Structure and Utility of Blind Speed Intervals Associated with Doppler Measurements of Range Rate

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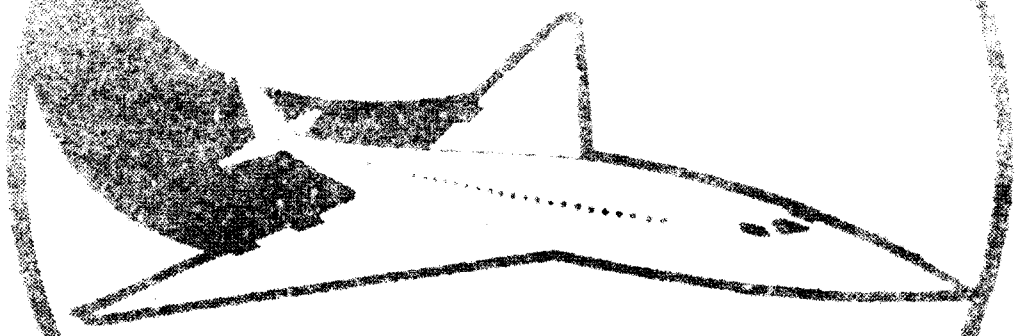

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16. Abstract

In the case of a coherent pulsed radar system, the time rate of change of the slant range of an aircraft may be determined to within an integer multiple of a known speed by measuring the pulse-to-pulse phase shift in the reflection of a transmitted wave train of electromagnetic energy. The integer multiplier is not necessarily a known, and lack of knowledge of the multiplier gives rise to an ambiguity. The ambiguity may be removed by appropriate processing of the pulse-to-pulse phase shift in the reflection of each of two wave trains that differ in one or both of the dimensions of interpulse period and carrier frequency. The processing is tantamount to a two-phase estimation procedure that is based on some properties of a collection of intervals of real numbers generated by two known speeds that serve as the moduli of distinct congruence relations. There is a connection between this procedure and the use of the Chinese remainder theorem in a multiple channel search system as a means for determining true slant range from several ambiguous range cell numbers.

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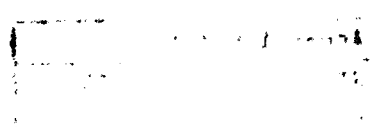

EXECUTIVE SUMMARY

There have been several investigations of the use of range rate (i.e., the time rate of change of the slant range of a target relative to a radar) as an aid in tracking an aircraft. Some of these studies dwell on the tracking problem under the tacit assumption that the range rate is available. Others are concerned with the determination of range rate itself, and, in particular, the evaluation of a system that provides an estimate of range rate from two Doppler measurements. Each of these is system specific in the sense that the evaluation is empirical in nature and the results are limited to the performance of a concrete design.

This report is not system specific. It deals with some basic properties of an abstract structure that is associated with the measurement of the pulse-to-pulse phase shift in each of two wave trains of electromagnetic energy. The wave trains are reflections from an aircraft of energy transmitted from a radar. The transmitted energy consists of two pulse trains that differ in at least one of the dimensions of carrier frequency and interpulse period. The structure is available for use in any case where the pulse-to-pulse phase shift in the reflection of each transmitted train is measured, and the properties of the structure can be exploited in the estimation of range rate.

The structure is a collection of intervals of numbers associated with two transmitted pulse trains that differ in one or both of the dimensions of carrier frequency and interpulse period. The intervals are mutually exclusive, the union of all intervals in the class is the real line, and the intervals form a periodic structure. The structure is useful in the sense that it is a vehicle for estimating range rate in two consecutive phases. The first phase consists of the selection of an interval deemed most likely to contain the range rate. This is an interval estimate of the rate. The second phase is the formulation of a point estimate within the closure of the selected interval.

There is a connection between the properties of the class of intervals and computer programming concepts of speed, memory, and data structures that can be exploited to fabricate efficient software realizations of two phase range rate estimators. There is also a connection between such estimators and the Chinese remainder theorem that is used to determine target slant range from ambiguous range cell numbers in a multiple channel search system.

1. INTRODUCTION.

The concept of using range rate (i.e., the time rate of change of the slant range of a target relative to a radar) as an aid in the automatic tracking of an aircraft has been a topic of interest for over a decade (references 1 through 4). Some investigations (references 1 and 2) of the subject assume that the range rate is available and that it can be transformed into a velocity consistent with the reference frame for tracker input data. These studies deal with the use of this parameter as a known input for acquiring and maintaining a track. Other investigations (references 3 and 4) are concerned with simulations and/or analyses that evaluate the performance of specific systems that are supposed to determine range rate by means of two congruence relations associated with Doppler measurements derived from a coherent pulsed radar.

The objective of this report is threefold. One goal is to document some fundamental properties of a partition of the real line that is generated by the moduli of two distinct congruence relations. Another goal is to show how these properties can be exploited to estimate range rate. The remaining goal is to show the connection between such estimators and the Chinese remainder theorem (reference 5, page 48) that is used as a means for determining true target slant range from several ambiguous range cell numbers derived from a multiple channel search system (reference 6, page 19-16). The hope is that this information will prove useful in the design and evaluation of computer systems that are supposed to determine range rate on the basis of two congruence relations associated with Doppler measurements.

From a theoretical point of view, coherent pulsed radars (e.g., the current operational ASR-9 airport surveillance radar and the developmental ARSR-4 air route surveillance radar) are capable of determining the range rate of an aircraft to within an integer multiple of a known speed. In effect, the known speed is the modulous of a congruence relation (reference 7, page 21), and the range rate is the sum of an observation and an integer multiple of the modulous. Any integer multiple of the modulous is commmonly referred to as a blind speed. The modulous itself is hereinafter referred to as the base blind speed or simply the base speed. The observation is a speed version of a measurement of the pulse-to-pulse phase shift introduced by the radial motion of an aircraft relative to the radar into the reflection of a wave train transmitted at the radar carrier frequency. The base speed is a positive number that is uniquely determined by the transmitter carrier frequency and the pulse repetition rate (i.e., the reciprocal of the interpulse period of the transmitted wave train). The observed speed is nonnegative and it is less than the base speed.

There is ambiguity associated with the observation if the integer multiplier of the base speed is unknown. Otherwise, assuming that the carrier frequency and pulse repetition rate are known, the observation uniquely determines the range rate.

The ambiguity does not exist if the base speed is sufficiently large to guarantee that the absolute value of range rate is less than half the base speed. In this case, the integer multiplier must be 0 when the observation is less than half the base speed. Otherwise, the integer multiplier is -1. The situation is analogous to that of a particle moving at constant speed along a circle. Range rate corresponds to the directed speed of the particle. The directed speed is positive if the particle is moving in one direction (i.e., the positive direction of motion)

along the circle, and it is negative if it is moving in the opposite direction. The available data consists of the radius of the circle, the location of the particle at each of two consecutive time epochs, and the interepoch period. The ratio of the circumference of the circle to the interepoch period can be viewed as a base speed, and it is known that the particle speed (i.e., the absolute value of the directed speed) is less than half this base speed. The observation corresponds to the result obtained by measuring the distance in the positive direction of motion along the circle from the particle location at the first epoch to the particle location at the second epoch and dividing it by the interepoch period. The quotient is a nonnegative number other than half the base speed, and it is less than the base speed. If it is less than half the base speed then the particle must be moving along the circle in the positive direction, and the directed speed is identical to the quotient. If the quotient is greater than half the base speed then the particle is moving in the opposite direction, and the directed speed is the same as the quotient less the base speed.

In practice, it may not be possible to generate a base speed that is greater than twice the absolute value of the range rate of every target of interest. It turns out in many cases of this nature that the ambiguity introduced by lack of knowledge of the integer multiplier may be removed by employing two base speeds.

Two base speeds may be realized by transmitting in succession one train of pulses at one pulse repetition rate and another train of pulses at the same frequency and a different pulse repetition rate (reference 3). Two base speeds can also be generated by other combinations of carrier frequency and pulse repetition rate in two pulse trains. Specifically, each of the ASR-9 and ARSR-4 radars is capable of transmitting two wave trains that differ in both carrier frequency and pulse repetition rate. Consequently, two base speeds may be realized with either one of the ASR-9 and ARSR-4 sensor systems.

At present, there exists some experimental evidence to the effect that it is possible to derive a reasonable unambiguous estimate of range rate from the measurement of the pulse-to-pulse phase shift in the reflection of each of two pulse trains that differ in one or both of the dimensions of carrier frequency and pulse repetition rate (reference 4). As will be seen, such estimates may be realized by exploiting the properties of an infinite number of disjoint continuums of real numbers that are generated by two base speeds. These continuums are herein referred to as blind speed intervals.

The document is divided into six parts. The first part consists of sections 2 and 3 which justify the use of a congruence relation as the model of the measurement equation providing range rate as a function of a base speed and a speed version of a phase measurement. The second part consists of sections 4 through 11. These eight sections are concerned with the properties of the blind speed intervals generated by the base speeds involved in two measurement equations. Sections 12 through 18 make up the third part of the document. These seven sections deal with the application of the properties of blind speed intervals to the estimation of range rate from the speed versions of two phase measurements. The problem is twofold in the sense that it involves a determination of range rate to within a blind speed interval (i.e., an interval estimate) followed by a point estimate of the location of the range rate within the closure of the selected interval. Section 19 is the fourth part of the document, and it deals with the connection between such two phase estimation procedures and the Chinese remainder

theorem. The fifth part of the document consists of some conclusions and recommendations appearing in section 20. The sixth and final part consists of 13 appendices containing the proofs of propositions appearing in sections 7 through 11, section 17, and section 19.

2. DOPPLER FILTERING.

The carrier frequency f of the electromagnetic energy transmitted from a radar and the carrier frequency f' of that portion of the transmitted energy that is reflected back to the radar by a target are not necessarily the same. The difference, when it exists, is a manifestation of the Doppler effect. If the target is moving at a constant speed v in the radial direction away from the antenna (i.e., v is the range rate of the target) then

$$f - f' = [2(v/c)f]/[1 + v/c] \tag{1}$$

where c is the speed of light (i.e., $5.8275X10^8$ knots). This difference (i.e., the Doppler frequency shift) is essentially the same as 2(v/c)f in the case where the absolute value of the ratio of v to c is much less than 1.

If the absolute value of v/c is much less than 1 and the transmitted energy is propagated in a wave train of N (N > 1) pulses at the radar carrier frequency, then it is possible to determine the Doppler frequency shift to within an integer multiple of the pulse repetition rate R. The measurement can be achieved if each transmitted pulse is initiated at the same phase of the carrier signal, and the analog portion of the radar receiver is equipped with a stable coherent oscillator that generates the transmitter carrier frequency at a constant phase (reference 8, pages 714-715). Under these conditions, the signal generated at the receiver from the reflected energy can be demodulated into two orthogonal signals from which it is possible to recover N pairs of numbers that provide a measurement of the Doppler shift to within an integer multiple of R. The components of the nth pair of numbers (n = 1, 2, ..., N) are generated by the reflection of the nth transmitted pulse from the target. The components of the nth pair are essentially the same as the sine and cosine of the angle

$$A(n) = 2(pi)[(f - f')/R](n - 1) + F \tag{2}$$

where F is a constant. The nth pair of numbers can be viewed as the real and imaginary parts of a complex number with magnitude 1 and argument A(n) (i.e., exp[iA(n)]). The angle in radians that is subtended at the origin of the complex plane by the phasors exp[iA(n)] and exp[iA(n-1)] and measured in the counter clockwise direction from exp[iA(n-1)] to exp[iA(n)] is a nonnegative number P that is less than 2(pi). The pulse-to-pulse phase shift (i.e., 2(pi)[(f - f')/R]) is equal to the sum P + k2(pi) for some integer k. It follows that

$$f - f' = P/[2(pi)/R] + kR, \tag{3}$$

as asserted.

EXAMPLE 1: Equation (1) implies that the pulse-to-pulse phase shift can be expressed in the form

$$4(pi)(f/R)(v/c)(1/[1 + (v/c)]).$$

Suppose that R is 927 hertz (Hz) and f is 2900 megahertz (MHz). If v is 400 knots then the pulse-to-pulse phase shift is 4.2946(2pi) radians, P is 0.2946(2pi) radians and k of (3) is 4. If the range rate is -400 knots then P is 0.7054(2pi) radians and the integer k is -5.

The preceding remarks ignore all sorts of anomalies that must be overcome in a practical situation. Nevertheless, there are digital techniques (reference 8, pages 715-753) that make it possible to acquire a nonnegative number P less than 2(pi) such that the left and right sides of (3) can be construed to be close to one another with a reasonable degree of certainty.

3. THE BASE BLIND SPEED.

If the target is an aircraft then the determination of the Doppler shift to within an integer multiple of R is equivalent to a determination of the range rate to within an integer multiple of the ratio of c to 2(f/R). This assertion is based on the observation that the absolute value of the ratio of the range rate of an aircraft to the speed of light is much less than 1. Indeed, if the absolute value of v/c is much less than 1, then the Doppler shift is close to 2(v/c)f, and the relationship (3) between the Doppler shift, P, R and the integer k is substantially equivalent to the expression

$$v = s + kS \tag{4}$$

where

$$S = c/[2(f/R)] \tag{5}$$

and

$$s = [PS]/[2(pi)]. \tag{6}$$

Equation (4) can be viewed as the range rate measurement equation in any case where the radar transmission parameters f and R are known (i.e., S is known and s, a speed determined by the phase P, has the attributes of an observation).

For future reference, it is important to recognize three characteristics of expression (4). First, k is an integer. Second, as implied by (5), S is a positive number. Third, since the phase P is a nonnegative number less than 2(pi), it follows from (6) that s is a nonnegative number that is less than S.

EXAMPLE 2: Suppose that f is 2331 MHz and R is 1200 Hz. The corresponding base speed S is 150 knots. Suppose now that the phase measurement P is 359.999° so that P/[2(pi)] is 359.999/360. Also, suppose that the speed version of P is to be stored for later use with a precision that is three digits to the right of the decimal point. Some care must be exercised in the final determination of s in order to acquire a stored speed less than 150 knots. If arithmetic operations are carried out in registers capable of dealing with 10 significant figures then the number stored as a result of truncation may be vastly different from that stored as a result of rounding. The speed version of P assigned to s under truncation is 149.999 knots. The result of rounding is 150.000 knots, and, in this case, one of the speeds 0.000 knots and 149.999 knots should be assigned to s.

In radar jargon, any integer multiple of S is called a blind speed. In this document, S itself is referred to as the base blind speed or simply the base speed. As already indicated, a radar may transmit two pulse trains that differ in one or both of the dimensions of carrier frequency and pulse repetition rate. In this way it is possible to realize two relationships of the form (4) corresponding to two distinct base speeds.

EXAMPLE 3: A prototype long range radar system is described in reference 3 that is capable of determining range rate from two pulse trains. Each train consists of eight pulses transmitted at a carrier frequency of 1285 MHz. The pulse repetition rate of one train is 345 Hz and the pulse repetition rate of the remaining train is 417 Hz. The base blind speeds associated with the 345 Hz and 417 Hz trains are 78.2 knots and 94.6 knots, respectively.

EXAMPLE 4: In the case of the ASR-9 radar it is possible to transmit two trains of pulses with distinct carrier frequencies and distinct pulse repetition rates. Among the many existing possibilities is a pulse train characterized by a carrier frequency of 2800 MHz and a pulse repetition rate of 1312 Hz combined with another pulse train in which the carrier frequency is 2700 MHz and the pulse repetition rate is 1020 Hz. The corresponding base speeds are 136.530 knots and 110.075 knots.

