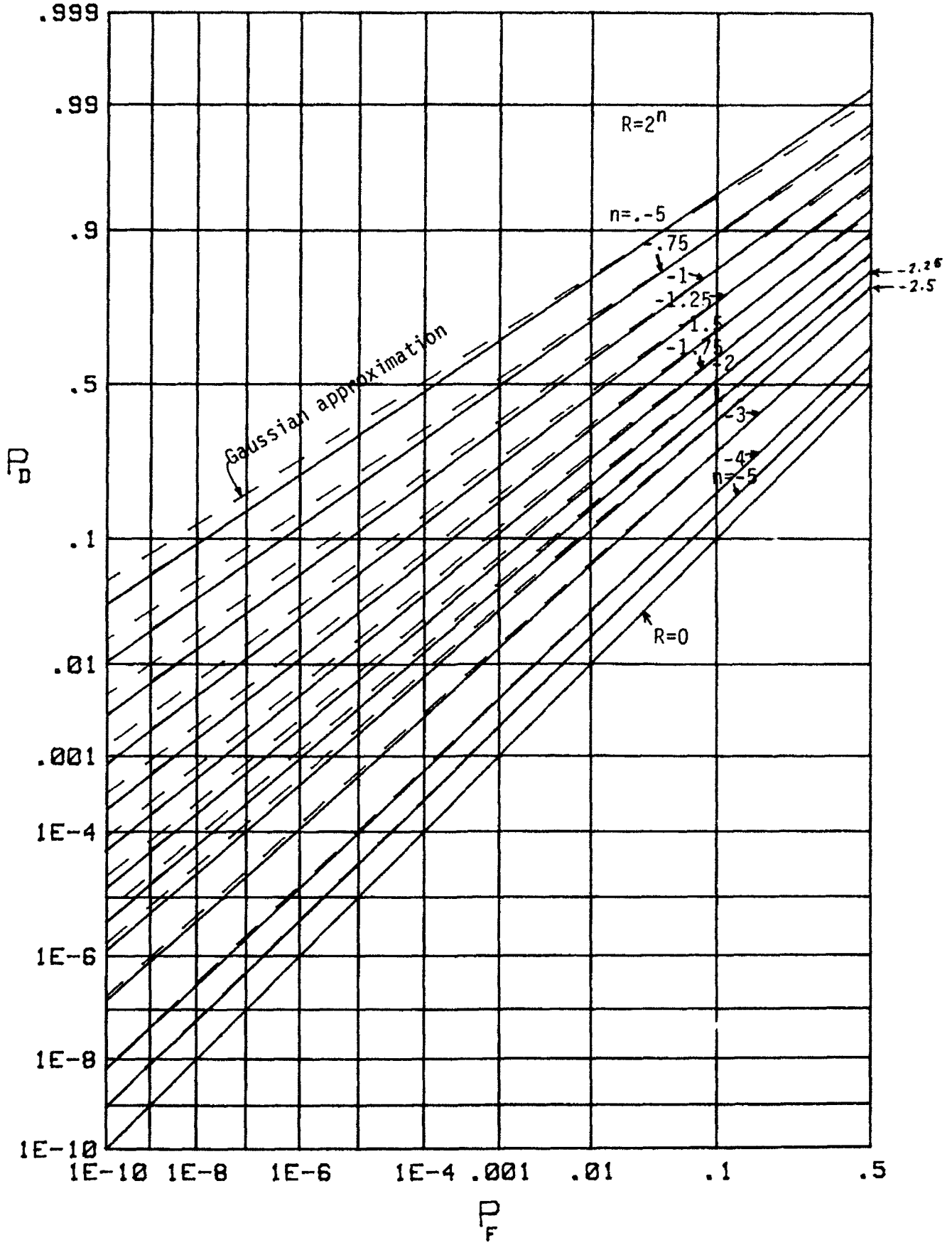

Figure 35. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=256, r=2

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APPENDIX A. CORRELATOR OUTPUT INDEPENDENCE OF MEANS

If we let scale factors $\alpha=1$ but $\beta\neq1$ in (11), we still get $\gamma=1$ from (13). This means that correlator output q in (12) is independent of μ_u, μ_v, β. To see this directly, let

$$u_n = \mu_u + y_n, \quad v_n = \mu_v + z_n \ , \tag{A-1}$$

where means

$$\overline{y_n} = \overline{z_n} = 0 \ . \tag{A-2}$$

Then (11) yields, with $\alpha=1$,

$$\tilde{u}_n = y_n - \frac{1}{N}\sum_{m=1}^{N} y_m \ , \tag{A-3}$$

which is obviously independent of the actual value of mean μ_u. Also, (11) yields

$$\tilde{v}_n = z_n - \frac{\beta}{N}\sum_{m=1}^{N} z_m + \mu_v(1-\beta) \ , \tag{A-4}$$

which still depends on μ_v and β. When (A-4) is substituted in (12), we get correlator output

$$q = \sum_{n=1}^{N} \tilde{u}_n \left[z_n - \frac{\beta}{N}\sum_{m=1}^{N} z_m + \mu_v(1-\beta) \right] . \tag{A-5}$$

But since

$$\sum_{n=1}^{N} \tilde{u}_n = 0 \tag{A-6}$$

from (A-3), (A-5) reduces to

$$q = \sum_{n=1}^{N} \tilde{u}_n z_n = \sum_{n=1}^{N} \left(y_n - \frac{1}{N} \sum_{m=1}^{N} y_m \right) z_n =$$

$$= \sum_{n=1}^{N} y_n z_n - \frac{1}{N} \sum_{m=1}^{N} y_m \sum_{n=1}^{N} z_n \qquad \text{(A-7)}$$

in terms of the ac components defined in (A-1) and (A-2). Correlator output (A-7) is obviously independent of means μ_u, μ_v and scale factor β in (11).

APPENDIX B. A USEFUL INTEGRAL OF EXPONENTIALS OF MATRIX FORMS

For symmetric KxK matrix M, with det M>0, the following K-fold integral is well known (see for example, [9, section 8-3]):

$$\int dX \exp[-\tfrac{1}{2}X^TMX+N^TX] = \left[\frac{(2\pi)^K}{\det M}\right]^{1/2} \exp\left[\tfrac{1}{2} N^TM^{-1}N\right]. \qquad (B-1)$$

Here X and N are Kx1 column matrices. We wish to extend this result to the case of double integral

$$I = (2\pi)^{-K} \iint dU\, dV \exp\left[-\tfrac{1}{2} U^TAU - \tfrac{1}{2} V^TBV + U^TCV + D^TU + E^TV\right], \qquad (B-2)$$

where A and B are symmetric without loss of generality, and the integral converges; here, matrices A, B, C are KxK while U, V, D, E are Kx1. Notice that if we had the apparently more-general term

$$U^TC_1V + V^TC_2U = U^T(C_1 + C_2^T)V\ , \qquad (B-3)$$

we would simply let $C = C_1 + C_2^T$, and thereby immediately have form (B-2).

To accomplish the evaluation of I in (B-2), identify M = B, $N^T = U^TC + E^T$, X = V in (B-1), and thereby evaluate the V-integral in (B-2), with result

$$\int dV \ldots = \left[\frac{(2\pi)^K}{\det B}\right]^{1/2} \exp\left[\tfrac{1}{2}(U^TC + E^T)B^{-1}(C^TU + E)\right]\ . \qquad (B-4)$$

Substituting (B-4) in (B-2), regrouping, and using the symmetry of B (and therefore B^{-1}), there follows

$$I = \frac{(2\pi)^{-K/2}}{(\det B)^{1/2}} \int dU \exp\left[-\tfrac{1}{2} U^T(A-CB^{-1}C^T)U + (D^T+E^TB^{-1}C^T)U + \tfrac{1}{2} E^TB^{-1}E\right]. \qquad (B-5)$$

Now reemploying (B-1) with identifications $M = A - CB^{-1}C^T$, $N = D + CB^{-1}E$, $X = U$, we get a closed form result for (B-5):

$$I = \left[\det(AB - CB^{-1}C^TB)\right]^{-1/2} \exp\left[\frac{1}{2} E^TB^{-1}E + \right.$$

$$\left. + \frac{1}{2}(D+CB^{-1}E)^T(A-CB^{-1}C^T)^{-1}(D+CB^{-1}E)\right]. \qquad \text{(B-6)}$$

This is the desired general result for integral (B-2).

As an aside, there is probably a more symmetric closed form result than (B-6), since if we represent (B-2) by I(A, B, C, D, E), we quickly see, by interchange of dummy variables U and V, that

$$I(A, B, C, D, E) = I(B, A, C^T, E, D). \qquad \text{(B-7)}$$

However, we have not discovered the symmetric form of (B-6). The present form follows as a result of the sequential integration of (B-2), first on V, then on U.

APPENDIX C. PROGRAM FOR CUMULATIVE AND EXCEEDANCE DISTRIBUTION FUNCTIONS VIA CHARACTERISTIC FUNCTION (23)-(24).

The numerical procedure employed in appendices C, D, and G here is heavily based on [3]. The choices of L, Δ, b in lines 90 to 110 to control truncation error and aliasing are also made according to the method of [3]. The parameters in (24) are evaluated once in lines 210-260 so as to minimize computation time. The FFT subroutine used in lines 1030 et seq is listed in [3, pp. B-11 - B-12], and employs a zero-subscripted array. A sample plot of the cumulative and exceedance distribution functions follows the program.

```
10 ! CROSS-CORRELATOR WITH SAMPLE MEAN REMOVAL; NUSC TR 7045
20    N=5                      !  Number of terms summed to yield output
30    Gamma=.5                 !  Scale factor in sample mean removal
40    Mu=-.4                   !  U channel mean
50    Mv=.3                    !  V channel mean
60    Su=.7                    !  U channel standard deviation
70    Sv=.9                    !  V channel standard deviation
80    Rho=.6                   !  Correlation coefficient
90    L=60                     !  Limit on integral of char. function
100   De'ta=.12                !  Sampling increment on char. function
110   Bs=.25*(2*PI/Delta)      !  Shift b, as fraction of alias interval
120   Mf=2^8                   !  Size of FFT
130   PRINTER IS 0
140   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
150   REDIM X(0:Mf-1),Y(0:Mf-1)
160   DIM X(0:1023),Y(0:1023)
170   Su2=Su*Su                                !  Calculate
180   Sv2=Sv*Sv                                !  parameters
190   T1=1-Gamma
200   T2=T1*T1
210   E1=2*Rho*Su*Sv
220   E2=Su2*Sv2*(1-Rho*Rho)
230   F1=E1*T1
240   F2=E2*T2
250   G1=N*Mu+Mv*T1
260   G2=.5*N*(Su2*Mv*Mv+Sv2*Mu*Mu-E1*Mu*Mv)*T2
270   N1=.5*(N-1)
280   Muq=G1+(N-Gamma)*Rho*Su*Sv               !  Mean of random  variable q
290   Muy=Muq+Bs                               !  Mean of shifted variable y
300   X(0)=0
310   Y(0)=.5*Delta*Muy
320   FOR Ns=1 TO INT(L/Delta)
330   Xi=Delta*Ns                              !  Argument xi of char. function
340   X2=Xi*Xi                                 !  Calculation
350   T1=-Xi*F1                                !  of
360   T2=1+X2*F2                               !  characteristic
370   CALL Div(-X2*G2,Xi*G1,T2,T1,A,B)         !  function
380   CALL Log(1+X2*E2,-Xi*E1,C,D)             !  fy(xi)
390   CALL Log(T2,T1,E,F)
400   CALL Exp(A-N1*C-.5*E,B-N1*D-.5*F+Xi*Bs,Fyr,Fyi)
```

```
410   Ms=Ns MOD Mf                          !  Collapsing
420   X(Ms)=X(Ms)+Fyr/Ns
430   Y(Ms)=Y(Ms)+Fyi/Ns
440   NEXT Ns
450   CALL Fft10z(Mf,X(*),Y(*))             !  0 subscript FFT
460   PLOTTER IS "GRAPHICS"
470   GRAPHICS
480   SCALE 0,Mf,-14,0
490   LINE TYPE 3
500   GRID Mf/8,1
510   PENUP
520   LINE TYPE 1
530   B=Bs*Mf*Delta/(2*PI)                  !  Origin for random variable q
540   MOVE B,0
550   DRAW B,-14
560   PENUP
570   FOR Ks=0 TO Mf-1
580   T=Y(Ks)/PI-Ks/Mf
590   X(Ks)=.5-T                            !  Cumulative probability in X(*)
600   Y(Ks)=Pr=.5+T                         !  Exceedance probability in Y(*)
610   IF Pr>=1E-12 THEN Y=LGT(Pr)
620   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
630   IF ABS(Pr)<1E-12 THEN Y=-12
640   PLOT Ks,Y
650   NEXT Ks
660   PENUP
670   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
680   FOR Ks=0 TO Mf-1
690   Pr=X(Ks)
700   IF Pr>=1E-12 THEN Y=LGT(Pr)
710   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
720   IF ABS(Pr)<1E-12 THEN Y=-12
730   PLOT Ks,Y
740   NEXT Ks
750   PENUP
760   PAUSE
770   DUMP GRAPHICS
780   PRINT LIN(5)
790   PRINTER IS 16
800   END
810   !
820   SUB Div(X1,Y1,X2,Y2,A,B)              !  Z1/Z2
830   T=X2*X2+Y2*Y2
840   A=(X1*X2+Y1*Y2)/T
850   B=(Y1*X2-X1*Y2)/T
860   SUBEND
870   !
880   SUB Log(X,Y,A,B)                      !  PRINCIPAL LOG(Z)
890   A=.5*LOG(X*X+Y*Y)
900   IF X<>0 THEN 930
910   B=.5*PI*SGN(Y)
920   GOTO 950
930   B=ATN(Y/X)
940   IF X<0 THEN B=B+PI*(1-2*(Y<0))
950   SUBEND
960   !
```

```
970   SUB Exp(X,Y,A,B)                    !  EXP(Z)
980   T=EXP(X)
990   A=T*COS(Y)
1000  B=T*SIN(Y)
1010  SUBEND
1020  !
1030  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 SUBSCRIPT
```

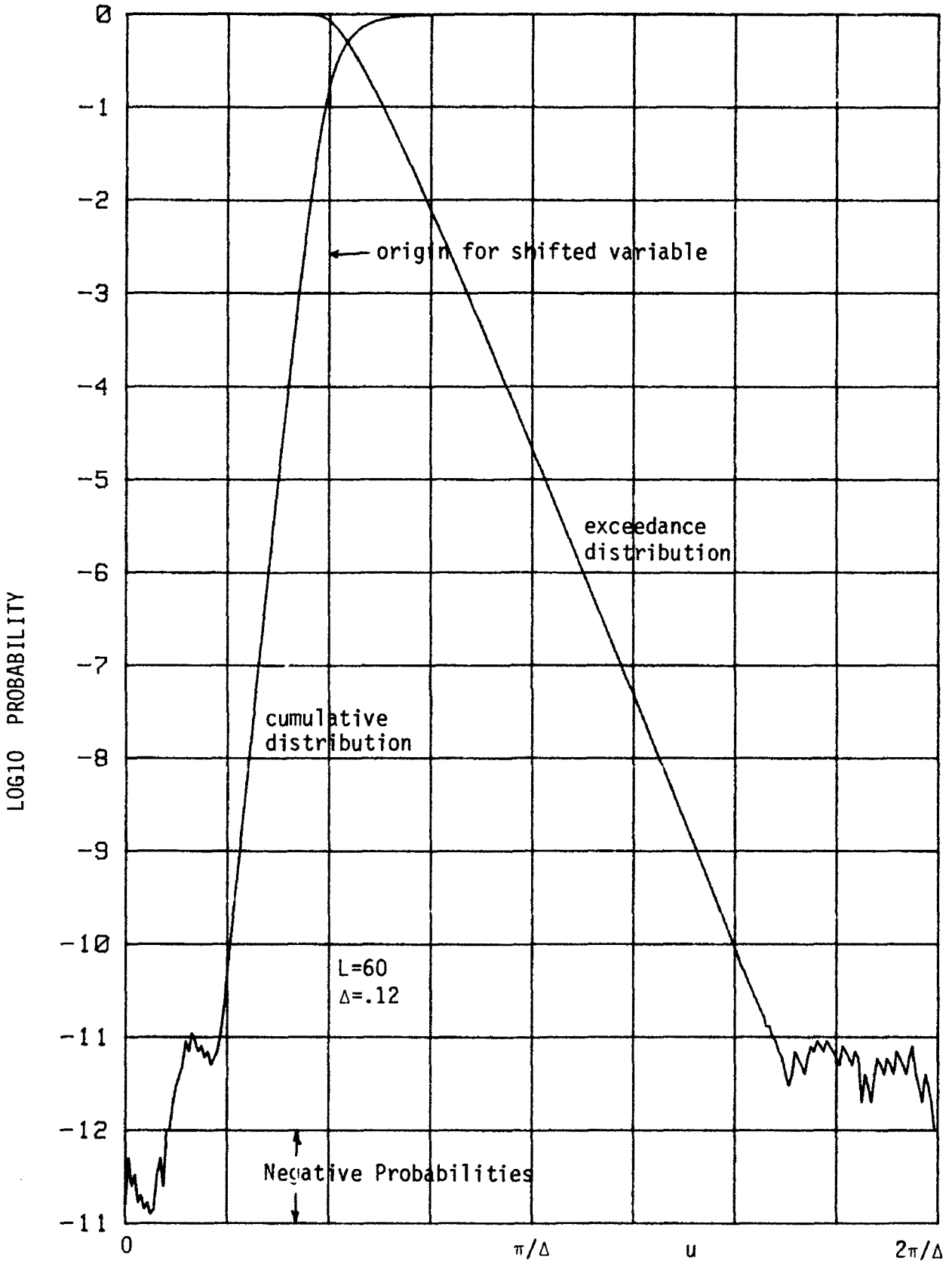

Figure C-1. Cumulative and Exceedance Distribution Functions

APPENDIX D. PROGRAM FOR EVALUATION OF OPERATING CHARACTERISTICS FOR γ=1

The comments in appendix C are relevant here also. The characteristic function used as the starting point is given by (81). Sampling increment Δ employed on the characteristic function can be coarse for small signal-to-noise ratio R, but must be finer for larger R. The quantity Delta0, Δ_0, in line 30 is that used for R=0; all other Δ values are sub-multiples, as indicated by lines 110-130. A table follows.

N	4	6	8	12	16	24	32	48	64	96	128	256
Δ_0	.10	.09	.08	.06	.05	.05	.05	.05	.04	.03	.03	.02

Table D-1. Values of Δ_0 for γ=1

Let Δ_1 denote the sampling increment employed for a particular value of signal-to-noise ratio $R_1>0$. The cumulative and exceedance distribution function values are available at spacing $s = 2\pi/(M_f\Delta)$ in general, where M_f is the FFT size. If, for example, $\Delta_1 = \Delta_0/2$, then $s_1 = 2s_0$, meaning that probability values occur twice as coarsely for R_1 as for R=0. Then in order to plot detection probability P_D vs false alarm probability P_F without interpolating points, it is necessary to skip every other P_F point available, and only plot those P_F, P_D pairs corresponding to the same threshold. More generally, if $\Delta_1 = \Delta_0/K$, where K is an integer, then $s_1 = Ks_0$, and we plot only every K-th point of the available P_F values. Here we have chosen K to be a power of 2, for the purpose of ease of plotting.

We choose bias (shift) b in line 40 in order to give a random variable y which is virtually always positive for R=0; see [3]. We then keep b fixed as R increases, which makes the probability of y>0 even greater. This feature of choosing the same b for all $R\geq 0$ enables an easy comparison of P_D and P_F, since common threshold values are then conveniently realized.

It was observed under (33) that the characteristic functions (39) or (81) have monotonically decreasing magnitudes for all $\xi \geq 0$. This makes the choice of L, the truncation value on the characteristic function integral in (63) or (64), rather simple; all we need to do is monitor $|f_h(\xi)|$ of (81) until it decreases below a tolerance, here taken as 1E-12. There is no trial-and-error procedure required as in [3] to guarantee negligible truncation error.

Subroutines Exp and Log have already been listed in appendix C, and so are not relisted here.

```
10 !  GAMMA = 1       SAMPLE MEAN REMOVAL
20    Nc=8                          !  N, Number of terms summed
30    Delta0=.02                    !  Initial delta
40    Bs=PI/Delta0                  !  Bias b
50    Mf=2^10                       !  Size of FFT
60    OUTPUT 0;"GAMMA = 1";"     N =";Nc
70    OUTPUT 0;""
80    DATA -2,-1,0,.5,1,1.5,2,2.5,3,3.5,4,5
90    READ Ns(*)                    !  SNR R=2^n
100   OUTPUT 0;Ns(*);
110   DATA 1,2,2,2,4,4,4,8,8,16,16,32,64
120   READ Idelta(*)
130   MAT Delta=(Delta0)/Idelta
140   OUTPUT 0;Delta(*);
150   DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,.001,.01,.1,.5,.9,.99,.999
160   READ Sc(*)
170   DIM Ns(1:12),Idelta(0:12),Delta(0:12),Sc(1:14)
180   DIM X(0:8191),Y(0:8191)
190   FOR I=1 TO 14
200   Sc(I)=FNInvphi(Sc(I))
210   NEXT I
220   S=Sc(1)
230   B=Sc(14)
240   Scale=(B-S)/(0-S)
250   X1=30
260   X2=170
270   Y1=35
280   Y2=Y1+(X2-X1)*Scale
290   PLOTTER IS "9872A"
300   LIMIT X1,X2,Y1,Y2
310   OUTPUT 705;"VS3"
320   SCALE S,0,S,B
330   FOR I=1 TO 14
340   MOVE S,Sc(I)
350   DRAW 0,Sc(I)
360   NEXT I
370   FOR I=1 TO 11
380   MOVE Sc(I),S
390   DRAW Sc(I),B
400   NEXT I
410   MOVE S,S
420   DRAW 0,0
430   PENUP
```

```
440   M1=Mf-1
450   M2=Mf/2
460   FOR In=0 TO 12
470   IF In>0 THEN 500
480   Rc=0
490   GOTO 510
500   Rc=2^Ns(In)                  !  SNR R=2^n
510   OUTPUT 0;"R =";Rc,"    Delta =";Delta(In)
520   ASSIGN #1 TO "ABSCIS"        !  Temporary storage for
530   Delta=Delta(In)              !  false alarm probability
540   R2=Rc+2
550   R21=R2+1
560   N12=(Nc-1)/2
570   Mux=(Nc-1)*Rc                !  Mean of random  variable h
580   Muy=Mux+Bs                   !  Mean of shifted variable y
590   REDIM X(0:M1),Y(0:M1)
600   MAT X=ZER
610   MAT Y=ZER
620   X(0)=0
630   Y(0)=.5*Delta+Muy
640   Ls=0
650   Ls=Ls+1
660   Xi=Delta*Ls                  !  Argument xi of char. fn.
670   CALL Log(1+Xi*Xi*R21,-Xi*R2,Ai,Bi)          !  Calculation
680   CALL Exp(-N12*Ai,Xi*Bs-N12*Bi,Fyr,Fyi)      !  of
690   Ms=Ls MOD Mf                                 !  characteristic
700   Ar=Fyr/Ls                                    !  function
710   Ai=Fyi/Ls                                    !  fy(xi)
720   X(Ms)=X(Ms)+Ar
730   Y(Ms)=Y(Ms)+Ai
740   Magsq=Ar*Ar+Ai*Ai
750   IF Magsq>1E-24 THEN 650
760   OUTPUT 0;"Xi =";Xi;"     Mag =";SQR(Magsq)
770   CALL Fft1c2(Mf,X(*),Y(*))
780   FOR Ms=0 TO M1
790   T=Y(Ms)/PI-Ms/Mf
800   X(Ms)=.5-T                   !  Cumulative distribution function
810   Y(Ms)=.5+T                   !  Exceedance distribution function
820   NEXT Ms
830   OUTPUT 0;Y(0);Y(1);Y(M1-1);Y(M1)
840   PLOTTER IS "GRAPHICS"
850   GRAPHICS
860   SCALE 0,Mf,-14,0
870   LINE TYPE 3
880   GRID Mf/8,1
890   PENUP
900   LINE TYPE 1
910   FOR Ms=0 TO M1
920   Pr=Y(Ms)
930   IF Pr>=1E-12 THEN Y=LGT(Pr)
940   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
950   IF ABS(Pr)<1E-12 THEN Y=-12
960   PLOT Ms,Y
970   NEXT Ms
980   PENUP
```

```
990   FOR Ms=0 TO M1
1000  Pr=Y(Ms)
1010  IF Pr>=1E-12 THEN Y=LGT(Pr)
1020  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1030  IF ABS(Pr)<1E-12 THEN Y=-12
1040  PLOT Ms,Y
1050  NEXT Ms
1060  PENUP
1070  DUMP GRAPHICS
1080  OUTPUT 0;""
1090  IF In>0 THEN 1200
1100  FOR Ms=M2 TO M1
1110  IF Y(Ms)<=0 THEN 1130
1120  NEXT Ms
1130  M3=Ms-1
1140  REDIM X(M2:M3)
1150  FOR Ms=M2 TO M3
1160  X(Ms)=FNInvphi(Y(Ms))
1170  NEXT Ms
1180  PRINT #1;X(*)              !  Store false alarm probability
1190  GOTO 1380
1200  REDIM X(M2:M3)
1210  READ #1,X(*)               !  Read in false alarm probability
1220  Id=Idelta(In)
1230  J2=M2/Id
1240  J3=INT(M3/Id)+1
1250  FOR J=J2 TO J3
1260  Y(J)=FNInvphi(Y(J))
1270  NEXT J
1280  PLOTTER IS "9872A"
1290  LIMIT X1,X2,Y1,Y2
1300  OUTPUT 705;"VS3"
1310  SCALE 5,0,5,8
1320  FOR J=J2 TO J3
1330  T=J*Id
1340  IF T>M3 THEN 1370
1350  PLOT X(T),Y(J)
1360  NEXT J
1370  PENUP
1380  NEXT In
1390  END
1400  !
1410  SUB Exp(X,Y,A,B)                    !  EXP(Z)
1460  !
1470  SUB Log(X,Y,A,B)                    !  PRINCIPAL LOG(Z)
1550  !
1560  DEF FNInvphi(X)       !  INVPHI(X) via AMS 55, 26.2.23
1570  IF (X>0) AND (X<1) THEN 1600
1580  P=9.9999999999E90*(2*X-1)
1590  GOTO 1670
1600  IF X=.5 THEN RETURN 0
1610  P=X
1620  IF X>.5 THEN P=.5-(X-.5)
1630  P=SQR(-2*LOG(P))
1640  T=1+P*(1.432788+P*(.189269+P*.001308))
1650  P=P-(2.515517+P*(.802853+P*.010328))/T
1660  IF X<.5 THEN P=-P
1670  RETURN P
1680  FNEND
1690  !
1700  SUB Fft13z(N,X(*),Y(*))    ! N <= 2^13, N=2^INTEGER, 0 SUBSCRIPT
```

APPENDIX E. ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS WHEN r>0

The characteristic function of interest is given by (100) and (102):

$$f_h(\xi) = (1+i\xi)^{-\nu}\,(1-i\xi\omega)^{-\nu}\,\exp\left[\frac{i\xi\eta}{1-i\xi\omega}\right], \tag{E-1}$$

where for notational convenience in this appendix, we let

$$\nu = \frac{N}{2}, \quad \eta = Nr^2, \quad \omega = 1+2R\ . \tag{E-2}$$

The cumulative distribution function is obtained by substitution of (E-1) in (63):

$$P_h(u) = \frac{-1}{i2\pi}\int_{C_+} d\xi\ \xi^{-1}(1+i\xi)^{-\nu}\,(1-i\xi\omega)^{-\nu}\,\exp\left[\frac{i\xi\eta}{1-i\xi\omega} - iu\xi\right]. \tag{E-3}$$

The ν-th powers are principal value, being positive real where C_+ crosses the positive imaginary axis.

Now let $z = 1+i\xi$ in (E-3), yielding

$$P_h(u) = \frac{-1}{i2\pi}\int_{C_1} dz\ (z-1)^{-1}\,z^{-\nu}\,(1+\omega-\omega z)^{-\nu}\,\exp\left[\frac{(z-1)\eta}{1+\omega-\omega z} - u(z-1)\right]. \tag{E-4}$$

The contours C_+ and C_1 in (E-3) and (E-4) are depicted as dashed lines in figure E-1. The pole at $\xi=0$ is moved to $z=1$; the remaining singularities are branch points (ν non-integer); the ν-th powers are positive real where C_1 crosses the positive real axis. For $u<0$, an equivalent contour to C_1 is that indicated by C_2 in figure E-1, since the exponential in (E-4) furnishes rapid decay in the left-half z-plane. We write (E-4) in the form

$$P_h(u) = \frac{\exp(u)}{i2\pi}\int_{C_2} dz\ z^{-\nu}\exp(-uz)\ g_1(z)\ , \tag{E-5}$$

where

$$g_1(z) = (1-z)^{-1}\ (1+\omega-\omega z)^{-\nu}\ \exp\left[\frac{(z-1)\eta}{1+\omega-\omega z}\right]. \qquad \text{(E-6)}$$

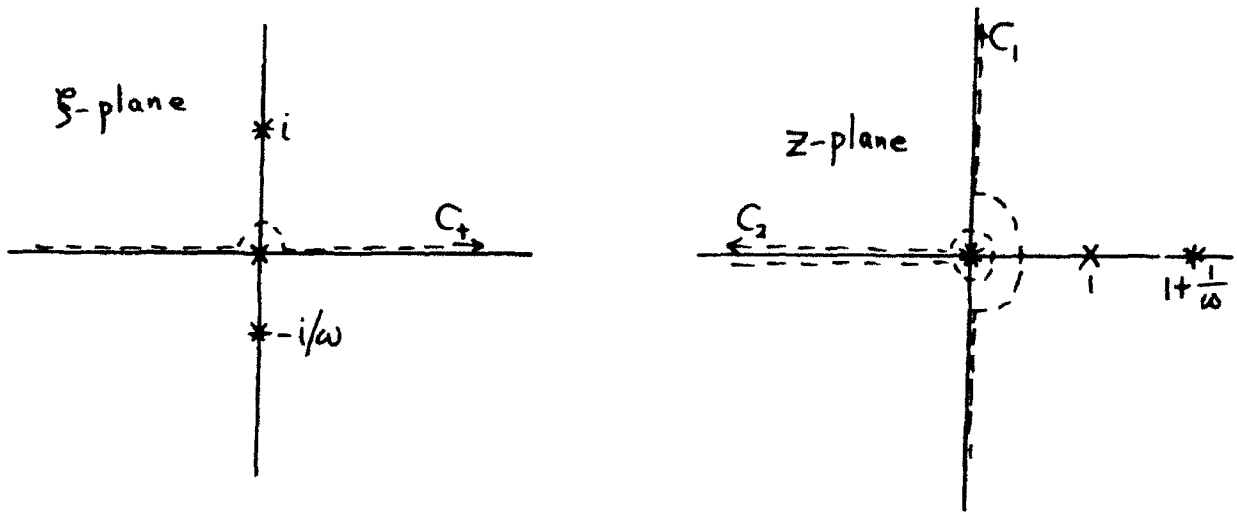

Figure E-1. Contours of Integration for Cumulative Distribution Function

In order to get the asymptotic development of (E-5), we expand $g_1(z)$ in a power series in z,

$$g_1(z) = g_1(0) + g_1'(0)\ z + \ldots\ , \qquad \text{(E-7)}$$

where

$$g_1(0) = (1+\omega)^{-\nu}\ \exp\left(\frac{-\eta}{1+\omega}\right),\ \frac{g_1'(0)}{g_1(0)} = 1 + \frac{\nu\omega}{1+\omega} + \frac{\eta}{(1+\omega)^2}\ . \qquad \text{(E-8)}$$

Appeal to [10, p. 96, (4)] then yields (for all ν)

$$P_h(u) \sim \exp(u)\left[\frac{(-u)^{\nu-1}}{\Gamma(\nu)}\ g_1(0) + \frac{(-u)^{\nu-2}}{\Gamma(\nu-1)}\ g_1'(0)\right] =$$

$$= \Gamma(\nu)^{-1}\ (1+\omega)^{-\nu}\ (-u)^{\nu-1}\ \exp\left[u - \frac{\eta}{1+\omega}\right]\left[1 - \frac{\nu-1}{u}\left(1 + \frac{\nu\omega}{1+\omega} + \frac{\eta}{(1+\omega)^2}\right)\right]$$

$$\text{as } u \to -\infty\ . \qquad \text{(E-9)}$$

Substitution of (E-2) in (E-9) then yields result (104).

When (E-1) is substituted into (64) instead, we obtain the exceedance distribution function in the form

$$1 - P_h(u) = \frac{1}{i2\pi} \int_{C_-} d\xi\, \xi^{-1} (1+i\xi)^{-\nu} (1-i\xi\omega)^{-\nu} \exp\left[\frac{i\xi\eta}{1-i\xi\omega} - iu\xi\right] . \qquad \text{(E-10)}$$

Now let $z = \frac{1}{\omega} - i\xi$, to get

$$1 - P_h(u) = \frac{-1}{i2\pi} \int_{C_3} dz\, (z - \frac{1}{\omega})^{-1} (1 + \frac{1}{\omega} - z)^{-\nu} (\omega z)^{-\nu} *$$

$$* \exp\left[\frac{(1-\omega z)\eta}{\omega^2 z} + u(z - \frac{1}{\omega})\right] . \qquad \text{(E-11)}$$

The contours C_- and C_3 in (E-10) and (E-11) are depicted as dashed lines in figure E-2. The pole at $\xi=0$ is moved to $z = 1/\omega$; the remaining singularities are branch points.

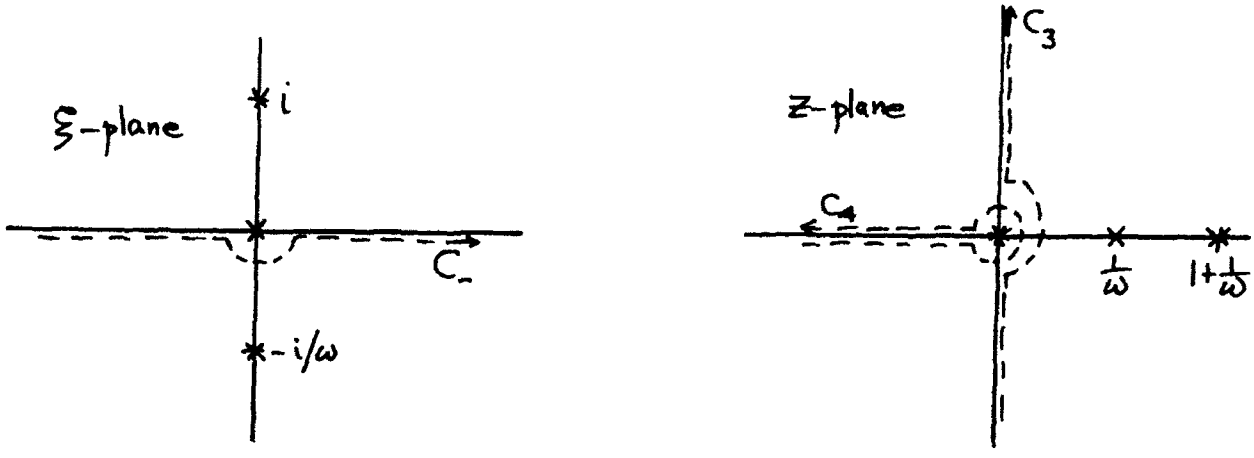

Figure E-2. Contours of Integration for Exceedance Distribution Function

For $u>0$, an equivalent contour to C_3 is that indicated by C_4 in figure E-2, since the exponential in (E-11) furnishes rapid decay in the left-half z-plane. We write (E-11) in the form

$$1-P_h(u) = \omega\,(1+\omega)^{-\nu} \exp\left(-\frac{u+\eta}{\omega}\right) \frac{1}{i2\pi} \int_{C_4} dz\; z^{-\nu} \exp\left[uz + \frac{\eta}{\omega^2 z}\right] g_2(z)\ , \quad \text{(E-12)}$$

where

$$g_2(z) = (1-\omega z)^{-1} \left(1 - \frac{\omega z}{1+\omega}\right)^{-\nu}\ . \quad \text{(E-13)}$$

As above, we expand $g_2(z)$ in a power series in z,

$$g_2(z) = 1 + \omega\left(1 + \frac{\nu}{1+\omega}\right) z + \ldots \quad \text{(E-14)}$$

and substitute it in (E-12); employment of [10, p. 105, (2)] now yields (for all ν)

$$1 - P_h(u) \sim \omega\,(1+\omega)^{-\nu} \exp\left(-\frac{u+\eta}{\omega}\right) *$$

$$*\left[\left(\omega\sqrt{\frac{u}{\eta}}\right)^{\nu-1} I_{\nu-1}\left(\frac{2\sqrt{\eta u}}{\omega}\right) + \omega\left(1 + \frac{\nu}{1+\omega}\right)\left(\omega\sqrt{\frac{u}{\eta}}\right)^{\nu-2} I_{\nu-2}\left(\frac{2\sqrt{\eta u}}{\omega}\right)\right]$$

$$\text{as } u \to +\infty, \quad \text{(E-15)}$$

where $I_\mu(z)$ is the modified Bessel function of the first kind. This is the general result for the exceedance distribution function; the various parameters given in (E-2) relate it back to the problem of interest in the main text.

As a check on this result, we let $r \to 0$; then $\eta \to 0$, and (E-15) reduces, via [6, 9.6.7], to

$$1 - P_h(u) \sim \Gamma(\nu)^{-1}\, \omega\,(1+\omega)^{-\nu}\, u^{\nu-1} \exp(-u/\omega)\left[1 + \frac{\nu-1}{u}\,\omega\left(1 + \frac{\nu}{1+\omega}\right)\right]$$

$$\text{as } u \to +\infty;\ r=0. \quad \text{(E-16)}$$

Employing the identifications in (E-2), there follows from (E-16)

$$1 - P_h(u) \sim \frac{1+2R}{2(1+R)\Gamma(N/2)} \left(\frac{u}{2(1+R)}\right)^{\frac{N-2}{2}} \exp\left(\frac{-u}{1+2R}\right) *$$

$$* \left[1 + \frac{1+2R}{1+R} \frac{(N-2)(N+4+4R)}{8u}\right] \quad \text{as } u \to +\infty;\ r=0\ . \qquad \text{(E-17)}$$

In order to compare this result with that for $\gamma=1$, sample mean removal, we must replace N here by N-1; see the paragraph under (101). When this is done, (E-17) reverts precisely to (61) and (62).

Returning to the general result for the exceedance distribution function in (E-15), if we keep $\eta>0$ and use [6, 9.7.1] for large arguments of $I_\mu(z)$, there follows the simpler (less accurate) result

$$1 - P_h(u) \sim \left[2\pi^{1/2}(1+\omega)^\nu \eta^{\frac{\nu}{2} - \frac{1}{4}}\right]^{-1} \omega^{\nu+\frac{1}{2}} *$$

$$* u^{\frac{\nu}{2} - \frac{3}{4}} \exp\left[-\frac{1}{\omega}\left(u^{1/2} - \eta^{1/2}\right)^2\right] \quad \text{as } u \to +\infty. \qquad \text{(E-18)}$$

When (E-2) is substituted in (E-18), the result quoted in (105) follows.

As a special case of (E-18), if $\nu=1$ (i.e. N=2), then

$$1 - P_h(u) \sim \frac{\omega^{3/2}}{2\pi^{1/2}(1+\omega)(\eta u)^{1/4}} \exp\left[-\frac{1}{\omega}\left(u^{1/2} - \eta^{1/2}\right)^2\right] \text{ as } u \to +\infty;\ \nu=1. \qquad \text{(E-19)}$$

APPENDIX F. EXCEEDANCE DISTRIBUTION FUNCTION FOR $\gamma=0$, $N=1$, $r>0$

CHARACTERISTIC FUNCTION APPROACH

When characteristic function (100) with N=1 is substituted in (64), the expression for the exceedance distribution function becomes

$$1 - P_h(u) = \frac{1}{i2\pi} \int_{C_-} d\xi\, \xi^{-1} (1+i\xi)^{-1/2} (1-i\xi\omega)^{-1/2} \exp\left[\frac{i\xi r^2}{1-i\xi\omega} - iu\xi\right], \quad \text{(F-1)}$$

where $\omega = 1+2R$ as in (102). The square roots are taken as +1 at $\xi=0$. For $u \geq 0$, the contour C_- can be modified to that indicated in figure F-1, where the contributions of the large circular arcs in the lower-half ξ-plane tend to zero. The small circle of radius ρ centered at branch point $\xi = -i/\omega$ must have $0 < \rho < 1/\omega$, since the latter is the distance to the pole at the origin.

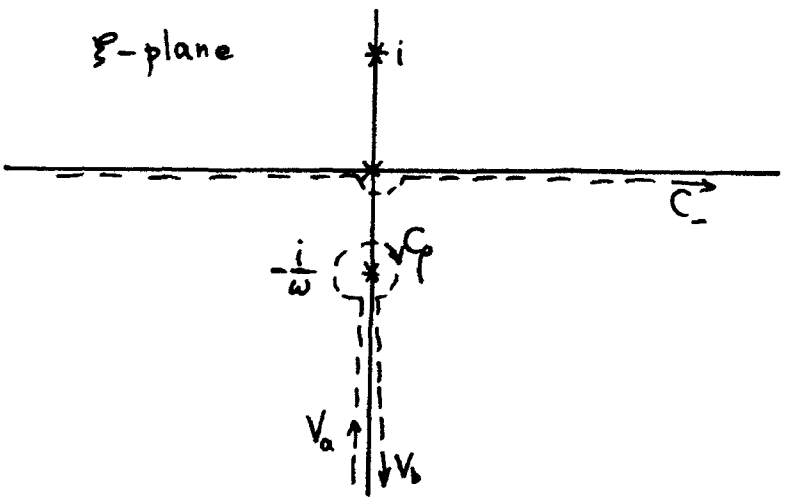

Figure F-1. Equivalent Contours for (F-1)

It is easy to show that the two vertical contributions in figure F-1 are equal. Under the change of variable

$$i\xi = \frac{1+t^2}{\omega}, \quad \text{(F-2)}$$

the sum of the two vertical contributions to exceedance distribution function (F-1) becomes

$$V(\rho) = \frac{2\omega^{1/2}}{\pi} \exp\left(-\frac{u+r^2}{\omega}\right) \int_{(\omega\rho)^{1/2}}^{+\infty} dt\ (1+t^2)^{-1} *$$

$$* (1+\omega+t^2)^{-1/2} \exp\left(-\frac{u}{\omega}t^2 - \frac{r^2}{\omega t^2}\right). \qquad \text{(F-3)}$$

This integral remains convergent even as $\rho \to 0+$. Furthermore, the integrand decays rapidly, has no cusps, and involves only elementary functions; also the integral is a sum of positive quantities and retains significance even for large u.

On the small circular contour C_ρ in figure F-1, let

$$i\xi = \frac{1}{\omega} - \rho \exp(i\theta), \qquad \text{(F-4)}$$

to obtain, for the circular contribution to the exceedance distribution function, the quantity

$$C(\rho) = \frac{\omega}{2\pi(1+\omega)^{1/2}} \exp\left(-\frac{u+r^2}{\omega}\right) \rho^{1/2} \int_{-\pi}^{\pi} d\theta\ (1-\omega\rho E)^{-1} *$$

$$*(1 - \frac{\omega\rho}{1+\omega} E)^{-1/2} \exp\left(i\frac{\theta}{2} + \frac{\lambda}{\rho}E^* + \rho u E\right), \qquad \text{(F-5)}$$

where we define in this appendix

$$\lambda = \frac{r^2}{\omega^2} = \left(\frac{r}{1+2R}\right)^2, \quad E = \exp(i\theta). \qquad \text{(F-6)}$$

The exceedance distribution function is given exactly by the sum of (F-3) and (F-5), for any $0 < \rho < 1/\omega$. It would be advantageous numerically to let $\rho \to 0+$ in these two equations; however, the limit of (F-5) is not obvious and can easily be done incorrectly.

AN ERRONEOUS APPROACH FOR $C(\rho)$

It is tempting to let $\rho \to 0+$ in those locations in (F-5) where it will "do no damage", obtaining for the integral with scale factor $\rho^{1/2}$ the quantity

$$I_\rho \equiv \rho^{1/2} \int_{-\pi}^{\pi} d\theta \exp\left[i \frac{\theta}{2} + \frac{\lambda}{\rho} \exp(-i\theta)\right] . \qquad \text{(F-7)}$$

(The fallacy of doing this for a residue calculation with an essential singularity is demonstrated in the next subsection.) Furthermore, the limit of (F-7) as $\rho \to 0+$ can in fact be determined in closed form, as follows. Observe that the integrand of (F-7) has a saddle point in the complex θ-plane at

$$\theta_s = -iL , \quad L = \ln\left(\frac{2\lambda}{\rho}\right) ; \qquad \text{(F-8)}$$

this is in fact the only saddle point in the $(-\pi,\pi)$ strip in the θ-plane. Now let $z = \theta - \theta_s$, getting for (F-7)

$$I_\rho = (2\lambda)^{1/2} \int_{-\pi+iL}^{\pi+iL} dz \exp\left[i \frac{z}{2} + \frac{1}{2} \exp(-iz)\right] . \qquad \text{(F-9)}$$

The radius ρ now appears only in the limit L of the integral, and the integrand has a saddle point at $z=0$.

The straight line contour for (F-9) can be deformed into contour C, depicted in figure F-2, which goes through the saddle point at z=0. Now if

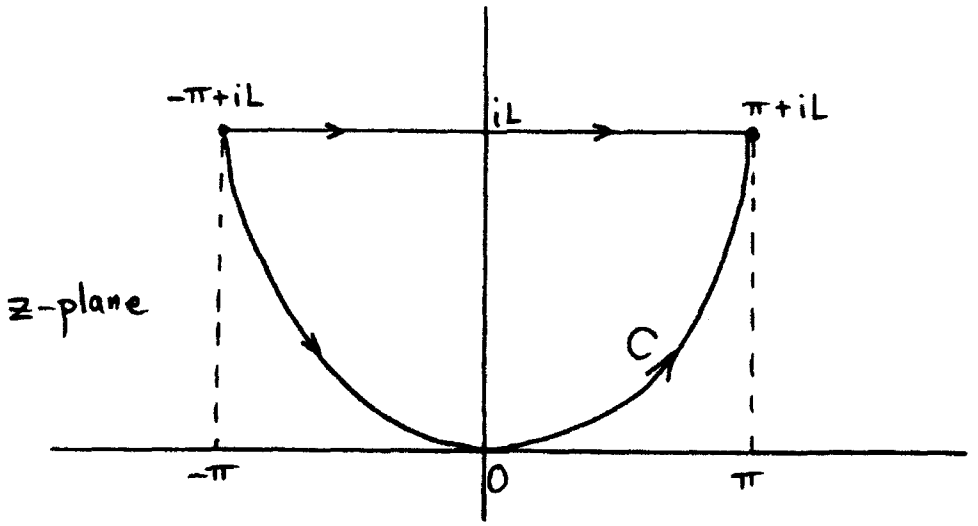

Figure F-2. Equivalent Contours for (F-9)

$\rho \to 0^+$, then $L \to +\infty$, and (F-9) yields

$$I_0 = (2\lambda)^{1/2} \int_{-\pi+i\infty}^{\pi+i\infty} dz \, \exp\left[i \frac{z}{2} + \frac{1}{2} \exp(-iz) \right] , \qquad \text{(F-10)}$$

where the contour is the limit of C in figure F-2 as $L \to +\infty$; that is, the integral is between the two valleys at $\pm\pi+i\infty$ and is connected by the saddle point at z=0.

The steepest descent curves out of the saddle point of the integrand of (F-10) are given explicitly by

$$y = -\ln\left(\frac{\sin x}{x}\right) \quad \text{for } -\pi < x < \pi . \qquad \text{(F-11)}$$

Thus if we let

$$z = x + iy = x - i\,\ell n\left(\frac{\sin x}{x}\right) ,$$

$$\frac{dz}{dx} = 1 - i\,\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) , \qquad \text{(F-12)}$$

on the steepest descent curves in (F-10), there follows

$$I_o = (2\lambda)^{1/2} \int_{-\pi}^{\pi} dx\left(1 - i\,\frac{\cos x}{\sin x} + i\,\frac{1}{x}\right) \exp\left[i\,\frac{x}{2} + \frac{1}{2}\,\ell n\left(\frac{\sin x}{x}\right) + \frac{1}{2}\,\frac{x}{\sin x}\,\exp(-ix)\right] =$$

$$= (2\lambda)^{1/2} \int_{-\pi}^{\pi} dx\left(1 - i\,\frac{\cos x}{\sin x} + i\,\frac{1}{x}\right) \left(\frac{\sin x}{x}\right)^{1/2} \exp\left[\frac{x\,\cos x}{2\,\sin x}\right] =$$

$$= (2\lambda)^{1/2}\, 2\int_{0}^{\pi} dx\,\left(\frac{\sin x}{x}\right)^{1/2} \exp\left(\frac{x\,\cos x}{2\,\sin x}\right) = (2\lambda)^{1/2}\, 2(2\pi)^{1/2} = 4(\lambda\pi)^{1/2} . \qquad \text{(F-13)}$$

(The integral value of $(2\pi)^{1/2}$ in (F-13) was deduced by numerical integration.)

Recalling the definition of I_ρ in (F-7), we then have the dubious result for the limit of (F-5):

$$C(0) \stackrel{?}{=} \left(\frac{2}{\pi}\right)^{1/2} \frac{r}{(1+R)^{1/2}} \exp\left(-\frac{u+r^2}{1+2R}\right), \qquad \text{(F-14)}$$

where we employed (102) and (F-6). Actual numerical evaluation of (F-14), combined with V(0) from (F-3), gives incorrect results for the exceedance distribution function (F-1); thus the replacement of ρ with 0 in (F-5) is invalid. The explanation for this pitfall is the essential singularity of (F-1) at $\xi = -i/\omega$; a simpler illustration follows.

RESIDUE OF ESSENTIAL SINGULARITY

The function

$$\exp(\tfrac{1}{z}) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \ldots \qquad \text{(F-15)}$$

has an essential singularity at z=0, with residue 1, as exemplified by this Laurent expansion. Now consider the function

$$f(z) = \exp(\tfrac{1}{z})\; g(z)\;, \qquad \text{(F-16)}$$

where g(z) is analytic at z=0. Then

$$f(z) = \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\right)\left(g(0) + g^{(1)}(0)\; z + \frac{1}{2!}g^{(2)}(0)\; z^2 + \ldots\right)\;. \qquad \text{(F-17)}$$

The coefficient of 1/z in (F-17) is the residue of f(z) at z=0; namely

$$\text{Res} = g(0) + \frac{1}{2!}\,g^{(1)}(0) + \frac{1}{3!}\,\frac{1}{2!}\,g^{(2)}(0) + \ldots = \sum_{n=0}^{+\infty} \frac{g^{(n)}(0)}{n!\,(n+1)!}\;. \qquad \text{(F-18)}$$

Thus the residue of f(z) at z=0 depends on the behavior of g(z) in a neighborhood of z=0, and not just the value g(0).

A couple of examples yield the following:

$$g(z) = (1-az)^{-1}\;, \quad \text{Res} = \frac{\exp(a)-1}{a}\;;$$

$$g(z) = \exp(a^2 z)\;, \quad \text{Res} = \frac{I_1(2a)}{a}\;. \qquad \text{(F-19)}$$

CORRECT APPROACH FOR C(ρ)

Reconsider the integral in (F-5) plus the scale factor $\rho^{1/2}$; making the substitution $z = \theta + iL$, where L is given in (F-8), there follows for this quantity

$$(2\lambda)^{1/2} \int_{-\pi+iL}^{\pi+iL} dz\ (1-2\omega\lambda\ e^{iz})^{-1}\ \left(1 - \frac{2\omega\lambda}{1+\omega}\ e^{iz}\right)^{-\frac{1}{2}} *$$

$$* \exp\left[i\ \frac{z}{2} + \frac{1}{2}\ e^{-iz} + 2\lambda u\ e^{iz}\right] . \qquad \text{(F-20)}$$

The uppermost singularity of the integrand in the z-plane (within the $-\pi$, π strip) is a pole at $z_p = i\ \ell n(2\omega\lambda)$; however, the straight line contour in (F-20) remains above this pole because $\rho < 1/\omega$; see (F-8). Furthermore, the total integrand of (F-20) has a saddle point on the imaginary axis of the z-plane above the pole location z_p, because the integrand is infinite at the pole and at $z = 0 + i\infty$. Thus the straight line contour in (F-20) can be modified so as to pass through the saddle point, and yet remain above z_p. Finally, letting $\rho \to 0+$, then $L \to +\infty$, and (F-20) combined with (F-5) yields the exact result for the circular component

$$C(0) = \frac{r}{\pi\ 2^{1/2}\ (1+\omega)^{1/2}} \exp\left(-\frac{u+r^2}{\omega}\right) \int_{-\pi+i\infty}^{\pi+i\infty} dz\ (1-2\omega\lambda\ e^{iz})^{-1} *$$

$$*\left(1 - \frac{2\omega\lambda}{1+\omega}\ e^{iz}\right)^{-1/2} \exp\left[i\ \frac{z}{2} + \frac{1}{2}\ e^{-iz} + 2\lambda u\ e^{iz}\right] , \qquad \text{(F-21)}$$

where the two valleys of the integrand at $\pm\pi+i\infty$ are joined with a contour through the saddle point lying above the pole at $z_p = i\ \ell n(2\omega\lambda)$. Here $\omega = 1+2R$.

The other component of the exceedance distribution function, corresponding to (F-21), is given by (F-3) at $\rho = 0+$:

$$V(0) = \frac{2\,\omega^{1/2}}{\pi} \exp\left(-\frac{u+r^2}{\omega}\right) \int_0^{+\infty} dt\ (1+t^2)^{-1} *$$

$$*(1+\omega+t^2)^{-1/2} \exp\left(-\frac{u}{\omega}\,t^2 - \frac{r^2}{\omega t^2}\right). \qquad \text{(F-22)}$$

Thus for $u \geq 0$, (F-1) and figure F-1 yield exceedance distribution function

$$1 - P_h(u) = C(0) + V(0) = \text{(F-21)} + \text{(F-22)}\ . \qquad \text{(F-23)}$$

Computationally, (F-21) is not too attractive, because of the complex integrand and/or the need to determine the steepest descent paths to $\pm\pi+i\infty$ numerically. Accordingly, an alternative direct procedure for determining the exceedance distribution function of random variable h is now presented.

DIRECT EVALUATION OF EXCEEDANCE DISTRIBUTION FUNCTION

For $\gamma=0$, N=1, (12) and (3) yield the crosscorrelator output for the signal and noise model as

$$q = u_1 v_1 = [u_u + u_s(1) + u_d(1)]\ [u_v + v_s(1) + v_d(1)]\ . \qquad \text{(F-24)}$$

The normalized crosscorrelator output is then, from (49) and (97),

$$h = \frac{q}{(D_u D_v)^{1/2}} = (r_u + u_s' + u_d')\ (r_v + v_s' + v_d') \equiv xy\ , \qquad \text{(F-25)}$$

where x and y are joint Gaussian random variables with statistics

$$\bar{x} = r_u , \quad \bar{y} = r_v ,$$

$$\sigma_x^2 = 1 + R_u, \quad \sigma_y^2 = 1 + R_v, \quad \overline{(x-\bar{x})(y-\bar{y})} = \rho_s (R_u R_v)^{1/2} . \qquad \text{(F-26)}$$

We now make the same assumptions as in (99); see also (56) et seq. Then (F-26) specializes to

$$\bar{x} = \bar{y} = r, \quad \sigma_x^2 = \sigma_y^2 = 1 + R \equiv \sigma^2, \quad \rho_{xy} = \frac{R}{1+R} . \qquad \text{(F-27)}$$

The joint probability density function of x, y is then given by

$$p_2(x,y) = \left[2\pi\sigma^2\left(1-\rho_{xy}^2\right)^{1/2}\right]^{-1} \exp\left[-\frac{(x-r)^2 + (y-r)^2 - 2\rho_{xy}(x-r)(y-r)}{2\sigma^2(1-\rho_{xy}^2)}\right] . \qquad \text{(F-28)}$$

We now have cumulative distribution function

$$P_h(u) = \iint_{R_2+R_4} dx\, dy\, p_2(x,y) \quad \text{for } u \leq 0 , \qquad \text{(F-29)}$$

and exceedance distribution function

$$1 - P_h(u) = \iint_{R_1+R_3} dx\, dy\, p_2(x,y) \quad \text{for } u \geq 0 , \qquad \text{(F-30)}$$

where regions R_1, R_2, R_3, R_4 are indicated in figure F-3.

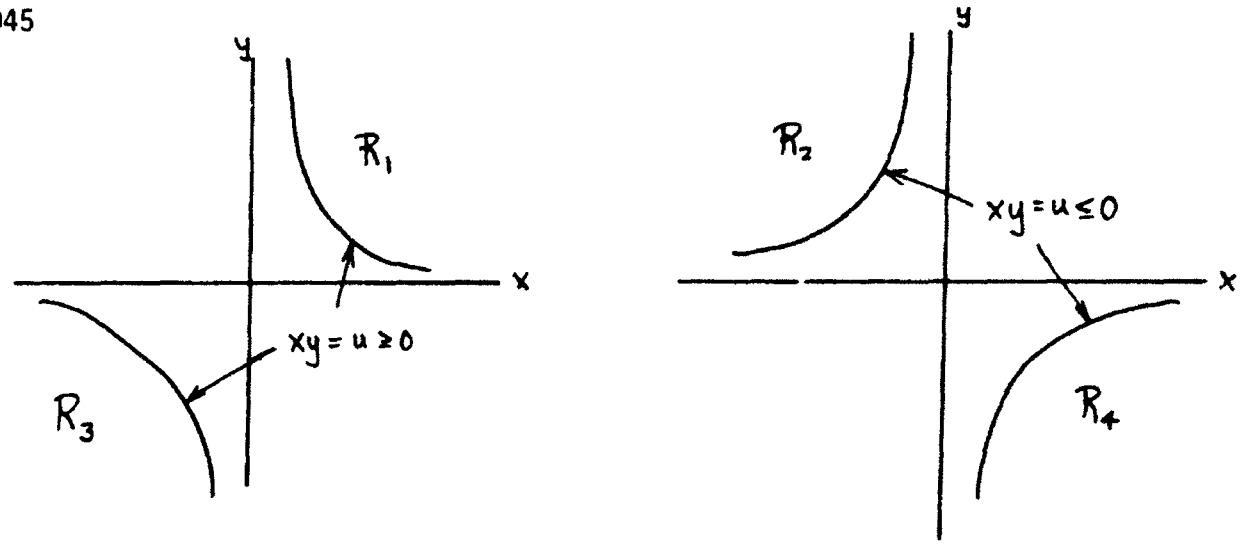

Figure F-3. Regions of Integration

If we now rotate axes according to

$$s = \frac{x+y}{2}, \quad t = \frac{x-y}{2} \tag{F-31}$$

and employ (84) and (F-27), there follows (after scale changes of the variables)

$$P_h(u) = 2\int_{-\infty}^{+\infty} dv\ \phi\left(v - r(2/\omega)^{1/2}\right)\Phi\left(-(\omega v^2 - 2u)^{1/2}\right) \quad \text{for } u \leq 0\ , \tag{F-32}$$

and

$$1 - P_h(u) = 2\int_{0}^{+\infty} dv\ \phi(v)\left[\Phi\left(\frac{r - \sqrt{u+v^2/2}}{\sqrt{\omega/2}}\right) + \Phi\left(\frac{-r - \sqrt{u+v^2/2}}{\sqrt{\omega/2}}\right)\right] \text{for } u \geq 0\ . \tag{F-33}$$

Here $\omega = 1+2R$. These real integrals are very useful for the evaluation of the distributions of h when $\gamma=0$, N=1. In fact, (F-33) is preferred over (F-21)-(F-23); but (106) is preferred over (F-32) since Φ need not be evaluated in (106). This is in fact the procedure utilized here to obtain numerical results for this case of $\gamma=0$, N=1.

APPENDIX G. PROGRAM FOR EVALUATION OF OPERATING CHARACTERISTICS FOR γ=0

The comments in appendices C and D are relevant here also. The characteristic function used as the starting point is given by (100). It was observed under (103) that $|f_h(\xi)|$ for (100) is monotonically decreasing for all $\xi \geq 0$; thus the choice of truncation value L is simplified; see appendix D comments. A table for sampling increment Δ_0 (when R=0) follows.

N	Δ_0 for r=1	Δ_0 for r=2
3	.07	.05
4	.07	.05
8	.05	.03
16	.04	.02
32	.03	.02
64	.025	.015
128	.020	.010
256	.012	.007

Table G-1. Values of Δ_0 for γ=0

```
10 !  GAMMA = 0        NO SAMPLE MEAN REMOVAL
20    Nc=32                        !  N, Number of terms added
30    Rs=1                         !  r, Normalized mean
40    Delta0=.03                   !  Initial delta
50    Bs=2*PI/Delta0*.375          !  Bias b (depends on r)
60    Mf=2^10                      !  Size of FFT
70    OUTPUT 0;"GAMMA = 0";"     N =";Nc;"     r =";Rs
80    OUTPUT 0;""
90    DATA -4,-3,-2.5,-2,-1.5,-1,-.5,0,.5,1,1.5,2
100   READ Ns(*)                   !  SNR R=2^n
110   OUTPUT 0;Ns(*);
120   DATA 1,2,2,2,2,2,2,2,2,4,4,4,8
130   READ Idelta(*)
140   MAT Delta=(Delta0)/Idelta
150   OUTPUT 0;Delta(*);
160   DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,.001,.01,.1,.5,.9,.99,.999
170   READ Sc(*)
180   DIM Ns(1:12),Idelta(0:12),Delta(0:12),Sc(1:14)
190   DIM X(0:8191),Y(0:8191)
200   FOR I=1 TO 14
210   Sc(I)=FNInvphi(Sc(I))
220   NEXT I
```

```
230   S=Sc(1)
240   B=Sc(14)
250   Scale=(B-S)/(0-S)
260   X1=30
270   X2=170
280   Y1=35
290   Y2=Y1+(X2-X1)*Scale
300   PLOTTER IS "9872A"
310   LIMIT X1,X2,Y1,Y2
320   OUTPUT 705;"VS3"
330   SCALE S,0,S,B
340   FOR I=1 TO 14
350   MOVE S,Sc(I)
360   DRAW 0,Sc(I)
370   NEXT I
380   FOR I=1 TO 11
390   MOVE Sc(I),S
400   DRAW Sc(I),B
410   NEXT I
420   MOVE S,S
430   DRAW 0,0
440   PENUP
450   M1=Mf-1
460   N2=Nc/2
470   Rsn=Nc*Rs*Rs                !  N r^2
480   FOR In=0 TO 12
490   IF In>0 THEN 520
500   Rc=0
510   GOTO 530
520   Rc=2^Ns(In)                 !  SNR R=2^n
530   OUTPUT 0;"R =";Rc,"    Delta =";Delta(In)
540   ASSIGN #1 TO "ABSCIS"       !  Temporary storage
550   Delta=Delta(In)             !  for false alarm probability
560   R2=Rc*2
570   R21=R2+1
580   Mux=Nc*Rc+Rsn               !  Mean of random  variable h
590   Muy=Mux+Bs                  !  Mean of shifted variable y
600   REDIM X(0:M1),Y(0:M1)
610   MAT X=ZER
620   MAT Y=ZER
630   X(0)=0
640   Y(0)=.5*Delta*Muy
650   Ls=0
660   Ls=Ls+1
670   Xi=Delta*Ls                 !  Argument xi of char. fn.
680   Ei=Xi*R21                                    !  Calculation
690   CALL Log(1+Xi*Ei,-Xi*R2,Ai,Bi)               !  of
700   CALL Div(0,Xi*Rsn,1,-Ei,Ci,Di)               !  characteristic
710   CALL Exp(Ci-N2*Ai,Di+Xi*Bs-N2*Bi,Fyr,Fyi)    !  function
720   Ms=Ls MOD Mf                                 !  fy(xi)
730   Ar=Fyr/Ls
740   Ai=Fyi/Ls
750   X(Ms)=X(Ms)+Ar
760   Y(Ms)=Y(Ms)+Ai
770   Magsq=Ar*Ar+Ai*Ai
780   IF Magsq>1E-24 THEN 660
790   OUTPUT 0;"Xi =";Xi;"    Mag =";SQR(Magsq)
800   CALL Fft13z(Mf,X(*),Y(*))
```

```
810   FOR Ms=0 TO M1
820   T=Y(Ms)/PI-Ms/Mf
830   X(Ms)=.5-T                    !  Cumulative distribution function
840   Y(Ms)=.5+T                    !  Exceedance distribution function
850   NEXT Ms
860   OUTPUT 0;Y(0);Y(1);Y(M1-1);Y(M1)
870   PLOTTER IS "GRAPHICS"
880   GRAPHICS
890   SCALE 0,Mf,-14,0
900   LINE TYPE 3
910   GRID Mf/8,1
920   PENUP
930   LINE TYPE 1
940   FOR Ms=0 TO M1
950   Pr=Y(Ms)
960   IF Pr>=1E-12 THEN Y=LGT(Pr)
970   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
980   IF ABS(Pr)<1E-12 THEN Y=-12
990   PLOT Ms,Y
1000  NEXT Ms
1010  PENUP
1020  FOR Ms=0 TO M1
1030  Pr=X(Ms)
1040  IF Pr>=1E-12 THEN Y=LGT(Pr)
1050  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1060  IF ABS(Pr)<1E-12 THEN Y=-12
1070  PLOT Ms,Y
1080  NEXT Ms
1090  PENUP
1100  DUMP GRAPHICS
1110  OUTPUT 0;""
1120  IF In>0 THEN 1270
1130  FOR Ms=0 TO M1
1140  IF Y(Ms)<.7 THEN 1160
1150  NEXT Ms
1160  M2=Ms-1
1170  FOR Ms=M2 TO M1
1180  IF Y(Ms)<=0 THEN 1200
1190  NEXT Ms
1200  M3=Ms-1
1210  REDIM X(M2:M3)
1220  FOR Ms=M2 TO M3
1230  X(Ms)=FNInvphi(Y(Ms))
1240  NEXT Ms
1250  PRINT #1;X(*)                 !  Store false alarm probability
1260  GOTO 1460
1270  REDIM X(M2:M3)
1280  READ #1;X(*)                  !  Read in false alarm probability
1290  Id=Idelta(In)
1300  J2=INT(M2/Id)
1310  J3=INT(M3/Id)+1
1320  FOR J=J2 TO J3
1330  Y(J)=FNInvphi(Y(J))
1340  NEXT J
```

```
1350  PLOTTER IS "9872A"
1360  LIMIT X1,X2,Y1,Y2
1370  OUTPUT 705;"VS3"
1380  SCALE S,0,S,B
1390  FOR J=J2 TO J3
1400  T=J*Id
1410  IF T<M2 THEN 1440
1420  IF T>M3 THEN 1450
1430  PLOT X(T),Y(J)
1440  NEXT J
1450  PENUP
1460  NEXT In
1470  END
1480  !
1490  SUB Div(X1,Y1,X2,Y2,A,B)          !  DIV(Z)
1540  !
1550  SUB Exp(X,Y,A,B)                  !  EXP(Z)
1600  !
1610  SUB Log(X,Y,A,B)                  !  PRINCIPAL LOG(Z)
1690  !
1700  DEF FNInvphi(X)           !  INVPHI(X) via AMS 55, 26.2.23
1830  !
1840  SUB Fft13z(N,X(*),Y(*))   !  N <= 2^13, N=2^INTEGER, 0 SUBSCRIPT
```

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Under-Ice Roughness: Shot Noise Model

A. H. Nuttall

ABSTRACT

The one-dimensional roughness of an under-ice profile of elliptical bosses is modeled in the time domain by a shot-noise process of elliptical pulses of random amplitude, duration, and time of occurrence. A sample realization of 8000 data points is generated and plotted for visual comparison with experimental under-ice data. Also, theoretical and simulation results for the power density spectrum, the autocorrelation function, the characteristic function, the cumulative distribution function, and the probability density function of the shot-noise process are plotted and compared.

