integrated systems

Final Report

Contract No: F49620-90-C-0064

Adaptive Control of Nonlinear Flexible Systems

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ISI Report No. 5878-2

January 18, 1993

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(U)

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

DTIC ELECTE MAR 08 1993 S B D

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS: None
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Unclassified — DISTRIBUTION STATEMENT A: Approved for public release; Distribution Unlimited
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): ISI Report # 5878-2	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 93 0081
6a. NAME OF PERFORMING ORGANIZATION: Integrated Systems Inc. — 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: USAF, Air Force Office of Scientific Research
6c. ADDRESS (City, State, and ZIP Code): 3260 Jay St. Santa Clara, CA 95054	7b. ADDRESS (City, State, and ZIP Code): Building 410 Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR — 8b. OFFICE SYMBOL (If applicable): NA	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620-90-C-0064
8c. ADDRESS (City, State, and ZIP Code): Bolling AFB Washington, DC 20332	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61102F	2302	B1	

11. TITLE (Include Security Classification): Adaptive Control of Nonlinear Flexible Systems (U)

12. PERSONAL AUTHOR(S): Dr. Robert L. Kosut and Dr. M. Guntekin Kabuli

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Final	FROM 8/1/90 TO 31/7/92	1/18/92	55

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The objective is the development of adaptive control methods which can significantly improve closed-loop performance for a broad class of nonlinear flexible systems. Towards this end, a nonlinear controller, applicable to a broad class of nonlinear systems, was devised. The controller consists of a feedforward signal generator which incorporates a model estimate together with a global feedback linearizer. There is an inner feedback controller which modifies control action in accordance with output errors between the feedforward ideal output and the actual sensed output. The adaptive scheme studied uses measured data to update the model in the feedforward signal generator. It was discovered in many simulations that this two-level approach to adaptive feedback linearization can perform significantly better than feedback linearizers using an observer network.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☒ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Dr Spencer T. Wu	22b. TELEPHONE (Include Area Code): 202-767-6962 — 22c. OFFICE SYMBOL: NA

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

(U)

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1 Introduction

1.1 Research Objectives

The objective of this work is the development of adaptive control methods which can significantly improve closed-loop performance for a broad class of nonlinear flexible systems.

Towards this end, the goals were as follows:

1. Form a parametric model representing a broad class of nonlinear flexible systems.
2. Design a nonlinear controller based on the parametric model which provides desired closed-loop performance if the parameters were known.
3. Identify the parameters of the system using measured data and use these estimates in the controller.
4. Analyze the stability and performance properties of the complete adaptive system using the method of averaging.

We concentrated most of our effort on items 1 and 2 as these turned out to be the most difficult and challenging tasks. These must first be resolved in order to proceed to adaptation (items 3 and 4).

In general, the final adaptive system described above fits into the generic adaptive scheme depicted in Figure 1.

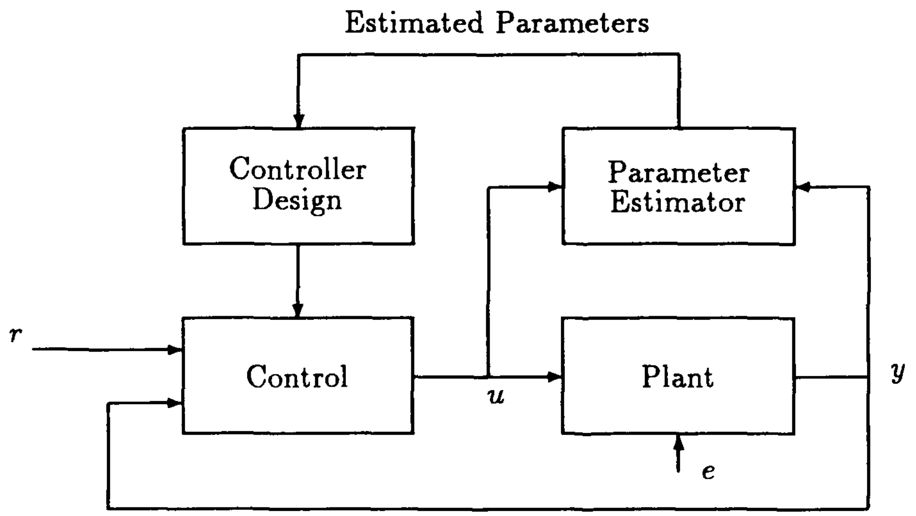

Figure 1: Adaptive control system with parameter estimator.

In this traditional adaptive control system, often referred to as the self-tuning-regulator (STR), [2], the identified model is usually selected out of a model set with unknown parameters. The controller is designed as if the parameter estimates were in fact the correct parameters for describing the plant. In the ideal case it is assumed that there exist parameters, which if known, would precisely account for the measured data. Even in this ideal case, the transient errors between the identified model and the true system can be so large as to completely disrupt performance. In the usual (non-ideal) case – the true system is not in the model set – both unacceptable transient or asymptotic behavior can occur *e.g.*, [1].

The above cited problems are well documented in the case of adaptive linear control, *i.e.*, when the adaptive parameters are held fixed the closed-loop system is linear. However, the case considered here is adaptive *nonlinear* control, *i.e.*, when the adaptive parameters are held fixed the closed-loop system is nonlinear. Naturally the same problems will arise in this case as in the adaptive linear case. Unfortunately many of the controller design and parameter estimator issues have not been developed sufficiently as yet. In consequence, the first step has been to resolve some of these issues.

1.2 Publications

See Appendices A and B .

1.3 Personnel

Dr. Robert Kosut and Dr. M. Güntekin Kabuli worked on the project during the reporting period.

1.4 Interactions

We have had several interactions with Prof. Crawley and students at MIT, Prof. Rock of Stanford, Prof. Sastry of Berkeley, and with Norm Coleman's group at ARDEC.

2 Status of Research Effort

This section is divided into subsections each corresponding to the stated goals in Section 1.1.

2.1 Modeling the Plant

2.1.1 Nonlinear Feedback Interconnection

A large class of nonlinear systems can be modeled as depicted in the following block diagram.

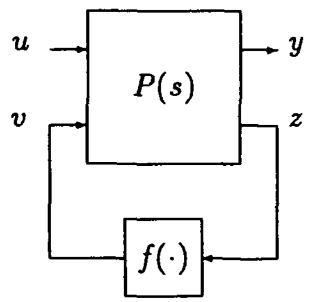

Figure 2: Plant model.

In this configuration $P(s)$ is a transfer function matrix representing the linear-time-invariant (LTI) part of the plant. The nonlinear part is represented by $f(\cdot)$, a memoryless nonlinear function, which appears as a feedback interconnection. The variable y is the sensed output and u is the control signal applied to the actuator inputs. The variables z and v are not usually measured, although some of them may be contained in y. This model is capable of representing all of the significant nonlinearities in large space structures.

The plant model can be described in both input-output as well as state space format. The input-output description is

$$\begin{aligned} \begin{bmatrix} y \\ z \end{bmatrix} &= P \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ v &= f(z) \end{aligned}$$

Assuming $P(s)$ is finite-dimensional, the state space format is:

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 v \\ y &= C_1 x + D_{11} u + D_{12} v \\ z &= C_2 x + D_{21} u + D_{22} v \end{aligned}$$

As a simple example of the model structure in Figure 2, consider the mechanical system depicted in Figure 3.

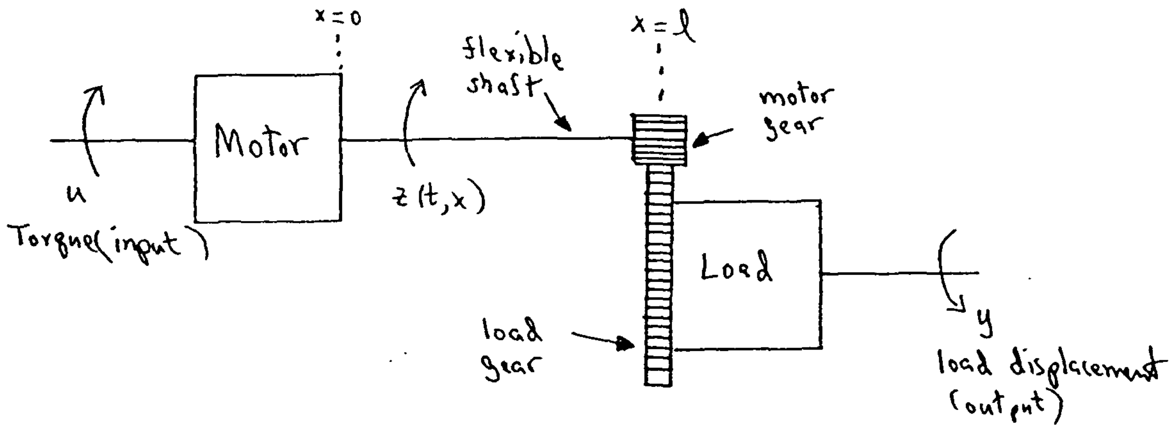

Figure 3: A flexible rotating system with backlash in the gear-train.

This system represents the case where tortional actuation is applied to a load through a flexible gear-train. The gearing is shown to occur at the end of the flexible member, although other combinations are certainly possible.

Neglecting any electronic dynamics, and assuming that the flexible rod is both uniform and damped, the motion of the rigid body and first tortional "mode" for small angular deflections can be approximated by the system of differential equations,

$$\begin{aligned} J_M \ddot{x}_1 &= u + D(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1) \\ J_G \ddot{x}_2 &= -v - D(\dot{x}_2 - \dot{x}_1) - K(x_2 - x_1) \\ J_L \ddot{x}_3 &= Nv \\ z &= x_2 - Nx_3 \\ v &= f(z) \\ y &= x_3 \end{aligned}$$

where u denotes the input applied torque, (x_1, x_2, x_3) are angular deflections as indicated in the figure, the load displacement $y = x_3$ is the sensed output, z is the relative gear angle, and $f(\cdot)$ is a memoryless nonlinearity arising from backlash in the gear train which depends on a parameter θ. The constants are defined as follows: J_M, J_G, and J_L are the motor, motor gear, and load inertias, respectively, N is the gear ratio which is greater than one, and D, K are the damping and stiffness, respectively, of the elastic rod.

In this example the transfer matrix $P(s)$ has the block structure described before with the scalar elements:

$$\begin{aligned} P_{11} &= \frac{Ds + K}{s^2[J_M J_G s^2 + (J_M + J_G)(Ds + K)]} \\ P_{12} &= \frac{N}{s^2 J_L} \\ P_{21} &= \frac{Ds + K}{s^2[J_M J_G s^2 + (J_M + J_G)(Ds + K)]} \\ P_{22} &= -\frac{N^2}{s^2 J_L} - \frac{J_M s^2 + Ds + K}{s^2[J_M J_G s^2 + (J_M + J_G)(Ds + K)]} \end{aligned}$$

A typical backlash nonlinearity is the cubic,

$$f(z) = bz^3 \tag{1}$$

where b is a constant.

Although this model is based on the rotating elastic system with a motor, it bears great similarities to nonlinear joint effects in LSS. Professor Crawley and colleagues from M.I.T. have identified models of this type. A distinguishing feature of this type of model is the large number of joints which need to be accounted. Referring to the model in Figure 2, this means that the dimension of the vectors (z, v) is also very large, that is, the nonlinear vector function $f(z)$ has a dimension essentially equal to the number of joints with significant nonlinearities. In addition, the linear part of the model, $P(s)$, which is a matrix of transfer functions, has a large number of states corresponding to the lightly damped flexible modes of the structure.

2.1.2 Model Uncertainty and Adaptation

In our initial efforts it is assumed that the LTI part $P(s)$ is known and the nonlinear function $f(z)$ depends on an unknown or uncertain parameters. For example, $f(z) = \theta z^3$ where θ is an unknown parameter. More generally,

$$f(z) = F(z)\theta$$

where $G(z)$ is a known matrix of nonlinear functions and θ is an unknown parameter vector.

The objective of adaptive control is to estimate θ from the input-output measurements (u, y) and then adapt the controller to the estimate. The first step then is to devise a nonlinear controller which depends on the parameter θ. The adaptive controller will be identical but with θ replaced by the adaptive estimate $\hat{\theta}$.

After forming the parameter adaptive algorithm or estimator, the plan is to analyze the complete adaptive system using the classical method of averaging analysis [3, 4]. Under previous AFOSR funding we refined the averaging method for application to *linear* flexible systems [1]. In a feasibility study for nonlinear systems, also supported by AFOSR, we used the Duffing oscillator to represent the nonlinear flexible system and adjusted a single parameter in the controller. The results, reported in [9], were very encouraging. We observed from simulations that the adaptive system performed very well, and achieved an almost optimum final tuning of the controller parameter. Moreover, we discovered that the resulting performance was directly predictable from the averaging analysis, and this, despite the fact that the system states passed in and out of chaotic and multi-periodic attractors.

Before all this can be accomplished, we must first be able to control the nonlinear plant when the parameters are known. This proved to be a major challenge, thus limiting our efforts with adaptation. As the more interesting results concern the controller design, we limit the remainder of the report to this part of the effort.

2.2 Nonlinear Controller Design via Feedback Linearization

In this section we describe our efforts to develop a nonlinear controller based on the nonlinear plant model just described. Our first assumption is that the plant parameters are known, *i.e.*, both $P(s)$ and $f(\cdot)$ are known. Methods for direct control design of nonlinear plants are few and far between without making additional assumptions. One such method, known as *feedback linearization* has received considerable attention in recent years, *e.g.*, [5]. This is our starting point.

The basic idea is to find a nonlinear feedback system which transform the system states to a new set of coordinates where the system appears linear.

To see how this approach works, suppose that all the elements of $P(s)$ are strictly proper. Then, the state-space description of the plant is becomes:

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 f(z) \\ y &= C_1 x \\ z &= C_2 x \end{aligned}$$

If $\det(C_1 B_1) \neq 0$, that is, the inverse of $C_1 B_1$ exists, then it is easy to show that the nonlinear control,

$$u = u_{NL} = (C_1 B_1)^{-1}[u_L - C_1 A x - C_1 B_2 f(z)]$$

yields

$$\dot{y} = u_L$$

An interpretation is that the nonlinear control u_{NL} globally linearizes the plant system from input u_L to output y. In this case the *feedback linearized* plant is a single integrator which is easy to control, *e.g.*, let

$$u_L = \alpha(r - y)$$

where r is a reference, usually a set-point or constant. Thus, under feedback linearization the output obeys the differential equation,

$$\dot{y} = \alpha(r - y)$$

Thus, for any positive constant α, if r is a constant, $y(t) \to r$ exponentially fast as $t \to \infty$. Moreover, for any bounded r, y is also bounded.

There are some difficulties with using this method:

1. *Stability*

   Although the y-system is stable, the remaining states in x may not be bounded.

2. *Implementation*

   The nonlinear feedback u_{NL} depends on all the system states, x, which in general, are not available.

3. *Robustness*

Feedback linearization requires precise knowledge of the plant, *i.e.*,knowledge of $P(s)$ and $f(\cdot)$.

To illustrate the above issues we make the further simplification that $y = z$, *i.e.*, the nonlinear function acts directly on the measured output. Furthermore, the system states can be partitioned so that

$$x = \begin{bmatrix} y \\ \zeta \end{bmatrix}$$

In this case the state description of the plant becomes,

$$\begin{aligned} \dot{y} &= A_{11}y + A_{12}\zeta + B_{11}u + B_{12}f(y) \\ \dot{\zeta} &= A_{21}y + A_{22}\zeta + B_{21}u + B_{22}f(y) \end{aligned}$$

The feedback linearizing control now becomes,

$$u_{NL} = B_{11}^{-1}[\alpha(r-y) - A_{11}y - A_{12}\zeta - B_{21}f(y)]$$

The closed-loop system is then,

$$\begin{aligned} \dot{y} &= \alpha(r-y) \\ \dot{\zeta} &= A_{21}y + B_{21}B_{11}^{-1}[\alpha(r-y) - A_{11}y - B_{21}f(y)] + B_{22}f(y) + [A_{22} - B_{21}B_{11}^{-1}A_{12}]\zeta \end{aligned}$$

2.2.1 Stability

In our example, it is obvious that bounded r implies bounded y. Since the y-system is decoupled from the ζ-system, ζ will also be bounded if and only if the eigenvalues of $A_{22} - B_{21}B_{11}^{-1}A_{12}$ all have negative real parts. Although we have made many simplifying assumptions, this test can be generalized [5].

