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FOUNDATIONS OF THE GENERAL THEORY OF VOLLEY FIRE

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Volley fire problems arise frequently in applied military operations research work, and this paper develops a powerful general theory whose systematic application to volley fire problems greatly aids in their solution. This provides US Army and other military analysts ready access to systematic methods whose application can greatly simplify the solution of volley fire attrition models. In the simpler cases, the theory leads directly to elegant formulas for the expectation and variance of the number of survivors. In more complicated situations, it provides algorithms useful for numerical calculations. After sampling previous work on the analysis of volley fire models, the general theory is developed and applied to a number of volley fire situations of practical interest. The theory powerfully unifies and extends previously used methods for solving volley fire problems and often provides simpler and more intuitive solutions than have previously appeared.

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DEPARTMENT OF THE ARMY

REPLY TO
ATTENTION OF

16 APR 1993

CSCA-MVM

MEMORANDUM FOR Deputy Under Secretary of the Army (OR), Headquarters,
Department of the Army, Washington, DC 20310

SUBJECT: Foundations of the General Theory of Volley Fire

1. The U.S. Army Concepts Analysis Agency (CAA) is pleased to publish this
Research Paper on the general theory of volley fire. It gathers together in one place
and provides a unifying theoretical basis for several past works in this field. As
such, it makes a unique contribution to the literature on the attrition processes
that take place in combat. The availability of this treatment will smooth the way
for future developments in this field.

2. Questions or inquiries should be directed to the Office of Special Assistant for
Model Validation, U.S. Army Concepts Analysis Agency (CSCA-MV), 8120
Woodmont Avenue, Bethesda, MD 20814-2797, (301) 296-1611.

E. B. VANDIVER III
Director

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CAA	FOUNDATIONS OF THE GENERAL THEORY OF VOLLEY FIRE	SUMMARY CAA-RP-92-1

THE REASON FOR PERFORMING THIS RESEARCH was that volley fire problems arise frequently in applied military operations research work. This paper develops a powerful general theory whose systematic application to volley fire problems greatly aids in their solution.

THE STUDY SPONSOR was the Director, US Army Concepts Analysis Agency.

THE STUDY OBJECTIVE was to provide the US Army and other military analysts ready access to systematic methods whose application can greatly simplify the solution of volley fire attrition models.

THE SCOPE OF THE STUDY involved reviewing a sample of the past work on volley fire problems, developing new and more powerful methods for their solution, and illustrating their application to several volley fire situations of practical interest.

THE PRINCIPAL FINDINGS of the work reported herein are that, in the simpler cases, the theory leads directly to elegant formulas for the expectation and variance of the number of survivors. In more complicated situations, it provides algorithms useful for numerical calculations. After sampling previous work on the analysis of volley fire models, the general theory is developed and applied to a number of volley fire situations of practical interest. The theory powerfully unifies and extends previously used methods for solving volley fire problems and often provides simpler and more intuitive solutions than have previously appeared. It also yields hitherto unpublished results. Our approach also shows that volley fire models generalize many of the classical probability problems in the theory of matchings, occupancy, and statistical mechanics. In addition, it suggests potentially important new concepts, such as those for equivalent and complementary volleys. It also provides a useful system for classifying volleys into a few "canonical forms" based on their common features, which facilitate their solution by avoiding the need for *ad hoc* methods. Several potential areas for further investigation are also suggested.

THE STUDY EFFORT was directed by Dr. Robert L. Helmbold, Office of the Special Assistant for Model Validation.

COMMENTS AND SUGGESTIONS may be sent to the Director, US Army Concepts Analysis Agency, ATTN: CSCA-MV, 8120 Woodmont Avenue, Bethesda, Maryland, 20814-2797.

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1. INTRODUCTION

Estimating the attrition that results when a battery of weapons shoots at an array of targets is one of the most characteristic activities of military operations analysts. In many cases the shots are fired as a volley, or nearly so. In this paper a group of shots is called a volley if all of them are fired before the weapons adjust their operations on the basis of any damage done to the target array. When the state of the target array is taken to specify which targets are still alive, a more precise restatement of this definition is that the weapons act only on the state of the target array at the start of the volley, and not on any change in state that occurs while the volley is in progress.

This definition generalizes those given in the prestigious Oxford English Dictionary [1971], which defines a volley as "A simultaneous discharge of a number of firearms or artillery; a salvo," a salvo as "A simultaneous discharge of artillery or other firearms, whether with hostile intent or otherwise," and a fusillade as "A simultaneous discharge of firearms; a wholesale execution by this means." When weapons are discharged simultaneously, the battery of weapons plainly does not have time during the volley to adjust to any damage done to the target array, and in that case our definition agrees with those given in the Oxford English Dictionary. However, in this paper we continue to speak of a volley fire situation whenever the weapons do not perceive (or if for any reason they do not heed) changes in the state of the target array. In such cases, whether the shots are fired simultaneously or over an extended period of time is clearly immaterial, because at the end of the volley their effects are the same.

In addition, the results presented in this paper can be applied not only to artillery or firearms, but to a wider class of weapons including rockets, antitank weapons, intercontinental ballistic missiles, machineguns, bombs, anti-aircraft artillery, and so forth. For this reason, volleys delivered by a battery of weapons against an array of targets are often used to model attrition in military operations research. They frequently appear as components of larger models, simulations, or war games, where they are used to assess the outcomes of individual volleys, or of several successive volleys, or of countervolleys fired alternately by one side and then by the other. For one example of such component volleys, see Ketron [1983]. When successive volleys are fired, the state of the system at the end of each successive volley usually depends only on the state of the system at the start of the volley, in which case the state vector evolves according to a Markov process, the importance of which in models of combat interactions has been emphasized by Koopman [1970], among others. In this paper, several volleys of practical military interest are presented and solved to illustrate the general theory's ability to yield previously unpublished results, as well as to provide simpler and more intuitive derivations of known results.

Our theory takes for its object the determination of the outcome when a battery of weapons volleys against a target array. It makes use of techniques borrowed from the fields of combinatorics, probability, algebra, and analysis (the latter principally in connection with limit laws of probability). Although

weapon batteries and target arrays consisting of a single element are technically within the purview of volley fire theory, they are normally viewed as special cases and are analyzed as duels. Thus, the theory of volley fire concentrates on cases where the battery of weapons and the target array both contain several elements. The scope of the theory is intended to include the analysis of multiple volleys by a battery against a single target array, and also exchanges of volleys where the weapons battery and the target array volley back and forth. Unfortunately, the current state of the theory does not provide a very satisfying treatment of multiple or counter volleys. Consequently, apart from a few passing remarks, this paper concentrates on single volleys with the understanding that they can be chained together in various (often *ad hoc*) ways to estimate the effects of successive volleys, if so desired.

The central object in this paper is the computation of the probability of various outcomes of a single volley. We want to know such things as:

1. How many targets can be expected to survive the volley?

2. How variable is the number of survivors?

3. What is the probability that 6 targets survive? That 12 survive? In general, what is the probability that t targets survive?

4. If the target array consists of two or more types of target, what are the correlations between the number of survivors of each type?

5. What is the probability that all targets of a specified type will be wiped out?

Until recently, volley fire problems were treated by *ad hoc* methods that gave limited results for special cases. Despite the ingenuity of some of these *ad hoc* methods, they concentrated so strongly on special cases that their general theoretical foundations tended to be hidden rather than revealed. Just very recently, it was recognized that there are some deeper and more general concepts whose systematic application to volley fire problems can greatly aid in their solution. In the simpler cases, these concepts lead directly to elegant formulas for the expectation, variance, and correlation of the number of survivors. In more complicated situations, they provide algorithms useful for numerical computations. The systematic application of these general analytical methods has led to simpler and more intuitive proofs of all of the known results in the theory and has clarified their interrelations. This approach has also led to several new and previously unknown results. This has put us in the position where, for the first time, it appears that such a thing as a theory of volley fire might exist.

It also turns out that the theory of volley fire includes as a special case all of the theory of random allocations. That familiar field of classical probability theory deals with the random allocations or distributions of r objects into n cells. Treatments of the problem of *rencontres* or random matchings; of the distribution of particles among energy states for Maxwell-Boltzman, Bose-Einstein, or Fermi-Dirac

statistics in the kinetic theory of matter; and of many other famous classical problems are examples of those dealt with in the theory of random allocations. Certain aspects of this classical theory of random allocations have recently been developed extensively by the Russian mathematician Kolchin [1978] and colleagues. However, they have dealt almost exclusively with the study of limiting distributions for certain classes of random allocations. In contrast, practically all of the extant work on the theory of volley fire has been devoted to solving certain difficult combinatorial problems in the theory of probability. In the future, it may be possible and desirable to extend to the theory of volley fire some of the asymptotic results from the theory of random allocations.

Several authors have studied special cases of volley fire, and Table 1 shows a sample of the earlier work on special cases. We will say more about these papers presently. The general theory presented in this paper supersedes these specialized approaches, because it can readily be used not only to reproduce all of the previous results, but to provide additional information not obtainable by the specialized methods. Because volley fire models have arisen in a variety of contexts, other works on them may not have come to our attention, and we apologize in advance to the authors of any volley fire analyses not listed in Table 1.

Dixon [1953] was apparently one of the first to analyze a volley fire situation. He showed by selected examples how the outcome of repeated volleys by a homogeneous battery of weapons (that is, one in which the weapons are all alike) against a homogeneous array of passive targets could be computed. He applied his results to calculate the distribution of the number of survivors for some situations in which successive waves of interceptor aircraft (the weapons battery) attack a formation of bombers that is attempting to reach its bomb release zone. Dixon's method for finding the distribution of the number of survivors at the end of a single volley requires the exhaustive enumeration of certain combinatorial configurations. This method is similar in principle to those later employed by Robertson [1956] and by Helmbold [1960]. Dixon works out a few specific examples involving no more than four weapons and only a handful of targets but does not present an explicit algorithm for enumerating the required configurations.

However, Dixon does find an important general formula for the expected number of survivors after a single volley. He argues correctly that the probability a particular weapon selects a particular target (say, target t) is $1/T$, where T is the number of targets alive at the start of the volley. Thus, the probability that target t survives the fire of this weapon is $1 - q/T$, where q is the kill probability.

Table 1. A Sampling of Some Early Work on Volley Fire Models

Principal author	Pub. date	Target array	Weapon battery	Allocation of fire	Provides explicit formulas for:		
					Expected no. of survivors	Variance of no. of survivors	Distribution of no. of survivors
Dixon	1953	Homog.	Homog.	Unif. random	Yes	No	No[a]
Lavin	1953	Homog.	Homog.	Unif. random	No	No	No[a]
Wegner	1954	Homog.	Homog.	Unif. random	Yes	No	No[a]
Thomas	1956	Homog.	Homog.	Unif. random	Yes	No	Yes
Robertson	1956	Homog.	Heterog.	Unif. random	No	No	Yes[b]
Helmbold	1960	Homog.	Heterog.	Unif. random	Yes	No	Yes[b]
Rau	1964	Homog.	Homog.	Unif. random	Yes	Yes	Yes
Rau	1965	Homog.	Homog.	Unif. random	Yes	No	Yes
Ancker	1965	Homog.	Homog.	Unif. random	No	No	Yes
Helmbold	1966	Homog.	Heterog.	Unif. random	Yes	No	Yes
Helmbold	1968	Heterog.	Heterog.	Random	Yes	No	No
Karr	1974	Homog.	Homog.	Compound[c]	Yes	No	Yes
Karr	1974	Heterog.	Heterog.	Compound[c]	Yes	No	No

Notes:

a. Distributions are provided only for a few examples involving a small number of weapons and targets. A general algorithm is not explicitly stated.

b. Although a computationally well-defined algorithm for obtaining the distribution is provided, it requires as an intermediate step the cumbersome generation of certain combinatorial configurations.

c. Allocation of fire is determined by a compound process in which targets are first acquired, and then fire is allocated uniformly at random over the subarray of acquired targets.

Consequently, the probability that target t survives the fires of all of the W weapons in the battery is $(1 - q/T)^W$. Since this probability is the same for each target, the expected number of survivors after one volley is

$$E(T^1) = T(1 - q/T)^W.$$

This elegant result will be called "Dixon's Formula." As indicated in Table 1, it applies when a homogeneous battery of weapons volleys against a homogeneous array of passive targets, provided the weapons allocate their fire to targets selected independently and uniformly at random from the target array. Such volleys, including their generalization to volleys by a heterogeneous battery of weapons against a homogeneous target array, will be called Dixon-Robertson-Rau (DRR) volleys after three who have contributed substantially to the theory of volley fire, although in actuality none of these three provided closed form solutions for the general case where the weapons battery is heterogeneous or the allocation of fire may be nonuniform. However, such formulas are easily found using our general approach.

By using the same methods as Dixon [1953]. Lavin and Wegner [1953] generate distributions of the number of survivors for additional examples of DRR volleys. They obtain expressions for the cases where $W = 1. 2. 3, 4.$ and 8. and for T up to 9. They also apply the then new electronic digital computer technology to compute the matrix products required in the Markov process approach. Wegner [1954] continues in this vein and also introduces a process in which the two sides exchange volleys simultaneously.

Robertson [1956] provides an explicit algorithm for computing the distribution of survivors when a homogeneous battery of weapons volleys against a homogeneous target array and illustrates by example a method for obtaining the distribution of the number of survivors when the battery is heterogeneous. She applies the results to situations in which an infantry rifle squad (the battery) is defending its position against an assault conducted by another rifle squad. She makes no reference to the earlier work of Dixon. Lavin, and Wegner and seems to have arrived independently at her results.

Thomas [1956] derives and solves in closed form a partial difference equation for the distribution of the number of survivors in a DRR volley and obtains Dixon's formula from it. He has in mind the case of interceptors (weapons) against bombers (targets). He also presents computationally convenient recursive formulas and a generating function for the distribution of survivors, and suggests various approximations to that distribution. However, Thomas does not explicitly cite a formula for the variance of the number of survivors. He does note the Markov chain solution for successive volleys, and analyzes a volley in which each weapon in the battery may have kill probability q_1 or q_2 (with probabilities v_1 and $v_2 = 1 - v_1$, respectively). He also considers a heterogeneous target array consisting of just two types of target (bombers and decoys): for this case he analyzes the allocation of a fixed budget to bombers and decoys to maximize the expected number of surviving bombers.

Also in the mid 1950s Helmbold (following up ideas originated jointly by him and his colleagues Martin N. Chase. John C. Flannagan. and Hunter M. Woodall, Jr.) was examining the use of volley fire models to represent situations in which antitank weapons (the battery) are defending against tank assaults. This work was conducted in ignorance of the work of Dixon. Lavin. Wegner. Robertson, and Thomas. Some of it was later recorded in Helmbold [1960], which contains the following Generalized Dixon Formula for the case where the weapons are not all the same:

$$E(T^1) = T \prod_{w=1}^{W} (1 - q_w/T),$$

where q_w is the kill probability of weapon w. This result can be obtained by an argument similar to Dixon's but with minor modifications to allow different kill probabilities for different weapons. By the time Helmbold [1960] was published, he had become aware of Robertson [1956], but not of Dixon [1953]. Lavin [1953]. Wegner [1954], or Thomas [1956].

By applying the principle of inclusion and exclusion to DRR volleys, Rau [1964] not only derives Dixon's Formula, but also finds the distribution and variance of the number of survivors. Subsequently, Rau [1965] obtained the same results by an ingenious and entirely different argument. Rau provides explicit and relatively simple formulas for the number of survivors. His formulas are much more convenient than the algorithms proposed by Robertson [1956] or Helmbold [1960]. Rau's formulas will not be repeated here, because they can be obtained by particularizing more general results given later in this paper. Rau applies his formulas to situations where surface-based air defense weapons (the battery) shoot at an intruding formation of aircraft. There is no indication in either of Rau's reports that he was aware of any of the earlier work on volley fire models.

Ancker and Williams [1965] obtain both iterative and closed form solutions for the distribution of the number of survivors for DRR volleys. They do this by setting up and solving an appropriate partial difference equation. They do not state Dixon's Formula, although it is derivable from their expressions for the distribution of the number of survivors, nor do they cite any of the works listed in Table 1. Helmbold [1966] pointed out that their argument is easily extended to obtain the distribution of survivors when the weapon battery is heterogeneous, and showed that this result leads quickly to the Generalized Dixon Formula.

Later, Helmbold [1968] further generalized these results to the case where the target array as well as the weapons battery is heterogeneous, and where, in addition, weapons select targets independently (but not necessarily uniformly) at random. Unfortunately, no results on the distribution or the variance of the number of survivors can be obtained with the methods used by Helmbold [1968]. This lack will be corrected by the results to be given later in this paper. Helmbold [1968] does not cite the earlier papers of Dixon, Lavin, Wegner, Thomas, or Rau because he was not then aware of their existence.