INTRODUCTION

The under-ice profile has been observed to appear like a random collection of superposed elliptical bosses, each of random amplitude, length, and location. An analogous model in the time domain is shot noise composed of overlapping pulses of random amplitude, duration, and time of occurrence. Accordingly, we have generated a sample realization of a shot noise process for visual comparison with experimental under-ice data, and for possible corroboration of this model. The particular realization generated has 8000 data points, although the number of effectively-independent samples is far fewer, as will be demonstrated.

A number of analytical results for shot noise have been derived in the past [1]; however, they did not cover the case of random duration modulation. We have extended the analyses to include random durations (as well as random amplitudes and random time occurrences) and evaluated the spectrum of the shot noise process, as well as the autocorrelation function and the first-order characteristic function of the instantaneous amplitude. From the latter, the first-order probability density function and cumulative distribution function of shot noise have been evaluated via a generalized Laguerre expansion employing 32 cumulants or moments. Comparisons of all these theoretical results with the corresponding sample quantities, obtained from the 8000 data point realization above, reveal excellent agreement.

A REALIZATION OF A SHOT-NOISE PROCESS

Shot noise is characterized by a superposition of pulses, each located independently and uniformly on the time scale. A sample pulse is illustrated in figure 1. The time of occurrence t_k (center of symmetrical pulse, for

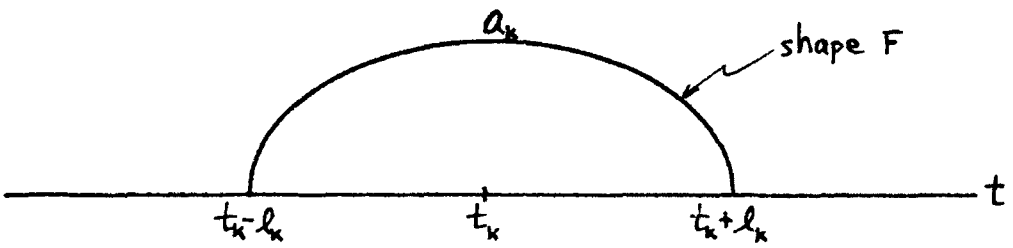

Figure 1. Sample Pulse of Shot Noise

example) is uniformly distributed in time t, with an average number of pulses per second, ν. The amplitude a_k and half-duration ℓ_k of an individual pulse are also all independent and are each identically randomly-distributed with arbitrary probability density functions. Finally the fundamental pulse shape F in figure 1 is arbitrary.

A realization of shot noise is given by

$$I(t) = \sum_k a_k \, F\left(\frac{t-t_k}{\ell_k}\right) , \qquad (1)$$

where the summation extends over all k. The particular data we generate here employs the following example; unscaled pulse shape F is circular:

$$F(x) = \left\{ \begin{array}{ll} (1-x^2)^{1/2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{array} \right\} . \qquad (2)$$

This pulse is continuous; however, it has cusps (infinite slope) at $x = \pm 1$. The reason for this selection will become apparent when we discuss the spectrum of shot noise process (1).

The amplitude probability density function for random variable a_k is Rayleigh,

$$p(a) = \frac{a}{\sigma_a^2} \exp\left(\frac{-a^2}{2\sigma_a^2}\right) U(a) , \tag{3}$$

and the duration probability density function for random variable ℓ_k is also Rayleigh,

$$p(\ell) = \frac{\ell}{\sigma_\ell^2} \exp\left(\frac{-\ell^2}{2\sigma_\ell^2}\right) U(\ell) . \tag{4}$$

Here, step function

$$U(x) = \left\{\begin{matrix} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{matrix}\right\}. \tag{5}$$

The mean values of random variables a_k and ℓ_k are given respectively by

$$\bar{a} = \bar{a}_k = \left(\frac{\pi}{2}\right)^{1/2} \sigma_a , \quad \bar{\ell} = \bar{\ell}_k = \left(\frac{\pi}{2}\right)^{1/2} \sigma_\ell , \tag{6}$$

in terms of the parameters σ_a and σ_ℓ of probability density functions (3) and (4). Alternatively, the mean square values are given by

$$\overline{a^2} = \overline{a_k^2} = 2\sigma_a^2 , \quad \overline{\ell^2} = \overline{\ell_k^2} = 2\sigma_\ell^2 . \tag{7}$$

Three typical component pulses are depicted in figure 2, and can range from circular through various elongated elliptical shapes. The total length of an individual pulse is $L_k = 2\ell_k$. An important parameter of this time-

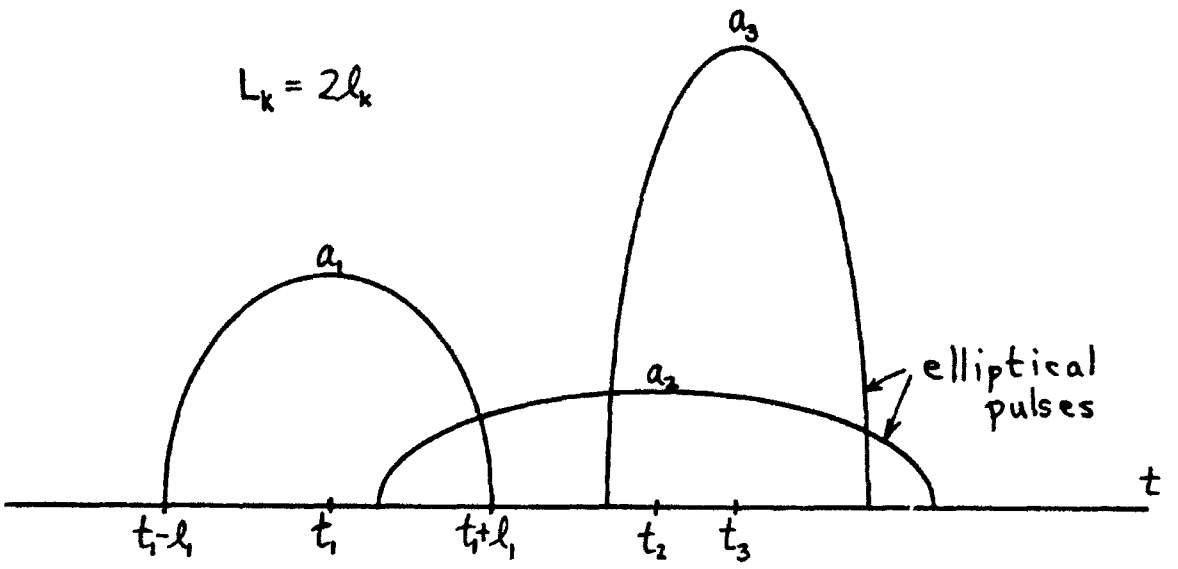

Figure 2. Three Component Pulses

limited pulse shape in figures 1 and 2 is the (dimensionless) overlap factor

$$\Gamma_k \nu = 2\bar{\ell}_k \nu = 2\left(\frac{\pi}{2}\right)^{1/2} \sigma_\ell \nu \ . \tag{8}$$

This is the average number of pulses that are overlapping at any one instant of time, and is a partial measure of the applicability of the central limit theorem. A more meaningful measure are the cumulants, for probability density function (3) and pulse shape (2), the normalized third and fourth cumulants are

$$\frac{1.017}{(\Gamma_k \nu)^{1/2}} \quad \text{and} \quad \frac{1.2}{\Gamma_k \nu} \ , \tag{9}$$

respectively. In the sample realization generated here, the overlap factor in (8) was 6.2, leading to normalized cumulant values in (9) of .58 and .39, respectively. Since a Gaussian probability density function would lead to zero cumulants above second-order, the shot noise realization dealt with here is distinctly non-Gaussian.

In the three parts of figure 3, a realization of shot noise model (1) is given for parameter values

$$\sigma_a = 1 \text{ sec}, \quad \sigma_\ell = 20 \text{ sec}, \quad \nu = .124 \text{ pulses/sec}. \quad (10)$$

The waveform (1) is sampled at unit time increments and connected by straight lines; thus the initial 100 data points illustrated in figure 3A have a jagged appearance for those component pulses with small ℓ_k, as for example at time instants 67-68. The larger duration pulses, like the one centered at t = 29, have a smoother appearance.

In figure 3B, the initial 1000 data points illustrate the very erratic character of shot noise; the waveform consists of some very sharp spiky pulses and other broader smooth components. The appearance of a downward trend in these 1000 data points is erased when the entire 8000 data point sequence is viewed in figure 3C. The possibility of shot noise process (1) reaching a zero value (when no pulses overlap) is confirmed by the waveform values near t = 6400.

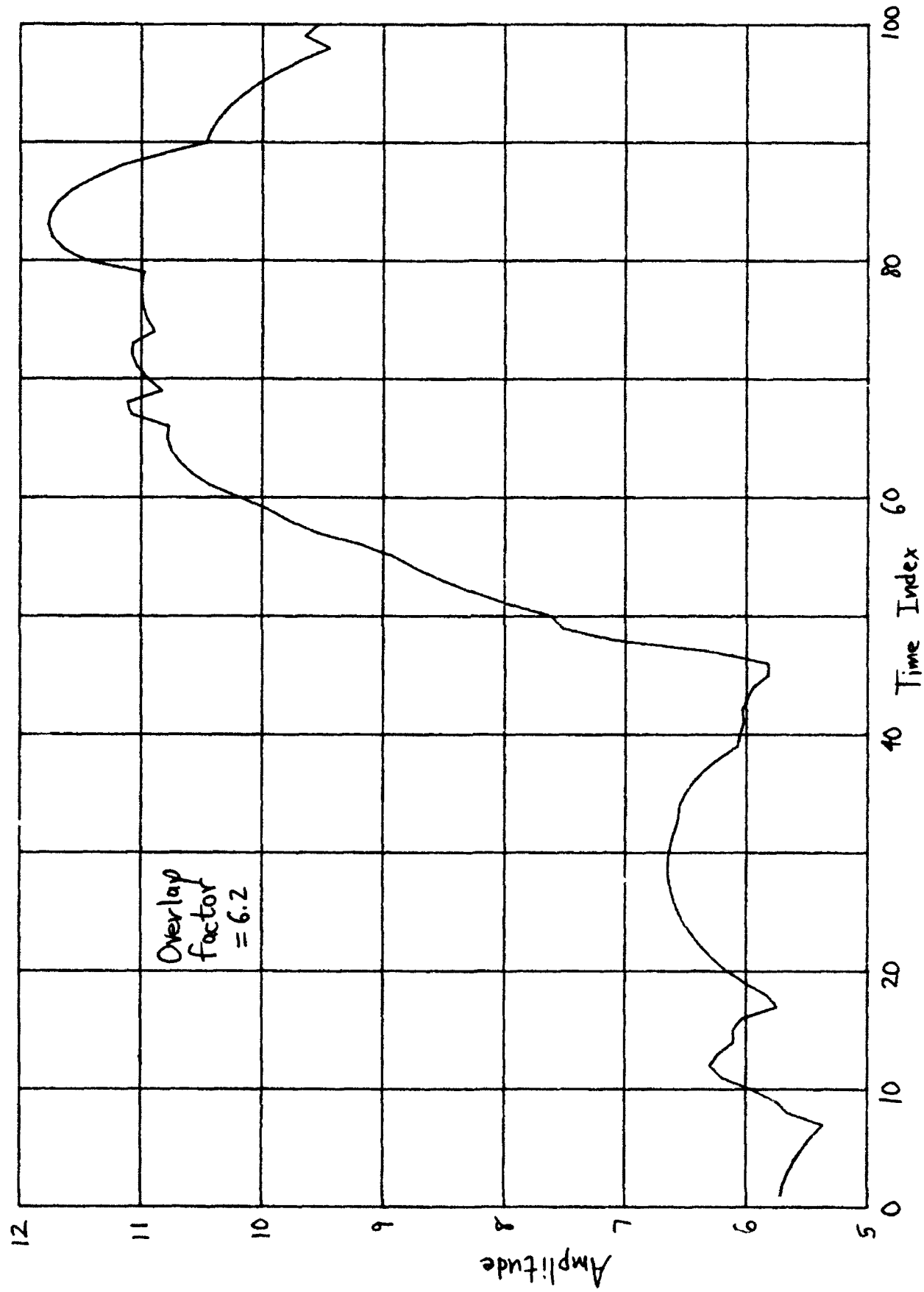

Figure 3A. Initial 100 Data Points

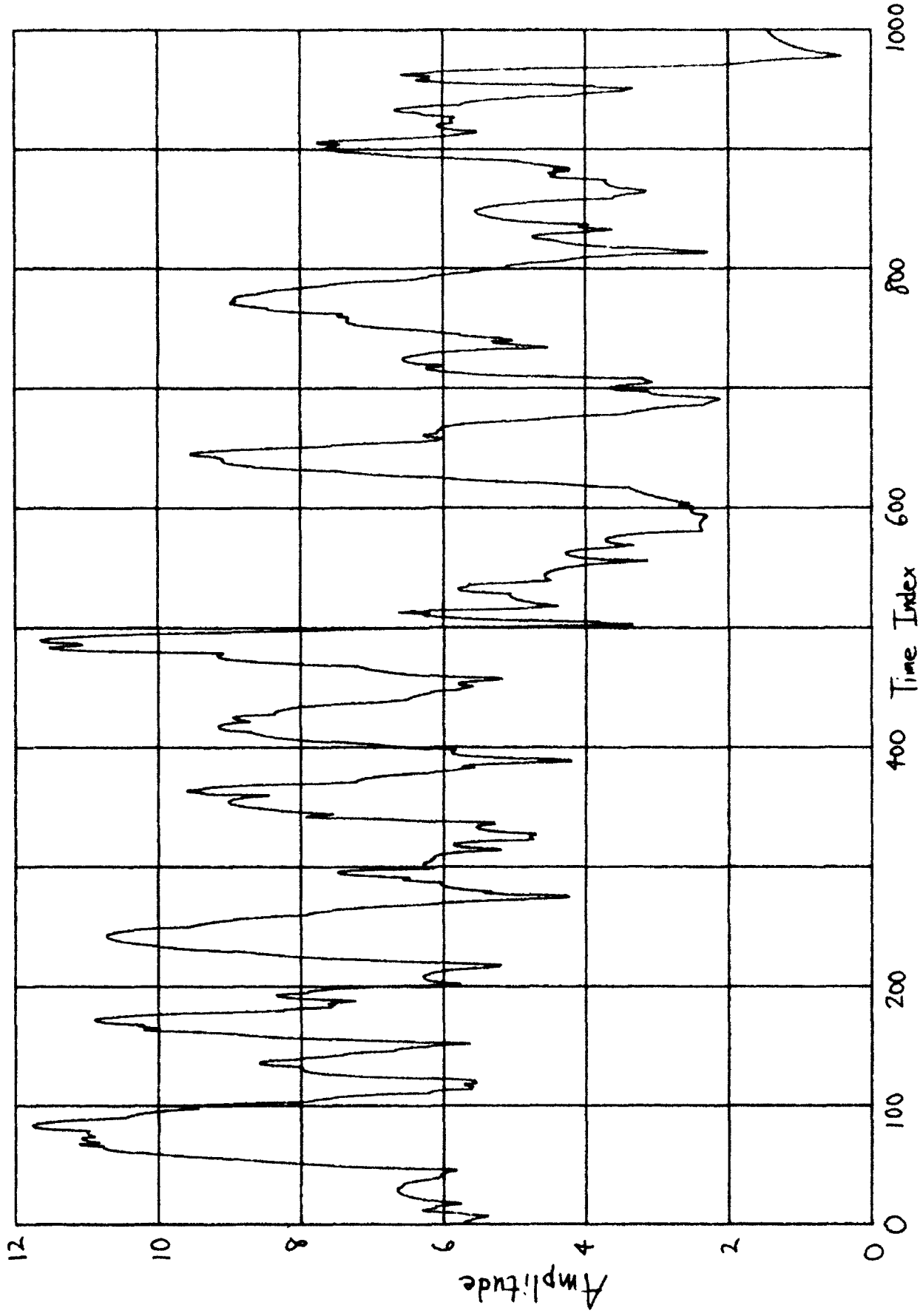

Figure 3B. Initial 1000 Data Points

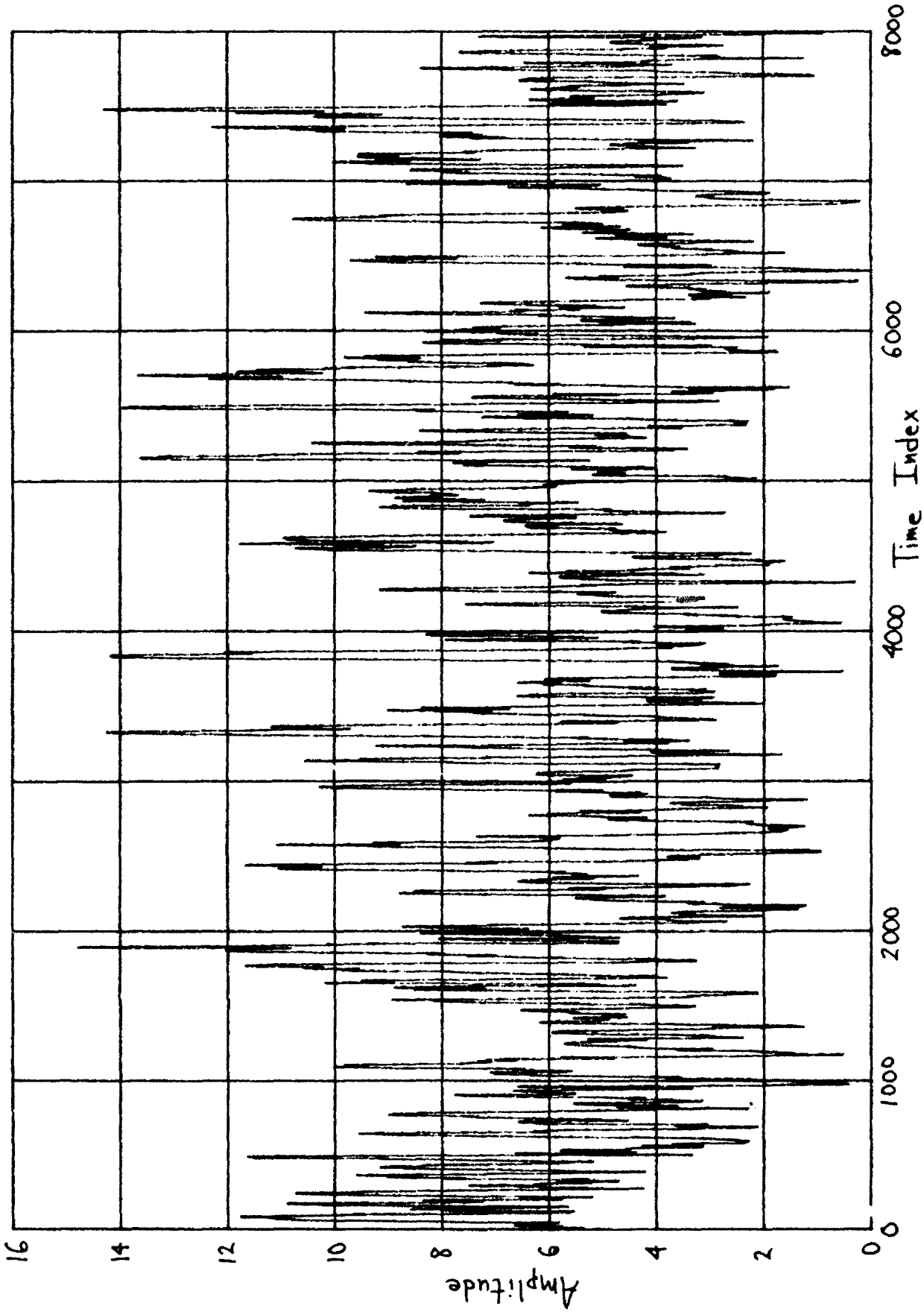

Figure 3C. Total Data Segment

CORRELATION AND SPECTRUM OF SHOT NOISE PROCESS

The derivations of the correlation and spectrum of the shot noise process (1) are given in appendix A; from (A-12), we have, in general, the correlation function at delay τ,

$$R_I(\tau) = \nu \overline{a^2} \int d\ell \; p(\ell)\ell \; \phi(\tau/\ell) + I_{dc}^2 \; , \tag{11}$$

where the dc component of I(t) is, from (A-13),

$$I_{dc} = \nu \bar{a} \bar{\ell} \int dx \; F(x) \; , \tag{12}$$

and

$$\phi(y) = \int dx \; F(x) \; F(x-y) \tag{13}$$

is the (aperiodic) correlation of an individual pulse F. (All integrals are over the range of non-zero integrand.)

Also, from (A-16), the general spectrum of process I(t) is, at frequency f,

$$G_I(f) = \nu \overline{a^2} \int d\ell \; p(\ell) \; \ell^2 \; |S(\ell f)|^2 + I_{dc}^2 \; \delta(f) \; , \tag{14}$$

where

$$S(f) = \int dx \; \exp(-i2\pi fx) \; F(x) \tag{15}$$

is the voltage density spectrum (Fourier transform) of pulse F. Thus $|S(f)|^2$ is the energy density spectrum corresponding to pulse F.

It should be observed that the entire probability density function $p(\ell)$ of half-duration random variable ℓ_k is required in order to evaluate the correlation or spectrum of shot noise. However, only the first two moments, $\overline{a}$ and $\overline{a^2}$, are required known about probability density function p(a) of amplitude random variable a_k. The only way that the dc term I_{dc} can be zero is if random variable a_k has zero mean ($\overline{a} = 0$), or if pulse F has zero area ($S(0) = 0$).

Example

The example of interest here was given earlier in (2) and (4), namely a circular pulse F and a Rayleigh probability density function for random variable ℓ_k. The spectrum $G_I(f)$ in (14) is evaluated in (A-17) through (A-22), with the results

$$S(f) = \frac{J_1(2\pi f)}{2f}, \quad S(0) = \frac{\pi}{2},$$

$$I_{dc} = \left(\frac{\pi}{2}\right)^{3/2} \nu\, \overline{a}\, \sigma_\ell,$$

$$G_I(f) = 2\pi^2\, \nu\, \overline{a^2}\, \sigma_\ell^2\, \frac{2\exp(-z)\, I_1(z)}{z} + \left(\frac{\pi}{2}\right)^3 \nu^2\, \overline{a}^2\, \sigma_\ell^2\, \delta(f)$$

$$\text{with } z = (2\pi\sigma_\ell f)^2. \tag{16}$$

The asymptotic behavior of spectrum (16) is [2, eq. 9.7.1]

$$G_I(f) \sim \frac{\nu\, \overline{a^2}}{(2\pi)^{3/2}\, \sigma_\ell}\, f^{-3} \quad \text{as} \quad f \to +\infty. \tag{17}$$

That is, the spectrum decays at a -30 dB/decade rate at large frequencies; this is due to the square root singularities at $x = \pm 1$ of pulse F given in (2). This decay rate has been observed in some spectral analyses of under-ice profiles, and was one of the reasons for choosing the specific circular pulse in (2) for this investigation.

The spectrum in (16) is plotted in figure 4, for the choice of parameters earlier in (10), as a dashed line, normalized to 0 dB at $f = 0$. Superposed is a linear-predictive spectral analysis result with predictive order 10, for the 8000 data points of figure 3C. The two results are in excellent agreement, even at the -50 dB level, with the inevitable 3 dB aliasing effect at the Nyquist frequency, as indicated.

The correlation $R_I(\tau)$ in (11) is evaluated in (A-23) through (A-33), for the example (2) and (4), with the result

$$R_I(\tau) = \frac{8}{3}(2\pi)^{1/2}\, \nu\, \overline{a^2}\, \sigma_\ell\, s \exp(-s)\, [(1+4s)\, K_1(s) - (3+4s)\, K_0(s)] +$$

$$+ \left(\frac{\pi}{2}\right)^3 \nu^2\, \overline{a}^2\, \sigma_\ell^2, \quad \text{with } s = \left(\frac{\tau}{4\sigma_\ell}\right)^2 . \tag{18}$$

This quantity, exclusive of the I_{dc}^2 term, and normalized at the origin, is plotted in figure 5 as a dashed line, for delays (lags) τ up to 100. It is seen to decay monotonically to zero as τ increases, and reach its 1/e value at approximately $\tau = 30$.

The remaining solid curve on figure 5 is the normalized sample autocorrelation function of the 8000 data point sequence in figure 3C, where the sample mean was subtracted from the given data. The agreement with theoretical result (18) is excellent. The dotted horizontal lines at $\pm 2\sigma$ in figure 5 are the ±2 sigma values of the correlation estimate at delays where the true correlation is presumed zero; the details of this analysis are given in appendix B.

This procedure is duplicated in figure 6, where the correlation function estimate out to lag $\tau = 1000$ is plotted. The drifting of the estimate outside the $\pm 2\sigma$ limits (at $\tau = 470$ and 820) is consistent with an occasional excursion of a random variable outside its $\pm 2\sigma$ range. The correlation estimate (used for figures 5 and 6) at time separation k is

$$R_k = \frac{1}{N} \sum_{n=k+1}^{N} x_n \, x_{n-k} \quad \text{for } k \geq 0 \, , \tag{19}$$

where $\{x_n\}_1^N$ is the available data in figure 3C, with its sample mean removed.

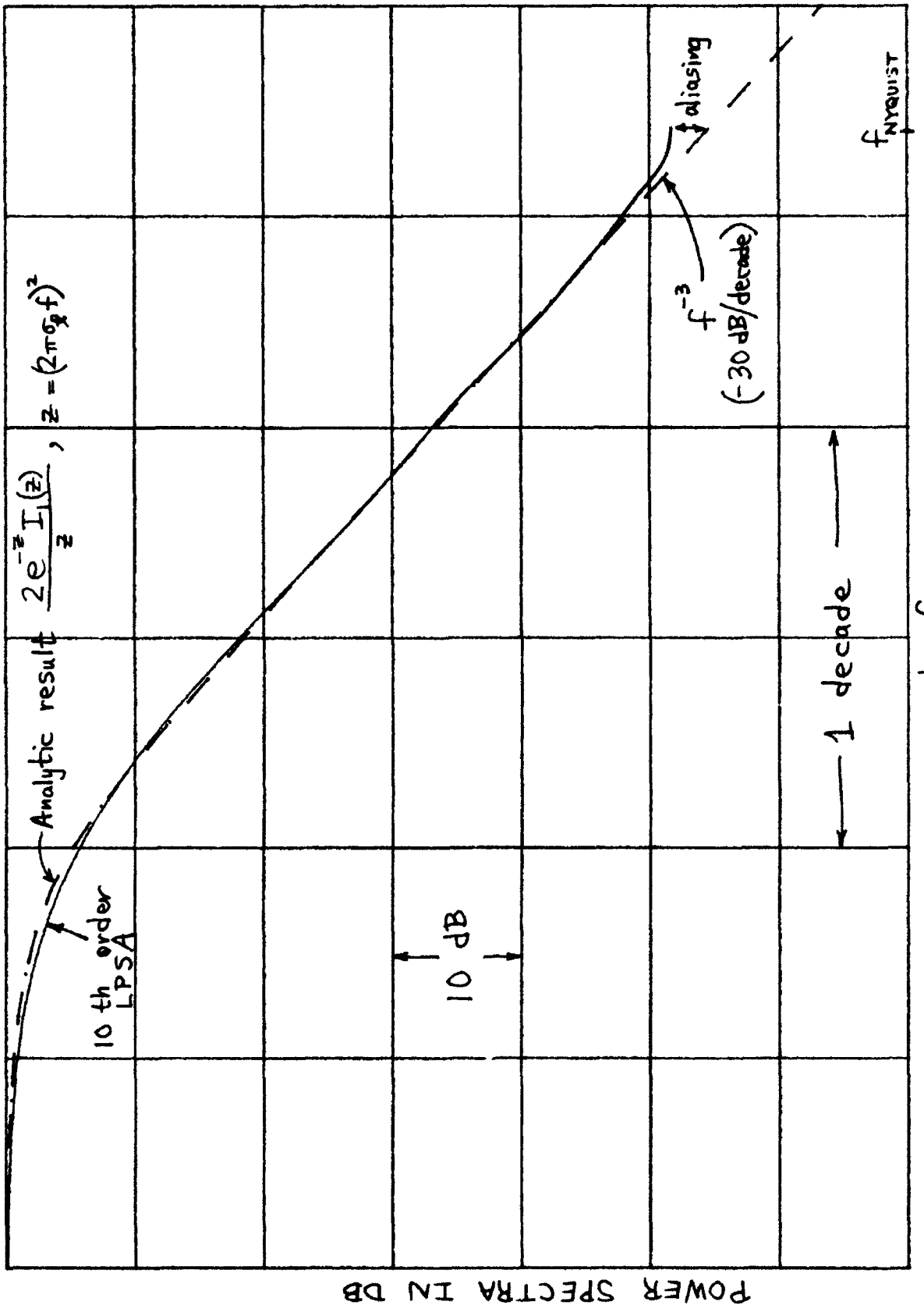

Figure 4. Comparison of Power Spectra

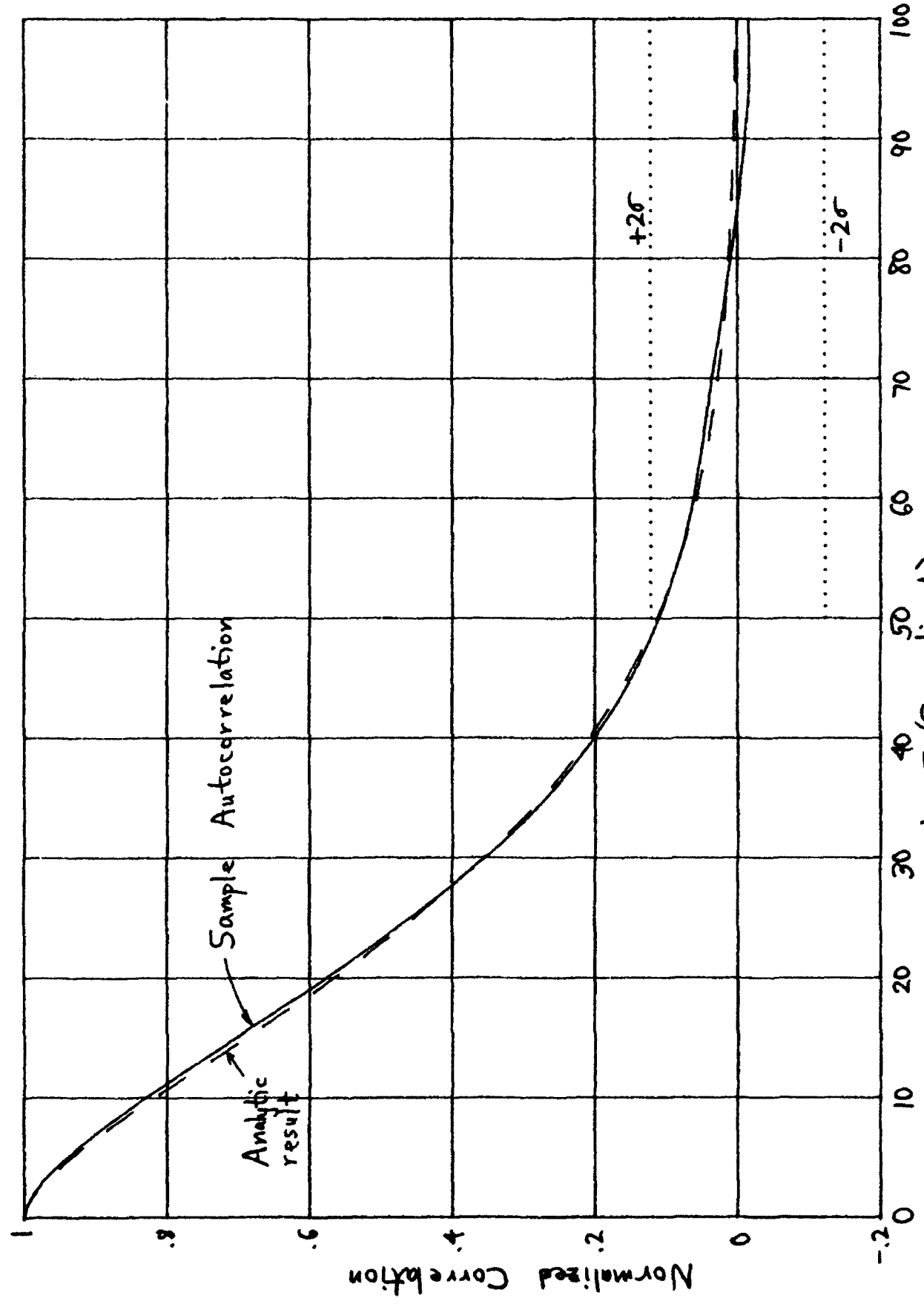

Figure 5. Estimate of Autocorrelation Function near origin

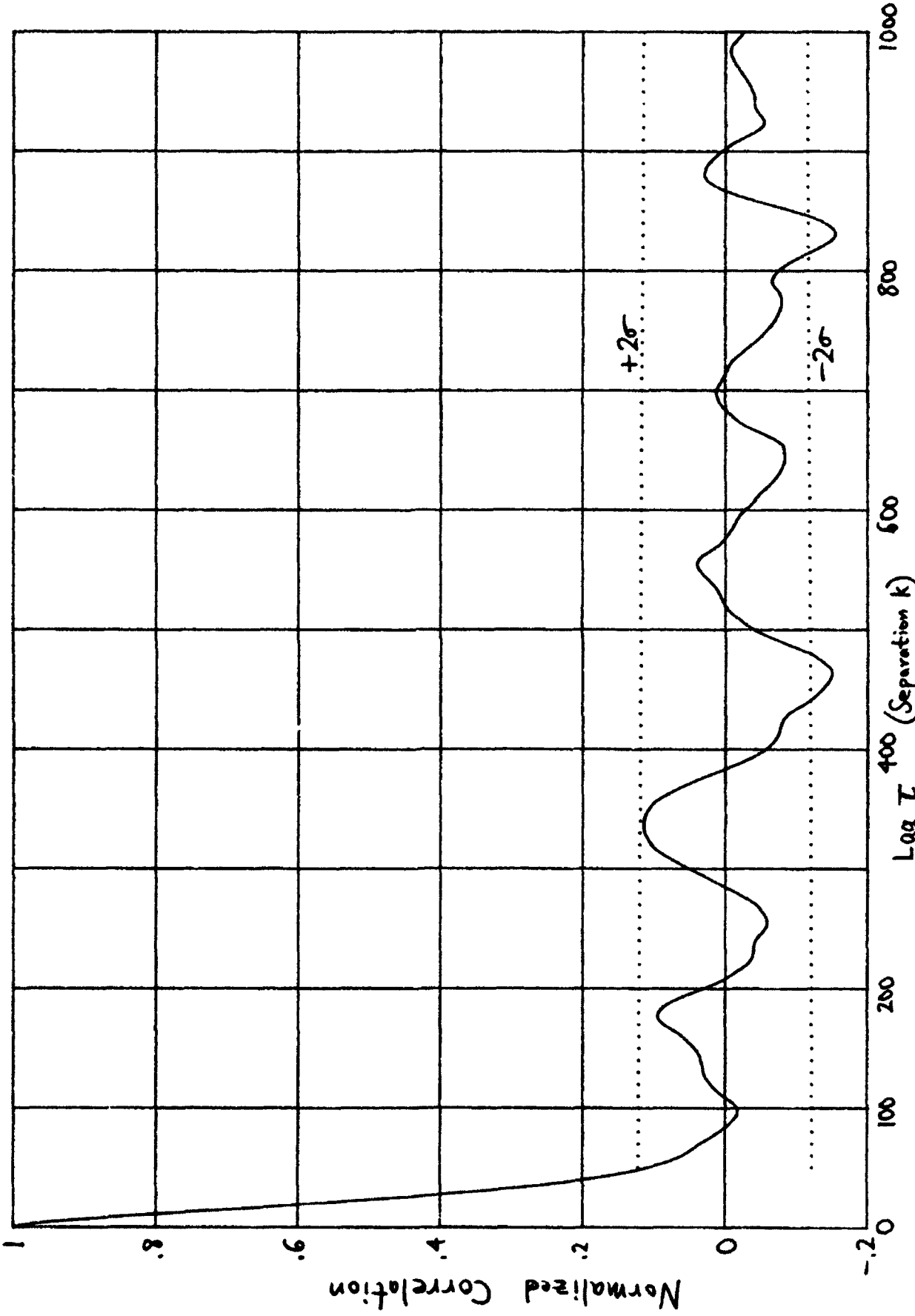

Figure 6. Estimate of Autocorrelation Function

AMPLITUDE STATISTICS OF SHOT NOISE

The first-order characteristic function of shot noise process I(t) is derived in appendix C; it is given by (C-9) as

$$f_I(\xi) = \exp\left[\nu \bar{\ell} \int dx \left\{ f_a[\xi F(x)] - 1 \right\}\right] . \tag{20}$$

Here f_a is the first-order characteristic function of amplitude random variable a_k. Observe that the probability density function $p(\ell)$ of duration ℓ_k is irrelevant to characteristic function f_I, except for its mean $\bar{\ell}$; this is in contrast to the spectrum and correlation results in (11) and (14), where p(a) was irrelevant except for parameters $\bar{a}$ and $\overline{a^2}$. (For $\ell_k = 1$ for all k, (20) reduces to a simplified version of [1, eq. 4.5-4].)

The characteristic function of the amplitude random variable a_k can be expanded in terms of its moments

$$\mu_a(n) = \overline{a^n} = \int da\, a^n\, p(a) \quad \text{for } n \geq 0 , \tag{21}$$

according to

$$f_a(\xi) = \sum_{n=0}^{\infty} \mu_a(n)(i\xi)^n/n! \quad . \tag{22}$$

This result is useful if the ℓn of (20) is expanded in a series in ξ; namely

$$\ell n\, f_I(\xi) = \nu \bar{\ell} \sum_{n=1}^{\infty} \mu_a(n)(i\xi)^n \int dx\, F^n(x)/n! \quad , \tag{23}$$

giving immediately the cumulants of I(t) as

$$\chi_I(n) = \nu \bar{\ell}\, \mu_a(n) \int dx\, F^n(x) \quad \text{for } n \geq 1 . \tag{24}$$

That is, the n-th cumulant of I(t) is proportional to the n-th moment of random variable a_k as well as the n-th "moment" of pulse F. (For $\ell_k = 1$ for all k, (24) reduces to [1, eq. 1.5-2].)

The normalized cumulant of I(t) is

$$\gamma_I(n) = \frac{\chi_I(n)}{[\chi_I(2)]^{n/2}} = \frac{1}{(\nu \bar{\ell})^{\frac{n}{2}-1}} \frac{\mu_a(n) \int dx\, F^n(x)}{\left[\mu_a(2) \int dx\, F^2(x)\right]^{n/2}} . \qquad (25)$$

In particular, the coefficients of skewness and excess [3, pp. 184 and 187] are

$$\gamma_I(3) = \frac{1}{(\nu \bar{\ell})^{1/2}} \frac{\mu_a(3) \int dx\, F^3(x)}{\left[\mu_a(2) \int dx\, F^2(x)\right]^{3/2}} \qquad (26)$$

and

$$\gamma_I(4) = \frac{1}{\nu \bar{\ell}} \frac{\mu_a(4) \int dx\, F^4(x)}{\left[\mu_a(2) \int dx\, F^2(x)\right]^2} . \qquad (27)$$

These quantities are very important measures of the approach of I(t) to a Gaussian process; if $\nu\bar{\ell}$ is very large, the normalized cumulants $\gamma_I(n)$ are all substantially zero for $n \geq 3$, meaning that I(t) is nearly Gaussian. Thus although probability density function $p(\ell)$ is not directly relevant to the probability density function or characteristic function (20) of I(t), the exact probability density function of I(t) is critically dependent on the mean $\bar{\ell}$ through the dimensionless parameter $\nu\bar{\ell}$. More precisely, (26) and (27) are the critical quantities; see also [1, eq. 1.6-3].

If either the third moment of random variable a_k is zero, or if the third moment of pulse F is zero, then $\gamma_I(3) = 0$. In that case, $\gamma_I(4)$ is the most important statistic measuring the applicability of the central limit theorem; $\gamma_I(4)$ can never be zero for shot noise, since neither the fourth moment of random variable a_k or pulse F can be zero (except in a trivial case).

The first moment of shot noise I(t) is the mean

$$I_{dc} = \overline{I(t)} = \chi_I(1) = \nu \bar{\ell} \bar{a} \int dx\ F(x) \qquad (28)$$

and has already been encountered in (12). It can be zero only if the first moment of random variable a_k or of pulse F is zero.

Example

Numerous cases have been considered in appendix C; in the main body here, we limit attention to example (2) and (3) presented earlier. We find

$$\mu_a(n) = 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) \sigma_a^n \quad \text{for } n \geq 0 ,$$

$$\int dx\ F^n(x) = \frac{2^{n+1} \Gamma^2\left(\frac{n}{2} + 1\right)}{\Gamma(n+2)} \quad \text{for } n \geq 0 . \qquad (29)$$

Then (26) and (27) yield result (9) quoted earlier.

The realization of shot noise process I(t) in figure 3C employed the parameters in (10). The sample cumulative distribution function of these 8000 data points is depicted in figure 7, on a normal probability ordinate; thus a

truly Gaussian random variable would have the straight line character indicated. The significant deviation of the sample cumulative distribution function from the Gaussian line is due to the small value of the overlap factor in (8), namely

$$\Gamma_k \nu = 2 \mathcal{I}_k \nu = 6.2 \ . \qquad (30)$$

The moments in (29) are all positive and are easily numerically evaluated via recursion; hence the cumulants in (24) can be accurately evaluated for high-order n. When these cumulants are employed in a generalized Laguerre expansion of the cumulative distribution function of I(t), using 32 moments of (29), the solid curve in figure 8 is obtained. The sample cumulative distribution function of figure 7 is duplicated here, although the abscissa is scaled differently. The agreement between theory and experiment in figure 8 is excellent, considering the fact that we only have about 8000/30 = 270 effectively independent samples of I(t) in figure 3C; the denominator factor of 30 here is the effective correlation duration, previously identified in figure 5 at the 1/e point.

Finally, when the same 32 moments are used in a generalized Laguerre expansion of the probability density function of I(t), the result in figure 9 is obtained. The small bump near the origin is real and accurate; it and the non-symmetric tails of the probability density function confirm the distinctly non-Gaussian character of I(t). The method for the determination of the cumulative distribution function and probability density function in figures 8 and 9 will be presented in a NUSC Technical Report [4] by the author; the programs are listed here in appendix D, along with an example of the sequence of Laguerre coefficients.

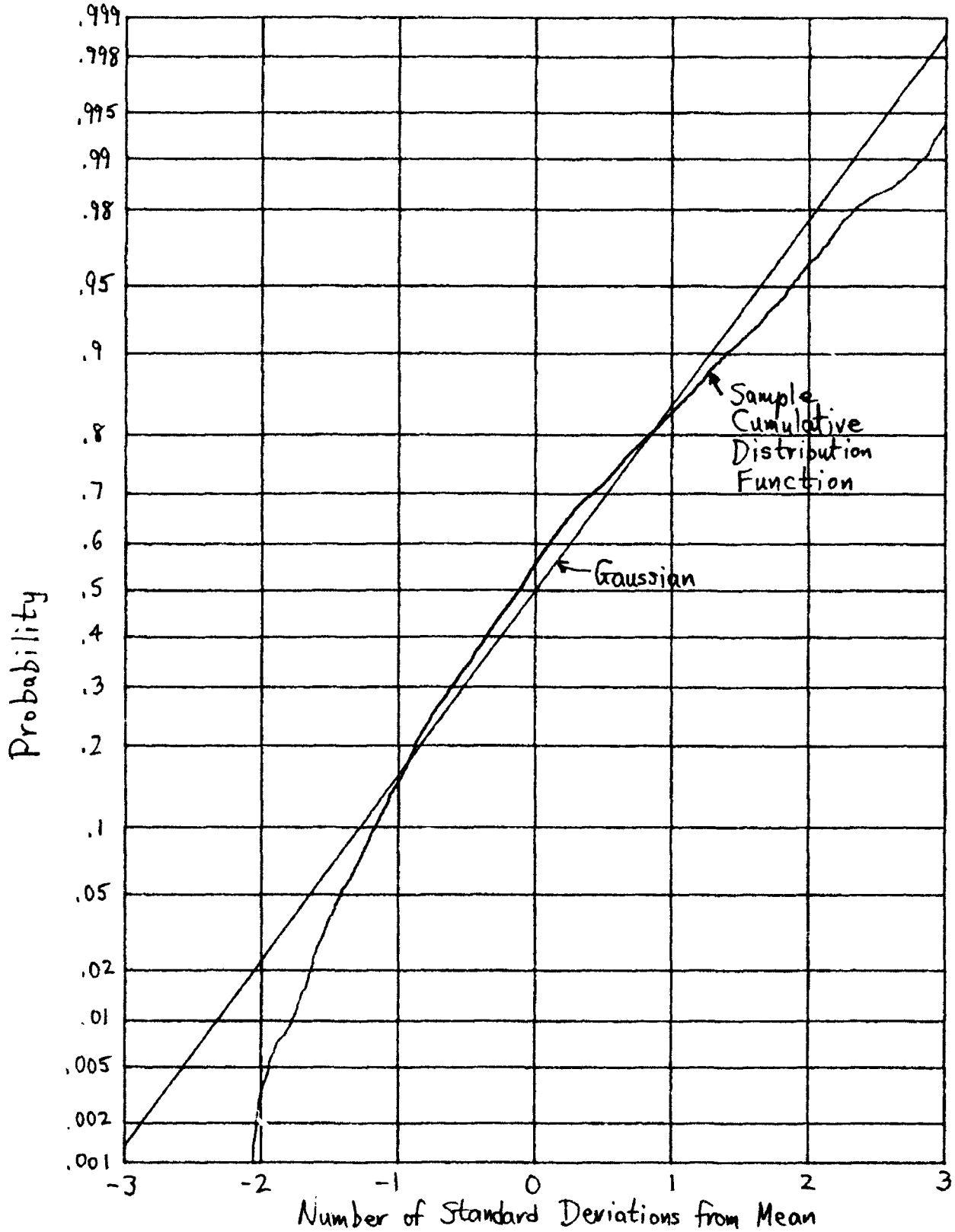

Figure 7. Estimate of Cumulative Distribution Function

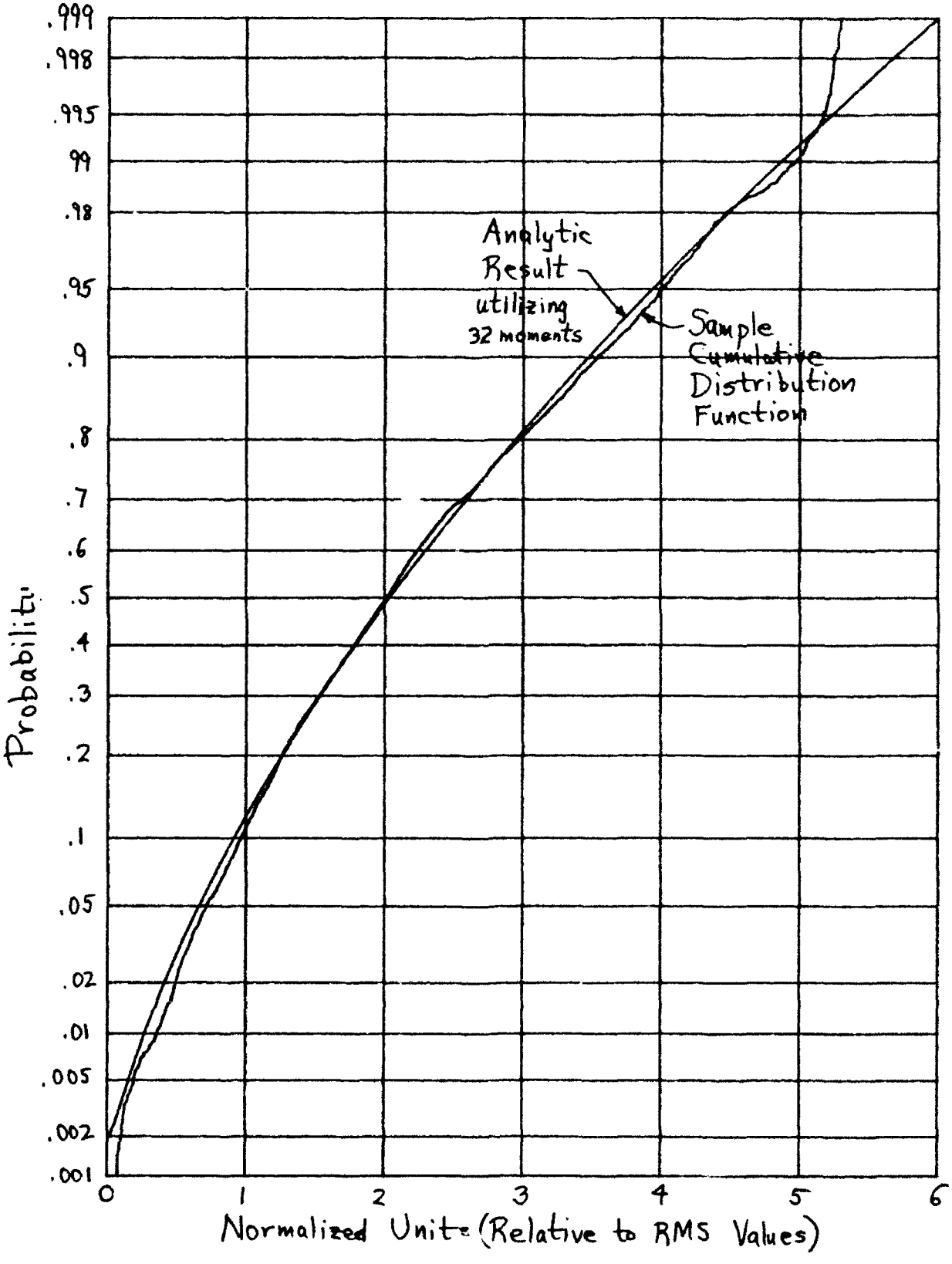

Figure 8. Comparison of Cumulative Distribution Functions

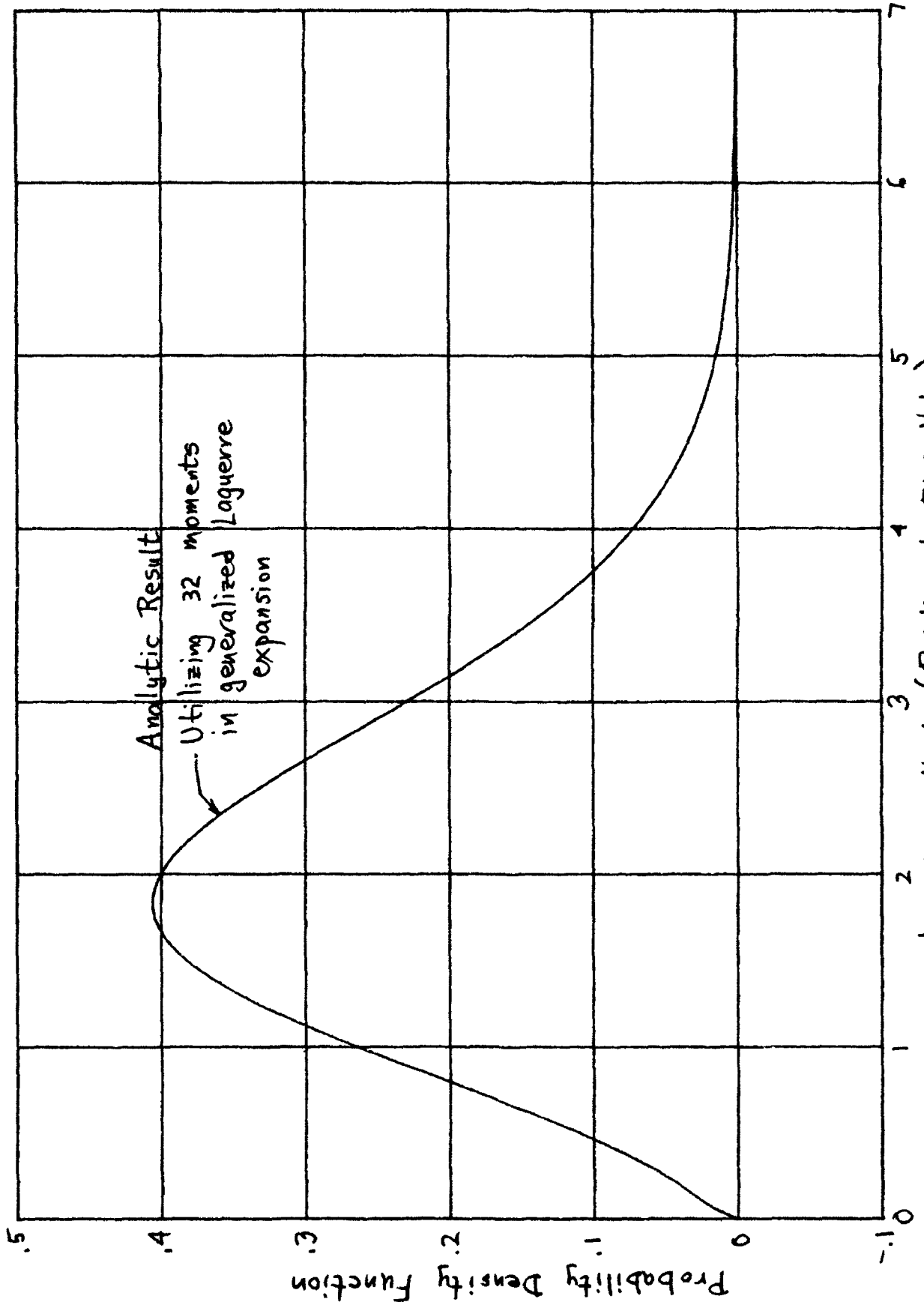

Figure 9. Theoretical Probability Density Function (Continuous Portion)

APPENDIX A. DERIVATION OF SPECTRUM AND CORRELATION

The method employed below follows that given by Rice [1, sections 1.4 and 1.5] rather closely. We generalize [1, eq. 1.3-1] to the current form introduced in (1):

$$I_K(t) = \sum_{k=1}^{K} a_k \, F\left(\frac{t-t_k}{\ell_k}\right) \quad , \tag{A-1}$$

where $\{a_k\}$, $\{t_k\}$, $\{\ell_k\}$ are all independent random variables. K is the presumed number of pulses to occur in a large time interval T, and a_k is a random amplitude as in [1, eq. 1.5-1]; but random duration ℓ_k is new. Then product

$$I_K(t)\; I_K(\cdot-\tau) = \sum_{k=1}^{K} a_k^2 \, F\left(\frac{t-t_k}{\ell_k}\right) \, F\left(\frac{t-\tau-t_k}{\ell_k}\right) +$$

$$+ \sum_{\substack{k=1 \\ k\neq m}}^{K} \sum_{m=1}^{K} a_k \, a_m \, F\left(\frac{t-t_k}{\ell_k}\right) \, F\left(\frac{t-\tau-t_m}{\ell_m}\right) . \tag{A-2}$$

Holding random variables $\{a_k\}$ and $\{\ell_k\}$ fixed for now, the statistical average of (A-2) over $\{t_k\}$ is

$$\sum_{k=1}^{K} a_k^2 \, \frac{1}{T} \int_T dt_k \, F\left(\frac{t-t_k}{\ell_k}\right) \, F\left(\frac{t-\tau-t_k}{\ell_k}\right) +$$

$$+ \sum_{\substack{k=1 \\ k\neq m}}^{K} \sum_{m=1}^{K} a_k \, a_m \, \frac{1}{T} \int_T dt_k \, F\left(\frac{t-t_k}{\ell_k}\right) \frac{1}{T} \int_T dt_m \, F\left(\frac{t-\tau-t_m}{\ell_m}\right) =$$

$$\cong \frac{1}{T} \sum_{k=1}^{K} a_k^2 \, \ell_k \, \phi(\tau/\ell_k) + \frac{1}{T^2} \sum_{\substack{k=1 \\ k\neq m}}^{K} \sum_{m=1}^{K} a_k \, a_m \, \ell_k \, \ell_m \, S^2(0) \; , \tag{A-3}$$

where T is an arbitrary large (but finite) time interval, and

$$\phi(y) = \int dx\ F(x)\ F(x-y) \tag{A-4}$$

is the aperiodic autocorrelation of pulse F, while

$$S(f) = \int dx\ \exp(-i2\pi fx)\ F(x) \tag{A-5}$$

is the voltage density spectrum of F.

The remaining averages over independent random variables $\{a_k\}$ and $\{\ell_k\}$ in (A-3) now yield

$$\frac{1}{T} K\ \overline{a^2} \int d\ell\ p(\ell)\ell\ \phi(\tau/\ell) + \frac{1}{T^2}\ (K^2-K)\ [\bar{a}\ \bar{\ell}\ S(0)]^2, \tag{A-6}$$

where $p(\ell)$ is the probability density function of random variable ℓ_k.

Now K is itself a random variable, with discrete probability (in an interval T) of [1, eq. 1.1-3]

$$\frac{(\nu T)^K}{K!} \exp(-\nu T) \quad \text{for } K = 0, 1, 2, \ldots . \tag{A-7}$$

There then follows the characteristic function of random variable K as

$$f_K(\xi) = \exp\left(\nu T[\exp(i\xi) - 1]\right), \tag{A-8}$$

with series expansion

$$\ln f_K(\xi) = \nu T\ [\exp(i\xi)-1] = \nu T \sum_{n=1}^{\infty} (i\xi)^n/n! \ . \tag{A-9}$$

Thus the cumulants of random variable K are all equal,

$$\chi_K(n) = \nu T \quad \text{for } n \geq 1 \ , \tag{A-10}$$

giving in particular the first two moments

$$\bar{K} = \nu T,\ \overline{K^2} = \nu T(\nu T + 1) \ . \tag{A-11}$$

The use of (A-11), to perform the remaining average of (A-6) with respect to random variable K, then yields the correlation function of the shot noise process I(t):

$$R_I(\tau) = \nu\, \overline{a^2} \int d\ell\; p(\ell)\, \ell\, \phi(\tau/\ell) + [\nu\, \bar{a}\, \bar{\ell}\, S(0)]^2. \qquad \text{(A-12)}$$

The dc component of I(t) is

$$I_{dc} = \nu\, \bar{a}\, \bar{\ell}\, S(0) = \nu\, \bar{a}\, \bar{\ell} \int dx\; F(x)\ . \qquad \text{(A-13)}$$

The spectrum of I(t) is the Fourier transform of (A-12):

$$G_I(f) = \nu\, \overline{a^2} \int d\ell\; p(\ell)\, \ell^2 \bar{\Phi}(\ell f) + I_{dc}^2\, \delta(f)\ , \qquad \text{(A-14)}$$

where

$$\bar{\Phi}(f) = \int d\tau\, \exp(-i2\pi f\tau)\, \phi(\tau) =$$

$$= \int d\tau\, \exp(-i2\pi f\tau) \int dx\; F(x)\, F(x-\tau) = |S(f)|^2, \qquad \text{(A-15)}$$

by use of (A-4) and (A-5). Thus (A-14) can be expressed as

$$G_I(f) = \nu\, \overline{a^2} \int d\ell\; p(\ell)\, \ell^2\; |S(\ell f)|^2 + I_{dc}^2\, \delta(f)\ . \qquad \text{(A-16)}$$

$|S(f)|^2$ is the energy density spectrum of pulse F.

Example

The example of interest here is given in (2) and (4):

$$F(x) = \left\{ \begin{matrix} (1-x^2)^{1/2} \text{ for } |x| < 1 \\ 0 \quad \text{for } |x| > 1 \end{matrix} \right\}\ ,$$

$$p(\ell) = \frac{\ell}{\sigma_\ell^2} \exp\left(\frac{-\ell^2}{2\sigma_\ell^2}\right) U(\ell)\ . \qquad \text{(A-17)}$$

Then from (A-5) and [5, eq. 3.752 2],

$$S(f) = \int_{-1}^{1} dx \exp(-i2\pi fx) \left(1-x^2\right)^{1/2} = \frac{J_1(2\pi f)}{2f} . \qquad (A-18)$$

Substitution of (A-17) and (A-18) in the integral in (A-16) yields, by use of [5, eq. 6.633 2],

$$\int d\ell \frac{\ell}{\sigma_\ell^2} \exp\left(\frac{-\ell^2}{2\sigma_\ell^2}\right) \frac{J_1^2(2\pi f)}{4f^2} = 2\pi^2\sigma_\ell^2 \frac{2 \exp(-z)\, I_1(z)}{z} , \qquad (A-19)$$

where

$$z = (2\pi\sigma_\ell f)^2. \qquad (A-20)$$

Then the spectrum (A-16) is given by

$$G_I(f) = 2\pi^2 \nu \overline{a^2} \sigma_\ell^2 \frac{2 \exp(-z)\, I_1(z)}{z} + I_{dc}^2 \delta(f) , \qquad (A-21)$$

where

$$I_{dc} = \nu \bar{a} \bar{\ell} S(0) = \left(\frac{\pi}{2}\right)^{3/2} \nu \bar{a} \sigma_\ell \qquad (A-22)$$

by means of (A-13), (A-18), and (6).

To determine the correlation of shot noise process I(t), we consider first the continuous portion of the spectrum in (A-21):

$$G_C(f) = 2\pi^2 \nu \overline{a^2} \sigma_\ell^2 \frac{2 \exp(-4\pi^2\sigma_\ell^2 f^2)\, I_1(4\pi^2\sigma_\ell^2 f^2)}{4\pi^2\sigma_\ell^2 f^2} . \qquad (A-23)$$

The corresponding correlation is

$$R_c(\tau) = \int_{-\infty}^{+\infty} df\, \exp(i2\pi f\tau)\, G_c(f) =$$

$$= 4\pi^2\, \nu\, \overline{a^2}\, \sigma_\ell^2 \int_0^\infty df\, 2\cos(2\pi f\tau)\, \frac{\exp(-4\pi^2\sigma_\ell^2 f^2)\, I_1(4\pi^2\sigma_\ell^2 f^2)}{4\pi^2\sigma_\ell^2 f^2} =$$

$$= 2\pi\, \nu\, \overline{a^2}\, \sigma_\ell \int_0^\infty dz \cos\left(\frac{\tau}{\sigma_\ell} z^{1/2}\right) \frac{\exp(-z)\, I_1(z)}{z^{3/2}} =$$

$$= 2^{3/2}\pi\, \nu\, \overline{a^2}\, \sigma_\ell \exp(-s)\, W_{-\frac{3}{2},\frac{1}{2}}(2s)\,, \qquad \text{(A-24)}$$

where we employed (A-20) and [5, eq. 6.755 2], and defined

$$s = \left(\frac{\tau}{4\sigma_\ell}\right)^2. \qquad \text{(A-25)}$$

The W-function in (A-24) is the Whittaker function [2, p. 505].

Now by [5, eqs. 9.232 1 and 9.222 1], we have

$$W_{-\frac{3}{2},\frac{1}{2}}(2s) = W_{-\frac{3}{2},-\frac{1}{2}}(2s) = \frac{\exp(-s)}{\Gamma(3/2)} \int_0^\infty dt\, \exp(-2st)\, t^{1/2}(1+t)^{-3/2} =$$

$$= \frac{2}{\pi^{1/2}} \exp(-s) \int_0^\infty dt\, \exp(-2st) \left[\frac{1}{t^{1/2}(1+t)^{3/2}} - \frac{1}{t^{1/2}(1+t)^{5/2}}\right]. \qquad \text{(A-26)}$$

But according to [5, eq. 3.364 3],

$$\int_0^\infty dt\, \frac{\exp(-2st)}{t^{1/2}(a+t)^{1/2}} = \exp(as)\, K_0(as)\,. \qquad \text{(A-27)}$$

Partial differention with respect to a then yields

$$-\frac{1}{2}\int_0^\infty dt\,\frac{\exp(-2st)}{t^{1/2}(a+t)^{3/2}} = s\exp(as)\,[K_0(as) - K_1(as)] \qquad (A\text{-}28)$$

and (repeated)

$$\frac{3}{4}\int_0^\infty dt\,\frac{\exp(-2st)}{t^{1/2}(a+t)^{5/2}} = s^2\exp(as)\left[2K_0(as) - 2K_1(as) + \frac{K_1(as)}{as}\right]. \qquad (A\text{-}29)$$

Here we used [2, eq. 9.6.28] in the forms

$$K_0'(z) = -K_1(z), \quad K_1'(z) = -K_0(z) - \frac{K_1(z)}{z}. \qquad (A\text{-}30)$$

If we now set a = 1 in (A-28) and (A-29), and then employ these results in (A-26), we obtain

$$W_{-\frac{3}{2},\frac{1}{2}}(2s) = \frac{2}{\pi^{1/2}}\frac{2}{3}s\left[(1+4s)K_1(s) - (3+4s)K_0(s)\right]. \qquad (A\text{-}31)$$

Finally, the use of (A-31) in (A-24) yields

$$R_c(\tau) = \frac{8}{3}(2\pi)^{1/2}\,\nu\,\overline{a^2}\,\sigma_\ell^2\; s\exp(-s)\left[(1+4s)K_1(s) - (3+4s)K_0(s)\right]. \qquad (A\text{-}32)$$

The Fourier transform of the impulsive part of the spectrum in (A-21) is simply the constant

$$I_{dc}^2 = \left(\frac{\pi}{2}\right)^3 \nu^2\,\overline{a}^2\,\sigma_\ell^2, \qquad (A\text{-}33)$$

which must be added to $R_c(\tau)$ in (A-32) to obtain $R_I(\tau)$. Here s is given by (A-25).

As $\tau \to 0+$, there follows from (A-32),

$$\lim_{\tau\to 0+} R_c(\tau) = \frac{8}{3}(2\pi)^{1/2}\,\nu\,\overline{a^2}\,\sigma_\ell^2. \qquad (A\text{-}34)$$

APPENDIX B. VARIANCE OF CORRELATION ESTIMATE

Let the available data be $\{x_n\}_1^N$, with zero mean and variance σ^2:

$$\overline{x}_n = 0, \quad \overline{x_n^2} = \sigma^2 \quad \text{for } 1 \leq n \leq N . \tag{B-1}$$

The autocorrelation estimate at delay k is defined here as

$$R_k = \frac{1}{N} \sum_{n=k+1}^{N} x_n \, x_{n-k} \quad \text{for } k \geq 0 . \tag{B-2}$$

At delay 0, the mean value of estimate R_o is

$$\overline{R_o} = \frac{1}{N} \sum_{n=1}^{N} \overline{x_n^2} = \sigma^2 . \tag{B-3}$$

We now want to evaluate the standard deviation of estimate R_k at delays k large enough that x_n and x_{n-k} are statistically independent. We have mean value

$$\overline{R_k} = \frac{1}{N} \sum_{n=k+1}^{N} \overline{x}_n \, \overline{x}_{n-k} = 0 , \tag{B-4}$$

using the independence at separation k. The mean square value of estimate R_k is

$$\overline{R_k^2} = \frac{1}{N^2} \sum_{m,n=k+1}^{N} \overline{x_m \, x_n \, x_{m-k} \, x_{n-k}} . \tag{B-5}$$

For the large separation values k of interest here, the only statistical dependence that contributes non-trivially to the double sum is the following:

$$\overline{R_k^2} = \frac{1}{N^2} \sum_{m,n=k+1}^{N} \overline{x_m x_n}\ \overline{x_{m-k} x_{n-k}} = \frac{\sigma^4}{N^2} \sum_{m,n=k+1}^{N} \rho^2(m-n) =$$

$$= \frac{\sigma^4}{N^2} \sum_{|n|<N-k} (N-k-|n|)\, \rho^2(n) \cong \frac{\sigma^4}{N^2} (N-k) \sum_{n} \rho^2(n) \ . \quad (B\text{-}6)$$

Here ρ is the correlation coefficient of data $\{x_n\}$, and we have assumed that N is moderately larger than the effective correlation length of ρ. The ratio of the standard deviation of estimate R_k to the mean value at k=0 is then the normalized standard deviation at separation k:

$$\sigma_k' \cong \frac{1}{N}\left[(N-k) \sum_{n} \rho^2(n)\right]^{1/2}. \quad (B\text{-}7)$$

Notice that no Gaussian assumptions on data $\{x_n\}_1^N$ have been employed in this analysis; however, $\rho(k)$ is essentially zero at the k values of interest.