2.2.2 Implementation

Since u_{NL} depends on states which cannot be measured, an obvious solution is to reconstruct these states from measurements with an observer network. Thus, an implementable feedback linearizing control is,

$$\begin{aligned} u_{NL} &= (C_1B_1)^{-1}[u_L - C_1A\hat{x} - C_1B_2f(z)] \\ \dot{\hat{x}} &= A\hat{x} + B_1u + B_2f(y) + L(y - C_1\hat{x}) \end{aligned}$$

where L is chosen so that A_LC is Hurwitz. To see that the resulting closed- loop system is stable, define the state error

$$\tilde{x} \stackrel{\text{def}}{=} x - \hat{x}$$

Then,

$$\begin{aligned} \dot{y} &= \alpha(r-y) + CA\tilde{x} \\ \dot{\tilde{x}} &= (A - LC)\tilde{x} \end{aligned}$$

Since $\tilde{x}(t) \to 0$ exponentially fast, the system is stable.

2.2.3 Robustness

Robustness is more difficult to determine. For example, suppose that $\hat{f}(z)$ is an estimate of $f(z)$. The feedback linearizing control now becomes,

$$\begin{aligned} u_{NL} &= (C_1B_1)^{-1}[u_L - C_1A\hat{x} - C_1B_2\hat{f}(z)] \\ \dot{\hat{x}} &= A\hat{x} + B_1u + B_2\hat{f}(y) + L(y - C_1\hat{x}) \end{aligned}$$

where as before L is chosen so that A_LC is Hurwitz. Again, it is convenient to introduce the error variables,

$$\begin{aligned} \tilde{x} &\stackrel{\text{def}}{=} x - \hat{x} \\ \tilde{y} &\stackrel{\text{def}}{=} y - \hat{y} \end{aligned}$$

where $\hat{y}$ is the ideal output which satisfies,

$$\dot{\hat{y}} = \alpha(r - \hat{y})$$

Then, the closed-loop system is:

$$\begin{aligned} \dot{\tilde{x}} &= (A - LC)\tilde{x} + B_2\tilde{f} \\ \dot{\tilde{y}} &= -\alpha\tilde{y} + CA\tilde{x} + CB_2\tilde{f} \\ \tilde{f} &= f(y) - \hat{f}(y), \quad y = \hat{y} + \tilde{y} \end{aligned}$$

When $f = \hat{f}$ we return to the previous case.

Although theoretical results, such as the Total Stability Theorem [1], indicate that the above system has robustness properties, we have discovered from simulations that these systems can be extremely sensitive to even small perturbations (see Appendix A) . This has motivated us to find another means to apply feedback linearization. This effort is described next.

2.2.4 Effect of Feedforward Control

The block diagram below shows a controller configuration which consists of a feedforward signal generator and an inner feedback loop.

Observe that the feedforward controller copies the plant using the estimate $\hat{f}$ in place of the unknown nonlinear function f. The feedforward block produces two signals, $\hat{y}$ and $\hat{u}$. To generate these we feedback linearize the "model" in the feedforward signal generator. Since this is a model, all states are available and feedback linearization is ideal. The signal $\hat{u}$ then tries to undo the effect of the unknown nonlinearity f in the actual plant. Any errors will appear in the error signal $\hat{y} - y$ which is acted upon by the feedback controller K in the inner loop. This controller, hopefully acting on small errors, can most likely be linear and independent of errors between f and $\hat{f}$.

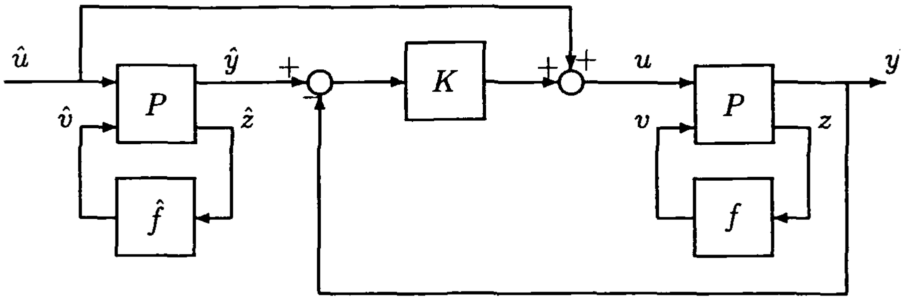

Figure 4: Nonlinear feedforward controller.

Using the previous example, the control is now:

$$\begin{aligned} u &= \hat{u} + \tilde{u} \\ \hat{u} &= (C_1B_1)^{-1}[\alpha(r - \hat{y}) - C_1A\hat{x} - C_1B_2\hat{f}(z)] \\ \dot{\hat{x}} &= A\hat{x} + B_1\hat{u} + B_2\hat{f}(\hat{y}) \\ \dot{\hat{y}} &= \alpha(r - \hat{y}) \end{aligned}$$

To examine the closed-loop system, it is again convenient to introduce the error variables,

$$\begin{aligned} \tilde{x} &\stackrel{\text{def}}{=} x - \hat{x} \\ \tilde{y} &\stackrel{\text{def}}{=} y - \hat{y} \end{aligned}$$

Here $\hat{x}$ is the state of the feedforward signal generator rather than the estimator state. Hence, $\hat{x}$ is bounded. As before $\hat{y}$ is the ideal output. Now the closed-loop system is:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B_1\tilde{u} + B_2\tilde{f} \\ \tilde{y} &= C_1\tilde{x} \\ \tilde{f} &= f(y) - \hat{f}(\hat{y}), \quad y = \hat{y} + \tilde{y} \end{aligned}$$

This system is very similar to that previously developed using feedback linearization with an observer. However, we do not encounter the same difficulties with this system in the simulations referred to in Appendix A .

2.3 Towards a Nonlinear Control Design Based on Measured Variables

Within the context of finite-dimensional linear time-invariant (LTI) feedback interconnections and LTI plants, full-order estimated state-feedback based design approaches are widely used in control design problems. After suitable augmentations, the design problem is reduced to the stabilization of $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ (can be taken as strictly proper, without any loss of generality) subject to a particular cost criterion (e.g., $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$). The analytical and/or numerical solutions to the associated optimization problem yields a state-feedback gain K and an output-injection gain L , where the matrices $(A-BK)$ and $(A-LC)$ are both strictly Hurwitz. The resulting controller is given by $\begin{bmatrix} A-BK-LC & L \\ -K & 0 \end{bmatrix}$. The relative ease of solutions associated with such design problems stems from the inherent decomposition into two sub-problems which can be dealt with separately. The estimation of the state and the control law based on the ideal state are solved separately, and then the estimated state is used instead of the true plant states in the control law. This two-step approach is referred to as the separation principle.

Before proceeding with nonlinear control design methods and why the separation principle fails in general, it is useful to observe the inherent properties that make the separation principle work in the LTI setting. Consider the plant $P : u \mapsto y$,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with a stabilizable pair (A, B) and a detectable pair (C, A) . If the plant state x were available, a control of the form $u = -Kx$ would solve the stabilization problem. Since x is not available, an estimate $\widehat{x}$ is constructed using (u, y) or perturbed versions thereof. Consider the estimator $E : (u_e, y_e) \mapsto \widehat{x}$,

$$\dot{\widehat{x}} = (A - LC)\widehat{x} + Bu_e + Ly_e \quad ,$$

where

$$\begin{aligned} u_e + d_u &= u \\ y + d_y &= y_e \end{aligned}$$

where the exogenous inputs d_u and d_y denote the unmeasured but bounded actuator and sensor disturbances, respectively. Letting $e_x := x - \widehat{x}$, and using the estimates in the control, i.e., $u_e = -K\widehat{x}$, the closed-loop system is described by

$$\begin{aligned} \dot{x} &= (A - BK)x + BKe_x + Bd_u \\ \dot{e}_x &= (A - LC)e_x + Bd_u - Ld_y \quad . \end{aligned}$$

The properties that make the separation principle work, can be listed as follows:

1. For bounded (d_u, d_y) , e_x is bounded. When $(d_u, d_y) = 0$, $e_x \to 0$.

2. For bounded estimation error e_x , x is bounded. As $e_x \to 0$, $x \to 0$.

Due to possibly persistently exciting disturbance (d_u, d_y) ; e_x is not expected to go to zero. Hence, the state-feedback (in terms of x) must be ***robust*** to perturbations e_x which does not necessarily correspond to initial conditions mismatches in P and E .

For the sake of argument, suppose that one indeed has an asymptotic observer and a "globally" asymptotically stabilizing full-state feedback law. Intuitively, for "small" errors in the state-estimate, the separation principle based control design will be stabilizing. Thus, after waiting "sufficiently long" for reliably reconstructing the states, one can use the estimated states instead of the true states in the control. Apart from the nontrivial assumption that such an asymptotic observer is available, another pitfall in such an argument is that one may not be able to wait without a nominal bounded-input bounded-output (BIBO)-stabilizing loop, since the plant may blow up before estimation errors are sufficiently small. (e.g., $\dot{x} = x^2 + u$, $u = 0$, $x(0) = x_0 > 0$, $x(t) = \frac{x_0}{1-tx_0}$) . The existence of such a nominal stabilizing loop is the same as requiring a control that robustly BIBO-stabilizes the plant, although the desired performance may not be achieved.

Imposing the algebraic nonlinearities in the ordinary differential equations to be globally Lipschitz in order to extend the appealing properties of linear vector fields, may be quite restrictive. Nonlinearities in one coordinate system may be Lipschitz, whereas in an another coordinate system they may not be. *Consider*

$$\begin{aligned} \dot{x_1} &= x_2 + f(x_1) \\ \dot{x_2} &= x_3 \\ \dot{x_3} &= u \quad . \end{aligned}$$

Under the nonlinear transformation

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_2 + f(x_1) \\ \xi_3 &= x_3 + (x_2 + f(x_1))f^{(1)}(x_1) \end{aligned}$$

the input to state map can be expressed in terms of

$$\begin{aligned} \dot{\xi_1} &= \xi_2 \\ \dot{\xi_2} &= \xi_3 \\ \dot{\xi_3} &= u + \underbrace{\xi_2^2 f^{(2)}(\xi_1) + \xi_3 f^{(1)}(\xi_1)}_{\alpha(\xi)} \quad . \end{aligned}$$

Even if $f(\cdot)$ is Lipschitz, $\alpha(\cdot)$ is not.

For a given nonlinear plant $\mathcal{P} : u \mapsto y$, consider the tracking problem: for a particular class of reference signals r , determine a control

$$u = \mathcal{K}(r, y)$$

such that $y \to r$ and for a reasonable class of bounded disturbances, under the perturbed control

$$u = d_u + \mathcal{K}(r, y + d_y)$$

the closed-loop signals remain bounded. In other words, the closed-loop (see Figure 5) is BIBO-stable (in the sense of a particular extended space) and achieves the desired tracking when disturbances are zero.

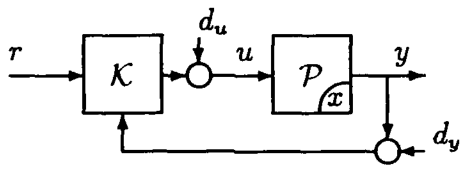

Figure 5: Desired dynamic controller $\mathcal{K}$

We will focus on a particular class of single-input nonlinear plants which admit a finite-dimensional state-space description of the form

$$\dot{x} = f(x) + g(x)u$$

for which the State-Space Exact Linearization Problem [5] is solvable over $\mathbb{R}^n$, i.e., controllability and involutivity conditions hold in $\mathbb{R}^n$. For such plants, when states are available, the origin can be rendered globally asymptotically stable. Moreover, when x is available for feedback, the asymptotic tracking problem can be solved for any output that does not introduce a nonminimum-phase zero dynamics. In the case where the output y to track r does introduce a nonminimum-phase zero dynamics, one may be required to introduce dynamic augmentation prior to full-state feedback. Such strong global results do rely on perfect state measurement and render controller candidates as shown in Figure 6 .

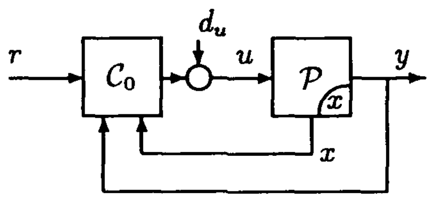

Figure 6: Solvable design problem, dynamic controller $\mathcal{C}_0$; $d_x = 0$, $d_y = 0$

Taking into account the control law in Figure 5 and the fact that one typically encounters number of sensors much less than the number of states, one might dismiss the plant state based control laws on the grounds that they are not implementable. As it is motivated by the separation principle based control design, for classes of plants that admit asymptotic observers or in the case of bounded state-estimation errors, the stability properties of the control law with $y = x$ in Figure 5 is of major importance (see also Figure 8).

Motivated by the separation principle based control design, a design approach can be taken as follows:

- **Assumption** : Imposing stability properties on (u, y) is equivalent to imposing them on x . The plant $\mathcal{P} : u \mapsto y$ does not have any "hidden modes". The input output pair (u, y) completely characterizes the internal dynamics.

- **Problem I** : An estimator that allows the decoupling of the design problem (see Figure 7). Recall from the motivating LTI setting, that the role of the estimator is to guarantee that for all bounded d_u, bounded d_y and possibly unbounded u_e , the error $(x - \hat{x})$ is bounded. Moreover, when $d_u = 0$ and $d_y = 0$, $e_x \to 0$. In other words, for the interconnection in Figure 7 , the error e_x must be uncontrollable from u and the bound on e_x must depend only on the bound on the exogenous variable d_y . This property is the crux of the separation principle, since it simplifies the design problem to Problem II .

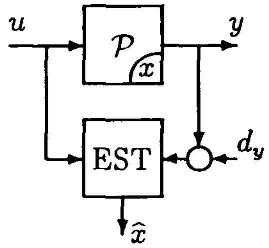

Figure 7: Problem I; estimator with $(x - \hat{x}) \in L_\infty$ for all u in $L_{\infty,e}$ and d_y in L_∞

- **Problem II** : A stabilizing state-feedback law. For bounded d_x, d_u and d_y, determine a controller $\mathcal{C}_1$ for which the closed-loop signals remain bounded. Moreover, when $d_u = 0$ and $d_y = 0$, as $d_x \to 0$, $y \to r$. Note that this step is nontrivial since the controller $\mathcal{C}_0$ in Figure 6 is typically based on $d_x = 0$ and $d_y = 0$. The local stability properties of the design in Figure 6 may not suffice for the specified bound on d_x obtained from Problem I. With a slight abuse of notation, from now on we will refer to $\mathcal{C}_1$ in Figure 8 as a robust controller, since its design does take into account the predetermined bounds on d_x and d_y . The abuse is due to the fact that in the LTI setting, robustness is attributed to closed-loop signal dependent perturbations, since exogenous additive disturbances cannot drive a stable LTI loop unstable. Since local and global stability results merge in the LTI setting and global stability results are too restrictive in the nonlinear setting, perhaps a better suited description for the controller $\mathcal{C}_1$ in Figure 8 is *semi-global* ; i.e., for the given bounds on d_u,d_x and d_y the closed-loop signals remain bounded.

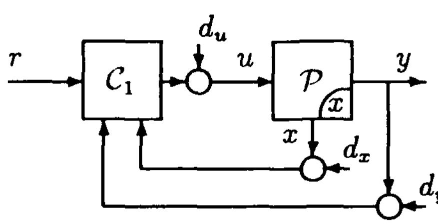

Figure 8: Problem II; dynamic controller $\mathcal{C}_1$, possibly different from $\mathcal{C}_0$

• **Merging Solutions to Problems I and II :** Recall that the solution to Problem I guarantees that bounded deviations about possibly unbounded input output pairs of $\mathcal{P}$ result in bounded e_x . Hence one can incorporate Figure 7 in Figure 8 as the bounded disturbance d_x generator to obtain Figure 9 .

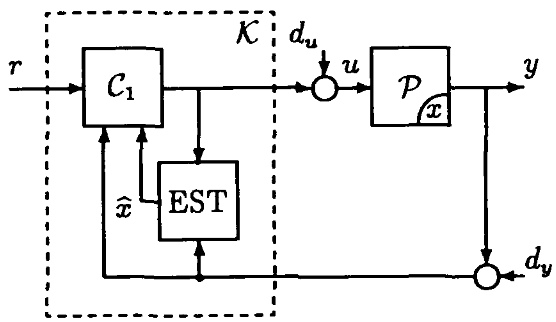

Figure 9: Candidate controller $\mathcal{K}$

Note that, in the case that the estimator in Problem I satisfies the property that $e_x \to 0$ for $d_u = 0$ and $d_y = 0$, that is when x is asymptotically reconstructed, the choice of the robust controller $\mathcal{C}_1$ in Problem II guarantees that the closed-loop signals remain bounded due to bounded d_x ; in addition, the nominal design constraint on $\mathcal{C}_1$ ensures that $y \to r$ as $d_x \to 0$.

As a preliminary step towards control design based on state-estimates, the following two studies show the importance of the choice of the nominal globally stabilizing (or tracking) control law based on perfect state measurements.