Karr [1974], motivated by problems in the penetration of aircraft through defended areas, considers a compound process in which each weapon of the battery independently acquires targets. After the acquisition process is completed, each weapon selects exactly one of the targets it has acquired and fires at it; however, a weapon that acquires no targets fires no shots. When the weapons battery and target array are both homogeneous, Karr derives the following formula for the expected number of survivors (we shall call it Karr's Formula, although it was originally proposed on the basis of intuition by LTG Glenn A. Kent, USAF):

$$E(T^1) = T\left[1 - (1 - (1-d)^T)q/T\right]^W,$$

where d is the probability that a particular weapon will acquire a particular target and is assumed to be the same for all weapon-target pairs. Karr also gives the distribution of the number of survivors when the battery and target array are both homogeneous. He obtains the expectation of the number of survivors when the battery and array are both heterogeneous, but not its distribution or variance. Later

we will show how our general theory can be used to provide that information. Karr cites none of the prior work on volley fire models.

Clearly the work just described has been disjointed, unsystematic, and failed to make the best use of earlier work. Results were usually obtained by *ad hoc* methods that (despite their other merits) had an unfortunate tendency to conceal common concepts and generally applicable principles, rather than to reveal them. The main contribution of this paper is to identify some general concepts whose systematic application to volley fire problems can greatly aid in their solution. These general concepts are natural and powerfully unify previously used methods. Several potential areas for further investigation are also suggested. The general approach developed here also reveals that volley fire models generalize in a natural way many of the classical probability problems in the theory of matchings, occupancy, and statistical mechanics. Moreover, they yield hitherto unpublished results, and often provide simpler and more intuitive solutions than have previously appeared. In the simpler cases, these concepts lead easily and directly to elegant formulas for the expectation and variance of the number of survivors. In more complicated situations, they provide algorithms useful for numerical calculations. Applications of the approach to several volley fire situations are presented to illustrate the specific combinatorial techniques that appear most effective in analyzing volley fire problems.

That some reasonably complex problems yield easily to the new methods developed here can be illustrated by considering a sample problem in which a heterogeneous battery of weapons volleys against a heterogeneous target array, and weapons select targets independently (but not uniformly) at random. Suppose, as shown in Figure 1, that three weapons volley against an array of six targets. Weapons 1 and 3 are medium antitank weapons, and each fires three shots during the volley. Weapon 2 is a heavy antitank weapon and fires two shots during the volley. The target array consists of three medium and three heavy tanks, alternating with each other as shown in Figure 1. The heavy tank labeled as target number 4 contains the tank unit's commander. The antitank weapons are 90% reliable. Let v_{wt} be the probability that weapon w ($w = 1$, 2, or 3) directs a reliable shot at target t ($t = 1$, 2, ..., 6). Let q_{wt} be the probability that target t will be killed if a reliable shot is directed at it by weapon w, and let the numerical values of these factors be as in Table 2. Note that, in this example problem,

$$\sum_{t = 1}^{6} v_{wt} = 0.90$$

for $w = 1$, 2, or 3 because the 90 percent reliability per shot has been included in the v_{wt} values. (The reliability factor would have the same effect on the results if, instead, it had been used to reduce the value of q_{wt}.) We assume that each weapon directs each of its shots at a target selected in accord with the v_{wt} values, but independently of all other shots fired. By employing some of the results to be
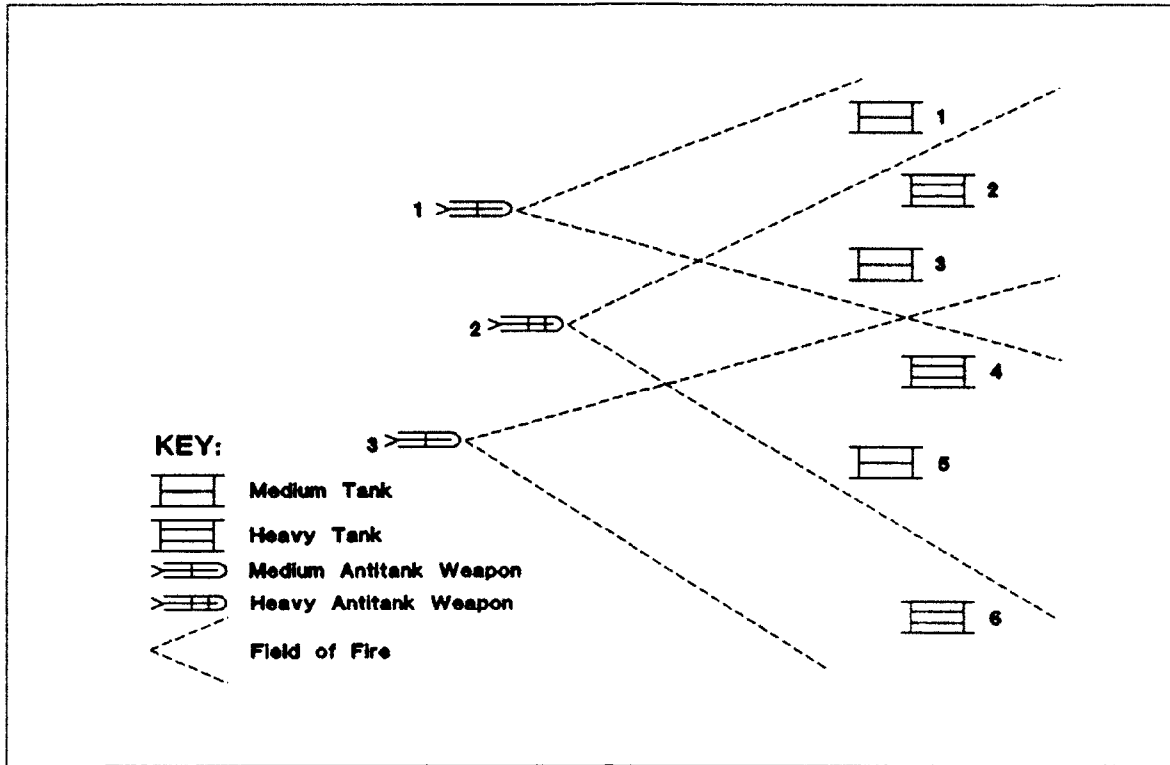
Figure 1: A Sample Volley Problem

Table 2. Values of v_{wt} and q_{wt} for the Sample Problem

	w					
	1		2		3	
t	v_{1t}	q_{1t}	v_{2t}	q_{2t}	v_{2t}	q_{1t}
1	0.23	0.70	0	N/A	0	N/A
2	0.44	0.30	0.30	0.90	0	N/A
3	0.23	0.70	0.15	0.60	0	N/A
4	0	N/A	0.30	0.90	0.35	0.30
5	0	N/A	0.15	0.60	0.20	0.70
6	0	N/A	0	N/A	0.35	0.30
No. shots	3		2		3	

presented in Section 7 for volleys by independently effective weapons and a small hand calculator of the type readily available nowadays, we found in a few minutes the following information.

1. The average and standard deviation of the number of tanks surviving the volley is 3.054 and 1.007, respectively.

2. The average and standard deviation of the number of heavy tanks surviving the volley is 1.447 and 0.759, respectively.

3.   The average and standard deviation of the number of medium tanks surviving the volley is 1.606 and 0.818, respectively.

4.   The probability that the commander's tank survives the volley is 0.382.

5.   The probability that exactly 0, 1, 2, or 3 heavy tanks survive the volley is 0.096, 0.429, 0.407, and 0.068, respectively.

6.   The probability that exactly 0, 1, 2, or 3 medium tanks survive the volley is 0.082, 0.363, 0.422, and 0.133, respectively.

7.   The correlation between the numbers of medium and heavy tanks surviving the volley is -0.187.

It should be emphasized that these values are *not* the result of any form of Monte Carlo simulation. Instead, they are exact values obtained by substituting the assumed values of $v_{w,t}$ and $q_{w,t}$ into exact formulas for the situation described. Consequently, they could be used to verify that a Monte Carlo simulation was operating correctly.

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2. NOTATION AND BASIC CONCEPTS

Suppose that at the start of a volley the target array consists of T targets. The state of the target array at the end of the volley will be represented by the *complexion*

$$(\tau_1, \tau_2, \ldots, \tau_T)$$

where $\tau_j = 1$ if target j is alive at the end of the volley and $\tau_j = 0$ otherwise. (A development in which additional target states are allowed is possible, but is more complex and will not be pursued here.) In principle, any properly posed question regarding the outcome of a volley can be answered if (and only if) the probabilities of each of these 2^T complexions is known. In many volley fire problems, however, a direct evaluation of the probabilities of the complexions is difficult, while the following indirect approach is more effective and in many ways more natural.

One of the characteristics of our approach is that it focuses on the probability that a target survives instead of on the probability that it is killed. This facilitates the theoretical development and yields more elegant formulas for the outcome of a volley. Consequently, we begin by defining z_j to be the event that target j survives, that is,

$$z_j = \left\{ \text{complexions} \mid \tau_j = 1 \right\} \ .$$

The event complementary to z_j is

$$\overline{z_j} = I - z_j = \left\{ \text{complexions} \mid \tau_j = 0 \right\} \ ,$$

where I is the set of all complexions and so carries a probability value of unity. In this paper, set-theoretic intersections are usually written as products, so that (for example) $z_j \overline{z_k} z_m \overline{z_n} z_r$ represents the event that targets $j, m,$ and r survive the volley while targets k and n do not.

Now consider the following *family of basic events:*

$I,$

z_j for $j = 1(1)T,$

$z_j z_k$ for $j = 2(1)T$ and $k = 1(1)(j-1),$

$z_j z_k z_l$ for $j = 3(1)T,\ k = 2(1)(j-1),$ and $l = 1(1)(k-1),$

$\ldots$

$z_1 z_2 z_3 \cdots z_T.$

Here and elsewhere in this paper the notation $m = a(b)c$ denotes that m is a variable that ranges over the set of values $a,\ a+b,\ a+2b,\ \ldots,\ c-b,\ c$. Call a basic event that is specified by the product of exactly r z's an r-th order basic event. For each $r = 0(1)T$, there are exactly $\binom{T}{r}$ r-th order basic

events, because that is the number of combinations of r z's that can be selected from the set of T z's. Consequently, there are a total of 2^T members in the family of basic events.

These basic events and their probabilities play so central a role in the general theory of volley fire that we define a volley to be *solved completely* if the probability of each basic event is known. It is well-known that the probabilities of the basic events suffice to determine the probability of any complexion, and therefore of any well-defined outcome of a volley (see Note 1 in Appendix A). In several cases of practical interest, the basic event probabilities are easily evaluated, as will be demonstrated by the examples presented later in this paper. However, their computation inescapably requires special knowledge or assumptions regarding the tactical behavior as well as the technical military capability of the weapons and targets, and so it is not feasible to provide a useful general formula for them. For the present, we simply take for granted that all or some of the basic event probabilities for the volley in question can be obtained.

Suppose we say that a volley is *solved to order m* if the probabilities of all basic events of order $r = 0(1)m$ are known. Many interesting and important questions can be answered easily, once a volley is solved to some low order. To illustrate this more fully, let A be an arbitrary but fixed collection or *subarray* of T_A targets, that is, subarray A consists of T_A targets "of type A." The subarray A may be identical to the full target array, or may be any proper subarray. Designate the targets in subarray A as $A_1, A_2, \ldots, A_{T_A}$. The probabilities of the following *subfamily of basic events associated with subarray A* are available whenever the volley has been solved to order T_A:

$$\begin{cases} 1, \\ z_{A_j} \text{ for } j = 1(1)T_A, \\ z_{A_j} z_{A_k} \text{ for } j = 2(1)T_A \text{ and } k = 1(1)(j-1), \\ \ldots \\ z_{A_1} z_{A_2} z_{A_3} \cdots z_{A_{T_A}}. \end{cases}$$

For each $r = 0(1)T_A$ this subfamily contains $\binom{T_A}{r}$ r-th order basic events, so in all there are 2^{T_A} basic events in this subfamily.

Now let $P_{A[m]}$ be the probability that exactly m targets of type A survive the volley. Then, by the principle of inclusion and exclusion as described in Feller [1950], Liu [1968], Riordan [1958], Netto [1927], Frechet [1940], Frechet [1943], Ryser [1963], and many other texts on probability and combinatorics, for $m = 0(1)T_A$

$$P_{A[m]} = \sum_{r=m}^{T_A} (-1)^{r-m} \binom{r}{m} S_{A_r} = \sum_{r=0}^{T_A - m} (-1)^r \binom{m+r}{m} S_{A_{m+r}}, \tag{1}$$

where S_{A_r} is the sum of the probabilities of all r-th order basic events in the subfamily associated with subarray A. That is, for $r = 0(1)T_A$,

$$S_{A_r} = \sum_r P(z_{A_{j_1}} z_{A_{j_2}} \cdots z_{A_{j_r}}) = \sum_{j_1 = r}^{T_A} \sum_{j_2 = 1}^{j_1 - 1} \sum_{j_3 = 1}^{j_2 - 1} \cdots \sum_{j_{r-1} = 1}^{j_{r-2} - 1} \sum_{j_r = 1}^{j_{r-1} - 1} P(z_{A_{j_1}} \cdots z_{A_{j_r}}). \quad (2)$$

When there is little chance of confusion, S_{A_r} will be referred to briefly as the *r-th order basic sum.* These basic sums appear frequently in the theory developed in this paper. Observe that the r-th order basic sum is the sum of $\binom{T_A}{r}$ basic event probabilities. Since r varies from 0 to T_A, there are exactly $T_A + 1$ basic sums. Of course, the value of the basic sum S_{A_0} is unity, since it is the probability of the basic event involving no z_{A_j} specification, that is, S_{A_0} is the probability of the set I of all complexions.

In many applications of the theory of volley fire, detailed information as to which *specific* targets survive is not essential and information regarding only the *number* of survivors is sufficient. In such cases, equation (1) shows that the problem reduces to finding the values of $T_A + 1$ basic sums, rather than the probabilities of 2^{T_A} basic events (or complexions). We proceed to show that further simplification is possible when the complete probability distribution of the number of survivors is not required, and only the values of its first few moments are needed.

Let $G_A(x)$ be the generating function for the distribution of the number of survivors, that is,

$$G_A(x) = \sum_{m=0}^{T_A} x^m P_{A[m]} \cdot \quad (3)$$

Note that the probability that at least m targets survive can easily be generated from $G_A(x)$—see Note 2 in Appendix A.

Replacing $P_{A[m]}$ by its value as given by equation (1) and then interchanging the order of summation (having due regard for the region in the (m, r) plane over which the summation extends) yields

$$G_A(x) = \sum_{r=0}^{T_A} \sum_{m=0}^{r} (-1)^r S_{Ar} \binom{r}{m}(-x)^m$$

$$= \sum_{r=0}^{T_A} (x-1)^r S_{Ar} \cdot \quad (4)$$

The expectation and variance of T_A^1, the number of type A targets that survive the volley, are easily obtained from $G_A(x)$ by taking derivatives, and we find:

$$E(T_A^1) = G'_A(1) = S_{A_1} \tag{5}$$

and

$$\mathrm{Var}(T_A^1) = G''_A(1) + G'_A - \left[G'_A(1)\right]^2$$

$$= 2S_{A_2} + S_{A_1} - S_{A_1}^2 \quad . \tag{6}$$

Since A may be any subarray of targets, formulas for the expectation and variance of the total number of survivors can be obtained simply by suppressing A in formulas (5) and (6). It is often important for applications that the expected number of survivors can be obtained from the first order basic sum, its variance from the first two basic sums, and (in general) its n-th order moment from the first n basic sums (see Note 3 in Appendix A).

Now let A and B be arbitrarily prescribed subarrays containing T_A and T_B targets, respectively. Designate the targets in these subarrays as A_j, where $j = 1(1)T_A$, and as B_k, where $k = 1(1)T_B$. The subarrays A and B may overlap in any way. By definition, the correlation between the number of survivors of type A and type B is

$$\rho_{AB} = \frac{E(T_A^1 T_B^1) - E(T_A^1)E(T_B^1)}{\sqrt{\mathrm{Var}(T_A^1)\mathrm{Var}(T_B^1)}} \quad . \tag{7}$$

The variance and expectation of T_A^1 and T_B^1 in (7) can be found from equations (5) and (6). To find $E(T_A^1 T_B^1)$, observe that

$$T_A^1 = \sum_{j=1}^{T_A} \tau_{A_j} \quad ,$$

and similarly for T_B^1, so that

$$T_A^1 T_B^1 = \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} \tau_{A_j} \tau_{B_k} \quad .$$

By well-known properties of the indicator functions and expectations, it then follows that

$$E(T_A^1 T_B^1) = \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} E(\tau_{A_j} \tau_{B_k})$$

$$= \sum_{j=1}^{T_A} \sum_{k=1}^{T_B} P(z_{A_j} z_{B_k}) \quad . \tag{8}$$

and we see that all of the quantities appearing in equation (7) are available whenever the volley has been solved to the second order.