4. CONGRUENCE OPERATORS.

Numbers J and J' are said to be congruent modulo a number B if the difference between J and J' is an integer multiple of B The number B is referred to as the modulous of the congruence.

EXAMPLE 5: Referring to expression (4), the range rate v of the target is congruent to the observation s modulo the base blind speed S.

EXAMPLE 6: The sum 21.5 + k(94.6) is congruent to 21.5 modulo 94.6 for any integer k.

There are two binary operators, @ and #, that are useful in the representation of any number J as the sum of an integer multiple of a positive number B and a nonnegative number that is less than B. J@B is the greatest integer less than or equal to J/B, and J#B is defined by the relationship

$$J\#B = J - (J@B)B. \qquad (7)$$

Needless to say, J and J#B are congruent modulo B. In fact, J and J#B + kB are congruent modulo B for any integer k. It can be verified that the representation of J as the sum of J#B and (J@B)B is unique in the sense that J can be represented as the sum of a nonnegative number less than B and an integer multiple of B in one and only one way (i.e, the nonnegative number must be J#B and the integer multiplier must be J@B).

EXAMPLE 7: There is one and only one way that -21.2 can be represented as the sum of an integer multiple of 94.6 and a nonnegative number less than 94.6 (i.e., 73.4 + (-1)94.6).

EXAMPLE 8: Referring to expressions (4)-(6), v#S is s and v@S is k.

5. NUMERICAL ASPECTS OF TWO BASE SPEEDS.

In all that follows, L and H represent numbers such that

$$0 < L < H. \tag{8}$$

These are hereinafter referred to as the base blind speeds or the base speeds.

The base speeds must satisfy exactly one of the following four mutually exclusive and exhaustive conditions.

B1. One of the base speeds is a rational number and the other is an irrational number.

B2. Each of the base speeds is an irrational number and neither one of the speeds can be expressed as a rational multiple of the other.

B3. Each of the base speeds is a rational number.

B4. Each of the base speeds is an irrational number and either one of the speeds can be expressed as a rational multiple of the other.

As will be seen, there is distinct difference between those situations in which the base speeds satisfy one of conditions B1 and B2 and those cases in which one of the conditions B3 and B4 apply.

EXAMPLE 9: Condition B1 is satisfied if L is pi and H is the rational number 17/2.

EXAMPLE 10: If L is the square root of 2 and H is the square root of 13 then the base speeds satisfy condition B2. This must be so. Otherwise, there would exist integers i and j such that $2i^2$ is the same as $13j^2$, a contradiction of the unique factorization theorem for integers (reference 8, page 19). Indeed, the integers $2i^2$ and $13j^2$ cannot be the same because there is an odd power of 2 that is a factor of $2i^2$ and not a factor of $13j^2$, and there is an odd power of 13 that is a factor of $13j^2$ and not a factor of $2i^2$.

EXAMPLE 11: Condition B4 is satisfied if H is the square root of 13 and L is H/2.

The distinction between conditions B3 and B4 is just a matter of a scale factor. If the base speeds satisfy condition B4 then there is a rational number t such that H is the same as tL. Except for the matter of the scale factor L, this is identical to the case in which the base speeds are the rational numbers 1 and t.

Condition B3 is significant from a practical point of view. It applies to any situation in which each of the base speeds is represented by a number consisting of a finite number of nonzero digits. Such numbers are rational and the operands of the arithmetic operations involved in common machine computation.

6. BLIND SPEED INTERVALS.

There is an important set of numbers associated with the base speeds and an ordered pair of integers i and j, denoted by the 2-tuple (i,j). The set is herein represented by I(i,j), and a number x is a member of the set if and only if it satisfies the expression

$$\max(iL, jH) \prec x < \min([i + 1]L, [j + 1]H). \tag{9}$$

The set I(i,j) may be empty or nonempty. A number x satisfies the expression (9) if and only if it satisfies each of the equations

$$x@L = i \text{ and } x@H = j. \tag{10}$$

Consequently, the set I(i,j) is nonempty if and only if there is at least one number x satisfying each of equations (10) or the equivalent condition (9), and a number x is a member of I(i,j) if and only if it satisfies each of equations (10). If the set I(i,j) is nonempty then it is hereinafter referred to as a blind speed interval.

EXAMPLE 12: Suppose that L is 78.2 and H is 94.6. The set I(2,0) is empty, and so it is not a blind speed interval. On the other hand, I(1,1) is the interval consisting of all numbers greater than or equal to 94.6 and less than 156.4, and I(2,1) is the interval containing all the numbers that are greater than or equal to 156.4 and less than 189.2. Thus, I(1,1) and I(2,1) are blind speed intervals.

EXAMPLE 13: The set I(0,0) is always nonempty, and it consists of all numbers greater than or equal to 0 and less than the low base speed L. Also, the set I(1,0) is always nonempty. If H is less than 2L then I(1,0) is the set of all numbers that are greater than or equal to L and less than H. Otherwise, it is the set of all numbers that are greater than or equal to L and less than 2L.

It is useful to keep two special kinds of sets in mind when dealing with the subject of blind speed intervals. One of these is referred to as a half open interval. As implied by (9) and demonstrated in examples 12 and 13, a blind speed interval always takes the form of a continuum of numbers of finite length with a greatest lower bound p and a least upper bound q, and a number x is a member of the continuum if and only if

$$p \prec x < q. \tag{11}$$

In other words, the lower bound is in the continuum and the upper bound is outside the continuum. This is often referred to as a half open interval that is closed on the left and open on the right. A shorthand notation for such a set is [p,q), and it is used freely throughout the remainder of this report. Obviously, if the half open interval [p,q) is a blind speed interval then it is identical to the set I(p@L,p@H), and the upper bound q is the minimum of the numbers (p@L+ 1)L and (p@H + 1)H. The remaining interval type is called a closed interval. Like the half open interval, a closed interval is a continuum of numbers. It is different from a half open interval in that the greatest lower bound and least upper bound of a closed interval are members of the interval. Thus, the union of [p,q) and the singleton set consisting of the number q is a closed interval. This union is commonly denoted by [p,q], and it is often referred to as the closure of [p,q). Needless to say, the length of [p,q) is the same as the length of its closure (i.e., q - p).

There are several rather obvious facts concerning the class of blind speed intervals that are worthy of special attention. First, distinct blind speed intervals are disjoint (i.e., the same number cannot be a member of each of two blind speed intervals). Second, it can be verified that the union of all blind speed intervals is the real line. In other words, the class of all blind speed intervals is a complete partition of the real line (i.e., the members of the class are disjoint and the union of the members of the class is identical to the set of all real numbers). Third, the length of a blind speed interval cannot exceed the low base speed L. Consequently, there are an infinite number of blind speed intervals.

7. KEY OF A BLIND SPEED INTERVAL.

The function d defined for any ordered integer pair (i,j) by the relationship

$$d(i,j) = iL - jH \tag{12}$$

is useful as means for identifing blind speed intervals. If the set I(i,j) is nonempty then d(i,j) is here referred to as the key of the set. In other words, every blind speed interval has a key. As will be seen, there are situations in which the key of a blind speed interval is unique and other situations in which the same number is the key of an infinite number of blind speed intervals.

The set of all numbers that qualify as keys is here referred to as the key set, and it is denoted by K(L,H). The number 0 is always a member of the key set. This is a direct result of the fact that d(0,0) is 0 and the key of the blind speed interval I(0,0). The number L is also a member of K(L,H). This assertion follows from the fact that the set I(1,0) is always a blind speed interval, and the number d(1,0) is the key of I(1,0).

EXAMPLE 14: If L is 1 and H is 2 then it turns out that the only members of K(L,H) are 0 and 1. This assertion is a direct result of the definition of a blind speed interval. Indeed, if I(i,j) is a blind speed interval then there exists a number satisfying relationship (9). This implies that iL is less than (j + 1)H, and jH is less than (i + 1)L. In other words, iL - jH is less than H and greater than -L whenever I(i,j) is a blind speed interval. In the case where L is 1 and H is 2, it follows that d(i,j) must be an integer greater than -1 and less than 2.

PROPOSITION 1: Blind speed intervals I(i,j) and I(r,s) share the same key if and only if

$$(r - i)L = (s - j)H, \tag{13}$$

and if distinct blind speed intervals I(i,j) and I(r,s) share the same key then each of the integers r - i and s - j is nonzero.

Proposition 1 provides some insight into the size of the key set. It implies that distinct blind speed intervals share the same key if and only if either one of the base speeds can be expressed as a rational multiple of the other. It follows that the key set must be infinite in size if the base speeds satisfy either one of the conditions B1 and B2. As will be seen, the key set is finite in size if the base speeds satisfy either one of the two remaining conditions (i.e., conditions B3 and B4).

8. WHOLE DIVISORS OF RATIONAL NUMBERS

The concept of a greatest common whole divisor of two numbers is useful in the description of the structure of blind speed intervals in the case where the base speeds satisfy condition B3. Hereafter, a number w is referred to as a whole divisor of a number y if y/w is an integer. A number that is a whole divisor of each of two numbers x and y is referred to as a common whole divisor of x and y. A common whole divisor of two numbers that exceeds any other common whole divisor of the same numbers is called the greatest common whole divisor of the numbers. If it exists, the greatest common whole divisor of the numbers x and y is hereinafter represented by g(x,y).

EXAMPLE 15: The numbers 3 and pi do not have a common whole divisor. The greatest common whole divisor of 3/2 and 5 is 1/2. The latter assertion can be established from the fact that a positive whole divisor of 5 must be the ratio of 5 to a positive integer M, and the ratio of 3/2 to 5/M is (3M)/10. Since (3M)/10 is not an integer when M is less than 10, it follows that 5/10 must be the greatest common whole divisor of 5 and 3/2.

The greatest commmon whole divisor of two positive integers always exists, and it is identical to the product of the prime powers that are common factors of the two integers. Positive integers x and y are said to be relatively prime if g(x,y) is 1 (i.e., 1 is the only prime power that is a common factor of x and y).

EXAMPLE 16: The prime powers that are factors of 12 are 3 and 2^2. The prime powers that are factors of 56 are 7 and 2^3. It follows that g(12,56) is 4.

PROPOSITION 2: If p, q, r and s are positive integers and

$$g(p,q) = g(r,s) = 1 \tag{14}$$

then

$$g(p/q,r/s) = [1/(qs)]g(p,r)g(q,s) \tag{15}$$

and the numbers (p/q)/g(p/q,r/s) and (r/s)/g(p/q,r/s) are relatively prime integers.

Proposition 2 implies that any two positive rational numbers have a greatest common whole divisor. This is a direct result of the fact that any positive rational number can always be expressed as the ratio of two relatively prime integers. Thus, if each of the base speeds L and H is rational then g(L,H) exists and the numbers

$$L' = L/g(L,H) \text{ and } H' = H/g(L,H) \tag{16}$$

are relatively prime integers.

EXAMPLE 17: In example 3, L is 78.2 (i.e., 782/10) and H is 94.6 (i.e., 946/10). The prime powers that are factors of 782 are 2, 17, and 23. The prime powers that are factors of 946 are 2, 11, and 43, and the prime powers that are factors of 10 are 2 and 5. It follows that L is the ratio of the relatively prime integers 391

and 5 (i.e., 391/5), and H is the ratio of the relatively prime integers 473 and 5 (i.e., 473/5). The prime powers that are factors of 391 are 17 and 23, and the prime powers that are factors of 473 are 11 and 43. Hence, g(391,473) is 1. Since g(5,5) is 5, it follows that g(L,H) is the ratio of 5 to 25, L' is 391, and H' is 473.

EXAMPLE 18: Suppose that L is 110.075 (i.e., 110075/1000) and H is 136.530 (i.e., 136530/1000), as in example 4. The prime powers that are factors of 110075 are 5^2, 7, 17, and 37. The prime powers that are factors of 136530 are 2, 5, 3^2, 37, and 41, and the prime powers that are factors of 1000 are 2^3 and 5^3. In other words, L is a ratio of the relatively prime numbers 4403 (i.e., 7X17X37) and 40 (i.e., 2^3X5), and H is a ratio of the relatively prime numbers 13653 (i.e., 3^2X37X41) and 100 (i.e., 2^2X5^2). It follows that g(L,H) is 37/200, and the integers 595 (i.e., L') and 738 (i.e., H') are relatively prime.

PROPOSITION 3: If p and q are relatively prime integers and r and s are integers such that rp is the same as sq then there is an integer k such that r is kq and s is kp.

The fact that L' and H' are relatively prime integers is significant. As will be seen, this fact and proposition 3 can be viewed as the foundation of the structure that is formed by the class of blind speed intervals generated by rational base speeds.

9. STRUCTURE OF RATIONAL BLIND SPEED INTERVALS.

There are 3 parameters that are particularly useful in the description of the properties of blind speed intervals generated by rational base speeds. Two of the parameters are the relatively prime integers L' and H' defined by equations (16). The remaining parameter is

$$T(L,H) = (LH)/g(L,H) = L'H = LH'. \qquad (17)$$

EXAMPLE 19: The parameter T(L,H) is 36988.6 knots in examples 3 and 17. It is 81235.35 knots in examples 4 and 18. In either case, T(L,H) far exceeds the maximum speed that can be achieved by any operational aircraft.

While there is only one blind speed interval (i.e., I(0,0)) associated with the key 0 in cases B1 and B2, there are infinitely many blind speed intervals that share 0 as a key in the case where each of the base speeds is a rational number. The class of all such intervals can be completely described by a simple application of proposition 3 to the relatively prime integers L' and H'. The proposition implies that a necessary and sufficient condition for integers i and j to satisfy the relationship

$$iL' = jH' \qquad (18)$$

is that there exist an integer k such that

$$i = kH' \text{ and } j = kL'. \qquad (19)$$

Since the definitions (16) of L' and H' imply that (18) is equivalent to equality of iL and jH, it follows that the same condition (i.e., (19)) is necessary and sufficient for an integer multiple of the base speed L to be identical to an integer multiple of the base speed H. In other words, the class of blind speed intervals that share 0 as a key is merely the class consisting of all sets of the form

$$I(kH',kL') = [kT(L,H),kT(L,H) + L) \tag{20}$$

where k is an integer.

Corresponding to any integer k there is a half open interval other than a blind speed interval that is an important structural feature in the case where both L and H are rational numbers. The interval is

$$D(k) = [kT(L,H),(k + 1)T(L,H)). \tag{21}$$

There are a multitude of reasons why this set is significant. Among these there are two that are based on the relationship of the interval to the real line and the relationship of the interval to the blind speed intervals that are subsets of itself. First, the union of all such intervals is the real line. Second, the least of the numbers in D(k) is the least of the numbers in the blind speed interval I(kH',kL'), and the least upper bound of D(k) is the greatest lower bound of the blind speed interval I((k+1)H',(k+1)L'). As a result, D(k) must be the union of all blind speed intervals that are subsets of itself.

EXAMPLE 20: If L is 3 and H is 4 then T(L,H) is 12. Each of the sets I(0,0), I(1,0), I(1,1), I(2,1), I(2,2) and I(3,2) is nonempty and a subset of D(0) (i.e., [0,12)). In addition, the union of these six blind speed intervals is identical to D(0).

The idea of a pure translation of a set of numbers is useful in the description of the structure of the class of all blind speed intervals. A set I of numbers is the same as a pure translation of a set J of numbers if there exists a single number n such that I is the set of all numbers that can be represented as the sum of n and a member of J. A shorthand notation for this relationship is

$$I = J + n. \tag{22}$$

The number n is here referred to as the translation distance. This distance is directed in the sense that it can be positive or negative.