In the first study, set-point asymptotic tracking problem is solved for an output that exhibits nonminimum-phase zero dynamics. Using dynamic augmentation and then applying state-feedback asymptotic tracking problem is solved. In other words, the controller $\mathcal{C}_0$ in Figure 6 is obtained. The derivations emphasize that the standard Lipschitz constraints, for ease of Lyapunov based derivations, on the resulting vector fields after possible changes in coordinates is too restrictive.

In the second study, stability properties of the origin is investigated when the feedback law is based on estimates from an asymptotic observer. It is shown that the globally stabilizing exact feedback linearization based controller $\mathcal{C}_0$ in Figure 6 is no longer globally stabilizing in Figure 7 . A globally stabilizing controller $\mathcal{C}_1$ that solves the Problem II is constructed. Using the asymptotic observer, the controller $\mathcal{K}$ in Figure 9 is obtained.

2.3.1 Example 1: Tracking and Nonminimum-Phase Zero Dynamics

In order to illustrate a strictly causal plant with a one-dimensional possibly nonminimum-phase zero dynamics, consider the following two-state u to y map.

$$\begin{aligned} \dot{x}_1 &= x_2 + f(x_1) \\ \dot{x}_2 &= 4(x_2 - x_1) + u \\ y &= x_2 - x_1 \end{aligned}$$

Note that when $f = 0$, the transfer function from u to y is $\frac{(s-1)}{(s-2)^2}$.

It is easy to see that a tracking control based on the coordinate transformation using derivatives of y will not be stabilizing. Note that

$$\dot{y} = 4y + u - x_2 - f(x_1) \quad .$$

For $\dot{r} = 0$,

$$u = -4y + x_2 + f(x_1) + \alpha(r - y)$$

results in

$$\dot{(y - r)} = -\alpha(y - r) \quad ;$$

hence for any $\alpha > 0$, $y \to r$. However, the asymptotic tracking constraint

$$x_2 = x_1 + y(0)e^{-\alpha t}$$

renders

$$\dot{x}_1 = x_1 + f(x_1) + r + (y(0) - r)e^{-\alpha t} \quad .$$

Take for example, $r = 0,\ f(\cdot) = (\cdot)^3$. Clearly, in the limit

$$\dot{x}_1 = x_1 + x_1^3 \quad ,$$

and the states are unbounded.

When f is zero, it is well known that step inputs can be asymptotically tracked if and only if plant and/or the compensator have at least one pole at zero. Since $\frac{(s-1)}{(s-2)^2}$ does not have a pole at zero, the standard LTI design procedure would involve augmentation with an integrator. Proceed with the same augmentation for the nonlinear plant at hand (see Figure 10).

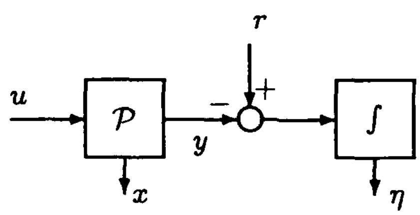

Figure 10: Augmented plant

If x were available, since η is part of the controller, the augmented plant states are available. If one can find a relative degree three output, i.e., solve the state-feedback linearization problem, then $\dot{\eta} \rightarrow 0$ would imply that $y \rightarrow r$. This requires that the augmented plant dynamics

$$\begin{aligned} \dot{\eta} &= r - x_2 + x_1 \\ \dot{x}_1 &= x_2 + f(x_1) \\ \dot{x}_2 &= 4(x_2 - x_1) + u \end{aligned}$$

satisfies the controllability and involutivity conditions. Both conditions are satisfied provided that

$$f^{(1)}(x_1) \neq -1$$

which we will assume for global results. It is interesting to note that this condition is in fact a generalization to the hidden mode concept in the LTI setting, i.e., if $f(x_1) = -x_1$, the linear plant from u to y would have a hidden unstable mode.

Now consider the relative degree three output

$$z = \eta + x_1$$

motivated by the $f = 0$ case. Note that

$$\begin{aligned} z^{(1)} &= r + x_1 + f(x_1) \\ z^{(2)} &= \underbrace{(1 + f^{(1)}(x_1))}_{\beta(x_1)}(x_2 + f(x_1)) \end{aligned}$$

and recall that the (global) controllability condition is equivalent to $\beta(x_1) \neq 0$ for all $x_1 \in \mathbb{R}$. Hence, under the coordinate transformation

$$\xi = \begin{bmatrix} z \\ z^{(1)} \\ z^{(2)} \end{bmatrix}$$

the augmented plant dynamics can be put into the controllable canonical form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \underbrace{f^{(2)} \cdot (x_2 + f)^2 + \beta(x_1)(u + 4(x_2 - x_1) + f^{(1)} \cdot (x_2 + f))}_{\alpha(x) + \beta(x_1)u} \end{aligned}$$

where dependencies on x_1 are suppressed in the f terms, for the sake of brevity. For any ψ_1, ψ_2 and ψ_3 such that the polynomial

$$s^3 + \psi_1 s^2 + \psi_2 s + \psi_3$$

is strictly Hurwitz, the control u determined by

$$\alpha(x) + \beta(x_1)u = \psi_3\xi_1 - \psi_2\xi_2 - \psi_1\xi_3$$

renders $\xi = 0$ globally asymptotically stable (recall that the transformation to ξ coordinates is one-to-one and onto) . Hence

$$\begin{aligned} 0 &= r + x_1 + f(x_1) \\ 0 &= \beta(x_1)(x_2 + f(x_1)) \end{aligned}$$

and $\beta(x_1) \neq 0$ imply that

$$r + x_1 - x_2 \rightarrow 0$$

i.e., $y \rightarrow r$. Note that the above exact cancellation based scheme is not the only way to stabilize the augmented plant in the ξ coordinates. In fact, as we will see later on, such an exact cancellation may make the global nature of the nominal result not robust, i.e., perturbations in the states (other than initial conditions) may render the result only locally stable. Hence, it is crucial that one studies the sensitivity of the full-state based control design and modify it if possible so that for the class of anticipated disturbances, the local nature of the control law is satisfactory.

2.3.2 Making the Separation Principle Work

Consider the plant

$$\begin{aligned} \dot{x} &= Ax + Bu + F(y) \\ y &= Cx \end{aligned}$$

with (A, B, C) a minimal triple. Suppose that there exists a nonlinear coordinate transformation

$$\xi = \Phi(x)$$

such that the plant can be equivalently represented as

$$\begin{aligned} \dot{\xi} &= A\xi + B(u + \alpha(\xi)) \\ y &= C\Phi^{-1}(\xi) \end{aligned}$$

For such a system, asymptotic state construction is easier in the x-coordinates, since

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + F(y) + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}$$

yields an estimation error $e_x := x - \hat{x}$, where

$$\dot{e_x} = (A - LC)e_x \quad .$$

The stabilizing control is easier to express in the ξ-coordinates since

$$u = -K\xi - \alpha(\xi)$$

renders $\xi = 0$ globally asymptotically stable since the above control yields

$$\dot{\xi} = (A - BK)\xi \quad .$$

The heuristic step of replacing ξ by $\Phi(\hat{x})$ can only guarantee *local* results. For "sufficiently" small e_x , the estimated-state feedback will be locally stabilizing. The crucial points are how large the basin-of-attraction is and if it is possible to enlarge it to yield a satisfactory design.

Let $\hat{\xi} := \Phi(\hat{x})$. The closed-loop dynamics under the feedback law

$$u = -K\hat{\xi} - \alpha(\xi)$$

can be expressed as

$$\begin{aligned} \dot{\hat{\xi}} &= (A - BK)\hat{\xi} + \underbrace{\left.\frac{\partial \Phi}{\partial \hat{x}}\right|_{\hat{x}=\Phi^{-1}(\hat{\xi})}}_{M(\hat{\xi})} \underbrace{\left(LCe_x + F(C\Phi^{-1}(\hat{\xi}) + e_x)) - F(C\Phi^{-1}(\hat{\xi})) \right)}_{G(\hat{\xi}, e_x)} \\ \dot{e_x} &= (A - LC)e_x \end{aligned}$$

Even if F is Lipschitz (hence $\|G(\hat{\xi}, e_x)\|$ depends only on $\|e_x\|$), $M(\hat{\xi})$ need not be Lipschitz. Hence establishing and modifying the basin-of-attraction are nontrivial design tasks. However, the crucial observation is that the stabilization problem is in fact the robustness of the full-state feedback law subject to L_2 disturbances.

The following example illustrates the need for a robust state-feedback law and the sensitivity of the exact-linearization based control law.

2.3.3 Example 2

Consider the following one state plant model $\mathcal{P} : u \mapsto y$

$$\dot{y} = y^3 + u$$

where the goal is to render $y = 0$ globally asymptotically stable. The perfect state (namely the plant output) is not available due to an unknown output disturbance d, where $d \in L_2$ and $\dot{d} \in L_\infty$. The fact that $d \to 0$ motivates the control law candidate

$$u = -\hat{y}^3 - \alpha\hat{y}$$

with $\alpha > 0$. We show that such an exact cancellation based control law yields only locally stable results, whereas choosing a nonlinear control law for the linearized plant significantly improves the results.

The motivation behind the choice of d above is possible asymptotic estimation error subject to initial condition mismatches. The over-simplicity of this one-state plant should not be misleading, since even if $y(0)$ is known perfectly, the estimator built for plants of the form $\dot{x} = Ax+Bu+F(y)$ with y available, will not necessarily guarantee zero state-estimation error. In fact, if one were to build an estimator

$$\dot{\hat{y}} = u + y^3 + (y - \hat{y})$$

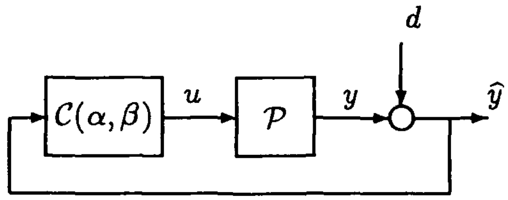

Figure 11: Control under perturbed state measurement

and base the control law on $\hat{y}$, by setting $d = \hat{y} - y$, one obtains

$$\dot{d} = -d \tag{2}$$

which implies the L_2 disturbance in the plant states (see Figure 11).

Consider the controller of the form

$$\mathcal{C}(\alpha,\beta) \ : \ \hat{y} \mapsto u \quad , \quad u = -\alpha\hat{y} - \beta\hat{y}^3 \quad .$$

for $\alpha > 0$ and $\beta \geq 1$ and the disturbance d as in (2). The dynamics of the resulting closed-loop system in Figure 11 is described by

$$\dot{y} = -\alpha(y+d) + y^3 - \beta(y+d)^3 \tag{3}$$
$$\dot{d} = -d \tag{2}$$

Without getting into extensive simulations to obtain the phase-portraits for different α and β values, it is useful to derive some qualitative properties of (3) and (2) analytically.

For $\alpha > 0$ and $\beta \geq 1$, $(y,d) = (0,0)$ is the only equilibrium point. The Jacobian at $(0,0)$ has eigenvalues $-\alpha$ and -1 , with associated eigenvectors $(y,0)$ and $(y, \frac{1-\alpha}{\alpha}y)$, respectively. Since $d(t)$ cannot change sign, the phase portrait can be decomposed into three invariant sets: $d > 0$, $d = 0$ and $d < 0$; i.e., the trajectories remain exclusively in the set that the initial condition belongs to. There is an odd symmetry in the flows since (y,d) and $(-y,-d)$ satisfy the same differential equations. Hence, it suffices to consider the flow for $d > 0$ since $d = 0$ case is asymptotically stable for all $y(0)$.

The purpose of the exercise is the choice of the nonlinear control determined by β ; although the basin-of-attraction will be affected by the choice of α , the qualitative properties for $\beta = 1$ and $\beta > 1$ are generic. In the following, we will consider the case where $\alpha = 1$, (i.e., the eigenvectors are colinear). Let the Lyapunov function candidate be $V = \frac{1}{2}(y^2 + d^2)$. Evaluating $\dot{V}$ along the solutions of (3) and (2) , we obtain

$$\dot{V} = (1-\beta)y^4 - y^2 - (1+3\beta y^2)d^2 - yd(1+3\beta y^2 + \beta d^2) \quad . \tag{4}$$

From (4), one can easily deduce the following:

- $\beta > 1$ $\quad \dot{V}$ is eventually negative, due to the dominating first term in (4) . Hence, y remains bounded for any $(y(0), d(0))$. Once d gets sufficiently small , $y \to 0$.

- $\beta = 1$ The first term in (4) drops. The second and third are nonnegative. The last term changes sign according to (y, d) in the first or second quadrant. Clearly, for any $(y(0), d(0))$ in the first quadrant, the volume V decreases, however, the trajectories might continue into the second quadrant, where the last term becomes positive. We now show that, finite escape-time behavior is indeed expected for sufficiently large $d(0)$. For this specific case, note that (3) becomes

$$\dot{y} \;=\; -y - d_0e^{-t} - 3y^2d_0e^{-t} - 3yd_0^2e^{-2t} - d_0^3e^{-3t} \quad . \tag{5}$$

Considering the quadratic term above, take the approximate solution

$$y(t) \approx \frac{a}{(t-a)} \tag{6}$$

about the approximate escape-time $a > 0$. Substituting (6) in (5), we obtain

$$\frac{-a}{(t-a)^2} \approx \frac{-a}{(t-a)} - d_0e^{-t} - 3\frac{a^2}{(t-a)^2}d_0e^{-t} - 3\frac{a}{(t-a)}d_0^2e^{-2t} - d_0^3e^{-3t} \quad .$$

Multiplying both sides by $(t-a)^2$ and evaluating at a, we obtain

$$ae^{-a} = \frac{1}{3d_0} \quad .$$

The above equation predicts possible finite-escape time for $d_0 > \frac{1}{3e}$. In fact for $y(0) = -1$ and $d_0 > 0.8$, simulations indicate finite escape-time behavior in the second quadrant.

3 References

[1] B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986.

[2] K.J. Åström and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989.

[3] N.N. Bogoliuboff and Y.A. Mitropolskii, *Asymptotic methods in the theory of Nonlinear Oscillators*, Gordon and Breach, New York, 1961.

[4] J.K. Hale, *Ordinary Differential Equations*, Kreiger, Molaban, FL, 1980; originally published (1969), Wiley (Interscience), New York.

[5] A. Isidori, *Nonlinear Control Systems*, 2nd ed., Springer-Verlag, Berlin, FRG, 1989.

[6] A. Isidori and C. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. on Aut. Contr.*, Vol. AC-32, No. 2, pp.131-140, Feb. 1990.

[7] I. Kanellakopoulos and P. V. Kokotovic, "Observer-based Adaptive Control of Nonlinear Systems Under Matching Conditions," *Proceedings of the American Control Conference*, pp. 549-555, San Diego, CA, May 1990.

[8] I. Kanellakopoulos, P. V. Kokotovic and A. S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241-1253, November 1991.

[9] R.L. Kosut, D. Meldrum, and G.F. Franklin, "Adaptive control of a nonlinear oscillating system," *Proc. 1989 ACC*, Pittsburgh, PA, June 21-23, 1989.

Appendix A

Appendix A includes a regular paper presented at the 31st IEEE Conference on Decision and Control, held in Tucson, Arizona on 16–18 December 1992 and published in the Proceedings of the 31st IEEE CDC pp. 251–256 . Appendix A is self-contained, hence, all section, equation and figure cross-referencing within these pages pertain only to Appendix A . An abridged version of the preliminary stages of this work has also been published in the Proceedings of the 1992 Yale Workshop on Adaptive and Learning Systems.

1992 CDC

Adaptive Feedback Linearization: Implementability and Robustness

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Abstract

The robustness to parameter mismatch in feedback linearization based nonlinear tracking systems is investigated. The certainty-equivalence principle gives rise to four possible feedback configurations: One is the widely used case, where the states are assumed to be available; two others are observer-based; the last one is a model-follower based on a feedforward/feedback implementation using a signal generator. It is noted that the unacceptable transient behaviour of the adaptive tracking scheme is closely related to the sensitivity of the underlying certainty-equivalence based control, which is analyzed through a perturbation approach. Simulations are performed on an example to illustrate the points.

*Research supported by AFOSR, Directorate of Aerospace Sciences under contract *F49620-90-C-0064* and ARO, Engineering Sciences Division under contract DAAL03-91-C-0011.

1 Introduction

Consider the following nonlinear tracking problem:

- For a given nonlinear plant with state-space description

$$\begin{aligned} \dot{x} &= f(x,u,\theta) \\ \dot{\theta} &= 0 \\ y &= h(x,u,\theta) \quad , \end{aligned}$$

  and a given constant reference signal r , determine a feedback law $u = \mathcal{C}(y,r)$ such that the closed-loop system is internally stable and the output y asymptotically tracks r .

The parameter θ is clearly uncontrollable. The desired control law has to be expressed in terms of the measured plant variables ; such control laws will be classified as *implementable* . The goal is in fact a robust tracking controller, since performance is achieved despite the uncertainty in the uncontrollable variable θ .