The preceding development can easily be extended to obtain expressions for the expectation, variance, and correlation between weighted survivor functions, such as

$$M_A^1 = M_{A_0} + \sum_{j=1}^{T_A} M_{A_j} \tau_j \quad , \tag{9}$$

where the constant term M_{A_0} is present only when non-zero weights are assigned to losses (see Note 4 in Appendix A). This slight generalization can be treated by a straightforward extension of the methods used to analyze the special case in which $M_{A_0} = 0$ and $M_{A_j} = 1$ for $j = 1(1)T_A$, to which we now return.

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3. EQUIVALENT VOLLEYS AND CANONICAL FORMS

Volleys frequently are described by specifying how targets are to be acquired, how fire is to be allocated among the acquired targets, and how the damage done to the target array by various allocations of fire is to be determined. For many purposes, such descriptions are absolutely essential.

On the other hand, the basic concepts introduced in the preceding section make no reference to the verbal description of a volley. Instead, they deal only with its basic event probabilities. When these basic event probabilities are obtained for a number of different volleys, it is observed that the mathematical expressions for them sometimes exhibit the same functional form. Recognition of this common functional form can be of capital importance, since all volleys whose basic event probabilities have the same functional form can be analyzed by the same mathematical methods. When the basic event probabilities for two volleys can be put in the same functional form, the volleys are said to be *equivalent*, and the common functional form is said to be their *canonical form*. We will not here attempt to formalize exactly when two mathematical expressions can be put into the same functional form. Instead, we go directly to examples of canonical forms, each of which is analyzed more fully later in this paper.

3-1. Independently Survivable Targets. We say that the targets within a subarray A of T_A targets are independently survivable if the z_{A_j}'s for $j = 1(1)T_A$ are independent events. In that case, the canonical form for the basic event probabilities is

$$P(z_{A_{j_1}} z_{A_{j_2}} \cdots z_{A_{j_r}}) = \prod_{n=1}^{r} P(z_{A_{j_n}}) \quad , \tag{10}$$

where the argument on the left is any r-th order basic event associated with subarray A. All volleys which are equivalent to a volley of this form are called volleys against independently survivable targets.

Some of the properties shared by all such volleys are as follows. All targets in the full array are independently survivable if, and only if, they are independently survivable within every subarray. Moreover, such a volley can be solved completely whenever it can be solved to the first order, since all of its basic event probabilities are known functions of the first order basic event probabilities.

3-2. Exchangeably Survivable Targets. The targets within a subarray A of T_A targets are called exchangeably survivable if the z_{A_j}'s for $j = 1(1)T_A$ are exchangeable events, that is, if the probability of any basic event in the subfamily of basic events associated with subarray A depends only on the *number* of z_{A_j}'s in its specification, but not on which *particular* z_{A_j}'s appear in it. Specifically, targets are exchangeably survivable within a subarray A whenever

$$P(z_{A_j}) = P(z_{A_1}) = P_{A1} \quad \text{for } j = 1(1)T_A,$$

$$P(z_{A_j}z_{A_k}) = P(z_{A_1}z_{A_2}) = P_{A2} \quad \text{for } j = 2(1)T_A \text{ and } k = 1(1)(j-1),$$

and so forth. Thus, the canonical form for a volley against exchangeably survivable targets is

$$P(z_{A_{j_1}}z_{A_{j_2}}\cdots z_{A_{j_r}}) = P(z_{A_1}z_{A_2}\cdots z_{A_r}) = P_{Ar} \quad , \tag{11}$$

where $j_1 = r(1)T_A$ and $j_n = 1(1)(j_{n-1}-1)$ for $n = 2(1)r$. The concept of exchangeability and some of its connections with other topics in the theory of probability and mathematical statistics can be found in Feller [1966], Loeve [1960], De Finetti [1974], and Frechet [1940], among others.

Some of the properties possessed by all volleys against exchangeably survivable targets are as follows. If all targets in the full array are exchangeably survivable, then they are exchangeably survivable within any subarray. Such a volley is solved completely once each of the T values P_r for $r = 1(1)T$ are known, where P_r is the probability of the r-th order basic event $z_1 z_2 \cdots z_r$.

In addition, when the targets in a subarray A of T_A targets are exchangeably survivable the general equations (1) through (6) immediately reduce to the following elegant forms:

$$S_{A_r} = \binom{T_A}{r}P_{Ar} \quad , \tag{12}$$

$$P_{A[m]} = \binom{T_A}{m}\sum_{r=0}^{T_A-m}(-1)^r\binom{T_A-m}{r}P_{A(m+r)}$$

$$= \binom{T_A}{m}\sum_{r=m}^{T_A}(-1)^{m+r}\binom{T_A-m}{r-m}P_{Ar} \quad . \tag{13}$$

$$G_A(x) = \sum_{r=0}^{T_A}(x-1)^r\binom{T_A}{r}P_{Ar} \quad . \tag{14}$$

$$E(T_A^1) = T_A P_{A1} \quad , \text{ and} \tag{15}$$

$$\text{Var}(T_A^1) = T_A(T_A-1)P_{A2} - T_A P_{A1}(T_A P_{A1}-1) \quad . \tag{16}$$

When A and B are subarrays of T_A and T_B targets, and if the targets are exchangeably survivable within their union subarray, $A \cup B$, then it can be shown that

$$E(T_A^1 T_B^1) = \sum_{j=1}^{T_A}\sum_{k=1}^{T_B}P(z_{A_j}z_{B_k}) = T_A T_B P_{A2} + T_{A \cap B}(P_{A1}-P_{A2}) \quad .$$

so that

$$\rho_{AB} = \frac{P_{A2}(T_A T_B - T_{A \cap B}) - (T_A T_B P_{A1} - T_{A \cap B})P_{A1}}{\sqrt{\mathrm{Var}(T_A^1)\mathrm{Var}(T_B^1)}} \qquad (17)$$

where $T_{A \cap B}$ is the number of targets in both A and B, that is, in $A \cap B$.

3-3. Independently versus Synergistically Effective Weapons. Suppose that a battery of W weapons volleys against an array of T targets. Suppose that we know the basic event probabilities when each of the W weapons acts alone and all other weapons are silent. Let $p_w(z_j)$, $p_w(z_j z_k)$, and so forth, be the basic event probabilities for a volley by weapon w acting alone against the target array. Then the canonical form for a volley by independently effective weapons is

$$P(z_j) = \prod_{w=1}^{W} p_w(z_j) \quad .$$

$$P(z_j z_k) = \prod_{w=1}^{W} p_w(z_j z_k) \quad . \qquad (18)$$

and so forth for each basic event probability. Consequently, volleys by independently effective weapons can be analyzed by temporarily setting aside all but one of the weapons in the battery, solving each of the resulting single weapon volleys, and recombining them via the independence of their individual effects. The volley used as an example in the Introduction is a volley by a battery of independently effective weapons.

If the weapons in a battery are not independently effective, then we say that they are *synergistically effective*. A volley by synergistically effective weapons cannot be solved completely by analyzing only its single weapon subvolleys.

Observe that a volley by independently effective weapons is against an array of independently (respectively, exchangeably) survivable targets whenever each of its single weapon subvolleys is against an array of independently (respectively, exchangeably) survivable targets.

3-4. Independently Effective Point Fire Weapons and Munitions. Suppose that a battery of W independently effective weapons volleys against an array of T targets, and consider

$$P(z_j z_k \cdots z_m) = \prod_{w=1}^{W} p_w(z_j z_k \cdots z_m)$$

$$= \prod_{w=1}^{W} \left[1 - p_w(\overline{z_j z_k \cdots z_m}) \right]$$

$$= \prod_{w=1}^{W} \left[1 - p_w(\overline{z_j} \cup \overline{z_k} \cup \ldots \cup \overline{z_m}) \right] \quad , \tag{19}$$

where $\cup$ indicates the set theoretic union of events. Now, in some volleys, a weapon may be unable to kill more than one target. A weapon that is unable to kill more than one target per volley will be called a point fire weapon. For each such weapon,

$$p_w(\overline{z_j}\,\overline{z_k}) \equiv p_w(\overline{z_j}\,\overline{z_k}\,\overline{z_r}) \equiv \ldots \equiv p_w(\overline{z_j}\,\overline{z_k}\,\ldots\,\overline{z_m}) \equiv 0 \quad .$$

so that

$$p_w(\overline{z_j} \cup \overline{z_k} \cup \ldots \cup \overline{z_m}) \equiv p_w(\overline{z_j}) + p_w(\overline{z_k}) + \ldots + p_w(\overline{z_m}) \quad . \tag{20}$$

Consequently, for volleys by batteries composed exclusively of independently effective point fire weapons. the canonical form is

$$P(z_{j_1} z_{j_2} \ldots z_{j_r}) = \prod_{w=1}^{W} \left\{ 1 - \sum_{n=1}^{r} p_w(\overline{z_{j_n}}) \right\} \quad . \tag{21}$$

A volley by a battery of independently effective point fire weapons can be solved completely by finding each of the WT values $p_w(\overline{z_j})$ for $w = 1(1)W$ and $j = 1(1)T$, where $p_w(\overline{z_j})$ is the probability that target j is killed during the subvolley in which weapon w acts alone against the full target array.

A weapon that is not a point fire weapon will be called an *area fire weapon*. Equations (18) or (19) give the canonical form for a volley by a battery of independently effective area fire weapons.

The definition of a point fire weapon needs to be broadened slightly to accommodate comfortably a number of important applications. For example, suppose that one of the weapons is a rifle that in the course of a volley may fire a number of shots and kill several targets. Under the definition given above. the rifle fails to qualify as a point fire weapon, although both common sense and conventional military terminology agree in ascribing "point fire" qualities to rifles. One appropriate response to this situation is to introduce the concept of independently effective *point fire munitions.* as follows.

Suppose that each weapon in a volley fires a certain number of shots. Let S_w be the number of shots fired by weapon w. Let $p_{ws}(z_j)$, $p_{ws}(z_j z_k)$, and so forth, be the basic event probabilities for a "volley" consisting of just shot number s from weapon w acting alone against the full target array. We say that weapon w fires independently effective munitions if

$$p_w(z_j) = \prod_{s=1}^{S_w} p_{ws}(z_j) \quad ,$$

$$P_w(z_j z_k) = \prod_{s=1}^{S_w} P_{ws}(z_j z_k) \quad . \tag{22}$$

and so on. If, in addition, shot s can kill at most one target, so that

$$P_{ws}(\overline{z_j}\ \overline{z_k}) \equiv P_{ws}(\overline{z_j}\ \overline{z_k}\ \overline{z_r}) \equiv \ldots \equiv P_{ws}(\overline{z_j}\ \overline{z_k}\ \ldots \overline{z_m}) \equiv 0 \quad .$$

then we say that shot s from weapon w is a point fire munition. In that case,

$$P_{ws}(z_{j_1} z_{j_2} \cdots z_{j_r}) = 1 - \sum_{n=1}^{r} P_{ws}(\overline{z_{j_n}}) \quad . \tag{23}$$

If all the shots fired by weapon w are independently effective point fire munitions, then

$$P_w(z_{j_1} z_{j_2} \cdots z_{j_r}) = \prod_{s=1}^{S_w} \left\{ 1 - \sum_{n=1}^{r} P_{ws}(\overline{z_{j_n}}) \right\} \quad . \tag{24}$$

If, in addition, the shots fired by the various weapons in the battery are independently effective, then so are the weapons. In that case, the canonical form for a volley of independently effective point fire munitions will be written as

$$P(z_{j_1} z_{j_2} \cdots z_{j_r}) = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{ 1 - \sum_{n=1}^{r} P_{ws}(\overline{z_{j_n}}) \right\} \quad . \tag{25}$$

Observe that a volley of independently effective point fire munitions is equivalent to a volley delivered by a battery of

$$W^0 = \sum_{w=1}^{W} S_w$$

independently effective point fire weapons, each of which fires exactly one shot. The equivalence is obtained by replacing the original battery of W weapons by the battery of W^0 weapons, and arranging things so that their kill probabilities correspond to those of the shots in the original volley. Therefore, in the theoretical treatment, we may freely replace a volley of independently effective point fire munitions by an equivalent volley of independently effective point fire weapons.

3-5. Summary of Canonical Forms. The foregoing suggests the taxonomy of canonical forms shown in Table 3. Each block in this table represents a canonical form possessing the combination of target and weapon attributes indicated by the column and row. The named volleys listed in the blocks of Table 3 are examples or special cases of canonical forms for the block. Volleys of independently effective point fire munitions are listed as if they were replaced by an equivalent volley of independently effective point fire weapons. Each of the examples listed in Table 3 is described at length and solved completely in

subsequent sections of this paper. If no examples are listed in a block, it indicates that we are not aware of any practically useful examples of that canonical form which can be solved completely. (In fact, the Bellwether Volley was contrived to provide a solvable example of a volley by synergistically effective weapons against an array of targets that are neither independently nor exchangeably survivable, rather than for its practical utility.) It appears that describing and solving volleys that not only fit the characteristics indicated by the lower right hand blocks of Table 3, but that also have a spectrum of valuable applications, is a worthwhile area of research.

Table 3. Taxonomy of Canonical Forms

Weapons	Targets		
	Independently survivable	Exchangeably survivable	Neither independently nor exchangeably survivable
Independently effective point fire	Gauntlet Volley* ICBM Volley*	Dixon-Robertson-Rau (DRR) Volley	Helmbold Volley* Burst Fire Volley* Hide-and-seek Volley
Independently effective area fire		Multishot Karr Volley	
Synergistically effective	Artillery Volley	Redundantly Survivable Target Volley	Bellwether Volley

* Since this is a volley of independently effective point fire munitions, or for other reasons is equivalent to a volley by independently effective point fire weapons, it is listed in that category.

4. COMPLEMENTARY VOLLEYS

Before analyzing these canonical forms and examples in detail, we introduce one last concept—namely, the idea of a complementary volley. Roughly speaking, two volleys are complementary if the outcome of each is the logical negation of the outcome of the other. Although this definition can be applied even when several target states are allowed, this paper considers only targets that either survive the volley, or do not. In that case, two volleys are complementary if targets that survive in one of them are killed in the other, and *vice versa*. Formally, we have the following

Definition.- Volley V^* is a complement of volley V if they have the same number of targets and if the probability P^* of each basic event $z_j^* z_k^* \ldots z_m^*$ of volley V^* is related to that of volley V by

$$P^*(z_j^* z_k^* \ldots z_m^*) = P(\overline{z_j}\ \overline{z_k} \ldots \overline{z_m}) \quad . \tag{26}$$

In some cases, the complementary volley may be much easier to solve than the original volley. Although no such cases have come to our attention in practice, that possibility is the most important reason for considering complementary volleys. They would be of theoretical interest in any case, because the notion and properties of complementary volleys lend a certain symmetry to the general theory. The following theorems and corollaries establish the basic properties of complementary volleys. Theorem 1 essentially states that if V^* is a complement of V, then V and V^* are mutually complementary. Corollary 1.1 states that the complement of the complement is (equivalent to) the original volley. Theorem 3 states that the number of survivors in volley V has the same distribution as the number of targets killed in volley V^*, and *vice versa*. Since the theory of complementary volleys is not completely developed in this paper, and has yet to show its promise in applications, some readers may wish to skip the remainder of this section, which is devoted to the mathematical statement and proofs o. these propositions. (In the following, the symbol □ denotes the end of a proof.)

Theorem 1: If volley V^* is a complement of volley V, then V is a complement of volley V^*, that is, for each basic event of volley V,

$$P(z_j z_k \ldots z_m) = P^*(\overline{z_j^*}\ \overline{z_k^*} \ldots \overline{z_m^*}) \quad .$$

Proof: Let A be any subarray of T_A targets. Then

$$P(z_{A1} z_{A2} \ldots z_{AT_A}) \equiv 1 - P(\overline{z_{A1}} \cup \overline{z_{A2}} \cup \ldots \cup \overline{z_{AT_A}})$$

$$\equiv 1 - \overline{S_{A1}} + \overline{S_{A2}} - \ldots + (-1)^{T_A} \overline{S_{AT_A}} \quad .$$

where

$$\overline{S_{Ar}} \equiv \sum_r P(\overline{z_{Aj_1}}\,\overline{z_{Aj_2}}\cdots\overline{z_{Aj_r}})$$

and, as usual, $\sum_r$ stands for the operation of taking the sum over all r-th order subarrays of subarray A (*cf.* equation (2)). But, since by hypothesis V^* is a complement of V, we may, by definition of a complementary volley, replace each term in the sum $\sum_r$ by its related complementary probability to obtain

$$\overline{S_{Ar}} = S^*_{Ar} \quad ,$$

where S^*_{Ar} is the r-th order basic sum for volley V^*. Consequently, we have

$$P(z_{A1}z_{A2}\cdots z_{AT_A}) = 1 - S^*_{A1} + S^*_{A2} - \ldots + (-1)^{T_A}S^*_{AT_A}$$

$$= 1 - P^*(z^*_{A1} \cup z^*_{A2} \cup \ldots \cup z^*_{AT_A})$$
$$= P^*(\overline{z^*_{A1}}\,\overline{z^*_{A2}}\cdots\overline{z^*_{AT_A}}) \quad .$$

Because A was any subarray of targets, this result holds for all basic events of volley V. □

Corollary 1.1: Let V^* be a complement of V and V^{**} be a complement of V^*. Then V^{**} is (equivalent to) V.