EXAMPLE 21: Referring to example 20 in which L is 3 and H is 4, I(0,0) is [0,3) and I(4,3) is [12,15). Hence, I(4,3) is the same as I(0,0) + 12. It can also be verified that I(-4,-3) is the sum of I(0,0) and -12.

PROPOSITION 4: If each of the base speeds L and H is a rational number, k is an integer and I(i,j) is a blind speed interval then I(i,j) + kT(L,H) is identical to I(r,s) where

$$r - i = kH' \text{ and } s - j = kL'. \tag{23}$$

The class of blind speed intervals generated by rational base speeds can be viewed as a periodic structure in the sense of pure translations of sets. This assertion is a direct result of proposition 4. If I(i,j) is a blind speed interval and a subset of D(0) then I(i,j) + kT(L,H) is surely a nonempty subset of D(k) and, by proposition 4, a blind speed interval. Since the lengths of D(0) and D(k) are the same (i.e., T(L,H)), it follows that every blind speed interval that is a subset of D(k) can be expressed as a pure translation by the directed distance kT(L,H) of a blind speed interval that is a subset of D(0).

EXAMPLE 22: Since each of the numbers L'H and LH' is identical to T(L,H), the set I(H'-1,L'-1) is identical to [T(L,H)-L,T(L,H)). This means that I(H'-1,L'-1) is a blind speed interval, and it is surely a subset of D(0). Thus, if k is an integer then the translation of the set I(H'-1,L'-1) through the directed distance kT(L,H) is a blind speed interval as well as a subset of D(k).

EXAMPLE 23: Referring to example 20 in which L is 3 and H is 4, there are six blind speed intervals that are subsets of D(-1) (i.e., [-12,0)), these are the same as pure translations of the six blind speed intervals that are subsets of D(0) (i.e., [0,12)), and the directed distance of each translation is -12.

PROPOSITION 5: If each of the base speeds L and H is a rational number then distinct blind speed intervals I(i,j) and I(r,s) share the same key if and only if there exists a nonzero integer k such that

$$r - i = kH' \text{ and } s - j = kL'. \qquad (24)$$

Proposition 5 is just a version of proposition 1 that is applicable to the special case in which each of the base speeds is a rational number. In fact, equation (13) of proposition 1 can be obtained by multiplying the first of equations (24) by L, multiplying the second equation by H and exploiting the fact that L'H and LH' are the same (i.e., T(L,H)).

The periodic nature of the class of blind speed intervals generated by rational base speeds is further emphasized by the relationship between the key set and the blind speed intervals that are subsets of D(k). The relationship is revealed by propositions 4 and 5. In effect, these propositions imply that two blind speed intervals share the same key if and only if either one of the intervals is equivalent to a pure translation of the other by an integer multiple of T(L,H). It follows that distinct blind speed intervals that are subsets of D(k) cannot share the same key, that the keys of the blind speed intervals that are subsets of D(k) are the same as the keys of the blind speed intervals that are subsets of D(0), and that the members of the key set K(L,H) are merely the keys of the blind speed intervals that are subsets of D(0). Thus, in a very real sense, the class of blind speed intervals can be viewed as a periodic structure with period T(L,H).

EXAMPLE 24: Referring to example 17 in which L is 78.2 and H is 94.6, proposition 5 implies that the keys of blind speed intervals I(i,j) and I(r,s) are distinct if the absolute value of i - r is less than 473 and the absolute value of j - s is less than 391.

EXAMPLE 25: If L is 4 and H is 7 then I(2,1) is a blind speed interval of length 4 with key 1. On the other hand, the key and length of the blind speed interval I(0,0) are 0 and 4, respectively. Since T(L,H) is 28, this is not a contradiction of propositions 4 and 5 (i.e., I(2,1) is a pure translation of I(0,0) by 8 - not 28).

EXAMPLE 26: Referring to example 25, I(9,5) is a blind speed interval of length 4, it is a pure translation of I(2,1) by 28, and 1 is the key shared by I(9,5) and I(2,1).

10. THE KEY SET GENERATED BY RATIONAL BASE SPEEDS.

In the case where each of the base blind speeds is a rational number, the size of the key set is identical to the number of blind speed intervals that are subsets of the half open interval D(0). As already pointed out, this assertion is a direct result of propositions 4 and 5. Proposition 6 provides additional insight into the structure and size of the key set.

PROPOSITION 6: If each of the base speeds L and H is a rational number then the key set (i.e., K(L,H)) consists of all numbers of the form kg(L,H) where k is an integer satisfying the inequality

$$-(L' - 1) \leq k \leq H' - 1. \qquad (25)$$

EXAMPLE 27: If L is 4 and H is 6 then g(L,H) is 2 and T(L,H) is 12. It can be verified that the blind speed intervals that are subsets of D(0) are I(0,0), I(1,0), I(1,1), and I(2,1). The corresponding keys are 0, 4, -2 and 2, and each of these is the product of g(L,H) and an integer that is at least -1 (i.e., -(L' - 1)) and at most 2 (i.e., H' - 1).

The size of the key set generated by rational base speeds can be obtained from proposition 6. The proposition implies that the number of distinct keys is given by the formula

$$N(L,H) = H' + L' - 1. \qquad (26)$$

EXAMPLE 28: The number of keys in the key set of examples 3 and 17 (i.e., where L is 78.2 and H is 94.6) is 863. In examples 4 and 18, where L is 110.075 and H is 136.530, the size of the key set is 1332.

11. ORDERING OF RATIONAL BLIND SPEED INTERVALS.

One blind speed interval is related to another blind speed interval in a very natural way. Indeed, every element of one of the intervals is less than every element in the remaining interval. Thus, if I(i,j) and I(r,s) are blind speed intervals such that each element of I(i,j) is less than each element of I(r,s) then it makes sense to say that I(i,j) is less than I(r,s). This relation is hereafter denoted by the shorthand notation $I(i,j) < I(r,s)$. It is transitive in the sense that if I(i,j), I(r,s) and I(p,q) are blind speed intervals such that $I(i,j) < I(r,s)$ and $I(r,s) < I(p,q)$ then $I(i,j) < I(p,q)$.

The relation < that exists between any two blind speed intervals can be used to identify a blind speed interval that is a subset of D(k) by the size of the elements of the interval relative to the size of the elements of the other blind speed intervals that are subsets of D(k). The minimal elements of the blind speed intervals that are subsets of D(k) can be arranged from left to right in an

ascending sequence. Each interval can be identified by the location in the sequence of the smallest number in the interval relative to the positions in the sequence of the minimal elements of the remaining intervals. In other words, the sum I(0,0) + kT(L,H) is an interval of order 1, I(1,0) + kT(L,H) is an interval of order 2, ..., and I(L'-1,H'-1) + kT(L,H) is an interval of order N(L,H). Using this ordering convention, it is natural to employ the notation I'(k,m) to identify the blind speed interval of order m that is a subset of D(k) where m is restricted to be a positive integer that is at most N(L,H). Clearly,

$$I'(k,1) < I'(k,2) < \ldots < I'(k,N(L,H)),$$

and it makes sense to employ the notation d'(m) to identify the key of I'(k,m).

EXAMPLE 29: Referring to example 27 in which L is 4 and H is 6, I'(0,1) is I(0,0), I'(1,2) is I(1,0) + 12, I'(0,3) is I(1,1) and I'(-3,4) is I(2,1) - 36. The corresponding keys are d'(1) (i.e., 0), d'(2) (i.e., 4), d'(3) (i.e., -2) and d'(4) (i.e., 2). Also, I'(-3,4) < I'(0,1) < I'(0,3) < I'(1,2).

PROPOSITION 7: If each of the base speeds L and H is a rational number then the order of the blind speed interval I(i,j) is given by the sum i#H' + j#L' + 1.

As will be seen, it may be of some practical interest to determine the order of a rational blind speed interval containing a prescribed number without determining other details such as the key of the interval or the least upper bound and the greatest lower bound of the interval. If J is the number in question then the order of the blind speed interval containing J is hereafter denoted by o(J). It follows from proposition 7 and equations (9) and (10) that

$$o(J) = (J@L)\#H' + (J@H)\#L' + 1. \tag{27}$$

EXAMPLE 30: Suppose J is 29 and the base speeds L and H are 4 and 6, respectively, as in example 27. It can be verified that L' is 2, H' is 3, J@L is 7, and J@H is 4 (i.e., the sum of the integers (J@L)#H', (J@H)#L', and 1 is 2). It can also be verified that the half open interval [28,30) is a blind speed interval of order 2. In fact, it is I'(2,2).

There exists a function defined on the set of integers that is useful in the determination of the key of a rational blind speed interval. The function is defined for any integer m by the formula

$$f(m) = [mL']\#[N(L,H)+1]. \tag{28}$$

The range of the function f is the set of N(L,H) + 1 nonnegative integers that are at most N(L,H). Also, since the sum N(L,H) + 1 is the same as L' + H', it follows from the definitions (16) of L' and H' that [(mL)#(L + H)]/g(L,H) is identical to f(m).

PROPOSITION 8: If each of the base speeds L and H is a rational number, S is the set of all nonnegative integers other than H' that are less than N(L,H) + 1, and k is an integer then the function f is a 1:1 mapping of the set of N(L,H) integers that are at least (k-1)(N(L,H) + 1) and at most k(N(L,H)+1) - 2 onto the set S and

$$f(k[N(L,H)+1]-1) = H'. \tag{29}$$

As implied by proposition 8, the function f has the curious property that it takes on the value H' if and only if the argument of the function is of the form kN(L,H) + k - 1 where k is an integer. In other words, if m is restricted to the nonnegative integers less than or equal to N(L,H) then f(m) is H' if and only if m is N(L,H).

EXAMPLE 31: In examples 27 and 29, where L is 4 and H is 6, the numbers 0, 2, 4, 1, and 3 are the values of f(m) as m increases through the nonnegative integers less than 5.

PROPOSITION 9: If each of the base blind speeds L and H is a rational number and m is an integer such that

$$1 \prec m \leq N(L,H) \tag{30}$$

then

$$d'(m)/g(L,H) = f(m-1) \tag{31}$$

whenever f(m-1) < H', and

$$d'(m)/g(L,H) = -[N(L,H) + 1] + f(m-1) \tag{32}$$

whenever f(m-1) > H'.

Propositions 8 and 9 provide an algorithm for calculating the members of the key set K(L,H) without detailed knowledge of the structure of the blind speed intervals. In essence, it is only necessary to determine f(m-1) for any positive integer m that is less than N(L,H) + 1. If f(m-1) is less than H' then d'(m), the key of I'(0,m), is the product of g(L,H) and f(m-1). Otherwise, it is the sum of this product and the product of -g(L,H) and the sum N(L,H) + 1. All members of the key set can be found by repeating this procedure as m increases from 1 through the positive integers. The entire process must be terminated when m reaches N(L,H) + 1 or, as implied by proposition 8, when f(m-1) is H'.

EXAMPLE 32: Suppose that L is 8 and H is 18. It can be verified that g(L,H) is 2, L' is 4, H' is 9, and N(L,H) is 12. The N(L,H) + 1 numbers 0 through N(L,H) arranged from left to right in ascending order are

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

The numbers f(0), f(1), ..., and f(N(L,H)) are

0, 4, 8, 12, 3, 7, 11, 2, 6, 10, 1, 5, 9.

Since H' is 9, the sequences

0, 4, 8, -1, 3, 7, -2, 2, 6, -3, 1, 5

and

d'(1)/g(L,H), d'(2)/g(L,H), ..., d'(N(L,H))/g(L,H)

are identical.

While the size of the key set is always at least 2, the case where N(L,H) is exactly 2 is not too interesting for the simple reason that d'(1) (i.e., the key of I(0,0)) is always 0 and d'(2) (i.e., the key of I(1,0)) is always L. Proposition 10 deals with the more complicated situation in which N(L,H) is greater than 2.

PROPOSITION 10: If each of the base blind speeds L and H is a rational number, N(L,H) exceeds 2, m is an integer such that

$$1 \prec m \leq N(L,H) - 2, \tag{33}$$

and

$$n = N(L,H) + 1 - m \tag{34}$$

then

$$d'(n)/g(L,H) = N(L,H) + 1 - d'(m+2)/g(L,H) \tag{35}$$

whenever d'(m+2)/g(L,H) exceeds L', and

$$d'(n) = -d'(m+2) \tag{36}$$

otherwise.

If the size N(L,H) of the key set is greater than 2 then there exists a relationship between the keys d'(3) and d'(N(L,H)), the keys d'(4) and d'((N(L,H)-1), d'(5) and d'(N(L,H)-2), etc. The relationship is provided by proposition 10. The last two sequences of example 32 illustrate this relationship. Other illustrations of the same relationship can be found in examples 33 and 34.

EXAMPLE 33: In examples 3 and 17, where L is 78.2 and H is 94.6, g(L,H) is 1/5, L' is 391, H' is 473 and N(L,H) is 863. Using proposition 9, it can be verified that d'(11)/g(L,H) is 454. The key of the blind speed interval located exactly nine blind speed intervals to the left of 0 (i.e., I'(-1,855)) is d'(855). Since 454 exceeds L', proposition 10 implies that

$$d'(855)/g(L,H) = N(L,H) + 1 - d'(11)/g(L,H) = 410. \tag{37}$$

Thus, d'(855) is 82.0.

EXAMPLE 34: Referring to example 33, it can be verified from proposition 9 that d'(10)/g(L,H) is 63, a number not exceeding L'. It follows from proposition 10 that d'(856) (i.e., the key of the blind speed interval located eight blind speed intervals to the left of 0) is -d'(10) (i.e., -12.6).

12. UTILITY OF KEYS IN A NOISELESS ENVIRONMENT.

There are four conditions under which keys have some utility in the identification of an unobservable number z. These are as follows.

C1. It is known that z must be a member of one interval in a prescribed class C of blind speed intervals.

C2. The keys of any two members of the class C are distinct.

C3. A set K'(C) of 2-dimensional records of the form (k,l) is available such that k is the key of a member of C and l is a list of attributes of the member of C that is identified by k. The greatest lower bound p of the interval identified by k is a member of the list l, and so the integers p@L and p@H can be obtained from the information stored in l.

C4. While z itself is unobservable, the numbers z#L and z#H are observables.

The utility of keys as a means for identifying an unobservable number z under conditions C1 through C4 is easy to see. First, the number z is a member of the blind speed interval I(z@L,z@H). Second, the definition (12) of the function d and the obvious relationship

$$z = z\#L + (z@L)L = z\#H + (z@H)H \tag{38}$$

imply that

$$d(z@L,z@H) = z\#H - z\#L. \tag{39}$$

Thus, by C4, the key of I(z@L,z@H) is available. Since distinct blind speed intervals are disjoint, the interval I(z@L,z@H) is the only blind speed interval containing z, and, by C1, it must be a member of C. Also, by C2, it is the only member of C that can be associated with the key that is equal to z#H less z#L. Thus, the observables provide a key, and this key matches the key component of one and only one 2-tuple in K'(C). By C3, the integers z@L (i.e., p@L) and z@H (i.e., p@H) can be obtained from the information stored in the list component of this 2-tuple, and either of these numbers together with the observables is sufficient to determine z via relationship (38).

EXAMPLE 35: Suppose that L is 4, H is 7, and C is the collection of the 4 blind speed intervals I(0,0), I(1,0), I(1,1), and I(2,1). The key components of the members of K'(C) are 0, 4, -3, and 1. If z is in fact 8.5 then the difference z#H - z#L is 1. This is the key of the blind speed interval I(2,1) (i.e., [8,12)), and it is clear that 8.5 is the sum of 2L (i.e., (8@L)L) and z#L as well as the sum of H (i.e., (8@H)H) and z#H.