As an example, for an exponential correlation of effective length K_e, there follows

$$\sum_{n} \rho^2(n) = \sum_{n} \exp\left(-2\frac{|n|}{K_e}\right) \cong \int dx \exp\left(-2\frac{|x|}{K_e}\right) = K_e \ , \quad (B\text{-}8)$$

where we assume that K_e is moderately larger than unity. Then (B-7) yields

$$\sigma_k' \cong \frac{1}{N}[(N-k)K_e]^{1/2}. \quad (B\text{-}9)$$

These results hold only for those values of k where $\rho(k)$ has substantially gone to zero. Larger values of K_e lead to larger relative standard deviations; this is consistent with the fact that there are then a lesser number of effectively-independent samples in the limited data set of length N.

For the 8000 data point example of interest here, inspection of figure 5 reveals that $K_e \cong 30$. Thus

$$\pm 2\sigma'_k = \pm \frac{(8000-k)^{K_2}}{730}. \qquad \text{(B-10)}$$

These confidence limits are superposed as dotted lines on figures 5 and 6.

APPENDIX C. PROPERTIES OF CHARACTERISTIC FUNCTION OF SHOT NOISE

Derivation of Characteristic Function

The method of derivation of the characteristic function of I(t) presented here parallels that of Rice [1, sections 1.4 and 1.5] very closely. We generalize [1, eq. 1.3-1] to

$$I_K(t) = \sum_{k=1}^{K} a_k \, F\left(\frac{t-t_k}{\ell_k}\right) \tag{C-1}$$

where $\{a_k\}$, $\{t_k\}$, $\{\ell_k\}$ are all independent random variables; see (A-1) and the ensuing discussion. The characteristic function of an individual component in (C-1) is

$$f_1(\xi) = \overline{\exp\left[i\xi \, a_k \, F\left(\frac{t-t_k}{\ell_k}\right)\right]} \, , \tag{C-2}$$

where the statistical average is over a_k , t_k , ℓ_k . The average over t_k (for fixed a_k, ℓ_k) is, for T a large but finite time interval [1, p. 152],

$$\frac{1}{T}\int_0^T dt_k \, \exp\left[i\xi \, a_k \, F\left(\frac{t-t_k}{\ell_k}\right)\right] =$$

$$= \frac{1}{T}\int_0^T dt_k \left\{\exp\left[i\xi \, a_k \, F\left(\frac{t-t_k}{\ell_k}\right)\right] - 1\right\} + 1 =$$

$$\cong \frac{1}{T}\int d\tau \left\{\exp\left[i\xi \, a_k \, F\left(\frac{t-\tau}{\ell_k}\right)\right] - 1\right\} + 1 \, , \tag{C-3}$$

for large T, where we have used the fact that

$$F(x) \to 0 \text{ as } x \to \pm\infty. \tag{C-4}$$

Let $x = (t-\tau_k)/\ell_k$ in (C-3) to get

$$\frac{1}{T}\ell_k \int dx\{\exp[i\xi a_k F(x)]-1\} + 1 . \tag{C-5}$$

Now performing the averages on random variables ℓ_k and a_k, we have, for the characteristic function of an individual component of (C-1),

$$f_1(\xi) = \frac{1}{T}\bar{\ell}\int da\ p(a)\int dx\{\exp[i\xi a\ F(x)]-1\} + 1 , \tag{C-6}$$

where p(a) is the probability density function of random variable a_k.

Interchanging integrals, (C-6) becomes

$$f_1(\xi) = \frac{1}{T}\bar{\ell}\int dx\{f_a[\xi F(x)]-1\} + 1 , \tag{C-7}$$

where f_a is the characteristic function of amplitude a_k. Then from (C-1), since all the individual random variables are independent, the characteristic function of $I_K(t)$ is

$$f_{I_K}(\xi) = [f_1(\xi)]^K . \tag{C-8}$$

Finally, the characteristic function of total shot noise process (1) is, by use of discrete probability distribution (A-7) for random variable K, given by the average

$$f_I(\xi) = \sum_{K=0}^{\infty} \frac{(\nu T)^K}{K!} \exp(-\nu T)\, f_{I_K}(\xi) =$$

$$= \exp[-\nu T + \nu T\, f_1(\xi)] =$$

$$= \exp[\nu \bar{\ell} \int dx \{f_a[\xi F(x)] - 1\}] \quad . \qquad \text{(C-9)}$$

The (imprecise) large time interval T has dropped out of the general result (C-9). Also, the only parameter required about the duration random variable ℓ_k is its mean. The exact characteristic function f_a of amplitude a_k and the exact pulse shape F directly affect the characteristic function of I(t). For $\ell_k = 1$ for all k, (C-9) reduces to a simplified version of [1, eq. 1.5-4].

Cumulants of I(t)

The characteristic function of random amplitude a_k can be expanded in a power series

$$f_a(\xi) = \sum_{n=0}^{\infty} \mu_a(n) \frac{(i\xi)^n}{n!} \quad , \qquad \text{(C-10)}$$

where $\mu_a(n)$ is the n-th moment of a_k:

$$\mu_a(n) = \overline{a^n} = \int da\, a^n\, p(a) \quad . \qquad \text{(C-11)}$$

Then from (C-9), we develop

$$\ell n\ f_I(\xi) = \nu \bar{\ell} \int dx \left\{ f_a[\xi F(x)] - 1 \right\} =$$

$$= \nu \bar{\ell} \sum_{n=1}^{\infty} \mu_a(n) \frac{(i\xi)^n}{n!} \int dx\ F^n(x)\ , \qquad \text{(C-12)}$$

allowing for immediate identification of the cumulants of I(t) as

$$\chi_I(n) = \nu \bar{\ell}\ \mu_a(n) \int dx\ F^n(x) \quad \text{for } n \geq 1;\ \chi_I(0) = 0\ . \qquad \text{(C-13)}$$

For $\ell_k = 1$ for all k, this reduces to [1, eq. 1.5-2].

The normalized cumulants of I(t) are

$$\gamma_I(n) = \frac{\chi_I(n)}{[\chi_I(2)]^{n/2}} = \frac{1}{(\nu\bar{\ell})^{\frac{n}{2}-1}} \frac{\mu_a(n) \int dx\ F^n(x)}{\left[\mu_a(2) \int dx\ F^2(x)\right]^{\frac{n}{2}}}\ . \qquad \text{(C-14)}$$

These quantities tend to zero rapidly for $\nu\bar{\ell} >> 1$; see also [1, eq. 1.6-3]. Thus $\nu\bar{\ell}$ has a pronounced effect on how Gaussian I(t) is.

Behavior of characteristic function $f_I(\xi)$ at $\xi = \pm\infty$

If pulse F(x) is non-zero only over (x_1, x_2), we have

$$\int dx \left\{ f_a[\xi F(x)] - 1 \right\} = \int_{x_1}^{x_2} dx \left\{ f_a[\xi F(x)] - 1 \right\}\ . \qquad \text{(C-15)}$$

Now if random variable a_k has a characteristic function f_a with the property that

$$f_a(\pm\infty) = 0\ , \qquad \text{(C-16)}$$

then

$$(C\text{-}15) \rightarrow \int_{x_1}^{x_2} dx \left\{0\text{-}1\right\} = -(x_2 - x_1) \text{ as } \xi \rightarrow \pm\infty, \qquad (C\text{-}17)$$

in which case (C-9) yields

$$f_I(\pm\infty) = \exp[-\nu\bar{\ell}(x_2 - x_1)] \ . \qquad (C\text{-}18)$$

If pulse extent $x_2 - x_1$ is infinite, as for the Gaussian or exponential pulses,

$$F(x) = \exp(-x^2) \text{ or } \exp(-x)U(x) \ , \qquad (C\text{-}19)$$

then (C-18) is zero. On the other hand, if $x_2 - x_1$ is finite, as for circular pulse

$$F(x) = \left\{ \begin{matrix} (1-x^2)^{1/2} \text{ for } |x| < 1 \\ 0 \quad \text{for } |x| > 1 \end{matrix} \right\}, \qquad (C\text{-}20)$$

then

$$f_I(\pm\infty) = \exp[-\nu\sqrt{2}] > 0 \quad \text{for circular pulse.} \qquad (C\text{-}21)$$

This non-zero characteristic function value corresponds to an impulse at the origin of probability density function p_I, with area (C-21). Physically, this means that there are occasionally regions of the t-scale where no pulses overlap, and there $I(t) = 0$. The probability of this happening is, generally,

$$P_0 = \text{Prob}\left\{I(t) = 0\right\} = f_I(\pm\infty) = \exp[-\nu\bar{\ell}(x_2 - x_1)] \ . \qquad (C\text{-}22)$$

On the other hand, for the Gaussian or exponential pulses cited in (C-19), $x_2 - x_1 = +\infty$, and $f_I(\pm\infty) = 0$, meaning that there is no impulse at the origin of probability density function p_I. Physically, the infinite tails (even if single-sided, as for the exponential pulse) disallow I(t) ever from becoming zero.

Cumulants of Continuous Portion of p_I.

The impulse at the origin means that probability density function p_I and cumulative distribution function P_I can be expressed respectively as

$$p_I(u) = P_0\,\delta(u) + p_c(u)\ ,$$

$$P_I(u) = P_0 + \int_0^u dt\ p_c(t) \quad \text{for } u > 0, \tag{C-23}$$

where $p_c(u)$ is a continuous function of u, with area $1-P_0$. The characteristic function relation corresponding to (C-23) is

$$f_I(\xi) = P_0 + f_c(\xi)\ , \tag{C-24}$$

and the moments are related according to

$$\mu_c(n) = \begin{cases} \mu_I(0) - P_0 & \text{for } n=0 \\ \mu_I(n) & \text{for } n \geq 1 \end{cases}\ . \tag{C-25}$$

The cumulants of f_c or p_c can then be found from these moments (C-25), by recursive relations; see [4] or [6]. This procedure is necessary to get accurate series expansions for the probability density function p_c and its cumulative distribution function, without having to approximate a delta function.

Overlap Factor

In the case where pulse extent $x_2 - x_1$ is finite, it is possible to find the average number of overlapping pulses at any one time instant; this statistic, denoted by $\overline{K_1}$, is called the overlap factor. In order to determine it in a simple fashion, we concoct a very special shot noise process: let

$$a_k = 1 \quad \text{for all } k \ ,$$

$$F(x) = 1 \quad \text{for } x_1 < x < x_2 \ . \tag{C-26}$$

Then I(t) is a step function with amplitudes limited to the values 0, 1, 2, ... Then obviously, the average number of overlapping pulses at one time instant is just

$$\overline{K_1} = \overline{I(t)} = \nu \overline{\ell} \, u_a(1) \int dx \, F(x) = \nu \overline{\ell} (x_2 - x_1) \ , \tag{C-27}$$

upon use of (C-13) with n=1 and (C-26). If we let

$$\overline{L} = \overline{\ell} (x_2 - x_1) \tag{C-28}$$

denote the average pulse duration, we have the overlap factor in the form

$$\overline{K_1} = \nu \, \overline{L} \ . \tag{C-29}$$

For the Gaussian or exponential pulses in (C-19), we have $x_2 - x_1 = +\infty$, giving $\overline{L} = +\infty$, $\overline{K_1} = +\infty$. This is in fact true, since all the infinite tails overlap; however, it is not then an informative statistic.

Closed Form Characteristic Function Examples

There are a couple of examples of the circular pulse shape F and amplitude characteristic function f_a, where (C-9) can be evaluated in closed form. This furnishes an alternative to the moment approach [4] used here.

Consider the circular pulse in (C-20); then the integral in (C-9) is (using (C-15))

$$\int_{-1}^{1} dx \left\{ f_a\left[\xi\left(1-x^2\right)^{\frac{1}{2}}\right]-1\right\} = 2\int_{0}^{\frac{\pi}{2}} d\theta \cos\theta\, f_a[\xi\cos\theta] - 2 , \qquad \text{(C-30)}$$

which holds for any characteristic function f_a. Now first let the probability density function of a_k be exponential:

$$p(a) = \frac{1}{\mu_a}\exp\left(\frac{-a}{\mu_a}\right) U(a) , \quad f_a(\xi) = (1 - i\xi\mu_a)^{-1} . \qquad \text{(C-31)}$$

Substitution in (C-30) yields

$$2\int_{0}^{\frac{\pi}{2}} \frac{d\theta\cos\theta}{1-i\xi\mu_a\cos\theta} - 2 . \qquad \text{(C-32)}$$

But we know that

$$2\int_{0}^{\frac{\pi}{2}} \frac{d\theta\cos\theta}{1-z\cos\theta} = -\frac{\pi}{z} + \frac{4}{z\left(1-z^2\right)^{\frac{1}{2}}} \arctan\left[\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}\right] =$$

$$= -\frac{\pi}{z} + \frac{2}{z\left(1-z^2\right)^{\frac{1}{2}}} \arccos(-z) , \qquad \text{(C-33)}$$

via [5, eqs. 2.554 2 and 2.553 3]. Then letting $z = i\xi\mu_a$ and using [2, eqs. 4.4.2 with 4.4.26], (C-32) becomes

$$\frac{-2\,\ell n(s-\mu_a\xi) + i\pi(s-1)}{\mu_a\xi s} - 2, \quad \text{with } s = \left(1 + \mu_a^2\xi^2\right)^{1/2}. \tag{C-34}$$

Combining these results in (C-9), the closed form characteristic function is

$$f_I(\xi) = \exp\left[-\frac{2\nu\ell}{\mu_a\xi s}\left\{\mu_a\xi s + \ell n(s-\mu_a\xi) - i\,\frac{\pi}{2}(s-1)\right\}\right], \tag{C-35}$$

which holds for a circular pulse F and an exponential probability density function p(a).

The second example is the one considered in detail here, namely the Rayleigh probability density function p(a) given in (3). First substituting (C-30) in (C-9), we have characteristic function

$$f_I(\xi) = \exp[2\nu\ell(J(\xi)-1)], \tag{C-36}$$

where integral $J(\xi)$ is defined as

$$J(\xi) = \int_0^{\frac{\pi}{2}} d\theta\, \cos\theta\, f_a[\xi\cos\theta]. \tag{C-37}$$

For Rayleigh probability density function (3), (C-37) can be expressed as follows:

$$J(\xi) = \int_0^{\frac{\pi}{2}} d\theta\, \cos\theta \int_0^{\infty} da\, \exp(ia\xi\cos\theta)\,\frac{a}{\sigma_a^2}\exp\left(\frac{-a^2}{2\sigma_a^2}\right). \tag{C-38}$$

Transform to rectangular coordinates according to $a\cos\theta = \sigma_a x$, $a\sin\theta = \sigma_a y$, and obtain

$$J(\xi) = \int_0^{\infty}\int dx\, dy\,\frac{x}{(x^2+y^2)^{1/2}}\exp\left(i\xi\sigma_a x - \frac{x^2+y^2}{2}\right). \tag{C-39}$$

But the integral on y here is, via $y = x\,u^{1/2}$, equal to

$$\frac{1}{2}\int_0^\infty \frac{du}{u^{1/2}(1+u)^{1/2}} \exp\left(-\frac{1}{2}x^2u\right) = \frac{1}{2}\exp\left(\frac{x^2}{4}\right) K_0\left(\frac{x^2}{4}\right), \qquad (C\text{-}40)$$

the latter by means of [5, eq. 3.364 3]. Thus (C-39) becomes

$$J(\xi) = \int_0^\infty dx\; x \exp\left(i\xi\sigma_a x - \frac{x^2}{2}\right) \frac{1}{2}\exp\left(\frac{x^2}{4}\right) K_0\left(\frac{x^2}{4}\right) =$$

$$= \int_0^\infty du \exp(i\,2^{1/2}\sigma_a \xi\, u)\; u \exp\left(-\frac{u^2}{2}\right) K_0\left(\frac{u^2}{2}\right). \qquad (C\text{-}41)$$

At this point, we have two alternatives. First, (C-41) could be efficiently evaluated for all ξ via an FFT; the decay of the integrand is according to $\exp(-u^2)$ for large u. Secondly, $J(\xi)$ can be expressed in a closed form in terms of a hypergeometric function; specifically

$$J(\xi) = {}_2F_2\left(1, 1; \frac{1}{2}, \frac{3}{2}; -2b^2\right) +$$

$$+ i\left(\frac{\pi}{2}\right)^{3/2} b \exp(-b^2)\left[I_0(b^2) - I_1(b^2)\right], \qquad (C\text{-}42)$$

where $b = \sigma_a\xi/2$. The upper line follows from [5, eq. 6.755 6], while the lower line used [5, eq. 6.755 9] with an application of partial derivative $\partial/\partial a$ to both sides. The characteristic function f_I is finally obtained by employing (C-42) in (C-36).

Still another alternative is afforded by use of the closed form for the characteristic function of the Rayleigh probability density function, as given in [7, eq. 6].

Moments for Some Particular Pulse Shapes F

The moments of pulse shape F were encountered in evaluating the cumulants $\chi_I(n)$ of shot noise I(t), according to (24) or (C-13) as

$$\mu_F(n) = \int dx\ F^n(x) \quad \text{for } n \geq 1 . \tag{C-43}$$

For circular pulse (C-20), [5, eq. 3.621 1] yields moments

$$\mu_F(n) = \int_{-1}^{1} dx\ (1-x^2)^{n/2} = 2\int_0^{\frac{\pi}{2}} d\theta\ (\cos\theta)^{n+1} = \frac{2^{n+1}\Gamma^2\left(\frac{n}{2}+1\right)}{\Gamma(n+2)} , \tag{C-44}$$

a result already quoted in (29).

More generally, for

$$F(x) = \begin{cases} (1-x^2)^{\alpha} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}, \tag{C-45}$$

[5, eq. 3.621 1] yields, with a trigonometric substitution,

$$\mu_F(n) = 2^{2n\alpha+1}\frac{\Gamma^2(n\alpha+1)}{\Gamma(2n\alpha+2)} . \tag{C-46}$$

For

$$F(x) = \begin{cases} (\cos\ x)^{\alpha} & \text{for } |x| < \frac{\pi}{2} \\ 0 & \text{for } |x| > \frac{\pi}{2} \end{cases}, \tag{C-47}$$

[5, eq. 3.621 1] yields directly

$$\mu_F(n) = 2^{n\alpha}\frac{\Gamma^2\left(\frac{n\alpha+1}{2}\right)}{\Gamma(n\alpha+1)} . \tag{C-48}$$

For

$$F(x) = x^{\alpha} \exp(-x)\, U(x) ,$$

$$\mu_F(n) = \frac{\Gamma(n\alpha+1)}{n^{n\alpha+1}} , \tag{C-49}$$

while for

$$F(x) = x^{\alpha} \exp(-x^2/2)\, U(x) ,$$

$$\mu_F(n) = \frac{2^{\frac{n\alpha-1}{2}} \Gamma\left(\frac{n\alpha+1}{2}\right)}{n^{\frac{n\alpha+1}{2}}} . \tag{C-50}$$

Both relations follow directly from the definition of the Γ function.

Some Probability Density Functions for Amplitude a_k

For probability density function

$$p(a) = \frac{(a/\alpha)^{\gamma} \exp(-a/\alpha)}{\alpha\Gamma(\gamma+1)}\, U(a) , \tag{C-51}$$

we have characteristic function

$$f_a(\xi) = (1-i\xi\alpha)^{-\gamma-1} \tag{C-52}$$

with moments

$$\mu_a(n) = (\gamma+1)_n\, \alpha^n \quad \text{for } n \geq 0 \tag{C-53}$$

and cumulants

$$\chi_a(n) = (n-1)!\, (\gamma+1)\, \alpha^n \quad \text{for } n \geq 1 . \tag{C-54}$$

This example subsumes the exponential probability density function, upon setting $\gamma = 0$.

For probability density function

$$p(a) = \frac{2(a/\alpha)^{\gamma} \exp(-a^2/\alpha^2)}{\alpha \Gamma\left(\frac{\gamma+1}{2}\right)} U(a) , \qquad (C\text{-}55)$$

we have moments

$$\mu_a(n) = \frac{\Gamma\left(\frac{n+\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \alpha^n \quad \text{for } n \geq 0 . \qquad (C\text{-}56)$$

This example subsumes respectively the one-sided Gaussian for $\gamma=0$, the Rayleigh for $\gamma=1$, and the Maxwell probability density functions for $\gamma=2$. The result in (29) follows immediately by setting $\gamma=1$, $\alpha = 2^{1/2}\sigma_a$.

Convergence of Series for $\ell n\ f_I(\xi)$

A power series expansion for $\ell n\ f_I(\xi)$ was developed in (C-12), namely,

$$\ell n\ f_I(\xi) = \nu\bar{\imath} \sum_{n=1}^{\infty} \mu_a(n)\ \mu_F(n)\ \frac{(i\xi)^n}{n!} ; \qquad (C\text{-}57)$$

here we employed (C-43). Since the moments in (C-57) can be easily evaluated via recursion, according to results in the above two subsections, it might be thought that (C-57) could be employed to evaluate the characteristic function of I(t) directly, without recourse to the more difficult approaches required in (C-9) or (C-35) or (C-36)-(C-42).

To see the drawbacks of this approach, consider first a circular pulse F and a generalized exponential probability density function p(a) as in (C-51); then a combination of (C-44) and (C-53) yields, for the n-th term of the sum in (C-57),

$$T_n = \frac{2(\gamma+1)_n \Gamma^2\left(\frac{n}{2}+1\right)}{n!(n+1)!} (i\xi\alpha 2)^n. \tag{C-58}$$

Then ratio

$$\frac{T_n}{T_{n-2}} = -\frac{(n+\gamma)(n+\gamma-1)}{n^2-1} \xi^2\alpha^2 \sim -\xi^2\alpha^2 \quad \text{as } n \to +\infty, \tag{C-59}$$

regardless of the value of γ. Therefore

$$\left| \frac{T_n}{T_{n-1}} \right| \sim |\xi|\alpha \quad \text{as } n \to +\infty, \tag{C-60}$$

meaning that series (C-57) only converges for $|\xi| < 1/\alpha$. So (C-57) is not a viable approach for the calculation of the characteristic function in this case.

As a second example, we consider the circular pulse F with the generalized Rayleigh probability density function in (C-55). Combination of (C-44) with (C-56) yields for the n-th term of series (C-57),

$$T_n = \frac{2\Gamma^2\left(\frac{n}{2}+1\right) \Gamma\left(\frac{n+\gamma+1}{2}\right)}{n!(n+1)! \Gamma\left(\frac{\gamma+1}{2}\right)} (i\xi\alpha 2)^n . \tag{C-61}$$

Then ratio

$$\frac{T_n}{T_{n-2}} = -\frac{n+\gamma-1}{n^2-1} \frac{\xi^2\alpha^2}{2} \sim -\frac{\xi^2\alpha^2}{2n} \quad \text{as } n \to +\infty, \tag{C-62}$$

regardless of γ. Therefore

$$\left| \frac{T_n}{T_{n-1}} \right| \sim \frac{|\xi|\alpha}{(2n)^{1/2}} \quad \text{as } n \to +\infty, \tag{C-63}$$

meaning that series (C-57) converges for all ξ. However, direct numerical evaluation of (C-57) via (C-61) and (C-62) loses all its significant digits for large ξ, long before $\ln f_I(\xi)$ reaches its final value of $-2\bar{\ell}\nu + i0$, due to the alternating character of the series. So (C-57) is not a useful approach for evaluation of the characteristic function, except for small ξ. By contrast, the series expansion technique employed in [4] uses the moments to directly estimate the desired probability density function and cumulative distribution function of interest, for large arguments as well as small.

APPENDIX D. PROGRAMS FOR CUMULATIVE DISTRIBUTION FUNCTION AND PROBABILITY DENSITY FUNCTION

The programs used here to evaluate the cumulative distribution function and probability density function of shot noise are listed below. The n-th coefficient in a generalized Laguerre expansion of orthonormal polynomials is denoted by b_n and is plotted in figure D-1 for n = 0(1)70. It is seen to oscillate and decay with n until n = 32, at which point round-off error becomes important; however, by this time, $|b_n|$ has decayed below the 1E-5 level. The round-off error is so dominant beyond n = 35, that no useful results for b_n can be obtained then. The particular parameter values (α, β) used for the Laguerre weighting are indicated in the listings.

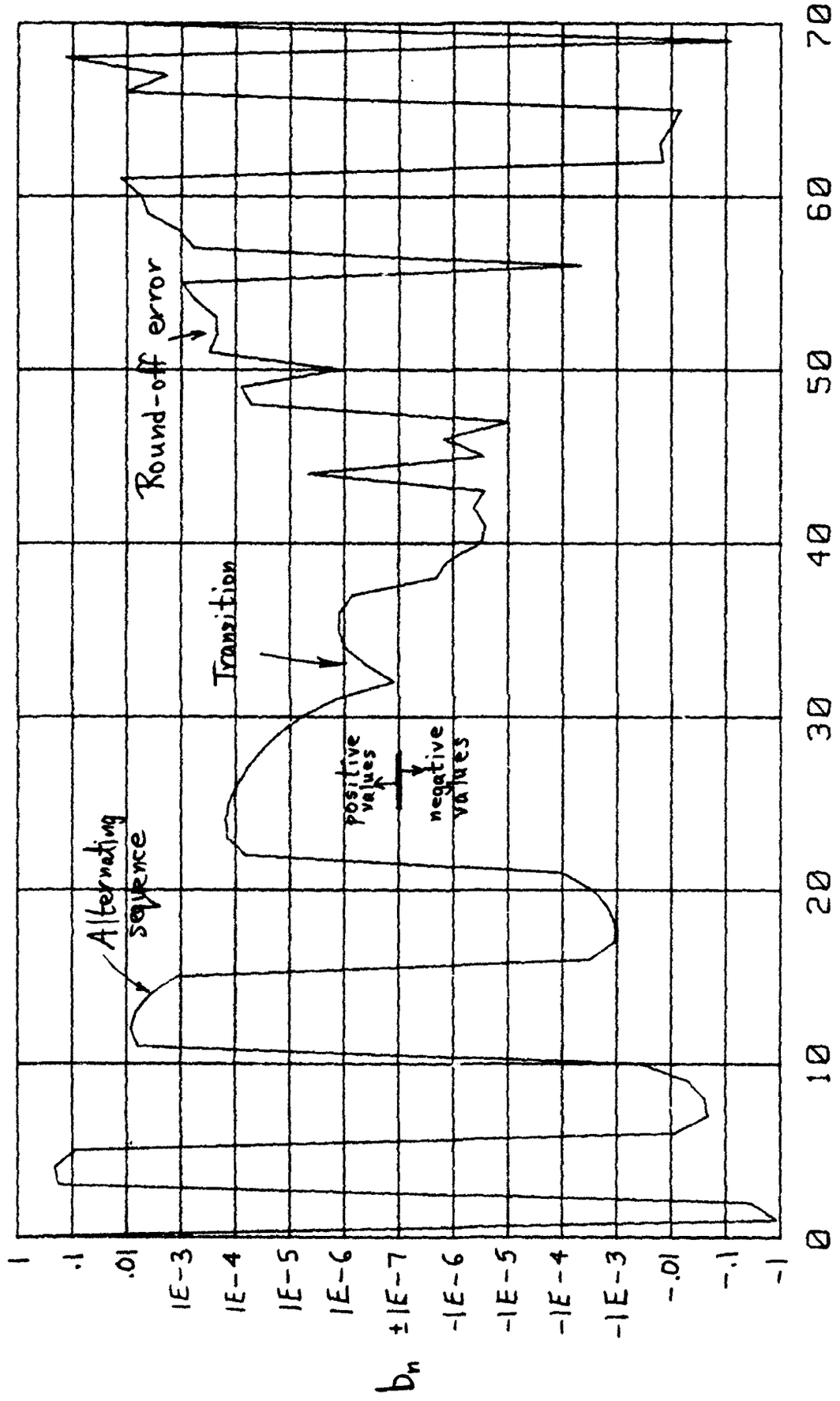

Figure D-1. Coefficient b_n in Laguerre Expansion

```
 10 !  STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u),
 20 !  VIA GENERALIZED LAGUERRE EXPANSION AND MOMENTS
 30    M=70    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
 40    DOUBLE M,I,N,K                    !  INTEGERS  -2^31 = -2,147,483,648
 50    REDIM Mom(0:M),A(0:M),L(0:M)
 60    REAL Mom(0:100),A(0:100),L(0:100),Y(1:21)
 70    CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
 80    Center=Mom(1)/Mom(0)              !  CENTER OF pc(u)
 90    R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110    Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120    Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=.74
140      Beta=2.1
150    CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))    !   DIRECT    MOMENTS
160 !  CALL Coeffldr_via_mom(M,Alpha,Beta,Mom(*),A(*))   !   RECURSIVE MOMENTS
170    PRINT "Center = ";Center
180    PRINT "Rms =";Rms
190    A1=Alpha+1.
200    O1=1./A1
210    F1=1./FNGamma(A1)
 20    DATA .001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
 30    DATA .6,.7,.8,.9,.95,.98,.99,.995,.998,.999
240    READ Y(*)
250    FOR I=1 TO 21
260    Y(I)=FNInvphi(Y(I))
270    NEXT I
280    Y1=Y(1)
290    Y2=Y(21)
300      INPUT "ORDER AND LIMITS:",N,U1,U2
310    PRINT "ORDER AND LIMITS:",N;U1;U2
320    Du=(U2-U1)/100.
330    PLOTTER IS "GRAPHICS"
340    GRAPHICS ON
350    WINDOW U1,U2,Y1,Y2
360    FOR U=U1 TO U2 STEP (U2-U1)*.1
370    MOVE U,Y1
380    DRAW U,Y2
390    NEXT U
400    FOR I=1 TO 21
410    MOVE U1,Y(I)
420    DRAW U2,Y(I)
430    NEXT I
440    PENUP
450    FOR I=1 TO 100
460    U=U1+Du*I
470    T=U/Beta
480    CALL Laguerre(N-1,A(*),T,L(*))
490    Sum=A(0)*FNF11(A1,T)*O1
500    FOR K=1 TO N
510    Sum=Sum+A(K)*L(K-1)/K
520    NEXT K
530    P=P0+F1*EXP(-T+A1*LOG(T))*Sum     !  PROBABILITY THAT RV < U
540    IF P>0. AND P<1. THEN 570
550    PENUP
560    GOTO 580
570    PLOT U,FNInvphi(P)
580    NEXT I
590    PENUP
600    GOTO 300
610    END
620    !
```

```
630     DEF FNInvphi(X)  !  INVPHI(X) via 26.2.23 with modification
640     D=X-.5
650     IF ABS(D)>.01 THEN 680
660     P=2.50662827463*D*(1.+D*D*1.04719755120)
670     RETURN P
680     P=X
690     IF X>.5 THEN P=1.-X
700     P=SQR(-2.*LOG(P))
710     T=1.+P*(1.432788+P*(.189269+P*.001308))
720     P=P-(2.515517+P*(.802853+P*.010328))/T
730     IF X<.5 THEN P=-P
740     RETURN P
750     FNEND
760     !
770     DEF FNGamma(X)  !  Gamma(X) via HART, page 282, #5243
780     DOUBLE N,K
790     N=INT(X)
800     R=X-N
810     IF N>0 OR R<>0. THEN 840
820     PRINT "FNGamma(X) IS NOT DEFINED FOR X = ";X
830     STOP
840     IF R>0. THEN 870
850     Gamma2=1.
860     GOTO 940
870     P=439.330444060025676+R*(50.1086937529709530+R*6.74495072459252899)
880     P=8762.71029785214896+R*(2008.52740130727912+R*P)
890     P=42353.6895097440896+R*(20886.8617892698874+R*P)
900     Q=499.028526621439048-R*(189.498234157028016-R*(23.081551524580125-R))
910     Q=9940.30741508277090-R*(1528.60727377952202+R*Q)
920     Q=42353.6895097440900+R*(2980.38533092566439-R*Q)
930     Gamma2=P/Q                    !  Gamma(2+R) for 0 < R < 1
940     IF N>2 THEN 980
950     IF N<2 THEN 1030
960     Gamma=Gamma2
970     RETURN Gamma
980     Gamma=Gamma2
990     FOR K=1 TO N-2
1000    Gamma=Gamma*(X-K)
1010    NEXT K
1020    RETURN Gamma
1030    R=1.
1040    FOR K=0 TO 1-N
1050    R=R*(X+K)
1060    NEXT K
1070    Gamma=Gamma2/R
1080    RETURN Gamma
1090    FNEND
1100    !
```

```
1110    DEF FNF11(A1,X)                    !  1F1(1;A1+1;X)
1120    DOUBLE K
1130    T=S=1.
1140    FOR K=1 TO 200
1150    T=T*X/(A1+K)
1160    S=S+T
1170    IF T<=1.E-17*S THEN RETURN S
1180    NEXT K
1190    PRINT "200 TERMS IN FNF11 AT";A1;X
1200    RETURN S
1210    FNEND
1220    !
1230    SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*))  !  Ln\alpha(X)
1240    DOUBLE K
1250    A1=Alpha-1.
1260    L(0)=1.
1270    L(1)=Alpha+1.-X
1280    FOR K=2 TO N
1290    L(K)=((K+K+A1-X)*L(K-1)-(K+A1)*L(K-2))/K
1300    NEXT K
1310    SUBEND
1320    !
1330    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
1340    DOUBLE K,J
1350    REAL Mom0
1360    Mom(0)=Mom0=EXP(Cum(0))
1370    FOR K=1 TO M
1380    T=1.
1390    S=Cum(K)*Mom0
1400    FOR J=1 TO K-1
1410    T=T*(K-J)/J
1420    S=S+T*Cum(K-J)*Mom(J)
1430    NEXT J
1440    Mom(K)=S
1450    NEXT K
1460    SUBEND
1470    !
```

```
1480    SUB Coeffld_via_mom(DOUBLE :,REAL Alpha,Beta,Mom(*),A(*)
1490    ALLOCATE B(0:M)
1500    DOUBLE K,K1,J,Mx
1510    T=1.
1520    FOR K=1 TO M
1530    T=T*(Alpha+K)*Beta
1540    Mom(K)=Mom(K)/T    !  NORMALIZED MOMENTS, RELATIVE TO Alpha AND Beta
1550    NEXT K
1560    Q=1.
1570    A(0)=B(0)=Mom(0)
1580    FOR K=1 TO M
1590    K1=K+1
1600    T=1.
1610    S=Mom(0)
1620    FOR J=1 TO K
1630    T=T*(J-K1)/J
1640    S=S+T*Mom(J)
1650    NEXT J
1660    Q=Q*(Alpha+K)/K
1670    A(K)=S
1680    B(K)=S*SQR(Q)
1690    NEXT K
1700    Mx=Mx+10
1710    IF Mx<M THEN 1700
1720    Threshold=-7.
1730    T2=Threshold*2.
1740    V=10.^Threshold
1750    GINIT
1760    PLOTTER IS "GRAPHICS"
1770    GRAPHICS ON
1780    WINDOW 0.,FLT(Mx),T2,0.
1790    LINE TYPE 3
1800    FOR J=0 TO Mx STEP 10
1810    MOVE J,T2
1820    DRAW J,0.
1830    NEXT J
1840    FOR J=T2 TO 0
1850    MOVE 0.,J
1860    DRAW Mx,J
1870    NEXT J
1880    PENUP
1890    LINE TYPE 1
1900    IMAGE 4D,2(4X,M.17DE)
1910    PRINT "   K                B(K)                              Sum"
1920    Sum=0.
1930    FOR K=0 TO M
1940    B=B(K)
1950    Sum=Sum+B*B
1960    PRINT USING 1900;K,B,Sum
1970    IF B<V THEN 2000
1980    Y=LGT(B)
1990    GOTO 2040
2000    IF B>-V THEN 2030
2010    Y=T2-LGT(-B)
2020    GOTO 2040
2030    Y=Threshold
2040    PLOT K,Y
2050    NEXT K
2060    PENUP
2070    SUBEND
2080    !
```

```
2090    SUB Coeffir_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
2100    ALLOCATE B(0:M)
2110    DOUBLE K,K1,J,Mx
2120    T=1.
2130    FOR K=1 TO M
2140    T=T*(Alpha+K)*Beta
2150    Mom(K)=Mom(K)/T    !  NORMALIZED MOMENTS, RELATIVE TO Alpha AND Beta
2160    NEXT K
2170    Q=1.
2180    A0=A(0)=B(0)=Mom(0)
2190    FOR K=1 TO M
2200    K1=K+1
2210    T=1.
2220    S=Mom(K)-A0
2230    FOR J=1 TO K-1
2240    T=T*(J-K1)/J
2250    S=S-T*A(J)
2260    NEXT J
2270    IF K MOD 2=1    IN S=-S
2280    Q=Q*(Alpha+K)/K
2290    A(K)=S
2300    B(K)=S*SQR(Q)
2310    NEXT K
2320    Mx=Mx+10
2330    IF Mx<M THEN 2320
2340    Threshold=-7.
2350    T2=Threshold*2.
2360    V=10.^Threshold
2370    GINIT
2380    PLOTTER IS "GRAPHICS"
2390    GRAPHICS ON
2400    WINDOW 0.,FLT(Mx),T2,0.
2410    LINE TYPE 3
2420    FOR J=0 TO Mx STEP 10
2430    MOVE J,T2
2440    DRAW J,0.
245     NEXT J
2460    FOR J=T2 TO 0
2470    MOVE 0.,J
2480    DRAW Mx,J
2490    NEXT J
2500    PENUP
2510    LINE TYPE 1
2520    IMAGE 4D,2(4X,M.17DE)
2530    PRINT "   K                B(K)                                Sum"
2540    Sum=0.
2550    FOR K=0 TO M
2560    B=B(K)
2570    Sum=Sum+B*B
2580    PRINT USING 2520;K,B,Sum
2590    IF B<V THEN 2620
2600    Y=LGT(B)
2610    GOTO 2660
2620    IF B>-V THEN 2650
2630    Y=T2-LGT(-B)
2640    GOTO 2660
2650    Y=Threshold
2660    PLOT K,Y
2670    NEXT K
2680    PENUP
2690    SUBEND
2700    !
```

```
2710    SUB Moments(DOUBLE M,REAL P0,Mom(*))        !  SHOT NOISE
2720    Overlap=6.2                 !  AV. NO. PULSES/SEC * AVERAGE PULSE DURATI
2730    Sigmaa=1.                   !  PARAMETER OF RAYLEIGH AMPLITUDE PDF
2740    P0=EXP(-Overlap)            !  PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
2750    ALLOCATE Cum(0:M)           !  ARRAY FOR CUMULANTS
2760    DOUBLE K
2770    S=Sigmaa*Sigmaa
2780    Cum(0)=0.
2790    Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
2800    Cum(2)=Overlap*S*4./3.
2810    FOR K=3 TO M
2820    Cum(K)=Cum(K-2)*S*K*K/(K+1)
2830    NEXT K
2840    CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
2850    Mom(0)=Mom(0)-P0
2860    SUBEND



 10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u), VIA
 20 !  GENERALIZED LAGUERRE EXPANSION AND MOMENTS
 30     M=90    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
 40     DOUBLE M,I,N,K                      !  INTEGERS < 2^31 = 2,147,483,648
 50     REDIM Mom(0:M),A(0:M),L(0:M)
 60     REAL Mom(0:100),A(0:100),L(0:100)
 70     CALL Moments(M,P0,Mom(*))           !  P0 IS STEP AT ORIGIN
 80     Center=Mom(1)/Mom(0)                !  CENTER OF pc(u)
 90     R2=Mom(2)/Mom(0)-Center*Center      !  MEAN SQUARE SPREAD OF pc(u)
100     Rms=SQR(R2)                         !  RMS SPREAD OF pc(u)
110     Alpha0=Center*Center/R2-1.          !  THE CHOICES   Alpha=Alpha0    AND
120     Beta0=R2/Center                     !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130       Alpha=.74
140       Beta=2.1
150     CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))    !   DIRECT    MOMENTS
160   ! CALL Coeffir_via_mom(M,Alpha,Beta,Mom(*),A(*))    !   RECURSIVE MOMENTS
170     PRINT "Center = ";Center
180     PRINT "Rms =";Rms
190     F1=1./(Beta*FNGamma(Alpha+1.))
200       INPUT "ORDER AND LIMITS:",N,U1,U2
210     PRINT "ORDER AND LIMITS:",N;U1;U2
220     Du=(U2-U1)/100.
230     H=4./(U2-U1)
240     PLOTTER IS "GRAPHICS"
250     GRAPHICS ON
260     WINDOW U1,U2,-H*.1,H
270     GRID (U2-U1)*.1,H*.1
280     PLOT 0.,0.
290     FOR I=1 TO 100
300     U=U1+Du*I
310     T=U/Beta
320     CALL Laguerre(N,Alpha,T,L(*))
330     Sum=A(0)
340     FOR K=1 TO N
350     Sum=Sum+A(K)*L(K)
360     NEXT K
370     P=F1*EXP(-T+Alpha*LOG(T))*Sum    !   PDF OF RV AT U
380     PLOT U,P
390     NEXT I
400     PENUP
410     GOTO 200
420     END
430     !
```

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Technical Report 7377
28 February 1985

Evaluation of Densities and Distributions via Hermite and Generalized Laguerre Series Employing High-Order Expansion Coefficients Determined Recursively via Moments or Cumulants

A. H. Nuttall

ABSTRACT

High-order series expansions of probability density functions and cumulative distribution functions, in Hermite as well as generalized Laguerre orthogonal polynomials, have been obtained, where the weighting functions in both cases can have arbitrary (mismatched) parameter values.

The high-order expansion coefficients for both the Hermite and generalized Laguerre series can each be obtained by any one of three fast recursive procedures (all of which have been programmed, and for which program listings are presented). The forms of these three recursive procedures differ in the Hermite versus Laguerre cases; however, they are basically either convolutions or finite alternating series with binomial coefficients. Accuracy of the three procedures is compared. Numerous examples of series expansions of probability density functions and cumulative distribution functions are given.

ABSTRACT

High-order series expansions of probability density functions and cumulative distribution functions, in Hermite as well as generalized Laguerre orthogonal polynomials, have been obtained, where the weighting functions in both cases can have arbitrary (mismatched) parameter values; that is, the two free parameters α and β in the weightings

$$w(u) = \frac{1}{\sqrt{2\pi}\,\beta} \exp\left(-\frac{(u-\alpha)^2}{2\beta^2}\right) \text{ for all } u, \qquad \text{Hermite}$$

$$w(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)} \text{ for } u > 0, \qquad \text{generalized Laguerre}$$

need not be chosen so that the first two expansion coefficients b_1 and b_2 in the orthonormal series are zero. (The zero-th order expansion coefficient b_0 is never zero.) Nonetheless, all the available N lowest-order moments of the approximating probability density function are maintained identical to those of the given probability density function, regardless of the weighting employed and any of its free parameter values.

It has been discovered that deliberate mismatch of α and β results in faster-decaying coefficient sequences $\{b_n\}_0^N$ than when α and β are chosen to make $b_1 = b_2 = 0$, which is a common choice. For example, the central limit theorem is just such a case, where α and β in the Hermite expansion are taken as the mean and standard deviation, respectively, and the number of moments employed is limited to just order N = 2.

A fast trial-and-error procedure is used in general to determine good values of weighting parameters α and β. The only statistics needed about the given probability density function or cumulative distribution function are either its moments or cumulants, through order N. Furthermore, all the results presented actually apply to functions which have arbitrary area (not necessarily equal to unity) and to functions which can become negative. In fact, one of the applications considered is to a shot noise process where the continuous part of the probability density function has area less than 1, and which is well approximated by a generalized Laguerre series expansion.

The high-order expansion coefficients for both the Hermite and generalized Laguerre series can each be obtained by any one of three fast recursive procedures (all of which have been programmed, and for which program listings are presented):

(a) recursively via cumulants,

(b) directly via moments,

(c) recursively via moments.

The forms of these three recursive procedures differ in the Hermite versus Laguerre cases; however, they are basically either convolutions or finite alternating series with binomial coefficients. The occurrence and quantitative value of round-off error for large N is easily discerned in a plot of the expansion coefficient sequence for each choice of α and β, and for each of the three procedures, as well as for both types of series expansions.

Comparisons of the accuracy of the three alternative recursive procedures reveals that expansion coefficients determined recursively via cumulants are generally most accurate and least susceptible to round-off error. Numerous examples of series expansions of probability density functions and cumulative distribution functions are given, including one with N = 150 terms, where the last expansion coefficient is of size 1E-10 relative to the leading coefficient b_0. Estimates of the error associated with the approximations obtained by the Hermite and generalized Laguerre series are derived and compared with results of several examples.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

TABLE OF CONTENTS (Cont'd)

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LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF SYMBOLS

$w(u)$	Weighting function at argument u
α,β	Parameters of weighting w; (1),(2)
b_n	n-th expansion coefficient in orthonormal polynomial series
N	Number of terms employed in series expansion
p	Probability density function
P	Cumulative distribution function; integral of p
1-P	Exceedance distribution function
f	Characteristic function corresponding to p
μ_n	n-th moment of p or f; (3)
χ_n	n-th cumulant of p or f; (7)
ν_n	n-th moment of weighting w; (11)
Q_n	n-th order orthonormal polynomial
p_N, P_N	N-th order approximations to p,P
E_N	Weighted squared error
A	Area of p; (24)
M	Mean location of p; (24)
R	Rms width of p; (24)
γ,ω	Parameters of probability density function p
ϕ	Normalized Gaussian function; (41)
Φ	Integral of ϕ; (41)
He_n	n-th Hermite polynomial; (50)
a_n, c_n	Auxiliary expansion coefficients; (44)-(47) or (90)-(92)
w,z	Auxiliary variables

LIST OF SYMBOLS (Cont'd)

$\hat{\chi}_n$	Normalized cumulants; (62)
$\hat{\mu}_n$	Normalized moments; (69)
$\tilde{\mu}_n$	Alternative normalized moments; (118)
$\hat{He}_n$	n-th normalized Hermite polynomial; (68)
Hi_n	n-th modified Hermite polynomial; (74)
$\hat{Hi}_n$	n-th normaiized modified Hermite polynomial; (80)
$L_n^{(\alpha)}$	n-th generalized Laguerre polynomial; (96)
${}_1F_1$	Confluent hypergeometric function
d_m	Auxiliary variables in (112)
RC	Recursively via Cumulants
DM	Directly via Moments
RM	Recursively via Moments
$\mathcal{J},\theta$	Parameters of p in (141)
Q_ν	Generalized Q-function; (165)
P_0	Area of impulse at origin of p, for shot noise
F	Hypergeometric function ${}_2F_1$
Env	Envelope function
$E_n(u;p)$	Estimated error of p_n; (178) or (183)
$E_n(u;P)$	Estimated error of P_n; (180) or (185)

EVALUATION OF DENSITIES AND DISTRIBUTIONS VIA HERMITE AND GENERALIZED LAGUERRE SERIES EMPLOYING HIGH-ORDER EXPANSION COEFFICIENTS DETERMINED RECURSIVELY VIA MOMENTS OR CUMULANTS

INTRODUCTION

In the theoretical analysis of performance of some systems with nonlinearities and/or memory, it often happens that the only statistics about the decision (or output) random variable of interest that can be easily found are the moments, or in other cases, the cumulants. Explicit relations for the low-order expansion coefficients in Edgeworth or Gram-Charlier series are available in terms of the available moments or cumulants [1, pp. 172 and 191], [2, pp. 223 and 226], [3, pp. 157 and 159]. However, for higher-order moments and cumulants, these explicit nonrecursive relations are very tedious to derive, become extremely lengthy, and are not practical to use.

We will address the problem of obtaining accurate high-order series expansion approximations of the probability density function and cumulative distribution function of a random variable of interest, in terms of the available moments or cumulants of that random variable. The necessity of being able to approximate probability density functions and cumulative distribution functions from knowledge of either the moments or the cumulants, is that some physical problems have these particular statistics as natural and convenient starting points. For example, if a physical processor sums together a number of independent Rician random variates, the characteristic function and cumulants of the individual random variables or their sum are not available in any useful analytic form; however, the nigh-order moments of an

individual Rician variate can be easily and accurately evaluated by recurrence, and thereby the moments of the sum can be obtained. Conversely, for shot noise with random amplitude and duration modulation, the probability density function is not readily available, whereas the characteristic function is, and the cumulants are simple to evaluate [4, appendix C].

The particular series expansions we employ are based on the two special classes of weighting functions

$$w(u) = \frac{1}{(2\pi)^{1/2}\beta} \exp\left(-\frac{(u-a)^2}{2\beta^2}\right) \quad \text{for all } u \qquad \text{Hermite} \qquad (1)$$

and

$$w(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \quad \text{for } u > 0 \qquad \text{generalized Laguerre.} \qquad (2)$$

The orthonormal polynomials associated with these weightings are directly related to the Hermite and generalized Laguerre polynomials, respectively [5, 22.2.15 and 22.2.13]. The weightings each have two free parameters, α and β, which can be manipulated to advantage in obtaining finite (high-order) series expansions which well approximate a given (unknown) probability density function and cumulative distribution function.

The question of when a set of moments uniquely determines the probability density function is a difficult one; see, for example, [3, pp. 109-112 and 179]. Also, the convergence of the series is very involved [2, pp. 223 and 258], [3, pp. 161-163]. But, even if the series is divergent, use of a limited number of expansion coefficients often gives a satisfactory approximation to the desired probability density function [3, p. 167]. We

presume here that the moments do uniquely determine the probability density function and are buoyed in that respect by the comment [3, p. 87] that most distributions in statistical practice do possess this property.

The main idea in the series expansion approach here is not necessarily to get as many terms as possible, but rather to get as rapid convergence as possible of the series. If a particular choice of weighting parameters α and β results in sufficiently small expansion coefficients, say, at order 10, this is better than another choice of α and β where 20 or 30 terms are required for the same size coefficients. In fact, if α and β could be chosen such that the series terminated (zero coefficients) after a few terms, that would be ideal; however, this is not the case, and in fact, the choice of α and β requires some trial-and-error to achieve rapidly decreasing coefficients.

The expansion coefficients of a given probability density function, in an orthonormal set of Hermite or generalized Laguerre polynomials, are denoted by $\{b_n\}_0^N$, where N is the number of available or known moments or cumulants. Very often, the choice of α and β in (1) or (2) has been made such that $b_1 = 0$ and $b_2 = 0$, for purposes of analytic simplicity and for hopeful early termination of the series; see for example [1, pp. 171 and 191], [2, p. 223], [3, p. 159]. However, it will be demonstrated that this is generally not the best choice, and that more rapidly decaying coefficients can be achieved by other (mismatched) values of α and β, which must be searched for numerically; this possibility is also mentioned in [3, p. 164]. In fact, an example will be given which illustrates that the choice of parameters α and β to make expansion coefficients b_1 and b_2 zero, can in fact, lead to a divergent Hermite series.

Depending on the available information about the probability density function, i.e., moments or cumulants, a variety of methods will be given for determining the expansion coefficients $\{b_n\}$. In particular, for both the Hermite and generalized Laguerre series, we can get the coefficients by three different procedures:

(a) recursively via cumulants,

(b) directly via moments,

(c) recursively via moments.

The reason for having these alternatives is that the calculation of expansion coefficients $\{b_n\}$ for high-order n invariably runs into large round-off error. In order to reduce this round-off error, the amount of number-crunching on the computer should be minimized, and any spurious transformations between moments and cumulants should be avoided if possible. Thus it is desireable to have techniques which can accomplish the desired goal of evaluating expansion coefficients $\{b_n\}$ as directly as possible from the available information. The use of different alternatives also enables comparisons of the computed expansion coefficients and thereby furnishes quantitative assessment of the amount of round-off error. Recursive inter-relationships between moments, central moments, and cumulants are given in [6], including cases of two dependent random variables.

FUNDAMENTAL EQUATIONS

DEFINITION OF STATISTICS

Suppose a function p has known moments*

$$\mu_n = \int du\ u^n\ p(u) \quad \text{for } 0 \leq n \leq N\ . \tag{3}$$

The function p need not have unit area, i.e., $\mu_0 \neq 1$ is allowed, and p can become negative at some arguments u. Nevertheless, for convenience, and since most of our applications are to random variables, we shall refer to p as a probability density function, and to its running integral

$$P(u) = \int_{-\infty}^{u} dt\ p(t) \tag{4}$$

as a cumulative distribution function. We shall presume that $\mu_0 > 0$ in all cases.

The characteristic function corresponding to probability density function p is the Fourier transform

$$f(i\xi) = \int du\ \exp(i\xi u)\ p(u)\ . \tag{5}$$

When f is expanded in a power series, the result is

$$f(i\xi) = \sum_{n=0}^{\infty} \mu_n (i\xi)^n / n! \tag{6}$$

---

* Integrals without limits are over the range of nonzero integrand.

in terms of the moments in (3). Alternatively, if $\ell n\ f$ is expanded in a power series,

$$\ell n\ f(i\xi) = \sum_{n=0}^{\infty} \chi_n\ (i\xi)^n/n! \ , \tag{7}$$

where the quantities $\{\chi_n\}$ are the cumulants of p or f. Observe that generally, to the lowest three orders,

$$\chi_0 = \ell n\ f(0) = \ell n\ \mu_0 \neq 0 \ ,$$

$$\chi_1 = \frac{\mu_1}{\mu_0} \ ,$$

$$\chi_2 = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 \ . \tag{8}$$

The available information on probability density function p will be either

$$\text{moments } \{\mu_n\}_0^N \quad \text{or cumulants } \{\chi_n\}_0^N \ . \tag{9}$$

Whichever is available, we wish to get high-order accurate approximations to p and cumulative distribution function P in (4); that is, values of N in the order of 10 to 100 are of interest.

WEIGHTING FUNCTION PROPERTIES

We select a nonnegative weighting function w such that

$$w(u) > 0 \text{ at least where } p(u) \neq 0 \ . \tag{10}$$

We also disallow any impulses in w. The moments of weighting w are defined analogously to (3) as

$$\nu_n = \int du\ u^n\ w(u) \quad \text{for } n \geq 0 \ ; \tag{11}$$

it is presumed that these quantities can be evaluated for as large n as required.

Suppose weighting w has r free parameters (plus a scaling parameter). It might then seem beneficial to choose them such that the moments of w and p are approximately equal,

$$\nu_n \cong \mu_n \quad \text{for } 1 \leq n \leq r \quad (\text{plus } \nu_0 \cong \mu_0) \ , \tag{12}$$

for then the abscissa scales of w and p would tend to match. However, (12) will turn out to be not so desireable, and the choice of the r weighting parameter values should be based on another criterion. The ordinate scale of w is actually immaterial, since the expansion coefficients $\{b_n\}$ will absorb this scaling; so henceforth we presume that $\nu_0 = 1$ with no loss of generality.

APPROXIMATION PROCEDURE

Let Q_n be any n-th order polynomial, and approximate probability density function p by function

$$p_N(u) \equiv w(u) \sum_{n=0}^{N} b_n Q_n(u) \quad \text{where } w(u) > 0 , \tag{13}$$

where $\{b_n\}_0^N$ are the expansion coefficients. Define weighted squared error

$$E_N = \int du\ \gamma(u)\ [p(u)-p_N(u)]^2 =$$

$$= \int du\ \gamma(u)\ [p(u)-w(u) \sum_{n=0}^{N} b_n Q_n(u)]^2 , \tag{14}$$

where error-weighting γ is nonnegative. If we minimize E_N by choice of expansion coefficients $\{b_n\}_0^N$, there follows the set of linear equations

$$\sum_{n=0}^{N} b_n \int du\ \gamma(u)\ w^2(u)\ Q_k(u)\ Q_n(u) = \int du\ \gamma(u)\ w(u)\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N. \tag{15}$$

In order to use only the available information in (9) about p, the right-hand side of (15) must simplify according to the selection

$$\gamma(u) = \frac{K}{w(u)} \quad \text{where } w(u) > 0 \text{ (and arbitrary elsewhere).} \tag{16}$$

Furthermore, since constant K merely scales error E_N, and appears on both sides of (15), we can set K = 1 without loss of generality. Then (14) becomes

$$E_N = \int du\ w(u)\left[\frac{p(u)}{w(u)} - \sum_{n=0}^{N} b_n\ Q_n(u)\right]^2 \quad \text{where } w(u) > 0\ , \tag{17}$$

and (15) reduces to

$$\sum_{n=0}^{N} b_n \int du\ w(u)\ Q_k(u)\ Q_n(u) = \int du\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N. \tag{18}$$

In general, this is N+1 simultaneous linear equations in the N+1 unknowns $\{b_n\}_0^N$. The choice $Q_k(u) = u^k$ would lead to an apparently simple set of equations, when (11) and (3) are used. However, a few numerical examples quickly reveals that they are very ill-conditioned, due to the character of the nondiagonal matrix with elements

$$\int du\ w(u)\ Q_k(u)\ Q_n(u) \quad \text{for } 0 \le k,n \le N \tag{19}$$

that appears on the left-hand side of (18). In order to avoid the significant round-off error associated with solving such a system for large N, we choose $\{Q_n\}_0^N$ to be a set of <u>orthonormal</u> polynomials with respect to weighting w; i.e., (19) is 1 for k = n, and 0 otherwise. Also recall that $\nu_0 = \int du\ w(u) = 1$ without loss of generality.

Equation (18) then reduces to an explicit relation for the expansion coefficients:

$$b_k = \int du\ p(u)\ Q_k(u) \quad \text{for } 0 \le k \le N\ , \tag{20}$$

and (17) for the error becomes merely

$$E_N = \int du\ \frac{p^2(u)}{w(u)} - \sum_{n=0}^{N} b_n^2\ . \tag{21}$$

It will be presumed that the integral in (21) is finite; otherwise, the error would be infinite, which is a meaningless problem. This will put some restrictions on the parameter choices of weighting w, since this error integral depends on these parameters as well as on the given probability density function p. The sum of squares in (21) must then be bounded, and in fact affords a measure of the adequacy of approximation (13), by saturating (at an apriori unknown value) for large N.

As N increases, the values of the lower-order expansion coefficients $\{b_k\}$ in (20) do not change. Therefore they only have to be computed once and do not have to be revised as more terms are added in series approximation (13), i.e., larger N.

EQUALITY OF PROBABILITY DENSITY FUNCTION MOMENTS

A very important property of expansion (13) is obtained as follows:

$$\int du\ Q_k(u)\ p_N(u) = \int du\ Q_k(u)\ w(u) \sum_{n=0}^{N} b_n\ Q_n(u) =$$

$$= b_k = \int du\ Q_k(u)\ p(u) \quad \text{for } 0 \leq k \leq N\ , \tag{22}$$

where we used, in order, (13), the orthonormality of (19), and (20). But since Q_k is a k-th order polynomial, relation (22) states that approximation p_N has exactly the same moments as given probability density function p, from order 0 through order N. This matching of moments between probability density functions p_N and p has been achieved regardless of the weighting w and its particular parameter values. Furthermore, (22) holds independently of whether the weighting-moment equalities in (12) are satisfied or not.

The cumulative distribution function corresponding to approximation p_N is defined as

$$P_N(u) \equiv \int_{-\infty}^{u} dt\ p_N(t) = \sum_{n=0}^{N} b_n \int_{-\infty}^{u} dt\ w(t)\ Q_n(t)\ . \tag{23}$$

Its utility depends on getting closed forms and simple recursions for the general integral on the right-hand side.

PARAMETERS OF GIVEN PROBABILITY DENSITY FUNCTION p

The moments of p were defined in (3). It is useful to define three important parameters of p:

$$\text{Area } A = \int du\ p(u) = \mu_0 \quad (\mu_0 > 0, \text{ but need not be } 1);$$

$$\text{Mean Location } M = \frac{\int du\ u\ p(u)}{\int du\ p(u)} = \frac{\mu_1}{\mu_0}\ ;$$

$$\text{RMS Width } R = \left[\frac{\int du (u-M)^2 p(u)}{\int du\ p(u)}\right]^{1/2} = \left[\frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2\right]^{1/2}. \tag{24}$$

(Conversely, $\mu_0 = A$, $\mu_1 = AM$, $\mu_2 = A(M^2+R^2)$.) These parameters depend on the probability density function p that we are trying to approximate and can be computed from the available information (9). They are useful for determining where the major concentration of p(u) lies on the u-scale, and have obvious physical interpretations.

In terms of the cumulants of p defined in (5)-(8), we have the alternative expressions

$$A = \exp(\chi_0), \qquad M = \chi_1, \qquad R = \chi_2^{1/2}, \tag{25}$$

or conversely

$$\chi_0 = \ln \mu_0 = \ln A, \quad \chi_1 = \frac{\mu_1}{\mu_0} = M, \quad \chi_2 = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 = R^2 . \tag{26}$$

GENERAL RESULTS FOR THREE LOWEST-ORDER POLYNOMIALS Q_n

The weighting function w and associated orthonormal polynomials satisfy the following equation:

$$\int du\, w(u)\, Q_k(u)\, Q_n(u) = \delta_{kn} . \tag{27}$$

Also we have weighting moments

$$\nu_n = \int du\, u^n\, w(u), \quad \text{with } \nu_0 = 1 . \tag{28}$$

It is then a straightforward matter to evaluate the three lowest-order orthonormal polynomials:

$$\begin{aligned} Q_0(u) &= 1 , \\ Q_1(u) &= \frac{1}{D_1}(u-\nu_1) , \\ Q_2(u) &= \frac{1}{D_2}\left[u^2(\nu_2-\nu_1^2) - u(\nu_3-\nu_2\nu_1) + (\nu_3\nu_1-\nu_2^2)\right] , \end{aligned} \tag{29}$$

where

$$D_1 = (\nu_2 - \nu_1^2)^{1/2} ,$$

$$D_2 = (\nu_2 - \nu_1^2)^{1/2} \left[(\nu_4 - \nu_2^2)(\nu_2 - \nu_1^2) - (\nu_3 - \nu_2\nu_1)^2\right]^{1/2} . \tag{30}$$

The general expansion coefficients in (20) then become

$$b_0 = \mu_0 ,$$

$$b_1 = \frac{1}{D_1}(\mu_1 - \nu_1\mu_0) ,$$

$$b_2 = \frac{1}{D_2}\left[\mu_2(\nu_2 - \nu_1^2) - \mu_1(\nu_3 - \nu_2\nu_1) + \mu_0(\nu_3\nu_1 - \nu_2^2)\right] . \tag{31}$$

All these results above are general and make no presumption about weighting moment equalities such as (12).

SPECIAL CHOICES OF WEIGHTING PARAMETERS

Suppose that weighting w has free parameters that can be varied so as to make the mean locations of w and p coincide (see (24)); that is,

$$\text{let} \quad \nu_1 = \frac{\mu_1}{\mu_0} . \tag{32A}$$

(The reason for the discrepancy with (12) is that we have set $\nu_0 = 1$ but have allowed $\mu_0 \neq 1$.) Inspection of (31) gives the following:

$$\text{then} \quad b_1 = 0 \quad \text{and} \quad b_2 = -\frac{\mu_0}{D_2}\left(\nu_2 - \frac{\mu_2}{\mu_0}\right)\left(\nu_2 - \frac{\mu_1^2}{\mu_0^2}\right). \tag{32B}$$

Conversely, (31) shows that requiring $b_1 = 0$ forces the choice in (32A) for ν_1. Thus equality of the first weighting moment ν_1 of w with the first (normalized) moment of probability density function p implies (and is implied by) the vanishing of the first expansion coefficient b_1. This may or may not be a useful choice, but, whether adopted or not, has no bearing on the equality of probability density function moments already demonstrated in (22).

As a second special choice, suppose that weighting w has enough free parameters that we can vary, so as to make the mean locations and rms widths of w and p coincide (see (24)); that is

$$\text{let} \quad \nu_1 = \frac{\mu_1}{\mu_0} \quad \text{and} \quad \nu_2 = \frac{\mu_2}{\mu_0} \, . \tag{33A}$$

(Again we have used $\nu_0 = 1$.) Manipulation of (31) yields the following conclusion:

$$\text{then} \quad b_1 = 0 \quad \text{and} \quad b_2 = 0 \, . \tag{33B}$$

Conversely, imposition of (33B) implies the results in (33A), as may be seen by reference to (31). (The apparent additional solution $\nu_2 = \mu_1^2/\mu_0^2 = \nu_1^2$ would yield an impulse for w and is disallowed.) Thus equality of the first two weighting moments of w with the first two (normalized) moments of probability density function p implies (and is implied by) the vanishing of the first two expansion coefficients b_1 and b_2. This common choice of weighting parameter values can be made if desired, but is not necessary (or recommended) for series approximations by orthonormal polynomials. The equality of probability density function moments in (22) will hold whether (33) is true or not.

EXAMPLE OF DIVERGENT ERROR INTEGRAL FOR $b_1 = 0$, $b_2 = 0$

As a demonstration of what forcing expansion coefficients b_1 and b_2 equal to zero can do, consider probability density function

$$p(u) = \frac{2\, u^{\gamma} \exp(-u^2/\omega^2)}{\omega^{\gamma+1} \Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{for } u > 0 \qquad (\gamma > -1,\ \omega > 0) \tag{34}$$

with moments

$$\mu_n = \omega^n \frac{\Gamma\left(\frac{\gamma+1+n}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} . \tag{35}$$

This class of probability density functions includes the one-sided Gaussian, Rayleigh, and Maxwell as special cases, for $\gamma = 0, 1, 2$, respectively.

Consider also the Hermite weighting given in (1), which has moments (11) equal to

$$\nu_0 = 1, \quad \nu_1 = \alpha, \quad \nu_2 = \alpha^2 + \beta^2 . \tag{36}$$

If we now insist on property (33B), then (33A) yields

$$\alpha = \omega \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} , \quad \beta^2 = \omega^2 \left[\frac{\gamma+1}{2} - \frac{\Gamma^2\left(\frac{\gamma}{2}+1\right)}{\Gamma^2\left(\frac{\gamma+1}{2}\right)} \right] . \tag{37}$$

But the leading integral in minimum error E_N in (21) is convergent only if $p^2(u)/w(u)$ decays sufficiently rapid for large u. We have from (34) and (1), the dominant behavior

$$p^2(u)/w(u) \propto\!\!\!\!\mathcal{C}\ \exp\left(-\frac{2u^2}{\omega^2} + \frac{u^2}{2\beta^2} \right) \quad \text{for large positive } u , \tag{38}$$

where $\propto\!\!\!\!\mathcal{C}$ denotes proportionality, but disregards the exact scale factor and subdominant behavior. Thus the integral in (21) is convergent only if

$$1 < \frac{4\beta^2}{\omega^2} = 2(\gamma+1) - 4\frac{\Gamma^2\left(\frac{\gamma}{2}+1\right)}{\Gamma^2\left(\frac{\gamma+1}{2}\right)} . \qquad (39)$$

However, calculation of (39) reveals that this inequality is never satisfied for any value of $\gamma > -1$; the function on the right-hand side starts at 0 when $\gamma = -1$, and increases monotonically towards 1 as $\gamma \to +\infty$, behaving like $1 - 1/(4\gamma)$ in this limit.

Thus expansion of probability density function (34) according to a Hermite weighting has an infinite error integral (21) (and perhaps a divergent series expansion) regardless of the values of γ and ω in the true probability density function, if we insist on expansion coefficients $b_1 = b_2 = 0$. Yet if we relax requirement (33B), and choose β according to (39) such that $\beta > \omega/2$, the error integral in (21) is certainly finite, regardless of α.