Proof: Because V^{**} is a complement of V^*,

$$P^{**}(z^{**}_j z^{**}_k \cdots z^{**}_m) = P^*(\overline{z^*_j}\,\overline{z^*_k}\cdots\overline{z^*_m}) = P(z_j z_k \cdots z_m) \quad .$$

where the last equality is supplied by Theorem 1. □

Theorem 2: Let volley V^* be a complement of volley V and let A be any subarray of T_A targets. Let S^*_{Ar} be the r-th order basic sum for volley V^* and S_{Ar} be the r-th order basic sum for volley V. Then

$$S^*_{Ar} = \sum_{m=0}^{r} (-1)^m \binom{T_A - m}{r - m} S_{Am} \quad \text{for } r = 0(1)T_A \quad .$$

Proof: Because the proof is longer than convenient to present here, it has been relegated to Note 5.

Corollary 2.1: With assumptions and notation as in Theorem 2,

$$S_{Am} = \sum_{r=0}^{m} (-1)^r \binom{T_A - r}{m - r} S^*_{Ar} \quad \text{for } m = 0(1)T_A \quad .$$

Proof: By assumption, V^* is a complement of V. By Theorem 1, it follows that V is a complement of V^*. Consequently, in the statement and proof of Theorem 2 the roles of V and V^* may be interchanged. $\square$

Theorem 3: Let V^* and V be complementary volleys, and let A and B be any two subarrays of T_A and T_B targets. Then the following assertions are true:

(i) $G_A^*(x) = x^{T_A} G_A(1/x)$,

(ii) $P_{A[m]}^* = P_{A[T_A - m]}$,

(iii) $E(T_A^{*\,1}) = T_A - E(T_A^1)$,

(iv) $\mathrm{Var}(T_A^{*\,1}) = \mathrm{Var}(T_A^1)$, and

(v) $\rho_{AB}^* = \rho_{AB}$.

Proof: See Note 6.

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5. VOLLEYS AGAINST INDEPENDENTLY SURVIVABLE TARGETS

These are the easiest volleys to analyze. Indeed, it must be admitted that the results in this section are well known and are traditionally obtained by elementary probability arguments that are simpler and more direct than those based on the general theory of volley fire. Nevertheless, we will rederive them using the general machinery developed above. Our purpose in doing so is to illustrate the application of the new methods in simple situations before using them in more complicated ones, where the elementary probability arguments do not apply. This also serves to demonstrate that the results obtained using the general methods do indeed agree with those reached by more familiar approaches. It will be found that the general methods are more precise and rigorous than the usual informal arguments. Moreover, the results are used later in this paper.

Recall that the canonical form for a volley against an array of T independently survivable targets is (*cf.* equation (10))

$$P(z_{j_1} z_{j_2} \cdots z_{j_r}) = \prod_{n=1}^{r} P(z_{j_n}) \quad .$$

Obviously, the targets within any subarray are also independently survivable. For this class of volleys, the complement takes a particularly simple form. In fact, we have the following.

Theorem 4: If V is a volley against independently survivable targets and V^* is a complement of V, then V^* is (equivalent to) a volley against independently survivable targets. Moreover, the basic event probabilities for volley V^* are

$$P^*(z_{j_1}^* z_{j_2}^* \cdots z_{j_r}^*) = \prod_{n=1}^{r} \left\{ 1 - P(z_{j_n}) \right\} \quad .$$

Proof: Let A be any subarray of T_A targets, and consider the collection of events $z_{A1}, z_{A2}, \cdots, z_{AT_A}$. By hypothesis, these events are independent with respect to P. But any collection of events is independent if, and only if, the collection of their complementary events is independent. Hence, the events $\overline{z_{A1}}, \overline{z_{A2}}, \cdots, \overline{z_{AT_A}}$ are independent with respect to the probability P. Hence, V^* is a volley against independently survivable targets, as asserted.

Now, by hypothesis, V^* is a complement of V, and so for every subarray A

$$P^*(z_{A1}^* z_{A2}^* \cdots z_{AT_A}^*) = P(\overline{z_{A1}} \, \overline{z_{A2}} \cdots \overline{z_{AT_A}}) = \prod_{j=1}^{T_A} P(\overline{z_{Aj}})$$

$$= \prod_{j=1}^{T_A} \left\{ 1 - P(z_{Aj}) \right\} = \prod_{j=1}^{T_A} P^*(z_{Aj}^*) \quad .$$

where the last equality follows because V^* is a complement of V. But since this holds for any subarray A, the events z_j^* for $j = 1(1)T$ are also independent with respect to the probability P^*. $\square$

Theorem 5: Let V be a volley against independently survivable targets and let A be any subarray of T_A targets. Then the generating function for the distribution of the number of type A survivors is

$$G_A(x) = \prod_{j=1}^{T_A} \left[1 + (x-1)P(z_{Aj}) \right] \quad . \tag{27}$$

Proof: Consider

$$\prod_{j=1}^{T_A} \left[1 + (x-1)P(z_{Aj}) \right] = \sum_{m=0}^{T_A} (x-1)^m C_{Am} \quad . \tag{28}$$

where

$$C_{Am} = \sum_m P(z_{Aj_1})P(z_{Aj_2})...P(z_{Aj_m}) = S_{Am} \quad .$$

where the last equality follows from equation (2) and the hypothesis that V is a volley against independently survivable targets. But substituting S_{Am} for C_{Am} on the right side of equation (28) and comparing the result with equation (4) gives the asserted result. $\square$

Corollary 5.1: Let V be a volley against independently survivable targets. Let A and B be disjoint subarrays of T_A and T_B targets, respectively. Let $A \cup B$ be the subarray consisting of the type A and type B targets. Then the following statements are true.

(i) $G_{A \cup B}(x) = G_A G_B$.

(ii) $\rho_{AB} = 0$.

(iii) $E(T_{A \cup B}^1) = E(T_A^1) + E(T_B^1)$.

(iv) $\text{Var}(T_{A \cup B}^1) = \text{Var}(T_A^1) + \text{Var}(T_B^1)$. and

(v) $P_{A \cup B}[m] = \sum_{k=0}^{m} P_{A}[m-k] P_{B}[k]$.

Proof: Part (i) is a general property of generating functions, which in the present context can be shown as follows. By Theorem 5,

$$G_{A \cup B}(x) = \prod_{j=1}^{T_{A \cup B}} \left\{ 1 + (x-1)P(z_{A \cup Bj}) \right\}$$

$$= \prod_{j=1}^{T_A} \left\{ 1 + (x-1)P(z_{Aj}) \right\} \prod_{k=1}^{T_B} \left\{ 1 + (x-1)P(z_{Bk}) \right\} \quad .$$

where the second equality follows from the hypothesis that A and B are disjoint. But by Theorem 5, the two factors on the right side are $G_A(x)$ and $G_B(x)$, proving assertion (i).

By the first assertion and a fundamental theorem of probability theory, T^1_A and T^1_B must be independent random variables. But then they are uncorrelated, which proves assertion (ii). Assertion (iv) is an immediate consequence of assertion (ii).

Assertion (iii) holds whenever A and B are disjoint, because in that case

$$T^1_{A \cup B} = T^1_A + T^1_B .$$

and because the expectation of a sum always equals the sum of the expectations. (When A and B are *not* disjoint, then $T^1_{A \cup B} \subseteq T^1_A + T^1_B$, and in that case $E(T^1_{A \cup B}) \leq E(T^1_A) + ET^1_B).)$

Assertion (v) follows either from the independence of T^1_A and T^1_B and standard results of probability theory, or by expanding $G_{A \cup B}(x)$ in powers of x in accord with equation (1) and comparing coefficients with those of the product of $G_A(x)$ and $G_B(x)$. □

Corollary 5.2: Let V be a volley against independently survivable targets and let A be any subarray of T_A targets. Then

$$E(T^1_A) = \sum_{j=1}^{T_A} P(z_{Aj})$$

and

$$\mathrm{Var}(T^1_A) = \sum_{j=1}^{T_A} P(z_{Aj})\left\{1 - P(z_{Aj})\right\} .$$

Proof: The first assertion is the same as equation (5). To prove the second, apply assertion (iv) of Corollary 5.1 repeatedly to show that

$$\mathrm{Var}(T^1_A) = \sum_{j=1}^{T_A} \mathrm{Var}(T^1_{Aj})$$

where T^1_{Aj} is the number of survivors in the subarray consisting of the single target Aj. But for a subarray of just one target, Theorem 5 gives

$$G_{Aj}(x) = 1 + (x - 1)P(z_{Aj}) .$$

and then equation (6) applies to give

$$\mathrm{Var}(T^1_{Aj}) = P(z_{Aj})\left\{1 - P(z_{Aj})\right\} . □$$

Corollary 5.3: Let V be a volley against independently survivable targets. Suppose that the targets in subarray A are exchangeably survivable. Then the following assertions are true:

(i) The probability that exactly m of the type A targets survive is

$$P_{A[m]} = \binom{T_A}{m} P_A{}^m (1 - P_A)^{T_A - m}$$

where

$$P_A = P(z_{A1}) \equiv P(z_{Am}) \text{ for } m = 1(1)T_A$$

is the survival probability of an arbitrarily selected target of type A.

(ii) $E(T_A^1) = T_A P_A$.

(iii) $\mathrm{Var}(T^1) = T_A P_A (1 - P_A)$.

Proof: When the targets in subarray A are exchangeably survivable, substituting for each $P(z_{Aj})$ its common value, P_A, in equation (27) yields

$$G_A(x) = \{1 + (x - 1)P_A\}^{T_A} = \{1 - P_A + x P_A\}^{T_A}$$
$$= \sum_{m=0}^{T_A} \binom{T_A}{m} P_A{}^m (1 - P_A)^{T_A - m} x^m .$$

and comparing this with equation (1) gives assertion (i). Assertions (ii) and (iii) follow by standard results in the theory of binomially distributed random variables, or from Corollary 5.2. □

Now we will consider some particular cases of volleys against independently survivable targets. Here, as in the discussion of other canonical forms, these examples illustrate the connection between the somewhat lifeless abstract canonical form of a volley and the animated, often colorful applied versions familiar to military operations analysts. In describing particular volleys, it is usually helpful to think of the battery of weapons as going in turn through the phases of target acquisition, allocation of fire, and achievement of effects. In the acquisition phase, candidates for attack by one or more weapons are obtained from the target array. In the allocation phase, fire from the weapons is allocated to the acquired targets. In the effects phase, the damage done by the allocated fire is determined. With this concept of the volley process in mind, we turn to the example of the Gauntlet Volley.

5-1. The Gauntlet Volley. The informal mental image of the action in a Gauntlet Volley is that each target separately "runs the gauntlet," that is, it faces and is subject to attack by each of the weapons in turn, with each weapon-target combination encounter being a separate engagement. Alternatively, we may think of each weapon as moving in turn from one target to the next, singlehandedly engaging

each target it comes to. Whichever intuitive picture is used, a Gauntlet Volley may be defined by the following postulates.

G-1: The probability that weapon w acquires target t is $a_w(t)$, and is independent of other acquisitions.

G-2: Each weapon may fire up to T shots at the target array, depending on how many targets it acquires. The probability that weapon w allocates one shot to target t is $v_w(t)$ if w acquires target t, and is zero otherwise, independent of what other events occur during the volley.

G-3: The probability that target t is killed by weapon w is $q_w(t)$ if a shot from weapon w is allocated to target t, and zero otherwise, independent of what other events occur during the volley.

The Gauntlet Volley is easily solved by observing that it is a volley against an array of independently survivable targets in which

$$P(z_t) = \prod_{r=1}^{W} \left\{ 1 - a_w(t)v_s(t)q_w(t) \right\} \quad .$$

so the results of Theorems 4 and 5 and their corollaries apply. The form of $P(z_t)$ shows that a Gauntlet Volley is also a volley by a battery of independently effective weapons. In general, the weapons of a Gauntlet Volley can kill more than one target, and so are area fire weapons. However, by postulate G-3, the munitions are independently effective point fire munitions, so a Gauntlet Volley is equivalent to a volley by point fire weapons. This justifies the location of the Gauntlet Volley entry in Table 3.

5-2. The ICBM Volley. The name of this volley was chosen because it has frequently been used to obtain quick estimates of the effects of a salvo of intercontinental ballistic missiles. It satisfies the following postulates.

ICBM-1: Each weapon acquires all of the targets in the target array.

ICBM-2: Each weapon fires exactly one shot. Shots are allocated as evenly as possible to the targets. More precisely, let $[W/T]$ be the greatest integer not larger than W/T, and let $R(W,T) = W - T[W/T]$ be the remainder when W is divided by T. Then $[W/T] + 1$ shots are allocated to each of the first $R(W,T)$ targets and $[W/T]$ shots are allocated to each of the remaining $T - R(W,T)$ targets.

ICBM-3: The probability that target t is killed by the shot from weapon w is $q(t)$ if weapon w's shot is allocated to target t, and zero otherwise, independent of what other events occur during the volley.

It is easily seen that an ICBM Volley is equivalent to the Gauntlet Volley V^0 in which a single weapon volleys against the target array, and in which the acquisition probabilities are $a^0(t) = 1$ for

$t = 1(1)T$, the allocation probabilities are $v^0(t) = 1$ for $t = 1(1)T$, and the conditional kill probabilities are

$$q^0(t) = \begin{cases} 1 - \left\{1 - q(t)\right\}^{[W/T] + 1} & , \text{ for } t = 1(1)R(W,T) \\[2em] 1 - \left\{1 - q(t)\right\}^{[W/T]} & , \text{ for } t = (R(W,T) + 1)(1)T \end{cases}$$

This equivalence justifies the location of the ICBM Volley entry in Table 3.

When the target array is partitioned into two distinct subarrays A and B such that A contains the first $R(W,T)$ targets, then it is easily seen from the equivalent Gauntlet Volley V^0 that

$$P(z_t) = \left\{1 - q(t)\right\}^{[W/T] + 1} \quad \text{for } t \in A, \text{ and}$$

$$P(z_t) = \left\{1 - q(t)\right\}^{[W/T]} \quad \text{for } t \in B.$$

Corollary 5.1 applies to yield

$$E(T^1) = \sum_{t=1}^{R(W,T)} \left\{1 - q(t)\right\}^{[W/T] + 1} + \sum_{t = R(W,T) + 1}^{T} \left\{1 - q(t)\right\}^{[W/T]} \quad ,$$

$$\rho_{AB} = 0 \quad .$$

and so forth.

Now suppose that $q(t) = q_A$ for all $t \in A$ and $q(t) = q_B$ for all $t \in B$, and let

$$P_A = (1 - q_A)^{[W/T] + 1} \quad , \text{ and}$$

$$P_B = (1 - q_B)^{[W/T]} \quad .$$

Then Corollaries 5.1 and 5.3 apply to yield the familiar formulas

$$E(T^1) = R(W,T)P_A + \left\{T - R(W,T)\right\}P_B \quad ,$$

$$\mathrm{Var}(T^1) = R(W,T)P_A(1 - P_A) + \left\{T - R(W,T)\right\}P_B(1 - P_B) \quad ,$$

$$P_{A[m]} = \binom{R(W,T)}{m} P_A{}^m (1 - P_A)^{R(W,T) - m} \quad \text{for } m = 0(1)R(W,T) \quad ,$$

$$P_{B[m]} = \binom{T - R(W,T)}{m} P_B{}^m (1 - P_B)^{T - R(W,T) - m} \quad \text{for } m = 0(1)\left\{T - R(W,T)\right\} \quad , \text{ and}$$

$$P_{[m]} = \sum_{k=0}^{m} P_{A[m-k]} P_{B[k]} \quad \text{for } m = 0(1)T \quad .$$

5-3. The Artillery Volley. The Artillery Volley is often used to estimate the effects of fragmenting ordnance delivered by artillery, aircraft, mortars, rockets, and so forth. It may be described as follows.

A-1: Individual targets *per se* are not acquired. However, an area believed to contain targets is acquired.

A-2: There are W weapons. Weapon w fires S_w shots. Shots are not allocated to individual targets, but are allocated stochastically to particular ground zeros in such a way that $\sigma_{ws}(u,v)dudv$ is the probability that shot s from weapon w has its ground zero located almost exactly at the point (u,v). Each ground zero distribution $\sigma_{ws}(u,v)$ is independent of the actual ground zeros of other shots.