EXAMPLE 36: If L is irrational and H is rational then the key of a blind speed interval is unique, and so it is possible to locate any real number z from the observables z#L and z#H. In other words, the class C of conditions C1 and C2 can be the collection of all blind speed intervals. In this case, both C and K'(C) are infinite in size.

13. A COMPLETE CLASS OF BLIND SPEED INTERVALS.

The class C of conditions C1 and C2 is identified by three requirements. It is a class of blind speed intervals, distinct members of the class have distinct keys, and one of the intervals in the class contains the unobservable number z. Any such collection is hereinafter referred to as a complete class of blind speed intervals.

The idea of a complete class of blind speed intervals is closely related to the idea of a covering class of intervals for a given set of real numbers. A class of intervals is said to cover a given set of numbers if every member of the set is an element of at least one interval of the class. Thus, if the unobservable number z is known to be a member of some prescribed set then a class of blind speed intervals that covers the set is a complete class of blind speed intervals if and only if distinct intervals of the class have distinct keys.

EXAMPLE 37: Suppose that L is 4, H is 7 and it is known that z is a member of the closed set [4,10]. The class of all blind speed intervals covers [4,10]. It cannot be a complete class because there are distinct intervals of the class (e.g., I(0,0) and I(7,4)) that share the same key. On the other hand, the class of all blind speed intervals that are subsets of D(0) is a covering class of [4,10] and it is a complete class.

In any given situation there may be 0, 1 or more complete classes of blind speed intervals. If the base speeds satisfy either one of conditions B1 and B2 then there is at least one complete class of blind speed intervals. This assertion is a direct result of the fact that the key of a blind speed interval is unique under condition B1 or condition B2. In other words, the collection of all blind speed intervals is necessarily complete when either one of the conditions B1 and B2 applies. In fact, it is the only complete class of blind speed intervals if the unobservable number z can be any real number and one of the conditions B1 and B2 applies. On the other hand, the collection of all blind speed intervals is incomplete in any case where the base speeds satisfy condition B3 or condition B4.

EXAMPLE 38: Suppose that L is 1 and H is 3. If the unobservable number z can be any real number then a complete class does not exist. Since D(0) is the same as [0,3), the same is true if the possible values of z consist of all numbers in the interval [0,10]. If it is known that z is a member of the interval [0,0.5) then the number of complete classes of blind intervals is infinite. Indeed, if z is restricted to the set [0,0.5) and k is an integer then the class consisting of the two sets I'(0,1) and I'(k,2) is complete, and there are as many such classes as there are integers.

If a complete class of blind speed intervals exists then among all complete classes there exists one of minimal size. Indeed, distinct blind speed intervals are disjoint sets. It follows that z of condition C1 is an element of one and only one blind speed interval and this interval must be a member of every complete class. This observation applies to any conceivable value that might be assumed by z. It follows that the intersection of all complete classes is itself complete. Since the intersection is a subset of every complete class, a complete class of minimal size exists, it is unique and it is the intersection of all complete classes.

Minimal completeness is a desirable property. This assertion is a direct result of the fact that the size of the class C of conditions C1 and C2 is identical to the size of the set K'(C) of condition C3. In other words, the effort required to search through the members of K'(C) for a matching key increases with the size of the set.

The concept of a minimal complete class of blind speed intervals is related to the idea of a covering class of blind speed intervals of minimal size for a prescribed set of numbers. This is no more than the intersection of all classes of blind speed intervals that cover the set. Since distinct blind speed intervals are

disjoint, a blind speed interval is a member of the minimal covering class of blind speed intervals for a prescribed set if and only if it contains at least one number in the set. Needless to say, if ever element of the set is a possible value of z, z is restricted to the set and the minimal covering class of blind speed intervals for the set is complete, then the minimal covering class and the minimal complete class are identical.

EXAMPLE 39: Referring to example 38 in which L is 1 and H is 3, the minimal covering class of blind speed intervals for the set [0,1.5] is of size 2 (i.e., it consists of the blind speed intervals I(0,0) and I(1,0)). If it is known that the unobservable number z is generated by a random variable uniformly distributed on the interval [0,1.5] then this class is both complete and minimal complete.

EXAMPLE 40: Suppose that L is 4, H is 7 and it is known that the possible values of z consist of all the numbers in the closed interval [1,13]. It can be verified that D(0) is [0,28) and N(L,H) is 10. It can also be verified that the N(L,H) blind speed intervals that are subsets of D(0) are I(0,0), I(1,0), I(1,1), I(2,1), I(3,1), I(3,2), I(4,2), I(5,2), I(5,3), and I(6,3). Since the keys of any two of these intervals are distinct, the collection of all 10 intervals is complete. However, the collection is not a minimal covering class for [1,13], and so it is not minimal complete.

EXAMPLE 41: Referring to example 40, another complete class of blind speed intervals is the collection consisting of the five intervals I(0,0), I(1,0), I(1,1), I(2,1), and I(3,1). It is also the minimal covering class of blind speed intervals for the set [1,13], and so it is the minimal complete class for [1,13].

14. EFFECT OF A NOISY ENVIRONMENT.

The fourth condition (i.e., condition C4) for using keys as a means for identifying an unobservable number z cannot be met in a noisy measurement environment. As in the noiseless measurement case, the observables are numbers x and y, each of which is a speed version (e.g., s of (6)) of a phase (e.g., P of (6)), satisfying the inequalities

$$0 \prec\!= x < L \text{ and } 0 \prec\!= y < H. \tag{40}$$

Unlike the noiseless measurement case, x is not necessarily the same as z#L and y is not necessarily the same as z#H. In other words, there exist measurement errors x' and y' defined by

$$x' = x - z\#L \text{ and } y' = y - z\#H. \tag{41}$$

These errors are not necessarily the same nor is it necessarily likely that either one of them is identical to 0.

The measurement errors are bounded. This assertion is a direct result of (40), (41), and the bounded nature of the numbers z#L and z#H. The constraints

$$-(z\#L) \prec\!= x' < L - z\#L \quad \text{and} \quad -(z\#H) \prec\!= y' < H - z\#H \tag{42}$$

follow directly from (40) and (41). The right and left hand bounds of each of these inequalities are limited in extent. Specifically, z#L and z#H are nonnegative numbers less than L and H, respectively. As a result, the absolute value of x' is less than L and the absolute value of y' is less than H.

Some care must be exercised in the representation of the measurement errors as random variables. Neither error can be properly represented as a random variable with unbounded range. For example, a normally distributed variable is not a reasonable representation of either one of the measurement errors x' and y'. On the other hand, the objection of unboundedness certainly does not apply to representations of the measurements x and y that are of the form

$$x = (z + X)\#L \text{ and } y = (z + Y)\#H \qquad (43)$$

where X and Y are random variables, normal or otherwise. In other words, equations (41) and (43) can be used to provide a statistical representation of the errors that is consistent with the constraints (42) regardless of the nature of the ranges of the variables X and Y.

15. UTILITY OF KEYS IN A NOISY ENVIRONMENT.

Keys can be used as a means for estimating the location of a number z to within a blind speed interval under conditions C1-C3 from imperfect measurements x and y of z#L and z#H, respectively. According to condition C1, z is known to be a member of one of the blind speed intervals in the class C of blind speed intervals. A reasonable decision rule for discerning the interval containing z is to choose a key from among the key components of the members of K'(C) that is closest to the difference y - x. A precise statement of the rule follows.

NEAREST NEIGHBOR DECISION RULE: If x and y are measurements of z#L and z#H, respectively, then a key k is to be selected from among the key components of the members of K'(C) such that the absolute value of the difference between k and y - x is at most the absolute value of the difference between y - x and the key component of any member of K'(C).

There is a connection between the nearest neighbor decision rule and the concept of a minimal complete class of blind speed intervals. The decision rule involves comparisons of the difference y - x with the key components of the members of K'(C). Since the number of key components in this set is just the size of the class C, there is good reason to implement the decision rule in conjunction with a minimal complete class of blind speed intervals. If the minimal complete class exists but cannot be found, then the class of minimal size among all known complete classes should be used.

The nearest neighbor decision rule is imperfect on at least two counts. First, there is no guarantee of nonoccurrence of the event that y - x is identical to the arithmetic mean of the key components of two members of K'(C). In this event, the distance between y - x and one key, k(1), is the same as the distance between y - x and another key, k(2). If there does not exist a key in K'(C) between the two keys k(1) and k(2), then this common distance is less than or equal to the distance between y - x and the key component of any member of K'(C). Hence, some plan must

be devised for a course of action to be taken whenever a tie arises. The plan might be to choose one of the keys k(1) and k(2) in a randomized fashion as demonstrated in example 42. Another possibility is to refuse to select a key in the presence of a tie. Second, the measurements x and y are imperfect and so the rule can lead to the selection of a key of a blind speed interval that does not contain the number z. This is an erroneous decision, and the rule should be subjected to analysis or some experimental test that provides a potential user with a meaningful assessment of the likelihood of such decision errors. In the case of a plan for dealing with ties that allows for the possibility of nonselection of a key, a result of the analysis or experiment should be a measurement of the likelihood that a key will not be selected.

EXAMPLE 42: Referring to examples 40 and 41 in which L is 4, H is 7, and the possible values of z are all the numbers in the closed interval [1,13], suppose that C is the collection of the 5 blind speed intervals of orders 1 through 5 that are subsets of D(0). The key components of the members of K'(C) are 0, 4, -3, 1, and 5. If x is 1.5 and y is 4.0 then the minimal distance between y - x and any of these key components is 1.5. One might consider one flip of a fair coin as a means for choosing one of the two keys (i.e., 1 and 4) that are located at this distance from y - x.

16. TIES AND THE PRECISION OF MEASUREMENTS.

It is always possible to eliminate the possibility of the occurrence of a tie in the case where each of the base speeds is a rational number and the class C of conditions C1 and C2 is such that the collection of all the key components of the members of K'(C) is the key set K(L,H). This can be accomplished by means of a simple limitation on the precision of the measurements x and y. This assertion is the direct result of two observations. First, proposition 6 implies that the key set consists of numbers that are integer multiples of g(L,H). In fact, the members of the key set are the N(L,H) numbers obtained by multiplying each of the integers

$$-[L' - 1], \ldots, 0, 1, \ldots, H' - 1$$

by g(L,H). Second, if the precision of each of the measurements x and y is limited so that each measurement is expressed as an integer multiple of g(L,H), then the relationships (40) imply that x is one of the L' numbers formed by multiplying each of the integers

$$0, 1, \ldots, L' - 1$$

by g(L,H), and y is one of the H' numbers formed by multiplying each of the integers

$$0, 1, \ldots, H' - 1$$

by g(L,H). It follows directly from these two observations that if each of the measurements x and y is limited to an integer multiple of g(L,H) then the difference y - x must be a key. This precludes any possibility of a tie since the minimal distance between y - x and a key component of a member of K'(C) is 0 when the collection of the key components of all members of K'(C) is identical to K(L,H).

EXAMPLE 43: In examples 3 and 17, the low base speed L is 78.2, H is 94.6, and the greatest common whole divisor g(L,H) of L and H is 1/5. If the measurements x and y are integer multiples of 0.2 then there is no need to be concerned about the possibility of a tie when the size of the set K'(C) is N(L,H).

If the size of the set is less than N(L,H) then a tie is possible even though each of the measurements x and y is an integer multiple of g(L,H). This point is demonstrated in example 44.

EXAMPLE 44: Suppose that L is 4, H is 7, and it is known that z is a number in the closed interval [0,7]. The collection consisting of the three blind speed intervals I(0,0), I(1,0), and I(1,1) can serve as the class C of conditions C1 and C2. The corresponding keys (i.e., key components of the members of K'(C)) are 0, 4, and -3. If x is 2 and y is 4 then y - x is 2, the minimal distance between y - x and a key of an interval in C is 2, and each of the keys 0 and 4 is at the minimal distance from y - x. In other words, each of the measurements is an integer multiple of g(L,H) (i.e., 1) and a tie exists.

17. ACQUISITION OF A COMPLETE CLASS.

A method for automatic identification of the members of a specific complete class of blind speed intervals is addressed here under the following assumptions.

A1. Each of the base speeds L and H is a rational number.

A2. Numbers r and b are available, r may be any real number, b is nonnegative and it is known that the unobservable number z is an element of the closed interval [r-b,r+b].

A3. The nonnegative number b is at most

$$T'(L,H) = [T(L,H) - L]/2. \qquad (44)$$

The specific class of interest is just the minimal covering class for the interval [r-b,r+b]. This covering class is here denoted by C(r,b), and the size of the class is denoted by S(r,b).

EXAMPLE 45: Suppose that L is 4, H is 7, r is 0, and b is 5. It can be verified that T(L,H) is 28, T'(L,H) is 12, and the class C(0,5) consists of the blind speed intervals I(-2,-1), I(-1,-1), I(0,0), and I(1,0) (i.e., S(0,5) is 4).

Assumption A2 implies that z is an element of an interval in the minimal covering class for the closed interval [r-b,r+b]. The number r might be a rough estimate of z obtained by means of some auxilliary measurement other than the measurements x and y of z#L and z#H, respectively, and the number b might be a bound that overcomes any uncertainty associated with the auxillary measurement. Alternatively, there may be some theoretical or practical reason that guarantees that z must be at least r - b and at most r + b.

EXAMPLE 46: If z represents the range rate of a commercial airliner other than a supersonic transport then the absolute value of z is surely not greater than 650 knots. In other words, it is certain that z is contained in the closed interval [-650,650] (i.e., r is 0 and b is 650).

EXAMPLE 47: Suppose that successive measurements of the range of an airliner from a mechanically steered radar are obtained, the elapsed time between the measurements is approximately that of a single rotation of the beam through 360°, and the ratio of the amount by which the second measurement exceeds the first measurement to the elapsed time is 450 knots. Under these circumstances, there may be ample reason to conclude that the range rate of the aircraft at the time of the second measurement must be within some predetermined speed bound of the ratio. The bound may be the result of some analysis based upon characteristics of the radar and the capabilities of aircraft under surveillance (e.g., the accuracy of a slant range measurement, radar scan time, and an upper limit on aircraft acceleration). If the speed bound is 250 knots, then z must be a member of the closed interval [200,700] (i.e., r is 450 and b is 250).

Assumption A3 guarantees that the minimal covering class C(r,b) for [r-b,r+b] is complete regardless of what number is assigned to r. This assertion is a direct result of proposition 12. Proposition 11 implies that the guarantee does not extend to those situations in which assumption A3 is not satisfied. In fact, If b exceeds T'(L,H) and it is less than half of the period T(L,H) then it is always possible to find a value for r that precludes the existence of a complete class of intervals. If b is greater than or equal to half of the period then a complete class does not exist regardless of what number is assigned to r.

PROPOSITION 11: If each of the base speeds L and H is a rational number then

1. there exist infinitely many numbers that can be assigned to r such that C(r,b) is not complete in the case where

$$T'(L,H) < b < T(L,H)/2, \text{ and} \tag{45}$$

2. C(r,b) is not complete for any number r in the case where b is at least T(L,H)/2.

PROPOSITION 12. If each of the base speeds L and H is a rational number and b is a nonnegative number that is at most T'(L,H) then C(r,b) is a complete class and there exists an integer k such that the sum r + b is an element of D(k), the difference r - b is an element of one of the sets D(k) and D(k-1), o(r-b) is at most o(r+b) if r - b is an element of D(k), and o(r-b) exceeds o(r+b) if r - b is an element of D(k-1).

EXAMPLE 48: Suppose that L is 4 and H is 7. It follows that T(L,H) is 28 and T'(L,H) is 12. If b is 12.1 and r is a number in the half open interval [15.9,16.1) then there does not exist a covering class for [r-b,r+b] that is complete. Consequently, such values of r preclude the existence of a complete class of blind speed intervals.