When θ is exactly known, the tracking problem has been solved for special classes of plants that can be rendered linear under algebraic state-feedback and change of coordinates (e.g., [Isi1] and Section II of [Sas1]) .

When θ is unknown, the widely used approach to design an adaptive tracking scheme is the certainty-equivalence principle: the tracking problem is solved as if θ is known and then the above control is based on the estimate $\hat{\theta}$ (see [Sas1] and the references therein) .

There are two problems with the above approaches. First, the states must all be sensed variables, which is often not the case; hence the controllers are not implementable in our sense. Secondly, certainty-equivalence based designs are not guaranteed to be robust, and hence, the adaptive tracking performance may be unacceptable due to poor transient behaviour, a phenomenon well known in the adaptive linear control (e.g., [And1]) .

A solution to the implementability problem is to use an estimate of the state in the controller. Unfortunately, there is no dual theory of observer design for feedback linearizable systems. Adaptive controllers based on observers have been obtained in the literature for specific classes of nonlinear plants (e.g., under output-matching conditions in [Kan1]) .

Addressing the second problem requires designing a robust nonlinear tracking controller *before* any adaptation is introduced. Hence, prior information on the plant description and the parameter uncertainty are crucial to improve the transient behaviour.

In this paper we consider a special class of nonlinear plants. A direct interpretation of the certainty-equivalence principle gives rise to four possible feedback configurations.

The first one is the widely used case, where the states are assumed to be available. Despite the underlying implementability problem, we address the issue of sensitivity through a perturbation analysis of the approach. Simulations are performed on an example to illustrate the points. Based on this tracking scheme, we derive an adaptive tracking controller.

Performance of the tracking scheme is illustrated in an example. Although the responses are bounded, the transients are very large. This unacceptable transient behaviour is closely related to the sensitivity of the underlying (non-adaptive frozen parameter $\widehat{\theta}$) certainty-equivalence based controller.

The other three approaches are all implementable. The state availability assumption is dropped at the expense of global results. Two of the schemes use the state-estimate in two different coordinates. The third is a new model-follower scheme. An example system is used throughout to illustrate the system behaviour using the four schemes.

2 Plant Description

We consider a class of single-input single-output nonlinear plants which can be exactly input-output linearized under algebraic state-feedback and change of coordinates. The class under consideration has no zero dynamics. Specifically, we assume the following:

1. The plant dynamics has the state-space description

$$\begin{aligned} \dot{x} &= Ax + Bu + f_0(x) + F_1(x)\theta \\ y &= c^T x \end{aligned} \tag{1}$$

   where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the scalar control input, $y \in \mathbb{R}$ is the scalar sensed output.

2. The matrices A and B are in the Brunowsky canonical form. $c^T := [\ 1\ 0\ 0\ \dots\ 0\ ]$.

3. The smooth nonlinear functions
   $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $F_1 : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ are known and have the lower-triangular forms $f_{0(i)}(x) = \tilde{f}_{0(i)}(x_1, x_2, \dots, x_i)$ and $F_{1(i,j)}(x) = \tilde{F}_{1(i,j)}(x_1, x_2, \dots, x_i)$ for $i = 1, \dots, n$, $j = 1, \dots, p$.

4. $\theta \in \mathbb{R}^p$ is a vector of *unknown* constant parameters.

2.1 Remarks

For the above plant description, there exists a possibly parameter dependent, coordinate transformation, $\xi = \Phi(x, \theta)$, where $\Phi(\cdot, \theta)$ is a diffeomorphism for all θ. Under the coordinate transformation Φ, (1) can be equivalently described by

$$\begin{aligned} \dot{\xi} &= A\xi + B[u + g_0(x) + G_1(x)\psi] \\ y &= c^T \xi \end{aligned} \tag{2}$$

where the over-parametrization ψ consists of entries of appropriate tensor products of θ.

The class of plants under consideration are transformable to $y^{(n)} = u + g_0(x) + G_1(x)\psi$; hence, they are state-dependent perturbations of a string of n-integrators.

For notational convenience, we will refer to the linear part in (2), i.e., the triple (I, A, B) as the linear plant P . Let

$$\mathcal{F}(x) \; := \; g_0(x) + G_1(x)\psi \quad . \tag{3}$$

This class of plants satisfies the parametric strict-feedback condition in [Kan2] ; hence, provided that the state x is available for feedback, one can obtain a globally adaptive tracking system for the plant in (1) using the procedure in [Kan2] .

The class of plants under consideration include the so-called benchmark example , where $n = 3$, $p = 1$, $f_0(x) = 0$ and $F_1(x) = [f(x_1)\ 0\ 0]^T$, where $f(\cdot)$ is not necessarily Lipschitz in x_1 .

We will denote "state-x is available for feedback" rather than "state is available for feedback" , in order to emphasize the specific coordinate the dynamics is described in. Note that even if state-x in (1) is available, state-ξ is not available, since it depends on a θ-dependent transformation where θ is unknown.

Whenever we refer to a stable map, the map is causal and bounded-input bounded-output stable (defined over an appropriate extended space). With a slight abuse of notation, a stable proper transfer function $h(s)$ will be denoted as h in the input-output description where $hG(x)$ will denote the time-domain waveform obtained by convolving the impulse response corresponding to $h(s)$ and the signal $G(x)$. Parantheses will be used where ambiguity arises. Unless specifically emphasized, factorizations denote proper stable factorizations.

2.2 Example

A simple third-order example, satisfying the conditions in Section 2, is used to illustrate certain points throughout the paper. Consider

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta f(x_1) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \quad . \end{aligned} \tag{4}$$

Since we are interested in an input-output approach, all initial conditions in the simulations are assigned as zero.

Candidates for the function $f(\cdot)$ that we have used are:

- $f(x_1) = -\tanh(\alpha x_1)$, motivated by a simple saturating friction nonlinearity. Note that although f is Lipschitz, the corresponding G_1 is not.
- $f(x_1) = -x_1^2$, following the benchmark problem in [Kan2].
- $f(x_1) = -x_1^2/(\alpha x_1^2 + 1)$.

The function f used in a particular simulation will be emphasized. We introduce the following, for future reference.

$$P := \frac{1}{s^3}\begin{bmatrix} 1 & s & s^2 \end{bmatrix}^T$$

$$\Phi(x) = \begin{bmatrix} x_1 \\ x_2 + \theta f(x_1) \\ x_3 + \theta x_2 f^{(1)}(x_1) + \theta^2 f(x_1) f^{(1)}(x_1) \end{bmatrix} =: \xi$$

$$\Phi^{-1} : \xi \mapsto \begin{bmatrix} \xi_1 \\ \xi_2 - \theta f(\xi_1) \\ \xi_3 - \theta \xi_2 f^{(1)}(\xi_1) \end{bmatrix}$$

$$g_0(x) := 0$$

$$G_1(x) := \begin{bmatrix} x_2^2 f^{(2)}(x_1) \ + \ x_3 f^{(1)}(x_1) \\ 2x_2 f(x_1) f^{(2)}(x_1) \ + \ x_2 (f^{(1)}(x_1))^2 \\ f(x_1)(f^{(1)}(x_1))^2 \ + \ f^2(x_1) f^{(2)}(x_1) \end{bmatrix}^T$$

$$\psi := \begin{bmatrix} \theta & \theta^2 & \theta^3 \end{bmatrix}^T$$

Note that for this example,

$$\begin{aligned} G_1(x)\psi &= (f^{(2)}(\xi_1)\xi_2^2 + f^{(1)}(\xi_1)\xi_3)\theta \\ &=: \tilde{G}_1(\xi)\theta \quad , \end{aligned} \tag{5}$$

which emphasizes the issue of overparametrization in different coordinates.

Another observation which becomes useful in the perturbation description is the map $(\widehat{\Phi}\Phi^{-1} - I)$, where

$$\widehat{\Phi}\Phi^{-1}(\xi) = \widehat{\Phi} \begin{bmatrix} \xi_1 \\ \xi_2 - \theta f(\xi_1) \\ \xi_3 - \theta \xi_2 f^{(1)}(\xi_1) \end{bmatrix}$$

$$= I(\xi) \ + \ (\widehat{\theta} - \theta) \begin{bmatrix} 0 \\ f(\xi_1) \\ \xi_2 f^{(1)}(\xi_1) \ + \ \widehat{\theta} f(\xi_1) f^{(1)}(\xi_1) \end{bmatrix} . \tag{6}$$

3 State-x Available

3.1 On Sensitivity of State-x Based Stabilization

The contents of this particular subsection focuses on a slight generalization of the class in Section 2 , where the input u need not be scalar.

Consider the input u to state-x map described by (1) or equivalently, by (2) . Under a change of coordinates (Φ) , the input u to state-ξ map can be rendered linear under an algebraic state-feedback $\mathcal{F}$ (see (3)) . Hence u to state-x map can be realized as in the dashed-box in Figure 1 .

The map from v to ξ is linear (time-invariant finite dimensional). Let $\xi = Pv$. Since P admits coprime factorizations, $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ with $\begin{bmatrix} \widetilde{U} & \widetilde{V} \\ -\widetilde{D} & \widetilde{N} \end{bmatrix} \begin{bmatrix} N & -V \\ D & U \end{bmatrix} = I$, where all eight maps in the above identity are stable.

The nonlinear map Φ is stable and its inverse is stable.

The algebraic map $\mathcal{F}$ is stable. In fact, for the subsequent derivations in this subsection, Φ and $\mathcal{F}$ need not be algebraic.

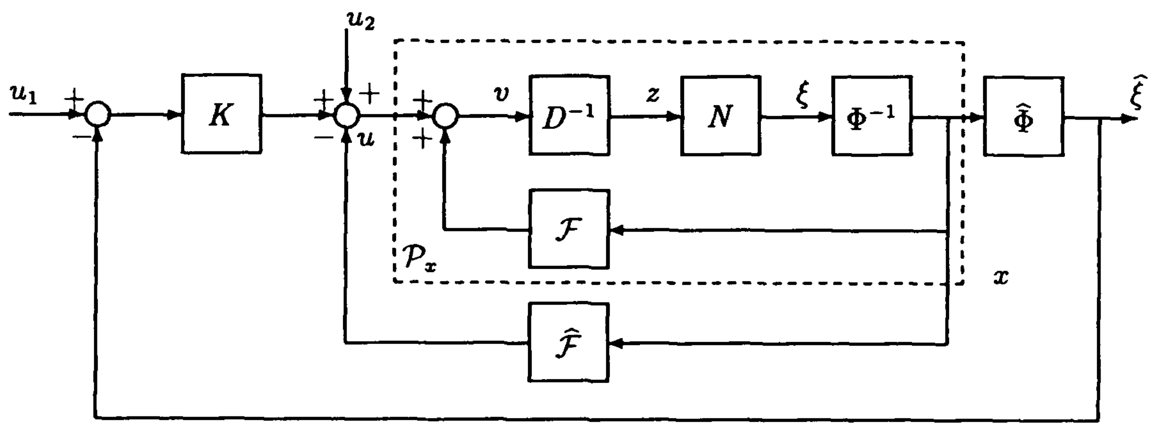

Figure 1: Stabilizing scheme based on the estimates $\widehat{\mathcal{F}}$ and $\widehat{\Phi}$

Note that, the input to state-x map in Figure 1 can be expressed as

$$\begin{aligned} \mathcal{P}_x &= \Phi^{-1}N(D - \mathcal{F}\Phi^{-1}N)^{-1} && (7) \\ &= (\widetilde{D}\Phi - \widetilde{N}\mathcal{F})^{-1}\widetilde{N} && (8) \end{aligned}$$

where

$$\begin{aligned} (\widetilde{U}\Phi + \widetilde{V}\mathcal{F})(\Phi^{-1}N) \;+\; (\widetilde{V})(D - \mathcal{F}\Phi^{-1}N) &= I && (9) \\ (\widetilde{D}\Phi - \widetilde{N}\mathcal{F})(\Phi^{-1}V) \;+\; (\widetilde{N})(U + \mathcal{F}\Phi^{-1}V) &= I && (10) \end{aligned}$$

Note that, (9)-(10) imply that (7)-(8) are in fact nonlinear right- and left-coprime factorizations of $\mathcal{P}_x$, respectively. The equations (9) and (10) also describe a stabilization scheme based on undoing the nonlinearities, since a stabilizing $\mathcal{C} \; : \; x \;\mapsto\; u$ is given by

$$\mathcal{C} \;=\; -K\Phi \;-\; \mathcal{F} \quad ,$$

where $v = -K\xi$ stabilizes P .

In order to incorporate the θ dependence in the u to x map, consider Φ and $\mathcal{F}$ both functions of θ whereas P is independent of θ. Note that such an assumption is not restrictive, since typically the choice of coordinates and algebraic state-feedback is constructed to render P as a string of integrators (in each channel). From now on, P and its associated terms in the Bezout-identity will be considered as θ-independent. The θ dependence in Φ and $\mathcal{F}$ will be emphasized by introducing the $\widehat{\ }$-versions when they're determined by the parameter estimate $\widehat{\theta}$.

We now can express the standard stabilization scheme based on certainty-equivalence approach in terms of the maps introduced so far. Since θ is not exactly known, the control law from x to u is realized as

$$u \;=\; K(\widehat{\xi}_r - \widehat{\Phi}(x)) - \widehat{\mathcal{F}}(x) \quad , \qquad (11)$$

where $\hat{\xi}_r$ denotes the desired reference in terms of the state-$\hat{\xi}$ (see Figure 1) . In other words, when $\theta = \hat{\theta}$, the tracking performance is determined by

$$\begin{aligned} \xi &= Pv \\ v &= K(\hat{\xi}_r - \xi) \quad . \end{aligned}$$

Note that the tracking scheme in (11) can be equivalently represented by the control law,

$$\hat{v} = K(\hat{\xi}_r - \hat{\xi}) \quad ,$$

where K is a $\hat{\xi}_r$-tracking compensator for $P : v \mapsto \xi$, and $\hat{v}$, $\hat{\xi}$ are as shown in Figure 2 .

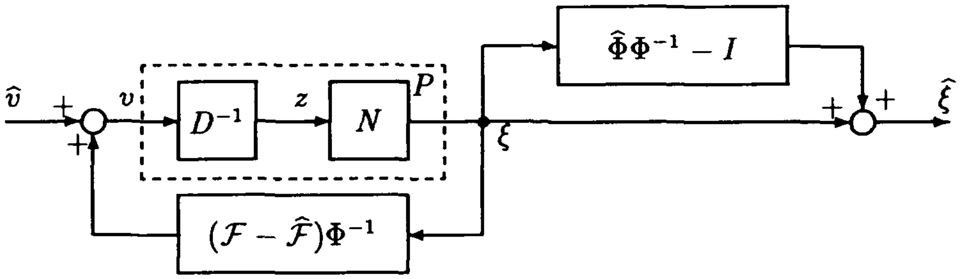

Figure 2: Perturbation of $P : v \mapsto \xi$ under θ mismatch

Note that the stable feedback perturbation in Figure 2 can be expressed as $(\mathcal{F} - \hat{\mathcal{F}})\Phi^{-1}$, since $x = \Phi^{-1}(\xi)$ is available, although Φ and ξ are not. As long as the feedback law from $\hat{\xi}$ to $\hat{v}$ is chosen as a linear time-invariant K stabilizing $P = ND^{-1}$, one obtains the precise conditions under which the certainty-equivalence based control law can withstand θ perturbations. Before stating the conditions, we will adopt the following definition.

- Let K stabilize P (i.e., $K = \tilde{V}^{-1}\tilde{U}$ and $P = ND^{-1}$, where $\tilde{U}N + \tilde{V}D = I$) . K is said to stabilize the perturbed plant in Figure 2 iff the control law $\hat{v} = u_2 + K(u_1 - \hat{\xi})$ yields a stable $(u_1, u_2) \mapsto z$, where z denotes the pseudo-state in Figure 2 (see also Figure 1) .

With this definition, one can work out the pseudo-state equation to derive the necessary and sufficient condition for K to robustly stabilize the perturbed v to ξ map, i.e.,

$$(I + \begin{bmatrix} \tilde{V} & \tilde{U} \end{bmatrix} \begin{bmatrix} (\hat{\mathcal{F}} - \mathcal{F})\Phi^{-1} \\ \hat{\Phi}\Phi^{-1} - I \end{bmatrix} N) \tag{12}$$

has a stable inverse. Note that P can be stabilized using only its first output, i.e., there exist $\tilde{U} =: [\ \tilde{U}_1\ 0\ 0\ \ldots\ 0\ ]$, such that $\tilde{U}N + \tilde{V}D = I$. Hence using a dynamic feedback compensator of the form $K =: [\ C\ 0\ 0\ \ldots\ 0\ ]$, C stabilizes P ; moreover, the map in (12) further simplifies to $(\ I + \tilde{V}(\hat{\mathcal{F}} - \mathcal{F})\Phi^{-1}N\ )$ since $c^T(\hat{\Phi}\Phi^{-1} - I) = 0$.