However, making the error integral in (21) finite is not the whole story, in so far as realizing useful approximations to the probability density function or cumulative distribution function. An example of probability density function (34) was taken with $\gamma = 3$, $\omega = 1$. When α and β were chosen according to (33) and (37) (giving $\beta = .48 < .5 = \omega/2$), the expansion coefficients $\{b_n\}$ initially decreased to approximately 1E-3 at n = 40 terms, and then diverged; yet a plot of the approximate exceedance distribution function obtained by a Hermite expansion overlaid the exact answer down to the 1E-16 level. On the other hand, when the weighting parameters in the Hermite expansion were chosen as* $\alpha = 0$, $\beta = .7 > .5 = \omega/2$, giving $b_1 \neq 0$ and

*This is example B in a later section

$b_2 \neq 0$, the expansion coefficients $\{b_n\}$ decreased to the 1E-4 level at $n = 70$ before round-off error became dominant; despite this apparent improvement in coefficient level, the approximate exceedance distribution function overlaid a plot of the exact result down to the 1E-10 probability level, which is several orders of magnitude worse than the above result. Thus emphasis on getting a convergent error integral in (21) may not always be desired.

For Hermite weighting (1) and the class of probability density functions which decay as $\exp(-u^q)$ as $u \to +\infty$, the error integral is always convergent if $q > 2$, and always divergent if $q < 2$. So an exponential probability density function, like $u^\gamma \exp(-u/\omega)$ for $u > 0$, always yields a divergent error integral when expanded in a Hermite series.

For generalized Laguerre weighting (2), it is necessary to consider $u = 0^+$ and $u = +\infty$ separately. If probability density function p behaves like u^γ as $u \to 0^+$, then a finite error integral requires that we choose $\alpha < 1 + 2\gamma$. Coupled with the finite area restriction on weighting w, a range of values of α is allowed, namely, $-1 < \alpha < 1 + 2\gamma$; this range always exists since $\gamma > -1$ is necessary for the probability density function itself to have finite area.

If also the probability density function behaves as $\exp(-u/\omega)$ as $u \to +\infty$, then a finite error integral with generalized Laguerre weighting requires that we choose $\beta > \omega/2$. So the range of choice of β is open on the large side, whereas that for α is a limited one, for this particular class of probability density functions.

HERMITE EXPANSION

In this section, we will deal exclusively with weighting (1),

$$w(u) = \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) \quad \text{for all } u \qquad (\beta > 0)\ , \tag{40}$$

where

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2), \quad \Phi(x) = \int_{-\infty}^{x} dt\ \phi(t)\ . \tag{41}$$

This weighting has two free parameters, α and β, and moments

$$\nu_0 = 1, \quad \nu_1 = \alpha, \quad \nu_2 = \alpha^2+\beta^2\ . \tag{42}$$

If ν_1 and ν_2 are specified, the parameters must then satisfy $\alpha = \nu_1$, $\beta = (\nu_2 - \nu_1^2)^{1/2}$. However, we shall keep α and β general and unspecified.

PROPERTIES OF POLYNOMIALS AND EXPANSIONS

The orthonormal polynomials associated with weighting (40) are the Hermite polynomials [5, 22.1.2 and 22.2.15]

$$Q_n(u) = He_n\left(\frac{u-\alpha}{\beta}\right)(n!)^{-1/2} \quad \text{for } n \geq 0\ . \tag{43}$$

The expansion coefficients are given by (20) as

$$b_n = \int du\ p(u)\ Q_n(u) = (n!)^{-1/2}\ c_n \quad \text{for } n \geq 0\ , \tag{44}$$

where we define

$$c_n = \int du\ p(u)\ He_n\left(\frac{u-\alpha}{\beta}\right) \quad \text{for } n \geq 0 \ . \tag{45}$$

The approximate probability density function then follows from (13) in the form

$$p_N(u) = w(u) \sum_{n=0}^{N} b_n\ Q_n(u) = \frac{1}{\beta}\ \phi\left(\frac{u-\alpha}{\beta}\right) \sum_{n=0}^{N} a_n\ He_n\left(\frac{u-\alpha}{\beta}\right) \ , \tag{46}$$

where we used (40), (43), (44), and defined

$$(n!)^{1/2}\ a_n = b_n = (n!)^{-1/2}\ c_n \quad \text{for } n \geq 0 \ . \tag{47}$$

These three different coefficients in (44)-(47) are introduced for convenience in further equation manipulations. Expansion coefficient b_n is the geometric mean of auxiliary coefficients a_n and c_n (with polarity). Expansion (46) is also called a Gram-Charlier series of type A [2, p. 222], [3, p. 156].

The approximate cumulative distribution function corresponding to (46) is

$$P_N(u) = \int_{-\infty}^{u} dt\ p_N(t) = \sum_{n=0}^{N} a_n \int_{-\infty}^{u} \frac{dt}{\beta}\ \phi\left(\frac{t-\alpha}{\beta}\right) He_n\left(\frac{t-\alpha}{\beta}\right) =$$

$$= \sum_{n=0}^{N} a_n \int_{-\infty}^{T} dx\ \phi(x)\ He_n(x) = a_0\ \Phi(T) - \phi(T) \sum_{n=1}^{N} a_n\ He_{n-1}(T) \ , \tag{48}$$

where

$$T = \frac{u-\alpha}{\beta} \tag{49}$$

and we used (41) and [5, 22.11.8].

The Hermite polynomials $\{He_n\}$ satisfy the recurrence [5, 22.7.14]

$$He_n(x) = x\, He_{n-1}(x) - (n-1)\, He_{n-2}(x) \quad \text{for } n \geq 2\,, \tag{50}$$

with starting values $He_0(x) = 1$, $He_1(x) = x$ [5, 22.3.11]. The highest-order term in $He_n(x)$ is x^n, with coefficient 1 [5, 22.1.2 and 22.3.11]. The magnitude of the term multiplying b_n in (46) has an envelope that decays approximately as $n^{-1/4}$ with n, regardless of argument u. This may be seen by using (47) and (49) to get

$$a_n\, He_n(T) = b_n (n!)^{-1/2}\, He_n(T) \propto b_n \left(n^{n+1/2}\, e^{-n}\right)^{-1/2} (n/e)^{n/2} = b_n\, n^{-1/4}$$

$$\text{as } n \to +\infty, \text{ for all } T, \tag{50A}$$

where we also used [5, 6.1.39 and 22.5.18] and [7, 8.22.8]. Here, $\propto$ denotes proportionality and we have taken the magnitude of the terms; the exact scale factor of proportionality will be presented in a later section where the errors of the approximations are estimated. So if b_n were to decay faster than $n^{-3/4}$, the probability density function series in (46) would converge absolutely.

Conditions are better for the cumulative distribution function series in (48); namely, based on the above result, there follows (for the envelope)

$$a_n\, He_{n-1}(T) = b_n (n!)^{-1/2}\, He_{n-1}(T) = b_n\, n^{-1/2}\, [(n-1)!]^{-1/2}\, He_{n-1}(T) =$$

$$\propto b_n\, n^{-1/2}\, n^{-1/4} = b_n\, n^{-3/4} \quad \text{as } n \to +\infty, \text{ for all } T\,. \tag{50B}$$

Thus if b_n decays faster than $n^{-1/4}$, the cumulative distribution function series converges absolutely. Furthermore, if the leading error integral in

(21) is finite, the sum of b_n^2 must be finite, meaning that b_n must decay faster than $n^{-1/2}$. So we can conclude that if the error integral is finite, the Hermite series for the cumulative distribution function in (48) converges. (Notice that this particular decay $n^{-1/2}$ of b_n is not sufficiently fast to make the same conclusion about the Hermite series for the probability density function in (46).) The above are sufficient conditions on expansion coefficients $\{b_n\}$, and are not necessary.

EXPANSION OF CHARACTERISTIC FUNCTION f

The coefficients a_n and c_n were defined in (45) and (47). Then the sum

$$\sum_{n=0}^{\infty} a_n w^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_n w^n = \sum_{n=0}^{\infty} \frac{w^n}{n!} \int du\, p(u)\, He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \int du\, p(u) \sum_{n=0}^{\infty} \frac{w^n}{n!} He_n\left(\frac{u-\alpha}{\beta}\right) = \int du\, p(u) \exp\left(\frac{u-\alpha}{\beta} w - \frac{1}{2} w^2\right) =$$

$$= \exp\left(-\frac{1}{2} w^2 - \frac{\alpha}{\beta} w\right) f\left(\frac{w}{\beta}\right), \tag{51}$$

where f is the characteristic function, and where we used (45), [5, 22.5.19 and 22.9.17], and (5). Letting $w = \beta z$, we have

$$f(z) \exp\left(-\alpha z - \frac{1}{2} \beta^2 z^2\right) = \sum_{n=0}^{\infty} a_n (\beta z)^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_n (\beta z)^n . \tag{52}$$

Thus $\{a_n\}$ and $\{c_n\}$ are the coefficients in these power series expansions of the function $f(z) \exp(-\alpha z - \beta^2 z^2/2)$, where f is the characteristic function corresponding to probability density function p, and α and β are arbitrary. A special case of (52) is given in [2, 17.6.10].

Collecting (46) and (52) together for comparison, and assuming that $p_n \to p$ as $N \to +\infty$, we have

$$p(u) = \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) \sum_{n=0}^{\infty} a_n\, He_n\left(\frac{u-\alpha}{\beta}\right),$$

$$f(i\xi) = \exp\left(i\alpha\xi - \frac{1}{2}\,\beta^2\xi^2\right) \sum_{n=0}^{\infty} a_n (i\beta\xi)^n \,. \tag{53}$$

Thus expansion of probability density function p in an infinite Hermite series is equivalent to an expansion of a modified form of the characteristic function in a power series, according to (53). Equations (51)-(53) will serve as very convenient starting points for the derivation of several alternative recurrences for the expansion coefficients $\{a_n\}$. Notice that weighting parameters α and β are completely unrestricted in (52) and (53), except that $\beta > 0$.

An analogous result holds for N finite, but must be derived in a different fashion, because we no longer can use infinite sum [5, 22.9.17]. Define the Fourier transform of (46) as the N-th order approximation to the characteristic function:

$$f_N(i\xi) = \int du\, \exp(i\xi u)\, p_N(u) =$$

$$= \sum_{n=0}^{N} a_n \int du\, \exp(i\xi u)\, \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \sum_{n=0}^{N} a_n \int dt\, \exp(i\xi\alpha + i\xi\beta t)\, \phi(t)\, He_n(t) =$$

$$= \exp(i\alpha\xi) \sum_{n=0}^{N} a_n \int dt\, \exp(i\beta\xi t) \left(-\frac{d}{dt}\right)^n \phi(t) =$$

$$= \exp(i\alpha\xi) \sum_{n=0}^{N} a_n (i\beta\xi)^n \int dt \exp(i\beta\xi t)\, \phi(t) =$$

$$= \exp\left(i\alpha\xi + \tfrac{1}{2}\beta^2(i\xi)^2\right) \sum_{n=0}^{N} a_n (i\beta\xi)^n , \qquad (54)$$

where we used [5, 22.11.8] in line 4, and repeated integration by parts in line 5. This result is the leading N terms of (53). As a by-product of this derivation, we have

$$\int dt \exp(zt)\, \phi(t)\, He_n(t) = \exp\left(\tfrac{1}{2} z^2\right) z^n . \qquad (55)$$

COEFFICIENTS RECURSIVELY VIA CUMULANTS

We are now in a position to obtain some useful recursive relations for the expansion coefficients $\{a_n\}$ in (51)-(54). The first one is obtained by taking the ℓn of (51):

$$\ell n\, f\left(\frac{w}{\beta}\right) - \frac{\alpha}{\beta} w - \frac{1}{2} w^2 = \ell n \left\{\sum_{n=0}^{\infty} a_n w^n\right\}. \qquad (56)$$

Then using (7) and identifying the right-hand side of (56) as a new power series, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \chi_n \left(\frac{w}{\beta}\right)^n - \frac{\alpha}{\beta} w - \frac{1}{2} w^2 = \sum_{n=0}^{\infty} \lambda_n w^n . \qquad (57)$$

There follows immediately

$$h_n = \begin{cases} \dfrac{\chi_n}{n!\ \beta^n} & \text{for } n \neq 1,2 \\ \dfrac{\chi_1 - \alpha}{\beta} & \text{for } n = 1 \\ \dfrac{1}{2}\left(\dfrac{\chi_2}{\beta^2} - 1\right) & \text{for } n = 2 \end{cases} . \tag{58}$$

But equality of the right-hand sides of (56) and (57) also requires that

$$\sum_{n=0}^{\infty} a_n w^n = \exp\left\{\sum_{n=0}^{\infty} h_n w^n\right\}. \tag{59}$$

It is shown in appendix A that a recursive solution to (59) for the $\{a_n\}$ is given by

$$a_n = \frac{1}{n} \sum_{m=1}^{n} m\, h_m\, a_{n-m} \quad \text{for } n \geq 1, \quad a_0 = \exp(h_0) . \tag{60}$$

Then eliminating $\{h_m\}$ by means of (58),

$$a_n = \frac{1}{n}\left[\left(\frac{\chi_1}{\beta} - \frac{\alpha}{\beta}\right) a_{n-1} + \left(\frac{\chi_2}{\beta^2} - 1\right) a_{n-2} + \sum_{m=3}^{n} \frac{\chi_m}{(m-1)!\ \beta^m}\, a_{n-m}\right] \quad \text{for } n \geq 1,$$

$$a_0 = \exp(\chi_0), \tag{61}$$

where $a_n \equiv 0$ for $n < 0$, and the sum is zero for $n < 3$.

Now define normalized cumulants (excluding n=0) according to

$$\hat{\chi}_n = \frac{\chi_n}{(n-1)!\ \beta^n} \quad \text{for } n \geq 1 . \tag{62}$$

Then (61) becomes

$$a_n = \frac{1}{n}\left[\left(\hat{\chi}_1 - \frac{\alpha}{\beta}\right) a_{n-1} + \left(\hat{\chi}_2 - 1\right) a_{n-2} + \sum_{m=3}^{n} \hat{\chi}_m \, a_{n-m}\right] \quad \text{for } n \geq 1 \,,$$

$$a_0 = \exp(\chi_0) \,. \qquad (63)$$

This convolution is the desired recursion for expansion coefficients $\{a_n\}$ via cumulants.

As particular cases, we have

$$a_1 = \frac{\chi_1 - \alpha}{\beta} a_0, \quad a_2 = \frac{1}{2}\left[\left(\frac{\chi_1 - \alpha}{\beta}\right)^2 + \frac{\chi_2}{\beta^2} - 1\right] a_0 \,. \qquad (64A)$$

Parameters α and β (>0) are completely arbitrary in the above three equations, and $\{\chi_n\}_0^N$ are the available cumulants of the probability density function under consideration.

Observe that if we choose $\alpha = \chi_1 = M$ and $\beta = \chi_2^{1/2} = R$ (see (24)-(25)), which is a very common choice, we have $a_1 = 0$ and $a_2 = 0$; this is a special case of the general property (33) stated earlier. This special choice of α and β corresponds to choosing the mean location and rms width of Hermite weighting (40) identical to those same parameters of the given probability density function. There then also follows, in this special case,

$$a_3 = \frac{1}{3}\hat{\chi}_3 \, a_0 \,, \quad a_4 = \frac{1}{4}\hat{\chi}_4 \, a_0 \,, \quad a_5 = \frac{1}{5}\hat{\chi}_5 a_0 \,,$$

$$a_n = \frac{1}{n}\left[\hat{\chi}_n \, a_0 + \sum_{m=3}^{n-3} \hat{\chi}_m \, a_{n-m}\right] \quad \text{for } n \geq 6 \quad \text{when } \alpha = \chi_1, \; \beta = \chi_2^{1/2} \,. \qquad (64B)$$

COEFFICIENTS DIRECTLY VIA MOMENTS

Before we begin this derivation, we present the following useful expansion [5, 22.9.17 and 22.5.19]:

$$\exp\left(-\frac{1}{2}y^2 + xy\right) = \sum_{n=0}^{\infty} \frac{1}{n!} He_n(x)\, y^n \ . \tag{65}$$

We now again refer to (51) and expand the terms as follows:

$$\sum_{n=0}^{\infty} a_n w^n = \exp\left(-\frac{1}{2}w^2 - \frac{\alpha}{\beta}w\right) f\left(\frac{w}{\beta}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} He_k\left(-\frac{\alpha}{\beta}\right) w^k \sum_{m=0}^{\infty} \frac{1}{m!} \mu_m \left(\frac{w}{\beta}\right)^m \ , \tag{66}$$

where we utilized (65) and (6). Equating coefficients of w^n on both sides of this equation, we have

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} He_k\left(-\frac{\alpha}{\beta}\right) \frac{\mu_{n-k}}{(n-k)!\ \beta^{n-k}} \quad \text{for } n \geq 0 \ . \tag{67}$$

We now define, for convenience, the normalized Hermite polynomials

$$\hat{He}_n(x) = \frac{1}{n!} He_n(x) \quad \text{for } n \geq 0 \ , \tag{68}$$

and the normalized moments

$$\hat{\mu}_n = \frac{\mu_n}{n!\ \beta^n} \quad \text{for } n \geq 0 \ . \tag{69}$$

(Notice the difference with the definition of the normalized cumulants (62).) Then (67) becomes

$$a_n = \sum_{k=0}^{n} \hat{He}_k\left(-\frac{\alpha}{\beta}\right) \hat{\mu}_{n-k} \quad \text{for } n \geq 0 , \tag{70}$$

which gives expansion coefficients $\{a_n\}$ directly in terms of the (normalized) moments of the given probability density function. The recurrence in (50) can be used to generate the Hermite factors needed in convolution (70). Parameters α and β (>0) of weighting (40) are arbitrary.

$$\hat{He}_0(x) = 1, \quad \hat{He}_1(x) = x, \quad \hat{He}_n(x) = \frac{1}{n}\left[x\,\hat{He}_{n-1}(x) - \hat{He}_{n-2}(x)\right] \text{ for } n \geq 2.$$

As particular cases, we have

$$a_0 = \mu_0, \quad a_1 = \frac{\mu_1 - \alpha\mu_0}{\beta}, \quad a_2 = \frac{\mu_2 - 2\alpha\mu_1 + (\alpha^2 - \beta^2)\,\mu_0}{2\beta^2} . \tag{71}$$

These agree with (64) which utilized cumulants. If we make the special choice of $\alpha = \mu_1/\mu_0$ and $\beta^2 = \mu_2/\mu_0 - (\mu_1/\mu_0)^2$, then $a_1 = 0$ and $a_2 = 0$.

An alternative more direct derivation of (67) is possible: from (47), (45), (B-3) in appendix B, and (3),

$$a_n = \frac{1}{n!}\, c_n = \frac{1}{n!} \int du\; p(u)\; He_n\left(\frac{u-\alpha}{\beta}\right) =$$

$$= \frac{1}{n!} \int du\; p(u) \sum_{k=0}^{n} \binom{n}{k} He_k\left(-\frac{\alpha}{\beta}\right)\left(\frac{u}{\beta}\right)^{n-k} =$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\, He_k\left(-\frac{\alpha}{\beta}\right) \frac{\mu_{n-k}}{(n-k)!\;\beta^{n-k}} \quad \text{for } n \geq 0 . \tag{72}$$

COEFFICIENTS RECURSIVELY VIA MOMENTS

Before we begin this derivation, we replace $x \to -ix$, $y \to iy$ in (65) to get

$$\exp\left(\frac{1}{2} y^2 + xy\right) = \sum_{n=0}^{\infty} \frac{1}{n!} He_n(-ix)\ (iy)^n = \sum_{n=0}^{\infty} \frac{1}{n!} Hi_n(x)\ y^n , \qquad (73)$$

where $Hi_n(x)$ is a real n-th order modified Hermite polynomial in x defined by

$$Hi_n(x) = i^n\ He_n(-ix) \quad \text{for } n \geq 0 . \qquad (74)$$

The recursion for these polynomials follows immediately from (50) as

$$Hi_n(x) = x\ Hi_{n-1}(x) + (n-1)\ Hi_{n-2}(x) \quad \text{for } n \geq 2 , \qquad (75)$$

with starting values $Hi_0(x) = 1$, $Hi_1(x) = x$. The difference with (50) is the polarity of the last term; thus for example, $Hi_2(x) = x^2 + 1$, $Hi_3(x) = x^3 + 3x$, versus $He_2(x) = x^2 - 1$, $He_3(x) = x^3 - 3x$.

We now rewrite (51) in the following form:

$$f\left(\frac{w}{\beta}\right) = \exp\left(\frac{1}{2} w^2 + \frac{\alpha}{\beta} w\right) \sum_{m=0}^{\infty} a_m\ w^m . \qquad (76)$$

Expanding in power series by means of (6) and (73),

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mu_n \left(\frac{w}{\beta}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!} Hi_k\left(\frac{\alpha}{\beta}\right) w^k \sum_{m=0}^{\infty} a_m\ w^m . \qquad (77)$$

Equating coefficients of w^n, there follows

$$\frac{\mu_n}{n!\ \beta^n} = \sum_{k=0}^{n} \frac{1}{k!} Hi_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 , \qquad (78)$$

or

$$\hat{\mu}_n = \sum_{k=0}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} = a_n + \sum_{k=1}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 , \tag{79}$$

where we have used normalized moments (69), and defined the normalized modified Hermite polynomials

$$\hat{Hi}_n(x) = \frac{1}{n!} Hi_n(x) \quad \text{for } n \geq 0 . \tag{80}$$

Finally, the desired recursion for expansion coefficients $\{a_n\}$ in terms of the moments follows as

$$a_n = \hat{\mu}_n - \sum_{k=1}^{n} \hat{Hi}_k\left(\frac{\alpha}{\beta}\right) a_{n-k} \quad \text{for } n \geq 0 . \tag{81}$$

Parameters α and β (>0) are arbitrary in (81) and (69).

SUMMARY

The approximations to the probability density function and cumulative distribution function are given by (46) and (48), respectively, where α and β are arbitrary constants, except that $\beta > 0$. The functions ϕ and Φ are defined in (41), while the Hermite polynomials $\{He_n\}$ are available via (50). The expansion coefficients $\{a_n\}$ are given by the three alternatives (63), (70), (81), in terms of normalized cumulants (62), normalized moments (69), normalized Hermite polynomials (68), and normalized modified Hermite polynomials (80) and (74). Programs for all three alternative procedures for determining expansion coefficients $\{a_n\}$ are listed in an appendix. The basis for these relations is the characteristics function expansion in (51)-(53).

GENERALIZED LAGUERRE EXPANSION

This section will treat weighting (2), namely,

$$w(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1} \Gamma(\alpha+1)} \quad \text{for } u > 0 \qquad (\alpha > -1,\ \beta > 0). \tag{82}$$

This weighting is a special case of the three-parameter weighting

$$\frac{(u-\gamma)^{\alpha} \exp\left(-\frac{u-\gamma}{\beta}\right)}{\beta^{\alpha+1} \Gamma(\alpha+1)} \quad \text{for } u > \gamma\ , \tag{83}$$

which is the most general scaled linear shift of the generalized Laguerre weighting [5, 22.2.12]

$$x^{\alpha} \exp(-x) \quad \text{for } x > 0\ . \tag{84}$$

We will consider only $\gamma = 0$ here. For a probability density function $p_0(u)$ which is known to be nonzero only for $u > u_0$, we would consider the modified probability density function $p(u) = p_0(u+u_0)$, because then $p(u) \neq 0$ only for $u > 0$, and the simpler weighting (82) would be directly applicable. This procedure is equivalent to choosing $\gamma = u_0$ in the three-parameter weighting (83) above, and requires knowledge of u_0. We presume that $p(u) \neq 0$ only for $u > 0$ henceforth in this section, and that any necessary shifting has already taken place.

Weighting (82) has two free parameters, α and β, and moments

$$\nu_n = (\alpha+1)_n \, \beta^n \quad \text{for } n \geq 0\ . \tag{85}$$

In particular,

$$\nu_0 = 1, \quad \nu_1 = (\alpha+1)\beta, \quad \nu_2 = (\alpha+2)(\alpha+1)\beta^2 . \tag{86}$$

If ν_1 and ν_2 are specified, then the parameters must satisfy

$$\alpha = \frac{\nu_1^2}{\nu_2 - \nu_1^2} - 1 , \quad \beta = \frac{\nu_2 - \nu_1^2}{\nu_1} . \tag{87}$$

However, we shall keep α and β general and unspecified except for the conditions in (82).

PROPERTIES OF POLYNOMIALS AND EXPANSIONS

The orthonormal polynomials associated with weighting (82) are the generalized Laguerre polynomials [5, 22.1.2 and 22.2.12]

$$Q_n(u) = L_n^{(\alpha)}\left(\frac{u}{\beta}\right)\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} \quad \text{for } n \geq 0, \quad u > 0 . \tag{88}$$

The expansion coefficients are given by (20) as

$$b_n = \int_0^\infty du\ p(u)\ Q_n(u) = \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} c_n \quad \text{for } n \geq 0 , \tag{89}$$

where we define

$$c_n = \int_0^\infty du\ p(u)\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \quad \text{for } n \geq 0 . \tag{90}$$

The approximate probability density function follows from (13) according to

$$p_N(u) = w(u) \sum_{n=0}^{N} b_n Q_n(u) =$$

$$= \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \quad \text{for } u > 0 , \tag{91}$$

where we used (82), (88), (89), and defined

$$\left(\frac{(\alpha+1)_n}{n!}\right)^{1/2} a_n = b_n = \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} c_n \quad \text{for } n \geq 0 . \tag{92}$$

These three different coefficients in (89)-(92) are introduced for convenience in further equation manipulations. Expansion coefficient b_n is the geometric mean of auxiliary coefficients a_n and c_n (with polarity).

The approximate cumulative distribution function corresponding to (91) is

$$P_N(u) = \int_0^u dt\, p_N(t) = \sum_{n=0}^{N} a_n \int_0^u dt \frac{t^\alpha \exp(-t/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)} L_n^{(\alpha)}\left(\frac{t}{\beta}\right) =$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n I_n\left(\frac{u}{\beta}\right) \quad \text{for } u > 0 , \tag{93}$$

where we define

$$I_n(y) = \int_0^y dx\, x^\alpha e^{-x} L_n^{(\alpha)}(x) \quad \text{for } n \geq 0, \quad y > 0 . \tag{94}$$

These quantities are evaluated in appendix C; when substituted in (93), they yield (95)

$$P_N(u) = \frac{(u/\beta)^{\alpha+1} \exp(-u/\beta)}{\Gamma(\alpha+1)} \left[\frac{a_0}{\alpha+1} {}_1F_1(1;\alpha+2;\frac{u}{\beta}) + \sum_{n=1}^{N} \frac{a_n}{n} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right)\right] \quad \text{for } u > 0,$$

where ${}_1F_1$ is the confluent hypergeometric function.

The generalized Laguerre polynomials $\left\{L_n^{(\alpha)}\right\}$ satisfy the recurrence [5, 22.7.12]

$$L_n^{(\alpha)}(x) = \frac{1}{n}\left[(\alpha-1+2n-x)\ L_{n-1}^{(\alpha)}(x) - (\alpha-1+n)\ L_{n-2}^{(\alpha)}(x)\right] \quad \text{for } n \geq 2\ , \tag{96}$$

with starting values $L_0^{(\alpha)}(x) = 1$, $L_1^{(\alpha)}(x) = \alpha+1-x$ [5, 22.4.7]. The highest order term in $L_n^{(\alpha)}(x)$ is $(-x)^n/n!$ [5, 22.1.2 and 22.3.9]; this is distinctly different from the coefficient 1 for the Hermite polynomials. Yet the envelope decay with n of the generalized Laguerre series for the probability density function and cumulative distribution function are identical to those of the Hermite series, for $u > 0$. To prove this, use (91) and (92) to get

$$a_n\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) = b_n\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \propto b_n\left(n^{-\alpha}\right)^{\frac{1}{2}}\ n^{\frac{\alpha}{2}-\frac{1}{4}} = b_n\ n^{-\frac{1}{4}}$$

$$\text{as } n \to +\infty, \quad \text{for } u > 0\ , \tag{97}$$

where we also used [5, 6.1.39] and [7, 8.22.1]. Again, $\propto$ denotes proportionality with n only; the exact scale factor will be presented in a later section where the errors of the approximations are estimated. So if b_n decays faster than $n^{-3/4}$, the probability density function series in (91) converges absolutely.

For the generalized Laguerre series of the cumulative distribution function in (95), we have, for the envelope of the general term,

$$\frac{1}{n}\, a_n\, L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right) = b_n\, \frac{1}{n} \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right) =$$

$$\propto b_n\, \frac{1}{n} \left(n^{-\alpha}\right)^{\frac{1}{2}} (n-1)^{\frac{\alpha+1}{2} - \frac{1}{4}} \sim b_n\, n^{-3/4} \text{ as } n \to +\infty, \quad \text{for } u > 0 . \tag{98}$$

Thus if b_n decays faster than $n^{-1/4}$, (95) converges absolutely. And if the error integral (21) is finite, this property of the $\{b_n\}$ is true. So if error integral (21) is finite, the generalized Laguerre series for the cumulative distribution function converges absolutely for $u > 0$; this is a sufficient, but not necessary, condition.

For zero argument, the generalized Laguerre polynomials behave differently for large n. From [5, 22.4.7 and 6.1.39],

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!} \sim \frac{n^\alpha}{\Gamma(\alpha+1)} \text{ as } n \to +\infty. \tag{99}$$

Then (97) and (98) are both replaced by $b_n\, n^{\alpha/2}$ as $n \to +\infty$. However, for $\alpha > 0$, the probability density function in (91) is zero at $u = 0$ due to the u^α term, so there is no need to perform the sum then. And the cumulative distribution function is always zero at $u = 0$, again eliminating the need to evaluate the sum in (95). So the difference in behavior at $u = 0$ is of no consequence.

EXPANSION OF CHARACTERISTIC FUNCTION f

The coefficients a_n and c_n for the generalized Laguerre series were defined in (90) and (92). Then the sum

$$\sum_{n=0}^{\infty} c_n w^n = \sum_{n=0}^{\infty} w^n \int_0^{\infty} du\ p(u)\ L_n^{(\alpha)}(u/\beta) =$$

$$= \int_0^{\infty} du\ p(u) \sum_{n=0}^{\infty} w^n\ L_n^{(\alpha)}(u/\beta) = \int_0^{\infty} du\ p(u)\ (1-w)^{-\alpha-1} \exp\left(-\frac{uw/\beta}{1-w}\right) =$$

$$= (1-w)^{-\alpha-1}\ f\left(\frac{-w/\beta}{1-w}\right), \tag{100}$$

where f is the characteristic function, and where we used (90), [5, 22.9.15], and (5). Thus $\{c_n\}$ are the expansion coefficients of the right-hand side of (100) in powers of w. If we let $w = \frac{-\beta z}{1-\beta z}$, we have the expansion for the characteristic function

$$f(z) = (1-\beta z)^{-\alpha-1} \sum_{n=0}^{\infty} c_n \left(\frac{-\beta z}{1-\beta z}\right)^n, \tag{101}$$

corresponding to given probability density function p. Weighting parameters α and β are arbitrary in (100) and (101).

Collecting (91) and (101) together for comparison, and assuming that $p_N \to p$ as $N \to +\infty$, we have, upon use of (92),

$$p(u) = \frac{u^{\alpha} \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha)}\left(\frac{u}{\beta}\right) \quad \text{for } u > 0,$$

$$f(i\xi) = (1-i\beta\xi)^{-\alpha-1} \sum_{n=0}^{\infty} a_n \frac{(\alpha+1)_n}{n!} \left(\frac{-i\beta\xi}{1-i\beta\xi}\right)^n. \tag{102}$$

Thus, expansion of probability density function p in an infinite generalized Laguerre series is equivalent to an expansion of the corresponding characteristic function in the series of the particular form in (102). Equations (100)-(102) will serve as very convenient starting points for the derivation of several alternative recurrences for expansion coefficients $\{a_n\}$. We reiterate that α and β are arbitrary in the above, except that $\alpha > -1$, $\beta > 0$.

An analogous result holds for N finite, but must be derived differently since we can no longer use infinite sum [5, 22.9.15]. Define the Fourier transform of (91) as the N-th order approximation to the characteristic function:

$$f_N(i\xi) = \int du \exp(i\xi u)\, p_N(u) =$$

$$= \int_0^\infty du \exp(i\xi u)\, \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\,\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n L_n^{(\alpha)}\left(\frac{u}{\beta}\right) =$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{N} a_n \int_0^\infty dt \exp(i\beta\xi t)\, t^\alpha e^{-t} L_n^{(\alpha)}(t) \,. \qquad (103)$$

In appendix D, it is shown that

$$\int_0^\infty dt\, e^{i\omega t}\, t^\alpha e^{-t} L_n^{(\alpha)}(t) = \frac{\Gamma(\alpha+1+n)}{n!} \frac{(-i\omega)^n}{(1-i\omega)^{\alpha+1+n}} \,. \qquad (104)$$

Substitution in (103) then yields

$$f_N(i\xi) = \sum_{n=0}^{N} a_n \frac{(\alpha+1)_n}{n!} \frac{(-i\beta\xi)^n}{(1-i\beta\xi)^{\alpha+1+n}} = \sum_{n=0}^{N} c_n \frac{(-i\beta\xi)^n}{(1-i\beta\xi)^{\alpha+1+n}} \,, \qquad (105)$$

where the last relation follows by use of (92). This result is the leading N terms of (102).

COEFFICIENTS RECURSIVELY VIA CUMULANTS

We can now obtain some useful recursive relations for expansion coefficients $\{a_n\}$ and/or $\{c_n\}$ in (100)-(105). We start by taking the ℓn of (100):

$$\ell n \left\{ \sum_{n=0}^{\infty} c_n w^n \right\} = -(\alpha+1)\, \ell n(1-w) + \ell n\, f\left(\frac{-w/\beta}{1-w}\right). \tag{106}$$

Identify the left-hand side as a new power series, and use (7) and [5, 15.1.8] to yield

$$\sum_{n=0}^{\infty} h_n w^n = (\alpha+1) \sum_{n=1}^{\infty} \frac{1}{n} w^n + \sum_{k=0}^{\infty} \frac{1}{k!} \chi_k \left(\frac{-w/\beta}{1-w}\right)^k =$$

$$= (\alpha+1) \sum_{n=1}^{\infty} \frac{1}{n} w^n + \sum_{k=0}^{\infty} \frac{(-1)^k \chi_k}{k!\ \beta^k} w^k \sum_{m=0}^{\infty} \frac{(k)_m}{m!} w^m . \tag{107}$$

Equating coefficients of w^n, there follows $h_0 = \chi_0$, while for $n \geq 1$,

$$h_n = \frac{1}{n}(\alpha+1) + \sum_{k=0}^{n} \frac{(-1)^k \chi_k}{k!\ \beta^k} \frac{(k)_{n-k}}{(n-k)!} =$$

$$= \frac{1}{n} \left[\alpha+1+ \sum_{k=1}^{n} (-1)^k \binom{n}{k} \hat{\chi}_k \right], \tag{108}$$

where we used the normalized cumulants defined in (62).

But since the left-hand sides of (106) and (107) are equal, we have

$$\sum_{n=0}^{\infty} c_n w^n = \exp\left\{ \sum_{n=0}^{\infty} h_n w^n \right\}, \tag{109}$$

or via appendix A, the recurrence

$$c_n = \frac{1}{n}\sum_{m=1}^{n} m\, h_m\, c_{n-m} \quad \text{for } n \geq 1, \quad c_0 = \exp(h_0) \ . \tag{110}$$

Finally, define

$$d_m = m\, h_m \quad \text{for } m \geq 1 \tag{111}$$

for notational convenience and thereby obtain

$$d_m = \alpha+1+ \sum_{k=1}^{m} (-1)^k \binom{m}{k} \hat{\chi}_k \quad \text{for } m \geq 1 \ ,$$

$$c_n = \frac{1}{n} \sum_{m=1}^{n} d_m\, c_{n-m} \quad \text{for } n \geq 1, \quad c_0 = \exp(\chi_0) \ , \tag{112}$$

by means of (108) and (110), respectively. Equation (112) is a recursive relation for expansion coefficients $\{c_n\}$ in terms of cumulants $\{\chi_n\}$ and auxiliary variables $\{d_m\}$. The $\{a_n\}$ are immediately available via (92).

As particular cases, we have, employing (62),

$$c_1 = \left(\alpha+1- \frac{\chi_1}{\beta}\right)c_0 \ ,$$

$$c_2 = \frac{1}{2}\left[(\alpha+2)(\alpha+1) - 2(\alpha+2)\frac{\chi_1}{\beta} + \frac{\chi_2 + \chi_1^2}{\beta^2}\right]c_0 \ . \tag{113}$$

Parameters α and β are completely arbitrary in all the above equations, except that $\alpha > -1$ and $\beta > 0$, and $\{\chi_n\}_0^N$ are the available cumulants.

Observe that if we

$$\text{let} \quad \alpha+1 = \frac{\chi_1^2}{\chi_2} = \frac{\mu_1^2}{\mu_2\mu_0 - \mu_1^2} = \frac{M^2}{R^2}$$

$$\text{and} \quad \beta = \frac{\chi_2}{\chi_1} = \frac{\mu_2\mu_0 - \mu_1^2}{\mu_1\mu_0} = \frac{R^2}{M} \ , \tag{114}$$

then $c_1 = 0$ and $c_2 = 0$ (here we also used (8) and (25)); this is a special case of general property (33) stated earlier. Since the probability density function p(u) is nonzero only for $u > 0$, then $\chi_1 > 0$ and $\chi_2 > 0$, giving allowable solutions to (114) in all cases. There then also follows, along with $d_1 = d_2 = 0$ in this special case, the explicit results

$$c_0 = \exp(\chi_0), \quad c_1 = 0, \quad c_2 = 0,$$

$$c_3 = \frac{\chi_1^2}{3!\,\chi_2^3}\,(2\chi_2^2 - \chi_3\,\chi_1)\,c_0,$$

$$c_4 = \frac{\chi_1^2}{4!\,\chi_2^4}\,(18\chi_2^3 - 12\chi_3\,\chi_2\,\chi_1 + \chi_4\,\chi_1^2)\,c_0,$$

$$c_5 = \frac{\chi_1^2}{5!\,\chi_2^5}\,(144\chi_2^4 - 120\chi_3\,\chi_2^2\,\chi_1 + 20\chi_4\,\chi_2\,\chi_1^2 - \chi_5\,\chi_1^3)\,c_0,$$

$$c_6 = \frac{\chi_1^2}{6!\,\chi_2^6}\Big(40(30\chi_2 + \chi_1^2)(\chi_2^2 - \chi_3\,\chi_1)\chi_2^2 + 10\chi_3^2\chi_1^4 +$$

$$+ 300\chi_4\,\chi_2^2\chi_1^2 - 30\chi_5\,\chi_2\,\chi_1^3 + \chi_6\,\chi_1^4\Big)\,c_0,$$

$$c_7 = \frac{\chi_1^2}{7!\,\chi_2^7}\Big(10800\chi_2^6 + 1260\chi_2^5\,\chi_1^2 - 12600\chi_3\,\chi_2^4\,\chi_1 - 1470\chi_3\,\chi_2^3\,\chi_1^3 +$$

$$+ 420\chi_3^2\,\chi_2\,\chi_1^4 + 4200\chi_4\,\chi_2^3\,\chi_1^2 + 70\chi_4\,\chi_2^2\,\chi_1^4 - 35\chi_4\,\chi_3\,\chi_1^5 -$$

$$- 630\chi_5\,\chi_2^2\,\chi_1^3 + 42\chi_6\,\chi_2\,\chi_1^4 - \chi_7\,\chi_1^6\Big)\,c_0. \qquad (115)$$

These relations have been confirmed by numerical comparison with (112).

These results greatly extend those of [1, (129)-(131)], where the equivalent of our c_3 is given (in terms of moments instead of cumulants, and with $\gamma_0 = 0$), along with the comment that "the higher-order coefficients are so complicated that the whole value of this type of series seems to depend on the fact that the first term alone (c_0) is often a good approximation." We find, on the other hand, that not only can we avoid the special choice in (114) and the corresponding complicated special results in (115), but we can handle any α,β pair and get very high-order coefficients c_n, simply by using the recurrence in (112), which is easily programmed. The only thing we lose are explicit results of the type given in (115); however, the latter are so complicated that they are of limited utility anyway.

COEFFICIENTS DIRECTLY VIA MOMENTS

We will need the following expression [5, 22.3.9]:

$$\frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x) = \frac{n!}{(\alpha+1)_n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{(\alpha+1)_k} . \qquad (116)$$

Then (92), (90), and (3) yield, for $n \geq 0$,

$$a_n = \frac{n!}{(\alpha+1)_n} c_n = \frac{n!}{(\alpha+1)_n} \int_0^\infty du\, p(u)\, L_n^{(\alpha)}\left(\frac{u}{\beta}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(\alpha+1)_k} \int_0^\infty du\, p(u)\, (-u/\beta)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\mu_k}{(\alpha+1)_k\, \beta^k} . \qquad (117)$$

It is useful, in this generalized Laguerre series case, to define an alternative set of normalized moments

$$\tilde{\mu}_n = \frac{\mu_n}{(\alpha+1)_n \, \beta^n} \quad \text{for } n \geq 0 \, . \tag{118}$$

(Although this seems to be very different from the earlier normalization in (69), (118) actually reduces to (69) for the α here equal to zero.) When (118) is utilized in (117), we have the desired expression for expansion coefficients $\{a_n\}$, directly in terms of (normalized) moments, in the surprisingly simple form

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \tilde{\mu}_k \quad \text{for } n \geq 0 \, . \tag{119}$$

Parameters α and β in (118) are arbitrary, except that $\alpha > -1$, $\beta > 0$.

As particular cases, (117)-(119) yield

$$a_0 = \mu_0, \quad a_1 = \mu_0 - \frac{\mu_1}{(\alpha+1)\beta} \, , \quad a_2 = \mu_0 - \frac{2\mu_1}{(\alpha+1)\beta} + \frac{\mu_2}{(\alpha+1)(\alpha+2)\beta^2} \, . \tag{120}$$

These agree with (113) which utilized cumulants. If we make the special choices of $\alpha+1 = \mu_1^2/(\mu_2\mu_0 - \mu_1^2)$ and $\beta = (\mu_2\mu_0 - \mu_1^2)/(\mu_1\mu_0)$, then $a_1 = 0$ and $a_2 = 0$; this is a common approach to the approximation problem, but totally unnecessary.

An alternative derivation of the direct moment relation (117) is possible: from (100), (6), and [5, 15.1.8],

$$\sum_{n=0}^{\infty} c_n w^n = (1-w)^{-\alpha-1} \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k \left(\frac{-w/\beta}{1-w}\right)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k \left(-\frac{w}{\beta}\right)^k (1-w)^{-\alpha-1-k} = \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k \left(-\frac{w}{\beta}\right)^k \sum_{m=0}^{\infty} \frac{(\alpha+1+k)_m}{m!} w^m . \quad (121)$$

Equating coefficients of w^n, we have, for $n \geq 0$,

$$c_n = \sum_{k=0}^{n} \frac{1}{k!} \mu_k \left(-\frac{1}{\beta}\right)^k \frac{(\alpha+1+k)_{n-k}}{(n-k)!} = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\mu_k}{(\alpha+1)_k \beta^k} , \quad (122)$$

which is equivalent to (117).

COEFFICIENTS RECURSIVELY VIA MOMENTS

The starting point for this case is the characteristic function expansion in (101):

$$f(z) = \sum_{m=0}^{\infty} c_m (-\beta z)^m (1-\beta z)^{-\alpha-1-m} = \sum_{m=0}^{\infty} c_m (-\beta z)^m \sum_{k=0}^{\infty} \frac{(\alpha+1+m)_k}{k!} (\beta z)^k \quad (123)$$

by use of [5, 15.1.8]. Now expand the left-hand side of (123) in powers of z, according to (6), and equate the coefficients of z^n to get, for $n \geq 0$,

$$\frac{1}{n!} \mu_n = \sum_{m=0}^{n} c_m (-\beta)^m \frac{(\alpha+1+m)_{n-m}}{(n-m)!} \beta^{n-m} = \beta^n \sum_{m=0}^{n} \frac{c_m (-1)^m}{(n-m)!} \frac{(\alpha+1)_n}{(\alpha+1)_m} . \quad (124)$$

Therefore

$$\frac{\mu_n}{(\alpha+1)_n \beta^n} = \sum_{m=0}^{n} (-1)^m \frac{n!}{(n-m)!} \frac{c_m}{(\alpha+1)_m} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} a_m \quad \text{for } n \geq 0 , \quad (125)$$

by use of (92). Then using normalized moment definition (118), (125) can be expressed as

$$a_n = (-1)^n \left[\tilde{\mu}_n - \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} a_m \right] \quad \text{for } n \geq 0 . \tag{126}$$

This is a recursive relation for expansion coefficients $\{a_n\}$ in terms of (normalized) moments. The parameters α and β in (118) are arbitrary, except that $\alpha > -1$, $\beta > 0$.

SUMMARY

The approximations to the probability density function and cumulative distribution function are given by (91) and (95), respectively, where α and β are arbitrary constants, except that $\alpha > -1$, $\beta > 0$. The generalized Laguerre polynomials are available via (96). The expansion coefficients $\{a_n\}$ are given by the three alternatives (112), (119), (126), in terms of normalized cumulants (62) and normalized moments (118); in the case of (112), the interrelationship between expansion coefficients $\{a_n\}$ and $\{c_n\}$ is given in (92). Programs for all three alternative procedures for determining the expansion coefficients $\{a_n\}$ are listed in an appendix. The basis for these relations is the characteristic function expansion in (100)-(102).

EXAMPLES OF HERMITE EXPANSION

EXAMPLE A

The first example is one which can be handled analytically, and thereby furnishes checks on numerical procedures and results. Consider the Gaussian probability density function

$$p(u) = \frac{1}{\omega}\,\phi\left(\frac{u-\gamma}{\omega}\right) \qquad (\omega > 0) \tag{127A}$$

with cumulative distribution function and characteristic function

$$P(u) = \Phi\left(\frac{u-\gamma}{\omega}\right), \quad f(i\xi) = \exp\left(i\xi\gamma - \frac{1}{2}\,\xi^2\omega^2\right). \tag{127B}$$

The cumulants are

$$\chi_0 = 0, \quad \chi_1 = \gamma, \quad \chi_2 = \omega^2, \quad \chi_n = 0 \quad \text{for } n \geq 3, \tag{128A}$$

while the moments are most easily evaluated by the recurrence

$$\mu_n = \gamma\,\mu_{n-1} + (n-1)\,\omega^2\,\mu_{n-2} \quad \text{for } n \geq 2, \quad \mu_0 = 1, \ \mu_1 = \gamma. \tag{128B}$$

It is obvious in this Hermite expansion case that the best choice of weighting parameters would be $\alpha = \gamma$, $\beta = \omega$, for then weighting w would match p perfectly and there would follow $b_n = 0$ for $n \geq 1$. We consider a mismatched choice of α and β to illustrate rapid decay of the expansion coefficients and some conditions on convergence.

Expansion coefficient c_n follows from (45) and (127A) according to

$$c_n = \int du\ p(u)\ He_n\left(\frac{u-\alpha}{\beta}\right) = \beta \int dx\ p(\alpha+\beta x)\ He_n(x) =$$

$$= \beta \int dx\ \frac{1}{\omega}\ \phi\left(\frac{\alpha-\gamma+\beta x}{\omega}\right)\ He_n(x) = \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ He_n\left(\frac{\gamma-\alpha}{\sqrt{\beta^2-\omega^2}}\right), \tag{129}$$

the last step via use of [5, 22.5.18] and [8, 7.374 10]. Then from (47),

$$b_n = (n!)^{-1/2} \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ He_n\left(\frac{\gamma-\alpha}{\sqrt{\beta^2-\omega^2}}\right) \quad \text{for } n \geq 0\ . \tag{130}$$

This equation is correct for all positive values of β and ω. However, for $\beta < \omega$, a more convenient form can be obtained by use of (74), if desired:

$$b_n = (n!)^{-1/2} \left(\frac{\sqrt{\omega^2-\beta^2}}{\beta}\right)^n\ Hi_n\left(\frac{\gamma-\alpha}{\sqrt{\omega^2-\beta^2}}\right), \tag{131}$$

where Hi_n is the modified Hermite polynomial. For $\beta = \omega$, a limit of (130) yields $b_n = (n!)^{-1/2}((\gamma-\alpha)/\beta)^n$.

If $\beta > \omega$, we can use the result in (50A) on (130) and obtain

$$b_n \propto \left(\frac{\sqrt{\beta^2-\omega^2}}{\beta}\right)^n\ n^{-1/4} \quad \text{as } n \to +\infty. \tag{132}$$

Since the quantity in parentheses is always less than 1 in this case of $\beta > \omega$, we have $b_n \to 0$ as $n \to +\infty$.

For $\beta < \omega$, we use [7, theorem 8.22.7] and find now that

$$b_n \propto \exp(\sqrt{2n}\ A) \left(\frac{\sqrt{\omega^2-\beta^2}}{\beta}\right)^n\ n^{-1/4} \quad \text{as } n \to +\infty, \tag{133}$$

where A is the absolute value of the argument of He_n in (130). This quantity (133) tends to zero with n, regardless of A, when $\beta > \omega/\sqrt{2}$.

Combining with the result above, we can conclude that

$$b_n \gtrless 0 \text{ as } n \gtrless +\infty \qquad \text{for } \frac{\omega}{\sqrt{2}} < \beta < +\infty. \tag{134}$$

Furthermore, b_n behaves as an n-th power, which is faster than $n^{-1/4}$, thereby guaranteeing convergence of the probability density function and cumulative distribution function series, according to the discussion in (50A) et seq. On the other hand, $\{b_n\}$ diverges when $0 < \beta < \omega/\sqrt{2}$, as may be seen from (133).

The error integral in (21) is, for Hermite weighting (40) and probability density function (127),

$$\int du\, \frac{p^2(u)}{w(u)} = \frac{\beta^2}{\omega\sqrt{2\beta^2-\omega^2}} \exp\left(\frac{(\gamma-\alpha)^2}{2\beta^2-\omega^2}\right) \quad \text{if } \frac{\omega}{\sqrt{2}} < \beta\,, \tag{135}$$

by use of [8, 3.323 2]; this integral is divergent if $\beta < \omega/\sqrt{2}$. Thus, for this particular example, the error integral and expansion coefficient sequence $\{b_n\}$ converge or diverge together, depending on the condition $\beta \gtrless \omega/\sqrt{2}$. The choice of α is irrelevant in this case.

A numerical example of sequence $\{b_n\}$ for

$$\gamma = 1.1,\ \omega = 2.3 \qquad \alpha = 1.14,\ \beta = 2.34 \tag{136}$$

is plotted in figure 1 on a logarithmic ordinate. Values of b_n less than 1E-7 in absolute value are all plotted at the ±1E-7 line. The critical ratio $\sqrt{\beta^2-\omega^2}/\beta$ in (130) is .184 for this example, leading to rapid decay of expansion coefficients $\{b_n\}$. The three sets of expansion coefficients in figure 1 are labelled according to the shorthand notation

RC: Recursively via Cumulants,
DM: Directly via Moments,
RM: Recursively via Moments. (137)

It is seen that the expansion coefficients determined recursively via cumulants, namely, the RC plot, decay rapidly and never encounter round-off error, whereas the DM and RM procedures both are subject to large round-off error for n > 70, as indicated by the large increasing oscillations. This example can be rather misleading, however, since all the cumulants (128A) of Gaussian probability density function (127A) are zero, except for $\chi_1 = \gamma$, $\chi_2 = \omega^2$; this leads to a very special form of the RC procedure unique to the Gaussian case.

In figure 2, the cumulative distribution function and exceedance distribution function, 1-P(u), as determined by Hermite expansion (48) using N = 50 terms, are plotted. The exact result, (127B), overlapped these curves over the full range plotted. The three procedures, RC, DM, and RM, all yielded identical distributions in figure 2, as inspection of figure 1 confirms, since the three sets of expansion coefficients are virtually the same for n < 50. Even though the three sets of expansion coefficients differ significantly for n > 60, the corresponding approximate probability density

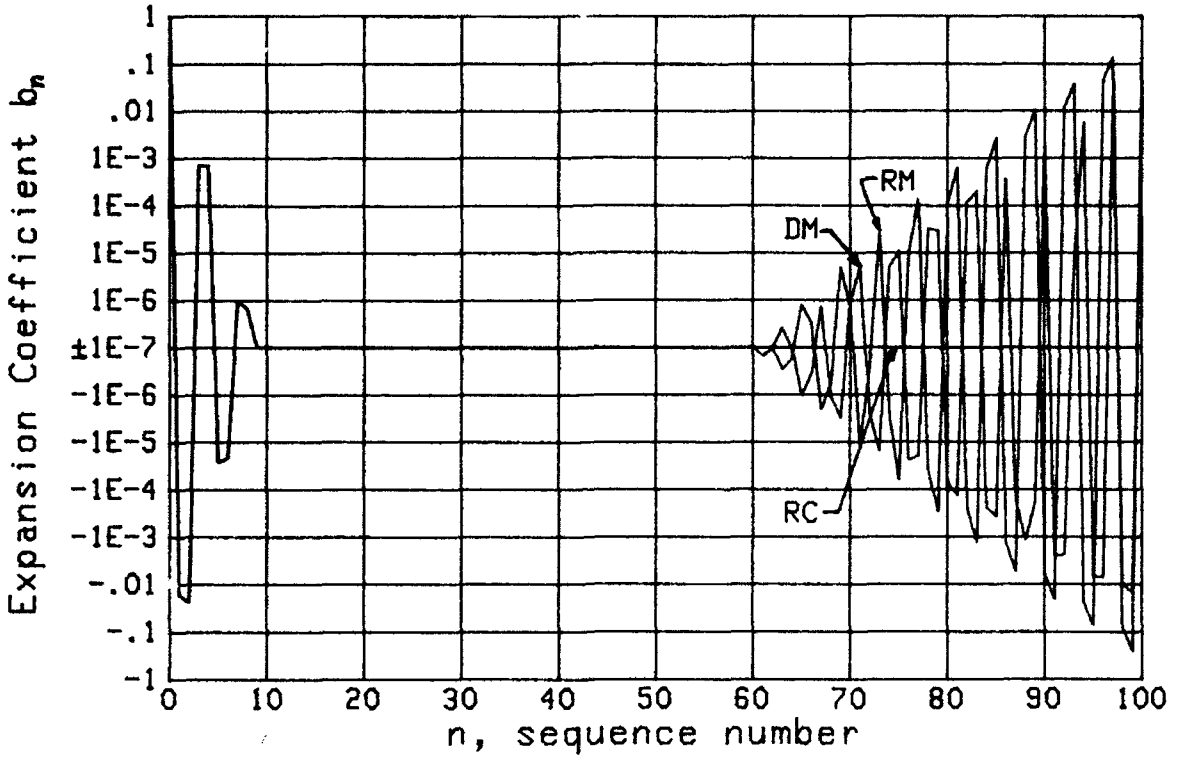

Figure 1. Hermite Coefficients for Example A

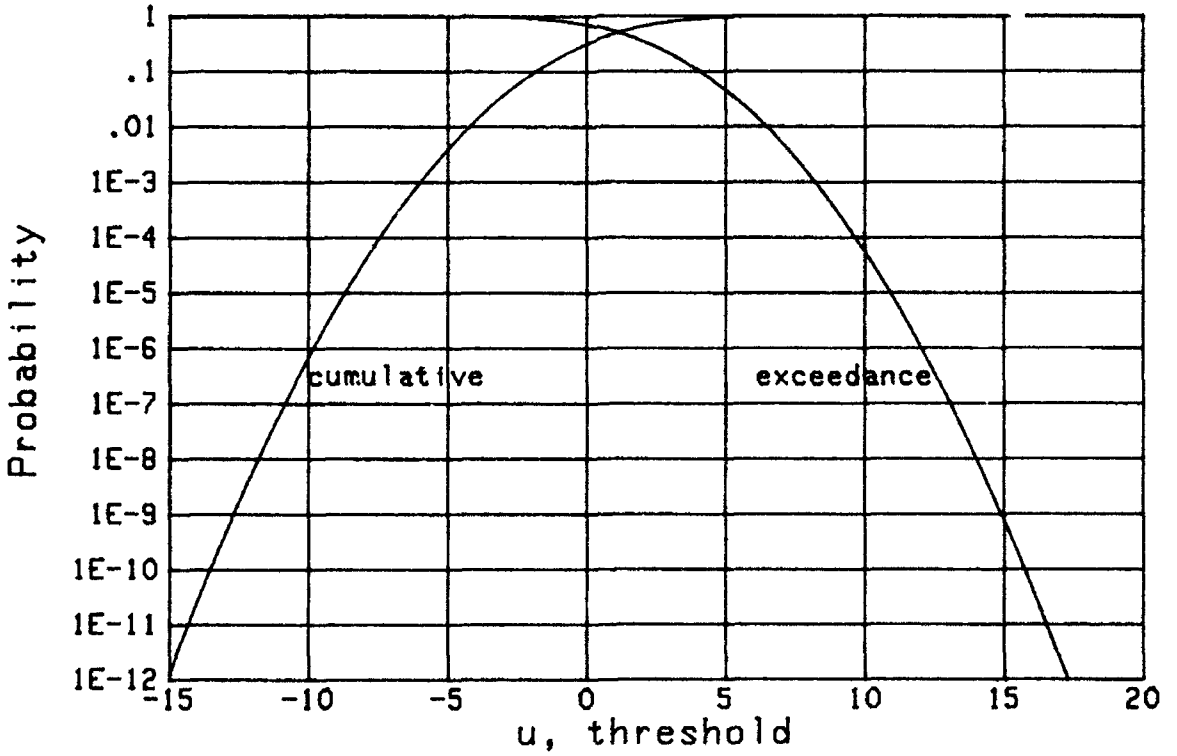

Figure 2. Distributions for Example A

functions and cumulative distribution functions for N = 70, say, would not be very different, because the relative differences in p and P are very small, somewhere in the 1E-5 range; see figure 1 for n = 70, and recall that $b_0 = 1$ for this example.

EXAMPLE B

The probability density function of interest here is the one previously considered in (34) et seq.:

$$p(u) = \frac{2\, u^{\gamma} \exp(-u^2/\omega^2)}{\omega^{\gamma+1} \Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{for } u > 0 \qquad (\gamma > -1,\ \omega > 0)\ . \tag{138}$$

This class of probability density functions includes, for $\gamma = 0, 1, 2$, respectively, the one-sided Gaussian, Rayleigh, and Maxwell as special cases. The characteristic function and cumulants are not easily determined directly for this function. However, the moments, as given already in (35), are readily evaluated via the simple recursion

$$\mu_n = \mu_{n-2} \frac{\omega^2}{2} (\gamma - 1 + n) \quad \text{for } n \geq 2, \quad \mu_0 = 1, \quad \mu_1 = \omega \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\ . \tag{139}$$

An example of the expansion coefficients for

$$\gamma = 3,\ \omega = 1 \qquad \alpha = 0,\ \beta = .7 \tag{140}$$

is depicted in figure 3. The values of b_n for n = 0, 1, 2, 3 are 1, 1.90, 2.18, 1.63, respectively, and lie above the top of the plotted region. The

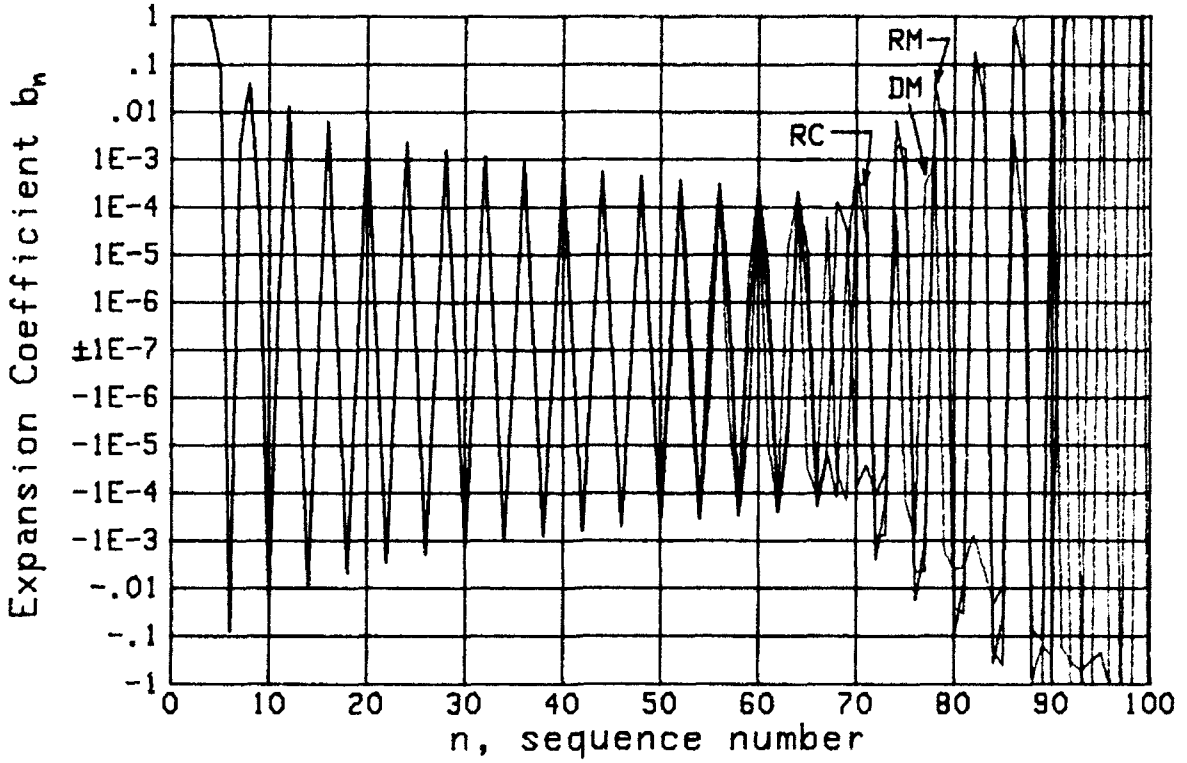

Figure 3. Hermite Coefficients for Example B

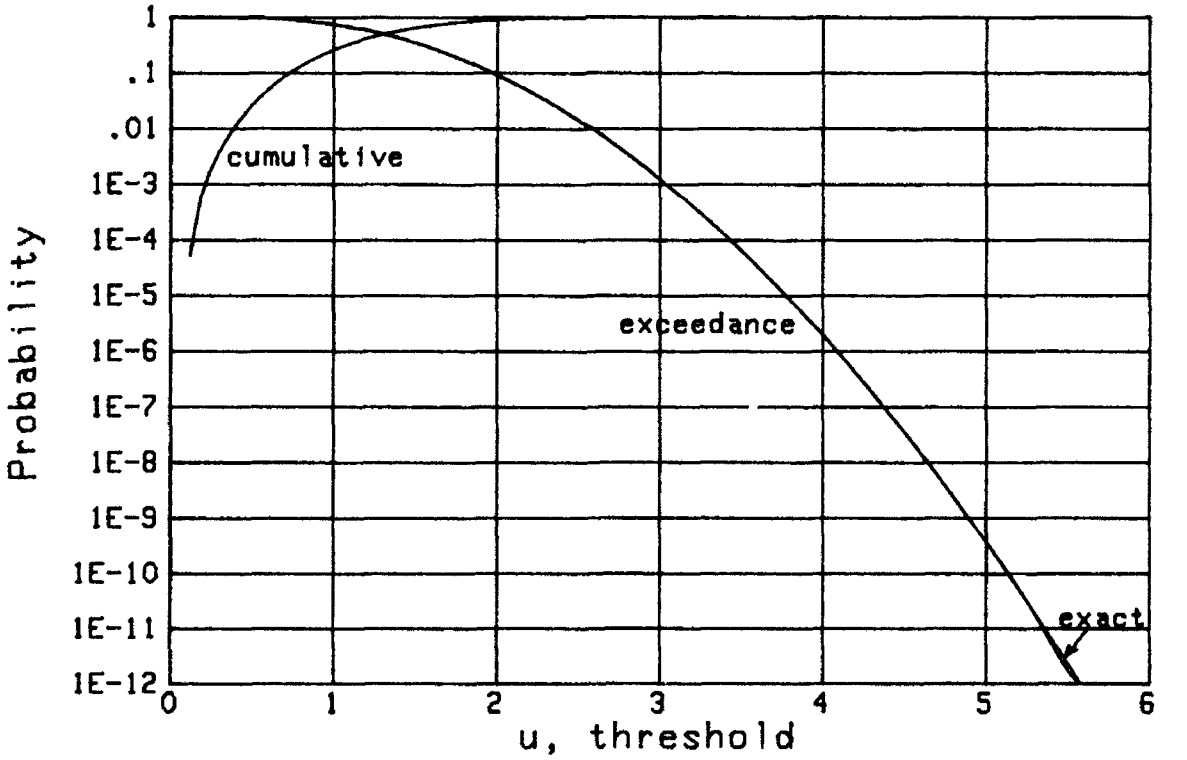

Figure 4. Distributions for Example B

coefficients obtained directly via moments, DM, decay to approximately 1E-4 near n = 70 and then encounter round-off error. The expansion coefficients corresponding to RC and RM are more noisy. The procedure used for RC was to determine the moments via (139), transform directly to cumulants according to (A-7), and then use (63).

A plot of the distributions using N = 65 terms is given in figure 4; the results are the same for all three sets of expansion coefficients, as may be seen by reference to figure 3. Furthermore, the exact cumulative distribution function, $P(u) = 1 - (1+u^2)\exp(-u^2)$ for $u > 0$, overlays these results except for the bow in the exceedance distribution function below 1E-11 near u = 5.5. Values of the cumulative distribution function for $u < 0$ as determined by series (48) are not zero, although they should be for this example; the generalized Laguerre series would fit this example better, since it is nonzero only for positive arguments.

EXAMPLE C

Consider the class of Bessel-function probability density functions

$$p(u) = \Lambda u^{\gamma} \exp(-u^2/\omega^2)\, I_{\beta}(\theta u) \quad \text{for } u > 0 , \tag{141}$$

which includes the Rice and generalized Q_M distributions, for example. The n-th moment is [8, 6.631 1]

$$\mu_n = \Lambda \frac{\theta^{\beta}\, \Gamma\left(\frac{n}{2}+h\right)\, \omega^{n+2h}}{2^{\beta+1}\, \Gamma(\beta+1)}\; {}_1F_1\left(\frac{n}{2}+h;\beta+1;\frac{1}{4}\omega^2\theta^2\right) \quad \text{for } n \geq 0 , \tag{142}$$

with $h = (\gamma+\delta+1)/2$; in order for μ_0 to be finite, we must have $h > 0$. The ${}_1F_1$ function in (142) can be evaluated via recursion; this leads to a recursion for the moments (see appendix E).

We consider here only the special case of the Rice probability density function, namely,

$$\lambda = \frac{2}{\omega^2} \exp\left(-\frac{1}{4}\omega^2\theta^2\right), \quad \gamma = 1, \quad \delta = 0, \tag{143}$$

for which

$$p(u) = \frac{2u}{\omega^2} \exp\left(-\frac{u^2}{\omega^2} - \frac{\omega^2\theta^2}{4}\right) I_0(\theta u) \quad \text{for } u > 0. \tag{144}$$

The moments in (142) then reduce to

$$\mu_n = \Gamma\left(\frac{n}{2}+1\right) \omega^n \, {}_1F_1\left(-\frac{n}{2};1;-\frac{1}{4}\omega^2\theta^2\right), \tag{145}$$

and can be easily determined by the recurrence presented in (E-5). The cumulative distribution function corresponding to (144) is the Q function [1]

$$P(u) = 1 - Q\left(\frac{\omega\theta}{\sqrt{2}}, \frac{\sqrt{2}\,u}{\omega}\right) \quad \text{for } u > 0; \tag{146}$$

the characteristic function is given in [9, appendix A] as an infinite series, meaning that the cumulants cannot be determined directly, except via the moments.