A-3: The probability that shot s from weapon w kills target t when target t is located almost exactly at (x,y) and the shot's ground zero is located almost exactly at (u,v) is given by the damage function $D_{wst}(x-u, y-v)$. Shots are independently effective given their ground zeros, that is, the probability that target t survives all shots when it is located almost exactly at (x,y) and the ground zero of shot s from weapon w is located almost exactly at (u_{ws}, v_{ws}) is equal to

$$F_t = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{ 1 - D_{wst}(x - u_{ws}, y - v_{ws}) \right\} \quad .$$

A-4: The probability that target t is located almost exactly at (x,y) is $p_t(x,y)dxdy$, independently of the locations of other targets and of the ground zeros of the shots.

Observe that, by virtue of the above postulates, an Artillery Volley is a volley against an array of independently survivable targets. Neither the weapons nor the munitions are point fire. As explained in Note 7, the munitions are independently effective only conditionally on their ground zeros, that is, in the sense specified in postulate A-3, and do not conform to the definition of independently effective munitions in the sense expressed by equation (18). These observations justify the location of the Artillery Volley entry in Table 3. Schroeter [1984] also develops expressions for the expectation and higher moments of T^1, the number of targets that survive an artillery volley.

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6. VOLLEYS AGAINST EXCHANGEABLY SURVIVABLE TARGETS

The canonical form of a volley against exchangeably survivable targets is given by equation (11). It is clear that if all the targets in an array are exchangeably survivable, so are the targets in any subarray. When the targets are exchangeably survivable, the elegant formulas (12) through (17) apply. We now prove the following.

Theorem 6: If V is a volley against exchangeably survivable targets and V^* is a complement of V, then V^* is also a volley against exchangeably survivable targets. Moreover, the complementary basic event probabilities are

$$P_r^* = \sum_{k=0}^{r} (-1)^k \binom{r}{k} P_k \quad \text{for } r = 0(1)T.$$

Proof: Let A be any subarray of T_A targets, and consider

$$P^*(z_{A1}^* z_{A2}^* \cdots z_{AT_A}^*) = P(\overline{z_{A1}} \ \overline{z_{A2}} \cdots \overline{z_{AT_A}})$$

$$= 1 - P(z_{A1} \cup z_{A2} \cup \ldots \cup z_{AT_A})$$

$$= 1 - S_{A1} + S_{A2} - S_{A3} + \ldots (-1)^{T_A} S_{AT_A} \ .$$

where

$$S_{Ar} = \sum_r P(z_{Aj_1} z_{Aj_2} \cdots z_{Aj_r})$$

and, as in equation (2), $\sum_r$ indicates that the sum is taken for indices $j_1, j_2, \ldots, j_r$ which are varied in such a way that the subarray consisting of the targets $A_{j_1}, A_{j_2}, \ldots, A_{j_r}$ sweeps over each of the r-th order subarrays of subarray A. But, because by hypothesis the events $z_{A1}, z_{A2}, \ldots, z_{AT_A}$ are exchangeable with respect to P,

$$S_{Ar} = \binom{T_A}{r} P_r$$

where

$$P_r = P(z_1 z_2 \ldots z_r) = P(z_{Aj_1} z_{Aj_2} \cdots z_{Aj_r})$$

is independent of which r events z_i appear. Thus,

$$P^*(z_{A1}^* z_{A2}^* \cdots z_{AT_A}^*) = \sum_{m=0}^{T_A} (-1)^m \binom{T_A}{m} P_m \ .$$

which depends only on T_A and not on which T_A events appear in the argument of P^*. Therefore the events $z^*_{A1}, z^*_{A2}, \ldots, z^*_{AT_A}$ are exchangeable with respect to P^*. But, since A was any subarray of targets, it follows that V^* is a volley against exchangeably survivable targets, and the complementary basic event probabilities are as given in the statement of the theorem. $\square$

In general, the state space for a volley consists of the 2^T possible complexions of the target array, and the sequence of complexions generated as successive volleys are fired is a Markov chain with 2^T states. However, when the targets are exchangeably survivable, only the number of survivors matters (that is, only those complexions which differ in the number of survivors are distinguishable), and the sequence of the number of survivors generated as successive volleys are fired is a Markov chain with only $T + 1$ states. The transition probabilities for the latter Markov chain are given by $P_{[m]}$, which may be calculated using equation (13). Consequently, volleys against exchangeably survivable targets are much easier to analyze than are volleys against arrays of targets that are neither independently nor exchangeably survivable. Particular examples of volleys against exchangeably survivable targets are given later in this paper (the names of these volleys are shown in Table 3).

7. VOLLEYS BY INDEPENDENTLY EFFECTIVE POINT FIRE WEAPONS OR MUNITIONS

In this section we show that the general theory developed earlier easily yields results for volleys by independently effective point fire weapons that are difficult to derive by other methods. In fact, we show that application of the general theory allows us to develop new results for some of these volleys. Recall that the canonical forms for volleys by batteries of independently effective weapons or munitions are given by equations (18) or (22), respectively. Before introducing and analyzing particular cases, we establish the following.

Theorem 7: Suppose that V is a volley by a battery of W independently effective weapons, and let V^* be a complement of V. Then, in general, V^* is *not* (equivalent to) the volley V^+, obtained by independently combining the complements of volley V's single weapon volleys.

Proof: In general,

$$P^*(z_1^*) = P(\overline{z_1}) = 1 - P(z_1) = 1 - \prod_{w=1}^{W} p_w(z_1) \quad .$$

On the other hand, the complements of volley V's single weapon volleys are

$$p_w^*(z_1^*) = p_w(\overline{z_1}) = 1 - p_w(z_1) \quad \text{for } w = 1(1)W .$$

and combining them independently leads to the volley V^+ in which

$$P^+(z_1^+) = \prod_{w=1}^{W} p_w^*(z_1^*) = \prod_{w=1}^{W} \{1 - p_w(z_1)\} \quad .$$

and so, in general, $P^+(z_1^+) \neq P^*(z_1^*)$. In fact, it is clear that $P^+(z_1^+) \leq P^*(z_1^*)$, and that the equality sign applies only in very special cases. Consequently, in general, volleys V^+ and V^* are not equivalent. □

Observe that, in general, the complements of volleys by independently effective point fire weapons appear to be volleys by synergistically effective weapons. This may provide a useful method for investigating and solving volleys by synergistically effective weapons. Specifically, some volleys by synergistically effective weapons may be most easily solved by recognizing that they are the complements of volleys by independently effective point fire weapons, solving the appropriate complementary volley, and translating its solution to the original volley.

7-1. The Dixon-Robertson-Rau Volley. The Dixon-Robertson-Rau (DRR) Volley occurs quite frequently in applications and also serves as a prototype for the study of other volleys by independently effective point fire weapons because it can be analyzed in considerable detail and has intuitively appealing solutions. It may be defined by the following postulates.

DRR-1: Each weapon in a battery of W weapons acquires all of the T targets in the array.

DRR-2: Each weapon fires exactly one shot, which it allocates to a target selected uniformly and independently at random from the target array. (That is, the probability that weapon w directs its shot at target t is equal to $1/T$ and is independent of the other events that occur during the volley.)

DRR-3: The probability that target t is killed by the shot from weapon w is q_w if w allocates its shot to t, and zero otherwise, independent of the other events that occur during the volley.

Observe that the DRR volley is a volley by independently effective point fire weapons, in which

$$p_w(\overline{z_t}) = \sum_{j=1}^{T} \text{Prob}(\text{weapon } w \text{ kills target } t \mid w \text{ fires at target } t) \times \text{Prob}(w \text{ fires at target } j)$$

$$= q_w/T \quad .$$

Because $p_w(\overline{z_t})$ is independent of t, the targets in a DRR volley are exchangeably survivable. Then, for any subarray A of T_A targets, equations (21) and (11) show that

$$P_{Ar} = \prod_{w=1}^{W} (1 - rq_w/T) \quad \text{for } r = 1(1)T_A.$$

Therefore, in general, $P_{A2} \neq P_{A1}^2$, and hence a DRR Volley generally is *not* a volley against independently survivable targets. These observations justify the location of the DRR Volley entry in Table 3.

Because equations (12) through (17) apply, we obtain immediately

$$S_{Ar} = \binom{T_A}{r} \prod_{w=1}^{W} (1 - rq_w/T) \quad \text{for } r = 0(1)T_A \quad .$$

$$E(T_A^1) = T_A \prod_{w=1}^{W} (1 - q_w/T) \quad .$$

$$\text{Var}(T_A^1) = T_A(T_A - 1) \prod_{w=1}^{W} (1 - 2q_w/T) + E(T_A^1)\left\{1 - E(T_A^1)\right\} \quad ,$$

$$P_{A[m]} = \binom{T_A}{m} \sum_{r=0}^{T_A} (-1)^r \binom{T_A - m}{r} \prod_{w=1}^{W} (1 - (m+r)q_w/T) \quad ,$$

and so forth. Observe that, when $q_w = 1$ for $w = 1(1)W$, the formulas for the DRR Volley give the expectation and variance of the number of empty cells in the classical occupancy problem, and $P_{A[m]}$

gives the probability that exactly m cells of an arbitrarily chosen collection of T_A cells are empty (compare this observation to Feller [1950]). (See also Note 8.)

Clearly, a volley in which there is but one weapon that fires a total of W shots, where each shot is allocated to a target independently at random from the target array and the w-th shot has kill probability q_w, is equivalent to a DRR volley. In fact, the number of weapons and the number of shots per weapon can be changed at will, subject only to the conditions that a total of W shots be fired, that each of the shots be allocated to a target selected independently at random from the target array, and that the kill probabilities of the shots correspond one-to-one with the q_w for $w = 1(1)W$. For example, a volley by W^0 weapons, each of which fires S shots with each shot fired at a target selected independently at random from the target array, in which the kill probability gradually improves on each shot, so that

$$q_{w^0 1} \leq q_{w^0 2} \leq \cdots \leq q_{w^0 S}$$

for each $w^0 = 1(1)W^0$, is equivalent to a DRR volley by a battery of $W = W^0 S$ weapons in which

$$q_w = \begin{cases} q_{w1} & \text{, for } w = 1(1)W^0 \\ q_{w-W^0,2} & \text{, for } w = (W^0+1)(1)(2W^0) \\ \vdots \\ q_{w-(S-1)W^0,S} & \text{, for } w = [(S-1)W^0+1](1)(SW^0) \end{cases}$$

(The assumed increase in kill probability is, of course, not essential. The important conditions are that the shots be independently effective, that each of them be directed at a randomly selected target, and that the targets are all alike.)

It might be conjectured that the complement V^* of a DRR Volley in which $q_w = q$ for $w = 1(1)W$ could be obtained by substituting $1 - q$ for q in the above formulas. However, this generally is false, because it results in the volley $V^\dagger$ for which

$$P_r^\dagger = \left\{1 - r(1-q)/T\right\}^W \quad .$$

but the correct expression for the complementary basic event probabilities as given by Theorem 6 is

$$P_r^* = \sum_{k=0}^{r} (-1)^k \binom{r}{k}(1 - kq/T)^W \quad .$$

and these expressions are not reducible to the same functional form. One way of demonstrating that is to observe that, because they both involve the same variables, the expressions for $P_r^\dagger$ and P_r^* are reducible

to the same functional form only if they are identically equal for all relevant values of r, q, T, and W. But when $r = 1$, $q = 1$, $T = 2$, and $W = 1$, we find $P_1^* = 1/2$, but $P_1^\dagger = 1$, so that the expressions are not identically equal. In fact, as shown in Theorem 8, the complement of a DRR Volley is not usually a DRR Volley.

Theorem 8: Let V be a DRR Volley by a battery of W weapons against an array of T targets in which $q_w = q$ for $w = 1(1)W$. Suppose that $W \leq T - 2$. Then V^*, the complement of V, is not (equivalent to) a DRR Volley by a battery of W weapons for which $q_w^* = q^*$ for $w = 1(1)W$.

Proof: If V^* were (equivalent to) a DRR Volley in which $q_w^* = q^*$, we would have to have

$$P_r^* = (1 - rq^*/T)^W \quad \text{for } r = 0(1)T \quad .$$

Equating this to the expression for P_r^* given by Theorem 6, we see that V^* is a DRR Volley only if the set of equations

$$(1 - rq^*/T)^W = \sum_{k=0}^{r} (-1)^k \binom{r}{k}(1 - kq/T)^W \quad \text{for } r = 1(1)T \tag{29}$$

has a solution q^* that lies between 0 and 1. But, as will be shown in a moment, when $W \leq T - 2$, the right side of equation (29) vanishes for $r = T$ and for $r = T - 1$. And then we would have to have $(1 - q^*)^W = 0$ and $[1 - (T-1)q^*/T]^W = 0$, which is impossible, because the first of these equations requires that $q^* = 1$ while the second requires that $q^* = T/(T-1) > 1$.

To show that when $w \leq T - 2$ the right side of equation (29) vanishes for $r = T$ and for $r = T - 1$, rewrite it as

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{m=0}^{W} (-1)^m \binom{W}{m}(q/T)^m k^m = \sum_{m=0}^{W} (-1)^{-m} \binom{W}{m}(q/T)^m C_r^m \quad ,$$

where

$$C_r^m = \sum_{k=0}^{r} (-1)^k \binom{r}{k}k^m \quad .$$

But it is well-known that C_r^m vanishes for $r > m$ (see, for example, Feller [1950, p 77]). Consequently, when $W \leq T - 2$ the right-hand side of equation (29) vanishes for $r = T$ and for $r = T - 1$, as asserted. □

Corollary 8.1: If $W \leq T - 1$, a complementary DRR Volley in which $q_w^* = q^*$ exists if, and only if, (i) $W = 1$ and $T = 2$, and (ii) $q^* = q = 1$.

Proof: When $W \le T - 1$, taking $r = T$ in equation (29) yields $(1 - q^*)^W = 0$, so that $q^* = 1$ is the only possible solution. But the complementary version of equation (29) reads

$$(1 - rq/T)^W = \sum_{k=0}^{r} (-1)^k \binom{r}{k}(1 - kq^*/T)^W \quad ,$$

and when $W \le T - 1$, putting $r = T$ in this equation yields $(1 - q)^W = 0$, so that $q^* = q = 1$ is the only possible solution. With these values of q and q^*, taking $r = 1$ in equation (29) yields the necessary condition for a solution as $(1 - 1/T)^W = 1/2$, or $1 - 1/T = 2^{-1/W}$.

If $W = 1$, then $T = 2$ is a possible solution. It is easily verified that the values $W = 1$, $T = 2$, and $q = q^* = 1$ do indeed provide a solution to equation (29). However, when $W > 1$, the right side of the last equation of the previous paragraph is irrational, while its left side is rational. Hence, when $W > 1$, equation (29) has no solution of the required type. □

When $T = 2$ and $W \ge T$, there are selected values of $q = q^*$ for which equation (29) has a solution, and therefore for which complementary DRR Volleys for which $q_w = q$ and $q_w^* = q^*$ exist. The values of $q = q^*$ which afford solutions of equation (29) when $T = 2$ and $W = 1(1)9$ are listed below.

W	$q = q^*$
1	1.000 000 000
2	0.585 786 438
3	0.412 598 948
4	0.318 207 170
5	0.258 898 874
6	0.218 202 564
7	0.188 552 672
8	0.165 991 914
9	0.148 250 576

Presumably, this list could be extended to higher values of W and T. Hence, for some $W \ge T$, complementary DRR Volleys with $q_w = q$ and $q_w^* = q^*$ exist for selected values of q and q^*.

The theory of complementary volleys is incomplete. This is true for the complements of volleys by independently effective weapons generally, and for the complements of DRR Volleys in particular. As such, this topic deserves additional study and research.

7-2. The Helmbold Volley. The DRR Volley can be generalized considerably at slight effort. For example, suppose a Helmbold Volley is defined by the following postulates.

H-1: Each weapon acquires all T targets in the target array.

H-2: Weapon w fires S_w shots during the volley, allocated at the rate of one target per shot. The probability that shot s from weapon w is allocated to target t is $v_{ws}(t)$ and is independent of the allocations made on other shots by the same or any other weapon.

H-3: The probability that shot s from weapon w kills target t is $q_{ws}(t)$ if that shot is allocated to target t, and is zero otherwise, independent of the other events that occur during the volley.

This obviously is a volley of independently effective point fire munitions, so that equation (25) applies with $p_{ws}(\overline{\tau_t}) = v_{ws}(t)q_{ws}(t)$. Hence, when A is any subarray of T_A targets,

$$S_{A1} = \sum_{j=1}^{T_A} \prod_{w=1}^{W} \prod_{s=1}^{S_w} \{1 - v_{ws}(A_j)q_{ws}(A_j)\} \quad , \text{ and}$$

$$S_{A2} = \sum_{j=2}^{T_A} \sum_{k=1}^{j-1} \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{ 1 - v_{ws}(A_j)q_{ws}(A_j) - v_{ws}(A_k)q_{ws}(A_k) \right\} \quad .$$

The antitank weapon volley presented in the introductory section is an example of a Helmbold Volley, and the equations above and others (such as equations (1), (5), (6), (7), and (8)) were used to calculate the numerical values for it.