If b is at most T'(L,H) and it is known that r + b is a member of D(k) then the size and composition of the minimal covering class C(r,b) can be determined directly from proposition 12. Indeed, if o(r-b) <= o(r+b) then

$$S(r,b) = o(r+b) - o(r-b) + 1 \tag{46}$$

and a blind speed interval is a member of C(r,b) if and only if it is a member of D(k) and the order of the interval is an integer k satisfying the expression

$$o(r-b) \prec k \leq o(r+b). \tag{47}$$

Otherwise, $o(r-b) > o(r+b)$,

$$S(r,b) = o(r+b) - o(r-b) + 1 + N(L,H), \tag{48}$$

and a blind speed interval is a member of C(r,b) if and only if it is a member of D(k) and the order of the interval is at most o(r+b) or it is a member of D(k-1) and the order of the interval is an integer k satisfying the expression

$$o(r-b) \prec k \leq N(L,H). \tag{49}$$

EXAMPLE 49: Suppose that L is 4 and H is 7. In addition, suppose b is 7 and r is 2. It can be verified that o(r-b) is 9, o(r+b) is 4, N(L,H) is 10, and the members of C(2,7) are the blind speed intervals of orders 9 and 10 that are subsets of the half open interval D(-1) and the blind speed intervals of orders 1, 2, 3, and 4 that are subsets of D(0).

There are some practical implications of proposition 12. This is especially true in those cases where r changes from time to time (e.g., as indicated in example 47). As will be seen, the proposition is a menu for the automatic identification of the orders of the members of the class C(r,b), and these orders, together with the nearest neighbor decision rule, can be used to establish a direct path from the measurements x and y to the attributes of the blind speed interval that is most likely to contain the unknown number z.

The orders of the extreme points of the interval [r-b,r+b] are simple functions of r-b and r+b, and these orders in turn determine the order of every member of C(r,b) in the case where b is at most T'(L,H). This assertion follows directly from the application of formula (27) to the two numbers r - b and r + b. Indeed, the relationships

$$o(r-b) = [(r-b)@L]\#H' + [(r-b)@H]\#L' + 1 \tag{50}$$

and

$$o(r+b) = [(r+b)@L]\#H' + [(r+b)@H]\#L' + 1 \tag{51}$$

are equivalent to the results obtained by substituting r - b and r + b for J in (27). If o(r-b) is at most o(r+b) then the members of the class C(r,b) must be blind speed intervals of orders o(r-b) through o(r+b). Otherwise, the members of C(r,b) are blind speed intervals of orders 0 through o(r+b) and orders o(r-b) through N(L,H).

Knowledge of the orders of the members of the class C(r,b) can be put to good use in the case where it is known that C(r,b) must be a subclass of some superclass of blind speed intervals such that distinct members of the superclass have distinct keys. Specifically, the attributes (e.g., key, greatest lower bound p, p@L and p@H) of each blind speed interval in the superclass can be stored in a record and all such records can be stored in computer memory in a sequence that is indexed by

the orders of the blind speed intervals that make up the superclass. Under these conditions, the order of a blind speed interval in C(r,b) serves as a pointer to the location in memory where the attributes of the interval are listed. The set consisting of the orders of the members of C(r,b) is just a list of pointers to the keys that must be compared to the measurement difference y - x in the application of the nearest neighbor decision rule. In this sense, it serves the same purpose as the set of keys of the class C of conditions C1 and C2 (i.e., the set consisting of all of the key components of the 2-dimensional elements of the collection K'(C) defined in condition C3).

EXAMPLE 50: Suppose that L is 78.2 knots and H is 94.6 knots, as in examples 3 and 17. Referring to example 17, L' is 391, H' is 473, N(L,H) is 863 and T(L,H) is 36988.6 knots. The parameter T'(L,H) of assumption A3 is 18455.2 knots. Suppose now that r represents a rough estimate of the range rate of a commercial airliner other than a supersonic transport and b is 250 knots, as in example 47. Except in the case of some gross error that should cause the invocation of a software exception, the maximum of the absolute values of r - b and r + b will be much less than T'(L,H). Suppose that such an exception is invoked if this maximum exceeds 1150 knots. This means that in the absence of an exception the minimal covering class C(r,b) for [r-b,r+b] will always be a subclass of the minimal covering class C(0,1150) for the interval [-1150,1150]. Since the orders of -1150 and 1150 are 837 and 27, respectively, the class C(0,1150) consists of the 54 blind speed intervals with orders 837 through 863 and 1 through 27. If it turns out that r is 100 knots then [r-b,r+b] is [-150,350] and the corresponding minimal covering class C(100,250) consists of the 11 blind speed intervals with orders 861 through 863 and 1 through 8.

There is good reason to keep the parameter b as close to 0 as is possible without violating the requirement that the unobservable number z be a member of the interval [r-b,r+b]. This assertion is a result of the fact that the nearest neighbor decision rule and modifications thereof involve some sort of comparison of the keys of the members of C(r,b) with the difference between the measurements x and y of z#L and z#H, respectively. Since the minimal covering class of disjoint intervals for a set must contain the members of the minimal covering class of disjoint intervals for a subset of that set, it follows that the size S(r,b) of C(r,b) must be a nondecreasing function of b. This means that cost in terms of one or more computational resources (e.g., time, speed, and memory) required to implement the nearest neighbor rule will tend to increase with increasing values of b.

EXAMPLE 51: Suppose that L is 78.2 and H is 94.6, as in example 50. It can be verified that the size of C(100,400) is 19. On the other hand, as pointed out in example 50, the size of C(100,250) is 11.

18. POINT ESTIMATION WITHIN A BLIND SPEED INTERVAL.

The determination of a key by means of the nearest neighbor decision rule only serves to locate the number z to within a blind speed interval subject to the uncertainty associated with the measurement noise. It does not provide a point estimate of the location of the number within the interval.

The determination of a point estimate of the unobservable number z can be viewed as the problem of forming a real valued statistic that is a function of the measurements x and y as well as the blind speed interval I(i',j') identified by the key that is selected by means of some decision rule (e.g., the nearest neighbor rule or some variation thereof). There are many approaches to this problem. Two of these are addressed here.

One approach to the point estimation problem is to define the estimate to be the arithmetic mean of the sums x + i'L and y + j'H. Unfortunately, as demonstrated in example 52, the mean of these two statistics may not fall in the interval I(i',j') with probability 1. Consequently, the use of the mean of x + i'L and y + j'H is somewhat at odds with the fact that the key selection process is supposedly a reasonable approach to identifying the blind speed interval that contains the very unknown that the point estimate is to locate.

EXAMPLE 52: Suppose that L is 4 and H is 7. Also, suppose that it is known that the unobservable number z must be an element of D(0) so that the collection of the 10 blind speed intervals that are subsets of D(0) is complete. It can be verified that the difference y - x is greater than 5.5 if the 2-dimensional measurement vector (x,y) is a member of the set

$$M = \{(x,y): 0 \leq x < 0.25,\ x + 5.5 < y < -x + 6.0\}. \qquad (52)$$

Now suppose that the measurement vector (x,y) is an element of M. Since the members of the key set are the 10 integers that are at least -3 and at most 6, it follows that the distance between y - x and the key 6 is less than the distance between y - x and any other key. This implies that the nearest neighbor rule will result in the selection of the blind speed interval I(5,2) (i.e., the half open interval [20,21)). However, half the sum of x + 5L and y + 2H is greater than 19.75 and less than 20 whenever (x,y) is a member of M (i.e., the mean of x + 5L and y + 2H is not a member of the blind speed interval that is selected as a result of the application of the nearest neighbor decision rule).

Another approach to the point estimation of z is to define the estimate to be the mean of two statistics that assume values in the closure of I(i',j'). The closure of I(i',j') is just the union of the blind speed interval I(i',j') and its least upper bound

$$u(i',j') = \min[(i'+1)L,(j'+1)H]. \qquad (53)$$

The problem with each of the statistics x + i'L and y + j'H is that it is not necessarily the same as u(i',j') or a member of I(i',j'), and so the mean of these statistics may fall above u(i',j') or below the greatest lower bound

$$l(i',j') = \max[i'L,j'H] \qquad (54)$$

of I(i',j'). This problem is circumvented by any statistic with a range that is a subset of the closure of I(i',j'). One such statistic defined for any random variable W is

$$Z(W) = \max\{l(i',j'),\min[W,u(i',j')]\}. \qquad (55)$$

It can be viewed as the result of passing W through a limiter that clips the input at the greatest lower bound and the least upper bound of I(i',j'). Needless to say, any convex combination of Z(x + i'L) and Z(y + j'H) can serve as a point estimate of z that will fall in the closure of I(i',j'). In other words, the range of the random variable

$$V(p) = pZ(x + i'L) + (1 - p)Z(y + j'H) \tag{56}$$

is a subset of the closure of I(i',j') for any nonnegative number p that is at most 1. As demonstrated in example 53, a value might be assigned to the parameter p on the basis of criteria that can be construed to summarize the accuracies of x and y as measurements of z#L and z#H, respectively.

EXAMPLE 53. If the standard deviations of the random variables X and Y of (43) are available then the ratio of the standard deviation of Y to the sum of the standard deviations of X and Y is worthy of consideration as a candidate value for p. If it is not possible to distinguish between the accuracies of x and y as estimates of z#L and z#H, respectively, then the statistics Z(x + i'L) and Z(y + j'H) might be weighted equally by setting p equal to 1/2.

19. POINT ESTIMATION AND THE CHINESE REMAINDER THEOREM.

As pointed out in section 16, it is always possible to eliminate the possibility of a tie in the selection of a key under the following conditions.

E1. The base speeds are rational numbers, the measurement x is an integer multiple of g(L,H) that is nonnegative and less than L, and the measurement y is an integer multiple of g(L,H) that is nonnegative and less than H.

E2. The class C of conditions C1 and C2 is such that the collection of key components of the members of K'(C) is the key set K(L,H).

As will be seen, it is also possible to eliminate the clipping operation in the formulation of a point estimate under the same conditions.

It is easy to see that the clipping operation is unnecessary under conditions E1 and E2. As shown in section 16, condition E1 implies that y - x is a key. By condition E2, it is a key of a blind speed interval in the class C and the only key component of a member of K'(C) that satisfies the requirements of the nearest neighbor rule. This means that under conditions E1 and E2 the nearest neighbor rule will result in the selection of a blind speed interval I(i',j') in the class C with key y - x, and y - x is the same as i'L - j'H. It follows that there is a number s such that

$$s = x + i'L = y + j'H, \tag{57}$$

and any convex linear combination of the sums x + i'L and y +j'H is the same as s. To show that the clipping operation is unnecessary, it is sufficient to show that s is at least as great as the greatest lower bound of I(i',j') (i.e., the maximum of the numbers i'L and j'H) and less than the least upper bound of I(i',j') (i.e., the minimum of the two numbers (i' + 1)L and (j' + 1)H). Since both x and y are

nonnegative, it is obvious that s cannot be less than the lower bound. By E1, the measurement x is less than L and the measurement y is less than H. This means that the sum x + i'L is less than (i' + 1)L and the sum y + j'H is less than (j' + 1)H. As a result, the number s must be less than the upper bound.

CHINESE REMAINDER THEOREM: If

a. n is an integer greater than 1,

b. a(1), ..., a(n) are n integers,

c. m(1), ..., m(n) are n positive integers that are pairwise relatively prime (i.e., g(m(i),m(j)) is 1 whenever i is unequal to j),

d. M is the product of the n integers m(1), ..., m(n), and

e. b(1), ..., b(n) are n integers such that b(j)[M/m(j)] is congruent to 1 modulo m(j) for j = 1, ...,n,

then an integer u is congruent to a(i) modulo m(i) for any integer i that is at least 1 and at most n if and only if it is congruent to the sum of the n integers a(1)b(1)[M/m(1)], a(2)b(2)[M/m(2)], ..., and a(n)b(n)[M/m(n)] modulo M.

Under conditions E1 and E2, there is a connection between the Chinese remainder theorem and a two-phase estimation procedure that first identifies a blind speed interval via the nearest neighbor rule and then formulates a point estimate s of the unknown within the selected interval. In order to see the connection, it is only necessary to recognize that (57) implies s/g(L,H) is an integer that is congruent to x/g(L,H) modulo L', s/g(L,H) is congruent to y/g(L,H) modulo H', and the integers L' and H' are relatively prime. According to the remainder theorem, an integer satisfies these two congruence relations if and only if it is congruer to the integer [xb(1)H' + yb(2)L']/g(L,H) modulo L'H' (i.e., T(L,H)/g(L,H)) where b(1) is any integer such that b(1)H' is congruent to 1 modulo L' and b(2) is any integer such that b(2)L' is congruent to 1 modulo H'. In other words, the estimate s must be of the form

$$S(k) = xb(1)H' + yb(2)L' + kT(L,H) \qquad (58)$$

for some integer k. Under condition E2, there is one and only one integer k' such that S(k') is an element of a blind speed interval in the class C, and s and S(k') are one and the same.

EXAMPLE 54: Suppose that L is 4 and H is 7, as in example 40. It can be verified that g(L,H) is 1, L' is 4, H' is 7, and T(L,H) is 28. Since 21 is congruent to 1 modulo 4, it is possible to use 3 in place of b(1) in (58). Also, since 8 is congruent to 1 modulo 7, the parameter b(2) in (58) can assigned the value 2. The corresponding expression for S(k) is 21x + 8y + 28k, and, for some integer k, it is the same as s whenever x is a nonnegative integer less than L and y is a nonnegative integer less than H.

EXAMPLE 55: Suppose that x is 3 and y is 1 in example 54. The difference y - x is -2, the key of the blind speed interval I(3,2) (i.e, [14,16)). The corresponding value of s is 15 (i.e., the common value of 3 + 3L and 1 + 2H). The sum S(k) reduces to 71 + 28k, and it is 15 when k is -2.

PROPOSITION 13: If L and H are rational numbers, L' is L/g(L,H), and H' is H/g(L,H), then there exists a nonnegative integer i less than H' and a nonnegative integer j less than L' such that

$$iL' - jH' = 1. \tag{59}$$

Proposition 13 can be used to identify a pair of integers that satisfy the requirements of b(1) and b(2) of (58). In fact, there are an infinite number of pairs of integers in this category. This assertion is based on two observations. First, it is clear that the integers i and j of proposition 13 are such that iL' is congruent to 1 modulo H' (i.e., i can serve as b(2)) and -jH' is congruent to 1 modulo L' (i.e., -j can serve as b(1)). In fact, it is clear that

$$(i + kH')L' - (j + kL')H' = 1 \tag{60}$$

for any integer k. It follows that i + mH' can serve as b(2) and -(j + nL') can serve as b(1) for any integers m and n. Second, the integers i and j of proposition 13 can be determined by a search routine. Indeed, j can be viewed as the smallest nonnegative integer less than L' such that L' is a whole divisor of 1 + jH', and i can be viewed as the corresponding integer quotient.

EXAMPLE 56: Suppose that L is 110.075 and H is 136.530, as in examples 4 and 18. It can be verified that g(L,H) is 0.185 (i.e., 37/200), L' is 595, H' is 738, and T(L,H) (i.e., L'H) is 81235.35. It can also be verified that 289L' - 233 H' is identical to 1. As a result, b(1) can be any integer that is congruent to -233 modulo L', and b(2) can be any integer that is congruent to 289 modulo H'. Moreover, if both of the measurements x and y are integer multiples of 0.185 (i.e., g(L,H)), then the estimate s of (57) must be equal to the right side of (58) for some integer k.