The particular example we consider here for the Hermite expansion is a sum of 8 independent random variables, each with Rice probability density function (144). For direct comparison with the exact results in [9], we also consider the normalized form of (144), namely $\omega^2 = 2$. Furthermore, we limit

numerical consideration in this particular example to evaluation of the cumulative and exceedance distribution functions for $\theta = 0$, which corresponds physically to the false alarm probability for the sum of eight normalized envelopes of narrowband Gaussian noise (i.e., a Rayleigh probability density function for the individual random variables).

For $\alpha = 4$, $\beta = 2.15$, the expansion coefficients $\{b_n\}$ are displayed in figure 5 for the RC, DM, and RM approaches. All the $\{b_n\}$ for $1 \leq n \leq 20$ are bigger than 1; the biggest is $b_6 = 12.25$. The $\{b_n\}$, for both moment approaches, have not been plotted for $n > 60$ because they continue to oscillate well beyond the ± 1 limits, while the RC coefficients decay exponentially with n. Despite the fact that the moments were the initially determined quantities for this example, the RC method far outperforms the DM and RM methods, as seen in figure 5. The reason for this is as follows: for the RC method, the procedure was to obtain moments via (145), cumulants via (A-7), cumulants of the sum of 8 independent random variables by simple scaling by a factor of 8, and then expansion coefficients via (63). For the DM and RM methods, the moments of the sum of 8 random variables were determined via [6, (14)] which progressively determined the moments of a sum of 2 random variables, then 3, 4,..., 8 in order, and then employed (70). This iterated procedure for moments requires more number-crunching and leads to considerably larger round-off error than the simple scaling required for the RC procedure. Thus it appears that when the random variable of interest is obtained as a sum of several independent random variables, the RC approach will be the prime candidate for expansion coefficient evaluation; this applies also if the individual random variables have different statistics, but remain independent.

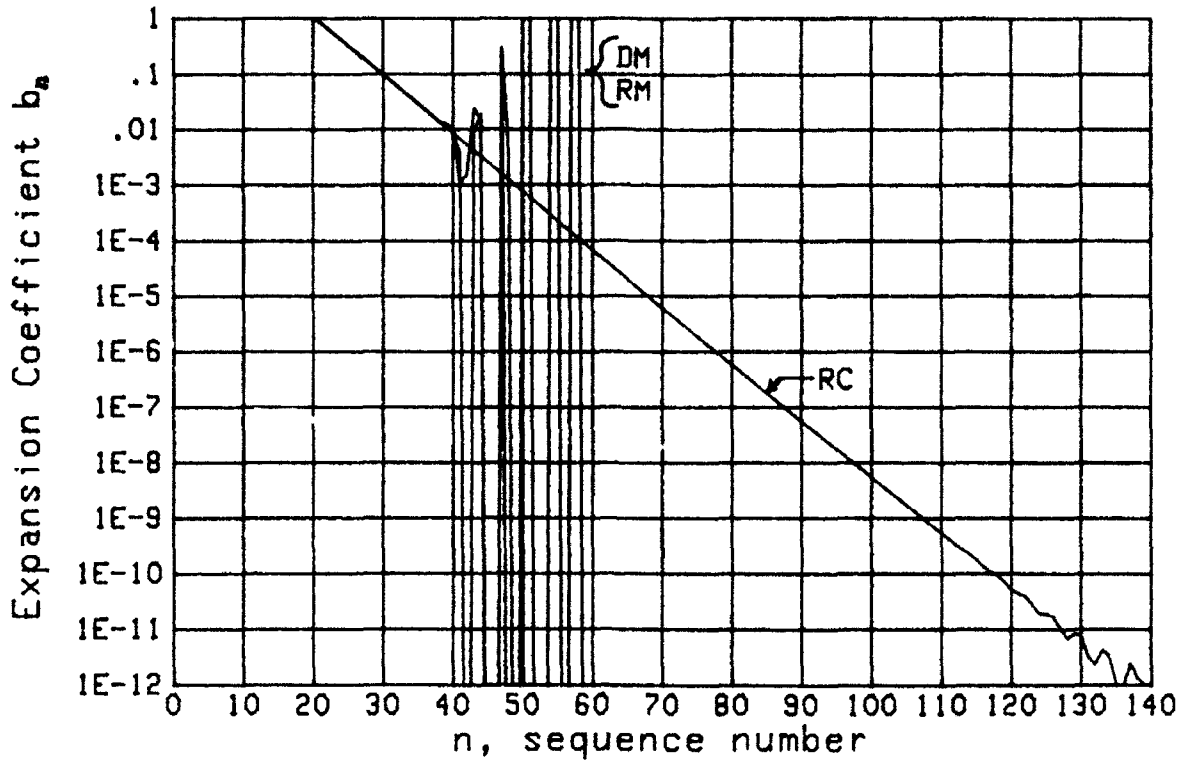

Figure 5. Hermite Coefficients for Example C

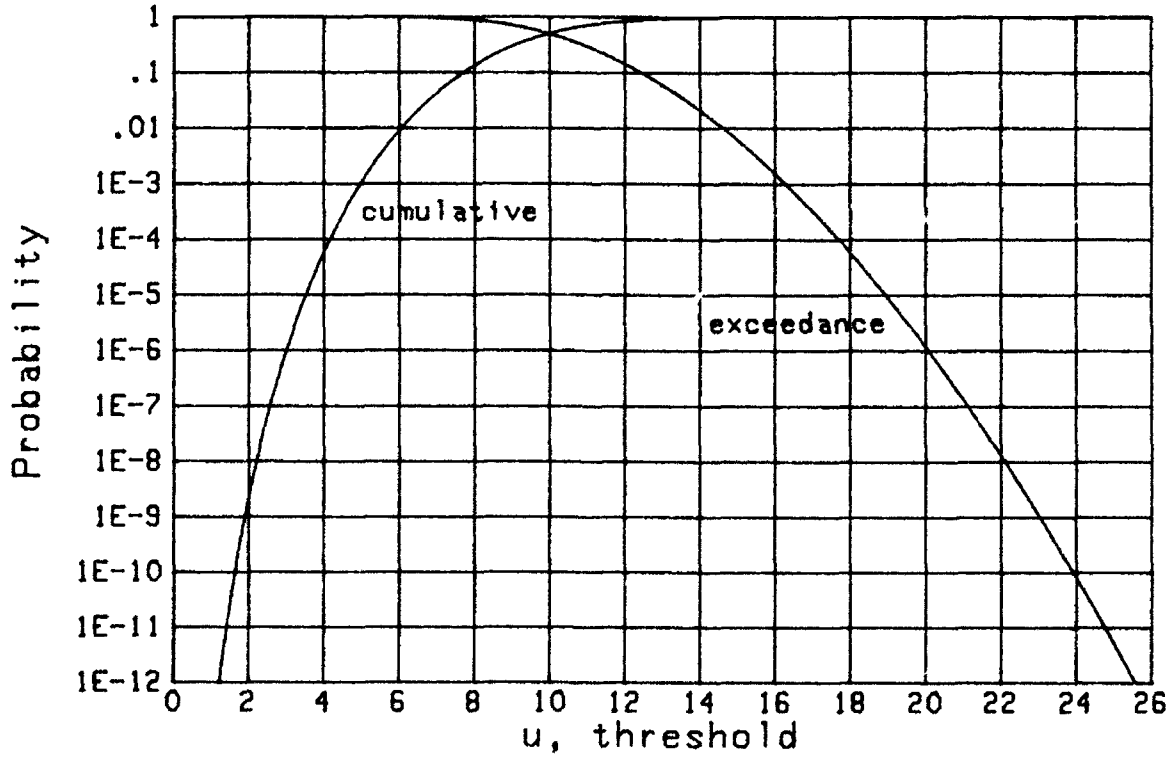

Figure 6. Distributions for Example C

The cumulative and exceedance distribution functions for this sum of 8 normalized Rayleigh variates are plotted in figure 6, for the N = 140 expansion coefficients of the RC procedure in figure 5. In order to make a precise determination of the accuracy of this Hermite series approach, the false alarm probabilities were computed at the eight thresholds listed under M = 8 in [9, table 1]. To the precision given in that table, the computed probabilities were exactly the specified values 1E-m for m = 1(1)8. Thus, as anticipated by figure 5, very accurate evaluation of false alarm probabilities are possible by this series approach.

A short search of values of the best weighting parameters α and β, to use with the DM approach, led to $\alpha = 5.84$, $\beta = 2.28$ and expansion coefficients b_n near 1E-4 at n = 28, before round-off error became dominant. This is better than the result of DM in figure 5 for $\alpha = 4$, $\beta = 2.15$. Evaluation of the false alarm probabilities at the thresholds in [9, table 1] gave 7 decimal accuracy at .1, and 4 decimal accuracy at 1E-8. This is adequate for most purposes, but is not as good as the RC approach.

EXAMPLE D

In [4, appendix C], the characteristic function for shot noise with random amplitude and duration modulation, and arbitrary individual pulse shape, is derived. (This result is then specialized to elliptical pulses and Rayleigh amplitude modulation [4, (C-36)-(C-42)].) Also, the cumulants are extracted, with general result [4, (24)], where ν is the average number of pulses/second, $\bar{\ell}$ is the average length of the duration modulation, $\mu_a(n)$ is

the n-th moment of the amplitude modulation, and F(x) is the individual pulse shape of the shot noise. Thus shot noise is a case where the cumulants are directly capable of evaluation, whereas the moments must be found indirectly.

For the special case of elliptical pulses and Rayleigh amplitude modulation, there follows for the cumulants [4, (29)]:

$$\chi_n = \nu\bar{\ell}\,\sigma_a^n\,2^{\frac{3}{2}n+1}\,\Gamma^3\left(\tfrac{n}{2}+1\right)/\Gamma(n+2) \quad \text{for } n \geq 1, \quad \chi_0 = 0\ . \tag{147}$$

These quantities are easily evaluated via recurrence

$$\chi_n = \chi_{n-2}\,\sigma_a^2\,n^2/(n+1) \quad \text{for } n \geq 3,\ \chi_1 = \left(\tfrac{\pi}{2}\right)^{3/2}\nu\bar{\ell}\,\sigma_a,\ \chi_2 = \tfrac{8}{3}\,\nu\bar{\ell}\,\sigma_a^2\ . \tag{148}$$

This procedure was used in [4, appendix D] to obtain the probability density function and cumulative distribution function results given there.

There is a nuance that arises in shot noise for pulse shapes of finite duration; see [4, pp. 40-42]. Namely, there is an impulse in the probability density function, at u = 0, of area

$$P_0 = \exp[-\nu\bar{\ell}\,(x_2 - x_1)]\ , \tag{149}$$

where (x_1,x_2) is the non-zero extent of an unmodulated individual pulse. Since an impulse is very difficult to approximate by a finite series of continuous functions, the effect of this quantity should be subtracted from the statistics (moments or cumulants), and the continuous portion of the probability density function should be approximated. Similarly, the

corresponding step in the cumulative distribution function at the origin should be eliminated from the approximation procedure.

This feature is easily incorporated if P_0 is subtracted from the zero-th order moment [4, p. 42]. The only undesireable side-effect of this manipulation is that the initially computed cumulants must be transformed to moments, then μ_0 corrected, and then all the new cumulants evaluated. This double transformation is necessary because the correction (subtraction) procedure can only be accomplished in the moment domain. Of course, when the DM or RM procedures are employed instead of RC, the last transformation to cumulants is unnecessary; this was, in fact, the procedure used in [4, p. 60].

When the individual pulse F(x) has infinite duration, as for an exponential or Gaussian waveform, then $x_2 - x_1$ is infinite and P_0 in (149) is zero. In that case, the considerations in the last two paragraphs can be disregarded, and the cumulants generated via (148) used as is. It is then very likely that even better accuracy in the expansion coefficients will be achieved than for this current example.

For overlap factor [4, p. 43]

$$\overline{K_1} = \nu\bar{\ell}(x_2 - x_1) = 6.2, \quad P_0 = \exp(-6.2) = .00203, \quad \sigma_a = 1 , \qquad (150)$$

and for weighting parameters $\alpha = 6.1$, $\beta = 4.3$, the expansion coefficients $\{b_n\}$ are displayed in figure 7 for the three recursive procedures. The RM results are considerably poorer than the RC and DM coefficients, which are

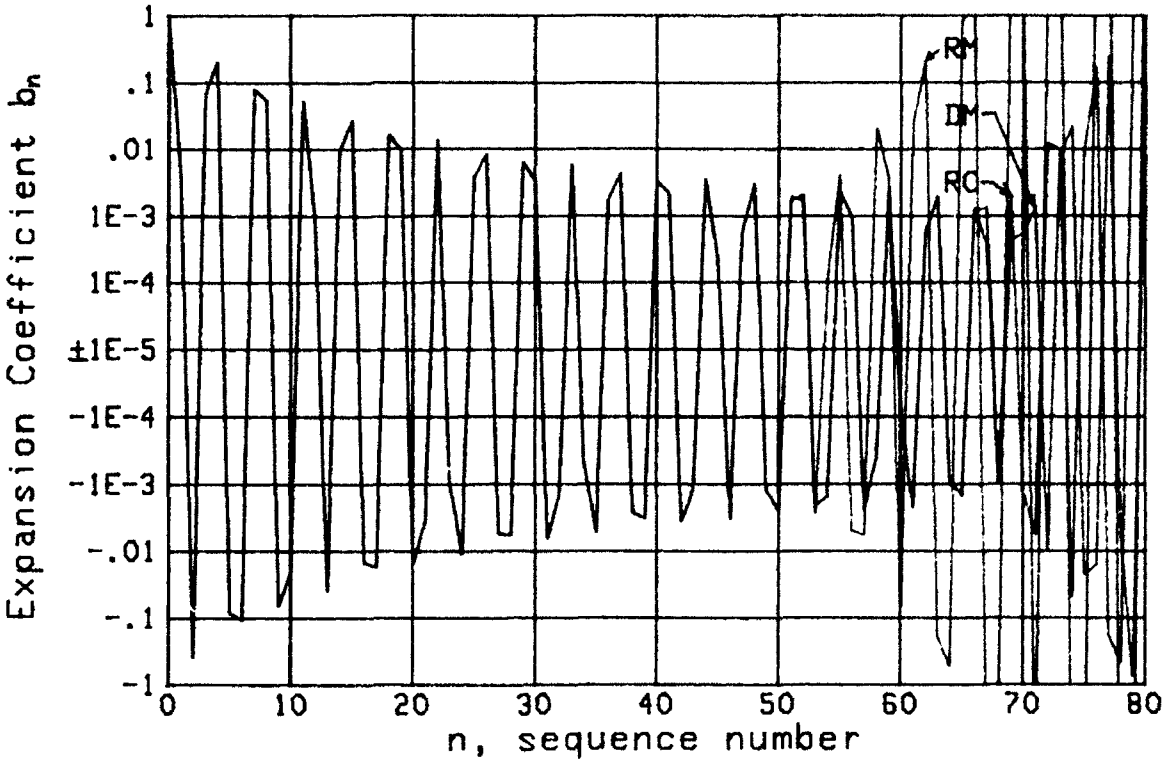

Figure 7. Hermite Coefficients for Example D

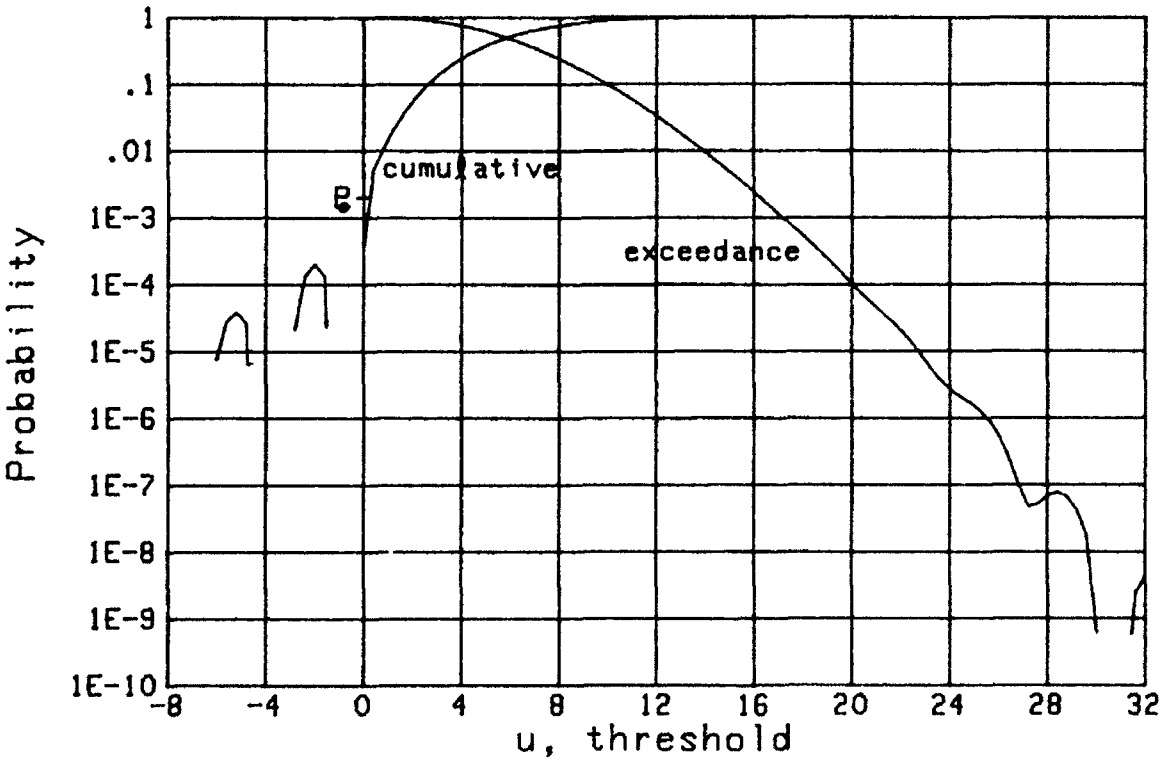

Figure 8. Distributions for Example D

comparable for n < 65. However, even here, the coefficients have only decayed to the 1E-3 level, which may not be sufficiently small for accurate results.

The distributions using N = 65 terms and the RC expansion coefficients are given in figure 8. Although the actual cumulative distribution function is zero for u < 0, the approximation oscillated around zero, reaching a positive peak of value .22E-3 at u = -2. Similarly, significant wiggles develop in the exceedance distribution function below the 1E-4 level. The reason for the inadequacy of these Hermite expansions near u = 0 is the abrupt zero behavior of the true probability density function for negative arguments, a feature inherently difficult to approximate by means of smooth continuous functions. The error of the approximations in figure 8 is estimated in a later section and superposed on the plot, for ease of ascertaining the reliability of the curves. The corresponding approximations for the generalized Laguerre series are better for this type of probability density function, as will be demonstrated in the next section.

The approximate probability density function for this example, again with N = 65 terms, is given in figure 9 on a linear ordinate. It reaches a negative peak of -8E-4, and crosses the u = 0 axis with value .004; both of these values should be zero, and will be for the generalized Laguerre series. To see how the approximate probability density function behaves for larger arguments, the logarithmic plot in figure 10 is used. Wiggles develop near the 1E-4 level and become large enough that negative values of the density are yielded near u = 28 and 31. It will be worthwhile to compare this Hermite series with the generalized Laguerre series to be presented in the next section. The estimated error associated with figure 10 is developed in a later section.

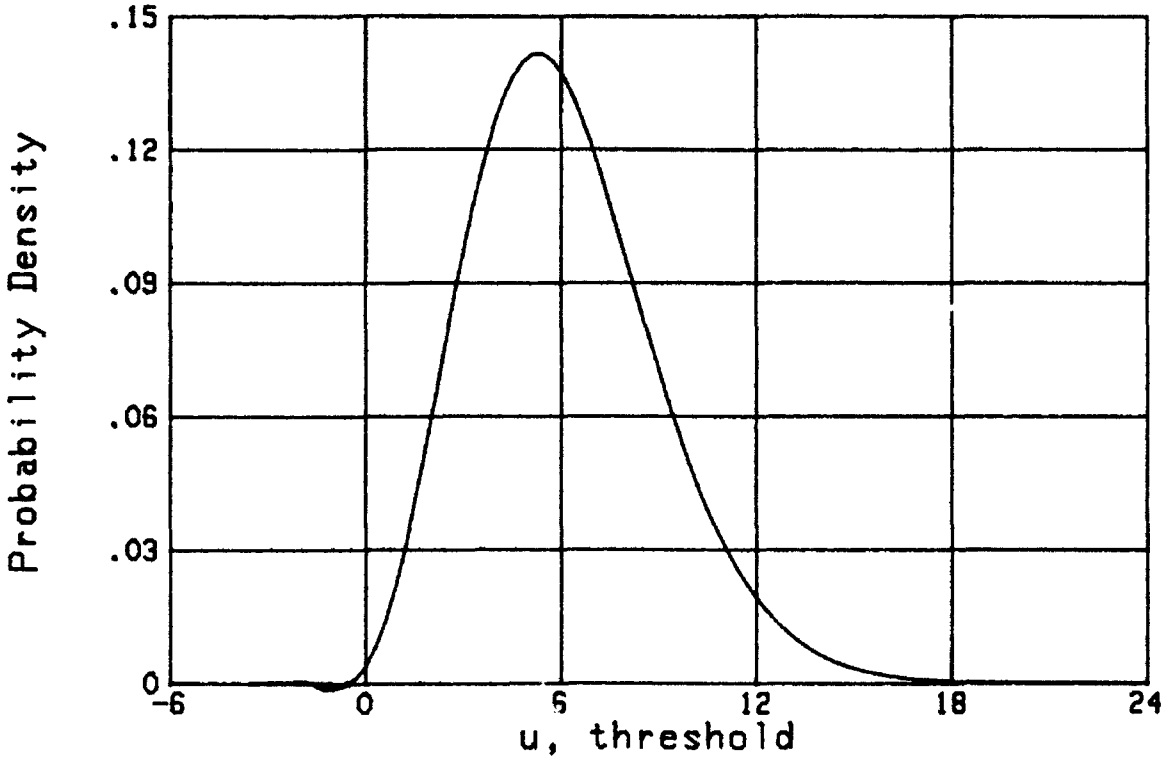

Figure 9. Linear Density for Example D

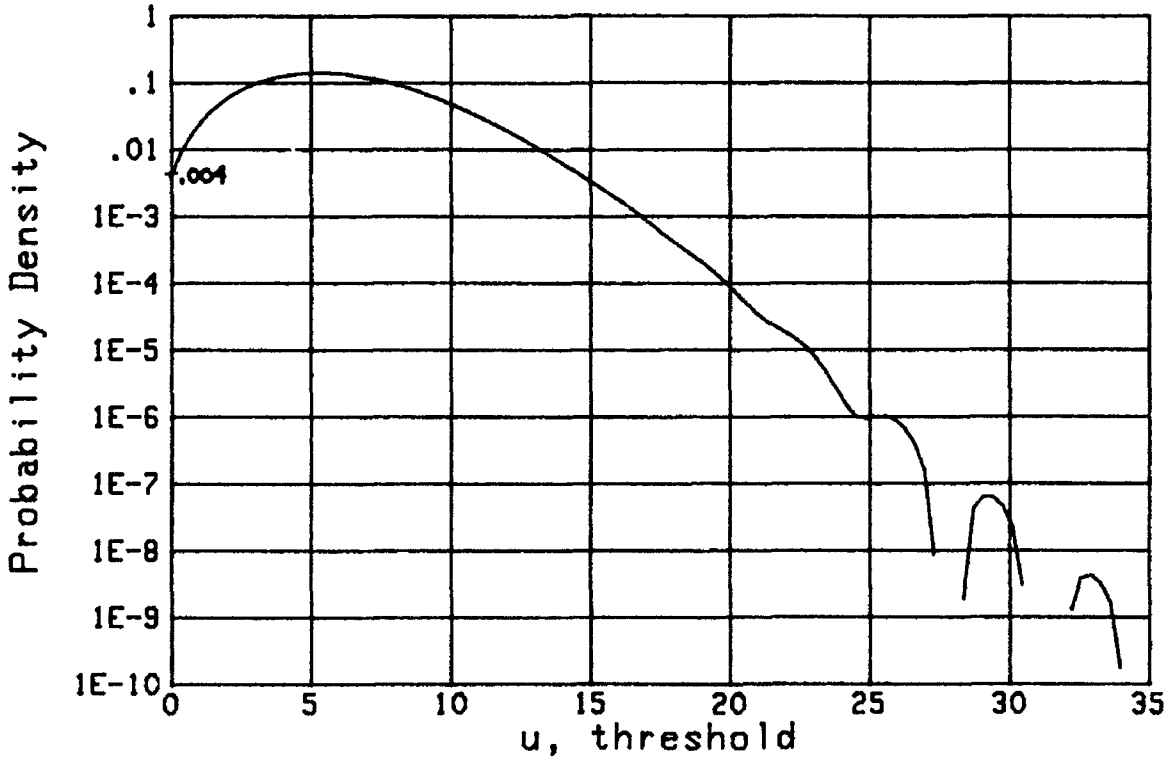

Figure 10. Log Density for Example D

EXAMPLES OF GENERALIZED LAGUERRE EXPANSION

EXAMPLE E

As with the earlier Hermite expansions, the first generalized Laguerre example here is one that can be evaluated analytically, for purposes of checking numerical procedures and results. Namely consider the Chi-square probability density function of $2(\gamma+1)$ degrees of freedom (which need not be integer):

$$p(u) = \frac{u^{\gamma} \exp(-u/\omega)}{\omega^{\gamma+1} \Gamma(\gamma+1)} \qquad (\gamma > -1,\ \omega > 0) \ . \tag{151}$$

All probability density functions and approximations are limited to $u > 0$ in this section, since they are zero for $u < 0$; this restriction will be presumed in the remainder of the presentation.

The exceedance distribution function is related to the incomplete Gamma function [5, 6.5.3]:

$$1 - P(u) = \int_u^{\infty} dt\ p(t) = \Gamma(\gamma+1,\ u/\omega) \big/ \Gamma(\gamma+1) \ . \tag{152}$$

The characteristic function follows from (151) as

$$f(i\xi) = (1 - i\xi\omega)^{-\gamma-1} \ , \tag{153}$$

with cumulants

$$\chi_k = (k-1)!\ (\gamma+1)\ \omega^k \quad \text{for } k \geq 1,\ \chi_0 = 0 \ , \tag{154}$$

and moments

$$\mu_k = (\gamma+1)_k\ \omega^k \quad \text{for } k \geq 0 \ . \tag{155}$$

Thus either set of statistics can be used as a starting position. The error integral in (21) is finite if

$$-1 < \alpha < 2\gamma + 1 \quad \text{and} \quad \beta > \omega/2 \ . \tag{156}$$

We will find the expansion coefficients by means of the characteristic function expansion (100), developed earlier for the generalized Laguerre series. Specifically, we utilize the power series expansion

$$(1-w)^{-\alpha-1} f\left(\frac{-w/\beta}{1-w}\right) = (1-w)^{\gamma-\alpha} \left(1-w\frac{\beta-\omega}{\beta}\right)^{-\gamma-1} =$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha-\gamma)_m}{m!} w^m \sum_{k=0}^{\infty} \frac{(\gamma+1)_k}{k!} \left(\frac{\beta-\omega}{\beta}\right)^k w^k \ , \tag{157}$$

where we used (153) and [5, 15.1.8] twice. The coefficient of a general term w^n is then immediately given by the closed form

$$c_n = \sum_{m=0}^{n} \frac{(\alpha-\gamma)_m}{m!} \frac{(\gamma+1)_{n-m}}{(n-m)!} \left(\frac{\beta-\omega}{\beta}\right)^{n-m} \quad \text{for } n \geq 0 \ . \tag{158}$$

Alternative expressions for the expansion coefficients are

$$c_n = \frac{(\gamma+1)_n}{n!} \left(\frac{\beta-\omega}{\beta}\right)^n F\left(\alpha-\gamma, -n; -n-\gamma; \frac{\beta}{\beta-\omega}\right) =$$

$$= \frac{(\gamma+1)_n}{n!} \left(\frac{-\omega}{\beta}\right)^n F\left(-n, -n-\alpha; -n-\gamma; \frac{\beta}{\omega}\right) =$$

$$= \frac{(\alpha+1)_n}{n!} F\left(-n, \gamma+1; \alpha+1; \frac{\omega}{\beta}\right) \qquad \text{for } n \geq 0 \ , \tag{159}$$

obtained by means of [5, 15.1.1, 15.3.5, 15.3.7] respectively. In fact, the last result can be obtained directly by using [8, 7.414 7] on (90) and (151):

$$c_n = \int_0^\infty du \, \frac{u^\gamma \exp(-u/\omega)}{\omega^{\gamma+1} \Gamma(\gamma+1)} L_n^{(\alpha)}\left(\frac{u}{\beta}\right) . \tag{160}$$

However, the latter two results in (159) are not numerically stable, whereas (158) and the first line of (159) are stable for large n, without encountering round-off error.

Some special cases of (158) are as follows:

$$\text{if } \alpha = \gamma, \text{ then } c_n = \frac{(\gamma+1)_n}{n!} \left(\frac{\beta-\omega}{\beta}\right)^n ;$$

$$\text{if } \beta = \omega, \text{ then } c_n = \frac{(\alpha-\gamma)_n}{n!} ;$$

$$\text{if } \alpha = \gamma \text{ and } \beta = \omega, \text{ then } c_n = \delta_{n0} . \tag{161}$$

The last case is to be expected, since the weighting exactly matches the probability density function (151) then.

A numerical example of sequence $\{b_n\}$ for

$$\gamma = 1.1, \quad \omega = 2.3 \qquad \alpha = 1.105, \quad \beta = 2.1 \tag{162}$$

is shown in figure 11, using the three recursive procedures developed earlier for the generalized Laguerre series in (112), (119), (126). In addition, exact result (158) is plotted for comparison. The expansion coefficients have a rapidly decaying transient for $n < 10$, and then a decay approximately proportional to $n^{-3/2}$ for large n. The abrupt change of character at $n = 5$ does not signify the onset of round-off error; rather, the latter is indicated

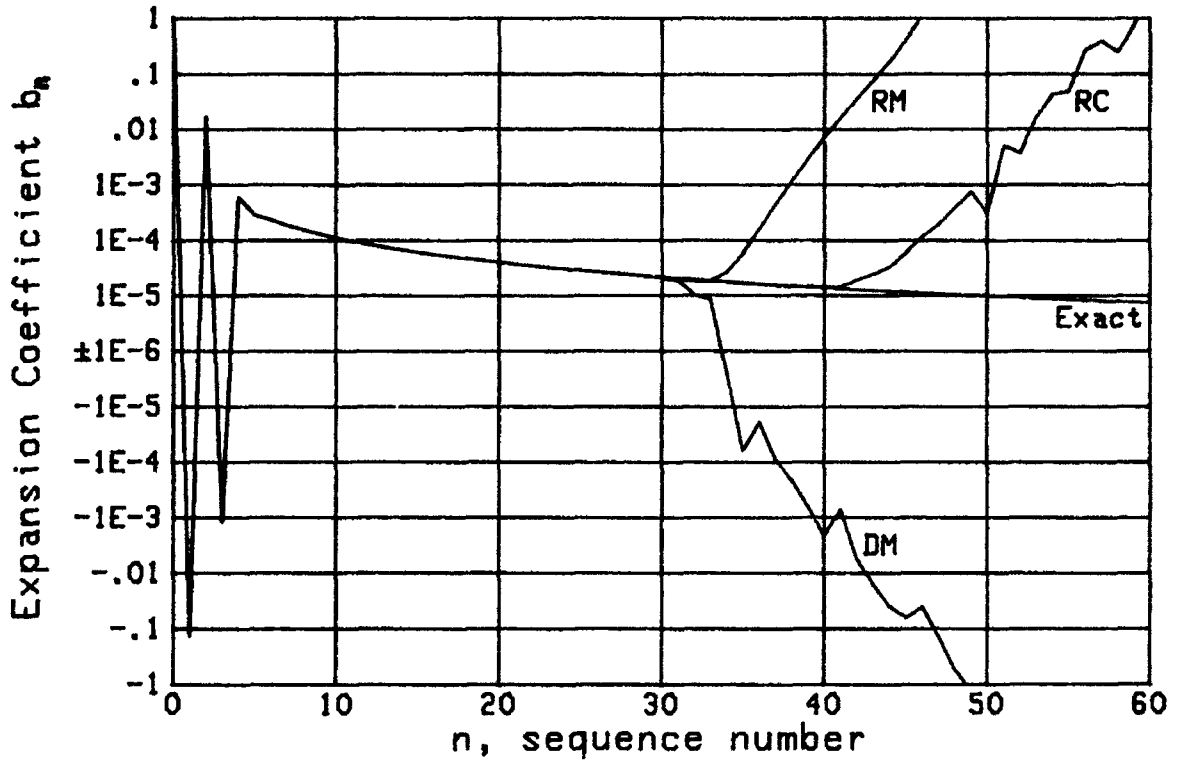

Figure 11. Generalized Laguerre; Example E

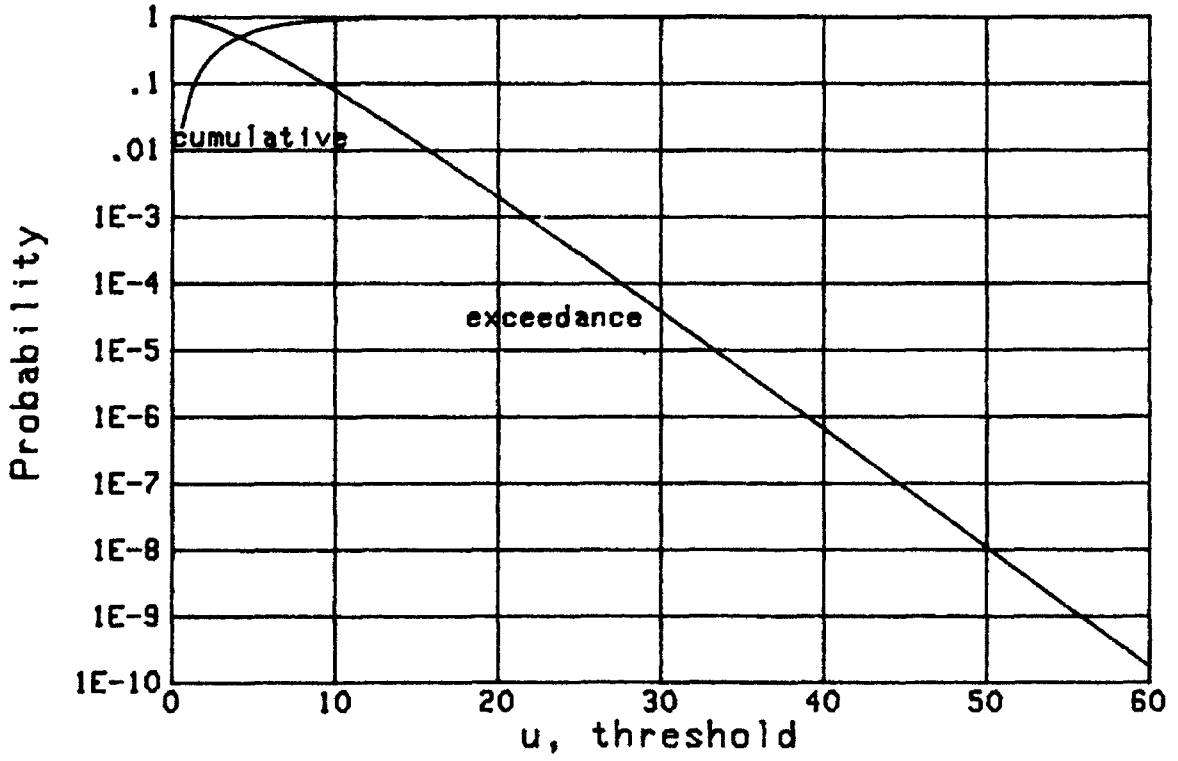

Figure 12. Distributions for Example E

by an erratic behavior, typically increasing exponentially with n (linear growth on a logarithmic ordinate).

A different plotting strategy will be adopted henceforth for the expansion coefficients $\{b_n\}$, in order not to clutter the diagrams with large oscillations as in figures 1, 3, 5, 7. Specifically, when the expansion coefficient b_n first exceeds the ±1 limits, the remainder of sequence $\{b_n\}$ will not be plotted, since this is a region of large round-off error. Thus, although the RM curve in figure 11 returns to the ±1 limits briefly at n = 52,53, these values are not displayed.

Round-off error for the RC procedure does not become as significant as for the two moment approaches until n has increased by almost 10, for this example in figure 11. In fact, the expansion coefficients for the RC procedure overlap the exact values until n = 40. The corresponding approximate distributions, using N = 40 terms in expansion (95) as determined by RC, are plotted in figure 12. The exact result (152) overlays these results over the entire range plotted.

EXAMPLE F

The following probability density function corresponds to a noncentral Chi-square variate of 2ν degrees of freedom:

$$p(u) = \frac{1}{2} \exp\left(-\frac{d^2+u}{2}\right)\left(\frac{\sqrt{u}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{u}) \qquad (\nu > 0) \; ; \qquad (163)$$

d is the noncentrality parameter, and 2ν need not be integer. The characteristic function is [8, 6.631 4]

$$f(i\xi) = (1-i2\xi)^{-\nu} \exp\left(\frac{id^2\xi}{1-i2\xi}\right), \tag{164}$$

and is the same as the one considered in [10, (50) et seq.]. The exceedance distribution function is the generalized Q-function:

$$1 - P(u) = \int_u^\infty dt\, \frac{1}{2} \exp\left(-\frac{d^2+t}{2}\right) \left(\frac{\sqrt{t}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{t}) =$$

$$= \int_{\sqrt{u}}^\infty dx\, x \exp\left(-\frac{d^2+x^2}{2}\right) \left(\frac{x}{d}\right)^{\nu-1} I_{\nu-1}(dx) = Q_\nu(d,\sqrt{u}) . \tag{165}$$

By expanding the ℓn of (164) in a power series in $i\xi$, the cumulants follow as

$$\chi_n = 2^n(n-1)! \left(\nu + \frac{1}{2} d^2\, n\right) \quad \text{for } n \geq 1, \quad \chi_0 = 0 . \tag{166}$$

And the moments are obtained from (163) as

$$\mu_n = 2^n (\nu)_n\, {}_1F_1(-n;\nu;-d^2/2) =$$

$$= 2^n\, n!\, L_n^{(\nu-1)} (-d^2/2) \quad \text{for } n \geq 0 , \tag{167}$$

by use of [8, 6.631 1] and [5, 13.6.9]. Both (166) and (167) lend themselves to simple recurrences which involve only positive quantities; thus the starting statistics can be quickly and accurately evaluated.

The numerical example we consider here will be compared with the exact results in [10, figure 11], namely,

$$\nu = 2.7, \qquad d = 3 \qquad\qquad \alpha = 1.7, \qquad \beta = 5.5 \; . \tag{168}$$

Since the probability density function in (163) behaves as $u^{\nu-1}$ as $u \to 0^+$, it is reasonable to choose weighting parameter α in (82) as $\nu-1$, as indicated in (168). And since (163) behaves as $\exp(-u/2)$ as $u \to +\infty$, we must choose $\beta > 1$ in order that the error integral in (21) is finite. The particular values in (168) approximately minimize the sum of $\{b_n^2\}_0^N$ in (21).

The expansion coefficients $\{b_n\}$ as determined by the three available recursive procedures are displayed in figure 13. The RC coefficients decrease to values less than 1E-10 near n = 50, before round-off error becomes significant. The two moment approaches deteriorate near n = 30, which is markedly poorer than the cumulant approach. The distributions, as determined by N = 50 terms of the RC approach, are given in figure 14, and agree with the d = 3 curve of [10, figure 11]. When the approximate probability density function for N = 50 was compared with exact result (163), 10 decimals of agreement were obtained; this is due to the ability to get very small $\{b_n\}$ in figure 13 via the RC method.

EXAMPLE G

This example is the Rice probability density function given in (144), with moments (145) and cumulative distribution function (146). The starting statistics are the moments as determined by recurrence (E-5)-(E-6).

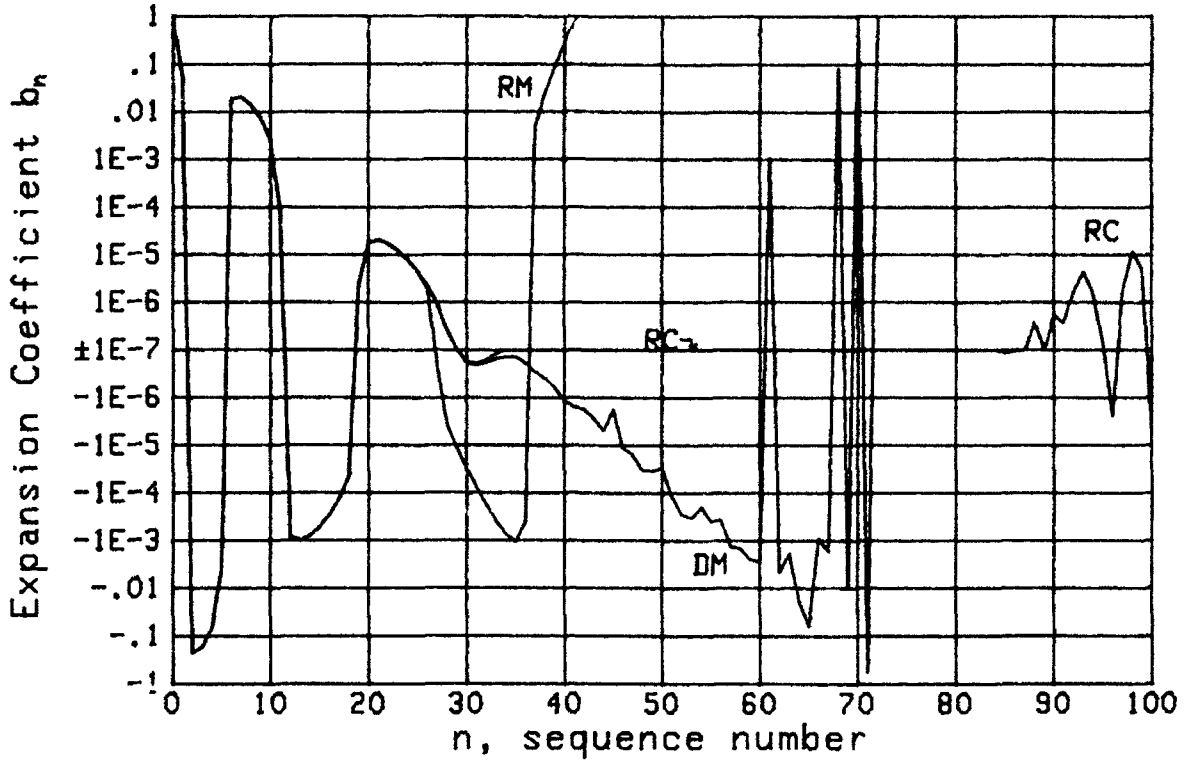

Figure 13. Generalized Laguerre; Example F

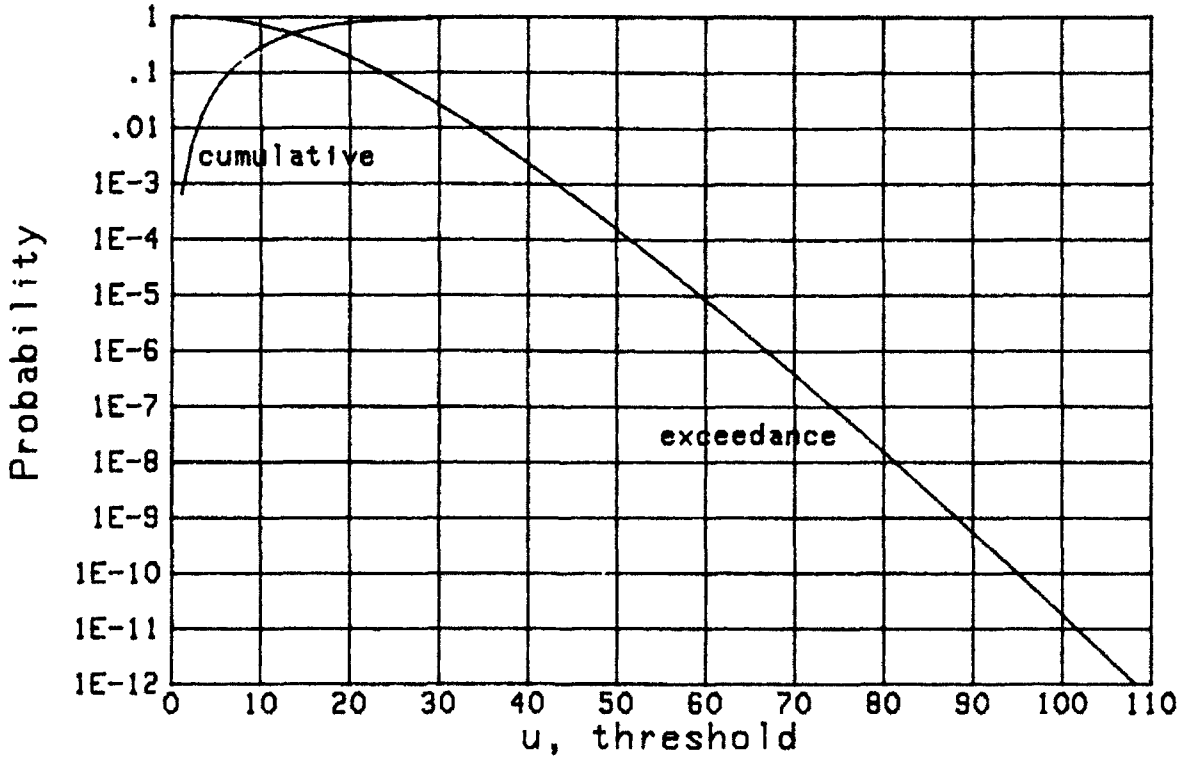

Figure 14. Distributions for Example F

The particular numerical case of interest is

$$\theta = 3, \qquad \omega^2 = 2 \qquad\qquad \alpha = 1, \qquad \beta = 1. \tag{169}$$

The values of α and β were found by the usual trial and error search procedure of observing plots of expansion coefficients $\{b_n\}$, looking for rapid decay and small round-off error; results for this example are displayed in figure 15. The RM procedure deteriorates rapidly at n = 30, whereas DM and RC are useable up to n = 55 and 65 approximately.

The cumulative and exceedance distribution functions for N = 65 terms of the RC procedure are plotted in figure 16, along with exact result (146). The approximate exceedance distribution function overlaps the exact one until slightly below the probability level 1E-4, which corresponds to the level of reliability of b_n in figure 15 at n = 65. Then the exceedance distribution function makes a positive (upward) turn below 1E-6, which is impossible for a physical density function which must remain positive; thus the approximation deteriorates rapidly for u > 7.

EXAMPLE H

This is a follow-on to the previous example, in that we consider a sum of 8 Rice variates, each with the statistics in (169). The expansion coefficients for

$$\theta = 3, \qquad \omega^2 = 2 \qquad\qquad \alpha = 26, \qquad \beta = 1 \tag{170}$$

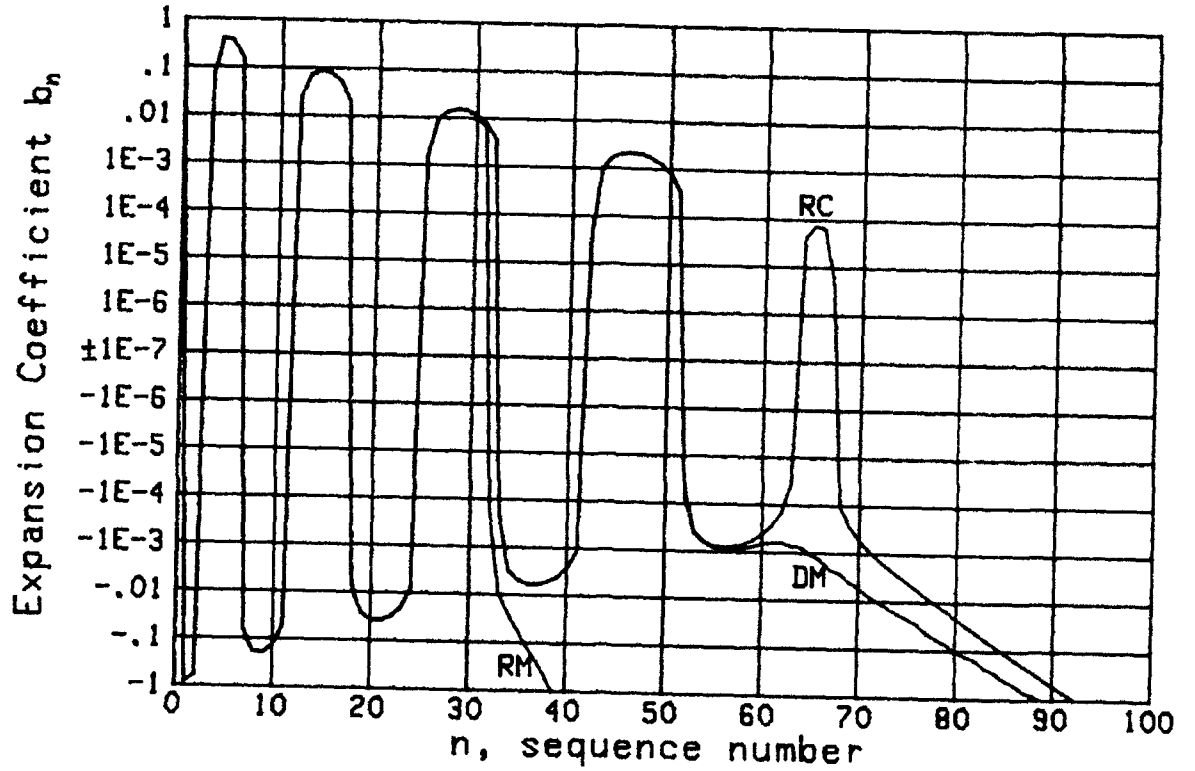

Figure 15. Generalized Laguerre; Example G

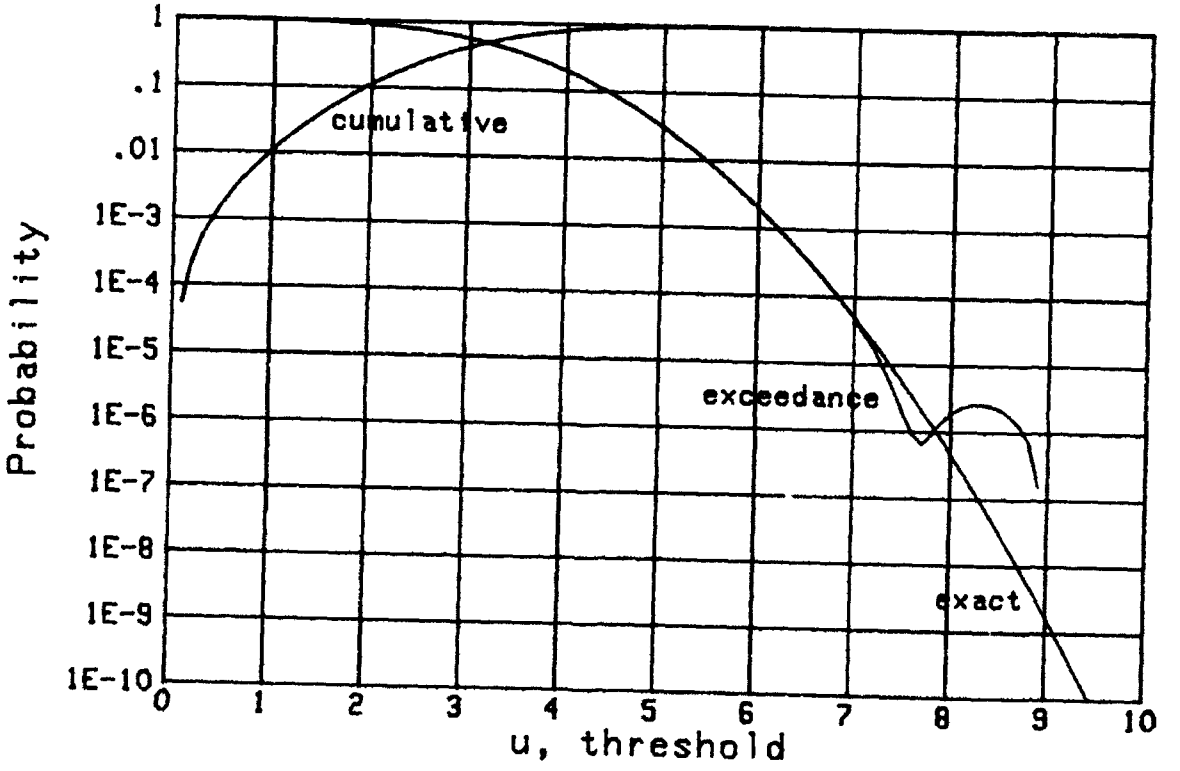

Figure 16. Distributions for Example G

are displayed in figure 17. Whereas both DM and RM are useless beyond n = 25, the expansion coefficients determined by RC decay down to the 1E-10 level at n ≅ 150 before round-off error becomes significant. The corresponding distributions in figure 18, using N = 143 terms of the expansion via RC, reveal accurate results down to the 1E-12 level of probability, except for a slight flare in the exceedance distribution function below 1E-11.

We also checked the example of the sum of 8 normalized Rayleigh variates considered earlier via a Hermite series in example C. For $\alpha = 10$, $\beta = .9$, the expansion coefficients $\{b_n\}$ decayed to the 1E-11 level at n = 100 for the RC approach and agreed with the false alarm probabilities calculated exactly in [9, table 1] for M = 8. By contrast, the DM expansion coefficients were subject to significant round-off error by the time n reached 30, and were useless for small probability calculations.

EXAMPLE I

We return to the shot noise process previously considered via a Hermite series in example D. The equations and discussions there should be reviewed, since they are directly relevant to the generalized Laguerre expansion here. For the choice of parameters in (150), the selection of generalized Laguerre weighting parameters

$$\alpha = .74, \qquad \beta = 2.1 \tag{171}$$

leads to the expansion coefficients plotted in figure 19. The DM and RC results agree to n = 32, and then begin to diverge from each other. By way of

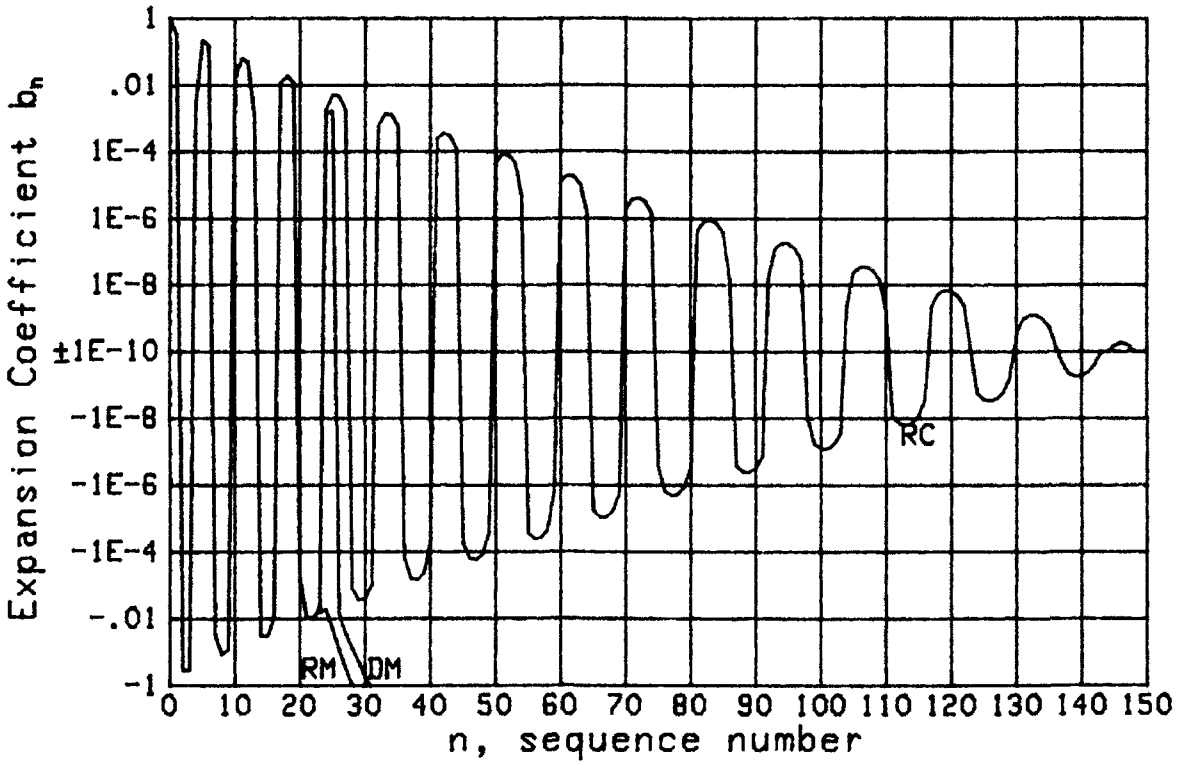

Figure 17. Generalized Laguerre; Example H

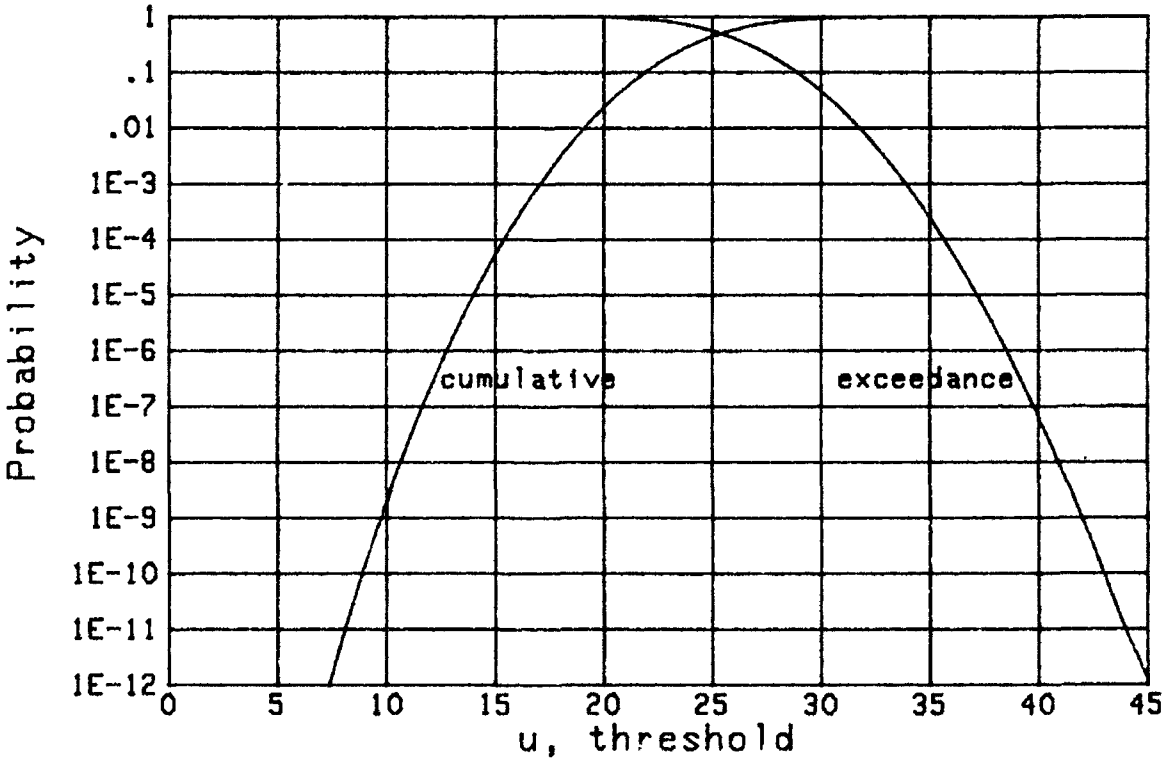

Figure 18. Distributions for Example H

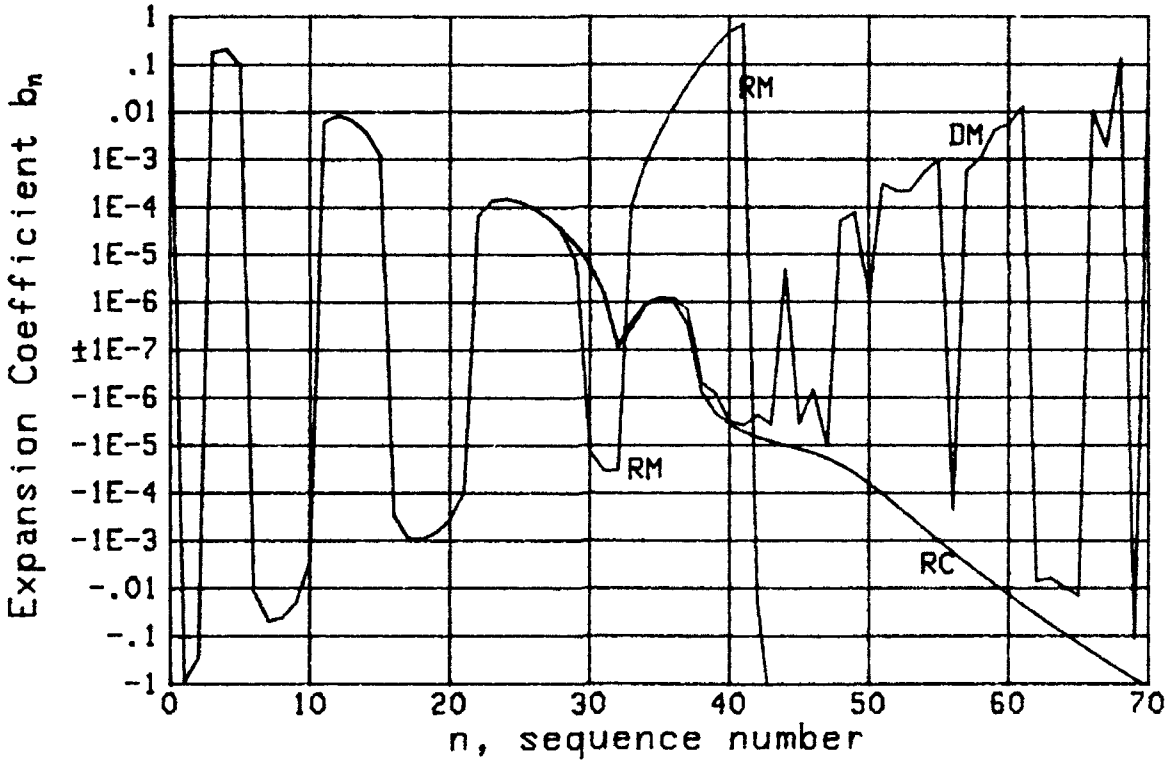

Figure 19. Generalized Laguerre; Example I

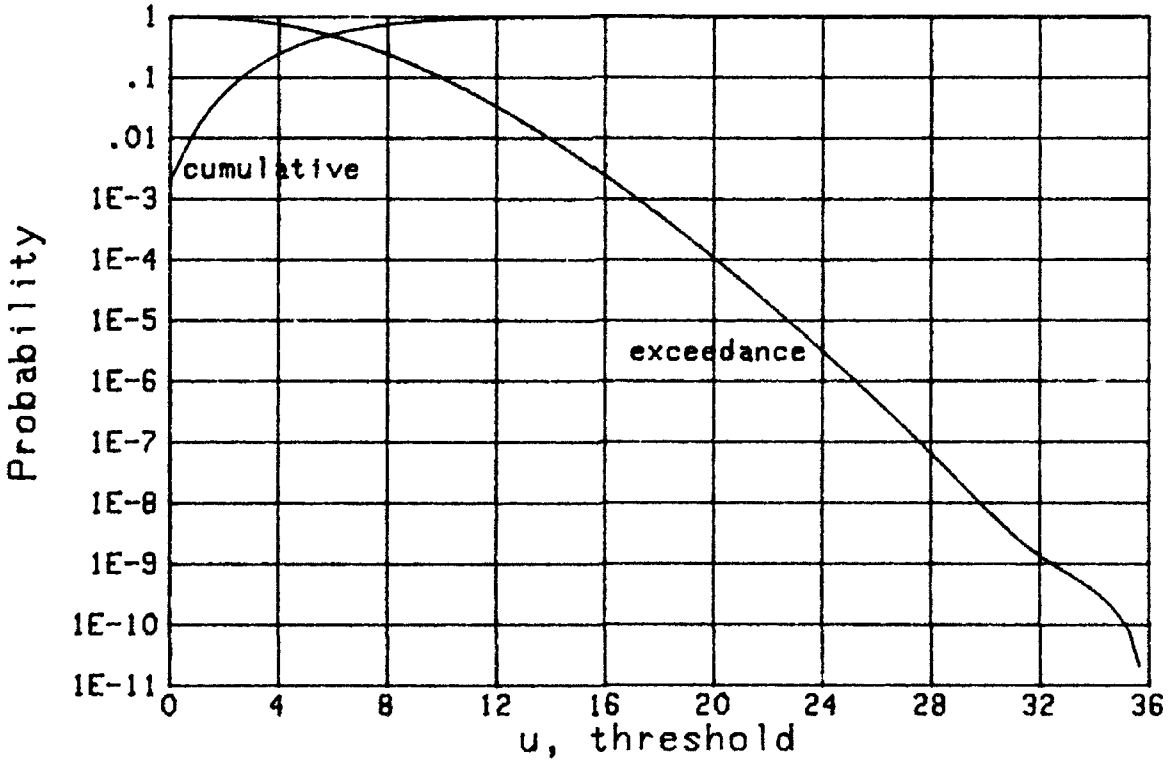

Figure 20. Distributions for Example I

contrast with the Hermite expansion coefficients in figure 7, where values in the 1E-3 range were achieved, values in the 1E-6 range can be obtained here for the generalized Laguerre expansion, for n in the mid-30s. The DM result was previously given in [4, figure D-1].

The distributions for N = 32 terms of the RC procedure are plotted in figure 20. This result is considerably better than the Hermite expansion in figure 8; instead of the wiggles which developed at 1E-4 in figure 8, the curve in figure 20 is smooth down to the 1E-8 probability level, and then develops a bump. Also, the cumulative distribution function is accurate at $u = 0$, where it takes on the value $P_0 = .002$ given in (150), and is zero for $u < 0$. This cumulative distribution function was previously given in [4, figure 8].

The probability density function for N = 32 terms of the RC procedure is given in figure 21; this result was previously given in [4, figure 9]. It is significantly better near the origin than the Hermite approximation given earlier in figure 9, which developed negative values for $u < 0$. In order to see what the probability density function does for larger u values, the same probability density function is plotted on a logarithmic ordinate in figure 22. It is accurate to the 1E-9 leve but then develops a hook that is incorrect; however, this approximation remains positive even at this very low value of the density, whereas the corresponding result via a Hermite expansion in figure 10 developed negative values. The estimated errors in figures 20 and 22 are evaluated in a later section.