Although Helmbold Volleys are volleys of independently effective point fire munitions, it is clear that in general they are not volleys against exchangeably survivable targets. This justifies the location of the Helmbold Volley in Table 3.

In a Helmbold Volley it is not necessary that $\sum_{t=1}^{T} v_{ws}(t) = 1$, that is, it is not required that each shot be allocated to some target. Allocation probabilities that do not sum to unity may be interpreted as indicating that some shots are allocated to false targets (either deliberately deceptive "dummies," or inadvertent spurious targets), or that some shots are lost as the result of malfunctions, duds, and/or human error, or in other ways.

Observe that the availability of explicit solutions to volleys such as the Helmbold Volley and others smooths the way to an investigation of various optimization problems in connection with volley fire problems. For example, tactical problems such as the best arrangement of overlapping fields of fire or the value of trading rate of fire for improved accuracy can be investigated using the expected number of survivors as an objective function that is to be minimized. Some force structure issues could be clarified by evaluating the impact of different small unit organizations on the number of survivors, and so forth.

If the targets and weapons are all different, so that the $v_{wt}q_{wt}$ products are all different, then the evaluation of S_r involves $\binom{T}{r}$ terms in its summation. So if there are many targets (say 100 targets), then to evaluate S_{50} involves the summation of $\binom{100}{50} \simeq 168,000$ moles of terms, each term being a product of W factors, each factor being of the form $1 - v_{w1}q_{w1} - \ldots - v_{w50}q_{w50}$. So when there are 100 targets, it is not practical to solve the volley *completely* by evaluating all of the basic sums. Fortunately, for most practical purposes, solving the volley completely is not really necessary, and we can make do with just the first few moments of the distribution of the number of survivors, such as the expectation

and variance. For 100 targets, the expected value can be found from S_1, which involves 100 terms in its summation. The variance can be found from S_2, which involves $\binom{100}{2} = 4,950$ terms; not a calculation one would cheerfully undertake to do by hand, but something that is very easy to do with modern computers. With 100 targets, the third moment involves the summation of $\binom{100}{3} = 161,700$ terms. Although this is feasible with modern computers, some numerical analysis may be in order to ensure adequate precision in the final result. But the main point is that the expectation and variance can be found without solving the volley completely, and for applications that is a very handy feature.

The Helmbold Volley can also be generalized to allow for some types of collateral damage. For example, this could be done by setting

$$p_{ws}(\overline{z_t}) = \sum_{t' \in T} q_{ws}(t \mid t') v_{ws}(t') \quad ,$$

where $v_{ws}(t')$ is as before, but where $q_{ws}(t \mid t')$ is the probability that a shot from weapon w aimed at or allocated to target t' actually kills target t instead. Because we continue to assume that

$$\text{Prob}(\overline{z_1} \cup \overline{z_2} \cup \ldots \cup \overline{z_n}) = \sum_{r=1}^{n} p_{ws}(\overline{z_r})$$

for all subsets of the target array, the resulting volley is still a point fire rather than an area fire volley.

7-3. The Burst Fire Volley. The Burst Fire Volley extends the Helmbold Volley to allow a burst of shots to be fired at an acquired target. It may be characterized by the following postulates.

BF-1: Each weapon acquires all of the T targets in the target array.

BF-2: Each weapon fires S_w bursts during the volley, allocated at the rate of one burst per target. The probability that weapon w allocates burst s to target t is $v_{ws}(t)$, and is independent of the allocations of other bursts from the same or any other weapon. The probability that weapon w fires b rounds during a burst allocated to target t is $f_{wst}(b)$, and is independent of the other events that occur during the volley. All rounds fired in a burst are allocated to the same target as the burst.

BF-3: The probability that target t is killed if it is allocated b rounds in burst s from weapon w is $q_{ws}(t,b)$, and is zero otherwise, independent of the other events that occur during the volley.

It is clear that a Burst Fire Volley is equivalent to a Helmbold Volley in which

$$q_{ws}(t) = \sum_{b=0}^{\infty} q_{ws}(t,b) f_{wst}(b) \quad .$$

provided that the "shots" of the Helmbold Volley are identified with the bursts of the Burst Fire Volley. This justifies the location of the Burst Fire Volley entry in Table 3.

If the rounds in a burst are independently effective given the weapon-target combination involved, then $q_{ws}(t,b) = 1 - \{1 - q_{ws}(t,1)\}^b$. In that case,

$$q_{ws}(t) = 1 - \sum_{b=0}^{\infty} \{1 - q_{ws}(t,1)\}^b f_{wst}(b) \quad.$$

Various simplifications are possible if, for example,

$$f_{wst}(b) = \frac{\lambda^b e^{-\lambda}}{b!} \qquad \text{(Poisson),}$$

$$f_{wst}(b) = (1 - \lambda)\lambda^b \qquad \text{(Geometric),}$$

$$f_{wst}(b) = \binom{B_{wst}}{b} \lambda^b (1 - \lambda)^{B_{wst} - b} \qquad \text{(Binomial),}$$

$$f_{wst}(b) = \begin{cases} 1 & \text{if } b = B_{wst} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Deterministic).}$$

In the Poisson case,

$$q_{ws}(t) = 1 - e^{-\lambda q_{ws}(t,1)} \quad.$$

In the Geometric case,

$$q_{ws}(t) = \frac{\lambda q_{ws}(t,1)}{1 - \lambda\{1 - q_{ws}(t,1)\}} \quad.$$

In the Binomial case.

$$q_{ws}(t) = 1 - \{1 - \lambda q_{ws}(t,1)\}^{B_{wst}} \quad.$$

In the Deterministic case,

$$q_{ws}(t) = 1 - \{1 - q_{ws}(t,1)\}^{B_{wst}} \quad.$$

In each of the above cases, the basic event probabilities are given by equation (25) with $p_{ws}(\overline{z_t}) = v_{ws}(t)q_{ws}(t)$, and many other quantities of interest can easily be calculated from these basic event probabilities.

7-4. The Hide-and-seek Volley. The Hide-and-seek Volley is a volley by a battery of W weapons against an array of T targets that satisfies the following postulates.

H-1: There are H hiding places and T targets, and $H \geq T$. Targets occupy hiding places uniformly at random and independently of each other, subject only to the condition that at most one target can occupy a given hiding place.

H-2: Each weapon fires exactly one shot, which it allocates to hiding place h with probability v_{wh}, independently of the actual location of the targets.

H-3: The probability that the shot from weapon w kills target t is $q_{wh}(t)$ when that shot is allocated to hiding place h and target t is occupying hiding place h, and is zero otherwise, independent of what other events occur during the volley.

Note that this is a volley by a battery of independently effective point fire weapons, and so it can be solved by finding the $p_w(\overrightarrow{z_t})$ values. To do that, observe that the probability that weapon w kills target t is $v_{wh}q_{wh}(t)$ if target t occupies hiding place h, and since the probability that target t occupies hiding place h is $1/H$ for each $h = 1(1)H$,

$$p_w(\overrightarrow{z_t}) = H^{-1}\sum_{h=1}^{H} v_{wh}q_{wh}(t) \quad .$$

Then

$$P(z_{t_1}z_{t_2}\cdots z_{t_r}) = \prod_{w=1}^{W}\left\{1 - H^{-1}\sum_{n=1}^{r}\sum_{h=1}^{H} v_{wh}q_{wh}(t_n)\right\}$$

gives the basic event probabilities, and so provides the complete solution to the Hide-and-seek Volley.

The Hide-and-seek Volley is equivalent to a Helmbold Volley V^0 by a battery of W weapons against an array of T targets in which each weapon fires exactly one shot, allocated according to a uniform distribution over the T targets, and in which the kill probabilities are defined by

$$q_w^0(t) = \left(\frac{T}{H}\right)\sum_{h=1}^{H} v_{wh}q_{wh}(t) \quad .$$

This justifies the placement of the Hide-and-seek Volley entry in Table 3.

For the special case in which each weapon allocates its shot to a hiding place selected from the H hiding places according to a uniform distribution, $v_{wh} = H^{-1}$. Then

$$q_w^0(t) = \left(\frac{T}{H}\right)\overline{q_w}(t) \quad , \text{ where}$$

$$\overline{q_w}(t) = H^{-1}\sum_{h=1}^{H} q_{wh}(t) \quad .$$

If, in addition, $\overline{q_w}(t)$ is independent of t, then the Hide-and-seek Volley becomes equivalent to a DRR Volley by a battery of W weapons against an array of T targets in which the kill probability is taken to be

$$q_w^0 = \left(\frac{T}{H}\right)\overline{q_w} \quad .$$

7-5. The Karr Volley. Karr [1974] has analyzed in some detail a volley that he proposed as a model of the penetration of aircraft through a defended area, and for certain other types of penetration processes. We will extend Karr's results by providing formulas for the variance and correlation of the number of survivors. We paraphrase Karr's postulates for this volley as follows.

K-1: A battery of W weapons volleys against an array of T targets. The probability that weapon w acquires target t is $d_w(t)$ and is independent of other acquisitions made by the same or any other weapon.

K-2: A weapon that acquires one or more targets fires exactly one shot, which it allocates to a target chosen uniformly at random from among those it acquired, independently of the other events that occur during the volley.

K-3: The probability that target t is killed by weapon w is $q_w(t)$ if w allocates its shot to target t, and is zero otherwise, independent of the other events that occur during the volley.

Since a Karr Volley clearly is by independently effective point fire weapons, equation (21) applies. To determine the $p_w(z_t)$ values, consider a Helmbold Volley with the same number of weapons and targets as in the Karr Volley. In this Helmbold Volley, let each weapon fire exactly one shot, which is allocated to target t with probability

$$v_w(t) = \text{Prob(Weapon } w \text{ of the Karr Volley both acquires and allocates its shot to target } t) \quad .$$

Furthermore, in this Helmbold Volley, let the kill probabilities $q_w(t)$ be the same as in the Karr Volley. Then this Helmbold Volley is equivalent to the Karr Volley, and the problem reduces to determining the values of $v_w(t)$. Now, as Karr [1974] points out,

$$v_w(t) = \sum_{m=0}^{T-1} (m+1)^{-1}\text{Prob(Weapon } w \text{ acquires target } t \text{ and exactly } m \text{ other targets)}$$
$$= \sum_{m=0}^{T-1} (m+1)^{-1}d_w(t)A_w(m,t) \quad ,$$

where we have written $A_w(m,t)$ as an abbreviation for the probability that weapon w acquires exactly m other targets, given that it acquires target t. But, because acquisitions are independent events by postulate K-1, $A_w(m,t)$ must also be equal to the probability that weapon w acquires exactly m of the

$T - 1$ targets in the subarray $C(t)$, where $C(t)$ is the subarray of $T - 1$ targets obtained by omitting target t.

Next, observe that $A_w(m, t)$ is the probability that exact'y m targets survive a Gauntlet Volley in which a battery of one weapon (corresponding to weapon w of the Karr Volley) volleys against an array of $C(t)$ targets, and in which the survival probabilities for the Gauntlet Volley are identified with the acquisition probabilities $d_w(t)$ of the Karr Volley. Let

$$G_{wt}(x) = \sum_{m=0}^{T-1} x^m A_w(m, t)$$

be the generating function for the distribution of the number of survivors for this Gauntlet Volley. Because

$$\int_0^1 G_{wt}(x)dx = \sum_{m=0}^{T-1} (m+1)^{-1} A_w(m, t) \quad .$$

it can be seen that

$$v_w(t) = d_w(t) \int_0^1 G_{wt}(x)dx$$

$$= d_w(t) \int_0^1 \sum_{m=0}^{T-1} (x-1)^m S_{wm}(t)dx \quad .$$

where $S_{wm}(t)$ is the m-th order basic sum for this Gauntlet Volley (see equation (2)). Carrying out the integration yields

$$v_w(t) = d_w(t) \sum_{m=0}^{T-1} (-1)^m (m+1)^{-1} S_{wm}(t)$$

where, by equations (2) and (10),

$$S_{wm}(t) = \sum_m \prod_{n=1}^m d_w(j_n)$$

$$= \sum_{j_1=m}^T \sum_{j_2=1}^{j_1-1} \cdots \sum_{j_m=1}^{j_{m-1}-1} \prod_{n=1}^m d_w(j_n) \quad ,$$

where the notation $\sum$ means that the index $j_k = t$ is to be omitted from the summation. This is equivalent to a result obtained by Karr [1974, p 19], using different methods.

In the case where the targets are all alike, $d_w(t) = d_w$ for all $t = 1(1)T$, so that

$$S_{wm}(t) = \binom{T-1}{m} d_w{}^m \quad ,$$

and hence

$$v_w(t) = d_w \sum_{m=0}^{T-1} (-1)^m (m+1)^{-1} \binom{T-1}{m} d_w{}^m$$

$$= \int_0^{d_w} \sum_{m=0}^{T-1} (-1)^m x^m \binom{T-1}{m} dx$$

$$= \int_0^{d_w} (1-x)^{T-1} dx$$

$$= T^{-1} \left\{ 1 - (1-d_w)^T \right\} \quad .$$

By virtue of the equivalence previously pointed out between the Karr and Helmbold Volleys, it follows that for the Karr Volley with exchangeably survivable targets, where A is any subarray of T_A targets,

$$P_{Ar} = \prod_{w=1}^W \left[1 - rq_w T^{-1} \left\{ 1 - (1-d_w)^T \right\} \right] \quad \text{for } r = 0(1)T_A \quad ,$$

$$E(T_A^1) = T_A P_{A1} \quad ,$$

$$\mathrm{Var}(T_A^1) = T_A(T_A - 1)P_{A2} + E(T_A^1)\left\{ 1 - E(T_A^1) \right\} \quad ,$$

$$\rho_{jk} = \frac{P_{A2} - P_{A1}{}^2}{P_{A1} - P_{A1}{}^2} \quad \text{for } j \neq k, \text{ and}$$

$$P_{A[m]} = \binom{T_A}{m} \sum_{r=0}^{T_A - m} (-1)^r \binom{T_A - m}{r} P_{A(m+r)} \quad \text{for } m = 0(1)T_A \quad .$$

Karr [1974] gives formulas for the distribution and expected number of survivors that are equivalent to (or special cases of) those given above, but does not provide formulas for the variance or correlation.

7-6. The Multishot Karr Volley. This volley generalizes the Karr Volley to allow each weapon to fire a number of shots, each allocated to a target chosen uniformly at random from among those it acquires. These results are new. In this volley, the weapons are independently effective. although they are area fire, rather than point fire weapons, as they are in the Karr Volley. Moreover, as will be apparent in the following development, the munitions are not independently effective point fire munitions. These observations justify the location of the Multishot Karr Volley in Table 3. Although this volley is not equivalent to a volley by independently effective point fire weapons, it seems appropriate to present and analyze it in the context of independently effective point fire weapons.

We treat only the case where the targets are all alike, and so write the acquisition probability as d_w and the kill probability as q_{ws}. When S_w is the number of shots fired by weapon w, the first order basic event probabilities can be found from

$$p_w(z_t) = (1 - d_w) + d_w \sum_{m=0}^{T-1} A_w(m, t) \prod_{s=1}^{S_w} \{1 - q_{ws}/(m+1)\} \quad .$$

where, as before, $A_w(m, t)$ is the probability that weapon w acquires exactly m additional targets, given that it acquires target t. By Corollary 5.3, for the case at hand,

$$A_w(m, t) = \binom{T-1}{m} d_w^m (1 - d_w)^{T-1-m}{}_k \quad .$$

That $p_w(z_t)$ is given by the indicated expression can be seen by reasoning as follows. Target t certainly will survive weapon w if it is not acquired by weapon w. This accounts for the $1 - d_w$ term. If target t and m other targets are acquired by weapon w, then t survives weapon w only if all of the S_w shots by weapon w fail to dispatch it. The expression for the probability of that event can readily be obtained from our earlier results for DRR Volleys and, when summed over m, it produces the second term in the equation for $p_w(z_t)$.

The second order basic event probabilities for a one-weapon battery are

$$p_w(z_j z_k) = (1 - d_w)^2 + 2(1 - d_w)d_w \sum_{m=0}^{T-2} A_w(m; j, k) \prod_{s=1}^{S_w} \{1 - q_{ws}/(m+1)\}$$

$$+ d_w^2 \sum_{m=0}^{T-2} A_w(m; j, k) \prod_{s=1}^{S_w} \{1 - 2q_{ws}/(m+2)\} \quad ,$$

where $A_w(m; j, k)$ is the probability that weapon w acquires exactly m targets other than j or k, given that it acquires both j and k. By Corollary 5.3, for the case at hand

$$A_w(m; j, k) = \binom{T-2}{m} d_w^m (1 - d_w)^{T-2-m} \quad .$$

The reasoning for each term in the expression for $p_w(z_j z_k)$ is that both j and k will survive the fire from weapon w if

(i)   Neither are acquired by weapon w,

(ii)   Exactly one of them is acquired by weapon w and the other is not, but the one that is acquired survives anyhow, or

(iii)   Both j and k are acquired by weapon w, but both survive anyhow.

