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# VARIATIONAL THEORY OF MOTION OF CURVED, TWISTED AND EXTENSIBLE ELASTIC RODS

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The variational theory of three dimensional motion of curved twisted and extensible elastic rods is obtained based entirely on the kinematical variables of position and rotations. The constitutive relations that define the resistive couples and the axial force as gradients of the strain energy function are established. A candidate for the strain energy function, derived on the basis of classical assumptions, is presented.

## INTRODUCTION

In the Historical Introduction of the "A Treatise on the Mathematical Theory of Elasticity," Love (1892) narrates that in 1742 Daniel Bernoulli wrote to Euler suggesting that the differential equation of the elastica could be found by making the integral of the work done or the square of the curvature a minimum. Acting on this suggestion Euler was able to obtain the differential equation of the elastica and the various forms of it. Thus the concept of the strain energy was born and the foundation of the variational theory of elastic rods were laid out. The equilibrium equations that were much later developed by Love are applicable to an initially bent and twisted rod.

Our aim in this paper is to establish a variational formulation for the title problem and in the process infer the existence of the strain energy function and determine the constitutive relations that relate this function to the bending and twisting couples and the axial force within the rod. This development together with equations of motion and the geometry of deformation define a direct approach and an exact nonlinear theory for the three dimensional motion of a one dimensional elastic medium capable of resisting bending twisting and extension. Going a step further, in order to actually construct an explicit form for the strain energy function, we enter the realm of hypothesis and use Kirchhoff's description of deformations in a thin rod. This view enables us to determine a strain energy function that can be used in engineering applications.

The recent history of investigations of the rod theories consists of developments along two separate streams, the direct approach and approximations from three-dimensional continuum. In the direct approach a one-dimensional continuum view is pursued and the medium is supposed endowed, in addition to its position, with vector fields, the directors,

that are to be interpreted appropriately to define bending, twist and extension properties of a rod. This approach has its origin in the work of E. and F. Cosserat (1909) and numerous investigations have contributed to it among them Naghdi (1982), Naghdi and Rubin (1984), Whitman and DeSilva (1970), Green and Laws (1966), and Eriksen (1970). Extensive investigation into the qualitative aspects of the nonlinear theory such as questions of existence of solutions and global behavior have been carried out by Antman (1976). His basic work entitled "The Theory of Rods" (1972) describes these theories both as approximations to the three-dimensional continuum theory and as a one-dimensional continuum with directors. The work presented here, although pertains to a one-dimensional continuum does not use directors, but is formulated entirely on the basis of kinematical quantities consisting of the position vector of points along the curve of centroids and the orientation angles of the cross sections of the rod relative to a fixed coordinate system. It is a generalization of the work of Tadjbakhsh (1966) in which the theory of planar motion of the extensible elastica was described.

The history of construction of approximate theories in the context of three-dimensional nonlinear continuum theory is also varied and to it many investigators including some of the above authors have contributed, see for example, Naghdi and Wenner (1974).

## KINEMATICS

An elastica is a nearly uniform slender rod of finite length. In the unstressed state the centroids of the cross section form a space curve  $C$  that is called the reference curve with an arc length  $s$ . The orientation of the principal axes of the cross section vary continuously along the rod. This means that in the unstrained state the rod has arbitrary twist and curvatures. With respect to an inertial Cartesian frame  $x$  the position of a point  $s$  in the unstrained state is denoted by  $X = X_i(s)n_i$ ,  $i = 1,2,3$ , with  $n_i$  being the dextral unit vectors of the frame  $x$ .

The cross sectional area can be slowly varying function of  $s$  and will be denoted by  $A(s)$ . As the rod deforms the curve  $C$  acquires new configuration  $c$  that changes with time. The arc length along  $c$  is denoted by  $\xi$  that depends upon  $s$  and  $t$ , i.e.  $\xi = \xi(s,t)$ . The position of a point  $s$  on  $c$  at an arbitrary time is  $x(s,t)$  so that

$$x(s,t_0) = X(s), \quad \xi(s,t_0) = s \quad (1a,b)$$

where  $t_0$  is a reference time at which the rod is in the unstrained state. Also

$$\frac{\partial \xi}{\partial s} = \xi' = (x'_i x'_i)^{\frac{1}{2}} \quad (2)$$

where prime denotes differentiation with respect to time and summation over a repeated index is implied. The strain  $e$  is defined by

$$e = \xi' - 1 \quad (3)$$

where  $e > 0$  denotes extension and  $e < 0$  contraction. The strict positivity of  $\xi'$  implies that  $-1 \leq e < \infty$ .