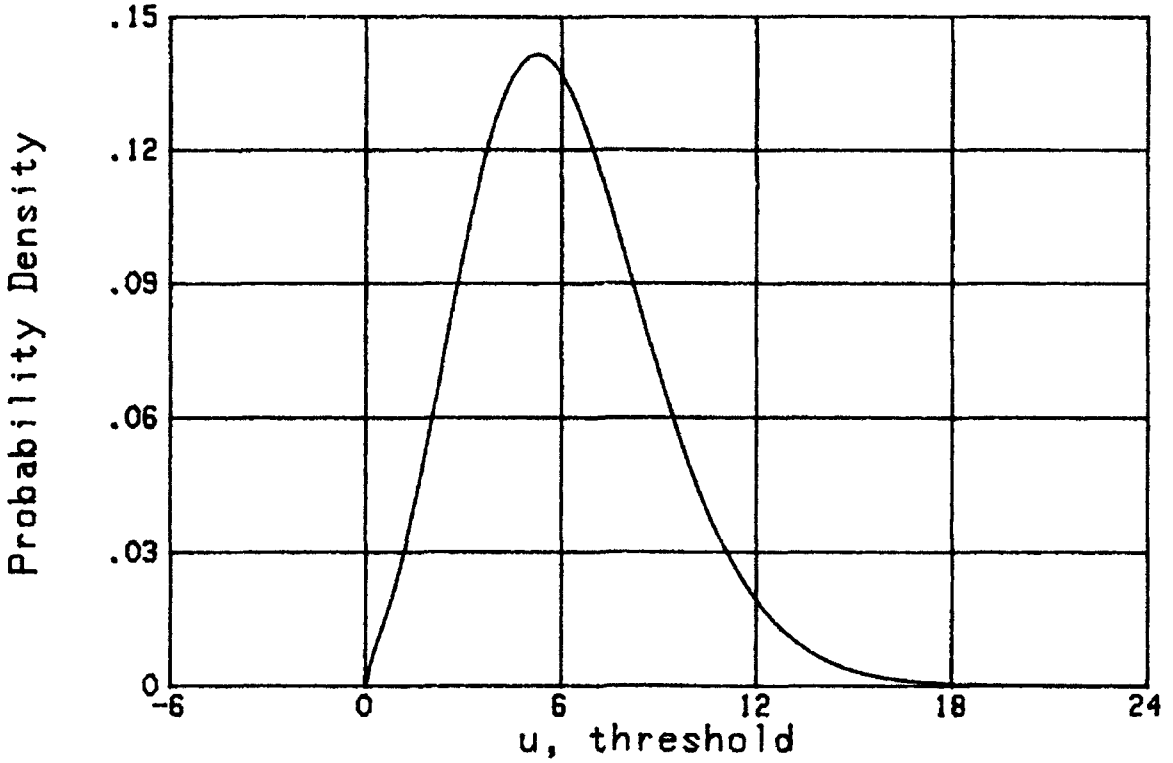

Figure 21. Linear Density for Example I

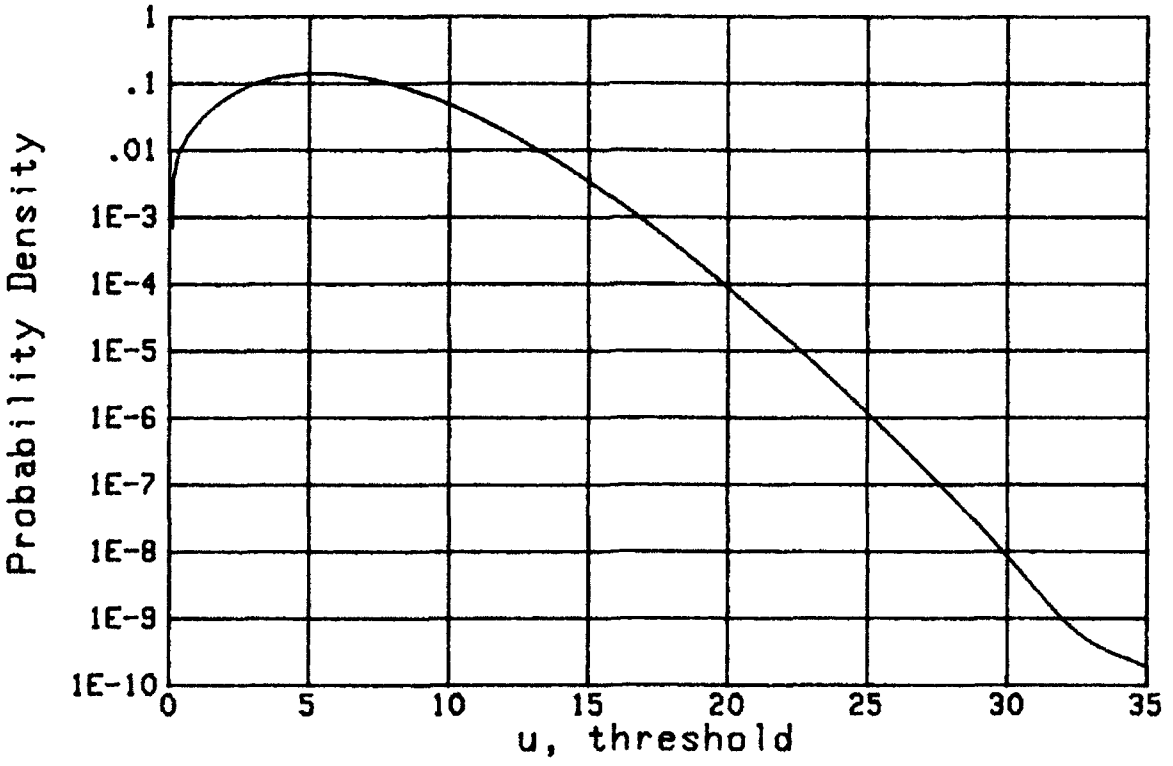

Figure 22. Log Density for Example I

EXAMPLE J

This last example is for probability density function

$$p(u) = \frac{1}{2} \exp\left(-u^{1/2}\right) \quad \text{for } u > 0 , \tag{172}$$

for which the moments are

$$\mu_n = (2n+1)! \quad . \tag{173}$$

The characteristic function and cumulants are not available in any convenient analytic form.

This is a particularly difficult example, since the characteristic function expansion in (6) has a zero radius of convergence; thus the moments do not uniquely determine the probability density function or cumulative distribution function. Also, the error integral in (21) is always infinite; in fact, regardless of the choice of weighting parameters α and β used in the generalized Laguerre series, the expansion coefficients $\{b_n\}$ always diverged. Nevertheless, a search of parameter values led to a pair of selections, namely,

$$\alpha = -.35, \qquad \beta = 30, \tag{174}$$

for which the expansion coefficients had an initial decay to the 1E-2 level before divergence took over; see figure 23. In fact, the identical same results were obtained for all three methods, RC, DM, RM; this is probably due to the fact that divergence of $\{b_n\}$ dominated before round-off error became significant.

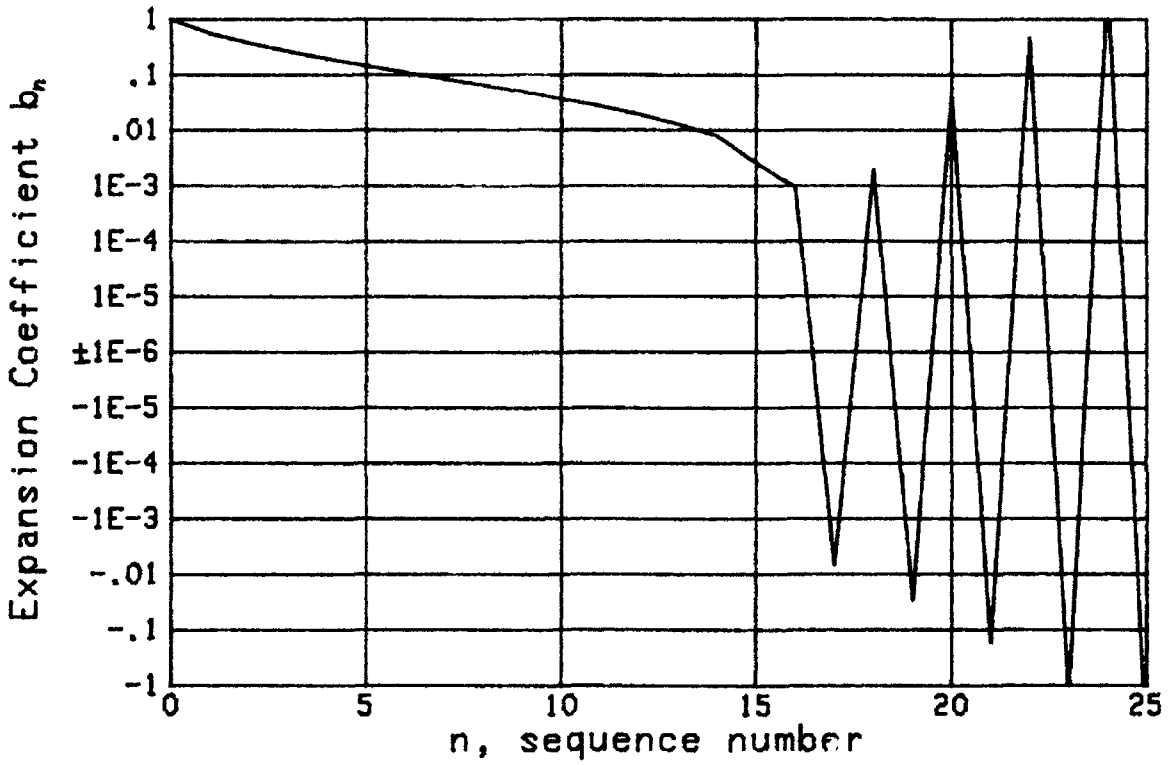

Figure 23. Generalized Laguerre; Example J

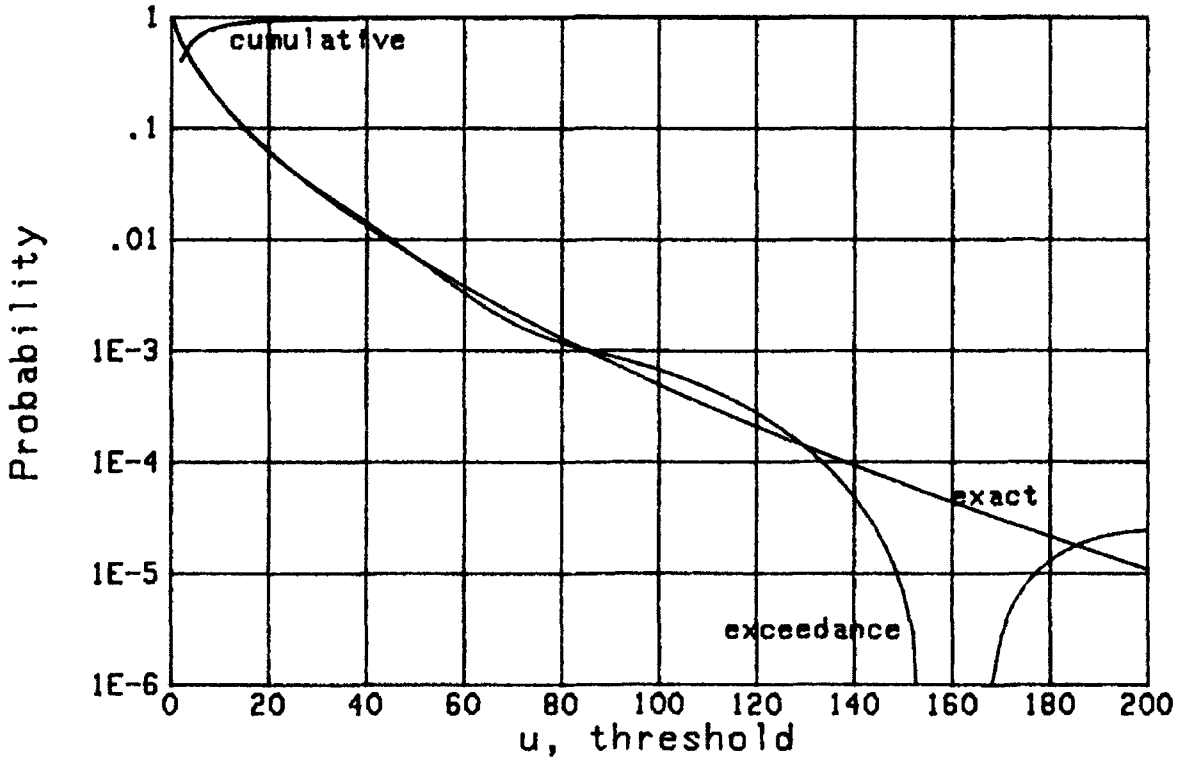

Figure 24. Distributions for Example J

The distributions are plotted in figure 24 for N = 15 terms of the generalized Laguerre series. Comparison with the exact exceedance distribution function

$$1 - P(u) = (1 + u^{1/2}) \exp(-u^{1/2}) \quad \text{for } u > 0 \tag{175}$$

reveals that the approximation is decent down to the .01 probability level, but then oscillates more and more violently as u increases. Thus even in this non-unique example, a limited-quality approximation is achieved by the generalized Laguerre series; this example confirms the comment in [3, p. 167] that, even for a divergent series, a limited number of expansion coefficients often gives a satisfactory approximation.

The exact and approximate probability density functions are plotted on a linear ordinate in figure 25, and on a logarithmic ordinate in figure 26, using N = 15 terms of the generalized Laguerre series, when the expansion coefficients were determined by the DM method. The approximate probability density function is negative for $150 < u < 190$, around the 1E-6 level. The estimated errors of the approximations in figures 24 and 26 will be developed in the next section.

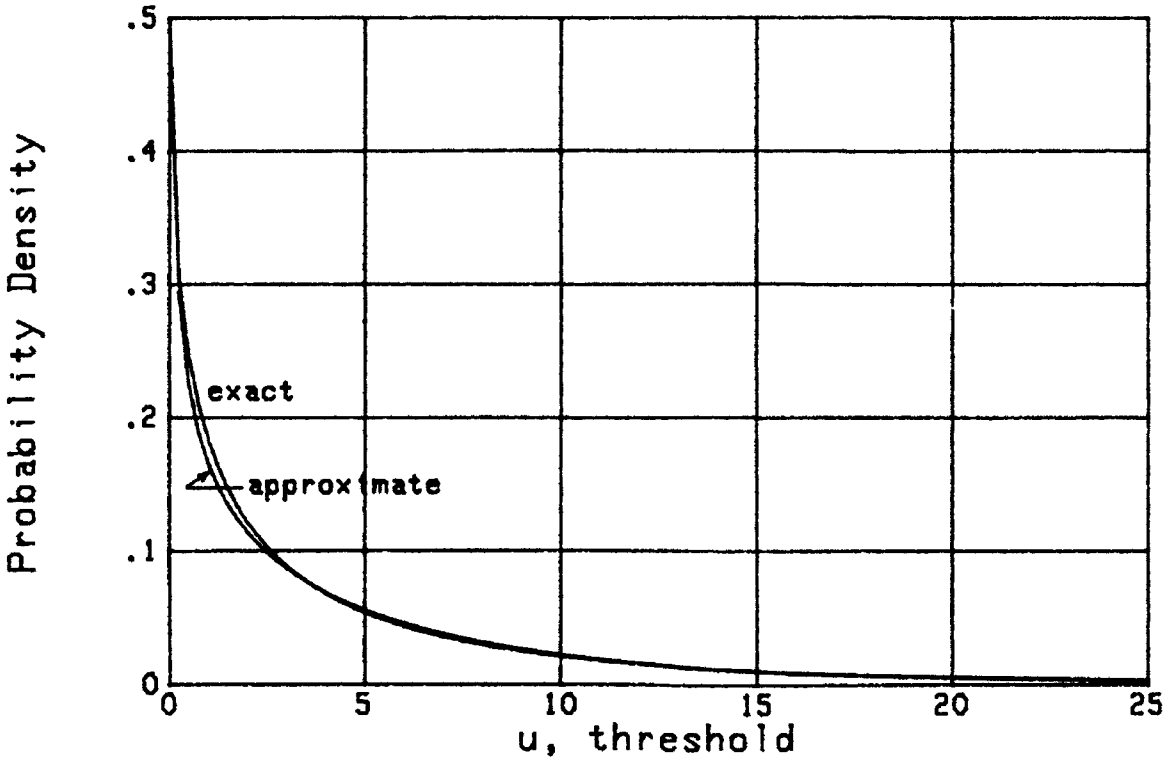

Figure 25. Linear Density for Example J

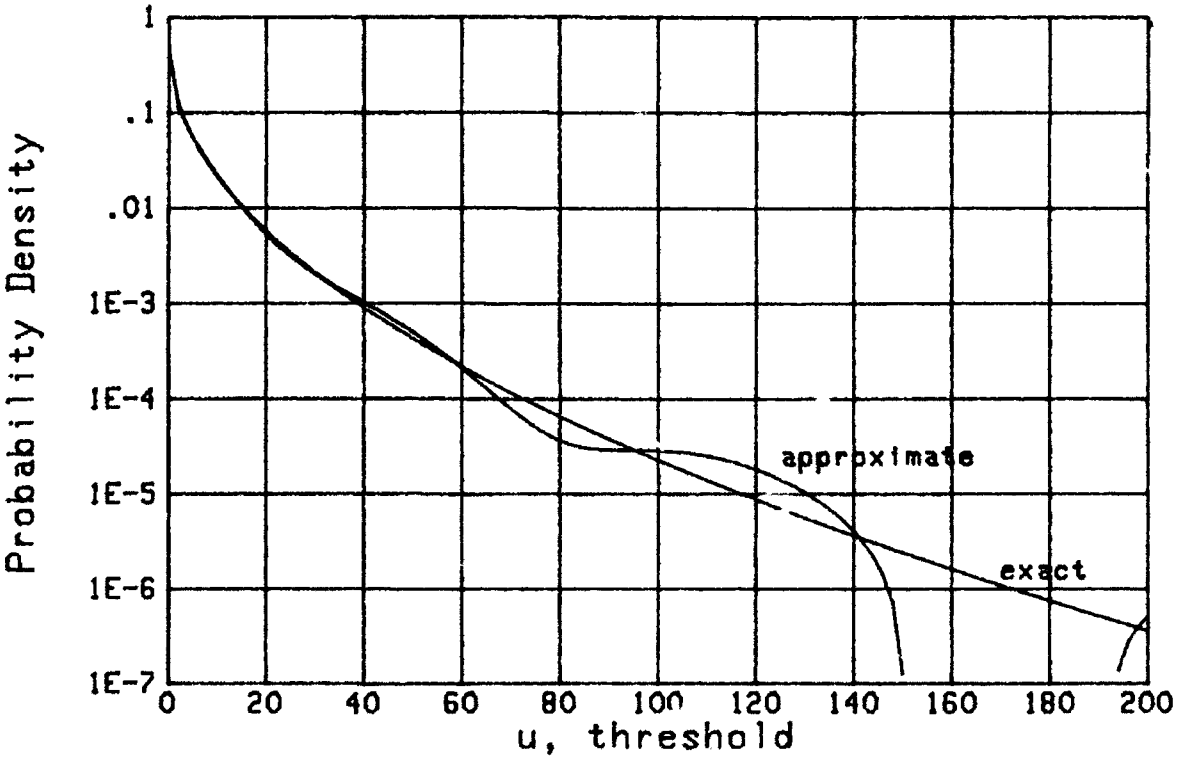

Figure 26. Log Density for Example J

ESTIMATED ERRORS OF APPROXIMATIONS

When the calculations of the approximate cumulative or exceedance distribution functions or the corresponding probability density function are made, it would be very useful to have a rough estimate of their reliability. One way, as discussed in the previous sections, is to look for nonsmooth or anomalous behavior on the tails of the functions. Here, we will develop a more quantitative estimate of the error and superpose it on some of the previous examples, for confirmation.

Both the Hermite and generalized Laguerre orthonormal polynomials oscillate with n and decay slowly. The same general behavior is true of expansion coefficients $\{b_n\}$. This leads to summations for the various functions with terms that also oscillate and decay. A rough estimate of the error is afforded by the envelope of these oscillations, evaluated at the first neglected term of the summation. This procedure will be pursued for both types of expansions; how useful it is will be indicated by numerical examples.

HERMITE EXPANSION

The following result for the envelope of the Hermite polynomial is obtained from [5, 6.1.39 and 22.5.18] and [7, 8.22.8]:

$$\text{Env}\left\{(n!)^{-1/2}\ He_n(x)\right\} \sim \exp(x^2/4)\left(\frac{2}{\pi n}\right)^{1/4} \quad \text{as } n \to +\infty. \tag{176}$$

Also, from (46) and (47), the n-th term of the approximate probability density function is

$$\frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right) b_n\,(n!)^{-1/2}\,He_n\left(\frac{u-\alpha}{\beta}\right)\,. \tag{177}$$

Then the magnitude of the error of the probability density function approximation, if the n-th term is the first one neglected, is roughly

$$E_n(\cdot;p) \equiv \frac{1}{\beta}\,\phi\left(\frac{u-\alpha}{\beta}\right)\,\mathrm{Env}\{b_n\}\,\mathrm{Env}\left\{(n!)^{-1/2}\,He_n\left(\frac{u-\alpha}{\beta}\right)\right\} =$$

$$\sim \left[2^{1/4}\,\pi^{3/4}\,\beta\right]^{-1}\,\exp\left(-\frac{(u-\alpha)^2}{4\beta^2}\right) n^{-1/4}\,\mathrm{Env}\{b_n\}\ \text{as}\ n \to +\infty. \tag{178}$$

Here we used (176).

As for the cumulative distribution function, we have from (47)-(49), the n-th term of the approximation as

$$-\phi\left(\frac{u-\alpha}{\beta}\right) b_n\,(n!)^{-1/2}\,He_{n-1}\left(\frac{u-\alpha}{\beta}\right)\,. \tag{179}$$

The magnitude of the error for the cumulative and exceedance distribution functions, if the n-th term is the first one neglected, is then defined as

$$E_n(u;P) \equiv \phi\left(\frac{u-\alpha}{\beta}\right)\,\mathrm{Env}\,\{b_n\}\,\mathrm{Env}\left\{(n!)^{-1/2}\,He_{n-1}\left(\frac{u-\alpha}{\beta}\right)\right\} =$$

$$\sim \left[2^{1/4}\,\pi^{3/4}\right]^{-1}\exp\left(-\frac{(u-\alpha)^2}{4\beta^2}\right) n^{-3/4}\,\mathrm{Env}\,\{b_n\}\ \text{as}\ n \to +\infty. \tag{180}$$

Again. (176) was of crucial importance in getting this result.

Since the above estimates are asymptotic in n, they will be most reliable for n large; their use for small n could be very misleading. The way to use these error estimates for the density and distribution approximations is as follows. First, a search on α and β, to find the fastest decaying expansion coefficients $\{b_n\}$, is conducted. The weighting parameter values, α and β, and the corresponding envelope value of the expansion coefficients $\{b_n\}$ at the point, n, where round-off error becomes dominant, are then noted. (For example, for figure 7, we observe that $\text{Env}\{b_n\} \approx 2E\text{-}3$ at $n = 65$, when $\alpha = 6.1$, $\beta = 4.3$; see example D.) Then (178) and (180) can be computed and plotted in the ranges of u of interest.

An example of this procedure for the shot noise process in example D is given in figures 27 and 28. In particular, the approximate results are repeated from figures 8 and 10, and error measures (180) and (178), respectively, are superposed as dashed lines, each on the appropriate figure. Just where the approximations develop large wiggles, the errors are of comparable magnitude, indicating unreliable estimates there.

It should be observed from these figures (or from (178) and (180)) that the absolute error is maximum at $u = \alpha$, but that the relative error is a minimum in that neighborhood. Also, although the absolute error decays with u, the correct answer decays faster, leading to an increasing relative error, which eventually becomes so excessive in the tails of the various functions that the approximations are useless.

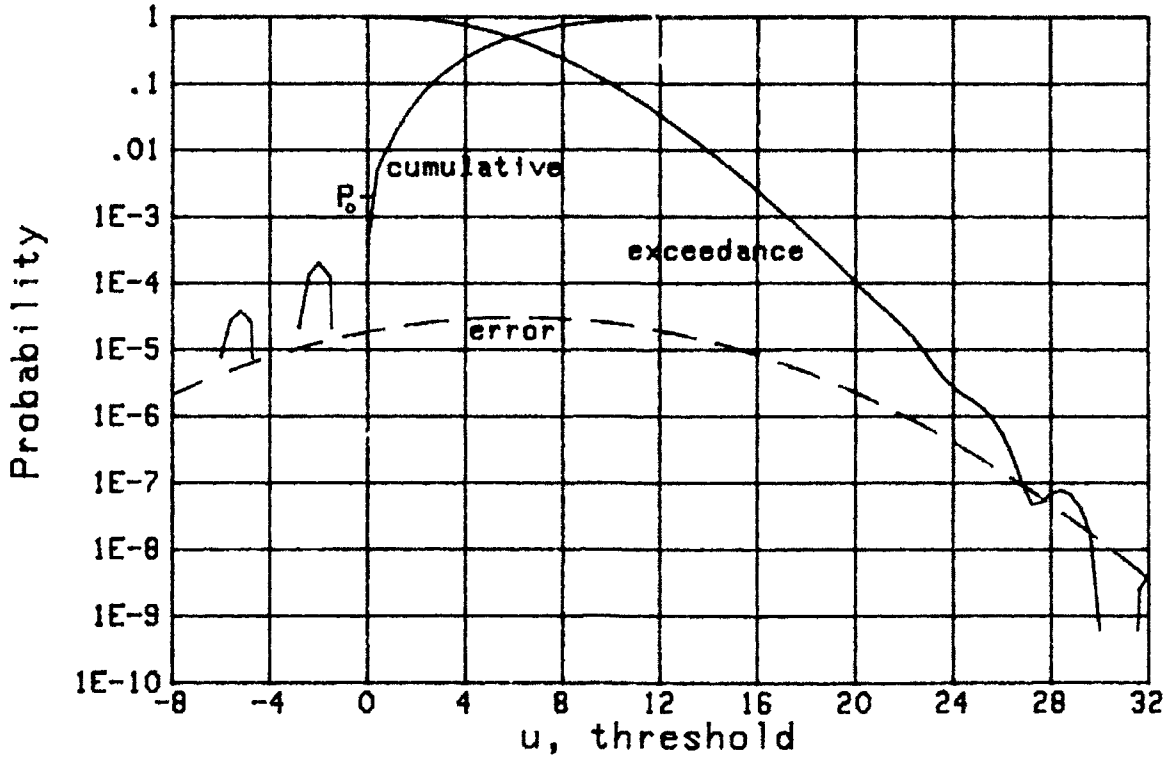

Figure 27. Estimated Error of Figure 8

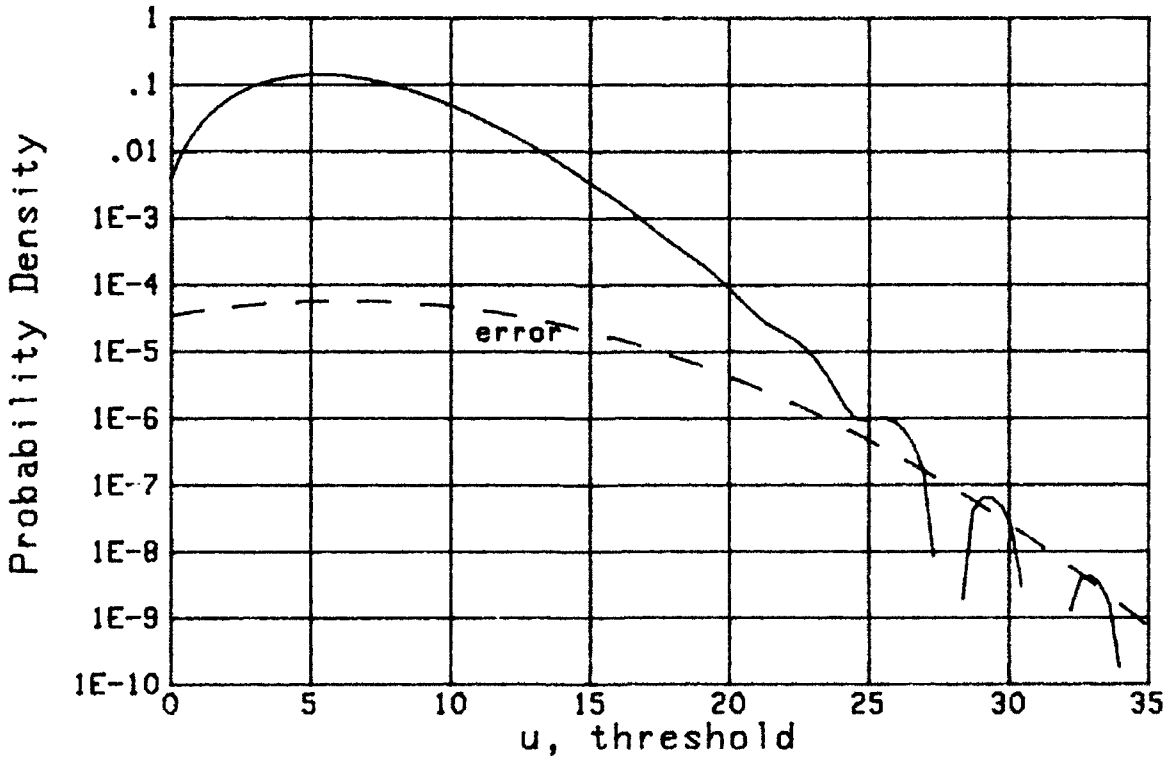

Figure 28. Estimated Error of Figure 10

The details for the generalized Laguerre series are very similar to those above and so will be abbreviated. The envelope of the generalized Laguerre polynomial is [7, 8.22.1]

$$\text{Env}\left\{L_n^{(\alpha)}(x)\right\} \sim \pi^{-\frac{1}{2}}\, e^{\frac{x}{2}}\, x^{-\frac{\alpha}{2}-\frac{1}{4}}\, n^{\frac{\alpha}{2}-\frac{1}{4}} \quad \text{as } n \to +\infty, \quad \text{for } x > 0\,. \tag{181}$$

From (91) and (92), the n-th term of the approximate probability density function is

$$\frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)}\; b_n \left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right). \tag{182}$$

Then the magnitude of the error of the probability density function approximation is, for $u > 0$,

$$E_n(u;p) \equiv \frac{u^\alpha \exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)}\; \text{Env}\left\{b_n\right\}\; \text{Env}\left\{\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_n^{(\alpha)}\left(\frac{u}{\beta}\right)\right\} =$$

$$\sim \left[\pi\Gamma(\alpha+1)\beta^2\right]^{-1/2} \left(\frac{u}{\beta}\right)^{\frac{\alpha}{2}-\frac{1}{4}} \exp\left(-\frac{u}{2\beta}\right) n^{-1/4}\, \text{Env}\left\{b_n\right\} \text{as } n \to +\infty, \tag{183}$$

where we used [5, 6.1.47] and (181). This quantity peaks at $u = \beta(\alpha-\frac{1}{2})$.

With regards to the cumulative distribution function, the n-th term of the approximation is, from (95) and (92),

$$\frac{u^{\alpha+1}\exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)}\ b_n\ \frac{1}{n}\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right). \tag{184}$$

Then the magnitude of the distribution error, for both the cumulative and the exceedance distribution functions, is roughly

$$E_n(u;P) \equiv \frac{u^{\alpha+1}\exp(-u/\beta)}{\beta^{\alpha+1}\Gamma(\alpha+1)}\ \mathrm{Env}\{b_n\}\ \mathrm{Env}\left\{\frac{1}{n}\left(\frac{n!}{(\alpha+1)_n}\right)^{1/2} L_{n-1}^{(\alpha+1)}\left(\frac{u}{\beta}\right)\right\} =$$

$$\sim\left[\pi\Gamma(\alpha+1)\right]^{-\frac{1}{2}}\left(\frac{u}{\beta}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\exp\left(-\frac{u}{2\beta}\right)n^{-\frac{3}{4}}\ \mathrm{Env}\{b_n\}\quad \text{as } n \to +\infty, \text{ for } u > 0, \tag{185}$$

upon use of (181). This quantity reaches its peak at $u = \beta(\alpha+\frac{1}{2})$.

An application of these results to the shot noise process, which was re-investigated in example I via the generalized Laguerre series, is given in figures 29 and 30. Specifically, the approximate results from figures 20 and 22 have been repeated, and error measures (185) and (183), respectively, superposed as dashed lines. They confirm the earlier observations that the distribution and density approximations are reliable until the anomalous behavior on the tails manifests itself.

The difficult example J is considered in figures 31 and 32. Since the expansion coefficient sequence $\{b_n\}$ in figure 23 diverged for large n, the selection of n = 15, as used in figures 24-26, is not the large value needed to justify the use of (183) and (185). Thus, the dashed curves on figures 31 and 32 must be considered only as ball-park estimates; in general, the approximate error appears to be too conservative in these two figures.

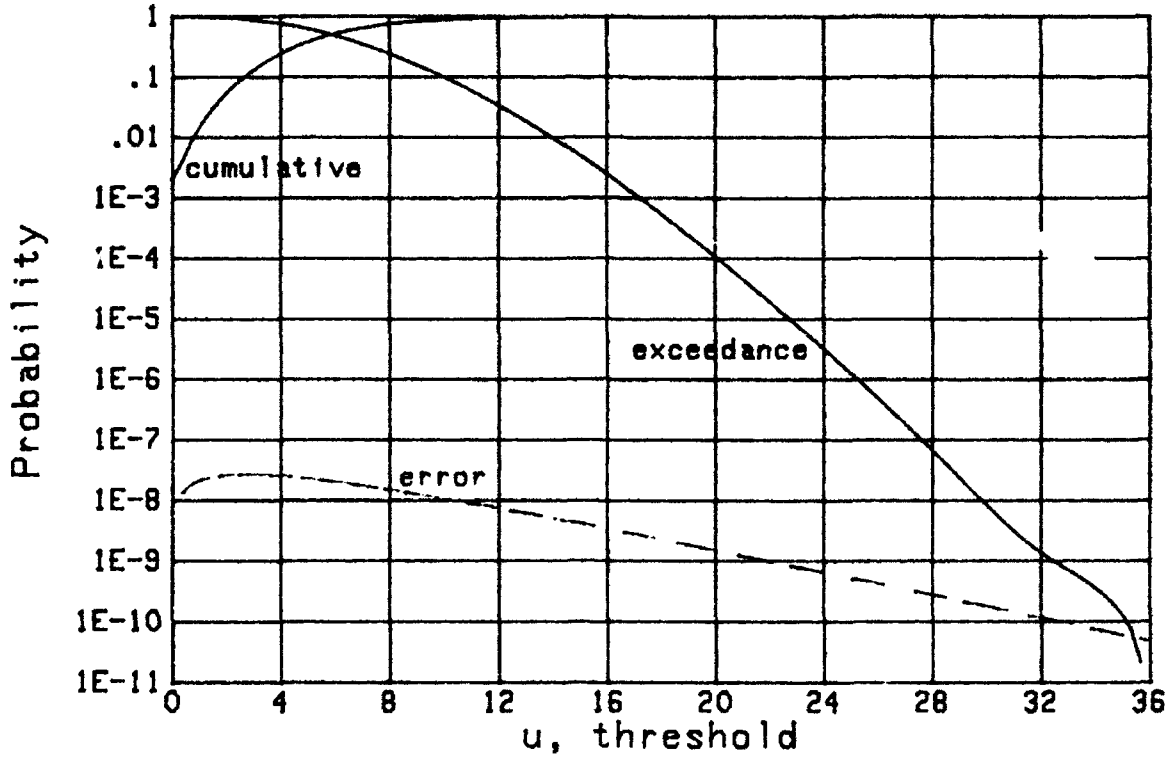

Figure 29. Estimated Error of Figure 20

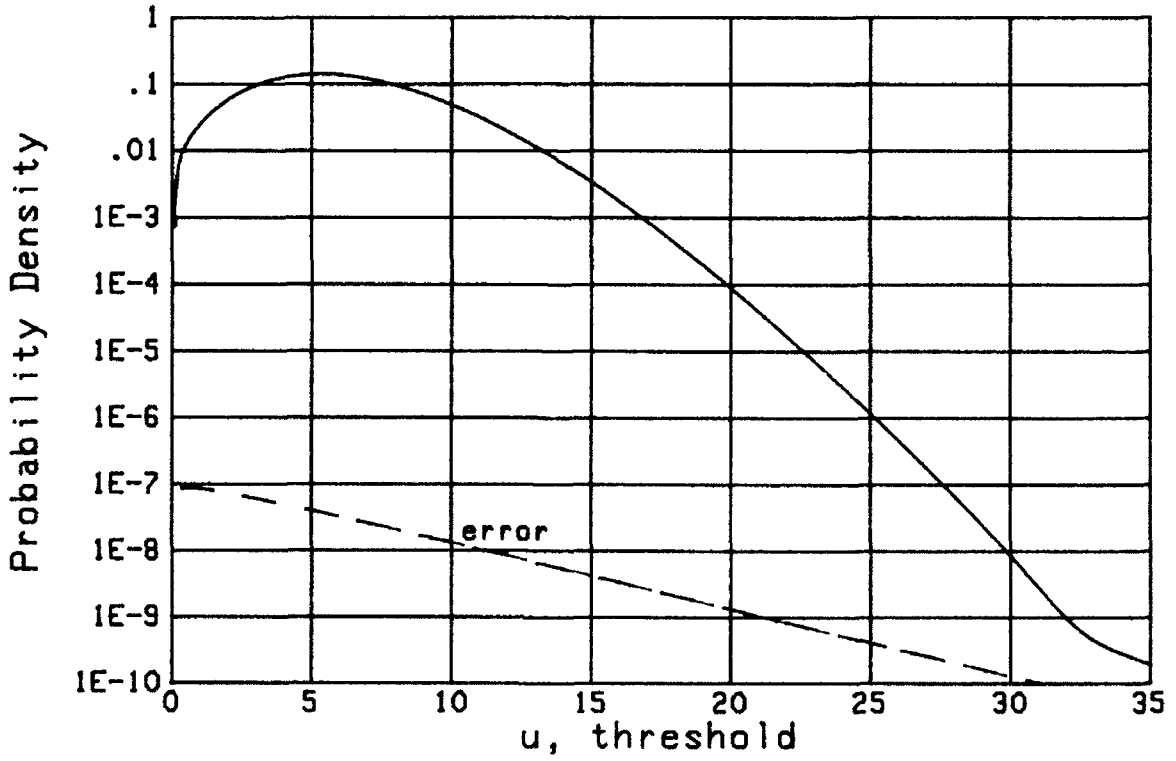

Figure 30. Estimated Error of Figure 22

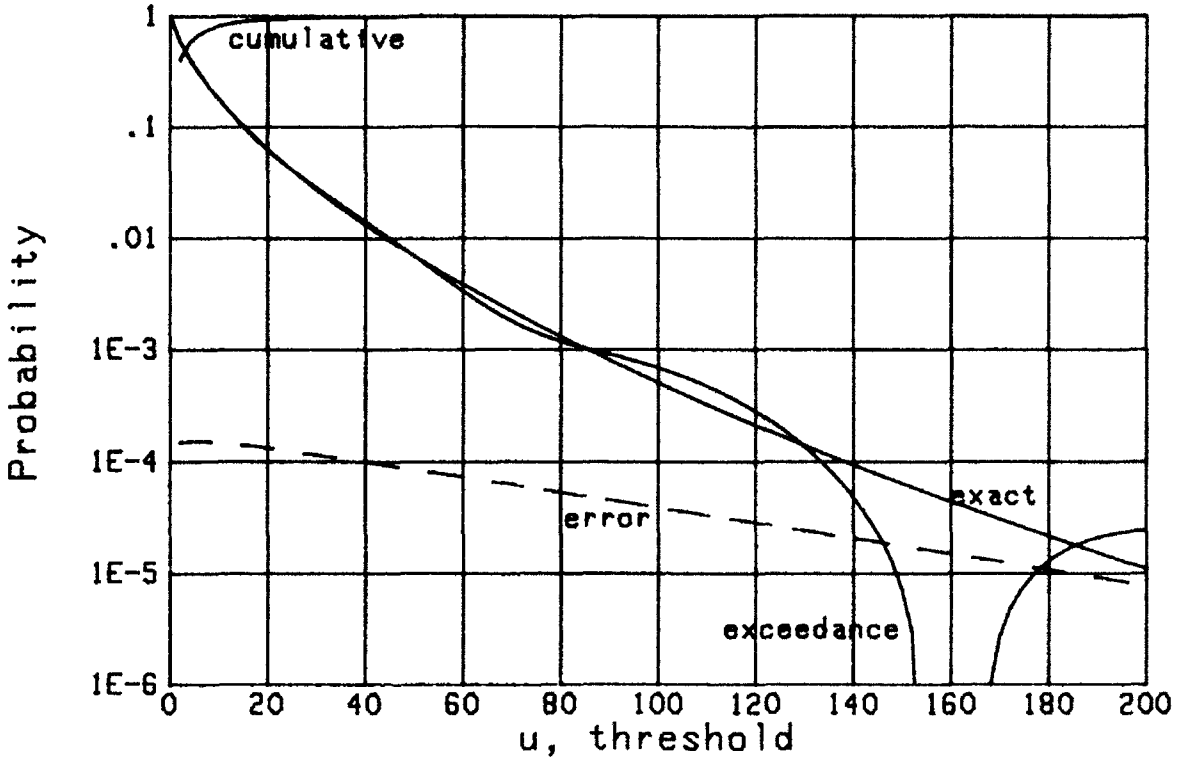

Figure 31. Estimated Error of Figure 24

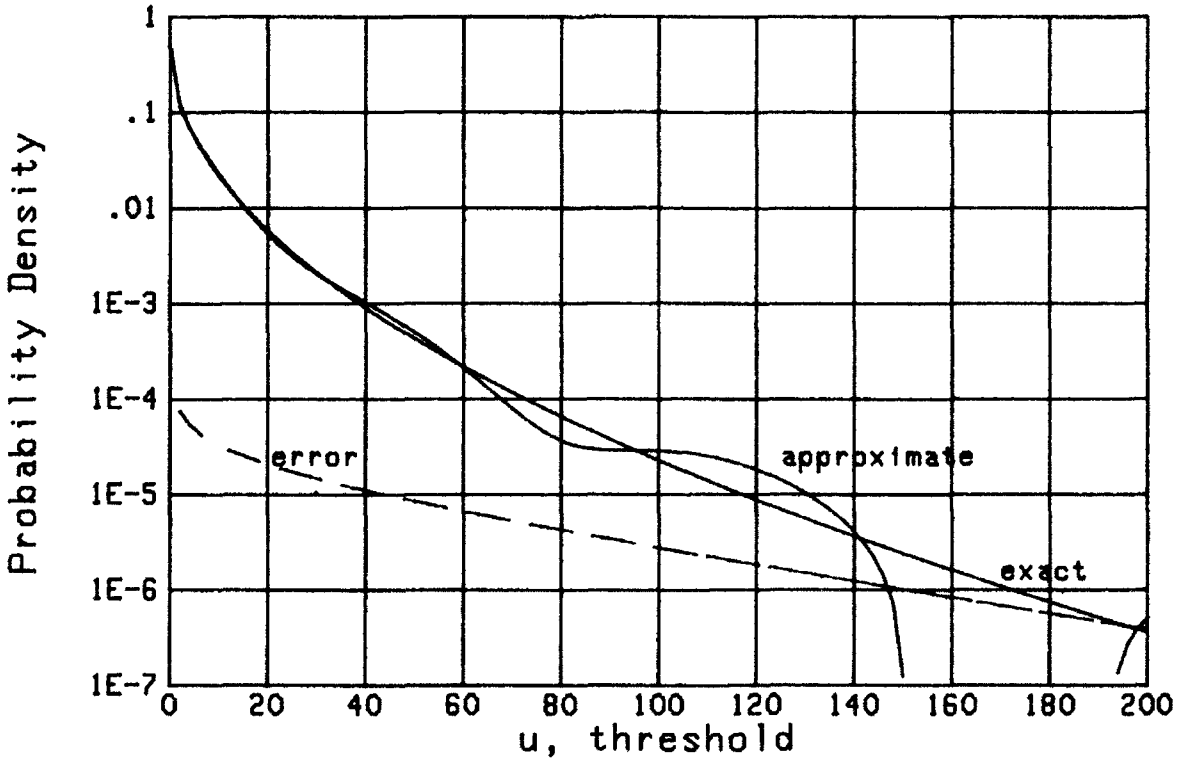

Figure 32. Estimated Error of Figure 26

Finally, the Rice variate of example G is re-considered in figure 33. We took $\mathrm{Env}\{b_n\} \approx 3E\text{-}4$ at $n = 65$, by extrapolating in figure 15 from smaller n, since round-off error is becoming significant by this value. It verifies the unreliability of the approximation in figure 33 for $u > 7$.

Although all the examples in this report have the capability of evaluating either the moments or the cumulants via recursion, this is by no means necessary. Any method whatsoever of accurately calculating the starting statistics, be they moments or cumulants, is acceptable. For example, if a random variable with known probability density function q is passed through a complicated nonlinearity g, the moments of the output are given by

$$\mu_n = \int du\ g^n(u)\ q(u)\ . \tag{186}$$

These quantities could be evaluated for $0 \leq n \leq N$ by brute-force numerical procedures if necessary. The limit value N will depend on the accuracy with which g and q can be evaluated; if $g(u) \geq 0$ for all u, these integrals can be accomplished to a high degree of accuracy, thereby allowing large values of N to be employed.

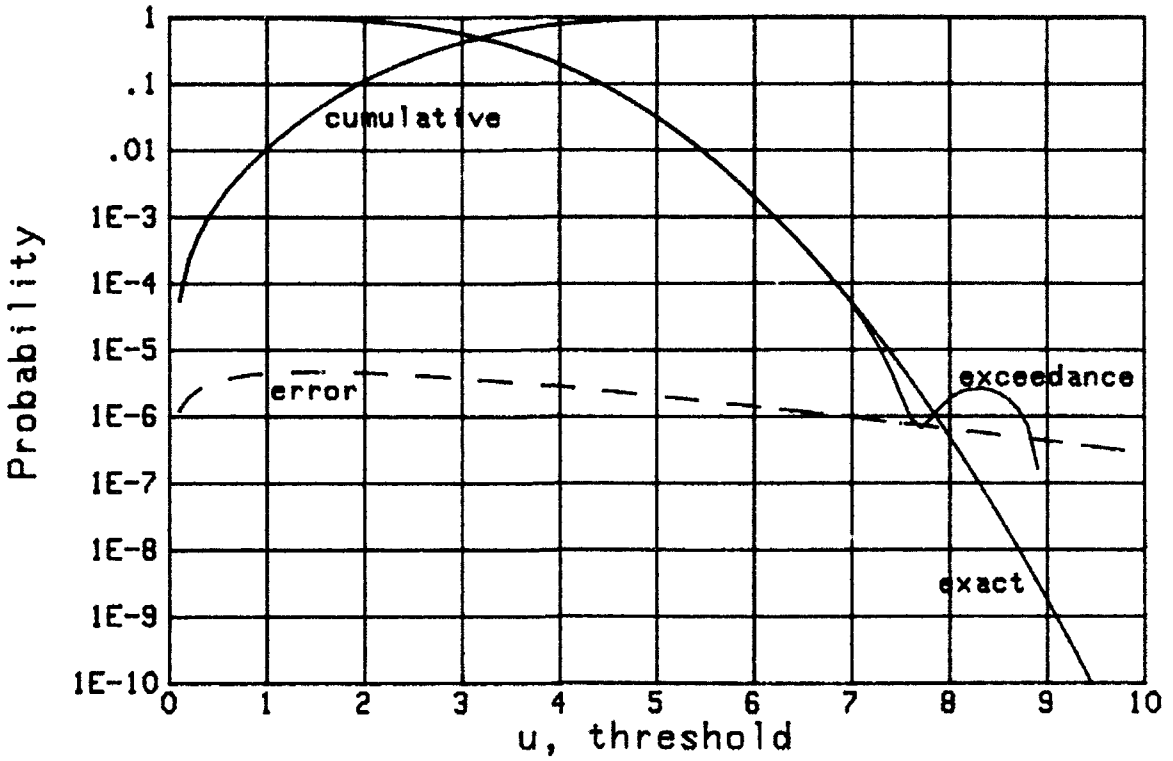

Figure 33. Estimated Error of Figure 16

DISCUSSION

Several alternative methods have been presented for obtaining either Hermite or generalized Laguerre series expansions of probability density functions or cumulative and exceedance distribution functions, by means of recursive relations involving either moments or cumulants. Furthermore, estimates of the errors of the approximations are furnished so that the reliability can be assessed. Comparisons between approximations obtained by either the Hermite or the generalized Laguerre series afford an assessment of the accuracy of each; also, the availability of three alternative recursive procedures for the expansion coefficients allows for selection of the best method and results, and determination of the amount of round-off error.

The key feature to this approach is the rapid calculation and observation of the orthonormal expansion coefficients $\{b_n\}$ for each particular guess of weighting parameters α and β. A trial and error procedure is suggested for determining α and β values that yield the set of fastest-decaying expansion coefficients. From observation of the expansion coefficients, the number of terms to retain in the series expansions is ascertained, being sure to avoid the effects of round-off error which dominates the calculated expansion coefficients $\{b_n\}$ for large n. Since the amount and location of round-off error on the plot of expansion coefficients also depends on α and β, a judicious search may be required to find acceptable weighting parameter values. Of course, a computer with a larger number of significant digits would greatly alleviate this drawback; the particular computer used for all the calculations reported here is the Hewlett-Packard 9000 Model 520 which

devotes 52 bits (15.65 decimal digits) to the mantissa and 11 bits to the exponent. Failure of the technique is indicated by divergence of the expansion coefficient sequence $\{b_n\}$.

Programs for the shot noise process considered in examples D and I are presented in appendix F. Times of execution are as follows. For the Hermite series, the 80 cumulants or 80 moments required as input for figure 7 took .7 or .35 seconds, respectively. The calculation, plotting, and display of the 80 expansion coefficients in figure 7 took 1.6 seconds via the RC approach and 1.75 seconds via the two moment approaches. The computation and display of the 100-point plots of the cumulative distribution function in figure 8 and the probability density function in figure 9, each using 65 terms in the series expansion, took 1.1 and .95 seconds, respectively.

For the generalized Laguerre series, the 70 cumulants or 70 moments required as input for figure 19 took .54 seconds or .28 seconds, respectively. The calculation and display of the 70 expansion coefficients in figure 19 took 1.8 seconds via the RC approach and 1.5 seconds via the two moment approaches. The computation and display of the 100-point plots of the cumulative distribution function in figure 20 and the probability density function in figure 21 took 1.1 and .7 seconds, respectively. These execution times are short enough to allow a human observer to conduct a rapid trial-and-error search of α,β space, determine adequate parameter values, and assess their accuracy.

Alternative exact procedures for determination of cumulative and exceedance distribution functions via characteristic functions have been presented in [9, 10, 11]. Those methods generally have the potential for

greater accuracy, are less subject to round-off-error, and would be preferred if possible. However, analysis of systems with nonlinearities and memory sometimes precludes or greatly hinders their application; in such cases, the current approach is a very good candidate for consideration.

The two weightings in (1) and (2), namely the Hermite and generalized Laguerre, have been investigated rather intensively here, because so many properties and recursions are available for the corresponding (orthonormal) polynomials. These properties have been utilized to derive simple recursive relations for the expansion coefficients and density and distribution functions, thereby realizing quick efficient procedures for numerical evaluation and observation.

It would be extremely useful to be able to extend these results to the weighting

$$u^{\alpha} \exp(-u^2/\beta^2) \text{ for } u > 0, \tag{187}$$

since this class of probability density functions is often encountered in nonlinear systems with Gaussian inputs. However, there are several pivotal recursive relations for the corresponding orthonormal polynomials that would be needed, and it is questionable if a fast procedure could be devised without them. Also, it is unknown if recursive procedures for the expansion coefficients in terms of moments or cumulants could be derived, as was done here for the Hermite and generalized Laguerre weightings. This is a topic worthy of further investigation.

APPENDIX A. COEFFICIENT RECURSION FOR EXPONENTIAL OF POWER SERIES

Suppose power series $\sum_{n=0}^{\infty} h_n z^n$ converges for some $|z| > 0$, and we exponentiate it, getting a new power series

$$\sum_{n=0}^{\infty} g_n z^n = \exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\} . \tag{A-1}$$

Then the lowest order coefficient is

$$g_0 = \exp(h_0) , \tag{A-2}$$

while for $k \geq 1$, we have

$$g_k = \frac{1}{k!}\left(\frac{d}{dz}\right)^k \left[\exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\left(\frac{d}{dz}\right)^{k-1}\left[\sum_{n=1}^{\infty} n\, h_n z^{n-1} \exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\sum_{p=0}^{k-1}\binom{k-1}{p}\left(\frac{d}{dz}\right)^p \left[\sum_{n=1}^{\infty} n\, h_n z^{n-1}\right]_{z=0} \left(\frac{d}{dz}\right)^{k-1-p} \left[\exp\left\{\sum_{n=0}^{\infty} h_n z^n\right\}\right]_{z=0} =$$

$$= \frac{1}{k!}\sum_{p=0}^{k-1}\binom{k-1}{p}(p+1)!\, h_{p+1}\,(k-1-p)!\, g_{k-1-p} =$$

$$= \frac{1}{k}\sum_{p=0}^{k-1}(p+1)\, h_{p+1}\, g_{k-1-p} = \frac{1}{k}\sum_{m=1}^{k} m\, h_m\, g_{k-m} . \tag{A-3}$$

Thus we have the recursion for coefficients $\{g_k\}$ in terms of the $\{h_m\}$:

$$g_k = \frac{1}{k}\sum_{m=1}^{k} m\, h_m\, g_{k-m} \quad \text{for } k \geq 1, \quad g_0 = \exp(h_0) . \tag{A-4}$$

If we now refer to (6) and (7) and identify

$$g_n = \mu_n/n! \ , \qquad h_n = \chi_n/n! \ , \tag{A-5}$$

there follows the moments in terms of the cumulants according to

$$\mu_k = \sum_{m=0}^{k-1} \binom{k-1}{m} \chi_{k-m} \mu_m \quad \text{for } k \geq 1, \quad \mu_0 = \exp(\chi_0) \ . \tag{A-6}$$

This is a slight generalization of [6, (10)]. This equation is immediately inverted, to yield the cumulants in terms of moments:

$$\chi_k = \frac{1}{\mu_0} \left[\mu_k - \sum_{m=1}^{k-1} \binom{k-1}{m} \chi_{k-m} \mu_m \right] \quad \text{for } k \geq 1, \quad \chi_0 = \ln \mu_0 \ , \tag{A-7}$$

which generalizes [6, (11)].

In terms of the normalized cumulants and moments defined in (62) and (69) respectively, we have

$$\hat{\mu}_k = \frac{1}{k} \sum_{m=0}^{k-1} \hat{\chi}_{k-m} \hat{\mu}_m \quad \text{for } k \geq 1, \quad \hat{\mu}_0 = \exp(\chi_0) \ , \tag{A-8}$$

and

$$\hat{\chi}_k = \frac{1}{\hat{\mu}_0} \left[k \hat{\mu}_k - \sum_{m=1}^{k-1} \hat{\chi}_{k-m} \hat{\mu}_m \right] \quad \text{for } k \geq 1, \quad \left(\chi_0 = \ln \mu_0 \right) . \tag{A-9}$$

APPENDIX B. EXPANSION OF $He_n(x+y)$

The quantity $He_n(x+y)$ is a polynomial of degree n in y. Therefore we can expand

$$He_n(x+y) = \sum_{m=0}^{n} \gamma_m \frac{y^m}{m!} , \tag{B-1}$$

where γ_m will also depend on n and x. In fact,

$$\gamma_m = \left(\frac{\partial}{\partial y}\right)^m \left[He_n(x+y)\right]_{y=0} = \left(\frac{\partial}{\partial t}\right)^m \left[He_n(t)\right]_{t=x} =$$

$$= \left(\frac{\partial}{\partial t}\right)^{m-1} \left[n\, He_{n-1}(t)\right]_{t=x} = n(n-1) \ldots (n-m+1)\, He_{n-m}(x) =$$

$$= \frac{n!}{(n-m)!} He_{n-m}(x) , \tag{B-2}$$

where we used [5, 22.8.8] repeatedly. Using (B-2) in (B-1), we have the alternative forms for the expansion,

$$He_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} He_{n-m}(x)\, y^m =$$

$$= \sum_{m=0}^{n} \binom{n}{m} He_{n-m}(y)\, x^m =$$

$$= \sum_{k=0}^{n} \binom{n}{k} He_k(y)\, x^{n-k} \quad \text{for } n \geq 0 . \tag{B-3}$$

Reverse Blank

APPENDIX C. EVALUATION OF $I_n(y)$ IN (94)

We have, from (94),

$$I_n(y) = \int_0^y dx\, x^\alpha\, e^{-x}\, L_n^{(\alpha)}(x) \quad \text{for } n \geq 0 . \tag{C-1}$$

Then

$$I_0(y) = \int_0^y dx\, x^\alpha\, e^{-x}\, 1 = \gamma(\alpha+1,y) = \frac{y^{\alpha+1}\, e^{-y}}{\alpha+1}\, {}_1F_1(1;\alpha+2;y), \tag{C-2}$$

using [5, 22.4.7, 6.5.2, and 6.5.12]. Also, we have from [5, 22.11.6],

$$x^\alpha\, e^{-x}\, L_n^{(\alpha)}(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left\{e^{-x}\, x^{\alpha+n}\right\} . \tag{C-3}$$

Then for $n \geq 1$, (C-1) can be developed as

$$I_n(y) = \int_0^y dx\, \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left\{e^{-x}\, x^{\alpha+n}\right\} =$$

$$= \frac{1}{n!} \int_0^y d\left(\frac{d}{dx}\right)^{n-1} \left\{e^{-x}\, x^{\alpha+n}\right\} = \frac{1}{n!}\left(\frac{d}{dy}\right)^{n-1} \left\{e^{-y}\, y^{\alpha+n}\right\} =$$

$$= \frac{1}{n}\, y^{\alpha+1}\, e^{-y}\, L_{n-1}^{(\alpha+1)}(y) , \tag{C-4}$$

where we set the lower limit of the evaluated integral to zero since $\alpha+n \geq \alpha+1 > 0$.

APPENDIX D. FOURIER TRANSFORM OF GENERALIZED LAGUERRE POLYNOMIAL

We wish to evaluate transform

$$A(\omega) = \int_0^\infty dt\, e^{i\omega t}\, t^\alpha\, e^{-t}\, L_n^{(\alpha)}(t) \ . \tag{D-1}$$

Now

$$n!\, t^\alpha\, e^{-t}\, L_n^{(\alpha)}(t) = \left(\frac{d}{dt}\right)^n \left\{e^{-t}\, t^{\alpha+n}\right\} \quad \text{for } n \geq 0 \ , \tag{D-2}$$

according to [5, 22.11.6]. Therefore for $n \geq 1$,

$$\begin{aligned} n!\, A(\omega) &= \int_0^\infty dt\, e^{i\omega t} \left(\frac{d}{dt}\right)^n \left\{e^{-t}\, t^{\alpha+n}\right\} = \\ &= \int_0^\infty e^{i\omega t}\, d\left(\frac{d}{dt}\right)^{n-1} \left\{e^{-t}\, t^{\alpha+n}\right\} = \\ &= -i\omega \int_0^\infty dt\, e^{i\omega t} \left(\frac{d}{dt}\right)^{n-1} \left\{e^{-t}\, t^{\alpha+n}\right\} \ , \end{aligned} \tag{D-3}$$

where we used integration by parts with the fact that the integrated part is zero at $t = 0$ and ∞, since $\alpha+n \geq \alpha+1 > 0$. Repeated integration by parts then yields

$$n!\, A(\omega) = (-i\omega)^n \int_0^\infty dt\, e^{i\omega t}\, e^{-t}\, t^{\alpha+n} = \Gamma(\alpha+1+n)\, \frac{(-i\omega)^n}{(1-i\omega)^{\alpha+1+n}} \ . \tag{D-4}$$

This is the result quoted in (104).

APPENDIX E. RECURRENCE FOR EXAMPLE C

The starting point is the moment expression in (142):

$$\mu_n = \Lambda \frac{\theta^{\gamma} \Gamma\left(\frac{n}{2}+h\right)}{2^{\gamma+1}\Gamma(\gamma+1)} \omega^{n+2h} \; {}_1F_1\left(\frac{n}{2}+h;\gamma+1;z\right) , \tag{E-1}$$

where $h = (\gamma+\gamma+1)/2$, $z = \omega^2\theta^2/4$. Denote the ${}_1F_1$ term in (E-1) by F_n, and the leading factor by G_n; thus $\mu_n = G_n F_n$. There follows immediately

$$G_n = G_{n-2}\, \omega^2\left(\frac{n}{2}+h-1\right) \quad \text{for } n \geq 2 . \tag{E-2}$$

For the ${}_1F_1$ function, we refer to [5, 13.4.1] to get

$$F_n = \frac{1}{\frac{n}{2}+h-1}\left[(n+2h-3-\gamma+z)F_{n-2} + \left(\gamma+2-\frac{n}{2}-h\right)F_{n-4}\right] . \tag{E-3}$$

If we substitute (E-2) and (E-3) into $\mu_n = G_n F_n$, and then re-apply (E-2) in the second term, we obtain

$$\mu_n = \omega^2(n+\gamma-2+z)\,\mu_{n-2} - \frac{1}{4}\omega^4\left[(n+\gamma-3)^2 - \gamma^2\right]\mu_{n-4} ; \tag{E-4}$$

we also eliminated h. Starting values for μ_n can be obtained from (E-1).

For the special case (143) and (144), (E-4) reduces to

$$\mu_n = \omega^2(n-1+z)\,\mu_{n-2} - \frac{1}{4}\omega^4(n-2)^2\,\mu_{n-4} , \tag{E-5}$$

with starting values

$$\mu_0 = 1, \quad \mu_1 = \frac{1}{2}\,\pi^{1/2}\,\omega\, e^{-z}\; {}_1F_1\left(\frac{3}{2};1;z\right) ,$$

$$\mu_2 = \omega^2(1+z), \quad \mu_3 = \frac{3}{4}\,\pi^{1/2}\,\omega^3\, e^{-z}\; {}_1F_1\left(\frac{5}{2};1;z\right) . \qquad \text{(E-6)}$$

Kummer's transformation [5, 13.1.27] was employed in this last equation; these forms afford accurate starting values for recursion (E-5).

APPENDIX F. PROGRAM LISTINGS

Eight programs are listed in this appendix. They are given in BASIC for the Hewlett Packard 9000 Model 520 computer. For ease of reference, a shorthand notation is adopted:

P	denotes	cumulative or exceedance distribution function
p	denotes	probability density function
H	denotes	Hermite expansion
L	denotes	generalized Laguerre expansion
RC	denotes	recursively via cumulants
DM	denotes	directly via moments
RM	denotes	recursively via moments

Table F-1. Shorthand Notation

Then, for example, the combination PHRC means that this program yields the cumulative or exceedance distribution function in terms of a Hermite expansion, by means of expansion coefficients determined recursively via cumulants. The eight programs listed here are, in order,

PHRC	Figures 7 and 8
pHRC	Figures 7, 9, and 10
PHDMandRM	Figures 7 (and 8)
pHDMandRM	Figures 7 (and 9, 10)
PLRC	Figures 19 and 20
pLRC	Figures 19, 21, and 22
PLDMandRM	Figures 19 (and 20)
pLDMandRM	Figures 19 (and 21, 22)

Table F-2. Program Abbreviations

The combination DMandRM means that this program gives the expansion coefficients directly via moments as well as recursively via moments; the user must select the procedure of interest.

The only input statistics we have given a listing for here is the shot noise process used in examples D and I; in particular, the cumulant and moment routines are listed at the very end of PHRC and PHDMandRM, respectively. The figure references given in table F-2 indicate where each particular program was used in this report; the parenthetical references are alternative ways of generating those figures. The remaining figures in this report require that the cumulant and moment subroutines be replaced by the appropriate statistics of interest.

To save space, no subroutines are listed more than once; instead, comments are made indicating where the needed routines are located, according to the coding in table F-2. For example, in program PHDMandRM, function subprogram FNPhi, line 570, the comment is made that this routine has already been listed in PHRC.

We now explain some of the details of the PHRC program, as an example, so that a user can apply these techniques and routines to his particular problem. The user must specify M in line 30, which is the maximum order of approximation desired, or the number of cumulants or moments that can be calculated. The notation DOUBLE in line 40 denotes INTEGER variables. The user must select α and β in lines 130,140; if they are chosen equal to α_0, β_0 which have been computed in lines 110,120, then expansion coefficients $a_1 = a_2 = 0$, or equivalently $b_1 = b_2 = 0$. However, this choice is recommended only as a starter on the search in α,β space.

The CALL in line 150 is to the subroutine which calculates the expansion coefficients for a Hermite series, recursively via cumulants, as can be deciphered from the abbreviated subroutine title. The expansion coefficients $\{b_n\}$ are calculated and the running sum of b_n^2 is calculated, both of which are printed on the CRT vs n. Also, a plot of the expansion coefficients $\{b_n\}$ is made in this subroutine, from which the user must decide on the order, N, to employ in the approximate cumulative and exceedance distribution function; alternatively, he can reject the sequence of $\{b_n\}$ so obtained, and re-run the program with different α,β values.

When a satisfactory α, β pair is obtained, the limits u_1, u_2 on the range of arguments of the distribution must also be specified; this selection is aided by the print-out of the center and rms width of the density under investigation. A plot of 100 values of the cumulative and exceedance distribution functions is then made on a logarithmic ordinate. The various subroutines are self-explanatory and are keyed to the equation numbers in this report.