Typical robustness results (small-gain, passivity based sufficient conditions) can be utilized together with assumptions on the nonlinear post-multiplicative and feedback perturbations in (12) (such as sector bounded nonlinearities, linear or multilinear θ dependence etc.) to generate classes of systems for which (12) can be justified. However, we are not interested in further restricting the set of plants under investigation for the sake of forcing some sufficient conditions. For the simple example in Section 2.2 , even if the nonlinearity $f(\cdot)$ is chosen to be globally Lipschitz, the same is no longer true for $\mathcal{F}$.

3.2 Sensitivity to e_θ : Simulation

The example in Section 2.2 with $f(x_1) = -x_1^2$ is used for the following simulation.

Recall that, since $\xi_1 = x_1 = y$, from (6) , $c^T(\widehat{\Phi}\Phi^{-1} - I) = 0$. Hence, by choosing a dynamic compensator C from y to $\hat{v}$, the error introduced by the output multiplicative term in Figure 2 is avoided.

The compensator $C : y \mapsto \hat{v}$ is chosen as an $\mathcal{H}_2$-optimal compensator for $1/s^3$ for a particular choice of weights. Throughout the simulations, $C(s) = \frac{70.222s^2+40.055s+10}{s^3+9.5219s^2+35.3333s+66.6297}$.

The sensitivity of the closed-loop to $e_\theta := \theta - \widehat{\theta}$ is illustrated as follows: The parameter θ is chosen as $\theta = 1$. For a unit-step reference input the parameter estimate $\widehat{\theta}$ is perturbed about the nominal value 1 . When $\theta = \widehat{\theta}$, the closed-loop system exhibits the desired tracking response of a linear system. When $\widehat{\theta}$ is perturbed to 1.0025 and 1.005 , the tracking performance of the loop is shown in Figure 3 .

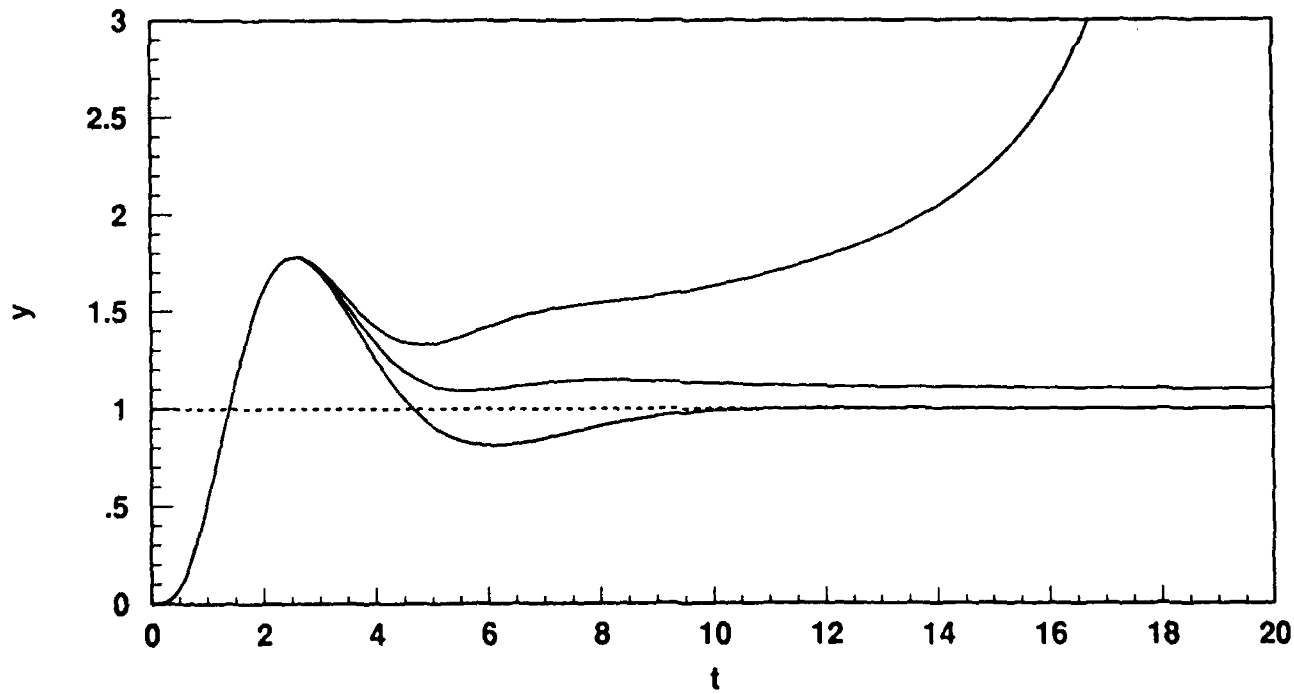

Figure 3: Sensitivity of certainty-equivalence based design using state-x for $\theta = 1$, $\widehat{\theta} = 1$, 1.0025 , 1.005 .

Clearly, within 0.5% relative error, the asymptotic tracking property is lost and the system goes unstable. An adaptation law based on this sensitive closed-loop is bound to exhibit unsatisfactory transient behaviour even if asymptotic tracking is achieved by parameter adaptation. Note that, Figure 3 shows the closed-loop responses for *frozen* $\widehat{\theta}$ values. Hence, one should definitely avoid very slow adaptation.

3.3 Adaptation

For the plant description in Section 2 , provided that the state-x is also available for exact linearization, one can bring an input-output approach to a particular case of certainty-equivalence based adaptive control design.

We now outline the design procedure:

1. For the class of plants in Section 2 , determine g_0 and G_1 , reducing zero columns of G_1 (if any) to cut down on unnecessary overparametrization in ψ .

2. Design a stable stabilizing compensator $C(s)$ for $1/s^n$; i.e., $C =: n_c/d_c$, $1/s^n =: n_p/d_p$ and $n_p n_c + d_p d_c = 1$.

3. An adaptive tracking control law candidate is determined by applying the control

$$u \;=\; C(r-y) - g_0(x) - (G_1 x)\widehat{\psi} \quad , \tag{13}$$

where the parameter estimate $\widehat{\psi}$ is updated using the error equation

$$y - n_p n_c r \;=\; d_c n_p((G_1 x)e_{\psi}) \quad , \tag{14}$$

so that $y \rightarrow n_p n_c r$. ($e_{\psi} = \psi - \widehat{\psi}$ and r denotes a reference signal that tracks a step input.)

Note that $(y - n_p n_c r)$ in (14) denotes the error between the plant output and the ideal tracking performance (if the nonlinear cancellations were exact) .

Consider the nonlinear plant described in Section 2 shown in Figure 1. For the scalar input case, let $D = d_p$.

The pseudo-state equations can be written as

$$\begin{aligned} d_p z &= u + g_0(x) + G_1(x)\psi \\ N z &= \xi \\ \Phi x &= \xi \\ y &= \xi_1 \quad , \end{aligned}$$

where N and d_p are stable; $c^T N =: n_p$ and (n_p, d_p) is a right coprime factorization of $1/s^n$. Since n_p is minimum-phase, $n_p d_p^{-1}$ is strongly stabilizable. Hence there exist n_c,d_c with d_c^{-1} stable, satisfying $n_c n_p + d_c d_p = 1$.

Since N is strictly proper, $\mathcal{G}$ is strictly causal, where

$$\begin{aligned} \mathcal{G} &= \mathcal{G}_0 + (\mathcal{G}_1(\cdot))\psi \\ \mathcal{G}_0 z &= g_0 x \qquad (15) \\ \mathcal{G}_1 z &= G_1 x \quad . \qquad (16) \end{aligned}$$

Hence,

$$d_p z = u + \mathcal{G} z$$

describes a well-posed u to z map. Since Φ and Φ^{-1} are stable maps and (n_p, d_p) is a right-coprime pair, the plant can be equivalently represented by the following pseudo-state equation

$$\begin{aligned} d_p z &= u + \mathcal{G} z \\ n_p z &= y \quad . \end{aligned}$$

Note that although z is not available, $\mathcal{G}_0 z$ and $\mathcal{G}_1 z$ are available since state-x is available ((15),(16)) .

Hence the plant description from u to y can be expressed as

$$y = n_p (d_p - \mathcal{G})^{-1} u \quad .$$

Now apply the certainty equivalence based control law

$$u = u_2 + \frac{n_c}{d_c}(u_1 - y) - \mathcal{G}_0 z - (\mathcal{G}_1 z)\widehat{\psi}$$

where u_1 and u_2 denote the exogenous additive inputs at the plant input and output in the standard unity-feedback system, respectively.

For $u_2 = 0$, $u_1 = r$, where r denotes the desired reference, we obtain

$$y - n_p n_c r = d_c n_p ((\mathcal{G}_1 z) e_\psi) \quad .$$

Since $d_c n_p$ is minimum-phase, one can utilize an augmented error scheme to update e_ψ .

In order to cut down on the number of states introduced by the filters in the augmented error scheme, we use the filtered error e_y ,

$$e_y := d_c^{-1}(y - n_p n_c r) \quad . \qquad (17)$$

Since n_p is minimum-phase, one can adapt $\widehat{\psi}$ using the error form

$$e_y = n_p((\mathcal{G}_1 z) e_\psi) \quad . \qquad (18)$$

Since $(d_c d_p)(0) = 0$, when cancellation is exact, step inputs can be asymptotically tracked.

We now derive the update law, based on the error equation in (18) . Since it utilizes the standard augmented error scheme, we briefly outline the steps.

Since n_p is minimum-phase and strictly proper, it can always be expressed as $n_p =: h_1 h_2$, where h_1 is strictly positive real, strictly proper and h_2 is proper stable. Note that $h_1 = \frac{1}{s+1}$ and $h_2 = (s+1)n_p$ will always work, since n_p is strictly proper. However, to cut down on the number of states introduced by filtering by h_2 in the augmented error implementation, it is advisable to factor out a strictly positive real factor of n_p, instead.

For ease of notation, let $W := \mathcal{G}_1 z$. Then (18) can be written as $e_y = h_1 h_2(W e_\psi)$. After adding, subtracting terms and using the fact that $\dot{\psi} = 0$, we obtain

$$\eta = h_1((h_2 W)e_\psi) \quad ,$$

where $\eta := e_y - h_1((h_2 W)\hat{\psi} - h_2(W\hat{\psi}))$. Since h_1 is strictly positive real, the update law

$$\dot{\hat{\psi}} = \gamma(h_2 W)^T \eta \quad , \quad \gamma > 0$$

guarantees that $\eta \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $e_\psi \in \mathcal{L}_\infty$. Provided that the closed-loop signals are bounded, $W \in \mathcal{L}_\infty$; hence, $\eta \to 0$, $\dot{\hat{\psi}} \to 0$ and $e_y \to 0$. This concludes that set point tracking is achieved in the limit provided that the closed-loop signals remain bounded.

3.3.1 Adaptive Tracking Scheme : Simulation

The adaptation scheme above is applied to the example in Section 2.2 for two choices of f. In both adaptive schemes, the closed-loop signals remain bounded; hence asymptotic tracking is achieved. However, the transient responses are unacceptable. Note that in both cases, even without adaptation, state-x based certainty-equivalence control performance was extremely sensitive to constant parameter errors.

- $f(x_1) = -x_1^2/(\alpha x_1^2 + 1)$, $\alpha = 0.05$, $\gamma = 10$, reference $r = 1$, $\theta = 1$. This function approximates $f(x_1) = -x_1^2$ reasonably for $|x_1| \leq 2$. The tracking performance is shown in Figure 4.
- $f(x_1) = -\tanh(\alpha x_1)$, $\theta = -1$, $\alpha = 4$, $\gamma = 1$, reference $r = 0.1$. The tracking performance is shown in Figure 5.

4 On Implementable Certainty-Equivalence Based Controllers

Recall that the example in Section 2.2 can be expressed in x or ξ coordinates as in (1) and (2), respectively. Note also that, due to (5), there is no need for overparametrization in the ξ coordinates.

If the parameter vector θ and either the state x or ξ are known, then either the control

$$u = v - \theta \tilde{G}_1(\xi) \tag{19}$$

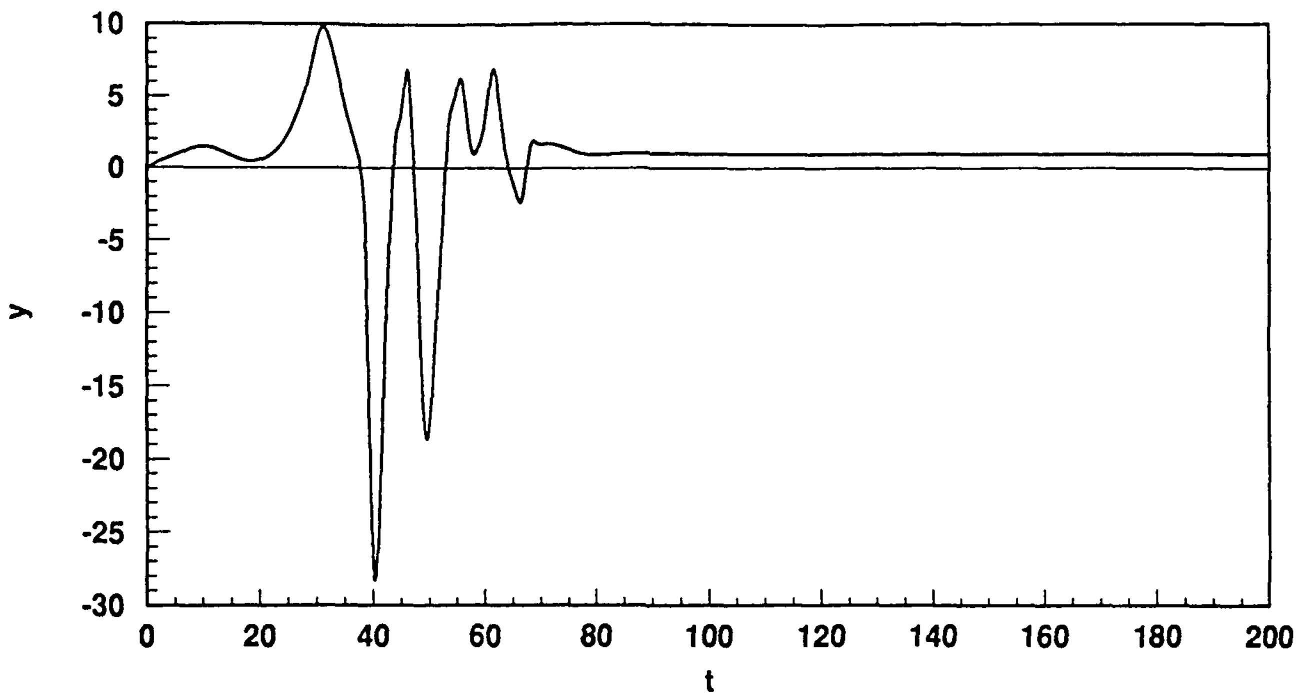

Figure 4: Asymptotic tracking of $r = 1$ from zero initial conditions

or, in terms of the state x, the control

$$u = v - G_1(x)\psi \tag{20}$$

provides exact *input-output linearization.* That is, the system $v \mapsto y$ is linear time-invariant, where

$$\begin{aligned} \dot{\xi} &= A\xi + Bv \\ y &= \xi_1 \end{aligned}$$

Let $C(s)$ be the same stabilizing compensator (for (c^T, A, B)) as in Section 3.2 . Hence $v = C(r - y)$, where r denotes the desired reference, achieves the desired tracking performance when $\widehat{\theta} = \theta$.

Since the parameter vector θ and neither state x nor ξ are known, it is natural to replace them in (19) and (20) with estimates $\widehat{\theta}$, $\hat{x}$, and $\hat{\xi}$. This is the *certainty equivalence principle.* As it does make a difference which observer state is used, we consider them separately. As in Section 3.2 , the following simulations are based on $f(x_1) = -x_1^2$.