20. CONCLUDING REMARKS.

The blind speed intervals generated by two rational base speeds form a periodic structure with useful properties that can be exploited in the estimation of range rate from two Doppler measurements. From a statistical point of view, these properties are the foundation for the construction of decison rules of the nearest neighbor variety that are reasonable formulas for identifying the key of the blind speed interval that most likely contains the range rate. The application of such a rule can be followed by a point estimate of range rate that falls in the closure of the selected interval. From the point of view of implementation, there is a natural connection between the properties of blind speed intervals and the concepts of memory, speed, pointers, arrays, and records employed in computer programming. This connection is pertinent to the fabrication of efficient software realizations of range rate estimators.

There has not been any attempt in this report to quantify the performance of range rate estimators that are functions of two Doppler measurements. In fact, there does not exist an analytic approach of a general nature that can be directly applied to the assessment of performance criteria such as error probability and speed of execution. This means that the performance of each specific realization of a range rate estimator must be individually evaluated by simulation or other

means that is consistent with the application. Nevertheless, in terms of performance, it is to be expected that a specific system that is designed to take advantage of the inherent properties of blind speed intervals will fare better than a system that is based on a design that ignores these properties.

There is a definite connection between the Chinese remainder theorem and a two-phase estimation procedure based on the properties of blind speed intervals when the precision of measurements is limited to an integer multiple of the greatest common whole divisor of the base speeds. This suggests that the concept of a blind speed interval as determined by two base speeds can be extended to any number of base speeds, and that there exists a natural extension of the material in sections 6 through 18 to coherent pulsed radar systems capable of handling three or more wave trains that give rise to an equal number of distinct congruence relations. The likely restriction under which such an extension may be possible is that each base speed is an integer multiple of the same rational number and there does not exist a prime power other than 1 that is a common factor of two or more of the integer multipliers.

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APPENDIX A. PROOF OF PROPOSITION 1

The first part of the proposition is a direct result of the fact that blind speed intervals I(i,j) and I(r,s) share the same key if and only if d(i,j) is the same as d(r,s) (i.e., iL - jH is the same as rL - sH).

The second part of the proposition is based on the fact that if the sets I(i,j) and I(r,s) are distinct then at least one of the differences i - j and r - s must be nonzero. If these sets are distinct blind speed intervals sharing the same key then equation (13) is valid, and the fact that the base speeds are nonzero implies that both differences are nonzero.

APPENDIX B. PROOF OF PROPOSITION 2

If x is a common whole divisor of the rational numbers p/q and r/s then x can always be expressed as a ratio, n/d, of relatively prime integers n and d. Indeed, the fact that (1/x)(p/q) is an integer implies that x is rational. Hence, x can be expressed as the ratio of n to d where n and d are relatively prime integers.

The integer n must be a common whole divisor of p and r, and the ratio of d to either one of the integers q and s must be an integer. This assertion is a direct result of the fact that each of the numbers g(n,d), g(p,q) and g(r,s) is 1. Since (d/n)(p/q) is an integer and each of the numbers g(n,d) and g(p,q) is 1 (i.e., n and d are relatively prime integers, and p and q are relatively prime integers), p must be an integer multiple of n, and d must be an integer multiple of q. Similarly, r must be an integer multiple of n, and d must be an integer multiple of s. In other words, n is a common whole divisor of p and r, and each of the integers q and s is a whole divisor of d.

The integer d must be an integer multiple of

$$v = (qs)/g(q,s). \qquad \text{(B-1)}$$

This assertion is a direct result of the fact that both q and s are whole divisors of d. Since q is a whole divisor of d, the prime powers of q must be factors of d. Likewise, the prime powers of s must be factors of d. It follows that d is an integer multiple of v.

Putting everything together, a common whole divisor x of p/q and r/s is the ratio of a common whole divisor of p and r to the product of v and some positive integer k. It follows that x cannot exceed the ratio of g(p,r) to v. The proposition follows directly from the fact that g(p,r)/v is a common whole divisor of p/q and r/s.

APPENDIX C. PROOF OF PROPOSITION 3

Since g(p,q) is 1, there is only one prime power that is a common factor of p and q, and that prime power is 1. It follows from the equality of rp and sq that there exist integers k and k' such that r is kq and s is k'p. The equality of rp and sq also implies that kqp is the same as k'pq. Since p and q are nonzero, the product pq cannot be zero. Hence, k and k' must be the same (i.e., r is the same as kq and s is the same as kp).

APPENDIX D. PROOF OF PROPOSITION 4

Some special parameters are useful in establishing the validity of the proposition. These are

$$a = \max[iL, jH], \tag{D-1}$$

$$b = \min[(i+1)L, (j+1)H], \tag{D-2}$$

$$c = \max[(i+kH')L, (j+kL')H], \tag{D-3}$$

and

$$d = \min[(i+kH'+1)L, (j+kL'+1)H] \tag{D-4}$$

where k represents an arbitrary integer.

The parameters c and d can be expressed as functions of a, b, and the parameter T defined by

$$T = T(L,H) = LH' = L'H. \tag{D-5}$$

Indeed, the relationships

$$c = \max[iL+kT, jH+kT] = a + kT \tag{D-6}$$

and

$$d = \min[(i+1)L+kT, (j+1)H+kT] = b + kT \tag{D-7}$$

are direct consequences of (D-5) and the definitions (D-1)-(D-4) of a, b, c, and d.

It is now a simple matter to establish the validity of the proposition. The blind speed interval I(i,j) is identical to [a,b), and it follows from (D-6) and (D-7) that

$$I(i,j) + kT = [a,b) + kT = [a+kT, b+kT) = [c,d). \tag{D-8}$$

Moreover, (D-3) and (D-4) imply that [c,d) is just I(r,s) where r is the same as i + kH' and s is j + kL' (i.e., the integers i, j, r, s, and k satisfy relationships (23)).

APPENDIX E. PROOF OF PROPOSITION 5

Equations (24) imply the equality of the keys of I(i,j) and I(r,s). This assertion is based on the definition (17) of T(L,H). The result of multiplying the left hand equation of (24) by L is that (r - i)L is the same as kT(L,H) (i.e., kLH'). Similarly, it can be shown that (s - j)H is the same as kT(L,H). It follows that (r - i)L and (s - j)H are equal. By proposition 1, this equality implies that the blind speed intervals I(i,j) and I(r,s) share the same key.

It remains to show that if distinct blind speed intervals I(i,j) and I(r,s) share the same key then there exists a nonzero integer k such that equations (24) are valid. Suppose that two distinct blind speed intervals I(i,j) and I(r,s) do share the same key. It follows from proposition 1 that the integers r - i and s - j are nonzero, and (r - i)L and (s - j)H are identical. Since, by assumption, L and H are rational, the greatest common whole divisor (i.e., g(L,H)) of L and H exists, and equality of the numbers (r - i)L and (s - j)H implies that (r i)[L/g(L,H)] and (s - j)[H/g(L,H)] are the same. But L/g(L,H) (i.e., L') and H/g(L,H) (i.e., H') are relatively prime integers. It follows from proposition 3 that there is an integer k such that r - i is kH' and s - j is kL' (i.e., equations (24) are satisfied). Finally, the integer k must be nonzero in view of the fact that each of the integers r - i and s - j is nonzero.

APPENDIX F. PROOF OF PROPOSITION 6

The proof is based on two equations involving L, H, and the reciprocal s of g(L,H) (i.e., the greatest common whole divisor of L and H). The equations are direct consequences of the fact that sL (i.e, L') and sH (i.e., H') are relatively prime integers. It follows that there exist integers m and n such that the sum of mLs and nHs is 1 (I. Herstein, Topics in Algebra, Blaisdell Publishing Company, New York, 1964, p. 18). This implies that

$$(km)L - (-kn)H = k/s \tag{F-1}$$

for any integer k, and, in turn, (F-1) implies that k/s is the key of the set I(km,-kn) whenever the latter is nonempty (i.e., it is a blind speed interval). Now let v represent a nonnegative integer. If k is the same as v then a transposition of the terms in (F-1) leads to the equation

$$(vm)L = (-vn)H + v/s. \tag{F-2}$$

Similarly, if k is -v then the equation

$$(vn)H = (-vm)L + v/s \tag{F-3}$$

can be obtained from (F-1). Relationships (F-2) and (F-3) are the equations of interest.

Equation (F-2) can be used to show that I(vm,-vn) is a blind speed interval with key v/s whenever the nonnegative integer v satisfies the inequality

$$0 \prec v \leq sH - 1. \tag{F-4}$$

Indeed, (F-2) implies that (vm)L is greater than or equal to (-vn)H. It also implies that

$$\begin{aligned}[(-vn) + 1]H &= (vm)L - v/s + H \\ &= [(vm) + 1]L - v/s + (H - L).\end{aligned} \tag{F-5}$$

This means that [(-vn) + 1]H is at most [(vm) + 1]L when v/s is at least H - L, and it exceeds [(vm) + 1]L when v/s is less than H - L. It follows that

$$I(vm,-vn) = [(vm)L,(-vn)H + H] \tag{F-6}$$

when

$$H - L \prec v/s < H, \tag{F-7}$$

and

$$I(vm,-vn) = [(vm)L,(vm)L + L] \tag{F-8}$$

when

$$0 \prec v/s < H - L. \tag{F-9}$$

Moreover, (F-5) implies that $[-(vn) + 1]H$ exceeds $(vm)L$ by the difference $H - v/s$ when v/s is less than H. Hence, $I(vm,-vn)$ is a nonempty interval of length $H - v/s$ with key v/s when v satisfies relationship (F-7), and it is a nonempty interval of length L with key v/s when v satisfies relationship (F-9).

Equation (F-3) can be used to show that $I(-vm,vn)$ is a blind speed interval with key $-v/s$ when v satisfies the relationship

$$0 \prec v \leq sL - 1. \tag{F-10}$$

By (F-3),

$$(vn)H \succeq (-vm)L \tag{F-11}$$

and

$$[(-vm) + 1]L = (vn)H - v/s + L. \tag{F-12}$$

Moreover, since v/s is nonnegative and L is less than H,

$$(vn)H - v/s + L \prec (vn)H + L < [(vn) + 1]H. \tag{F-13}$$

Relationships (F-11)-(F-13) imply that

$$I(-vm,vn) = [(vn)H, (-vm)L + L], \tag{F-14}$$

and equation (F-12) implies that $(vn)H$ is less than $(-vm)L + L$ by the amount $L - v/s$ when

$$0 \prec v/s < L. \tag{F-15}$$

In other words, $I(-(vm),vn)$ is a nonempty interval whenever the integer v satisfies (F-10), and the key and length of this blind speed interval are $-v/s$ and $L - v/s$, respectively.

It can now be shown that the key set $K(L,H)$ contains at least

$$N = s(H + L) - 1 \tag{F-16}$$

numbers. It is known that $I(vm,-vn)$ is a blind speed interval with key v/s whenever v is an integer that satisfies the relationship (F-4). Moreover, there are exactly sH integers satisfying (F-4). It is also known that $I(-vm,vn)$ is a blind speed interval with key $-v/s$ whenever v satisfies (F-10), and there are exactly sL integers satisfying (F-10). Since $-v/s$ and v/s are identical when v is 0, it follows that there are at least N distinct keys.

It remains to show that there are exactly N keys in the key set $K(L,H)$. This fact can be established by showing that the sum of the lengths of the blind speed intervals in the classes

$$A = \{I(vm,-vn): 1 \prec v \leq sH - 1\} \tag{F-17}$$

and

$$B = \{I(-vm,vn):\ 0 =< v <= sL - 1\} \qquad \text{(F-18)}$$

is equal to the length of the interval D(k) (i.e., T(L,H) = sLH) where D(k) is the set defined by (21). The length of each member of either one of the classes A and B has already been derived. Indeed, if w(x,y) is used to denote the length of the blind speed interval I(x,y) then

$$w(vm,-vn) = L \qquad \text{if } 0 =< v < s(H - L), \qquad \text{(F-19)}$$

$$w(vm,-vn) = H - v/s \qquad \text{if } s(H - L) =< v <= sH - 1, \qquad \text{(F-20)}$$

and

$$w(-vm,vn) = L - v/s \qquad \text{if } 0 =< v <= sL - 1. \qquad \text{(F-21)}$$

Using equations (F-19)-(F-21), it can be verified that the sum of the lengths of the blind speed intervals in classes A and B is sHL.

APPENDIX G. PROOF OF PROPOSITION 7

The parameter r(n) defined for any integer n by the formula

$$r(n) = (nH)@L \tag{G-1}$$

is a useful entity in establishing the validity of proposition 7. It follows from the definition of the operator @ that r(n) is an integer, r(n)L is at most nH, and r(n) is at least as large as any integer i for which iL is less than or equal to nH.

There are four observations concerning the integer r(n) that are worthy of special mention. First, the fact that base speed L is less than base speed H implies that

$$(r(n) + 1)L \leq nH + L < (n+1)H. \tag{G-2}$$

This means that r(n+1) exceeds r(n) + 1 when r(n+1)L is (n+1)H. Second, since r(n) is less than r(n) + 1, the definition of r(n) implies that

$$(r(n) + 1)L > nH. \tag{G-3}$$

This means that the set I(r(n),n) is nonempty (i.e., it is a blind speed interval) for any integer n, and nH is the greatest lower bound of I(r(n),n). Third, it follows from (G-2) and the definition of r(n+1) that

$$r(n) < r(n) + 1 \leq r(n+1). \tag{G-4}$$

Fourth, the relationship

$$nH < (r(n) + 1)L \prec r(n+1)L \leq (n + 1)H. \tag{G-5}$$

is a direct result of (G-3), (G-4), and the definition of r(n+1). This means that I(r(n+1),n) is a blind speed interval when r(n+1)L is less than (n + 1)H, and in this situation the number (n + 1)H is the least upper bound of I(r(n+1),n). It also follows from (G-5) that the set I(r(n)+m,n) is a blind speed interval in the case where the integers r(n+1) - r(n) - 1 and m are positive and the latter is not greater than the former.

In the remainder of this appendix, the letters k and n are used to represent integers that are closely tied to one another and the integers L' and H' defined by (16) (i.e., H' is H/g(L,H) and L' is L/g(L,H)). Specifically, k is any integer and n is subject to the constraint

$$kL' \prec n < (k + 1)L'. \tag{G-6}$$

As emphasized by (17), L'H and H'L represent the same number (i.e., T(L,H)). This means nH is the same as kT(L,H) when n is kL', and (n+1)H is the same as (k + 1)T(L,H) when n is the same as (k + 1)L' - 1.

There are two important sets that are useful in the description of the periodic nature of blind speed intervals generated by rational base speeds. One of these is the half open interval D(k) defined by (21) (i.e., [kT(L,H),(k+1)T(L,H))). The remaining set is the half open interval

$$O(n) = [nH,(n+1)H). \tag{G-7}$$

Under the constraint (G-6), there are L' sets of the form (G-7), and the union of these L' sets is D(k).

The set D(k) is the union of the blind speed intervals that are subsets of itself and these intervals can be identified by identifying the blind speed intervals that are subsets of O(n) for any integer n that satisfies the constraint (G-6). This assertion is a result of two observations. First, the set D(k) is the union of the L' sets O(kL'), ..., and O((k+1)L'-1). Second, since any integer multiple of H is the greatest lower bound of a blind speed interval, O(n) is the union of the blind speed intervals that are subsets of itself.