PROGRAM PHRC

```
10  !  STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u)   TR 7377, FIGURE 8
20  !  COEFFICIENTS OF HERMITE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30     M=80   !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40     DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50     REDIM Cum(0:M),A(0:M),He(0:M)
60     REAL Cum(0:100),A(0:100),He(0:100),P(0:100)
70     CALL Cumulants(M,P0,Cum(*))       !  P0 IS STEP AT ORIGIN
80     Center=Cum(1)                     !  CENTER OF PDF pc(u)
90     R2=Cum(2)                         !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110    Alpha0=Center                     !  THE CHOICES   Alpha=Alpha0   AND
120    Beta0=Rms                         !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=Center
140      Beta=Rms*1.5
150    CALL Coeffhr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160    PRINT "Center = ";Center
170    PRINT "Rms =";Rms
180    F1=1./SQR(2.*PI)
190      INPUT "ORDER AND LIMITS:",N,U1,U2
200    PRINT "ORDER AND LIMITS:",N;U1;U2
210    Du=(U2-U1)/100.
220    PLOTTER IS "GRAPHICS"
230    GRAPHICS ON
240    WINDOW U1,U2,-10.,0.
250    GRID Du*10.,1.
260    FOR I=0 TO 100
270    U=U1+Du*I
280    T=(U-Alpha)/Beta
290    CALL H    te(N,T,He(*))
300    Sum=0.
310    FOR K=1 TO N
320    Sum=Sum+A(K)*He(K-1)
330    NEXT K
340    P=A(0)*FNPhi(T)-F1*EXP(-.5*T*T)*Sum     !  PROBABILITY THAT RV < U
350    IF U>=0. THEN P=P+P0                    !  ADDITION OF STEP AT ORIGIN
360    P(I)=P
370    IF P>0. THEN 400
380    PENUP
390    GOTO 410
400    PLOT U,LGT(P)
410    NEXT I
420    PENUP
430    FOR I=0 TO 100
440    U=U1+Du*I
450    P1=1.-P(I)
460    IF P1>0. THEN 490
470    PENUP
480    GOTO 500
490    PLOT U,LGT(P1)
500    NEXT I
510    PENUP
520    GOTO 190
530    END
540    !
```

PROGRAM PHRC (cont'd)

```
   550     DEF FNPhi(X)          !  HART, page 140, #5708 & #5725              eq. 41
   560     Y=ABS(SQR(.5)*X)
   570     SELECT Y
   580     CASE <8.
   590     P=1631.76026875371470+Y*(456.26145870609263l+Y*(86.0827622119485951+Y*
(10.0648589749095425+Y*.564189586761813614)))
   600     P=3723.5079815548067 2+Y*(7113.663246954049 87+Y*(6758.21696411048589+Y*
(4032.26701083004974+Y*P)))
   610     Q=7542.47951019347576+Y*(2968.00490148230872+Y*(817.622386304544077+Y*
(153.077710750362216+Y*(17.8394984391395565+Y))))
   620     Q=3723.5079815548065 4+Y*(11315.192081854405 5+Y*(15802.535999402042 5+Y*
(13349.34656128445 74+Y*Q)))
   630     Phi=.5*EXP(-Y*Y)*P/Q
   640     CASE <26.6
   650     P=2.97886562639399289+Y*(7.40974060596474179+Y*(6.160209853109630 54+Y*
(5.019049726784267 46+Y*(1.275366644729965 95+Y*.564189583547755074))))
   660     Q=3.36907520698275277+Y*(9.608965327192787 87+Y*(17.0814407474660043 +Y*
(12.0489519278551290+Y*(9.39603401623505415+Y*(2.26052852076732697+Y)))))
   670     Phi=.5*EXP(-Y*Y)*P/Q
   680     CASE ELSE
   690     Phi=0.
   700     END SELECT
   710     IF X>0. THEN Phi=1.-Phi
   720     RETURN Phi
   730     FNEND
   740     !
   750     SUB Hermite(DOUBLE N,REAL X,He(*))     !   He/n(X)                  eq. 50
   760     DOUBLE K
   770     He(0)=1.
   780     He(1)=X
   790     FOR K=2 TO N
   800     He(K)=X*He(K-1)-(K-1)*He(K-2)
   810     NEXT K
   820     SUBEND
   830     !
   840     SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))              !  eq. A-6
   850     DOUBLE K,J
   860     REAL Mom0
   870     Mom(0)=Mom0=EXP(Cum(0))
   880     FOR K=1 TO M
   890     T=1.
   900     S=Cum(K)*Mom0
   910     FOR J=1 TO K-1
   920     T=T*(K-J)/J
   930     S=S+T*Cum(K-J)*Mom(J)
   940     NEXT J
   950     Mom(K)=S
   960     NEXT K
   970     SUBEND
   980     !
```

PROGRAM PHRC (cont'd)

```
 990     SUB Cumnt_via_momnt(DOUBLE M,REAL Mom(*),Cum(*))              !   eq. A-7
1000     DOUBLE K,J
1010     REAL Mom0
1020     Mom0=Mom(0)
1030     Cum(0)=LOG(Mom0)
1040     FOR K=1 TO M
1050     T=1.
1060     S=Mom(K)
1070     FOR J=1 TO K-1
1080     T=T*(K-J)/J
1090     S=S-T*Mom(J)*Cum(K-J)
1100     NEXT J
1110     Cum(K)=S/Mom0
1120     NEXT K
1130     SUBEND
1140     !
1150     SUB Coeffhr_via_cum(DOUBLE M,REAL Alpha,Beta,Cum(*),A(*))
1160     ALLOCATE B(0:M)
1170     DOUBLE K,J,Mx
1180     F=Beta*Beta
1190     Cum(1)=(Cum(1)-Alpha)/Beta        !  MODIFIED NORMALIZED
1200     Cum(2)=Cum(2)/F-1.                !  CUMULANTS FOR K=1 & 2; eq. 63
1210     FOR K=3 TO M
1220     F=F*Beta*(K-1)
1230     Cum(K)=Cum(K)/F                   !  NORMALIZED CUMULANTS; eq. 62
1240     NEXT K
1250     A(0)=B(0)=EXP(Cum(0))
1260     F=1.
1270     FOR K=1 TO M
1280     S=0.
1290     FOR J=1 TO K
1300     S=S+Cum(J)*A(K-J)
1310     NEXT J
1320     A(K)=S/K
1330     F=F*K
1340     B(K)=A(K)*SQR(F)
1350     NEXT K
1360     Mx=Mx+10
1370     IF Mx<M THEN 1360
1380     Threshold=-7.
1390     T2=Threshold*2.
1400     V=10.^Threshold
1410     GINIT
1420     PLOTTER IS "GRAPHICS"
1430     GRAPHICS ON
1440     WINDOW 0.,FLT(Mx),T2,0.
1450     LINE TYPE 3
```

PROGRAM PHRC (cont'd)

```
1460    FOR J=0 TO Mx STEP 10
1470    MOVE J,T2
1480    DRAW J,0.
1490    NEXT J
1500    FOR J=T2 TO 0
1510    MOVE 0.,J
1520    DRAW Mx,J
1530    NEXT J
1540    PENUP
1550    LINE TYPE 1
1560    IMAGE 4D,2(4X,M.17DE)
1570    PRINT "   K                B(K)                              Sum"
1580    Sum=0.
1590    FOR K=0 TO M
1600    B=B(K)
1610    Sum=Sum+B*B
1620    PRINT USING 1560;K,B,Sum
1630    IF B<V THEN 1660
1640    Y=LGT(B)
1650    GOTO 1700
1660    IF B>-V THEN 1690
1670    Y=T2-LGT(-B)
1680    GOTO 1700
1690    Y=Threshold
1700    PLOT K,Y
1710    NEXT K
1720    PENUP
1730    SUBEND
1740    !
1750    SUB Cumulants(DOUBLE M,REAL P0,Cum(*))   !  SHOT NOISE   eqs. 147-150
1760    Overlap=6.2              !  AV. NO. PULSES/SEC * AVERAGE PULSE DURATION
1770    Sigmaa=1.                !  PARAMETER OF RAYLEIGH AMPLITUDE PDF
1780    P0=EXP(-Overlap)         !  PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
1790    ALLOCATE Mom(0:M)        !  ARRAY FOR MOMENTS
1800    DOUBLE K
1810    S=Sigmaa*Sigmaa
1820    Cum(0)=0.
1830    Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
1840    Cum(2)=Overlap*S*4./3.
1850    FOR K=3 TO M
1860    Cum(K)=Cum(K-2)*S*K*K/(K+1)
1870    NEXT K
1880    CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
1890    Mom(0)=Mom(0)-P0         !  MOMENT CORRECTION FOR IMPULSE AT ORIGIN
1900    CALL Cumnt_via_momnt(M,Mom(*),Cum(*))
1910    SUBEND
```

PROGRAM pHRC

```
10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u)              TR 7377, FIGURE 9
20 !  COEFFICIENTS OF HERMITE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30    M=80   !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Cum(0:M),A(0:M),He(0:M)
60    REAL Cum(0:100),A(0:100),He(0:100)
70    CALL Cumulants(M,P0,Cum(*))       !  P0 IS STEP AT ORIGIN
80    Center=Cum(1)                     !  CENTER OF PDF pc(u)
90    R2=Cum(2)                         !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center                     !  THE CHOICES   Alpha=Alpha0   AND
120   Beta0=Rms                         !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=Center
140     Beta=Rms*1.5
150   CALL Coeffhr_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160   PRINT "Center = ";Center
170   PRINT "Rms =";Rms
180   F1=1./(Beta*SQR(2.*PI))
190     INPUT "ORDER AND LIMITS:",N,U1,U2
200   PRINT "ORDER AND LIMITS:",N;U1;U2
210   Du=(U2-U1)/100.
220   PLOTTER IS "GRAPHICS"
230   GRAPHICS ON
240   WINDOW U1,U2,0.,.15
250   GRID 6.,.03
260   FOR I=0 TO 100
270   U=U1+Du*I
280   T=(U-Alpha)/Beta
290   CALL Hermite(N,T,He(*))
300   Sum=A(0)
310   FOR K=1 TO N
320   Sum=Sum+A(K)*He(K)
330   NEXT K
340   P=F1*EXP(-.5*T*T)*Sum             !    PDF OF RV AT U
350   PLOT U,P
360   NEXT I
370   PENUP
380   GOTO 190
390   END
400 ! USE ROUTINES IN PHRC
```

PROGRAM PHDMandRM

```
10  ! STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);  COEFFICIENTS OF
20  ! HERMITE EXPANSION FOUND DIRECTLY VIA MOMENTS OR RECURSIVELY VIA MOMENTS
30    M=80     !   MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Mom(0:M),A(0:M),He(0:M)
60    REAL Mom(0:100),A(0:100),He(0:100),P(0:100)
70    CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
80    Center=Mom(1)/Mom(0)              !  CENTER OF PDF pc(u)
90    R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center                     !  THE CHOICES   Alpha=Alpha0   AND
120   Beta0=Rms                         !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=Center
140     Beta=Rms*1.5
150   CALL Coeffhd_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   DM
160 ! CALL Coeffhr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   F1=1./SQR(2.*PI)
200     INPUT "ORDER AND LIMITS:",N,U1,U2
210   PRINT "ORDER AND LIMITS:",N;U1;U2
220   Du=(U2-U1)/100.
230   PLOTTER IS "GRAPHICS"
240   GRAPHICS ON
250   WINDOW U1,U2,-10.,0.
260   GRID Du*10.,1.
270   FOR I=0 TO 100
280   U=U1+Du*I
290   T=(U-Alpha)/Beta
300   CALL Hermite(N,T,He(*))
310   Sum=0.
320   FOR K=1 TO N
330   Sum=Sum+A(K)*He(K-1)
340   NEXT K
350   P=A(0)*FNPhi(T)-F1*EXP(-.5*T*T)*Sum     !   PROBABILITY THAT RV < U
360   IF U>=0. THEN P=P+P0                    !   ADDITION OF STEP AT ORIGIN
370   P(I)=P
380   IF P>0. THEN 410
390   PENUP
400   GOTO 420
410   PLOT U,LGT(P)
420   NEXT I
430   PENUP
440   FOR I=0 TO 100
450   U=U1+Du*I
460   P1=1.-P(I)
470   IF P1>0. THEN 500
480   PENUP
490   GOTO 510
500   PLOT U,LGT(P1)
510   NEXT I
520   PENUP
530   GOTO 200
540   END
550   !
```

PROGRAM PHDMandRM (cont'd)

```
  560   DEF FNPhi(X)          !  HART, page 140, #5708 & #5725
  570 ! LISTED IN PHRC
  740   FNEND
  750   !
  760   SUB Hermite(DOUBLE N,REAL X,He(*))     !   He/n(X)
  770 ! LISTED IN PHRC
  830   SUBEND
  840   !
  850   SUB Hermite_i(DOUBLE N,REAL X,Hi(*)) ! Hi/n(X)=(-i)^n He/n(iX) eq.74-5
  860   DOUBLE K                              !    MODIFIED HERMITE POLYNOMIALS
  870   Hi(0)=1.
  880   Hi(1)=X
  890   FOR K=2 TO N
  900   Hi(K)=X*Hi(K-1)+(K-1)*Hi(K-2)
  910   NEXT K
  920   SUBEND
  930   !
  940   SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
  950 ! LISTED IN PHRC
 1070   SUBEND
 1080   !
 1090   SUB Coeffhd_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
 1100   ALLOCATE He(0:M),F(0:M),B(0:M)
 1110   DOUBLE K,J,Mx
 1120   CALL Hermite(M,-Alpha/Beta,He(*))
 1130   T=F(0)=1.
 1140   FOR K=1 TO M
 1150   F=F(K)=F(K-1)*K
 1160   T=T*Beta
 1170   He(K)=He(K)/F              !   NORMALIZED HERMITE POLYNOMIALS; eq. 68
 1180   Mom(K)=Mom(K)/(F*T)        !   NORMALIZED MOMENTS re Beta; eq. 69
 1190   NEXT K
 1200   FOR K=0 TO M
 1210   S=0.
 1220   FOR J=0 TO K
 1230   S=S+He(J)*Mom(K-J)
 1240   NEXT J
 1250   A(K)=S
 1260   NEXT K
 1270   MAT F=SQR(F)
 1280   MAT B=A.F
 1290   Mx=Mx+10
 1300   IF Mx<M THEN 1290
 1310   Threshold=-7.
 1320   T2=Threshold*2.
 1330   V=10.^Threshold
 1340   GINIT
 1350   PLOTTER IS "GRAPHICS"
 1360   GRAPHICS ON
 1370   WINDOW 0.,FLT(Mx),T2,0.
 1380   LINE TYPE 3
```

PROGRAM PHDMandRM (cont'd)

```
1390    FOR J=0 TO Mx STEP 10
1400    MOVE J,T2
1410    DRAW J,0.
1420    NEXT J
1430    FOR J=T2 TO 0
1440    MOVE 0.,J
1450    DRAW Mx,J
1460    NEXT J
1470    PENUP
1480    LINE TYPE 1
1490    IMAGE 4D,2(4X,M.17DE)
1500    PRINT "   K                B(K)                                Sum"
1510    Sum=0.
1520    FOR K=0 TO M
1530    B=B(K)
1540    Sum=Sum+B*B
1550    PRINT USING 1490;K,B,Sum
1560    IF B<V THEN 1590
1570    Y=LGT(B)
1580    GOTO 1630
1590    IF B>-V THEN 1620
1600    Y=T2-LGT(-B)
1610    GOTO 1630
1620    Y=Threshold
1630    PLOT K,Y
1640    NEXT K
1650    PENUP
1660    SUBEND
1670    !
1680    SUB Coeffhr_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1690    ALLOCATE Hi(0:M),F(0:M),B(0:M)
1700    DOUBLE K,J,Mx
1710    CALL Hermite_i(M,Alpha/Beta,Hi(*))
1720    T=F(0)=1.
1730    FOR K=1 TO M
1740    F=F(K)=F(K-1)*K
1750    T=T*Beta
1760    Hi(K)=Hi(K)/F  ! NORMALIZED MODIFIED HERMITE POLYNOMIALS; eqs. 80 & 74
1770    Mom(K)=Mom(K)/(F*T)          !   NORMALIZED MOMENTS re Beta; eq. 69
1780    NEXT K
1790    FOR K=0 TO M
1800    S=Mom(K)
1810    FOR J=1 TO K
1820    S=S-Hi(J)*A(K-J)
1830    NEXT J
1840    A(K)=S
1850    NEXT K
1860    MAT F=SQR(F)
1870    MAT B=A.F
```

PROGRAM PHDMandRM (cont'd)

```
1880    Mx=Mx+10
1890    IF Mx<M THEN 1880
1900    Threshold=-7.
1910    T2=Threshold*2.
1920    V=10.^Threshold
1930    GINIT
1940    PLOTTER IS "GRAPHICS"
1950    GRAPHICS ON
1960    WINDOW 0.,FLT(Mx),T2,0.
1970    LINE TYPE 3
1980    FOR J=0 TO Mx STEP 10
1990    MOVE J,T2
2000    DRAW J,0.
2010    NEXT J
2020    FOR J=T2 TO 0
2030    MOVE 0.,J
2040    DRAW Mx,J
2050    NEXT J
2060    PENUP
2070    LINE TYPE 1
2080    IMAGE 4D,2(4X,M.17DE)
2090    PRINT "   K                B(K)                              Sum"
2100    Sum=0.
2110    FOR K=0 TO M
2120    B=B(K)
2130    Sum=Sum+B*B
2140    PRINT USING 2080;K,B,Sum
2150    IF B<V THEN 2180
2160    Y=LGT(B)
2170    GOTO 2220
2180    IF B>-V THEN 2210
2190    Y=T2-LGT(-B)
2200    GOTO 2220
2210    Y=Threshold
2220    PLOT K,Y
2230    NEXT K
2240    PENUP
2250    SUBEND
2260    !
2270    SUB Moments(DOUBLE M,REAL P0,Cum(*))    !  SHOT NOISE    eqs. 147-150
2280    Overlap=6.2              !  AV. NO. PULSES/SEC * AVERAGE PULSE DURATION
2290    Sigmaa=1.                !  PARAMETER OF RAYLEIGH AMPLITUDE PDF
2300    P0=EXP(-Overlap)         !  PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
2310    ALLOCATE Cum(0:M)        !  ARRAY FOR CUMULANTS
2320    DOUBLE K
2330    S=Sigmaa*Sigmaa
2340    Cum(0)=0.
2350    Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
2360    Cum(2)=Overlap*S*4./3.
2370    FOR K=3 TO M
2380    Cum(K)=Cum(K-2)*S*K*K/(K+1)
2390    NEXT K
2400    CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
2410    Mom(0)=Mom(0)-P0         !  MOMENT CORRECTION FOR IMPULSE AT ORIGIN
2420    SUBEND
```

PROGRAM pHDMandRM

```
10  !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u); COEFFICIENTS OF HERMITE
20  !  EXPANSION FOUND DIRECTLY VIA MOMENTS OR RECURSIVELY VIA MOMENTS
30     M=80     !   MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40     DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50     REDIM Mom(0:M),A(0:M),He(0:M)
60     REAL Mom(0:100),A(0:100),He(0:100)
70     CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
80     Center=Mom(1)/Mom(0)              !  CENTER OF PDF pc(u)
90     R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110    Alpha0=Center                     !  THE CHOICES   Alpha=Alpha0   AND
120    Beta0=Rms                         !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=Center
140      Beta=Rms*1.5
150    CALL Coeffhd_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   DM
160 !  CALL Coeffhr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   RM
170    PRINT "Center = ";Center
180    PRINT "Rms =";Rms
190    F1=1./(Beta*SQR(2.*PI))
200      INPUT "ORDER AND LIMITS:",N,U1,U2
210    PRINT "ORDER AND LIMITS:",N;U1;U2
220    Du=(U2-U1)/100.
230    PLOTTER IS "GRAPHICS"
240    GRAPHICS ON
250    WINDOW U1,U2,0.,.15
260    GRID 6.,.03
270    FOR I=0 TO 100
280    U=U1+Du*I
290    T=(U-Alpha)/Beta
300    CALL Hermite(N,T,He)
310    Sum=A(0)
320    FOR K=1 TO N
330    Sum=Sum+A(K)*He(K)
340    NEXT K
350    P=F1*EXP(-.5*T*T)*Sum                !    PDF OF RV AT U
360    PLOT U,P
370    NEXT I
380    PENUP
390    GOTO 200
400    END
410 !  USE ROUTINES IN PHDM&RM
```

PROGRAM PLRC

```
 10 ! STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);  TR 7377, FIGURE 20
 20 ! COEFFICIENTS OF GEN. LAGUERRE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
 30    M=70   !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
 40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
 50    REDIM Cum(0:M),A(0:M),L(0:M)
 60    REAL Cum(0:100),A(0:100),L(0:100),P(0:100)
 70    CALL Cumulants(M,P0,Cum(*))       !  P0 IS STEP AT ORIGIN
 80    Center=Cum(1)                     !  CENTER OF PDF pc(u)
 90    R2=Cum(2)                         !  MEAN SQUARE SPREAD OF pc(u)
100    Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110    Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120    Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130      Alpha=.74
140      Beta=2.1
150    CALL Coeffir_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160    PRINT "Center = ";Center
170    PRINT "Rms =";Rms
180    A1=Alpha+1.
190    O1=1./A1
200    F1=1./FNGamma(A1)
210      INPUT "ORDER AND LIMITS:",N,U1,U2
220    PRINT "ORDER AND LIMITS:",N;U1;U2
230    Du=(U2-U1)/100.
240    PLOTTER IS "GRAPHICS"
250    GRAPHICS ON
260    WINDOW U1,U2,-11.,0.
270    GRID 4.,1.
280    P(0)=P0
290    PLOT 0.,LGT(P0)
300    FOR I=1 TO 100
310    U=U1+Du*I
320    T=U/Beta
330    CALL Laguerre(N-1,A1,T,L(*))
340    Sum=A(0)*FNF1(A1,T)*O1
350    FOR K=1 TO N
360    Sum=Sum+A(K)*L(K-1)/K
370    NEXT K
380    P(I)=P=P+F1*EXP(-T+A1*LOG(T))*Sum    !  PROBABILITY THAT RV < U
390    IF P>0. THEN 420
400    PENUP
410    GOTO 430
420    PLOT U,LGT(P)
430    NEXT I
440    PENUP
450    FOR I=0 TO 100
460    U=U1+Du*I
470    P1=1.-P(I)
480    IF P1>0. THEN 510
490    PENUP
500    GOTO 520
510    PLOT U,LGT(P1)
520    NEXT I
530    PENUP
540    GOTO 210
550    END
560    !
```

PROGRAM PLRC (cont'd)

```
570     DEF FNGamma(X)  !  Gamma(X) via HART, page 282, #5243            eq. 2
580     DOUBLE N,K
590     N=INT(X)
600     R=X-N
610     IF N>0 OR R<>0. THEN 640
620     PRINT "FNGamma(X) IS NOT DEFINED FOR X = ";X
630     STOP
640     IF R>0. THEN 670
650     Gamma2=1.
660     GOTO 740
670     P=439.330444060025676+R*(50.1086937529709530+R*6.74495072459252899)
680     P=8762.71029785214896+R*(2008.52740130727912+R*P)
690     P=42353.6895097440896+R*(20886.8617892698874+R*P)
700     Q=499.028526621439048-R*(189.498234157028016-R*(23.081551524580125-R))
710     Q=9940.30741508277090-R*(1528.60727377952202+R*Q)
720     Q=42353.6895097440900+R*(2980.38533092566499-R*Q)
730     Gamma2=P/Q                      !  Gamma(2+R) for  0 < R < 1
740     IF N>2 THEN 780
750     IF N<2 THEN 830
760     Gamma=Gamma2
770     RETURN Gamma
780     Gamma=Gamma2
790     FOR K=1 TO N-2
800     Gamma=Gamma*(X-K)
810     NEXT K
820     RETURN Gamma
830     R=1.
840     FOR K=0 TO 1-N
850     R=R*(X+K)
860     NEXT K
870     Gamma=Gamma2/R
880     RETURN Gamma
890     FNEND
900     !
910     DEF FNF1(A1,X)                  !  1F1(1;A1+1;X)                  eq. C-2
920     DOUBLE K
930     T=S=1.
940     FOR K=1 TO 200
950     T=T*X/(A1+K)
960     S=S+T
970     IF T<=1.E-17*S THEN RETURN S
980     NEXT K
990     PRINT "200 TERMS IN FNF1 AT";A1;X
1000    RETURN S
1010    FNEND
1020    !
```

PROGRAM PLRC (cont'd)

```
1030    SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*))                    !  eq. 96
1040    DOUBLE K
1050    A1=Alpha-1.
1060    L(0)=1.
1070    L(1)=Alpha+1.-X
1080    FOR K=2 TO N
1090    L(K)=((K+K+A1-X)*L(K-1)-(K+A1)*L(K-2))/K
1100    NEXT K
1110    SUBEND
1120    !
1130    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
1140  ! LISTED IN PHRC
1260    SUBEND
1270    !
1280    SUB Cumnt_via_momnt(DOUBLE M,REAL Mom(*),Cum(*))
1290  ! LISTED IN PHRC
1420    SUBEND
1430    !
1440    SUB Coeffir_via_cum(DOUBLE M,REAL Alpha,Beta,Cum(*),A(*))
1450    ALLOCATE B(0:M),C(0:M),D(1:M)
1460    DOUBLE K,J,J1,Mx
1470    T=Beta
1480    Cum(1)=Cum(1)/T
1490    FOR K=2 TO M
1500    T=T*Beta*(K-1)
1510    Cum(K)=Cum(K)/T          !   NORMALIZED CUMULANTS; eq. 62
1520    NEXT K
1530    A1=Alpha+1.
1540    FOR J=1 TO M
1550    J1=J+1
1560    T=1.
1570    S=A1
1580    FOR K=1 TO J
1590    T=T*(K-J1)/K
1600    S=S+T*Cum(K)
1610    NEXT K
1620    D(J)=S
1630    NEXT J
1640    A(0)=B(0)=C(0)=EXP(Cum(0))
1650    Q=1.
1660    FOR K=1 TO M
1670    S=0.
1680    FOR J=1 TO K
1690    S=S+D(J)*C(K-J)
1700    NEXT J
1710    C(K)=C=S/K
1720    Q=Q*K/(Alpha+K)
1730    A(K)=C*Q
1740    B(K)=C*SQR(Q)
1750    NEXT K
```

PROGRAM PLRC (cont'd)

```
1760    Mx=Mx+10
1770    IF Mx<M THEN 1760
1780    Threshold=-7.
1790    T2=Threshold*2.
1800    V=10.^Threshold
1810    GINIT
1820    PLOTTER IS "GRAPHICS"
1830    GRAPHICS ON
1840    WINDOW 0.,FLT(Mx),T2,0.
1850    LINE TYPE 3
1860    FOR J=0 TO Mx STEP 10
1870    MOVE J,T2
1880    DRAW J,0.
1890    NEXT J
1900    FOR J=T2 TO 0
1910    MOVE 0.,J
1920    DRAW Mx,J
1930    NEXT J
1940    PENUP
1950    LINE TYPE 1
1960    IMAGE 4D,2(4X,M.17DE)
1970    PRINT "   K                B(K)                              Sum"
1980    Sum=0.
1990    FOR K=0 TO M
2000    B=B(K)
2010    Sum=Sum+B*B
2020    PRINT USING 1960;K,B,Sum
2030    IF B<V THEN 2060
2040    Y=LGT(B)
2050    GOTO 2100
2060    IF B>-V THEN 2090
2070    Y=T2-LGT(-B)
2080    GOTO 210
2090    Y=Threshold
2100    PLOT K,Y
2110    NEXT K
2120    PENUP
2130    SUBEND
2140    !
2150    SUB Cumulants(DOUBLE M,REAL P0,Cum(*))       !  SHOT NOISE
2160  ! LISTED IN PHRC
2310    SUBEND
```

PROGRAM pLRC

```
10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u)              TR 7377, FIGURE 21
20 !  COEFFS. OF GENERAL. LAGUERRE EXPANSION FOUND RECURSIVELY VIA CUMULANTS
30    M=70    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF CUMULANTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Cum(0:M),A(0:M),L(0:M)
60    REAL Cum(0:100),A(0:100),L(0:100)
70    CALL Cumulants(M,P0,Cum(*))       !  P0 IS STEP AT ORIGIN
80    Center=Cum(1)                     !  CENTER OF PDF pc(u)
90    R2=Cum(2)                         !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0   AND
120   Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=.74
140     Beta=2.1
150   CALL Coeffir_via_cum(M,Alpha,Beta,Cum(*),A(*))     !   RC
160   PRINT "Center = ";Center
170   PRINT "Rms =";Rms
180   F1=1./(Beta*FNGamma(Alpha+1.))
190     INPUT "ORDER AND LIMITS:",N,U1,U2
200   PRINT "ORDER AND LIMITS:",N;U1;U2
210   Du=(U2-U1)/100.
220   PLOTTER IS "GRAPHICS"
230   GRAPHICS ON
240   WINDOW U1,U2,0.,.15
250   GRID 6.,.03
260   FOR I=0 TO 100
270   U=U1+Du*I
280   IF U<0. THEN 400
290   IF U>0. THEN 320
300   PLOT 0.,0.
310   GOTO 400
320   T=U/Beta
330   CALL Laguerre(N,Alpha,T,L(*))
340   Sum=A(0)
350   FOR K=1 TO N
360   Sum=Sum+A(K)*L(K)
370   NEXT K
380   P=F1*EXP(-T+Alpha*LOG(T))*Sum     !   PDF OF RV AT U
390   PLOT U,P
400   NEXT I
410   PENUP
420   GOTO 190
430   END
440 ! USE ROUTINES IN PLRC
```

PROGRAM PLDMandRM

```
10 !  STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u);    COEFFICIENTS OF
20 !  GENERALIZED LAGUERRE EXPAN. FOUND DIRECTLY AND RECURSIVELY VIA MOMENTS
30    M=70    !  MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Mom(0:M),A(0:M),L(0:M)
60    REAL Mom(0:100),A(0:100),L(0:100),P(0:100)
70    CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
80    Center=Mom(1)/Mom(0)              !  CENTER OF PDF pc(u)
90    R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120   Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=.74
140     Beta=2.1
150   CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   DM
160 ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   A1=Alpha+1.
200   O1=1./A1
210   F1=1./FNGamma(A1)
220     INPUT "ORDER AND LIMITS:",N,U1,U2
230   PRINT "ORDER AND LIMITS:",N;U1;U2
240   Du=(U2-U1)/100.
250   PLOTTER IS "GRAPHICS"
260   GRAPHICS ON
270   WINDOW U1,U2,-11.,0
280   GRID 4.,1.
290   P(0)=P0
300   PLOT 0.,LGT(P0)
310   FOR I=1 TO 100
320   U=U1+Du*I
330   T=U/Beta
340   CALL Laguerre(N-1,A1,T,L(*))
350   Sum=A(0)*FNF1(A1,T)*O1
360   FOR K=1 TO N
370   Sum=Sum+A(K)*L(K-1)/K
380   NEXT K
390   P(I)=P=P0+F1*EXP(-T+A1*LOG(T))*Sum    !   PROBABILITY THAT RV < U
400   IF P>0. THEN 430
410   PENUP
420   GOTO 440
430   PLOT U,LGT(P)
440   NEXT I
450   PENUP
460   FOR I=0 TO 100
470   U=U1+Du*I
480   P1=1.-P(I)
490   IF P1>0. THEN 520
500   PENUP
510   GOTO 530
520   PLOT U,LGT(P1)
530   NEXT I
540   PENUP
550   GOTO 220
560   END
570   !
```

PROGRAM PLDMandRC (cont'd)

```
  580    DEF FNGamma(X)  !  Gamma(X) via HART, page 282, #5243
  590  ! LISTED IN PLRC
  900    FNEND
  910    !
  920    DEF FNF1(A1,X)                    !  1F1(1;A1+1;X)
  930  ! LISTED IN PLRC
 1020    FNEND
 1030    !
 1040    SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*))  !  Ln\alpha(X)
 1050  ! LISTED IN PLRC
 1120    SUBEND
 1130    !
 1140    SUB Momnt_via_cumnt(DOUBLE M,REAL Cum(*),Mom(*))
 1150  ! LISTED IN PHRC
 1270    SUBEND
 1280    !
 1290    SUB Coeffld_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
 1300    ALLOCATE B(0:M)
 1310    DOUBLE K,K1,J,Mx
 1320    T=1.
 1330    FOR K=1 TO M
 1340    T=T*(Alpha+K)*Beta                !  NORMALIZED MOMENTS re
 1350    Mom(K)=Mom(K)/T                   !  Alpha and Beta; eq. 118
 1360    NEXT K
 1370    Q=1.
 1380    A(0)=B(0)=Mom(0)
 1390    FOR K=1 TO M
 1400    K1=K+1
 1410    T=1.
 1420    S=Mom(0)
 1430    FOR J=1 TO K
 1440    T=T*(J-K1)/J
 1450    S=S+T*Mom(J)
 1460    NEXT J
 1470    Q=Q*(Alpha+K)/K
 1480    A(K)=S
 1490    B(K)=S*SQR(Q)
 1500    NEXT K
 1510    Mx=Mx+10
 1520    IF Mx<M THEN 1510
 1530    Threshold=-7.
 1540    T2=Threshold*2.
 1550    V=10.^Threshold
 1560    GINIT
 1570    PLOTTER IS "GRAPHICS"
 1580    GRAPHICS ON
 1590    WINDOW 0.,FLT(Mx),T2,0.
 1600    LINE TYPE 3
```

PROGRAM PLDMandRM (cont'd)

```
1610    FOR J=0 TO Mx STEP 10
1620    MOVE J,T2
1630    DRAW J,0.
1640    NEXT J
1650    FOR J=T2 TO 0
1660    MOVE 0.,J
1670    DRAW Mx,J
1680    NEXT J
1690    PENUP
1700    LINE TYPE 1
1710    IMAGE 4D,2(4X,M.17DE)
1720    PRINT "   K                B(K)                                Sum"
1730    Sum=0.
1740    FOR K=0 TO M
1750    B=B(K)
1760    Sum=Sum+B*B
1770    PRINT USING 1710;K,B,Sum
1780    IF B<V THEN 1810
1790    Y=LGT(B)
1800    GOTO 1850
1810    IF B>-V THEN 1840
1820    Y=T2-LGT(-B)
1830    GOTO 1850
1840    Y=Threshold
1850    PLOT K,Y
1860    NEXT K
1870    PENUP
1880    SUBEND
1890    !
1900    SUB Coeffir_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1910    ALLOCATE B(0:M)
1920    DOUBLE K,K1,J,Mx
1930    T=1.
1940    FOR K=1 TO M
1950    T=T*(Alpha+K)*Beta              !  NORMALIZED MOMENTS re
1960    Mom(K)=Mom(K)/T                 !  Alpha and Beta; eq. 118
1970    NEXT K
1980    Q=1.
1990    A(0)=B(0)=Mom(0)
2000    FOR K=1 TO M
2010    K1=K+1
2020    T=1.
2030    S=Mom(K)-A(0)
2040    FOR J=1 TO K-1
2050    T=T*(J-K1)/J
2060    S=S-T*A(J)
2070    NEXT J
2080    IF K MOD 2=1 THEN S=-S
2090    A(K)=S
2100    Q=Q*(Alpha+K)/K
2110    B(K)=S*SQR(Q)
2120    NEXT K
```

PROGRAM PLDMandRM (cont'd)

```
2130    Mx=Mx+10
2140    IF Mx<M THEN 2130
2150    Threshold=-7.
2160    T2=Threshold*2.
2170    V=10.^Threshold
2180    GINIT
2190    PLOTTER IS "GRAPHICS"
2200    GRAPHICS ON
2210    WINDOW 0.,FLT(Mx),T2,0.
2220    LINE TYPE 3
2230    FOR J=0 TO Mx STEP 10
2240    MOVE J,T2
2250    DRAW J,0.
2260    NEXT J
2270    FOR J=T2 TO 0
2280    MOVE 0.,J
2290    DRAW Mx,J
2300    NEXT J
2310    PENUP
2320    LINE TYPE 1
2330    IMAGE 4D,2(4X,M.17DE)
2340    PRINT "   K                B(K)                          Sum"
2350    Sum=0.
2360    FOR K=0 TO M
2370    B=B(K)
2380    Sum=Sum+B*B
2390    PRINT USING 2330;K,B,Sum
2400    IF B<V THEN 2430
2410    Y=LGT(B)
2420    GOTO 2470
2430    IF B>-V THEN 2460
2440    Y=T2-LGT(-B)
2450    GOTO 2470
2460    Y=Threshold
2470    PLOT K,Y
2480    NEXT K
2490    PENUP
2500    SUBEND
2510    !
2520    SUB Moments(DOUBLE M,REAL P0,Mom(*))        !  SHOT NOISE
2530  ! LISTED IN PHDM&RM
2670    SUBEND
```

PROGRAM pLDMandRM

```
10 !  CONTINUOUS PART OF SHOT NOISE PDF, pc(u); COEFFICIENTS OF GENERALIZED
20 !  LAGUERRE EXPANSION FOUND DIRECTLY AND RECURSIVELY VIA MOMENTS
30    M=70   !   MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40    DOUBLE M,I,N,K                    !  INTEGERS < 2^31 = 2,147,483,648
50    REDIM Mom(0:M),A(0:M),L(0:M)
60    REAL Mom(0:100),A(0:100),L(3:100)
70    CALL Moments(M,P0,Mom(*))         !  P0 IS STEP AT ORIGIN
80    Center=Mom(1)/Mom(0)              !  CENTER OF PDF pc(u)
90    R2=Mom(2)/Mom(0)-Center*Center    !  MEAN SQUARE SPREAD OF pc(u)
100   Rms=SQR(R2)                       !  RMS SPREAD OF pc(u)
110   Alpha0=Center*Center/R2-1.        !  THE CHOICES   Alpha=Alpha0    AND
120   Beta0=R2/Center                   !  Beta=Beta0   WOULD MAKE A(1)=A(2)=0
130     Alpha=.74
140     Beta=2.1
150   CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   DM
160 ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*))     !   RM
170   PRINT "Center = ";Center
180   PRINT "Rms =";Rms
190   F1=1./(Beta*FNGamma(Alpha+1.))
200     INPUT "ORDER AND LIMITS:",N,U1,U2
210   PRINT "ORDER AND LIMITS:",N;U1;U2
220   Du=(U2-U1)/100.
230   PLOTTER IS "GRAPHICS"
240   GRAPHICS ON
250   WINDOW U1,U2,0.,.15
260   GRID 6.,.03
270   PLOT 0.,0.
280   FOR I=1 TO 100
290   U=U1+Du*I
300   T=U/Beta
310   CALL Laguerre(N,Alpha,T,L(*))
320   Sum=A(0)
330   FOR K=1 TO N
340   Sum=Sum+A(K)*L(K)
350   NEXT K
360   P=F1*EXP(-T+Alpha*LOG(T))*Sum    !   PDF OF RV AT U
370   PLOT U,P
380   NEXT I
390   PENUP
400   GOTO 200
410   END
420 ! USE ROUTINES IN PLDM&RM
```

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Technical Memorandum 831078
1 June 1983

Error of Zero-Crossing Location for Straight-Line Interpolation of Sampled Sinewave in Noise

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ABSTRACT

The maximum error, rms error, and the average-magnitude error are evaluated for the approximate zero-crossing location obtained by using linearly-interpolated values of a sampled sinewave. If the sampling rate is greater than four times the sinewave frequency, all three errors are less than 1.1 percent of the period of the sinewave. In the presence of additive noise, the rms error is derived and plotted for different signal-to-noise ratios and noise-bandwidth to signal-frequency ratios. Limitations on the approximate analytic result are pointed out and compared with simulation results.

INTRODUCTION

Sampling of a bandlimited waveform, at a frequency greater than twice the highest frequency contained in the waveform, loses no information according to the sampling theorem. However, perfect reconstruction of the original waveform from the samples requires that two conditions be met: all the samples over $(-\infty, +\infty)$ must be available, and $\sin(x)/x$ interpolation must be used. The first condition can never be met exactly and the second is often too time-consuming to be practical. Furthermore, determination of the zero crossings requires solution of a transcendental equation, which is very time-consuming. Accordingly, simpler schemes for waveform reconstruction, such as sample-and-hold or linear interpolation, are often employed.

Here we investigate the error incurred by using linear interpolation of the samples, in order to determine the zero-crossing locations of the original waveform. In particular, we first consider a pure sinewave and sample it at a rate greater than twice the sinewave frequency. The error in zero-crossing locations, afforded by linear interpolation between adjacent samples, is evaluated as a function of the ratio of sampling period to sinewave period. When noise is added, the error depends additionally on the signal-to-noise ratio and the noise correlation. Plots of the maximum, rms, and mean-magnitude errors are given in the noise-free case, whereas the rms error is presented for the noise-present cases.

NOISE-FREE ZERO-CROSSING ESTIMATION

A sinewave of amplitude A and frequency F is sampled at intervals T seconds apart. The sampling is not synchronous with the sinewave frequency; thus a random phase is associated with the occurrence of sampling relative to the sinewave. The situation is depicted in figure 1, where we are trying to estimate the actual zero-crossing of $y(t)$ at $t=(2F)^{-1}$; we must have $y(b) \leq 0 \leq y(a)$ for this plot to be relevant.

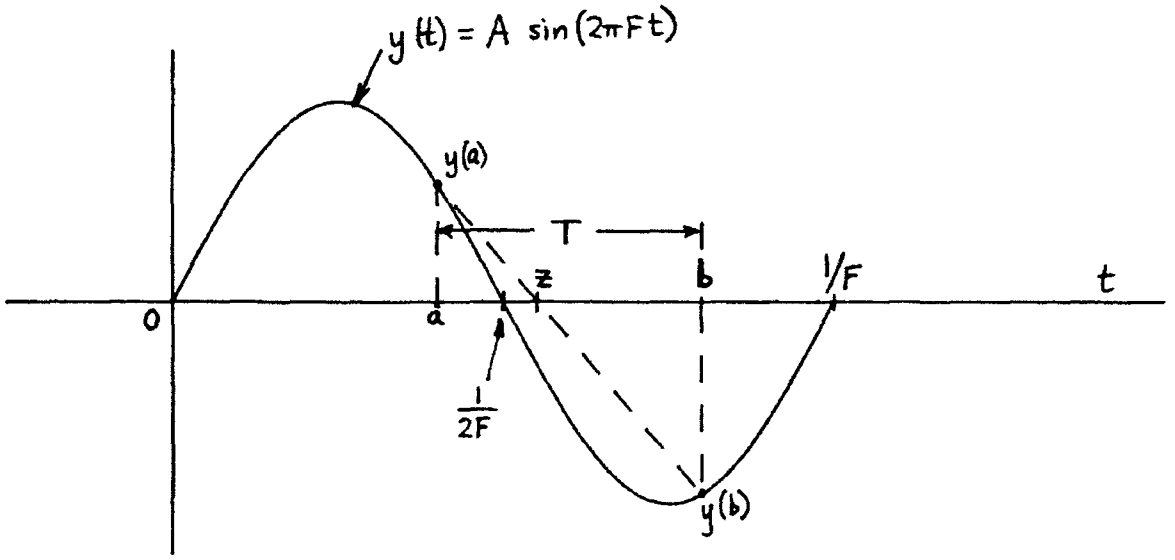

Figure 1. Estimated Zero-Crossing Location z

The interval between the samples at t=a and t=b is

$$T = b-a. \tag{1}$$

Since the sampling is not synchronized with the sinewave, the sampling instant, a, is uniformly distributed in an interval such that

$$a \leq \frac{1}{2F} \leq b = T + a, \quad \text{i.e.,} \quad \frac{1}{2F} - T \leq a \leq \frac{1}{2F} \quad . \tag{2}$$

The zero-crossing location, z, obtained by linearly-interpolating between the sample values at a and b, is

$$z = \frac{b\,y(a) - a\,y(b)}{y(a) - y(b)} = \frac{b\,\sin(2\pi Fa) - a\,\sin(2\pi Fb)}{\sin(2\pi Fa) - \sin(2\pi Fb)} \quad , \tag{3}$$

where the sinewave amplitude cancels out. If we eliminate b via use of (1), (3) becomes

$$z = a + T \frac{\sin(2\pi Fa)}{\sin(2\pi Fa) - \sin(2\pi F(a+T))} \quad , \tag{4}$$

in terms of sampling interval T.

Instead of using the non-symmetric interval for a as given by (2), we let

$$a = \frac{1}{2F} - \frac{T}{2} + c \quad , \tag{5}$$

where now c is uniformly distributed in the interval

$$-\frac{T}{2} \leq c \leq \frac{T}{2} \quad . \tag{6}$$

Physically, c is the displacement of the midpoint of interval (a,b) from the true zero-crossing at $(2F)^{-1}$. Under the substitution (5), (4) simplifies to

$$z = \frac{1}{2F} + c - \frac{T}{2} \frac{\tan(2\pi Fc)}{\tan(\pi FT)} \quad . \tag{7}$$

The error in estimation of the true zero-crossing location is then

$$e = z - \frac{1}{2F} = c - \frac{T}{2} \frac{\tan(2\pi Fc)}{\tan(\pi FT)} \quad . \tag{8}$$

It is seen from (8) that the error e is odd in c. Furthermore, e is zero for c=0 and c= ± T/2. A plot of error e in (8), normalized relative to the sinewave period, is given in figure 2 for different values of FT. It will be observed that the error increases significantly, for all c, as FT approaches .5-. Also, the maximum error, e = .25/F, occurs for FT = .5-, which corresponds to two samples per period. For small FT, the error is given from (8) by

$$e \cong \frac{(\pi FT)^2}{3} c\left(1 - \frac{4c^2}{T^2}\right) \quad \text{for} \quad FT << 1 \quad . \tag{9}$$

The maximum error in (9) occurs at $c \cong T/\sqrt{12}$ with value

$$\frac{e(max)}{1/F} \cong \frac{\pi^2}{9\sqrt{3}} (FT)^3 = .633\ (FT)^3 \quad \text{for} \quad FT << 1 \quad . \tag{10}$$

Thus the maximum error behaves as the cube of FT for small FT; FT is the ratio of sampling period T to sinewave period 1/F.

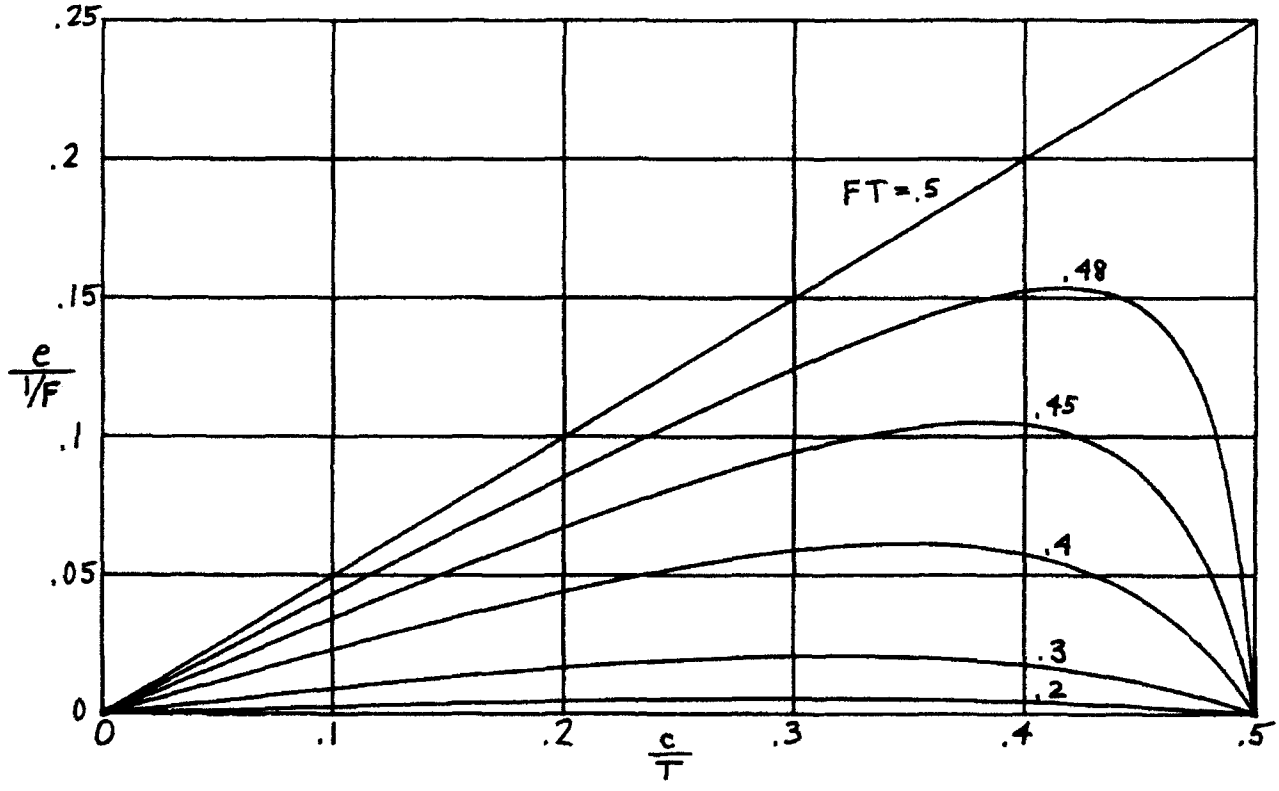

Figure 2. Error e, Relative to Sinewave Period 1/F

Since c is a random variable and uniformly distributed over ± T/2 according to (6), we can compute rms or mean-magnitude errors. But first observe from (8) that the average error, $\bar{e}$, is zero, since (8) is odd in c, which is itself symmetrically distributed about c=0.

We are interested in the following three error measures:

$$e(\max) = \max_{|c|<T/2} |e| \quad ,$$

$$e(\text{rms}) = \left[\int_{-T/2}^{T/2} dc\, e^2\, p(c) \right]^{1/2} \quad ,$$

$$e(\text{mag}) = \int_{-T/2}^{T/2} dc\, |e|\, p(c) \quad , \tag{11}$$

where p(c) is the uniform probability density function of random variable c. These are respectively the maximum error, root-mean-square error, and mean-magnitude error. We always have

$$e(\text{mag}) \leq e(\text{rms}) \leq e(\max); \tag{12}$$

the first inequality follows from use of Schwartz's inequality on (11).

Maximum Error

To determine the location of the maximum of the error e in (8), we differentiate with respect to c and set the derivative equal to zero, getting location c_o where

$$\cos^2(2\pi F c_o) = \frac{\alpha}{\tan(\alpha)} \quad , \tag{13}$$

and where

$$\alpha \equiv \pi F T \quad . \tag{14}$$

Then the location is explicitly

$$c_o = \frac{1}{2\pi F} \text{ arc } \tan(\sqrt{r-1}) \quad , \tag{15}$$

where

$$r \equiv \frac{\tan(\alpha)}{\alpha} = \frac{\tan(\pi F T)}{\pi F T} \quad . \tag{16}$$

Substitution of (15) in (8) now yields

$$\frac{e(max)}{1/F} = \frac{1}{2\pi}\left[\arctan(\sqrt{r-1}) - \frac{\sqrt{r-1}}{r}\right] . \qquad (17)$$

As FT→0, r→1+, and the bracket of (17) approaches

$$\frac{2}{3}(r-1)^{3/2} \cong \frac{2}{3}\left(\frac{\alpha^2}{3}\right)^{3/2} , \qquad (18)$$

the last step via use of (16). Then (17) yields

$$\frac{e(max)}{1/F} \cong \frac{\pi^2}{9\sqrt{3}}(FT)^3 = .633\ (FT)^3 \text{ as } FT\to 0 , \qquad (19)$$

in agreement with (10).

As FT→.5-, r→+∞, and (17) yields

$$\frac{e(max)}{1/F} \to \frac{1}{4} \text{ as } FT\to .5- , \qquad (20)$$

in agreement with figure 2.

RMS Error

We have already shown that $\bar{e}=0$ by use of (8) and (6). The mean-square value of e is

$$\overline{e^2} = \frac{2}{T}\int_0^{T/2} dc\left[c - \frac{T}{2}\frac{\tan(2\pi Fc)}{\tan(\alpha)}\right]^2 . \qquad (21)$$

Alternatively, using (14),

$$\frac{e(rms)}{1/F} = \frac{\alpha}{2\pi}\left[\int_0^1 dx\left(x - \frac{\tan(\alpha x)}{\tan(\alpha)}\right)^2\right]^{1/2} . \qquad (22)$$

This integral cannot be evaluated in closed form, due to the cross-product term in the integrand; see ref. 1, eq. 2.646 1.

For small FT, we have $\alpha << 1$, and then

$$x - \frac{\tan(\alpha x)}{\tan(\alpha)} \sim \frac{1}{3}\alpha^2 x(1-x^2) \quad , \tag{23}$$

leading to

$$\frac{e(rms)}{1/F} \sim \frac{\pi^2}{3}\sqrt{\frac{2}{105}}(FT)^3 = .454\ (FT)^3 \text{ as } FT \to 0. \tag{24}$$

Just as for e(max) in (19), this is cubic in FT; however, the scale factor is smaller here, as anticipated in (12).

As $FT \to .5-$, $\alpha \to \pi/2-$, and $\tan(\alpha) \to +\infty$ in (22); then

$$\frac{e(rms)}{1/F} \to \frac{1}{4\sqrt{3}} = .144 \text{ as } FT \to .5- \quad . \tag{25}$$

Mean-Magnitude Error

The mean-magnitude error is, from (8) and (6),

$$\overline{|e|} = \frac{2}{T}\int_0^{T/2} dc \left| c - \frac{T}{2}\frac{\tan(2\pi Fc)}{\tan(\alpha)} \right| . \tag{26}$$

Alternatively, using (14) and ref. 1, eq. 2.526 17,

$$\frac{e(mag)}{1/F} = \frac{\alpha}{2\pi}\int_0^1 dx \left| x - \frac{\tan(\alpha x)}{\tan(\alpha)} \right| = \frac{1}{2\pi}\left[\frac{\alpha}{2} + \frac{\ell n(\cos(\alpha))}{\tan(\alpha)}\right] . \tag{27}$$

We are able to eliminate the absolute value, by reference to figure 2.

For small FT, $\alpha << 1$, and we use (23) again to find

$$\frac{e(mag)}{1/F} \sim \frac{\pi^2}{24}(FT)^3 = .411\ (FT)^3 \quad \text{as} \quad FT \to 0 \quad . \tag{28}$$

The scale factor is smaller than in (24), in keeping with (12).

And as FT→.5-, $\alpha \rightarrow \pi/2-$, $\tan(\alpha) \rightarrow +\infty$, giving rise to

$$\frac{e(mag)}{1/F} \rightarrow \frac{1}{8} \quad \text{as FT} \rightarrow .5- \quad . \tag{29}$$

Plot of Three Error Measures

A plot of the three errors (normalized by the sinewave period) as given by (17), (22), and (27), is given in figure 3. The errors are monotonically increasing in FT, the ratio of sampling period to sinewave period. They are all cubic in FT for small FT, and increase rapidly as FT→.5-. For FT=.25, the errors are 1.13%, .81%, and .73% of the sinewave period; thus increasing the sampling frequency to double the minimum required for the sampling theorem allows for linear interpolation with an error in zero-crossing location only about 1% of the sinewave period.

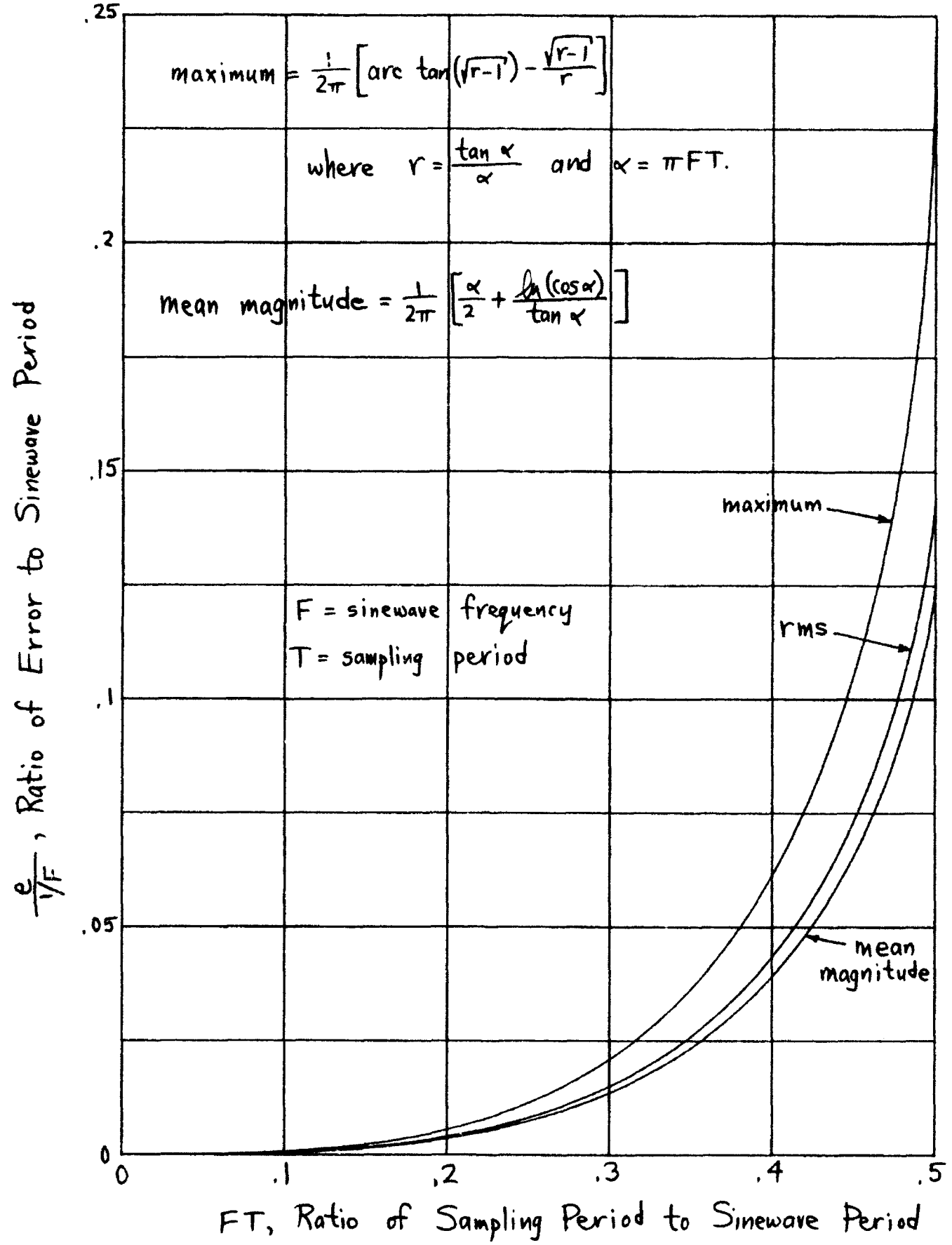

Figure 3. Three Error Measures for Noise-Free Case

ZERO-CROSSING ESTIMATION IN ADDITIVE NOISE

The observed waveform in this case is

$$y(t) = s(t) + n(t) = A \sin(2\pi Ft) + n(t), \tag{30}$$

where the additive independent noise n(t) is stationary and zero-mean. The noise need not be Gaussian; in fact, the only statistics required of the noise are the noise power and correlation coefficient respectively:

$$\sigma^2 = \overline{n^2(t)} \quad , \quad \rho(\tau) = \frac{1}{\sigma^2}\,\overline{n(t)\,n(t+\tau)} \quad . \tag{31}$$

(Actually, these are sufficient only for the high signal-to-noise ratio case considered below.)

The estimated zero-crossing is still given by the first equation in (3). When (30) is substituted, there follows, for the zero-crossing, the nonlinear functional

$$z = \frac{[b\,s(a) - a\,s(b)] + [b\,n(a) - a\,n(b)]}{[s(a) - s(b)] + [n(a) - n(b)]} \cong \frac{u_1 + v_1}{u_2 + v_2} \quad . \tag{32}$$

The signal terms are u_1, u_2, while the noise terms are v_1, v_2. Since random variables v_1 and v_2 can take on any value, there is no limit on the maximum error, as there was in the noise-free case. And since the rms error is intermediate to the three errors considered above and easier to evaluate than the mean-magnitude error, we will concentrate solely on the rms error in the noise case.

We begin by making a high signal-to-noise ratio assumption; that is, we assume

$$\frac{\text{signal power}}{\text{noise power}} = \frac{A^2/2}{\sigma^2} \equiv R >> 1 \quad . \tag{33}$$

Then (32) becomes approximately

$$z = \frac{u_1}{u_2}\,\frac{1 + v_1/u_1}{1 + v_2/u_2} \cong \frac{u_1}{u_2}\left(1 + \frac{v_1}{u_1}\right)\left(1 - \frac{v_2}{u_2}\right)$$

$$\cong \frac{u_1}{u_2}\left(1 + \frac{v_1}{u_1} - \frac{v_2}{u_2}\right) = \frac{u_1}{u_2} + \frac{v_1}{u_2} - \frac{u_1 v_2}{u_2^2} \quad . \tag{34}$$

The u_1/u_2 term in (34) is exactly the last term in (3) et seq.; the additional two terms in (34) are noise terms that are <u>linearly</u> dependent on the noise samples n(a) and n(b), under the high signal-to-noise ratio assumption. If we did not make this assumption, we would need the joint probability density function of the noise at the two sample instants a and b.

To evaluate the mean and variance of z in (34), we will temporarily hold sample times a and b fixed, and average over the noise statistics, getting conditional properties. Then we will employ (1), (5), and (6) and average over random variable c which reflects the random nature of the sampling mechanism. The preliminary average over the noise will be denoted by a wavy overline; thus from (34), the conditional mean

$$\tilde{z} = \frac{u_1}{u_2} + \frac{1}{u_2}\,\tilde{v}_1 - \frac{u_1}{u_2^2}\,\tilde{v}_2 = \frac{u_1}{u_2} \quad , \tag{35}$$

since noise n(t) has zero-mean for all t. Then appeal to (3)-(7) gives

$$\bar{z} = \frac{1}{2F} \quad , \tag{36}$$

just as in the noise-free case; this conclusion is heavily based on the high signal-to-noise ratio assumption.

We now consider the zero-mean random variable

$$w \equiv z - \bar{z} = c - \frac{T}{2}\frac{\tan(2\pi Fc)}{\tan(\alpha)} + \frac{v_1}{u_2} - \frac{u_1 v_2}{u_2^2}$$

$$= e + \frac{v_1}{u_2} - \frac{u_1 v_2}{u_2^2} \quad , \tag{37}$$

which is the perturbation of the estimated zero-crossing location from the true value $(2F)^{-1}$; here we used (34), (36), (7), (8), and (14). The conditional mean-square value of w is

$$\widetilde{w^2} = e^2 + \frac{1}{u_2^2}\widetilde{v_1^2} - \frac{2u_1}{u_2^3}\widetilde{v_1 v_2} + \frac{u_1^2}{u_2^4}\widetilde{v_2^2}, \tag{38}$$

since v_1 and v_2 have zero-mean. Now from (32),

$$v_1 = b\, n(a) - a\, n(b), \qquad v_2 = n(a) - n(b) \quad . \tag{39}$$

Therefore

$$\widetilde{v_1^2} = \sigma^2(b^2 + a^2 - 2ab\rho) = 2\sigma^2\left[(1-\rho)\left(c + \frac{1}{2F}\right)^2 + (1+\rho)\frac{T^2}{4}\right] \quad ,$$

$$\widetilde{v_1 v_2} = \sigma^2(b - b\rho - a\rho + a) = \sigma^2(1-\rho)(a+b) = 2\sigma^2(1-\rho)\left(c + \frac{1}{2F}\right) \quad , \tag{40}$$

$$\widetilde{v_2^2} = \sigma^2(1 + 1 - 2\rho) = 2\sigma^2(1 - \rho) \quad ,$$

where we used (31), (1), (5), and defined

$$\rho = \frac{1}{\sigma^2}\overline{n(a)n(b)} = \frac{1}{\sigma^2}\overline{n(a)n(a+T)} = \rho(T) \quad . \tag{41}$$

Before substituting (40) in (38), we derive simpler expressions for u_1 and u_2. From (32), (1), (5), and (30), we find

$$u_1 = 2A\left(c + \frac{1}{2F}\right)\sin(\alpha)\cos(2\pi Fc) - AT\cos(\alpha)\sin(2\pi Fc),$$

$$u_2 = 2A\sin(\alpha)\cos(2\pi Fc) \quad . \tag{42}$$

Then expressing (38) as the sum of four terms,

$$\tilde{\tilde{w}}^2 = T_1 + T_2 + T_3 + T_4 \quad , \tag{43}$$

and defining

$$\beta = 2\pi Fc \quad , \tag{44}$$

we have, upon use of (33),

$$T_1 = \left[c - \frac{T}{2}\frac{\tan(\beta)}{\tan(\alpha)}\right]^2 \quad ,$$

$$T_2 = \frac{1}{4R}\,\frac{(1-\rho)\left(c + \frac{1}{2F}\right)^2 + (1+\rho)\cdot\frac{T^2}{4}}{\sin^2(\alpha)\cos^2(\beta)} \quad ,$$

$$T_3 = \frac{-1+\rho}{4R}\,\frac{\left(c + \frac{1}{2F}\right)\left[2\left(c + \frac{1}{2F}\right)\sin(\alpha)\cos(\beta) - T\cos(\alpha)\sin(\beta)\right]}{\sin^3(\alpha)\cos^3(\beta)} \quad ,$$

$$T_4 = \frac{1-\rho}{16R}\,\frac{\left[2\left(c + \frac{1}{2F}\right)\sin(\alpha)\cos(\beta) - T\cos(\alpha)\sin(\beta)\right]^2}{\sin^4(\alpha)\cos^4(\beta)} \quad . \tag{45}$$

We now have to average the sum of the four terms in (45) over the symmetric distribution for c given in (6). However, each of the terms T_2, T_3, T_4 have even and odd components in c, and the odd components will average to zero. When we retain only the even functions of c in (45) and then add them together, a number of terms cancel, after considerable manipulation, leading to the significantly simpler expression

$$\text{even}\left\{T_2 + T_3 + T_4\right\} = \frac{T^2}{16R}\,\frac{(1-\rho)\cos^2(\alpha)\sin^2(\beta) + (1+\rho)\sin^2(\alpha)\cos^2(\beta)}{\sin^4(\alpha)\cos^4(\beta)} \quad . \tag{46}$$

When we employ (44) in (46) and average over c, we find, upon use of ref. 1, eqs. 2.526 30 and 2.526 10, the mean-square error of the zero-crossing location (due to the additive noise) as the surprisingly brief expression

$$\frac{e^2(\text{noise})}{1/F^2} = \frac{1}{4\pi^2}\,\frac{1}{R}\,\frac{2+\rho(T)}{3}\,\frac{\alpha}{\sin(2\alpha)} \quad , \tag{47}$$

upon simplification and use of (41). To this term must be added the averaged T_1 term in (45), which is simply the square of (22). The total normalized rms error is then

$$\frac{e(\text{rms})}{1/F} = \frac{1}{2\pi}\left[\alpha^2 \int_0^1 dx \left(x - \frac{\tan(\alpha x)}{\tan(\alpha)}\right)^2 + \frac{1}{R}\,\frac{2+\rho(T)}{3}\,\frac{\alpha}{\sin(2\alpha)}\right]^{1/2} . \tag{48}$$

Here signal-to-noise ratio R and parameter α are given by

$$R = \frac{A^2/2}{\sigma^2} \quad , \quad \alpha = \pi FT \quad . \tag{49}$$

We must emphasize that (48) is only valid for large signal-to-noise ratio, $R \gg 1$, so that approximation (34) is valid.

As $FT \to .5-$, $2\alpha \to \pi$, and the $\sin(2\alpha)$ term in the denominator of (48) approaches zero. This appears to cause the rms error to tend to $+\infty$ at $FT = .5-$. However, this obvious shortcoming of (48) is due to a breakdown of the approximation in (34). Specifically, (34) presumes that u_2 is not zero and that it is large compared to v_2; but (32) and (42) show that this quantity, given exactly by

$$u_2 = s(a) - s(b) = 2A \sin(\alpha) \cos(2\pi Fc) = 2A \sin(\pi FT) \cos(2\pi Fc) \quad , \tag{50}$$

can be small for some values of c. In particular, the minimum of u_2 is

$$\min_c u_2 = 2A \sin(\pi FT) \cos(\pi FT) = A \sin(2\pi FT) \quad , \tag{51}$$

which tends to zero as FT→.5. Thus the occasional values of c near ± T/2 cause u_2 to be small if FT is near .5, and the approximation in (34) to be invalid. Thus (48) is invalid as FT→.5, regardless of the signal-to-noise ratio.

At the same time, we observe that the sin(πFT) term in (50) reaches a maximum of 1 at FT=.5, but becomes small for FT near zero. Thus approximation (34) is not expected to be valid, nor (48) too accurate, for small FT. In fact, the maximum value of the minimum in (51) is A, at FT=.25. Thus (48) is expected to be most accurate in the neighborhood of FT=.25. Simulation results bear this conclusion out. Furthermore, FT values in the region about .25, like (.15, .35), encompass the range of most practical interest anyway; we cannot get away with two samples per cycle and we don't want to oversample if we don't have to.

As T→0, the first term in the bracket of (48) tends to zero; see (24). However, the second term does not, and we get

$$\frac{e(\text{rms})}{1/F} \sim \frac{1}{2\pi}\frac{1}{\sqrt{2R}} \text{ as } T\to 0 \quad . \tag{52}$$

This term constitutes a lower bound on the zero-crossing error, even as sampling period T tends to zero. (Notice that the error decays inversely with the square root of the signal-to-noise ratio R.) However, the precautionary note in the above paragraph makes (52) suspect, since u_2→0 then.

Plot of RMS Error

In order to plot (48) versus FT, we must have a functional form for the noise correlation coefficient ρ(T). We take here the example of a Gaussian correlation:

$$\rho(\tau) = \exp(-\pi^2 W^2 \tau^2) \quad . \tag{53}$$

The spectrum corresponding to (53) is proportional to

$$\exp(-f^2/W^2) \quad . \tag{54}$$

Thus at $f = \pm W$, this low-pass spectrum has dropped to 1/e of its peak value at f=0. Other correlations and spectra are possible, including bandpass cases if desired.

Equation (48) requires the quantity

$$\rho(T) = \exp(-\pi^2 W^2 T^2) = \exp(-\alpha^2 W^2/F^2) \quad , \tag{55}$$

where α is given by (49) as usual. The parameter W/F measures the ratio of the noise bandwidth (at the 1/e point) to the signal frequency and must be specified in order to plot (48) vs FT. In figures 4-6 are plotted the normalized rms error (48) for W/F = .1, 1, and 10 respectively, for various signal-to-noise ratios.

The accuracy of the analytic results in figures 4-6 for the relatively low signal-to-noise ratio of R = 10 = 10 dB is not too good, because (34) is a marginal approximation in this situation. Simulation results in the next section verify that (48) is a good approximation for R=20 dB and an excellent approximation for R=30 dB, provided that FT is not near 0 or .5.