Attached to any point  $s$  of  $c$  a Cartesian coordinate frame  $y$  will be assumed and will be referred to as the body reference frame. The coordinate axes of the body frame are  $y_1, y_2, y_3$  with the  $y_3$  axis pointing in the direction of increasing  $s$  and  $y_1$  and  $y_2$  being the principal axes of the cross section. The dextral unit vectors of the  $y$  frame will be denoted by  $e_i, i = 1, 2, 3$ , with their orientations at the reference time  $t_0$  being  $E_i$ .

Denoting by  $l_{ij}(s, t)$  and  $L_{ij}(s)$  the elements of the matrices of direction cosines of the dextral sets  $e_i$  and  $E_i$  one has

$$n_i = l_{ij} e_j = L_{ij} E_j \quad (4)$$

and

$$e_i = l_{ij} n_j, \quad E_i = L_{ij} n_j \quad (5a, b)$$

The angles  $\varphi_i$  represent rotations between the corresponding pairs of  $E_i$  and  $e_i$  when these directions are assumed to issue from a common origin, Fig. 1. These angles are determined through

$$\cos \varphi_i = e_i \cdot E_i = l_{ji} L_{ji}, \quad i=1, 2, 3 \text{ (sum only on } j) \quad (6)$$

The direction cosines  $l_{ij}$  are characterized non-uniquely by three orientations angles  $\theta_1(s, t), \theta_2(s, t)$  and  $\theta_3(s, t)$ . These angles can be selected in a variety of ways and represent three finite rotations about unit vectors  $e_i$  or  $n_i$ . If these rotations are properly selected the fixed orientation  $n_i$  may be brought to any arbitrary body orientation  $e_i$ . Kane et al. (1983) list at least 24 possibilities for the order of rotations of the angles  $\theta_1, \theta_2, \theta_3$  about the body set of unit vectors  $e_i$  or the fixed set of unit vectors  $n_i$ .

Regardless of the particular choice of orientation angles the angular velocity  $\omega = \omega_i e_i$  of a cross sectional element  $Ad_s$ , Fig. 1 of the rod is determined uniquely from

$$\omega_i = \eta_{igh} \dot{l}_{ig} l_{ih}, \quad (\dot{\phantom{x}} = \frac{\partial}{\partial t}) \quad (7)$$

where  $\eta_{ijk} = \epsilon_{ijk} (\epsilon_{ijk} + 1)/2$ , (no sum on  $i, j, k$ ) and  $\epsilon_{ijk}$  is the alternator tensor with the non-zero components  $\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = +1$  and  $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$ .

The curvature vector  $K = K_i e_i$  of the rod can be defined in a similar way with  $K_3$  representing twist and  $K_1$  and  $K_2$  representing bending curvatures about the principal directions of the cross section. Using the dynamical analogy of E.I. Routh, Love (1944) has noted that if the frame  $y$  were to move with unit speed along the curve  $c$  such that at any point  $\xi$  of  $c$  it has the orientation of the  $y$  frame at that point then the angular velocities  $\omega_1$  and  $\omega_2$  will be the principal curvatures  $K_1$  and  $K_2$  of the rod. Also the angular velocity  $\omega_3$

will be the twist curvature  $K_3$  of the rod. Thus the formulas that define the angular velocities from direction cosines can be used to determine curvatures, provided time differentiation is replaced by differential with respect to  $\xi$ . Therefore one has

$$K_i = \eta_{igh} \frac{\partial l_{ig}}{\partial \xi} l_{ih} = \frac{1}{1+e} \eta_{igh} l'_{ig} l_{ih} \equiv \frac{k_i}{1+e}, \quad (8)$$

where differentiation with respect to  $\xi$  has been replaced with differentiation with respect to  $s$  and curvature parameters  $k_i = (1+e)K_i$  is also introduced. For future use one may note the formulas for derivatives of direction cosines  $l_{ij}$  and the unit vectors  $e_i$ .

$$l'_{ij} = (1+e) \epsilon_{ghj} l_{ig} k_h \quad \dot{l}_{ij} = \epsilon_{ghj} l_{ig} w_h \quad (9a,b)$$

$$\dot{e}_i = \epsilon_{kji} k_j e_k \quad \dot{e}_i = \epsilon_{kji} w_j e_k \quad (10a,b)$$

if  $K_i$  be the curvatures of the rod in the unstrained state ( $e = 0$ ) then from (8)

$$K_i = \eta_{igh} L'_{ig} L_{ih} \quad (11)$$

Since  $e_3$  and  $\partial \mathbf{x} / \partial \xi$  are both unit tangents to the central line one has

$$\mathbf{x}'_i = (1+e)l_{i3} \quad (12)$$

We assume that the center of mass of the cross sections coincide with the centroids. The linear and the central angular momentum per unit length are then given by

$$\mathbf{p} = \rho A \mathbf{x}_i \mathbf{n}_i \quad (13)$$

and

$$\mathbf{H} = \rho I \boldsymbol{\omega} = \rho (I_{11} \omega_1 \mathbf{e}_1 + I_{22} \omega_2 \mathbf{e}_2 + I_{33} \omega_3 \mathbf{e}_3) \quad (14)$$

where  $\rho$  is the mass density per unit unstrained length and  $I$  is the diagonal moment of inertia tensor with components

$$I_{11} = \int_{\Delta} y_2^2 dA, \quad I_{22} = \int_{\Delta} y_1^2 dA, \quad I_{33} = I_{11} + I_{22} \quad (15)$$