4.1 Observer in the State-x

The control law is

$$\begin{aligned} \dot{\hat{x}} &= (A - Lc^T)\hat{x} + Bu + F_1(\hat{x})\hat{\theta} + Ly \\ u &= v - G_1(\hat{x})\widehat{\psi} \quad , \end{aligned} \tag{21}$$

where $(A - Lc^T)$ is strictly Hurwitz. Since all simulations are performed with zero initial

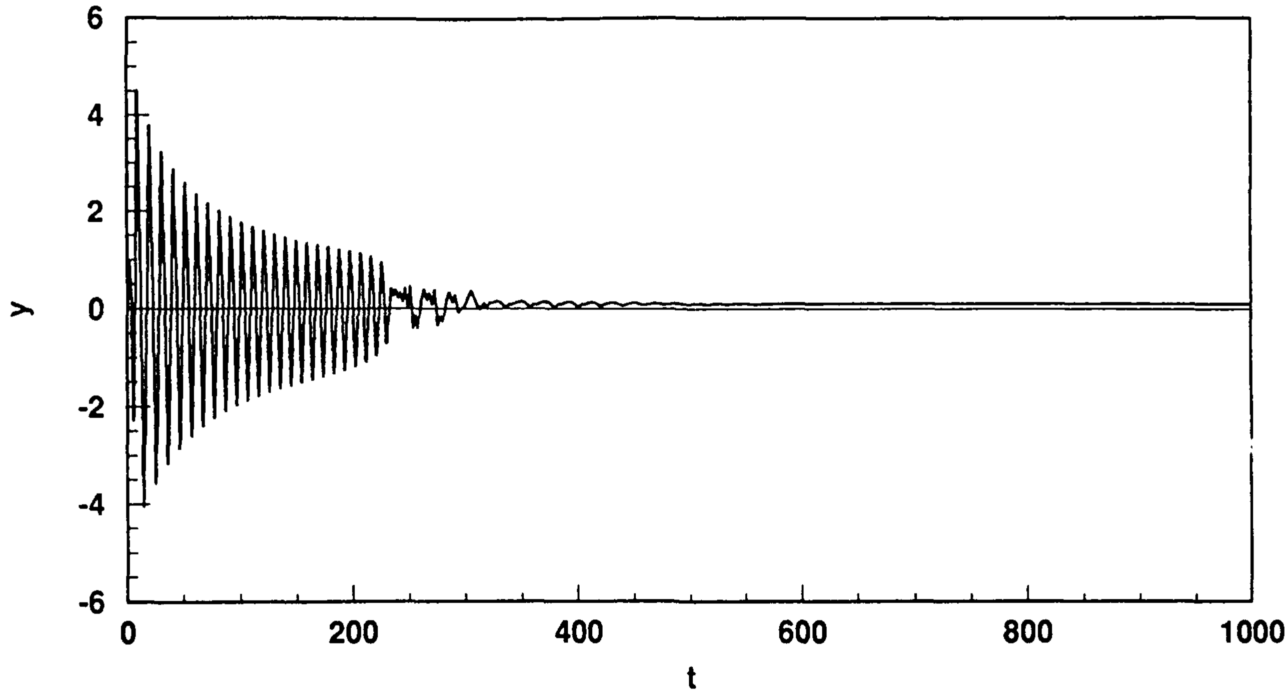

Figure 5: Asymptotic tracking of $r = 0.1$ from zero initial conditions

conditions, provided that $\widehat{\theta} = \theta$, the nominal performance is identical to that of the linear equivalent.

The sensitivity of the closed-loop to $e_\theta := \theta - \widehat{\theta}$ is illustrated as follows: The parameter θ is chosen as $\theta = 1$. For a unit-step reference input the parameter estimate $\widehat{\theta}$ is perturbed about the nominal value 1 . When $\widehat{\theta}$ is perturbed to 1.1 and 1.2 , the tracking performance of the loop is shown in Figure 6 .

4.1 Observer in the State-ξ

The control law is

$$\begin{aligned} \dot{\widehat{\xi}} &= (A - Lc^T)\widehat{\xi} + B[u + \widehat{\theta}\widetilde{G}_1(\widehat{\xi})] + Ly \\ u &= v - \widehat{\theta}\widetilde{G}_1(\widehat{\xi}) \quad , \end{aligned} \tag{22}$$

where $(A - Lc^T)$ is strictly Hurwitz. Since all simulations are performed with zero initial conditions, provided that $\widehat{\theta} = \theta$, the nominal performance is identical to that of the linear equivalent.

For a unit-step reference, $\widehat{\theta}$ is perturbed 1.25, 1.5, 1.75 and 2 ; the tracking performance of the loop is shown in Figure 7 .

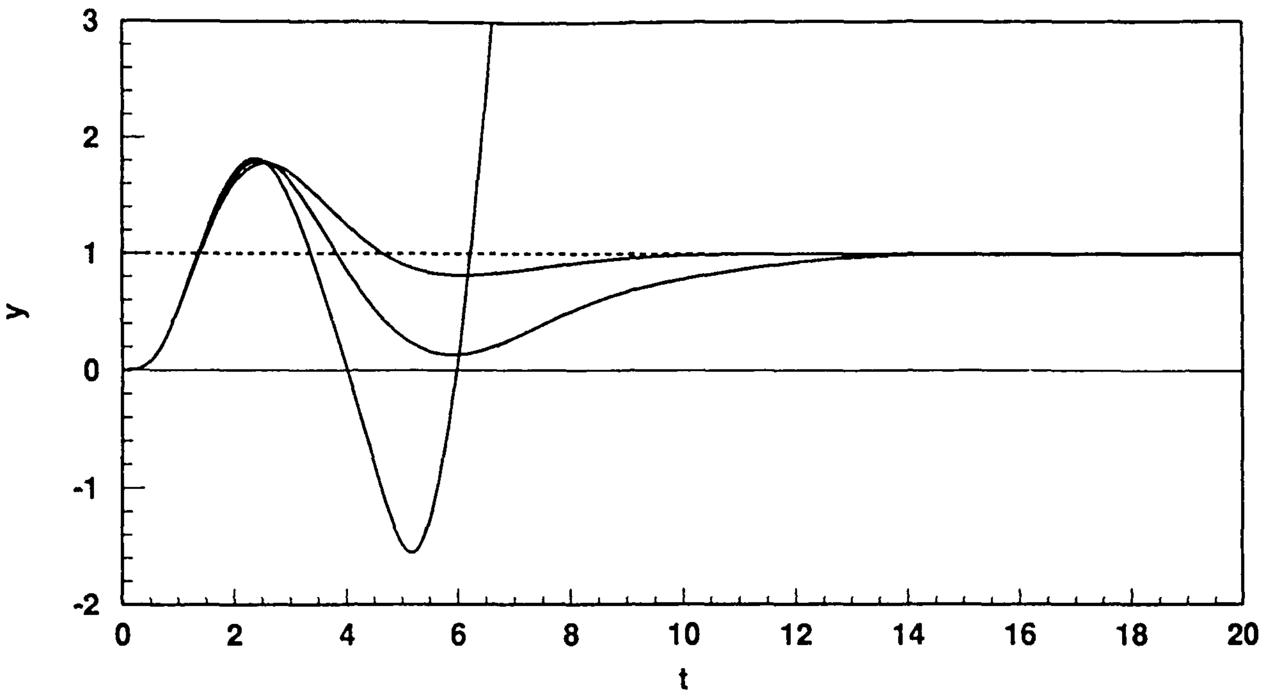

Figure 6: Sensitivity of certainty-equivalence based design using the estimate of state-x for $\theta = 1$, $\widehat{\theta} = 1$, 1.1 , 1.2 .

4.2 Model-Follower

This feedback-feedforward control scheme is based on a locally stable unity-feedback system. For the given nonlinear plant $\mathcal{P} : u \mapsto y$, let $C_{\mathcal{P}}(s)$ locally stabilize the plant in the unity-feedback system about the reference r . A model is used to generate the feedforward signals. The model is a copy of the plant description, where θ is replaced by $\widehat{\theta}$. Since all states are available for this model, exact linearization can be performed. Let $C(s)$ be the compensator that is being used so far in the previous three certainty-equivalence based designs. Subscript-m denotes the model variables.

The control law is (see Figure 8) :

$$\begin{aligned} \dot{\xi}_m &= A\xi_m + Bv_m \\ y_m &= c^T \xi_m \\ v_m &= C(r - y_m) \\ u_m &= v_m - \widehat{\theta}\widetilde{G}_1(\xi_m) \\ u &= u_m + C_{\mathcal{P}}(y_m - y) \quad . \end{aligned} \tag{23}$$

For the specific example, $C_{\mathcal{P}} = C$ achieved local stabilization about the unit-step reference. $\widehat{\theta}$ is perturbed to 1.25, 1.5, 1.75 , 2 , 2.5 , 2.75 and 3 ; the tracking performance of the loop is shown in Figure 9 . Note that this scheme exhibits the least sensitive tracking design.

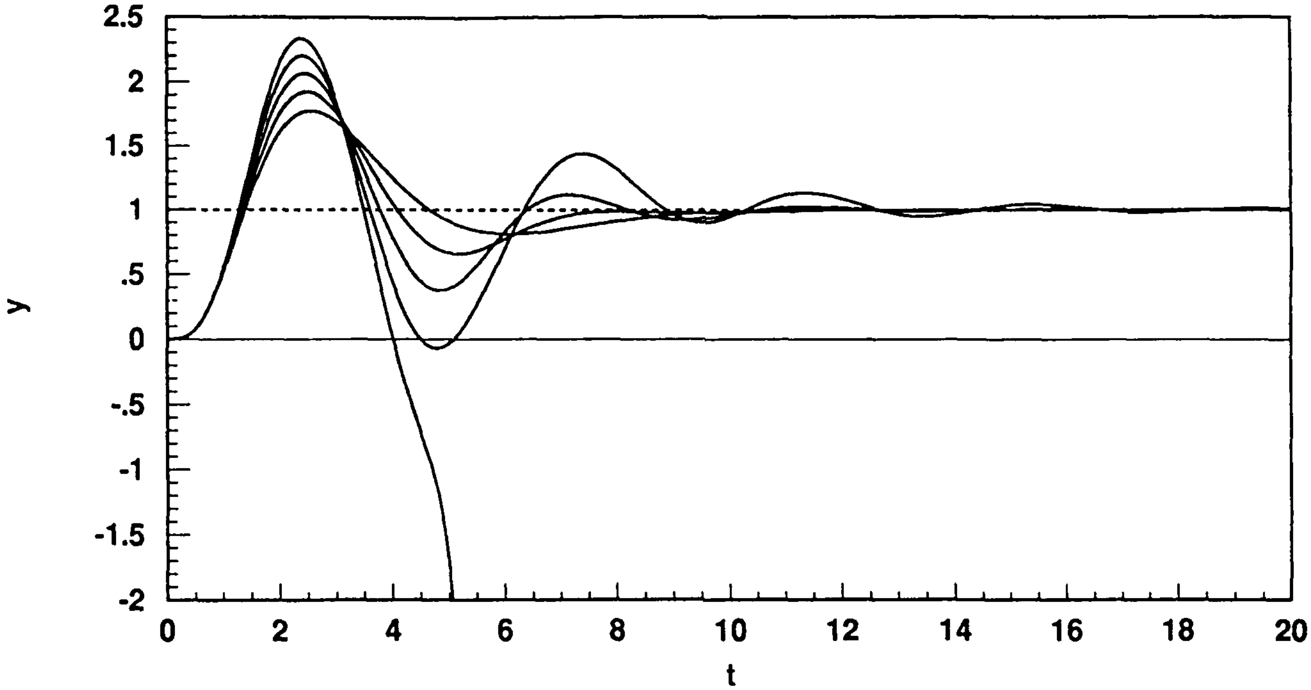

Figure 7: Sensitivity of certainty-equivalence based design using the estimate of state-ξ for $\theta = 1$, $\widehat{\theta} = 1$, 1.25 , 1.5 , 1.75 , 2 .

5 Concluding Remarks

An input-output approach has yielded some insight into robustness of feedback linearizable schemes under parameter mismatch. Our simulations of adaptive nonlinear systems show a phenomenon familiar in adaptive linear systems. Namely, even though the system is theoretically globally stable, the transient can be quite large. The problem is inherent in the parameter sensitivity of the system when the adaptation is frozen. The observer-based and model-follower schemes are most likely more robust to parameter changes then the full-state available scheme, which in general, is not implementable unless the states are measured.

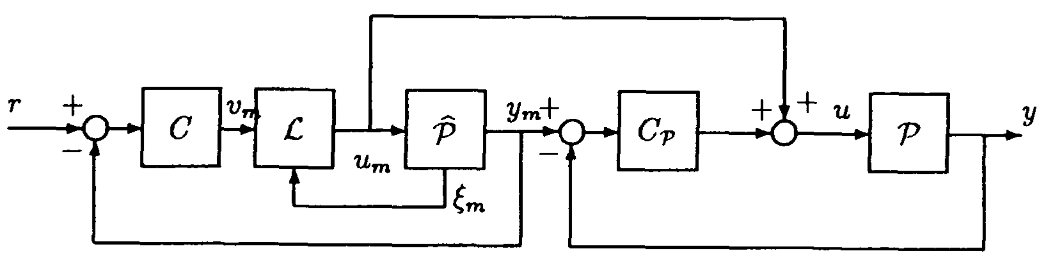

Figure 8: Model-follower described by (23), where $\mathcal{L}(v_m, \xi_m) := v_m - \widehat{\theta}\widetilde{G}_1(\xi_m)$.

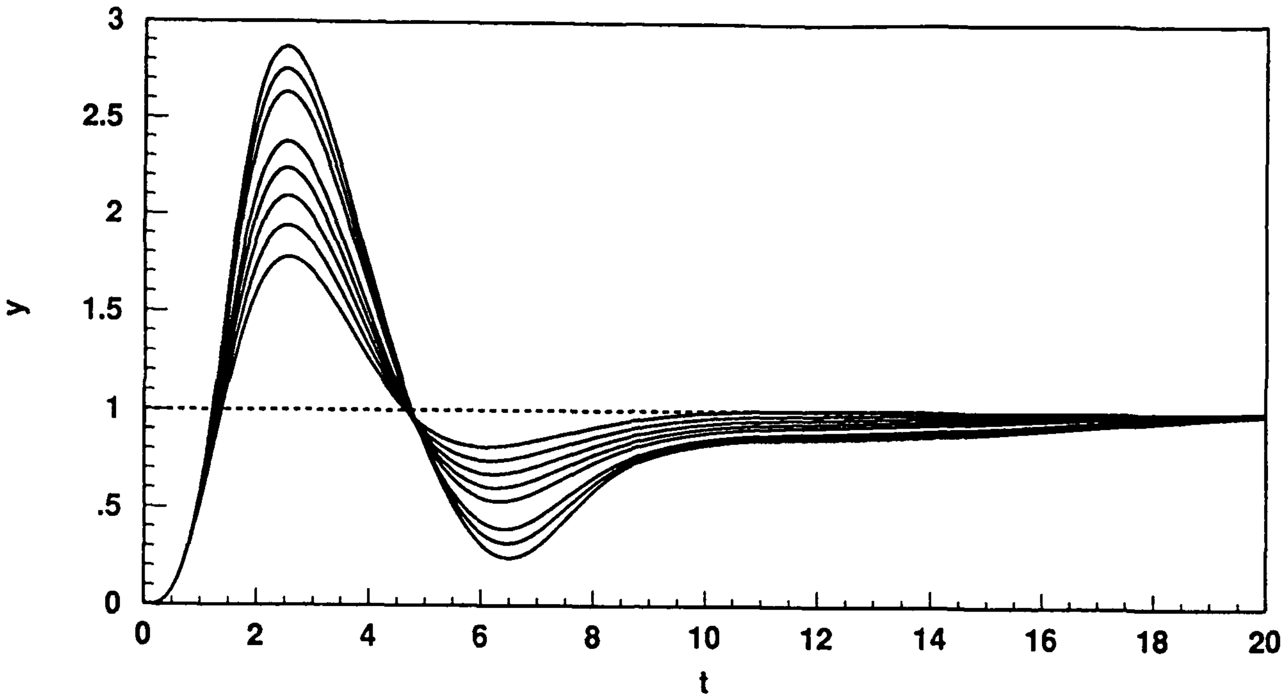

Figure 9: Sensitivity of the model-follower based certainty-equivalence design for $\theta = 1$, $\hat{\theta} = 1$, 1.25 , 1.5 , 1.75 , 2 , 2.5 , 2.75 , 3 .

6 References

[And1] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly and B. D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis* , MIT Press, 1986.

[Isi1] A. Isidori, *Nonlinear Control Systems* , Springer-Verlag, 1989.

[Kan1] I. Kanellakopoulos and P. V. Kokotovic, "Observer-based Adaptive Control of Nonlinear Systems Under Matching Conditions," *Proceedings of the American Control Conference*, pp. 549-555, San Diego, CA, May 1990.

[Kan2] I. Kanellakopoulos, P. V. Kokotovic and A. S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241-1253, November 1991.

[Sas1] S. S. Sastry and A. Isidori, "Adaptive Control of Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. 34, pp. 1123-1131, November 1989.

Appendix B

Appendix B includes a regular paper submitted for possible presentation at the 1993 Automatic Control Conference to be held in San Francisco, California. Appendix B is self-contained; hence, all section, equation and figure cross-referencing within these pages pertain only to Appendix B .