There is a definite relationship between the greatest lower bound and the least upper bound of the interval O(n) and the numbers r(n) and r(n+1). This relationship is a direct result of proposition 3, the constraint (G-6) on the integer n and the fact that the integers L' (i.e., L/g(L,H)) and H' (i.e., H/g(L,H)) are relatively prime. Indeed, nH is identical to r(n)L if and only if n is kL', and in this situation r(n) is kH'. Otherwise, nH is greater than r(n)L. Also, the two products (n + 1)H and r(n+1)L are identical if and only if n is (k + 1)L' - 1, and in this situation r(n+1) is (k+1)H'. Otherwise, (n + 1)H exceeds r(n+1)L. It is also worth noting that these observations and the fact that r(n) is a strictly monotonic increasing function of n imply that

$$kH' \prec r(n) < (k + 1)H' \tag{G-8}$$

under the tacit assumption that n is an integer subject to the constraint (G-6).

There are four types of blind speed intervals that are subsets of O(n). One of these is I(r(n),n) corresponding to the situation in which r(n)L is the same as nH (i.e., n is the same as kL'). Another is I(r(n+1),n) corresponding to the situation in which r(n+1)L is less than (n+1)H (i.e., L' exceeds 1 and the integer n is greater than or equal to kL' and less than (k + 1)L' - 1). Still another is I(r(n),n) where r(n)L is less than nH (i.e, L' exceeds 1 and n is greater than kL' and less than (k+1)L'). Finally, a blind speed interval can be of the form I(r(n)+m,n) where r(n+1) - r(n) - 1 is at least 1 and m is an integer such that

$$0 < m \prec r(n+1) - r(n) - 1. \tag{G-9}$$

This situation may arise in any case where n is at least kL' and less than (k + 1)L'.

There are two cases corresponding to two mutually exclusive restrictions on the parameter L' that need to be considered in the identification of the blind speed intervals that are subsets of D(k). One of the restrictions is that L' is 1. The other restriction is that L' exceeds 1.

In the case where L' is 1, k is the only possible value of n, the interval D(k) is identical to O(k), and O(k) is the union of r(k+1) - r(k) blind speed intervals. These intervals arranged in a row in ascending order are

$$I(r(k),k) < \ldots < I(r(k+1)-1,k) \qquad \text{(G-10)}$$

where the notation I(i,j) < I(r,s) means that any number in the blind speed interval I(i,j) is less than every number in the blind speed interval I(r,s).

In the case where L' exceeds 1, (k + 1)L' exceeds kL' by at least 2, and so D(k) is the union of 2 or more sets of the type O(n). If n is at least kL' and at most (k + 1)L' - 2 then O(n) is the union of r(n+1) - r(n) + 1 blind speed intervals and these intervals arranged in a row in ascending order are

$$I(r(n),n) < I(r(n)+1),n) < \ldots < I(r(n+1),n). \qquad \text{(G-11)}$$

If n is (k + 1)L' - 1 then O(n) is the union of r(n+1) - r(n) blind speed intervals, and the expression

$$I(r(n),n) < \ldots < I(r(n+1)-1,n). \qquad \text{(G-12)}$$

is a representation of the relationship between these intervals when arranged in a row in ascending order.

Proposition 7 can be established in two phases. The first phase of the proof establishes the proposition for the case in which L' is 1. The second phase of the proof confirms the validity of the proposition when L' is an integer greater than 1.

In the case where L' is 1, the order of each blind speed interval that is a subset of D(k) can be derived from an examination of the sequence (G-10). Indeed, r(k) is kH', r(k+1) is (k + 1)H', the number of blind speed intervals in the sequence (G-10) is H', and the order of I(r(k)+m,k) is m + 1 where m is any nonnegative integer less than H'. Since L' is 1, this is the same as saying that the order of I(r(k)+m,k) is the sum [r(k)+m]#H' + k#L' + 1.

The order of each blind speed interval that is a subset of D(k) can be expressed in terms of an integer j satisfying the constraint

$$0 \prec j <= L' - 1. \qquad \text{(G-13)}$$

In fact, the constraint (G-6) on n implies that n is an integer of the form kL' + j. In the case where L' exceeds 1, two mutually exclusive situations need to be examined. One of these is the situation in which j is less than the upper bound L' - 1 of relationship (G-13), and the other is the situation in which j is the same as the upper bound.

A blind speed interval that is a subset of D(k) is a member of one of L' lists. Each list can be identified by an integer j satisfying the constraint (G-13). In the case where L' exceeds 1, there are two types of lists. Lists 0 through L' - 2 are of one type and list L' - 1 is of another type.

List L' - 1 corresponds to the sequence obtained from (G-12) when n is (k + 1)L' - 1. The sets

$$I(r((k+1)L'-1),(k+1)L'-1)$$
$$\vdots$$
$$I(r((k+1)L')-1,(k+1)L'-1). \qquad \text{(G-14)}$$

are the members of this sequence arranged in a vertical list in ascending order. If m is an integer satisfying the relationship

$$0 \prec m < r((k+1)L') - r((k+1)L'-1) \qquad \text{(G-15)}$$

then $I(r((k+1)L'-1)+m,(k+1)L'-1)$ is a member of list L' - 1, and all members of the list are sets of this form.

In the case where L' exceeds 1 and j is a nonnegative integer less than L' - 1, the jth list corresponds to the sequence obtained from (G-11) when n is the same as the sum kL'+j. The sets

$$I(r(kL'+j)+0,kL'+j)$$
$$I(r(kL'+j)+1,kL'+j)$$
$$\vdots$$
$$I(r(kL'+j+1),kL'+j). \qquad \text{(G-16)}$$

are the members of this sequence arranged in a vertical list in ascending order. If m is an integer subject to the constraint

$$0 \prec m \leq r(kL'+j+1) - r(kL'+j) \qquad \text{(G-17)}$$

then the set $I(r(kL'+j)+m,kL'+j)$ is a member of the list, and every member of the list is a set of this form.

It remains to show that

$$x(j,m) = [kL'+j]\#L' + [r(kL'+j)+m]\#H' + 1 \qquad \text{(G-18)}$$

is the order of the set $I(r(kL'+j)+m,kL'+j)$ under each of two assumptions. The assumptions are

(a) j is L' - 1 and m is an integer subject to the constraint (G-15), and

(b) L' exceeds 1, j is a nonnegative integer less than L' - 1 and m is an integer subject to the constraint (G-17).

The remainder of the appendix deals with the evaluation of the right side of (G-18) under each of these assumptions.

Under assumption (b), it is easy to show that x(0,m) is the order of the blind speed interval I(r(kL')+m,kL') (i.e., the mth member of the list 0 formed by setting j equal to 0 in (G-16)). The constraint (G-17) on m implies

$$kH' = r(kL') \prec r(kL') + m \preceq r(kL'+1), \tag{G-19}$$

and the relationship

$$r(kL'+1) < r((k+1)L') = (k+1)H'. \tag{G-20}$$

is a result of the fact that L' exceeds 1 under assumption (b). It follows that

$$[r((kL')+m]\#H' = m. \tag{G-21}$$

Hence, x(0,m) is just m + 1.

If assumption (b) applies and x(j,r(kL'+j+1)-r(kL'+j)) is the order of the largest member of list j then x(j+1,0) is the order of the smallest member in list j + 1. This assertion can be established from the fact that the restriction on j under assumption (b) implies

$$kL' \prec kL' + j < kL' + j + 1 \preceq (k+1)L' - 1. \tag{G-22}$$

It follows that

$$[kL'+j+1]\#L' = [kL'+j]\#L' + 1. \tag{G-23}$$

Hence, x(j+1),0) is the sum of x(j,r(kL'+j+1)-r(kL'+j)) and 1.

If assumption (b) applies and x(j,0) is the order of the smallest member of list j then x(j,m) is the order of the mth member of list j. This assertion is the result of the fact that the restrictions imposed by assumption (b) on m and j imply

$$kH' = r(kL') \prec r(kL'+j) \prec r(kL'+j) + m \prec r(kL'+j+1) \tag{G-24}$$

and

$$r(kL'+j+1) \preceq r((k+1)L'-1) < r((k+1)L') = (k+1)H'. \tag{G-25}$$

It follows that

$$[r(kL'+j)+m]\#H' = r(kL'+j)\#H' + m. \tag{G-26}$$

Hence, x(j,m) is the same as x(j,0) + m.

Two results can now be established by way of induction on the integer j under assumption (b). First, x(j,m) is the order of the mth member of list j (i.e, the blind speed interval I(r(kL'+j)+m,kL'+j)). Second, x(L'-1,0) is the order of the smallest member of list L' - 1 (i.e., the blind speed interval I(r(k+1)L'-1),(k+1)L'-1)).

It is now a simple matter to show that x(L'-1,m) is the order of the mth member of list L' - 1 (i.e., the mth member of list j under assumption (a)). The constraint imposed by assumption (a) on m implies that

$$kH' = r(kL') \prec r((k+1)L'-1) \prec r((k+1)L'-1) + m \qquad \text{(G-27)}$$

and

$$r((k+1)L'-1) + m < r((k+1)L') = (k+1)H'. \qquad \text{(G-28)}$$

It follows that

$$[r((k+1)L'-1)+m]\#H' = [r((k+1)L'-1)]\#H' + m. \qquad \text{(G-29)}$$

Thus, x(L'-1,m) is the same as x(L'-1,0) + m.

APPENDIX H. PROOF OF PROPOSITION 8

The proof is based on the fact that the integers L' and H' defined by (16) are relatively prime and the size N(L,H) of the key set is one less than L' + H'.

The proof is divided into two parts. The first part of the proof establishes that a necessary and sufficient condition for f(m) to be H' is that there exists an integer k such that the integer m is the same as k(H' + L') - 1. The second part of the proof establishes the fact that if m and n are any integers such that f(m) and f(n) are identical then there is an integer k such that m is the same as n + k(H' + L'). This means that if the absolute value of the difference m - n is less than H' + L' then f(m) and f(n) are the same if and only if m is the same as n. It follows from the definition (28) of f(m) (i.e., f(m) is the same as [mL']#[H'+L']) that f(m) assumes every nonnegative integer value less than H' + L' as m increases through the H' + L' integers in the closed interval [(k-1)(H'+L'],k(H'+L')-1] defined for any integer k. The proposition is a direct result of the fact that f(m) must be H' when m is k(H'+ L') - 1.

If f(m) is H' then there exists an integer k such that m is identical to k(H' + L') - 1. Suppose that f(m) is H'. Since (28) implies that f(m) and (mL')#(L' + H') are the same, it follows that there is an integer x such that

$$mL' = H' + x(L' + H'). \tag{H-1}$$

This in turn implies that (m - x)L' is the same as (1 + x)H'. Proposition 3 and the fact that the integers L' and H' are relatively prime imply the existence of an integer k such that

$$m - x = kH' \text{ and } 1 + x = kL'. \tag{H-2}$$

The equation

$$m = k(L' + H') - 1 \tag{H-3}$$

can be obtained by solving the second of equations (H-2) for x and substituting the result into the first equation. Since the sum of L' and H' exceeds the size of the key set by 1, it follows that m is k[N(L,H) + 1] - 1.

If k is an integer and

$$m = k(L' + H') - 1 \tag{H-4}$$

then f(m) must be H'. The relationship

$$mL' = [k(L' + H') - 1]L' \tag{H-5}$$

is the result of multiplying (H-4) by L'. Equation (28) (i.e., the definition of f(m)) implies that there is an integer x such that

$$mL' = f(m) + x(L' + H'). \tag{H-6}$$

The equation

$$(kL' - x)(L' + H') = f(m) + L'. \tag{H-7}$$

is an obvious result of equating the right hand members of equations (H-5) and (H-6). The relationship

$$L' \leq (kL' - x)(L' + H') < H' + 2L' \tag{H-8}$$

is a direct result of (H-7) and the fact that (28) forces f(m) to be a nonnegative number less than L' + H'. In view of the fact that L' and H' are positive integers, it follows that the integer kL' - x is positive and identical to 1. Hence, by (H-7), f(m) has to be identical to H'.

If m and n are integers such that f(m) is the same as f(n) then there is an integer k such that m is the same as the sum of n and k(H' + L'). Suppose that f(m) and f(n) are the same. The definition (28) of f(m) implies the existence of integers i and j such that

$$mL' - i(H' + L') = nL' - j(H' + L'). \tag{H-9}$$

The equation

$$[m - n - (i - j)]L' = (i - j)H' \tag{H-10}$$

is the result of an obvious collection of terms that are multiples of L' and another collection of terms that are multiples of H'. Since L' and H' are relatively prime integers, it follows from proposition 3 that there exists an integer k such that

$$m - n - (i - j) = kH' \text{ and } (i - j) = kL'. \tag{H-11}$$

These equations imply that m is identical to the sum of n and k(H' + L').

APPENDIX I. PROOF OF PROPOSITION 9

The proof of proposition 9 utilizes nomenclature introduced in appendix G in the proof of proposition 7. Specifically, L' and H' are the relatively prime integers defined by equations (16), k represents any integer, n is an integer satisfying the constraint

$$kL' =\!< n < (k+1)L', \tag{I-1}$$

O(n) is the half open interval [nH,(n+1)H), and r(n) is the integer (nH)@L (i.e, r(n)L is at most nH, and r(n) is at least as great as any integer x such that xL is less than or equal to nH).

Four results are employed in the proofs of four lemmas appearing at the end of this appendix. First, the integer r(n) is less than r(n+1). Second, the product r(n)L is the same as nH if and only if n is kL', and in this case r(n) is kH'. Third, the product r(n+1)L is the same as (n+1)H if and only if n is the same as (k + 1)L' - 1, and in this case r(n+1) is (k+1)H'. Fourth, the relationship

$$nH < (r(n) + 1)L =\!< r(n+1)L <= (n+1)H \tag{I-2}$$

is valid for any integer n satisfying the constraint (I-1). The validity of each of these four assertions is established in appendix G.

Proposition 9 is a direct result of the four lemmas appearing at the end of this appendix and three additional facts established in appendix G. First, D(k) is the union of the L' sets

$$O(kL'), \ldots, O((k+1)L'-1).$$

Second, the set O(n) is the union of blind speed intervals that are subsets of itself. Third, a blind speed interval that is a subset of O(n) is a set of the form I(r(n+1),n) where r(n+1)L is less than (n + 1)L or else it is a set of the form I(r(n)+m,n) where m is a nonnegative integer that is less than or equal to r(n+1) - r(n) - 1. In other words, if n is k(L' + 1) - 1 then there are r(n+1) - r(n) blind speed intervals that are subsets of O(n). Otherwise, n is another integer satisfying the constraint (I-1), and there are r(n+1) - r(n) + 1 blind speed intervals that are subsets of O(n).

The following four lemmas establish the relationship between the function f defined by expression (28) and the key and order of a blind speed interval. Needless to say, if the integer k of expression (I-1) is 0 then the integer n is at least 0 and at most L' - 1, r(kL') is 0, and r((k+1)L') is H'. In other words, by proposition 7, the order of the blind speed interval I(r(n)+m,n) is identical to the sum r(n) + m + n + 1 for any nonnegative integer m that is at most r(n+1) - r(n) - 1, and the order of I(r(n+1),n) is the same as the sum r(n+1) + n + 1 whenever r(n+1)L is less than (n+1)H (i.e., L' exceeds 1 and n is less than L' - 1).

LEMMA 1. If r(n+1) - r(n) - 1 exceeds 0 and m is an integer such that

$$0 < m <= r(n+1) - r(n) - 1 \tag{I-3}$$

then f(r(n)+m+n) is positive and less than H', and the key of I(r(n)+m,n) is the product of g(L,H) and f(r(n)+m+n).