In the last case, we know from the analysis of DRR Volleys that the probability that both of two preselected targets survive when a volley of S_w shots is directed at random against an array of $m + 2$ exchangeably survivable targets is

$$\prod_{s=1}^{S_w} \{1 - 2q_{ws}/(m+2)\} \quad .$$

and the last term in the expression for $p_w(z_j z_k)$ follows easily. A similar argument can be used to obtain the expression for the second term.

In general, we will have

$$p_w(z_{j_1} z_{j_2} \cdots z_{j_r}) = \sum_{k=0}^{r} \binom{r}{k} d_w^k (1 - d_w)^{r-k} \sum_{m=0}^{T-r} A_w(m; j_1, j_2, \ldots, j_r) \prod_{s=1}^{S_w} \{1 - kq_{ws}/(m+k)\} \quad .$$

The reasoning is that, by Corollary 5.3,

$$\binom{r}{k} d_w^k (1 - d_w)^{r-k}$$

is the probability that exactly k of the targets $j_1, j_2, \ldots, j_r$ are acquired by weapon w, that from the analysis of DRR Volleys

$$\prod_{s=1}^{S_w} \{1 - kq_{ws}/(m+k)\}$$

is the probability that all of those k targets survive anyway (given that m additional targets, other than any of the $j_1, j_2, \ldots, j_r$, are also acquired, making a total of $m + k$ targets acquired by weapon w), and that, by Corollary 5.3,

$$A_w(m; j_1, j_2, \ldots, j_r) = \binom{T-r}{m} d_w^m (1 - d_w)^{T-r-m}$$

is the probability that exactly m targets other than the $j_1, j_2, \ldots, j_r$ are acquired. Observe that the expression given above for $p_w(z_{j_1} z_{j_2} \cdots z_{j_r})$ shows that in the Multishot Karr Volley, the munitions

generally are not independently effective point fire munitions (that is, neither equation (21) nor (25) applies).

Since the targets are exchangeably survivable, we may write the basic event probabilities for each of the single weapon volleys more briefly as

$$p_{wr} = p_w(z_{j_1} z_{j_2} \cdots z_{j_r}).$$

The basic event probabilities for a volley by the full weapons battery may then be obtained from

$$P_r = \prod_{w=1}^{W} P_{wr} \quad ,$$

in accord with equation (18). This formula for P_r does not seem to reduce to any substantially simpler expression. However, it provides a complete solution to the Multishot Karr Volley and can be used in conjunction with equations (11) through (17) to compute numerical values for quantities of interest, such as the expectation and variance of the number of survivors.

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8. VOLLEYS BY SYNERGISTICALLY EFFECTIVE WEAPONS

Volleys by synergistically effective weapons usually are more difficult to analyze than volleys by independently effective weapons, because no convenient general principles are available for expressing the effects of the whole battery of weapons in terms of smaller and more easily analyzed batteries. Of course, volleys by synergistically effective weapons against arrays of independently survivable targets often can be solved rather easily, as illustrated by the Artillery Volley. Even in that case, however, the effect of all the weapons in the entire battery had to be considered simultaneously. We now present two examples of volleys by synergistically effective weapons against targets that are not independently survivable.

8-1. The Bellwether Volley. This volley has been contrived to provide a solvable example of a volley by a battery of synergistically effective weapons against an array of targets that are neither independently nor exchangeably survivable. It is not put forward as having any important practical applications. It can be defined by the following postulates.

> **B-1:** All weapons acquire all of the targets in the array.

> **B-2:** One of the weapons is selected at random from among the battery of weapons to be the "bellwether" weapon. Let b_w be the probability that weapon w is chosen to be the bellwether weapon. The bellwether weapon then allocates its fire to a single target, which is chosen from the target array according to the probabilities $v_w(t)$ when weapon w is the bellwether weapon. All other weapons in the battery then allocate their fire to the same target as the bellwether weapon.

> **B-3:** The probability that target t is killed during the volley is $q(t)$ if all weapons concentrate their fire against it, and is zero otherwise, independently of what other events occur during the volley.

The Bellwether Volley is easily analyzed. The probability that target t survives, given that weapon w is selected as the bellwether weapon, is

$$1 - v_w(t) + v_w(t)\{1 - q(t)\} = 1 - v_w(t)q(t) \quad .$$

This is true because target t survives if the bellwether weapon does not allocate fire to it, or if the bellwether weapon allocates fire to it, but it survives anyway. Because the bellwether weapon is selected at random, the probability that weapon w will be chosen as the bellwether weapon is W^{-1}, and so the first order basic event probabilities are

$$P(z_t) = \sum_{w=1}^{W} b_w\{1 - v_w(t)q(t)\} = 1 - v(t)q(t) \quad , \text{ where}$$

$$v(t) = W^{-1} \sum_{w=1}^{W} v_w(t)$$

is the average probability that the battery will allocate all of its fire to target t. Now, the battery in a Bellwether Volley could be called a point fire *battery*, because it can kill at most one target per volley, and so the higher-order basic event probabilities are obtainable in terms of the first-order basic event probabilities, as follows.

$$P(z_{j_1} z_{j_2} \ldots z_{j_r}) = 1 - P(\overline{z_{j_1} z_{j_2} \ldots z_{j_r}})$$

$$= 1 - P(\overline{z_{j_1}} \cup \overline{z_{j_2}} \cup \ldots \cup \overline{z_{j_r}})$$

$$= 1 - \sum_{n=1}^{r} P(\overline{z_{j_n}}) \quad ,$$

because

$$P(\overline{z_{j_1}}\, \overline{z_{j_2}}) = P(\overline{z_{j_1}}\, \overline{z_{j_2}}\, \overline{z_{j_3}}) = \ldots = 0 \quad .$$

Therefore,

$$P(z_{j_1} z_{j_2} \ldots z_{j_r}) = 1 - \sum_{n=1}^{r} v(j_n) q(j_n)$$

$$= 1 - \sum_{n=1}^{r} \left\{ 1 - P(z_{j_n}) \right\}$$

$$= \sum_{n=1}^{r} P(z_{j_n}) - r + 1 \quad .$$

Consequently, summing both sides over all possible r-th order subarrays, and recalling the argument used in the proof of Theorem 2, we find

$$S_r = \sum_r P(z_j z_k \ldots z_r) = \binom{T-1}{r-1} S_1 - (r-1)\binom{T}{r} \quad .$$

which can be written as

$$S_r = \binom{T}{r} \left\{ (r/T) S_1 - (r-1) \right\}$$

$$= \binom{T}{r} \left\{ 1 - r(1 - S_1/T) \right\} \quad ,$$

where

$$S_1 = \sum_{t=1}^{T} P(z_t) = T - \sum_{t=1}^{T} v(t) q(t) \quad .$$

Then we can write the generating function as

$$G(x) = \sum_{r=0}^{T} (x-1)^r S_r.$$

$$= (T - S_1)x^{T-1} + \left\{1 - (T - S_1)\right\}x^T \quad .$$

Comparing this result with equation (3) shows that

$$P_{[T]} = 1 - (T - S_1) = 1 - \sum_{t=1}^{T} v(t)q(t) \quad ,$$

$$P_{[T-1]} = T - S_1 = \sum_{t=1}^{T} v(t)q(t) \quad ,$$

and

$$P_{[m]} = 0 \quad \text{for } m = 0(1)(T-2) \quad .$$

Also,

$$E(T^1) = T - \sum_{t=1}^{T} v(t)q(t)$$

and

$$\mathrm{Var}(T^1) = 2S_2 + S_1 - S_1{}^2$$

$$= (T - S_1)\left\{1 - (T - S_1)\right\} \quad .$$

The Bellwether Volley is not a volley by independently effective weapons. For if it were, then its first order basic event probabilities would be

$$P(z_t) = \prod_{w=1}^{W} \left\{1 - v_w(t)q_w(t)\right\} \quad ,$$

where $q_w(t)$ is the probability that target t would be killed if the fire of weapon w acting alone were allocated to it. But the required equality obviously does not hold in general. Nor would it hold even if it were assumed that

$$1 - q(t) = \prod_{w=1}^{W} \left\{1 - q_w(t)\right\} \quad ,$$

that is, that the weapons are independently effective, conditional on the selection of target t as the one against which the entire battery's fire is concentrated. Moreover, in a Bellwether Volley

$$P(z_1 z_2) = P(z_1) + P(z_2) - 1 \quad ,$$

which is not generally equal to $P(z_1)P(z_2)$. Hence, the Bellwether Volley is not a volley against independently survivable targets. Furthermore, it is clear that, in general, the Bellwether Volley is not a volley against exchangeably survivable targets, because there is no reason why the first order basic event probabilities $P(z_t) = 1 - v(t)q(t)$ should be independent of t. The observations of this paragraph justify the placement of the Bellwether Volley entry in Table 3.

8-2. A Redundantly Survivable Target Volley. By a redundantly survivable target we mean one that is able to survive several hits. More precisely, we assume that there is a redundancy number R that gives the maximum number of hits a target can tolerate without ill effect. That is, a target survives if it takes R or fewer hits during the volley and is killed otherwise (recall that in this paper we deal only with targets that are in one or the other of two possible states—dead or alive). With this notion of a redundantly survivable target in mind, we define the following version of a Redundantly Survivable Target Volley.

RST-1: Each weapon acquires all of the targets in the array.

RST-2: Each weapon fires exactly one shot, which it allocates to a target selected from the target array uniformly at random.

RST-3: The probability that the shot from weapon w hits target t is q if the shot is allocated to target t, and is zero otherwise, independently of what other events occur during the volley.

RST-4: Target t survives if, and only if, it takes no more than R hits in the course of the volley.

These postulates clearly describe a volley by a battery of synergistically effective weapons against an array of exchangeably survivable targets, confirming the placement of this volley in Table 3. To determine the basic event probabilities, observe that, for each $j = 1(1)r$, the probability that target t receives exactly n_j hits during the volley is determined by the multinomial distribution (see, for example, Feller [1950], Abramowitz and Stegun [1964], or Loeve [1960])

$$\frac{W!}{n_0! n_1! \ldots n_r!} q_0^{n_0} (q/T)^{n_1} \ldots (q/T)^{n_r} \quad ,$$

where

$$n_0 = W - \sum_{j=1}^{r} n_j$$

and

$$q_0 = 1 - \sum_{j=1}^{r} (q/T) = 1 - rq/T \quad .$$

Nevertheless, we have

$$P_r = P(z_1 z_2 \ldots z_r)$$

$$= \text{Prob}\{(n_1 \leq R) \cap (n_2 \leq R) \cap \ldots \cap (n_r \leq R)\}$$

$$= \sum_{n_1 = 0}^{R} \sum_{n_2 = 0}^{R} \cdots \sum_{n_r = 0}^{R} \frac{W!}{n_0! n_1! \ldots n_r!} \, q_{n_0}^{0} \, (q/T)^{W - n_0} \quad . \tag{30}$$

which provides the complete solution to this volley.

For the special case in which the targets survive if, and only if, they receive no hits, the redundancy number $R = 0$. In that case, the above expression for P_r reduces to

$$P_r = q_0^{W} = (1 - rq/T)^{W} \quad .$$

which (as it should be) is identical to that for a DRR Volley in which the kill probability $q_w = q$ for all $w = 1(1)W$.

When the targets are singly redundant (that is, when $R = 1$), put

$$m = \sum_{j=1}^{r} n_j \quad .$$

so that m ranges over the values $0(1)r$. Observe that, because in this case $n_j! = 1$ for each $j = 1(1)r$, equation (30) can be written as

$$P_r = \sum_{n_1 = 0}^{1} \sum_{n_2 = 0}^{1} \cdots \sum_{n_r = 0}^{1} \binom{W}{m} m! (1 - rq/T)^{W - m} (q/T)^{m}$$

$$= \sum_{k=0}^{r} \binom{r}{k} \binom{W}{k} k! (1 - rq/T)^{W - k} (q/T)^{k} \quad .$$

where the last equality follows by observing that in the multiple summation exactly $\binom{r}{k}$ terms are such that

$$m = \sum_{j=1}^{r} n_j = k \quad .$$

This is true because $\binom{r}{k}$ is the number of ways in which exactly k 1's can be assigned to the r n_j's (the other $r - k$ of the n_j's having the value of zero).

In general, when $R \geq 2$, the right side of equation (30) is not easily reduced to any substantially more compact expression. However, for the case $r = 1$ we note that

$$P_1 = \sum_{n=0}^{R} \binom{W}{n} (q/T)^n (1 - q/T)^{W-n} \quad .$$

which is sometimes useful, because the average number of survivors is given by $E(T^1) = TP_1$.

Observe that, by equations (13) and (30), when shots are allocated uniformly at random to the target array, we have

$$P_{[m]}(R) = \binom{T}{m} \sum_{r=m}^{T} (-1)^{m+r} \binom{T-m}{r-m} \frac{W!}{(w-Rr)!(R!)^r} (q/T)^{Rr} (1 - rq/T)^{W-Rr}$$

for the probability that exactly m of T targets each receive exactly R hits from a total of W shots. When $q = 1$, this is the same as the probability that exactly m cells each contain exactly R balls when a total of W balls are tossed randomly into T cells (compare this to Feller [1950]). Thus, the above generalizes the well-known occupancy problem of classical combinatorial probability theory.

9. CONCLUDING REMARKS

This concludes our presentation of the foundations of a general theory of volley fire models. In the course of it, we have reviewed previous work in this area and demonstrated that our approach not only powerfully unifies and extends previously used methods for solving volley fire problems, but often provides simpler and more intuitive solutions than have previously appeared. This general approach also shows that volley fire models generalize many of the classical probability problems in the theory of matchings, occupancy, and statistical mechanics. It also provides a useful system for classifying volleys into a few major categories to facilitate their solution by indicating the most appropriate solution method. In addition, it suggests potentially important new concepts, such as those for equivalent and complementary volleys. Moreover, it yields hitherto unpublished results.

In addition, various specific opportunities to extend or apply this treatment of the foundations of the general theory of volley fire were identified. Among them are the following.

1.   In general, the complements of volleys by independently effective weapons are volleys by synergistically effective weapons. How are those volleys by synergistically effective weapons characterized? What properties do they possess? What insights regarding the solution of volleys by synergistically effective weapons do they afford?

2.   How can the effects of successive volleys best be approximated? What error bounds apply to this approximation?

3.   What limiting forms do volleys approach as various parameters (such as the number of targets, the number of weapons, and so forth) tend toward large or small values?

4.   What optimization problems regarding volleys are most important, and what are their solutions?

Some larger issues which deserve attention in future research on volley fire models are as follows.

1.   How can volleys against arrays of targets that may be in more than two states at the end of the volley be most efficiently analyzed?

2.   What are the necessary and/or sufficient conditions under which explicit, closed-form solutions for the effect of successive volleys against arrays of targets be obtained?

3.   What are the outcomes of volleys in which the target array is active, that is, returns fire? As far as we are aware, the deepest results on this have been reported by Gafarian and Manion [1989]. Versions in which the targets can countervolley have been treated by Helmbold [1966] (who, in a heuristic manner, derived the Lanchester square law equations from the limit of an alternating volley), Helmbold [1968] (although under rather restrictive assumptions), Bashyam [1970], and Zinger [1980].

Hopefully, calling attention to these challenging problems will stimulate analysts to devise original and imaginative solutions to them.

APPENDIX A

NOTES

Note 1. The probability of any complexion, and therefore of any event concerning the outcome of a volley, can be expressed in terms of sums and differences of basic event probabilities. For example,

$$P(z_j \overline{z_k} z_l \overline{z_m} z_n) = E\left\{ \tau_j (1 - \tau_k) \tau_l (1 - \tau_m) \tau_n \right\}$$

$$= E\left\{ \tau_j \tau_l \tau_n (1 - \tau_k - \tau_m + \tau_k \tau_m) \right\}$$

$$= P(z_j z_l z_n) - P(z_j z_l z_n z_k) - P(z_j z_l z_n z_m) + P(z_j z_l z_n z_k z_m) \quad .$$

This illustrates the following useful prescription given by Loeve [1960] for finding the probability of an event: (i) express the event as a sum of intersections of z's and $\overline{z}$'s (note that complexions are already in this form); (ii) replace the z's by τ's and the $\overline{z}$'s by $(1 - \tau)$'s; (iii) express the result in the form of sums and differences of terms involving only products of the τ's; and then (iv) take the expectation. In taking the expectation, recall that the expectation of a sum is the sum of the expectations, and that

$$E\left(\prod_{n=1}^{N} \tau_{j_n} \right) = P\left(\prod_{n=1}^{N} z_{j_n} \right) \quad ,$$

because the τ's are the indicators of the z's. Following this prescription produces for the probability of any event an expression involving only the sums and differences of basic event probabilities and justifies the assertion that, in principle, this is always possible. Of course, when there are many targets and the event involves complicated sums and differences of basic events, it may not be feasible to perform the calculations, even though explicit algorithms for them may exist.