1993 ACC

On Feedback Linearizable Plants

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Abstract

An input-output approach is used to investigate stabilization of a class of nonlinear plants that can be rendered linear time-invariant under nonlinear stable dynamic feedback. Results are obtained for the standard unity-feedback configuration subject to bounded disturbances. It is shown that the design method based on applying the inverse transformation on *any* stabilizing controller for the transformed linear time-invariant plant, does not necessarily yield a stabilizing controller for the nonlinear plant. Conditions under which the method is justified are derived.

*Research supported by ARO, Engineering Sciences Division under contract DAAL03-91-C-0011 and AFOSR, Directorate of Aerospace Sciences under contract F49620-90-C-0064.

1 Introduction

Due to the relative ease of solving control problems associated with finite-dimensional linear time-invariant (LTI) plants, the class of nonlinear plants that can be rendered LTI by algebraic state-feedback has received considerable attention in the literature (e.g., [Isi1, Sas1, Kan1] and references therein) . This transformation based approach provides one of the few systematic means of designing nonlinear control laws and has been applied to numerous research problems involving stabilization, tracking, adaptive stabilization and adaptive tracking. While most of these applications might be valid nominal design approaches, the stability of resulting closed-loop systems subject to plant uncertainties and/or disturbance models has not received comparable attention.

Towards the goal of establishing robust control design methodologies for classes of nonlinear plants, a crucial step is guaranteeing stability when the system is subject to persistently exciting bounded disturbances and determining bounds on such disturbances for which stability is guaranteed. In other words, design methods and the subsequent analyses must be able to justify stability (be it local or global) in the presence of sensor and/or actuator noise, even before bringing in more demanding robustness to plant uncertainties.

In this paper, motivated by the class of nonlinear plants that can be rendered LTI under nonlinear algebraic state-feedback, we focus on a particular class of nonlinear plants that can be rendered LTI under nonlinear *stable*, possibly dynamic, feedback. We study the stabilization of this particular class of nonlinear plants. The study is based on an input-output framework, where the closed-loop results are stated for the standard unity-feedback configuration subject to bounded actuator and sensor disturbances [Des1] .

The results are organized as follows. The relationships between the graphs of the nonlinear plant and its transformed LTI counterpart are derived; the fact that the bounded input-output pairs of the two maps are related by stable maps motivates the generally adopted design procedure based on undoing the nonlinearities, which consists of the following steps:

1. transform the nonlinear plant to an LTI one,
2. design a stabilizing controller (possibly nonlinear) for the LTI plant and
3. using the inverse transformation and the designed controller for the LTI plant, determine the transformed controller for the nonlinear plant.

It is shown that this straightforward three step procedure is *not* a unified stabilizing control design method for the class of plants at hand *when the plant is subject to bounded actuator/sensor disturbances* . In other words, given any member of the particular class of plants under study, taking the three steps mentioned above does not necessarily guarantee a stabilizing controller. As might be expected, the problem arises due to the second step of the design procedure, since it inherently assumes that the design for the transformed LTI plant and the nonlinear plant are separate problems. In other words, *any* stabilizing controller for the LTI plant may not yield a stabilizing controller in the third step. Subsequent results formalize this issue and state conditions under which such a design approach is justified.

While the notion of incremental stability, which generalizes the properties of linear stable maps, can be easily imposed to guarantee sufficient conditions for the method to work in general, the extreme conservatism in expecting *any* controller in the second design step to work is shown by establishing that incremental stability has to be *necessary* , as well. Unless such restrictions happen to be satisfied for the specific problem at hand, all of these cautionary results imply the following: The design problem for the transformed LTI plant need not be decoupled from the original nonlinear design problem; hence, one should not rule out intentionally nonlinear controller design for the transformed LTI plant. Although this might seemingly defy the purpose of the transformation, since the transformed design problem is yet another nonlinear control design problem, it also emphasizes the need for more results in nonlinear control design for LTI plants, before nonlinear plants are transformed to linear ones. Moreover, despite the global linearizing transformation, one might end up with only a locally stable interconnection. Simple examples with analytical derivations are used to emphasize the points.

1.1 Notation and Preliminaries

- All nonlinear maps in this study are causal, multi-input multi-output and defined over appropriate products of extended $L_\infty[0,\infty)$ spaces. For a thorough treatment of general extended spaces within the input-output approach to nonlinear systems, see e.g., [Des1] . For notational convenience, for functions from $\mathbb{R}_+$ to $\mathbb{R}^n$, the associated set of bounded signals and the extended space will ve denoted by L_∞^n and $L_{\infty e}^n$, respectively.

  While most of the observations in this study do generalize to other extended spaces, the particular choice of sup bounded functions on $\mathbb{R}_+$ is motivated by studying the stability properties of nonlinear interconnections subject to persistently exciting disturbances.

  An n_i-input n_o-output causal nonlinear map $\mathcal{P}$ will be considered as

$$\mathcal{P} : \mathcal{U} \subset L_{\infty e}^{n_i} \rightarrow L_{\infty e}^{n_o} \quad ,$$

  where $\mathcal{U}$ denotes the domain. The extended space is a means of incorporating unbounded signals in the study; however, one must note that although $L_\infty \subset L_{\infty e}$, $(L_\infty)^C \setminus L_{\infty e} \neq \emptyset$, where $(\cdot)^C$ denotes the complement of the set with respect to the set of all functions on $\mathbb{R}_+$. The nonempty intersection arises due to discontinuities which are not jump-discontinuities. The scope of the extended spaces do not cover signals which exhibit "finite escape time"; hence the plant description over a strictly proper subset of the input extended space might be necessary. Hence, L_∞^n describes the set of bounded signals and $L_{\infty e} \setminus L_\infty$ denotes the set of unbounded signals (unbounded at infinity).

- Calligraphic capital letters will be used to denote nonlinear maps. Italic capital letters will denote linear time-invariant maps that admit finite-dimensional state-space descriptions; for this class, with a slight abuse of notation, the map and its associated transfer function representation will be used interchangeably. In the case that italic letters are used for nonlinear algebraic maps, parantheses will be included to emphasize the evaluation (i.e., Ax vs. $F(x)$) . For two nonlinear causal maps $\mathcal{F}$ and $\mathcal{G}$ the map $\mathcal{F}\mathcal{G}$ will denote the composition of the two maps.

- In an input-output approach to the analysis and design of nonlinear interconnections, the notions of boundedness and stability are crucial for subsequent results. Unlike the finite-dimensional linear time-invariant case, most of these properties depend on the particular framework. The following four definitions set up the particular framework in this paper. For a treatment of related topics in fractional representations of nonlinear casual maps and stability of nonlinear interconnections see e.g., [Ham1, Ver1, Vid1, Kab1] and references therein.

- The set of all input output pairs of a given map does admit a special partitioning which exhibits particular properties.

Definition 1 (graph of $\mathcal{P}$, $\mathcal{G}(\mathcal{P})$)
For a given map $\mathcal{P} : \mathcal{U} \subset L_{\infty e}^{n_i} \to L_{\infty e}^{n_o}$, the graph of $\mathcal{P}$ is denoted by $\mathcal{G}(\mathcal{P})$, where

$$\mathcal{G}(\mathcal{P}) = \mathcal{U} \times \mathcal{P}(\mathcal{U}) .$$

Moreover, $\mathcal{G}(\mathcal{P})$ admits the following unique partitioning

$$\mathcal{G}(\mathcal{P}) = \underbrace{\mathcal{U}_B \times (\mathcal{P}(\mathcal{U}))_B}_{\mathcal{G}_{BB}(\mathcal{P})} \cup \underbrace{\mathcal{U}_B \times (\mathcal{P}(\mathcal{U}))_U}_{\mathcal{G}_{BU}(\mathcal{P})} \cup \underbrace{\mathcal{U}_U \times (\mathcal{P}(\mathcal{U}))_B}_{\mathcal{G}_{UB}(\mathcal{P})} \cup \underbrace{\mathcal{U}_U \times (\mathcal{P}(\mathcal{U}))_U}_{\mathcal{G}_{UU}(\mathcal{P})} ,$$

where the subscripts B and U denote the bounded and unbounded components, respectively.
◇

In the bounded and unbounded partitioning of $\mathcal{G}(\mathcal{P})$, $\mathcal{G}_{BB}(\mathcal{P})$ denotes the desired i/o pairs, $\mathcal{G}_{BU}(\mathcal{P})$ describes the instabilites of the plant ("unstable poles") and $\mathcal{G}_{UB}(\mathcal{P})$ describes the instabilites in the inverse relation ("unstable zeros") . Clearly, a feedback system stabilizes $\mathcal{P}$ subject to bounded disturbances if and only if the closed-loop generates plant i/o pairs in $\mathcal{G}_{BB}(\mathcal{P})$. In other words, $\mathcal{G}_{BB}(\mathcal{P})$ must be nonempty. For example, for the particular map $\mathcal{P}$ from u to y where

$$\dot{y} = 1 + u^2$$

has $\mathcal{G}_{BB}(\mathcal{P}) = \emptyset$, hence there cannot exist any stabilizing scheme.

- Unless specifically emphasized, maps are not considered to be stable or algebraic.

Definition 2 (stable)
A causal map $\mathcal{H} : L_{\infty e}^{n_i} \to L_{\infty e}^{n_o}$ is said to be stable iff there exists a continuous nondecreasing $\phi_{\mathcal{H}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\mathcal{H}u\| \leq \phi_{\mathcal{H}}(\|u\|)$ for all $u \in L_{\infty}^{n_i}$.
◇

For a stable map $\mathcal{H}$,

- $\mathcal{G}_{BB}(\mathcal{H}) = L_\infty^{n_i} \times \mathcal{H}(L_\infty^{n_i})$,
- $\mathcal{G}_{BU}(\mathcal{H}) = \emptyset$,
- $\mathcal{G}_{UB}(\mathcal{H})$ may or may not be empty.

Definition 3 (unimodular)
A causal map $\mathcal{H} : L_{\infty e}^{n_i} \rightarrow L_{\infty e}^{n_i}$ is said to be unimodular iff $\mathcal{H}$ is stable, bijective and $\mathcal{H}^{-1}$ is stable.
◇

For a unimodular map $\mathcal{H}$,

- $\mathcal{G}_{BB}(\mathcal{H}) = L_\infty^{n_i} \times L_\infty^{n_i}$,
- $\mathcal{G}_{BU}(\mathcal{H}) = \emptyset$,
- $\mathcal{G}_{UB}(\mathcal{H}) = \emptyset$,
- $\mathcal{G}_{UU}(\mathcal{H}) = (L_{\infty e}^{n_i} \backslash L_\infty^{n_i}) \times (L_{\infty e}^{n_i} \backslash L_\infty^{n_i})$,

Definition 4 (incrementally stable)
A causal map $\mathcal{H} : L_{\infty e}^{n_i} \rightarrow L_{\infty e}^{n_o}$ is said to be incrementally stable iff $\mathcal{H}$ is stable and there exists a continuous nondecreasing $\tilde{\phi}_\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $u \in L_{\infty e}^{n_i}$, $\|\mathcal{H}(u+v) - \mathcal{H}u\| \leq \tilde{\phi}_\mathcal{H}(\|v\|)$ for all $v \in L_\infty^{n_i}$.
◇

For an incrementally stable map, bounded deviations in the input result in bounded deviations at the output. The bound on the output deviation is *independent* of the nominal input signal u .

In the case of an *algebraic* incrementally stable map $\mathcal{H}$ where the bounding function $\tilde{\phi}_\mathcal{H}$ is *linear* , the algebraic nonlinearity $\mathcal{H}$ is also referred to as Lipschitz continuous.

- From now on, the words bounded and stable will be interpreted in the sense of L_∞ .

- The nonlinear unity-feedback system $S(\mathcal{P},\mathcal{C})$ denotes the interconnection

$$\begin{aligned} y &= \mathcal{P}u \\ u &= u_1 - \mathcal{C}(u_2 + y) \quad , \end{aligned}$$

  where u_1 and u_2 denote the exogenous inputs perturbing the actuator and sensor signals u and y , respectively (see Figure 1) .

- The feedback system $S(\mathcal{P},\mathcal{C})$ is said to be stable iff the map from (u_1, u_2) to (u, y) is stable.

  Note that the stability of $S(\mathcal{P},\mathcal{C})$ requires that the closed-loop map exists and it is stable. The well-posedness condition that ensures the existence of the map is almost always satisfied in practice, since $\mathcal{P}$ and/or $\mathcal{C}$ are strictly causal.

2 A Class of Feedback Linearizable Plants

Definition 5 ($\mathcal{L}$)
A causal nonlinear plant $\mathcal{P}$ is said to belong to the class $\mathcal{L}$ iff there exist a unimodular map $\mathcal{M}$, a linear time-invariant map P with a finite-dimensional state-space representation, and a stable map $\mathcal{F}$ such that

$$\mathcal{P} = \mathcal{M}^{-1}P(I - \mathcal{F}\mathcal{M}^{-1}P)^{-1} \quad . \tag{1}$$

In order to emphasize the particular triple $(\mathcal{M}, P, \mathcal{F})$ used, equation (1) will be denoted by

$$\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F}) \quad .$$

◇

Note that if $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$ (see Figure 1) , the map from u to y can be expressed as

$$\mathcal{P} : u \mapsto y \begin{cases} \mathcal{M}y &= Pv \\ v &= u + \mathcal{F}y \end{cases} \quad .$$

Applying the stable output-feedback

$$u = \bar{v} - \mathcal{F}y$$

the map from $\bar{v}$ to $\mathcal{M}y$ is rendered linear. Clearly, for a given $\mathcal{P} \in \mathcal{L}$, the triple $(\mathcal{M}, P, \mathcal{F})$ is not uniquely determined. The well-posedness of the feedback loop in the description of $\mathcal{P}$ is inherently assumed by the existence of the map from u to y ; in general, strict causality of P and/or $\mathcal{F}$ suffice for well-posedness of the feedback loop in $\mathcal{L}(\mathcal{M}, P, \mathcal{F})$.

This particular class of nonlinear plants in Definition 5 is in fact motivated by the special class of nonlinear plants that can be put in the controllable canonical form

$$\begin{aligned} \dot{x} &= f(x) + Bu \\ \zeta &= \Phi(x) \\ \dot{\zeta} &= A\zeta + B(u + F(x)) \quad , \end{aligned}$$

where Φ is an algebraic change of coordinates and F is an algebraic stable map. Note that for this particular case, $y = x$, $\mathcal{M}y = \Phi(y)$, $\mathcal{F}y = F(y)$ and $P = (sI - A)^{-1}B$.

For plants in $\mathcal{L}$, the linear part P admits coprime factorizations; let

$$P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N} \tag{2}$$

with the Bezout identity

$$\begin{bmatrix} \tilde{U} & \tilde{V} \\ -\widetilde{D} & \widetilde{N} \end{bmatrix} \begin{bmatrix} N & -V \\ D & U \end{bmatrix} = I \quad , \tag{3}$$

where all of the eight maps are stable. By linearity, stable maps are also incrementally stable. Note also that for the description in (2) for a full rank P , the numerators drop rank at the zeros and the denominators drop rank at the poles. Since P is not necessarily single-input single-output, poles and zeros (except blocking zeros) may coincide. Since it is easier to describe the unstable zeros and unstable poles of P due to the transform algebra, one can easily describe the partitions $\mathcal{G}_{UB}(P)$ and $\mathcal{G}_{BU}(P)$. The following result establishes the connection between the graphs of P and $\mathcal{P} \in \mathcal{L}$.

Fact 6 (on graphs of P and $\mathcal{P}$)
If $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$, then there exists a unimodular map $\mathcal{T}$ such that

1. $\mathcal{G}_{BB}(\mathcal{P}) = \mathcal{T}\mathcal{G}_{BB}(P)$,
2. $\mathcal{G}_{UB}(\mathcal{P}) = \mathcal{T}\mathcal{G}_{UB}(P)$.

Proof: See Appendix.
◇

Note that Fact 6 establishes that the desired i/o pairs and the unstable dynamics of the inverse relations are related in the graphs of P and $\mathcal{P}$. Such a simple description between the rest of graph partitions cannot be established in general.

We now consider the stabilization of plants in $\mathcal{L}$; i.e., determining a causal nonlinear map C such that the feedback system $S(\mathcal{P}, \mathcal{C})$ (see Figure 1) is stable.

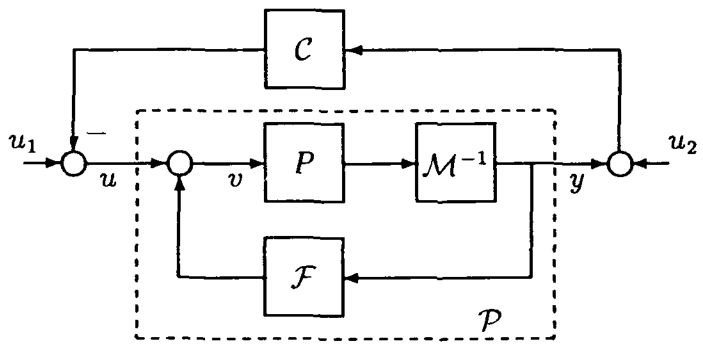

Figure 1: Feedback system $S(\mathcal{P}, \mathcal{C})$.