The reason for the dip near the origin in figure 6 is that $\rho(T)$ in (48) decreases rapidly when the signal bandwidth is large. The eventual rise of the $\alpha/\sin(2\alpha)$ term has not taken effect before $\rho(T)$ has essentially decayed to zero. Physically, noise samples that are highly correlated cause more of a zero-crossing perturbation than uncorrelated noise samples. However, the region near FT=0 in figures 4-6 is not expected to be reliable because the approximation in (34) is invalid.

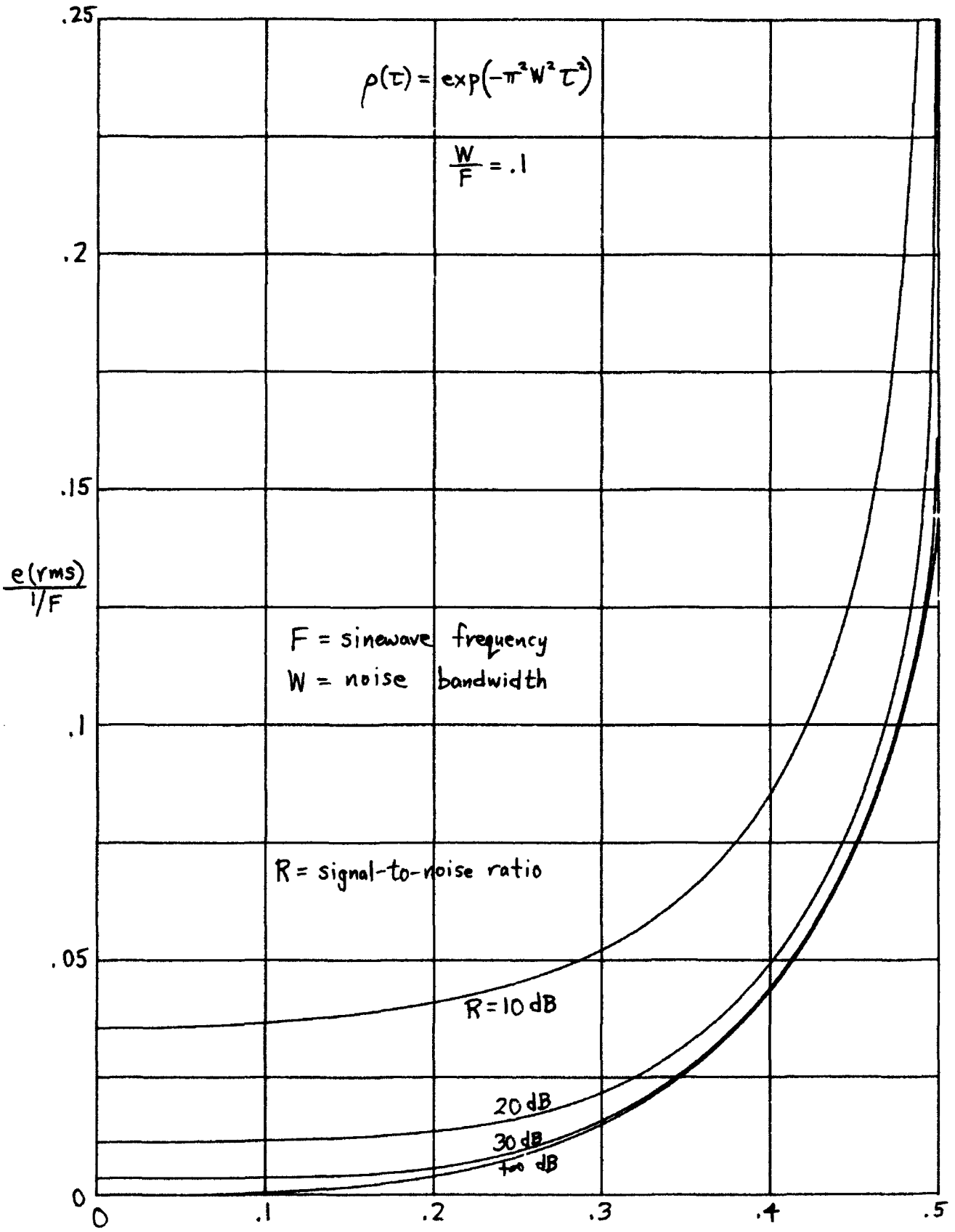

Figure 4. Normalized RMS Error for W/F = .1

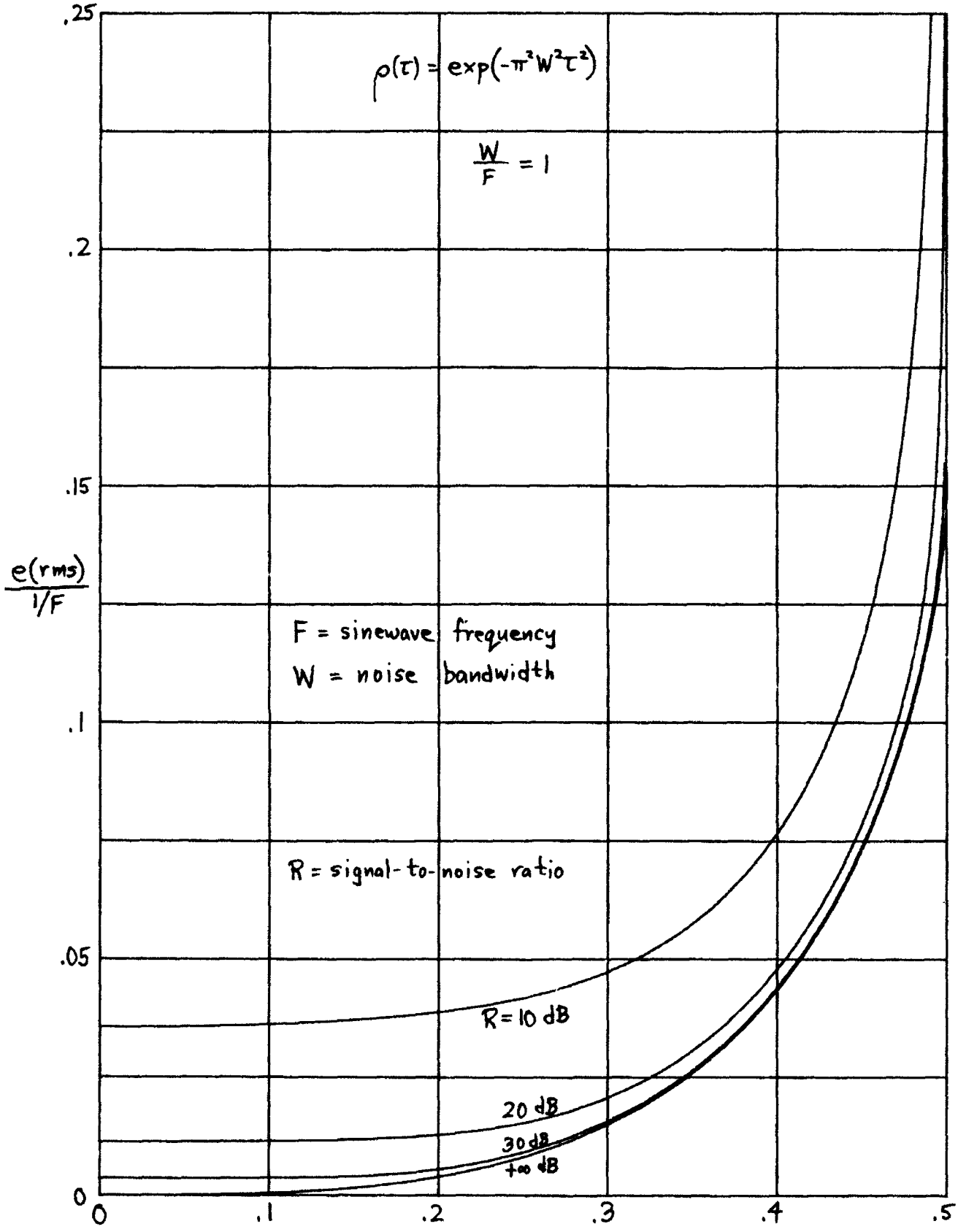

Figure 5. Normalized RMS Error for W/F = 1

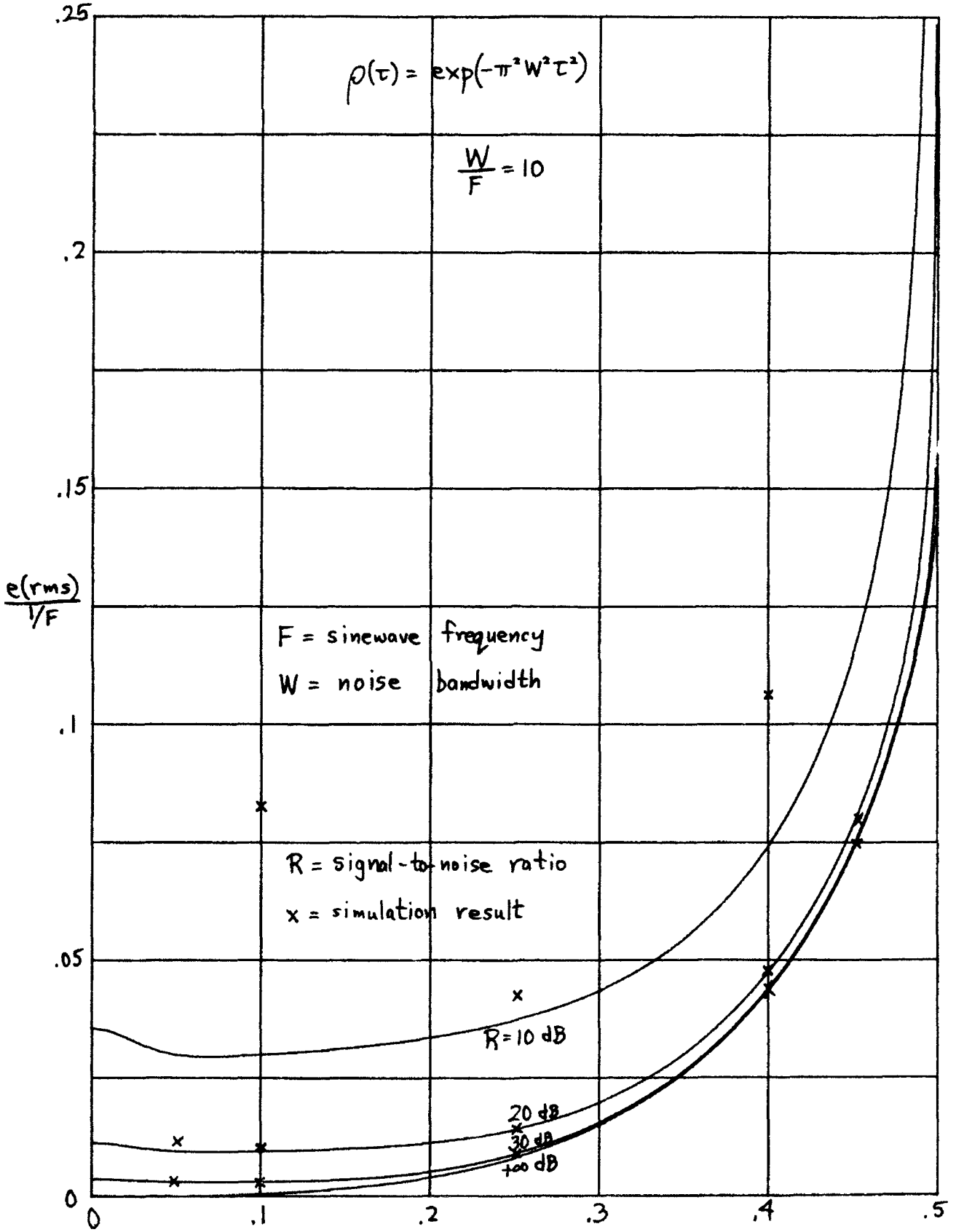

Figure 6. Normalized RMS Error for W/F = 10

SIMULATION RESULTS

A program for simulation of exact relation (32) is given in the appendix. A result for the noise-free case for F=1, T=.3, and 100,000 trials in given in table 1, and compared with the analytical results in (17), (22), and (27) respectively. The agreement is excellent for all three error measures.

Error	Analytical	Simulation
Maximum	.020936	.020936
RMS	.014978	.014995
Mean-Magnitude	.013553	.013572

Table 1. Noise-Free Comparison for F=1, T=.3, 100,000 trials

When noise is added, a simulation result for F=1, T=.3, ρ=.4, R=1000=30 dB, and 100,000 trials is given by (see appendix)

$$\frac{e(rms)}{1/F} = .015621 \quad . \tag{56}$$

The corresponding analytic result from (48) is .015634.

A check on (52) is afforded by a simulation with F=1, T=.001, ρ=1, R=30 dB. The contribution of the first term in (48) is .454 $(.001)^3$ = .454E-9, as given by (24), and is negligible compared with the second term. Whereas (52) yields .0035588, the simulation yielded .0035660, which is excellent agreement, considering that FT << 1 for this example. The saving feature here is the very large signal-to-noise ratio of R=1000.

Several simulation results, at 100,000 trials each, are displayed as Xs in figure 6. They were run at signal-to-noise ratio R equal to 10 dB, 20 dB, and 30 dB, for values of FT equal to .05, .1, .25, .4, and .45. The agreement between analysis and simulation is: excellent for all R=30 dB results; very good for R=20 dB except when FT is very small or near .5; and poor for R=10 dB except when FT is near .25. In fact, the R=10 dB results for FT=.05 and .45 were off the graph, and those for FT=.1 and .4 are rather inaccurate. These

results are consistent with the expectations given above for the approximate analytical result.

Finally, a simulation result for F=1, T=.5, W/F=.5, R=30 dB, and 100,000 trials gave 1.9488, which is much larger than any of the ordinates in figures 4-6. There is no check on this result since (48) is not valid at FT=.5. The value of ρ in this case is, from (55), $\exp(-(\pi/2)^2(1/2)^2) = \exp(-\pi^2/16) = .54$.

SUMMARY

In the noise-free case, the maximum error, rms error, and mean-magnitude error have been derived, and the normalized error has been plotted (in figure 3) as a function of FT, the ratio of sampling period T to sinewave period 1/F. The normalization is with respect to the sinewave period.

For the additive noise case, the rms error has been evaluated and plotted (in figures 4-6) for a variety of signal-to-noise ratios and ratios of noise bandwidth to signal frequency. The result is not correct for small signal-to-noise ratio or as FT→0 or .5-, due to approximations adopted in the analysis, in order to obtain a tractable result. Simulation results corroborate the analysis for FT values not near 0 or .5, and for large signal-to-noise ratio.

REFERENCE

1. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press Inc., NY, 1980.

APPENDIX. PROGRAM FOR SIMULATION

```
10    T=.3                        ! SAMPLING PERIOD; TAKE F=1
20    Rho=.4                      ! CORRELATION COEFFICIENT
30    R=1000                      ! SIGNAL-TO-NOISE RATIO
40    Tt=100000                   ! NUMBER OF TRIALS
50    Th=.5*ACS(Rho)
60    Ct=COS(Th)
70    St=SIN(Th)
80    Fac=SQR(6/R)
90    Tm=.5-T/2
100   Tp=.5+T/2
110   T2=2*PI
120   M1=M2=M3=M4=0
130   FOR It=1 TO Tt
140   C=(RND-.5)*T
150   A=Tm+C                      ! SAMPLING
160   B=Tp+C                      !    TIMES
170 ! GO TO 240          INSERT THIS LINE FOR NOISE-FREE CASE
180   X=(RND-.5)*Fac
190   Y=(RND-.5)*Fac
200   Tc=Ct*X
210   Ts=St*Y
220   G=Tc+Ts                     ! ADDITIVE NOISE WITH DESIRED
230   H=Tc-Ts                     !       LEVEL AND CORRELATION
240   Ya=SIN(T2*A)+G
250   Yb=SIN(T2*B)+H
260   Z=(B*Ya-A*Yb)/(Ya-Yb)       ! ZERO-CROSSING ESTIMATION
270   E=Z-.5                      ! ERROR
280   M1=M1+E                     ! AVERAGE ERROR
290   M2=M2+E*E                   ! SQUARED ERROR
300   M3=M3+ABS(E)                ! MAGNITUDE ERROR
310   M4=MAX(M4,ABS(E))           ! MAXIMUM ERROR
320   NEXT It
330   PRINT T;M1/Tt;SQR(M2/Tt);M3/Tt;M4
340   END
```

```
 .3 -1.11951611792E-05  1.56208258187E-02  1.38355727005E-02  .0302450328
```

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Mean and Variance of Product Array Response: Application to a Cross-Line Array

A. H. Nuttall

ABSTRACT

The mean and variance of the product of the narrowband responses of two arrays steered to the same look direction is evaluated and shown to depend on the complex coherence between the two array outputs. For a cross-line array of perpendicular equi-spaced elements, the stability is much poorer than for a sum array processor, due to the largely uncommon volume of intersection of the two cones of response of each line. When each array is extended to be planar, the degradation in stability is lessened, tending towards the sum array performance as the number of common elements increases.

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INTRODUCTION

A planar array with a grid structure of $M_1 \times M_2$ elements requires a large number of receivers and considerable signal processing when all the elements are employed and actively utilized. In an effort to conserve on the number of elements and amount of signal processing, the possibility of using a sparse array seems to have merit. In particular, it has been found that an equi-weighted planar array has the same auto spectral density response as the cross spectrum of a pair of perpendicular lines of double the length and with triangular weighting on each line. However, without investigating the variances of these sum and cross-line arrays, respectively, it is impossible to decide on their relative merits.

Here we will consider two arbitrary planar arrays (which may have some common elements and may even be linear arrays), both of which are steered to the same look direction and employ weightings for sidelobe control. The sample cross-spectral density of the two array outputs is the output variable of interest. Thus the combination of arrays is resolving in both spatial angles (wavenumber) as well as in temporal frequency. We will evaluate the mean and variance of the cross-spectral density estimate at the system output in terms of the statistical properties of the impingent noise field and the array parameters, such as look direction and weighting.

As a special case, by choosing the two planar arrays identical in element usage and weighting, we will reduce to the sum array since the cross-spectrum then becomes the auto-spectrum. Thus we can compare the performances of product arrays and sum arrays in terms of the mean and variance of their responses.

CROSS-SPECTRAL DENSITY ESTIMATE

Let $x(t)$ and $y(t)$ be any two array outputs which are stationary in time, zero-mean, and have auto-spectra and cross-spectrum $G_x(f)$, $G_y(f)$, $G_{xy}(f)$, respectively. f is temporal frequency in Hz. Sections of each waveform are gated out by multiplying by temporal weightings and then subjected to Fourier analysis according to*

$$X_m(f) = \int dt\ \exp(-i2\pi ft)\ w_m(t)\ x(t) \quad \text{for } 1 \leq m \leq N,$$

$$Y_n(f) = \int dt\ \exp(-i2\pi ft)\ w_n(t)\ y(t) \quad \text{for } 1 \leq n \leq N. \tag{1}$$

These temporal weightings are generally taken as delayed versions of a basic weighting $w(t)$ according to

$$w_n(t) = w(t-nS), \tag{2}$$

where S is a shift or time delay; however, we keep the more general case in (1) for the time being.

The cross-spectral density estimate at frequency f is obtained by multiplying outputs (1) and averaging in time according to

$$v = \sum_{n=1}^{N} X_n(f)\ Y_n^*(f). \tag{3}$$

This is the product array output; it has resolution capability in the spatial angles by virtue of each array output being steered to the same desired look direction, and it has frequency resolution governed by the lengths of the weightings in (1) or (2).

The mean of a general product term of components of (1) is

*An integral without limits is over the entire range of its non-zero integrand.

$$\overline{X_m(f)Y_n^*(f)} = \iint dt\, du\, \exp[-i2\pi f(t-u)]\, w_m(t)\, w_n(u)\, R_{xy}(t-u) =$$

$$= \int d\tau\, \exp(-i2\pi f\tau)\, R_{xy}(\tau)\, \phi_{mn}(\tau) = G_{xy}(f) \bullet \Phi_{mn}(f) =$$

$$= \int du\, G_{xy}(u)\, \Phi_{mn}(f-u), \tag{4}$$

where

$$R_{xy}(\tau) = \overline{x(t)\, y(t-\tau)} \tag{5}$$

is the cross-correlation of array outputs,

$$\phi_{mn}(\tau) = \int dt\, w_m(t)\, w_n(t-\tau) \tag{6}$$

is the aperiodic correlation of w_m and w_n, $\bullet$ denotes convolution, and

$$\Phi_{mn}(f) = \int d\tau\, \exp(-i2\pi f\tau)\, \phi_{mn}(\tau) = W_m(f)\, W_n^*(f) \tag{7}$$

is the product of windows of the individual temporal weightings.

If cross-spectrum G_{xy} does not vary significantly in the width of window Φ_{mn}, (4) yields approximation

$$\overline{X_m(f)Y_n^*(f)} \cong G_{xy}(f) \int du\, \Phi_{mn}(f-u) = G_{xy}(f)\, \phi_{mn}(0). \tag{8}$$

Now we apply these results to find the mean of the system output cross-spectral density estimate in (3):

$$\mu_v = \bar{v} = G_{xy}(f) \sum_{n=1}^{N} \phi_{nn}(0) = G_{xy}(f) \sum_{n=1}^{N} \int dt\, w_n^2(t), \tag{9}$$

which is proportional to the true cross-spectral density G_{xy} between the two array outputs.

In order to evaluate the variance of system output v in (3), we need magnitude-square value (suppressing f)

$$\overline{|v|^2} = \sum_{m,n=1}^{N} \overline{X_m Y_m^* X_n^* Y_n}. \tag{10}$$

We assume that the filtered outputs X_m and Y_n in (1) are complex Gaussian (as, for example, if x(t) and y(t) were Gaussian). We then observe that

$$\overline{X_m Y_n} = \iint dt\, du\, \exp[-i2\pi f(t+u)]\, w_m(t)\, w_n(u)\, R_{xy}(t-u) =$$

$$= \int dv\, G_{xy}(v)\, W_m(f-v) W_n(f+v). \tag{11}$$

Now for f removed from zero by at least the reciprocal of the segment length, the two windows in (11) do not overlap, and we get

$$\overline{X_m Y_n} \cong 0 \text{ for } f \neq 0 . \tag{12}$$

Then using the factoring property of zero-mean Gaussian random variables, (10) becomes

$$\overline{|v|^2} = \sum_{m,n=1}^{N} \left[\overline{X_m Y_m^*}\ \overline{X_n^* Y_n} + \overline{X_m X_n^*}\ \overline{Y_m^* Y_n} \right], \tag{13}$$

where we used (12). The variance of random variable v is

$$\sigma_v^2 = \overline{|v-\bar{v}|^2} = \overline{|v|^2} - |\bar{v}|^2 = \sum_{m,n=1}^{N} \overline{X_m X_n^*}\ \overline{Y_m^* Y_n} =$$

$$= \sum_{m,n=1}^{N} [G_x(f) \circledast \Phi_{mn}(f)][G_y(f) \circledast \Phi_{mn}^*(f)] =$$

$$\cong G_x(f)\, G_y(f) \sum_{m,n=1}^{N} \phi_{mn}^2(0) , \tag{14}$$

where we used (9), (4), and (8).

We now specialize the general weightings in (1) to the particular case in (2), obtaining from (6)

$$\phi_{mn}(0) = \int dt\ w(t-mS)\ w(t-nS) = \phi_w((n-m)S), \tag{15}$$

where

$$\phi_w(\tau) = \int dt\ w(t)\ w(t-\tau) \tag{16}$$

is the aperiodic correlation of basic temporal weighting w. Then variance σ_v^2 in (14) becomes

$$\sigma_v^2 = G_x(f)\ G_y(f) \sum_{m,n=1}^{N} \phi_w^2((n-m)S) = G_x(f)\ G_y(f) \sum_{p=1-N}^{N-1} (N-|p|)\ \phi_w^2(pS)\ . \tag{17}$$

We are now in position to formulate a quality ratio for the output of the product array. Namely we define the complex (voltage) quality ratio

$$\frac{\mu_v}{\sigma_v} = \sqrt{N}\, \gamma_{xy}(f) \left[\sum_{p=1-N}^{N-1} \left(1 - \frac{|p|}{N}\right) \frac{\phi_w^2(pS)}{\phi_w^2(0)} \right]^{-1/2}\ , \tag{18}$$

where we define the complex coherence between the array outputs as

$$\gamma_{xy}\ (f) = \frac{G_{xy}(f)}{[G_x(f)\ G_y(f)]^{1/2}}\ , \tag{19}$$

and have employed (9), (17), and (15). The quantity in (18) is desired large; it has leading factor $\sqrt{N}$, which however is partially compensated by the last factor of (18) if shift S is less than the segment length of temporal weighting w. A detailed investigation of the temporal processing factors in (18) is given in [1]; for present purposes, the factor is nearly maximized if shift S is taken about 50% of the segment length.

However, the most important quantity in quality ratio (18) is the complex coherence at frequency f, $\gamma_{xy}(f)$. It is always bounded in magnitude by 1, and can be significantly less than 1 if the two array outputs x(t) and y(t) are incoherent at frequency f of interest.

On the other hand, for identical arrays and array weightings, we have y(t) = x(t), and the quality ratio is again given by (18), where $\gamma_{xy}(f)$ is replaced by

$$\gamma_{xx}(f) = 1. \tag{20}$$

That is, the sum array and product array differ in their complex quality ratios simply by the factor $\gamma_{xy}(f)$, which is the complex coherence of the two arrays in the product formulation. Thus the relative performance of a product array can be investigated by determining the coherence of its component array outputs.

With this information, we can now give a qualitative measure of performance of the cross-line array. Consider two line arrays lying along the horizontal and vertical axes, respectively. Suppose both lines are steered to the same look direction in three-dimensional space. Since a line array must inherently have maximum response everywhere in a cone of symmetry centered on the line, the two cones will intersect at the desired look direction, but will have largely non-overlapping cones at other angles. Thus only a small fraction of the output of each array is in common; in fact, most of each array output comes from uncommon arrival angles.

Thus the two line array outputs will be largely independent of each other, meaning low coherence. Furthermore, the longer the line arrays, the finer becomes the angular resolution, and the common intersection volume of the cones decreases. Thus the performance of the product array relative to the sum array becomes poorer as the line arrays become longer. In the next section, these conclusions will be verified by a detailed quantitative analysis of the coherence between two general array outputs.

CROSS-SPECTRUM OF INDIVIDUAL ARRAY OUTPUTS

Let the pressure field at time t and general location x,y in a planar array be denoted by p(t,x,y). Let the field be stationary and homogeneous, with temporal-spatial correlation

$$\overline{p(t_1,x_1,y_1)\ p(t_2,x_2,y_2)} = R_p(t_1-t_2,x_1-x_2,y_1-y_2) \ . \tag{21}$$

The frequency-wavenumber spectrum corresponding to R_p is $\Phi_p(f,\mu,\nu)$, where

$$R_p(\tau,u,v) = \iiint df\ d\mu\ d\nu\ \exp(i2\pi f\tau+iu\mu+iv\nu)\ \Phi_p(f,\mu,\nu) \ . \tag{22}$$

We also define a partial transform of (22) as

$$\mathcal{G}_p(f,u,v) = \iint d\mu\ d\nu\ \exp(iu\mu+iv\nu)\ \Phi_p(f,\mu,\nu) \ . \tag{23}$$

This mixed function of temporal frequency and spatial separations will be of prime importance later, especially if it can be evaluated in closed form.

The grid structure of the planar array is such that the elements are equi-spaced, being located at positions md,nd in the x,y plane, for m,n integer. If a particular element is absent or is not used in an array output, the weighting of that element output is simply set equal to zero. For polar angle ϕ as measured from the z-axis, and azimuthal angle θ as measured from the x-axis, the time delay τ_{mn} employed at location md,nd, in order to steer in desired look direction ϕ_ℓ, θ_ℓ, is [2, Appendix A]

$$\tau_{mn} = -\frac{d}{c}(\alpha m+\beta n) \ , \tag{24}$$

where d is the element spacing, c is the speed of propagation, and

$$\alpha = \sin\phi_\ell\ \cos\theta_\ell \ , \ \beta = \sin\phi_\ell\ \sin\theta_\ell \ . \tag{25}$$

Broadside to the planar array corresponds to $\phi_\ell = 0$.

Array output x(t) is synthesized by choosing a particular subset of elements in the planar grid structure, weighting their outputs, and time-delay steering to the desired look direction ϕ_ℓ, θ_ℓ . Thus*

$$x(t) = \sum_{k\ell} w_x(k,\ell)\ p(t-\tau_{k\ell},\ kd,\ \ell d)\ . \tag{26}$$

By simply setting some weights to zero, a line array or a cross-line, or any desired array configuration, can be realized. Similarly, a second array output, which may employ some or all of the same elements, is

$$y(t) = \sum_{mn} w_y(m,n)\ p(t-\tau_{mn},\ md,\ nd)\ . \tag{27}$$

The cross-correlation of array outputs (26) and (27) is, upon use of (21),

$$R_{xy}(\tau) = \overline{x(t)y(t-\tau)} = \sum_{k\ell mn} w_x(k,\ell)\ w_y(m,n)\ R_p(\tau+\tau_{mn}-\tau_{k\ell},(k-m)d,(\ell-n)d). \tag{28}$$

Using (23) and (24), the cross-spectrum of x and y is

$$G_{xy}(f) = \sum_{k\ell mn} w_x(k,\ell)\ w_y(m,n)\ \exp[i2\pi f(\tau_{mn}-\tau_{k\ell})]\ \mathcal{G}_p(f,(k-m)d,(\ell-n)d) =$$

$$= \sum_{k\ell mn} w_x(k,\ell)\ w_y(m,n)\ \exp[i2\pi f\tfrac{d}{c}(\alpha k+\beta\ell-\alpha m-\beta n)]\ \mathcal{G}_p(f,(k-m)d,(\ell-n)d) =$$

$$= \sum_{qr} \phi_{xy}(q,r)\ \exp\left[i2\pi f\tfrac{d}{c}(\alpha q+\beta r)\right]\ \mathcal{G}_p(f,qd,rd)\ , \tag{29}$$

where we let q=k-m, r= -n, and defined

$$\phi_{xy}(q,r) = \sum_{k\ell} w_x(k,\ell)\ w_y(k-q,\ell-r) \tag{30}$$

*A summation without limits is over the entire range of its non-zero summand.

as the two-dimensional cross-correlation of the weight structure employed to yield array outputs x(t) and y(t).

As the first special case of the above, a pair of perpendicular cross-lines can be realized by setting

$$w_x(k,\ell) = 0 \text{ except for } \ell = 0,$$
$$w_y(m,n) = 0 \text{ except for } m = 0; \tag{31}$$

in this case, (30) yields

$$\phi_{xy}(q,r) = w_x(q,0)\ w_y(0,-r), \tag{32}$$

and (29) becomes the cross-spectrum of the cross-line array,

$$G_{xy}^{(c)}(f) = \sum_{qr} w_x(q,0)\ w_y(0,-r)\ \exp[i2\pi f\tfrac{d}{c}(\alpha q+\beta r)]\ \mathcal{G}_p(f,qd,rd). \tag{33}$$

A second special case is obtained by considering the sum array, which corresponds to setting

$$w_x(k,\ell) = w_y(k,\ell) = w(k,\ell), \tag{34}$$

thereby getting from (30)

$$\phi_{xy}(q,r) = \sum_{k\ell} w(k,\ell)\ w(k-q,\ell-r) \equiv \phi_s(q,r)\ , \tag{35}$$

and from (29), the auto-spectrum of the sum array,

$$G_{xy}^{(s)}(f) = \sum_{qr} \phi_s(q,r)\ \exp[i2\pi f\tfrac{d}{c}(\alpha q+\beta r)]\ \mathcal{G}_p(f,qd,rd)\ . \tag{36}$$

Comparison of special cases (33) and (36) reveals that the mean responses of the cross-line and sum arrays can be made equal by setting

$$w_x(q,0)\ w_y(0,-r) = \phi_s(q,r) = \sum_{k\ell} w(k,\ell)\ w(k-q,\ell-r) \text{ for all } q,\ r\ . \qquad (37)$$

So, for example, if the sum weighting $w(k,\ell)$ is flat over a rectangle, $\phi_s(q,r)$ is triangular in q as well as r, over a rectangle twice the size. Thus, if line weightings $w_x(q,0)$ and $w_y(0,-r)$ are each triangular over this double-length, (37) is satisfied, and $G_{xy}^{(c)}(f) = G_{xy}^{(s)}(f)$. This conclusion holds irrespective of the pressure field statistics $\mathscr{G}_p(f,qd,rd)$.

Returning now to the general case of (29) and (30), the auto-spectrum $G_x(f)$ of array output $x(t)$ in (26) is easily obtained by replacing w_y by w_x in (30) and using that result for ϕ_{xy} in (29). A similar procedure, but now replacing w_x by w_y, yields the auto-spectrum $G_y(f)$ of array output $y(t)$ in (27). Combined with (29) itself, we now have the capability of calculating coherence $\gamma_{xy}(f)$ as given earlier by (19).

An alternative illuminating form for cross-spectrum $G_{xy}(f)$ is obtained by substituting, for $\mathscr{G}_p$ in the second line of (29), the expression (23) and interchanging summation and integration:

$$G_{xy}(f) = \iint d\mu\ d\nu\ \Phi_p(f,\mu,\nu)*$$

$$*W_x(2\pi f\tfrac{d}{c}\alpha+d\mu,\ 2\pi f\tfrac{d}{c}\beta+d\nu)\ W_y^*(2\pi f\tfrac{d}{c}\alpha+d\mu,\ 2\pi f\tfrac{d}{c}\beta+d\nu)\ , \qquad (38)$$

where

$$W_x(a,b) = \sum_{k\ell} w_x(k,\ell)\ \exp(ika+i\ell b),$$

$$W_y(a,b) = \sum_{mn} w_y(m,n)\ \exp(ima+inb)\ , \qquad (39)$$

are the response patterns of the x and y arrays. Since the patterns in (39) peak at $a = b = 0$, the integral in (38) is dominated by the contribution at spatial frequencies

$$\mu = -2\pi\frac{f}{c}\alpha = -\frac{2\pi}{\lambda}\sin\phi_\ell\cos\theta_\ell ,$$

$$\nu = -2\pi\frac{f}{c}\beta = -\frac{2\pi}{\lambda}\sin\phi_\ell\sin\theta_\ell . \qquad (40)$$

That is, the cross-spectrum in (38) is influenced mainly by frequency-wavenumber spectrum value $\Phi_p(f,\mu,\nu)$ at the values given by (40). $\lambda=c/f$ is the wavelength at the frequency f of interest.

EXAMPLES OF FREQUENCY-WAVENUMBER SPECTRA

Square Support

Suppose that at some frequency $f=f_o$, the frequency-wavenumber spectrum is flat over a square:

$$\Phi_p(f_o,u,v) = \left\{ \begin{array}{l} \Phi_1(f_o)\,\dfrac{1}{4k_L^2} \quad \text{for } |u| < k_L,\ |v| < k_L \\ \\ 0 \quad \text{otherwise} \end{array} \right\}, \tag{41}$$

where k_L is the common limit of wavenumbers in u and v. Then from (23),

$$\mathcal{P}_p(f_o,u,v) = \Phi_1(f_o)\,\frac{\sin(k_L u)}{k_L u}\,\frac{\sin(k_L v)}{k_L v}\,. \tag{42}$$

This closed-form expression is separable in u and v and will lead to worthwhile simplifications when employed in (29), and especially cross-line result (33).

Circular Support

Suppose instead that

$$\Phi_p(f_o,u,v) = \left\{ \begin{array}{l} \Phi_1(f_o)\,\dfrac{1}{\pi k_L^2} \quad \text{for } u^2 + v^2 < k_L^2 \\ \\ 0 \quad \text{otherwise} \end{array} \right\}\,. \tag{43}$$

Then (23) yields

$$\mathcal{P}_p(f_o,u,v) = \Phi_1(f_o)\,\frac{2J_1\left(k_L\sqrt{u^2+v^2}\right)}{k_L\sqrt{u^2+v^2}}\,, \tag{44}$$

which has circular symmetry in separation space u,v. Although in closed form, it is not separable in u and v, and is more time-consuming to evaluate than (42).

CROSS-LINE ARRAY WITH SQUARE-SUPPORT SPECTRUM

In this section, we specialize to a cross-line array as depicted in figure 1 and couple it with the square-support frequency-wavenumber spectrum

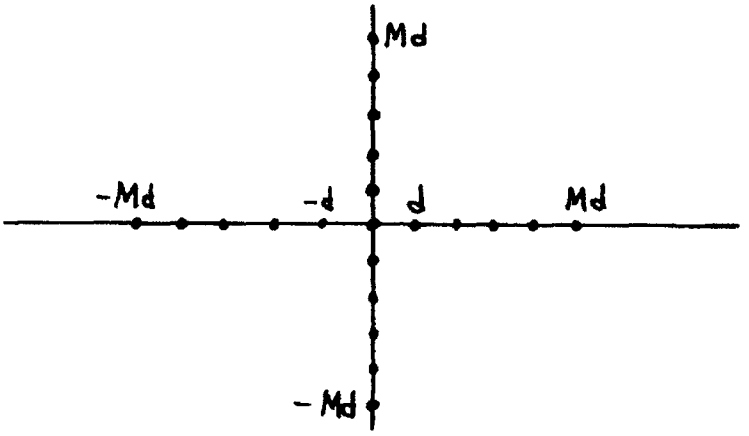

Figure 1. Cross-Line Array of 4M+1 Elements

of (41). The cross-line array has one element at the origin and M elements in each of the four perpendicular legs protruding from the origin, for a total of 4M+1 elements. All elements are spaced by d in both directions, consistent with (24)-(27).

The cross-spectrum at frequency f_o, between the two perpendicular lines, is given by (33) with (42):

$$G_{xy}^{(c)}(f_o) = \Phi_1(f_o) \sum_{qr} w_x(q,0)\, w_y(0,-r)\, \exp[ik_o d(\alpha q+\beta r)]\, \frac{\sin(k_L dq)}{k_L dq}\, \frac{\sin(k_L dr)}{k_L dr} =$$

$$= 4\Phi_1(f_o) \sum_{q=0}^{M} \varepsilon_q w_x(q,0)\cos(k_o d\alpha q)\, \frac{\sin(k_L dq)}{k_L dq} \sum_{r=0}^{M} \varepsilon_r w_y(0,r)\cos(k_o d\beta r)\, \frac{\sin(k_L dr)}{k_L dr}, \quad (45)$$

where

$$k_o \equiv \frac{2\pi}{\lambda_o} = 2\pi f_o/c\ , \qquad \varepsilon_q \equiv \begin{cases} 1/2 & \text{for} \quad q = 0 \\ 1 & \text{for} \quad q \geq 1 \end{cases}, \quad (46)$$

and the weight structure on each line has been assumed real and symmetric about the origin. The quantities in (45) are all real and require only two single-summations of size M at each value of the three dimensionless parameters $k_o d\alpha$, $k_o d\beta$, $k_L d$. Here, k_L is the common limit on allowed wavenumbers in square-support spectrum (41), and k_o is the wavenumber corresponding to frequency f_o of interest. Quantities α and β are given in terms of the look direction according to (25).

To find the auto-spectrum of line-array output x(t), we replace w_y by w_x in (30) and use the upper line of (31):

$$\phi_{xx}(q,r) = \sum_k w_x(k,0)\ w_x(k-q,-r) = \Psi_{xx}(q)\ \delta_{ro}, \tag{47}$$

where

$$\Psi_{xx}(q) \equiv \sum_k w_x(k,0)\ w_x(k-q,0) \tag{48}$$

is the auto-correlation of the x-array weights. Substitution of (47) in (29) yields

$$G_x^{(c)}(f_o) = \sum_q \Psi_{xx}(q)\ \exp(ik_o d\alpha q)\, \Phi_p(f_o, qd, 0) =$$

$$= 2\,\Phi_1(f_o) \sum_{q=0}^{2M} \varepsilon_q\, \Psi_{xx}(q)\, \cos(k_o d\alpha q)\, \frac{\sin(k_L dq)}{k_L dq}, \tag{49}$$

where we used (42) and (46). In a similar fashion, the auto-spectrum of line-array output y(t) is given by

$$G_y^{(c)}(f_o) = 2\Phi_1(f_o) \sum_{r=0}^{2M} \varepsilon_r\, \Psi_{yy}(r)\, \cos(k_o d\beta r)\, \frac{\sin(k_L dr)}{k_L dr}, \tag{50}$$

where

$$\Psi_{yy}(r) \equiv \sum_{\ell} w_y(0,\ell)\; w_y(0,\ell-r) \;, \tag{51}$$

in keeping with (31). The auto-spectral results in (49) and (50) each require real single-sums of size 2M. No double summations are required in the cross-spectral result of (45) or in the auto-spectral results of (49) and (50), although in the latter cases, the auto-correlations Ψ_{xx} and Ψ_{yy} must be pre-computed. The complex coherence at frequency f_o, $\gamma_{xy}^{(c)}(f_o)$, between individual array outputs x(t) and y(t) of the cross-line, is given by ratio (19) as usual; the factor $\Phi_1(f_o)$, as well as the absolute scales of the weight structures $\{w_x(k,0)\}$, $\{w_y(0,\ell)\}$, will cancel out in the ratio. The fundamental parameters are $k_o d\alpha$, $k_o d\beta$, $k_L d$.

For broadside steering of the cross-line array, we have $\phi_\ell = 0$ and (25) yields $\alpha = \beta = 0$. Then (45), (49), (50) are independent of k_o, and the coherence depends only on $k_L d$.

CROSS-LINE ARRAY WITH CIRCULAR-SUPPORT SPECTRUM

By combining (33) with the circular-support frequency-wavenumber spectrum of (44), and using assumed symmetry of the line-array weights, we obtain cross spectrum

$$G_{xy}^{(c)}(f_o) = 4\Phi_1(f_o) \sum_{q=0}^{M} \epsilon_q w_x(q,0)\cos(k_o d\alpha q) \sum_{r=0}^{M} \epsilon_r w_y(0,r)\cos(k_o d\beta r)\frac{2J_1(k_L d\sqrt{q^2+r^2})}{k_L d\sqrt{q^2+r^2}}. \quad (52)$$

This double sum can no longer be separated into the product of two single sums, as (45) was, due to the coupling caused by the Bessel function. However, the Bessel function need only be computed in a 45° sector of the q,r plane and then reflected about the 45° line; i.e., the same value is attained for q,r = m,n as for q,r = n,m.

The auto-spectrum of the x-array output is obtained by utilizing (44) in the top line of (49):

$$G_x^{(c)}(f_o) = 2\Phi_1(f_o) \sum_{q=0}^{2M} \epsilon_q \Psi_{xx}(q) \cos(k_o d\alpha q) \frac{2J_1(k_L dq)}{k_L dq}, \quad (53)$$

where Ψ_{xx} is again given by (48). In a similiar fashion, there follows

$$G_y^{(c)}(f_o) = 2\Phi_1(f_o) \sum_{r=0}^{2M} \epsilon_r \Psi_{yy}(r) \cos(k_o d\beta r) \frac{2J_1(k_L dr)}{k_L dr}. \quad (54)$$

The coherence is now available from (52)-(54). This example was not pursued numerically.

RESULTS

A program for the calculation of the coherence of a cross-line array, via (45)-(51), is given in appendix A. It was exercised to give the following results in figures 2-6. Although these plots look virtually identical, closer inspection of the numerical values (not included) reveals that there are small (insignificant) differences, even for the widely different values of $k_o d$, ϕ_ℓ, θ_ℓ considered here. The degradation of the cross-line array relative to the sum array is virtually independent of the particular look angle ϕ_ℓ, θ_ℓ. This can be partially explained by virtue of the fact that at broadside steering, a line array has a narrow beam but covers a full 360° angular sweep, whereas at endfire, the beam is broad but exists at only one angle. Thus the total angular coverage is essentially constant.

The overriding impression of figures 2-6 is that the degradation of the cross-line array is significant, relative to a sum array, in terms of the stability of the cross-spectral estimate. This is particularly so as the size of the array (4M+1 total elements) grows, or as $k_L d$ increases above .5. Further examples of interest may be obtained from the program in appendix A.

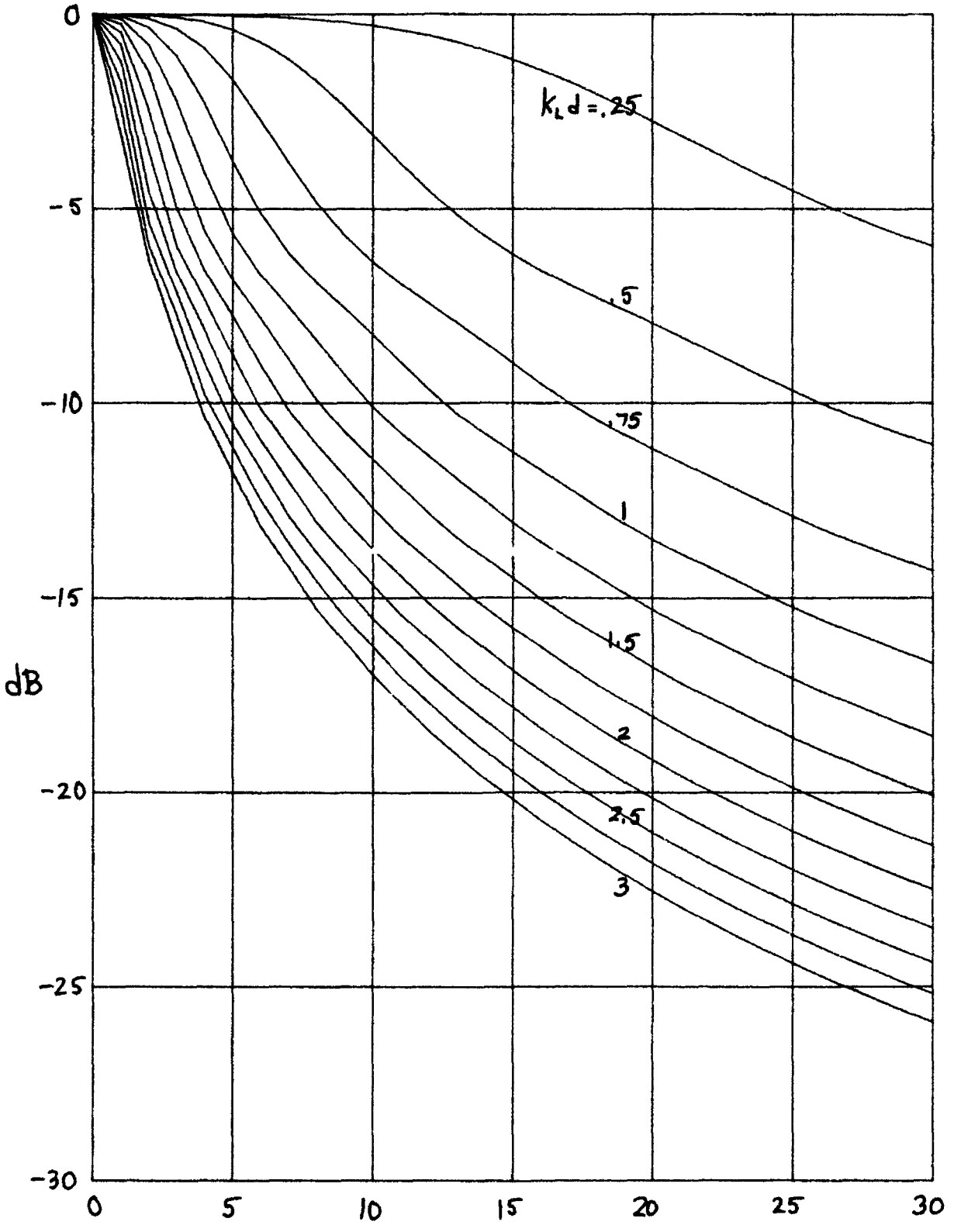

Figure 2. Degradation for $k_0 d = \frac{\pi}{60}$, $\phi_l = 0$, $\theta_l = 0$

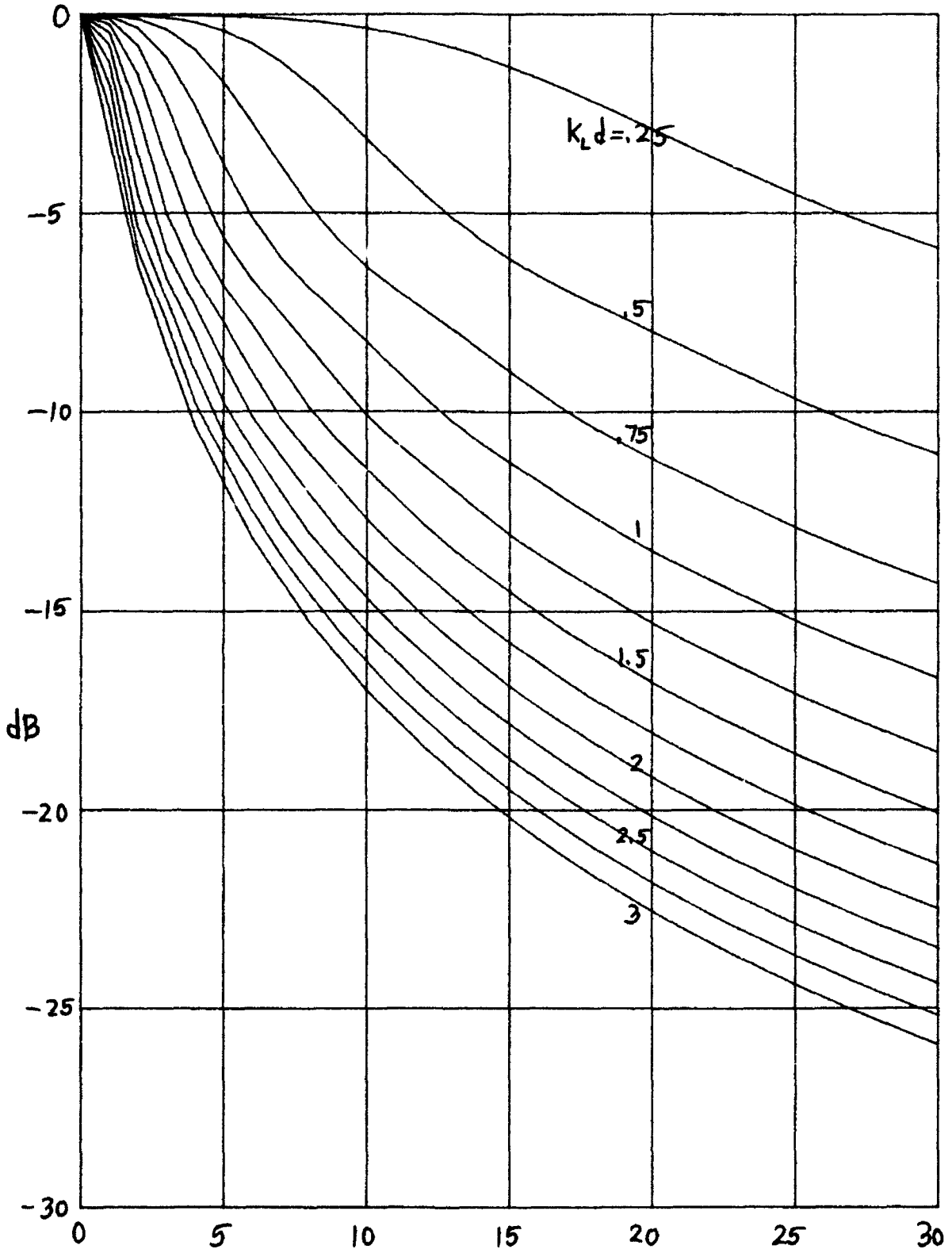

Figure 3. Degradation for $k_0 d = \frac{\pi}{60}$, $\phi_l = \frac{\pi}{2}$, $\theta_l = 0$

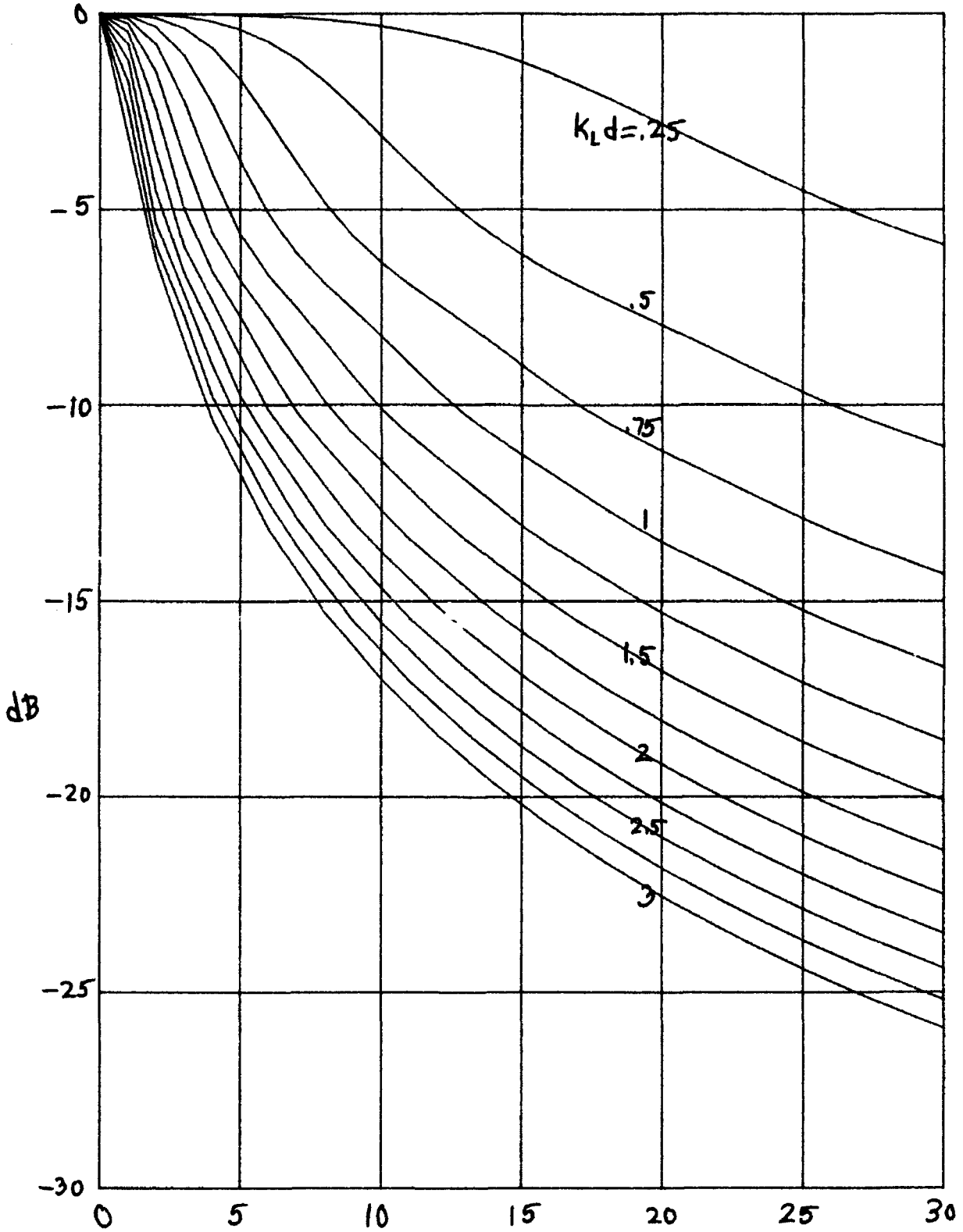

Figure 4. Degradation for $K_0 d = \frac{\pi}{60}$, $\phi_\ell = \frac{\pi}{4}$, $\theta_\ell = \frac{\pi}{4}$

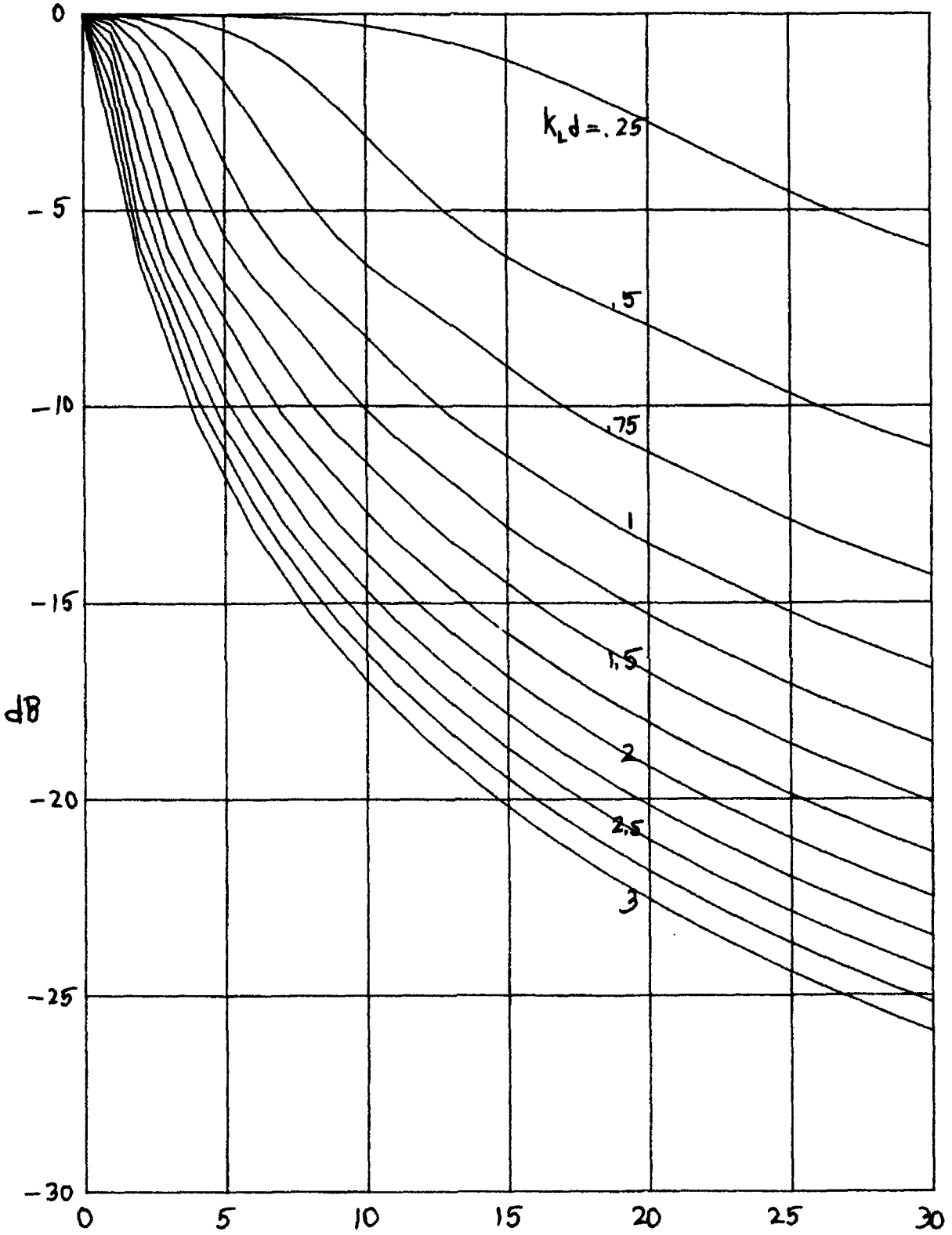

Figure 5. Degradation for $k_0 d = \frac{\pi}{150}$, $\phi_\ell = \frac{\pi}{4}$, $\theta_\ell = \frac{\pi}{4}$

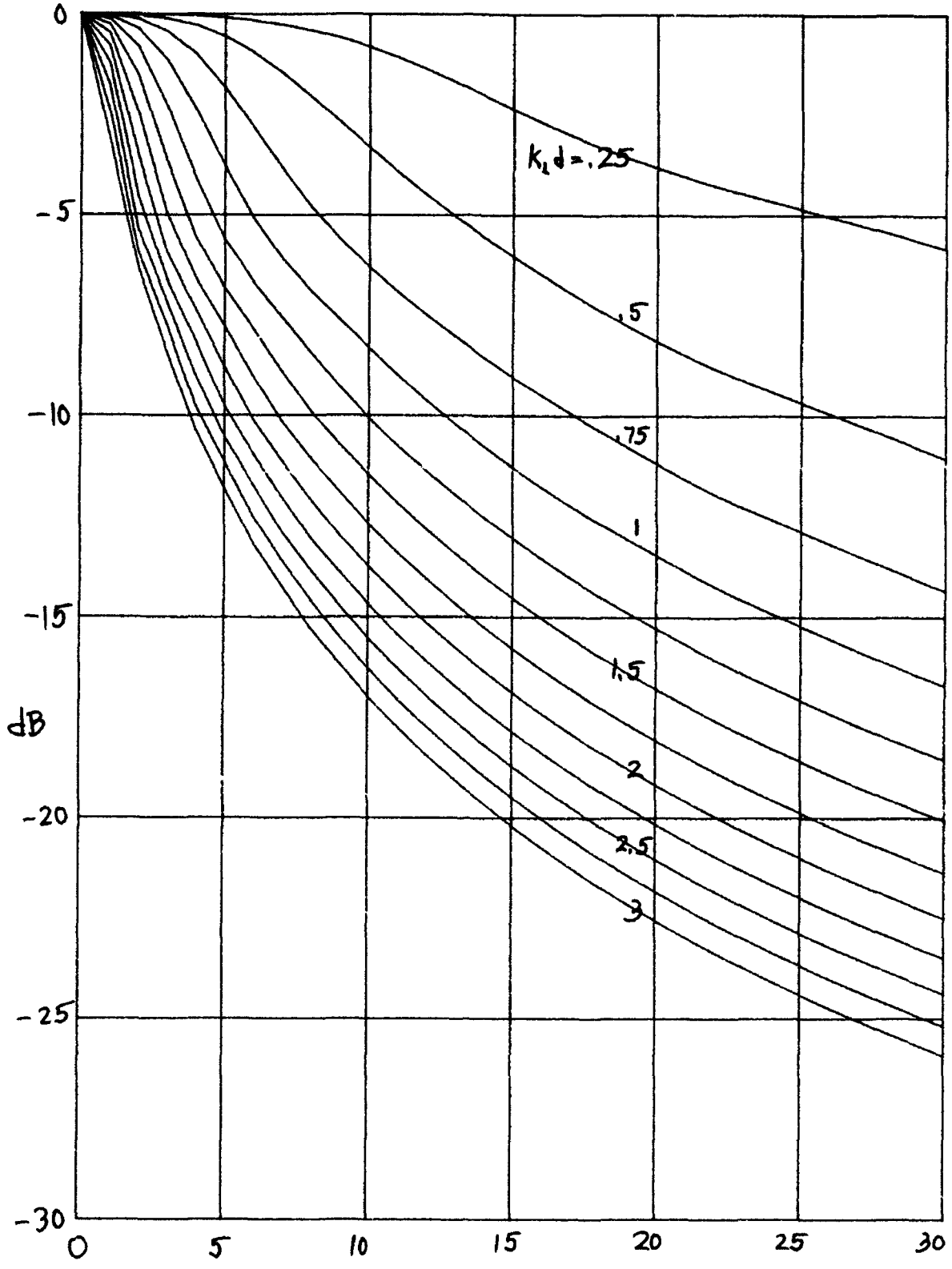

Figure 6. Degradation for $k_0 d = \frac{\pi}{15}$, $\phi_s = \frac{\pi}{4}$, $\theta_s = \frac{\pi}{4}$

SUMMARY

When the horizontal line array in figure 1 is time-delay steered to look in some desired direction ϕ_ℓ, θ_ℓ, it must also respond in a cone of symmetry centered on the axis of the line. Similarily, although the vertical line is steered to look in the same direction ϕ_ℓ, θ_ℓ, it too has a cone of equal response, but now centered on the vertical axis. These two cones will intersect at ϕ_ℓ, θ_ℓ and thereby lead to some common power at frequency f at their respective array outputs, prior to multiplication and averaging according to (3). However, both arrays also respond to uncommon (i.e. uncorrelated) power contributions from other directions, each within its own cone of response. The sharper the beams of each line array, the less common power will be intercepted, leading to less coherence between the two line outputs. Thus the stability of the cross-line array, relative to the sum array, degrades as the size of each line array increases.

If each of the product arrays were made of a parallel pair of lines (separated by some integer multiple of d), the responses of each would be rather complicated. However, there would again be some common overlap at ϕ_ℓ, θ_ℓ, but a great deal of uncommon response at other sidelobe regions, still causing a decreased coherence and unstable estimates. As more parallel lines are added to each product array, the performance should monotonically approach that of the sum array, being actually realized when all of the available parallel lines are employed, since the outputs of the horizontal and vertical arrays are then identical.

APPENDIX A. PROGRAM FOR CROSS-LINE ARRAY

The following program calculates the coherence of a cross-line array by means of (45)-(51). M is the number of elements in each of the four perpendicular legs protruding from the origin. The weighting in line 60 is assumed the same for both lines and is triangular. The autocorrelation of the weight structure is computed and stored in line 130. Inputs of $k_0 d$, ϕ_ℓ, θ_ℓ are required in lines 160-180 respectively. When these latter inputs are desired changed, the program can be continued at line 160 instead of 10, provided that the array weights have not also been changed. The input of $k_L d$ occurs in line 320. The dB output in line 520 is according to

$$dB = 10 \log_{10} \left| \gamma_{xy}(f_0) \right|^2 ,$$

since γ_{xy} is proportional to the voltage quality ratio.

```
10    D[illegible]
20    DI[illegible]
30    FO[illegible]
40    M1=M+1
50    FO[illegible]
60    [illegible]
70    NE[illegible]
80    FO[illegible]
90    S=0
100   FO[illegible]
110   S=[illegible]
120   NE[illegible]
130   R[illegible]
140   NE[illegible]
150   NE[illegible]
```

```
160  Kod=PI/5                    !  INPUT
170  Polar=PI/4                  !  INPUT
180  Azimuth=PI/4                !  INPUT
190  S=Kod*SIN(Polar)
200  Ta=S*COS(Azimuth)
210  Tb=S*SIN(Azimuth)
220  FOR Os=0 TO 60
230  Cosa(Os)=COS(Ta*Os)
240  Cosb(Os)=COS(Tb*Os)
250  NEXT Os
260  PLOTTER IS "9872A"
270  LIMIT 30,170,10,235
280  OUTPUT 705;"IN"
290  SCALE 0,30,-30,0
300  GRID 5,5
310  PENUP
320  FOR Kld=.25 TO 3 STEP .25
330  Sinc(0)=1
340  FOR Os=1 TO 60
350  S=Kld*Os
360  Sinc(Os)=SIN(S)/S
370  NEXT Os
380  FOR M=0 TO 30
390  S1a=S1b=.5*W(M,0)
400  FOR Os=1 TO M
410  S=W(M,Os)*Sinc(Os)
420  S1a=S1a+S*Cosa(Os)
430  S1b=S1b+S*Cosb(Os)
440  NEXT Os
450  S2a=S2b=.5*Ps(M,0)
460  FOR Os=1 TO M+M
470  S=Ps(M,Os)*Sinc(Os)
480  S2a=S2a+S*Cosa(Os)
490  S2b=S2b+S*Cosb(Os)
500  NEXT Os
510  Coherence=2*S1a*S1b/SQR(S2a*S2b)
520  Db(M)=10*LGT(Coherence^2)
530  NEXT M
540  FOR M=0 TO 30
550  PLOT M,Db(M)
560  NEXT M
570  PENUP
580  NEXT Kld
590  END
```

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1. Nuttall, A. H., "Spectral Estimation by Means of Overlapped FFT Processing of Windowed Data", NUSC Report No. 4169, 13 October 1971.

2. Nuttall, A. H., "A Two-Parameter Class of Bessel Weightings with Controllable Sidelobe Behavior for Linear, Planar-Circular, and Volumetric-Spherical Arrays; The Ideal Weighting-Pattern Pairs", NUSC Technical Report 6761, 1 July 1982.

Subject Matter Index

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