PROOF OF LEMMA 1

The key associated with the set I(r(n)+m,n) is given by the formula

$$d(r(n)+m,n) = (r(n) + m)L - nH. \tag{I-4}$$

Since L' is L/g(L,H) and H' is H/g(L,H), the result of adding nL and its additive inverse -nL to the right side of this equation is

$$d(r(n)+m,n) = g(L,H)F(m) \tag{I-5}$$

where the formula

$$F(m) = (r(n) + m + n)L' - n(H' + L') \tag{I-6}$$

defines F(m).

It remains to show that F(m) is f(r(n)+m+n) and that this integer is positive and less than H'. The constraint (I-3) on m implies that

$$r(n) + 1 \prec r(n) + m \leq r(n+1) - 1. \tag{I-7}$$

This and relationship (I-2) imply

$$nH < (r(n) + m)L < r(n+1)L \prec (n + 1)H. \tag{I-8}$$

It follows from (I-4) and (I-5) that

$$0 < g(L,H)F(m) < H. \tag{I-9}$$

This result implies

$$0 < F(m) < H'. \tag{I-10}$$

In view of the definition (I-6) of F(m), it follows that

$$F(m) = [(r(n)+m+n)L']\#[H'+L']. \tag{I-11}$$

In other words, F(m) is just f(r(n)+m+n), and, by (I-10), the latter is a positive integer less than H'.

LEMMA 2: If r(n)L is the same as nH (i.e., n is ths same as kL') then the key of I(r(n),n) is identical to f(r(n)+n) and the latter is 0.

PROOF OF LEMMA 2

Since r(n)L is the same as nH, the key associated with I(r(n),n) is

$$d(r(n),n) = r(n)L - nH = 0. \tag{I-12}$$

The lemma follows directly from the fact that r(n)L can assume the value nL if and only if n of constraint (I-1) is kL'.

Indeed, if n is kL' then r(n) is kH'. This means that

$$f(r(n)+n) = [(r(n)+n)L']\#[H'+L'] = 0. \qquad (I\text{-}13)$$

LEMMA 3: If r(n+1)L is less than (n + 1)H then f(r(n+1)+n) is a positive integer less than H', and the key of I(r(n+1),n) is the product of g(L,H) and f(r(n+1)+n).

PROOF OF LEMMA 3

The key of I(r(n+1),n) is given by the formula

$$d(r(n+1),n) = r(n+1)L - nH. \qquad (I\text{-}14)$$

The result of adding nL and its additive inverse -nL to the right side of (I-14) is

$$d(r(n+1),n) = g(L,H)F \qquad (I\text{-}15)$$

where the formula

$$F = (r(n+1) + n)L' - n(H' + L') \qquad (I\text{-}16)$$

defines F.

It remains to prove that F is f(r(n+1)+n) and that the latter is a positive integer less than H'. The assumption that r(n+1)L is less than (n+1)H implies that the right side of (I-14) is less than H. The expression

$$0 < d(r(n+1),n) < H \qquad (I\text{-}17)$$

follows from (I-14) and (I-2) (i.e., r(n+1)L is greater than nH). The relationships (I-15) and (I-17) imply that

$$0 < F < H'. \qquad (I\text{-}18)$$

It follows from (I-16) that

$$F = [(r(n+1) + n)L']\#(H' + L'). \qquad (I\text{-}19)$$

In other words, F is the same as f(r(n+1)+n) and f(r(n+1)+n) is a positive integer less than H'.

LEMMA 4. If r(n)L is less than nH (i.e., n is greater than kL' and less than (k+1)L') then f(r(n)+n) exceeds H' and the key of I(r(n),n) is the sum of -(H + L) and the product of g(L,H) and f(r(n)+n).

PROOF OF LEMMA 4

The key of I(r(n),n) is

$$d(r(n),n) = r(n)L - nH = -(H + L) + X \qquad (I\text{-}20)$$

where

$$X = (r(n) + 1)L - (n - 1)H. \qquad (I\text{-}21)$$

Also, it is clear that the result of adding (n - 1)L and its additive inverse -(n - 1)L to the right side of (I-21) is

$$X = g(L,H)F \tag{I-22}$$

where the formula

$$F = (r(n) + n)L' - (n - 1)(H' + L') \tag{I-23}$$

defines F.

It remains to show that F is f(r(n)+n) and that the latter exceeds H'. In view of (I-2), the product (r(n) + 1)L exceeds nH. Also, the assumption that r(n)L is less than nH implies that (r(n) + 1)L is less than the sum nH + L. It follows directly from (I-21) that

$$H < X < L + H. \tag{I-24}$$

This fact and (I-22) imply that

$$H' < F < H' + L'. \tag{I-25}$$

It follows from (I-23) that F is $[(r(n)+n)L']*[H'+L']$. This fact and (I-25) imply that F is f(r(n)+n) and that the latter is greater than H'.

APPENDIX J. PROOF OF PROPOSITION 10

It is well to keep in mind that the assumption that N(L,H) exceeds 2, the constraint (33) on the integer m and the definition (34) of n have some very definite implications with regard to the numbers f(n-1) and f(m+1) generated by the function f defined by (28), the key d'(m+2) of the blind speed interval I'(0,m+2) and the key d'(n) of the blind speed interval I'(0,n). In fact, each of the two integers f(m+1) and f(n-1) is positive and other than one of the relatively prime integers L' and H' defined by (16). Indeed, (33), (34), and the fact that N(L,H) exceeds 2 imply that

$$2 \prec m + 1 \leq N(L,H) - 1 \qquad (J\text{-}1)$$

and

$$2 \prec n - 1 \leq N(L,H) - 1. \qquad (J\text{-}2)$$

These inequalities in turn imply that each of f(m+1) and f(n-1) is other than one of the three integers f(0), f(1) and f(N(L,H). It can be verified from (28) that f(0) is 0 and f(1) is L'. It is a direct result of proposition 8 that f(N(L,H)) is H' and f(i) assumes N(L,H) + 1 distinct values as the integer i increases from 0 to N(L,H). Consequently, each of the integers f(m+1) and f(n-1) is positive and other than L' or H'. Hence, by proposition 9, each of the integers d'(m+2)/g(L,H) and d'(n)/g(L,H) is other than one of the numbers 0, L', and H'.

As will be seen, the following lemma is useful in the proof of the proposition.

LEMMA : If B is a positive number and x is any number that is not an integer multiple of B (i.e., x#B exceeds 0) then (-x)#B is identical to B - x#B and (-x)@B is -(x@B + 1).

PROOF OF LEMMA

Any number x can be written as

$$x = x\#B + (x@B)B. \qquad (J\text{-}3)$$

The formula

$$-x = B - x\#B - (x@B + 1)B. \qquad (J\text{-}4)$$

can be obtained by multiplying both sides of (J-3) by -1 and adding B and its additive inverse -B to the right side of the resulting equation. The lemma follows directly from the fact that B - x#B is a nonnegative number less than B whenever x is other than an integer multiple of B. In other words, -(x@B + 1) must be (-x)@B and the difference B - x#B must be (-x)#B when x is other than an integer multiple of B.

The fact that f(n-1) is other than 0 leads to a useful relationship between f(n-1) and f(m+1). The sum H' + L' is the same as N(L,H) + 1. Since f(n-1) is other than 0, it follows from the lemma and the definition (28) of the function f that

$$f(n-1) = L' + H' - f(1-n). \qquad (J\text{-}5)$$

The definition (28) of the function f implies

$$f((m+1)-(L'+H')) = f(m+1). \tag{J-6}$$

It is also clear from the definition (34) of n that 1 - n is the same as m - N(L,H) (i.e., m + 1 - (H' + L')). The formula

$$f(n-1) = L'+H' - f(m+1) \tag{J-7}$$

follows directly from (J-5) and (J-6).

Proposition 10 can be established by considering the implications of (J-7) and proposition 9 under each of three mutually exclusive constraints on f(m+1). This assertion is based on two obvious consequences of proposition 9. First, if d'(m+2)/g(L,H) exceeds 0 then, by proposition 9, f(m+1) must be less than H' and identical to d'(m+2)/g(L,H). In this situation, there are two possibilities, namely, the integer H' exceeds L' by at least 2 and

$$L' < f(m+1) < H', \tag{J-8}$$

or else L' exceeds 1 and

$$0 < f(m+1) < L'. \tag{J-9}$$

Second, if d'(m+2)/g(L,H) is negative then, by proposition 9,

$$f(m+1) > H'. \tag{J-10}$$

Consequently, it only remains to show that d'(n) is the same as the additive inverse of d'(m+2) when f(m+1) satisfies (J-9) or (J-10), and the ratio d'(n)/g(L,H) is the same as the sum of the term H' + L' and the additive inverse of d'(m+2)/g(L,H) when f(m+1) satisfies the constraint (J-8).

Suppose that f(m+1) satisfies constraint (J-8). It follows from (J-7) that f(n-1) is less than H'. By proposition 9, d'(n)/g(L,H) is the same as f(n-1), and the ratio d'(m+2)/g(L,H) is the same as f(m+1). It follows from (J-7) that d'(n)/g(L,H) is equal to the sum of the term H'+ L' and the additive inverse of d'(m+2)/g(L,H).

Suppose now that f(m+1) satisfies the constraint (J-9). Since f(m+1) is less than L', relationship (J-7) implies that f(n-1) is greater than H'. It follows from proposition 9 that d'(m+2)/g(L,H) is identical to f(m+1), and d'(n)/g(L,H) is the same as the sum of -(L' + H') and f(n-1). Hence, by (J-7), the ratio d'(n)/g(L,H) is the same as -d'(m+2)/g(L,H).

Finally, suppose that f(m+1) satisfies (J-10). Since f(m+1) is greater than H', equation (J-7) implies that f(n-1) is less than L', an integer less than H'. Hence, by proposition 9, d'(m+2)/g(L,H) is the sum of -(H' + L') and f(m+1), and the ratio d'(n)/g(L,H) is the same as f(n-1). By (J-7), it follows that d'(n)/g(L,H) is identical to -d'(m+2)/g(L,H).

APPENDIX K. PROOF OF PROPOSITION 11

If b exceeds T'(L,H) then the minimal covering class C(r,b) for the interval [r-b,r+b] is incomplete whenever r + b is an integer multiple of T(L,H). Indeed, suppose that k is any integer, r is the same as -b + kT(L,H), and b is greater than T'(L,H). The assumption that r is - b + kT(L,H) implies that the sum r + b (i.e., kT(L,H)) is a member of the blind speed interval I'(k,1). Since 2b is greater than T(L,H) - L under the assumption that b exceeds T'(L,H), it is also true that

$$r - b = r + b - 2b = kT(L,H) - 2b < (k - 1)T(L,H) + L \qquad (K\text{-}1)$$

In view of the fact that L exceeds 0 and r + b is kT(L,H) (i.e., a number greater than (k-1)T(L,H)), this implies that there is an element of [r-b,r+b] that is also an element of I'(k-1,1). It follows that [r-b,r+b] has elements in common with distinct blind speed intervals with the same key (i.e., I'(k,1) and I'(k-1,1)). As a result, the minimal covering class C(r,b) is incomplete, and so there does not exist a complete covering class for [r-b,r+b] in the case where r is the same as -b + kT(L,H).

If b is at least as great as T(L,H)/2, then the minimal covering class C(r,b) for [r-b,r+b] is incomplete for any real number r. This assertion is a direct result of the following three facts: the sum r + b is an element of the half open interval D(k) defined by (21) (i.e., [kT(L,H),(k+1)T(L,H)) for some integer k, the number

$$x = r + b - T(L,H) \qquad (K\text{-}2)$$

is a member of the set D(k-1), and the orders of r + b and x are the same. In other words, r + b and x are members of different blind speed intervals sharing the same key. Needless to say, if 2b is greater than or equal to T(L,H) then r + b and x must also be members of [r-b,r+b], and the class C(r,b) is incomplete.

APPENDIX L. PROOF OF PROPOSITION 12

If b is a nonnegative number less than or equal to T'(L,H), a positive number less than T(L,H)/2, then there exists an integer k such that r + b is an element of the half open interval D(k) defined by (21) (i.e., the interval [kT(L,H),(k+1)T(L,H)), and r - b is an element of one of the intervals D(k) and D(k-1). Indeed, it is clear that there exists an integer k such that the number r + b is an element of the half open interval D(k). This means that

$$r + b \geq kT(L,H). \tag{L-1}$$

Since b is at most T'(L,H), it follows that

$$\begin{aligned} r - b = r + b - 2b &\geq r + b - 2T'(L,H) \\ &\geq kT(L,H) - 2T'(L,H). \end{aligned} \tag{L-2}$$

The relationship

$$r - b \geq (k - 1)T(L,H) + L > (k - 1)T(L,H) \tag{L-3}$$

is a direct result of (L-2) and the definition of T'(L,H) (i.e., T'(L,H) is half of the difference T(L,H) - L). In other words, the difference r - b is in D(k) or D(k-1).

In the case where r - b is an element of D(k), the minimal covering class C(r,b) of [r-b,r+b] is a subclass of the class of all blind speed intervals that are subsets of D(k). Hence, C(r,b) is both complete and minimal complete. Moreover, in this case, it is clear that o(r-b), the order of r - b, is at most o(r+b).

It remains to show that C(r,b) is complete in the case where the number r - b is an element of D(k-1). This can be done by showing that o(x) exceeds o(r+b) when x is a number in D(k-1) that is at least as large as r - b. In other words, the number x satisfies the constraint

$$r - b \leq x < kT(L,H). \tag{L-4}$$

It is clear that the order of x is the same as the order of the sum of x and the period T(L,H), and that the sum is an element of D(k). Hence, if it can be shown that x + T(L,H) exceeds r + b, an element of D(k), and these numbers fall inside distinct blind speed intervals then it must be concluded that o(x) is greater than o(r+b). Since x is at least r - b and b is at most T'(L,H), the difference

$$X = [x + T(L,H)] - (r + b) \tag{L-5}$$

satisfies the expression

$$X \geq T(L,H) + (r - b) - (r + b) = T(L,H) - 2b. \tag{L-6}$$

It follows from (L-6), the definition of T'(L,H) and the fact that b is at most T'(L,H) that X is at least the low base speed L. In view of the definition (L-5) of X, this means that

$$x + T(L,H) \geq r + b + L. \qquad \text{(L-7)}$$

Since the length of a blind speed interval cannot be greater than L and every blind speed interval is closed on the left and open on the right, it follows that x + T(L,H) and r + b cannot be in the same blind speed interval, and so o(x) must exceed o(r+b).

APPENDIX M. PROOF OF PROPOSITION 13

The proof of the proposition is based on four consequences of the assumption that L and H are rational numbers. First, L is the product of an integer L' and the greatest common whole divisor g(L,H) of L and H, and H is the product of g(L,H) and an integer H'. This means that the key of a blind speed interval I(i,j) is given by

$$d(i,j) = iL - jH = g(L,H)(iL' - jH'). \tag{M-1}$$

Second, the key set K(L,H) consists of the keys of the blind speed intervals that are subsets of D(0) (i.e., [0,T(L,H)). Third, if I(i,j) is a blind speed interval that is a subset of D(0) then the integers i and j are nonnegative, i is less than H', and j is less than L'. Fourth, by proposition 6, the key of a blind speed interval is of the form kg(L,H) where k is an integer that is at least -(L' - 1) and at most H' - 1.

It is now an easy matter to establish the validity of the proposition. The fourth fact implies that g(L,H) is a key. The second fact implies that this is the key of a blind speed interval I(i,j) that is a subset of D(0). The first fact implies that

$$1 = iL' - jH'. \tag{M-2}$$

Finally, the constraints

$$0 \leq i < H' \text{ and } 0 \leq j < L' \tag{M-3}$$

are consequences of the third fact.