The existence of a transformation of $\mathcal{P}$ to a linear P , and Fact 6 suggests stabilization schemes based on designing controllers for P . The following conjectures emphasize this design approach.

Conjecture 7
For a given $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$, if $S(P, C_P)$ is stable then $S(\mathcal{P}, (C_P\mathcal{M} + \mathcal{F}))$ is stable.
◇

Conjecture 7 is based on the design method of undoing the nonlinear maps $\mathcal{F}$ and $\mathcal{M}$, which consists of the following steps:

1. transform the nonlinear plant to an LTI one,

2. design a stabilizing controller (possibly nonlinear) for the LTI plant and

3. using the inverse transformation and the designed controller for the LTI plant, determine the transformed controller for the nonlinear plant.

Conjecture 8
For a given $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$, if $S(\mathcal{P}, \mathcal{C})$ is stable then $S(P, (\mathcal{C} - \mathcal{F})\mathcal{M}^{-1})$ is stable.
$\diamond$

If Conjectures 7 and 8 were in fact true, one would end up with a necessary and sufficient condition for stabilizing plants in $\mathcal{L}$ by merely using the transformations and nonlinear controller design methods based on linear plants. We now investigate these conjectures and state conditions (other than trivial cases like $\mathcal{P} = P = \mathcal{L}(I, P, 0))$, under which related results can be established.

3 Stabilizing Feedback Linearizable Plants

The following fact (see e.g., [Kab1]) describes the set of all nonlinear stabilizing compensators $\mathcal{C}_P$ in $S(P, \mathcal{C}_P)$.

Fact 9 ($\mathcal{S}(P)$)
Let the causal linear map P satisfy equations (2) and (3) . Under these assumptions, the feedback system $S(P, \mathcal{C}_P)$ is stable if and only if $\mathcal{C}_P \in \mathcal{S}(P)$, where

$$\mathcal{S}(P) = \{(U + D\mathcal{Q})(V - N\mathcal{Q})^{-1} \mid \mathcal{Q} \text{ is stable and } (V - N\mathcal{Q}) \text{ is bijective with a causal inverse}\}$$

The following Lemma justifies the design approach in Conjecture 7 for a special subset of plants in $\mathcal{L}$.

Lemma 10 (A Sufficient Condition for Stabilizing $\mathcal{L}(\mathcal{M}, P, \mathcal{F})$)
Let $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$, where the maps $\mathcal{F}$ and $\mathcal{M}$ are both incrementally stable. Under these assumptions, if $S(P, \mathcal{C}_P)$ is stable then $S(\mathcal{P}, (\mathcal{C}_P\mathcal{M} + \mathcal{F}))$ is stable.
Proof: See Appendix.
$\diamond$

The existence of a unimodular map that relates the desired input output pairs of P and $\mathcal{P}$ (see Fact 6-1) , is the main motivator for Conjectures 7 and 8 . The second part of Fact 6 relates the "unstable zero dynamics" of P and $\mathcal{P}$. For a precise description of the notion of zero dynamics in nonlinear systems that extend the well-known case in finite-dimensional linear time-invariant maps, see [Isil] . Unlike the motivating controllable canonical form derivation, the class of nonlinear plants in $\mathcal{L}$ *can* have "unstable zero dynamics" . In fact, Lemma 10 does apply to such cases since one can easily stabilize P with unstable zeros. The advantage of the class $\mathcal{L}$ is to take the stabilization concept of algebraic-feedback linearizable systems under full-state feedback (which relies on constructing an output that has the sufficient relative degree to avoid "zero dynamics") to a larger class where the input output map already has "unstable zero dynamics" . The virtue of Fact 6-2 is that one does not need to determine the stability properties of the "zero dynamics" of the nonlinear map $\mathcal{P}$, which is a highly nontrivial task.

We now present a counterexample to Conjecture 7 .

Example 11
Consider the single state plant model $\mathcal{P}$ from u to y described by

$$\dot{y} \;=\; y^2 \;+\; u \quad .$$

Note that $\mathcal{P} = \mathcal{L}(I, 1/s, (\cdot)^2)$. Choose $\mathcal{C}_P = I \in \mathcal{S}(P)$. For $\mathcal{C} = \mathcal{C}_P\mathcal{M} + \mathcal{F} = I + (\cdot)^2$, the closedloop system $S(\mathcal{P}, \mathcal{C})$ is described by

$$\dot{y} \;=\; -(y+u_2) + y^2 - (y+u_2)^2 + u_1 \quad .$$

Clearly, for $u_2 = 0$, the closed-loop map from u_1 to (u, y) in $S(\mathcal{P}, \mathcal{C})$ is stable. However, for $u_2 \neq 0$, the closed-loop system $S(\mathcal{P}, \mathcal{C})$ is *not* stable. To see this, consider $u_1 = \delta_1$ and $u_2 = \delta_2$ where $\dot{\delta}_i = 0$, $i = 1, 2$. In other words, under constant sensor and actuator bias, the closed-loop system is described by

$$\dot{y} \;=\; -(1+2\delta_2)y \;+\; \delta_1 \;-\; \delta_2 \;-\; \delta_2^2 \quad .$$

Hence, for $\delta_2 \leq -0.5$, the output y is not bounded.
$\diamond$

Example 11 emphasizes the importance of checking the closed-loop map from (u_1, u_2) to (u, y) , rather than just u_1 to (u, y) . The same example also illustrates the effect of estimated state feedback and how the lack of a separation principle for nonlinear systems renders the design procedure heuristic, which happens to fail for this particular case. When an estimator with steady-state bounded bias is introduced in the ideal state-feedback, the closed-loop is no longer stable.

The problems with Conjectures 7 and 8 are mainly due to ignoring the exogenous input u_2 , and relying on the following fact.

Fact 12 (Conjectures 7 and 8 under no measurement noise)
For a given $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$,

$$S(P, \mathcal{C}_P)|_{u_2=0} \text{ is stable} \quad \Longleftrightarrow \quad S(\mathcal{P}, \mathcal{C}_P\mathcal{M} + \mathcal{F})|_{u_2=0} \text{ is stable} \quad .$$

$\diamond$

The instability mechanism in Example 11 can also be used to establish the lack of performance robustness under finite-energy disturbances, where u_1 and $u_2 \to 0$. Given any bound on the amplitude of y , there exists a finite-duration disturbance (hence finite-energy) such that the closed-loop output exceeds this bound before asymptotically reaching zero.

We now state the main result that shows the conservatism of basing the stabilization of $\mathcal{P}$ on the stabilization of P . The following theorem establishes that Conjecture 7 is not true in general and that incremental stability is of paramount importance.

Theorem 13 (Conservatism of Conjecture 7)
Let $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$ with $\mathcal{M}$ incrementally stable. Under these assumptions, $\mathcal{S}(\mathcal{P}, (\mathcal{C}_P\mathcal{M} + \mathcal{F}))$ is stable for all $\mathcal{C}_P \in \mathcal{S}(P)$, if and only if $\mathcal{F}$ is incrementally stable.
Proof: See Appendix.
$\diamond$

Note that Theorem 13 emphasizes that the design procedure is no longer decoupled (namely first stabilize P and then apply the transformed controller to $\mathcal{P}$) if the incremental stability of $\mathcal{F}$ fails. When incremental stability fails, Theorem 13 does not imply that there does not exist a compensator for P for which Conjecture 7 is valid. On the contrary, the selection of $\mathcal{C}_P$ is crucial as emphasized in the following example.

Example 14
Consider the plant $\mathcal{P}$ in Example 11 . Choose $\mathcal{C}_P = I + (\cdot)^3$. Note that $\mathcal{C}_P \in \mathcal{S}(1/s)$ (see Appendix) . Apply the heuristic design procedure in Conjecture 7 , i.e., choose $\mathcal{C} = \mathcal{C}_P\mathcal{M} + \mathcal{F}$. The closed-loop system $\mathcal{S}(\mathcal{P}, I + (\cdot)^2 + (\cdot)^3)$ is stable (see Appendix) .
$\diamond$

In order to show how restrictive it is to impose $\mathcal{M}$ and $\mathcal{F}$ are both incrementally stable, consider the following example.

Example 15
Consider the map $\mathcal{P}$ from u to y described by

$$\begin{aligned} \dot{x}_1 &= x_2 + f(x_1) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ x(0) &= 0 \\ y &= x \quad , \end{aligned}$$

where the algebraic nonlinearity f is bounded, i.e., there exists a $k > 0$ such that $|f(x)| \leq k$ for all $x \in \mathbb{R}$. Clearly, $f(\cdot)$, is stable and incrementally stable. Moreover, for any linear time-invariant finite-dimensional C such that $S(1/s^3[1 \;\; s \;\; s^2\,]^T, C)$ is stable, the interconnection $S(\mathcal{P}, C)$ is stable. Now consider the plant $\mathcal{P}$ in terms of the canonical form $\mathcal{L}$. Provided that f is at least twice differentiable, under the coordinate transformation

$$\mathcal{M} \; : \; x \; \mapsto \; \begin{bmatrix} x_1 \\ x_2 + f(x_1) \\ x_3 + x_2 f^{(1)}(x_1) + f(x_1) f^{(1)}(x_1) \end{bmatrix}$$

and algebraic feedback

$$\begin{aligned}\mathcal{F}x &= x_2^2 f^{(2)}(x_1) + x_3 f^{(1)}(x_1) + 2x_2 f(x_1) f^{(2)}(x_1) + \\ &\quad x_2 (f^{(1)}(x_1))^2 + f(x_1)(f^{(1)}(x_1))^2 + f^2(x_1) f^{(2)}(x_1)\end{aligned}$$

one has

$$\mathcal{P} = \mathcal{L}(\mathcal{M}, \frac{1}{s^3}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}, \mathcal{F}) \quad .$$

Clearly, $\mathcal{M}$ is unimodular and $\mathcal{F}$ is stable. However, unless $f^{(2)}$ vanishes, $\mathcal{F}$ is not incrementally stable.

◇

4 Conclusion

In the course of bounded-input bounded-ouput stabilization of nonlinear plants, those that can be transformed into an LTI one are of particular interest due to a vast pool of theoretical and computational tools associated with LTI design problems. The existing transformation immediately suggests a simple design approach: design a stabilizing control law for the transformed LTI plant and then use the transformed control law for the nonlinear plant. As it is shown in the study, such an approach is not a sure design procedure unless more *restrictions are imposed on the transformation.* The design for the transformed LTI plant is really not decoupled from the original nonlinear design problem; hence, one should not rule out intentionally nonlinear controller design for the transformed LTI plant. Although this might seemingly defy the purpose of the transformation, since the transformed design problem is yet another nonlinear control design problem, it also emphasizes the need for more results in nonlinear control design for LTI plants. Along this line, for a given LTI plant, one has a complete parametrization of all nonlinear stabilizing controllers ; however, the choice of the parameter subject to a particular cost is not straightforward. Another approach is based on optimal control with non-quadratic integrand cost criteria from which nonlinear state-feedback laws are derived.

5 Appendix

Proof of Fact 6

Let $\mathcal{P} = \mathcal{L}(\mathcal{M}, P, \mathcal{F})$. From Figure 1 , the equations relating the (u, y) pair and (v, Pv) pair define the map $\mathcal{T}$ as follows:

$$\begin{aligned}\mathcal{T}(v, Pv) &= (u, y) &&= (v - \mathcal{F}\mathcal{M}^{-1}Pv, \mathcal{M}^{-1}Pv) \\ (v, Pv) &= \mathcal{T}^{-1}(u, y) &&= (u + \mathcal{F}y, \mathcal{M}y) \quad .\end{aligned}$$

Since $\mathcal{F}$ is stable and $\mathcal{M}$ is unimodular, $\mathcal{T}$ defined above is also unimodular. By unimodularity of $\mathcal{T}$,

$$(u, y) \in \mathcal{G}_{BB}(\mathcal{P}) \iff (v, Pv) \in \mathcal{G}_{BB}(P) \quad ,$$

which establishes Fact 6 1) .

Let $(u, y) \in \mathcal{G}_{UB}(\mathcal{P})$. Since $\mathcal{M}$ is stable, $Pv = \mathcal{M}y$ is bounded. Since u is unbounded and $\mathcal{F}\mathcal{M}^{-1}Pv$ is bounded, $v = u + \mathcal{F}\mathcal{M}^{-1}Pv$ is unbounded. Hence $(v, Pv) \in \mathcal{G}_{UB}(P)$. Similarly, v unbounded, Pv bounded imply y bounded and $u + \mathcal{F}y$ unbounded, hence u is unbounded. This establishes part 2) .

□

Proof of Lemma 10

By Fact 9 , the stability of $S(P, \mathcal{C}_P)$ implies that $\mathcal{C}_P = (U + DQ)(V - NQ)^{-1}$, for some stable Q . Any input output pair corresponding to the map $\mathcal{C}_P$ can be expressed as the pair $((V - NQ)\xi, (U + DQ)\xi)$, where ξ denotes the pseudo-state of this particular factorization. For this particular factorization, the input output pair of $\mathcal{C}_P$ is bounded if and only if ξ is bounded. Moreover, since $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{M}^{-1}$ are stable, the closed-loop system $S(\mathcal{P}, (\mathcal{C}_P\mathcal{M} + \mathcal{F}))$ is stable if and only if the closed-loop map from (u_1, u_2) to ξ is stable. Writing the equations describing the feedback system $S(\mathcal{P}, (\mathcal{C}_P\mathcal{M} + \mathcal{F}))$ in terms of ξ, we obtain (see Figure 1 and equation (2))

$$\widetilde{D}\mathcal{M}y = \widetilde{N}(\, u_1 + \mathcal{F}y - \mathcal{F}(u_2 + y) - (U + DQ)\xi \,) \qquad (4)$$

$$(V - NQ)\xi = \mathcal{M}(u_2 + y) \quad . \qquad (5)$$

Adding $\widetilde{D}\mathcal{M}(u_2 + y)$ to both sides of (4), substituting (5) and (3) and using the linearity of $\widetilde{N}$ and $\widetilde{D}$, we obtain

$$\xi = \widetilde{N}u_1 - \widetilde{N}(\mathcal{F}(y + u_2) - \mathcal{F}y) + \widetilde{D}(\mathcal{M}(y + u_2) - \mathcal{M}y) \quad .$$

Clearly, if $\mathcal{F}$ and $\mathcal{M}$ are incrementally stable, ξ is bounded for all bounded u_1 and u_2 . □

Proof of Theorem 1

"if" Follows by Lemma 10.

"only if" Follows by Example 11 , by setting $\mathcal{M} = I$, $P = 1/s$, $\mathcal{F} = (\cdot)^2$ and $\mathcal{C}_P = 1$.

□

Proof of the claim in Example 14

The closed loop system $S(1/s, I + (\cdot)^3)$ is described by

$$\dot{x} = u_1 - (x + u_2) - (x + u_2)^3 \quad .$$

The closed-loop system $S(\mathcal{P}, I + (\cdot)^2 + (\cdot)^3)$ is described by

$$\dot{x} = u_1 + x^2 - (x + u_2) - (x + u_2)^2 - (x + u_2)^3 \quad .$$

For u_1 and u_2 in L_∞ , using the Lyapunov function candidate $V(x) = x^2$, in both cases for $|x|$ sufficiently large $\dot{V}$ along the solution trajectories is eventually negative. Hence the closed-loop map from (u_1, u_2) to (u, x) is stable in both cases.

6 References

[Des1] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, 1975.

[Ham1] J. Hammer, "Fraction Representations of Non-Linear Systems: A Simplified Approach," *International Journal of Control*, vol. 46, no. 2, pp. 455–472, February 1987.

[Isi1] A. Isidori, *Nonlinear Control Systems* , Springer-Verlag, 1989.

[Kab1] M. G. Kabuli, *Factorization Approach to Nonlinear Feedback Systems* , Ph.D. Dissertation, University of California, Berkeley, 1989.

[Kan1] I. Kanellakopoulos, P. V. Kokotovic and A. S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241–1253, November 1991.

[Sas1] S. S. Sastry and A. Isidori, "Adaptive Control of Linearizable Systems," *IEEE Transactions on Automatic Control*, vol. 34, pp. 1123–1131, November 1989.

[Ver1] M. S. Verma, "Coprime Fractional Representations and Stability of Nonlinear Feedback Systems," *International Journal of Control*, vol. 48, pp. 897–918, 1988.

[Vid1] M. Vidyasagar and N. Viswanadham, "Stabilization of Linear and Nonlinear Dynamical Systems using an Observer-Controller Configuration," *Systems and Control Letters*, vol. 1, no. 2, pp. 87–91, 1981.