Note 2. Let $H_A(x)$ be the generating function for the "tail" probabilities

$$P_{Am+} = \text{Prob(At least } m \text{ targets of subarray A survive the volley)}$$

$$= \sum_{r=m}^{T_A} P_{A[r]}$$

$$= P_{A[m]} + P_{A(m+1)+} \quad .$$

Thus,

$$H_A(x) = \sum_{m=0}^{T_A} x^m P_{Am} +$$

$$= G_A(x) + \sum_{m=1}^{T_A} x^{m-1} P_{Am} +$$

$$= G_A(x) + \frac{H_A(x) - 1}{x} \quad .$$

Solving for $H_A(x)$ yields

$$H_A(x) = \frac{xG_A(x) - 1}{x - 1} = G_A(x) + \frac{G_A(x) - 1}{x - 1} \quad .$$

__*Note 2.*__ The n-th moment of the number of survivors can be expressed in terms of the first n basic sums, S_{Ak}, where $k = 1(1)n$. This can be shown as follows. Comparing the k-th derivative of $G_A(x)$. evaluated at $x = 1$, obtained from equation (3), with the same value, obtained from equation (4), shows that

$$G_A^{(k)}(1) = k!E\binom{T_A^1}{k} = k!S_{Ak} \quad .$$

Since, by a standard result in combinatorial analysis (see section 24.1.4B of Abramowitz and Stegun [1964]),

$$\left(T_A^1\right)^n = \sum_{k=1}^{n} \mathfrak{S}_2(n,k)k!\binom{T_A^1}{k} \quad ,$$

where the coefficients $\mathfrak{S}_2(n,k)$ are Stirling numbers of the second kind. it follows that

$$E\left\{\left(T_A^1\right)^n\right\} = \sum_{k=1}^{n} \mathfrak{S}_2(n,k)k!S_{Ak}$$

$$= \sum_{k=1}^{n} \mathfrak{S}_2(n,k)G_A^{(k)}(1) \quad .$$

These results express the n-th moment in terms of the first n basic sums. or in terms of the first n derivatives of the generating function. Abramowitz and Stegun [1964] tabulate the values of the Stirling numbers $\mathfrak{S}_2(n,k)$ for $k = 1(1)n$ and $n = 1(1)25$.

Note 4. Suppose weights m_{Aj} and $\overline{m_{Aj}}$ are assigned to the events z_{Aj} and $\overline{z_{Aj}}$, respectively. Then the surviving weight of the targets in subarray A is

$$M_A^1 = \sum_{j=1}^{T_A} \left\{ m_{Aj}\tau_{Aj} + \overline{m_{Aj}}(1 - \tau_{Aj}) \right\}$$

$$= M_{A0} + \sum_{j=1}^{T_A} M_{Aj}\tau_{Aj} \quad ,$$

where

$$M_{A0} = \sum_{j=1}^{T_A} \overline{m_{Aj}}$$

and

$$M_{Aj} = m_{Aj} - \overline{m_{Aj}} \quad \text{for } j = 1(1)T_A.$$

If hostile elements in the target array are intermingled with friendly units or other elements of value to the side controlling the battery of weapons, survival of the items of value might be assigned positive weights while survival of hostile elements are assigned negative ones.

Note 5. The proof of Theorem 2 is as follows. By equation (2),

$$S_{Ar}^* = \sum_r P^*(z_{Aj_1}^* z_{Aj_2}^* \cdots z_{Aj_r}^*) \quad .$$

where $\sum_r$ signifies that the sum is taken for indices $j_1, j_2, \ldots, j_r$ which are to be varied in such a way that the subarray consisting of the targets $\{Aj_1, Aj_2, \ldots, Aj_r\}$ sweeps over each of the r-th order subarrays of subarray A. By definition of a complementary volley, each term in this sum can be written as

$$P^*(z_{Aj_1}^* z_{Aj_2}^* \cdots z_{Aj_r}^*) = P(\overline{z_{Aj_1}}\ \overline{z_{Aj_2}} \cdots \overline{z_{Aj_r}})$$

$$= 1 - P(z_{Aj_1} \cup z_{Aj_2} \cup \ldots \cup z_{Aj_r})$$
$$= 1 - S_{A1}^r + S_{A2}^r - \ldots + (-1)^r S_{Ar}^r \quad .$$

where S_{Ak}^r is the k-th order basic sum associated with the subarray $\{Aj_1, Aj_2, \ldots, Aj_r\}$ of r targets, that is,

$$S^r_{A1} = \sum_{n=1}^{r} P(z_{Aj_n}) \quad ,$$

$$S^r_{A2} = \sum_{n=2}^{r} \sum_{m=1}^{n-1} P(z_{Aj_n} z_{Aj_m}) \quad ,$$

and so forth.

Then we may write S^*_{Ar} as a sum of terms

$$S^*_{Ar} = \sum_r \left\{ \sum_{m=0}^{r} (-1)^m S^r_{Am} \right\} = \sum_{m=0}^{r} (-1)^m \left[\sum_r S^r_{Am} \right] \quad .$$

The square-bracketed term in the last expression can be evaluated by a combinatorial argument. Consider first the term with $m = 1$, that is,

$$\sum_r S^r_{A1} = \sum_r \sum_{n=1}^{r} P(z_{Aj_n}) \quad .$$

Observe that, as the indices $j_1, j_2, \ldots, j_r$ vary so as to sweep over all r-th order subarrays of subarray A, the index j_1 will vary over the values $1(1)T_A$. The index j_2 will also vary over the values $1(1)T_A$, and so on for each index j_n for $n = 1(1)r$. Thus, we may write

$$\sum_r \sum_{n=1}^{r} P(z_{Aj_n}) = \sum_{j=1}^{T_A} C_j P(z_{Aj}) \quad ,$$

where C_j counts the number of times the value $P(z_{Aj})$ appears in the sum on the left hand side of this equation. But each C_j must have the value

$$C_j = \binom{T_A - 1}{r - 1} \quad ,$$

because that is the number of r-th order subarrays of subarray A that contain target Aj (that is, it is the number of ways in which the additional $r - 1$ targets needed to fill out an r-th order subarray can be selected from among the $T_A - 1$ targets that remain after an initial target has been selected). Hence,

$$\sum_r S^r_{A1} = \sum_r \sum_{n=1}^{r} P(z_{Aj_n}) = \sum_{j=1}^{T_A} \binom{T_A - 1}{r - 1} P(z_{Aj}) = \binom{T_A - 1}{r - 1} S_{A1} \quad .$$

Similarly,

$$\sum_r S^r_{A2} = \sum_r \sum_{n=2}^{r} \sum_{m=1}^{n-1} P(z_{Aj_n} z_{Aj_m}) = \sum_{j=2}^{T_A} \sum_{k=1}^{j-1} C_{jk} P(z_{Aj} z_{Ak})$$

$$= \sum_{j=2}^{T_A} \sum_{k=1}^{j-1} \binom{T_A - 2}{r-2} P(z_{Aj} z_{Ak}) = \binom{T_A - 2}{r-2} S_{A2} \quad .$$

because

$$C_{jk} = \binom{T_A - 2}{r-2}$$

is the number of ways in which the $r - 2$ additional targets needed to fill out an r-th order subarray can be selected from the $T_A - 2$ targets that remain after an initial pair have been selected.

And, in general, it can be seen that

$$\sum_r S^r_{Am} = \binom{T_A - m}{r - m} S_{Am} \quad ,$$

because the coefficient on the right-hand side is the number of ways in which the $r - m$ additional targets needed to fill out an r-th order subarray can be selected from the $T_A - m$ targets that remain after an initial selection of m targets has been made. Hence,

$$S^*_{Ar} = \sum_{m=0}^r (-1)^m \left(\sum_r S^r_{Am} \right) = \sum_{m=0}^r (-1)^m \binom{T_A - m}{m - r} S_{Am} \quad . \square$$

Note 6. To demonstrate Theorem 3, we first prove assertion (i), using the method suggested by Thomas [1984]. By equation (4) and Theorem 2,

$$G^*_A(x) = \sum_{r=0}^{T_A} (x - 1)^r \sum_{m=0}^r (-1)^m \binom{T_A - m}{r - m} S_{Am} \quad .$$

Interchanging the order of summation, having due regard for the region of summation in the (r, m) plane, yields

$$G^*_A(x) = \sum_{m=0}^{T_A} S_{Am} (-1)^m \sum_{r=m}^{T_A} (x - 1)^r \binom{T_A - m}{r - m} \quad .$$

which can be written as

$$G^*_A(x) = \sum_{m=0}^{T_A} S_{Am} (-1)^m (x - 1)^m x^{T_A - m}$$

$$= x^{T_A} \sum_{m=0}^{T_A} S_{Am} (x^{-1} - 1)^m = x^{T_A} G_A(1/x) \quad .$$

where the last equality follows from equation (4). This gives assertion (i), and assertion (ii) immediately follows. The proof of assertions (iii) and (iv) follow immediately on observing that assertion (ii) implies that T_A^{*1}, the number of survivors in volley V^*, has the same distribution as $T_A - T_A^1$, the number of targets killed in volley V. Assertion (v) is demonstrated as follows. By equation (7), the correlation is equal to

$$\rho_{AB}^* = \frac{E(T_A^{*1}T_B^{*1}) - E(T_A^{*1})(E_B^{*1})}{\text{Var}(T_A^{*1})\text{Var}(T_A^{*1})} \quad,$$

where, by assertion (iv), the variances in the denominator can be replaced by those for T_A^1 and T_B^1. By the definition of a complementary volley, we may write

$$E(T_A^{*1}T_B^{*1}) = E\left\{\sum_{j=1}^{T_A}\sum_{k=1}^{T_A}\tau_{Aj}^*\tau_{Bk}^*\right\} = E\left\{\sum_{j=1}^{T_A}\sum_{k=1}^{T_A}(1-\tau_{Aj})(1-\tau_{Bk})\right\}$$

$$= T_AT_B - T_BE(T_A^1) - T_AE(T_B^1) + E(T_A^1T_B^1) \quad.$$

The other term in the numerator of ρ_{AB}^* is, by assertion (iii),

$$E(T_A^{*1})E(T_B^{*1}) = T_AT_B - T_BE(T_A^1) - T_AE(T_B^1) + E(T_A^1)E_B^1) \quad.$$

so that the numerator of ρ_{AB}^* is equal to

$$E(T_A^1T_B^1 - E(T_A^1T_B^1) \quad,$$

and therefore $\rho_{AB}^* = \rho_{AB}$.□

Note 7. By postulates A-2 and A-3, the probability that target t survives all shots when it is located almost exactly at (x,y) is

$$P\{z_t \mid (x,y)\} = \int F_t \prod_{w=1}^{W}\prod_{s=1}^{S_w}\{\sigma_{ws}(u_{ws},v_{ws})du_{ws}dv_{ws}\} \quad.$$

Here F_t is the probability that target t survives, given the ground zeros of all the shots, as defined by postulate A-3, and $\int$ stands for the multiple integral

$$\int = \prod_{w=1}^{W}\prod_{s=1}^{S_w}\int_{(u_{ws},v_{ws})} \quad.$$

This integral may be written as

$$P\{z_t \mid (x,y)\} = \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left[\int_{(u,v)} \{1 - D_{wst}(x-u, y-v)\}\sigma_{ws}(u,v)dudv \right]$$

$$= \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{ 1 - \int_{(u,v)} D_{wst}(x-u, y-v)\sigma_{ws}(u,v)dudv \right\} \quad ,$$

where the first equality follows from the general theorem for converting multiple integrals to products of single integrals, and the second follows from the assumption that σ is a probability density function.

Hence,

$$P(z_t) = \int_{(x,y)} P\{z_t \mid (x,y)\}\rho_t(x,y)dxdy$$

$$= \int_{(x,y)} \prod_{w=1}^{W} \prod_{s=1}^{S_w} \left\{ 1 - \int_{(u,v)} D_{wst}(x-u, y-v)\sigma_{ws}(u,v)dudv \right\}\rho_t(x,y)dxdy \quad .$$

and the formulas of the canonical volley against independently survivable targets apply to complete the solution for the basic event probabilities of an Artillery Volley.

Now, for the munitions to be independently effective, it is necessary that

$$P(z_t) = \prod_{w=1}^{W} \prod_{s=1}^{S_w} p_{ws}(z_t) \quad ,$$

where

$$p_{ws}(z_t) = \int_{(x,y)} \left\{ 1 - \int_{(u,v)} D_{wst}(x-u, y-v)\sigma_{ws}(u,v)dudv \right\}\rho_t(x,y)dxdy$$

is the first order basic even probability for a volley that consists solely of shot s from weapon w. But, in general, $P(z_t)$ is not of the required form, and so the munitions generally are not independently effective. A similar argument shows that in general the weapons are not independently effective, either.

It is interesting to note that under some circumstances, the weapons can become independently effective in the limit. To show this, suppose that the damage function D_{wst} does not change with the shot number, so that it may be written as $D_{wt}(x-u, y-v)$. Suppose also that the number of shots S_w increases, while at the same time the distribution σ_{ws} "flattens out" in such a way that the product $S_w\sigma_{ws}(u,v)$ approaches the constant density N_w/A, where A has the physical dimensions of an area. Then, as Helmbold [1970] has shown, $P(z_t)$ approaches

$$P(z_t) = \exp\left(-\sum_{w=1}^{W} L_{wt} N_w / A\right)$$

$$= \prod_{w=1}^{W} p_w(z_t) \quad,$$

where

$$p_w(z_t) = \exp(-L_{wt} N_w / A)$$

is the first order basic event probability for a volley by weapon w acting alone against the target array. In the above equations,

$$L_{wt} = \int\limits_{(u,v)} D_{wt}(u,v)\,du\,dv$$

is commonly called the lethal area for a munition of weapon w, when fired against target t.

Observe further that the targets are exchangeably survivable if L_{wt} is independent of t, in which case Corollary 5.3 applies to show that the expected fraction of targets that survive is

$$E(T^1/T) = P \quad,$$

where

$$P = \exp\left(-\sum_{w=1}^{W} L_w N_w / A\right)$$

does not depend on either t or w. This expression for the expected fraction of survivors is frequently used to obtain a quick estimate of the effect of fragmenting ordnance fired into a target area. We note that the variance of the fraction of survivors is

$$\mathrm{Var}(T^1/T) = T^{-2}\mathrm{Var}(T^1)$$

$$= T^{-1}P(1-P) \quad,$$

where the second equality follows from Corollary 5.3.

Note 8. Dixon [1953], Thomas [1956]. and Helmbold [1966] have suggested various approximations for the effect of successive DRR Volleys. Some of these are built around the idea of replacing the random variable T^1 with its expectation $E(T^1)$ and approximating the number of survivors after the n-th volley as

$$T^n \simeq T^{n-1} \prod_{w=1}^{W} \left(1 - q_w/T^{n-1}\right) \quad.$$

where T^n is the number of survivors after the n-th volley. This is called the expected value approximation. Although Helmbold [1966] presents some empirical support for this approximation, he provides no rigorous theoretical basis for it. Consequently, as noted by Karr [1974] and others, the conditions under which it is valid are not clear. In essence, the issue is one of establishing bounds on the error involved in making the expected value approximation. Although the present author has not pursued this matter, it may be that such bounds could be developed by using the formulas for the variance (or higher moments) of the number of survivors, together with Chebyshev's Inequality (or similar inequalities).

Another approach might be through the study of limiting forms. For example, we observe that the e :pected value approximation tends to be more accurate when the number of targets is large. This follows from the fact that when there are many targets, the argument given by Feller [1950, pp 69-74] applies to yield the following limiting Poisson approximations to the outcome of a DRR Volley:

$$E(T_A^1) \simeq \text{Var}(T_A^1) \simeq \lambda_A \quad ,$$

$$\rho_{jk} \simeq 0 \quad ,$$

and

$$P_{A[m]} \simeq \frac{\lambda_A{}^m}{m!} e^{-\lambda_A} \quad ,$$

where

$$\lambda_A = T_A \exp\left(-\sum_{w=1}^{W} q_w/T\right) \quad .$$

In the limit, then,

$$P_{Ar} \simeq \exp\left(-r\sum_{w=1}^{W} q_w/T\right) \simeq (P_{A1})^r \quad ,$$

which shows that the z's tend to become independent as the number of targets increases—in which case the DRR Volley approaches a volley against independently survivable targets, for which the expected value approximation clearly is exact. For additional material related to the limiting forms of random allocations, see Kolchin, et al. [1978], and Choi [1987].

The study of limiting forms of volleys, and of approximations to the effect of several successive volleys, are two areas deserving additional research.

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APPENDIX B

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