### Equations of Motion

Referring to the body set of axes  $e_i$  one can define the vector  $\mathbf{F}$  of the resultant shear stresses  $F_1$  and  $F_2$  and the axial stress resultant  $F_3$ . Similarly, one may define the couple stress vector  $\mathbf{M}$  consisting of the bending moments  $M_1$  and  $M_2$  and the torque  $M_3$ . Explicitly we have

$$\mathbf{F} = F_i \mathbf{e}_i \quad \text{and} \quad \mathbf{M} = M_i \mathbf{e}_i \quad (16)$$

The well known dynamic equilibrium of the rod can be expressed by the equations of the balance of linear momentum

$$\mathbf{F}' + \mathbf{f} = \dot{\mathbf{p}} \quad (17)$$

and of the balance of angular momentum

$$\mathbf{M}' + \mathbf{x}' \times \mathbf{F} + \mathbf{m} = \dot{\mathbf{H}} \quad (18)$$

wherein  $\mathbf{f}$  and  $\mathbf{m}$  represent distributed force and moment acting on the rod. The scalar components of these equations can be referred to the body set of axes. For this purpose one needs to express all vector quantities in terms of unit vectors  $\mathbf{e}_i$  and use (9)–(10). Then (17) becomes

$$F'_1 + k_2 F_3 - k_3 F_2 + f_1^y = \rho A \bar{x}_j l_{j1} \quad (19a)$$

$$F'_2 + k_3 F_1 - k_1 F_3 + f_2^y = \rho A \bar{x}_j l_{j2} \quad (19b)$$

$$F'_3 + k_1 F_2 - k_2 F_1 + f_3^y = \rho A \bar{x}_j l_{j3} \quad (19c)$$

while (18) assumes the form

$$M'_1 + k_2 M_3 - k_3 M_2 - (1+e)F_2 + m_1^y = \rho I_1 (\omega_1 + \omega_2 \omega_3) \quad (20a)$$

$$M'_2 + k_3 M_1 - k_1 M_3 - (1+e)F_1 + m_2^y = \rho I_2 (\omega_2 + \omega_1 \omega_3) \quad (20b)$$

$$M'_3 + k_1 M_2 - k_2 M_1 + m_3^y = \rho [J \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2] \quad (20c)$$

where  $I_1 = I_{11}$ ,  $I_2 = I_{22}$  and  $I = J_{33} = I_1 + I_2$ . The superscript  $y$  on the components of  $\mathbf{f}$  and  $\mathbf{m}$  denote the components of these vectors in the body reference frame.

To express the equations of motion in the inertial frame we introduce the components of the stress resultants in that frame. Thus

$$F_i^x = l_{ij} F_j \quad M_i^x = l_{ij} M_j \quad (21a,b)$$

Then (19) becomes

$$F_i^x + f_i^x = \rho A \bar{x}_i \quad (22)$$

and (20) assumes the form

$$M_1^x - (1+e)F_2 + m_1^x = \rho (I_{rj} l_{1j} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{1g} \omega_s \omega_r) \quad (23a)$$

$$M_2^x - (1+e)F_1 + m_2^x = \rho (I_{rj} l_{2j} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{2g} \omega_s \omega_r) \quad (23b)$$

$$M_3^{x'} + m_3^x = \rho(I_{rj} l_{3j} \dot{\omega}_r + \epsilon_{grj} I_{sj} l_{3g} \omega_s \omega_r) \quad (23c)$$

Either of the set of equations (19)–(20) or (22)–(23) can be considered as the governing differential equations of motion. These equations will have to be supplemented with constitutive relations that define resultant axial stress  $F_3$  and the resultant bending and twisting couples  $M_1, M_2, M_3$  in terms of the axial strain  $e$  and curvatures  $k_1, k_2, k_3$ . In the next section we consider the derivation of these constitutive relations.

### Constitutive Relations

We assume that the motion of the elastic rod is equivalent to the stationarity of the Hamiltonian  $H$  which is defined by

$$\begin{aligned} H[l_{ij}(\varphi), x_i, e] &= \int_{t_1}^{t_2} \int_{s_1}^{s_2} \mathcal{L} ds dt + \\ &\int_{t_1}^{t_2} \left[ \bar{F}_i \bar{X}_i + \bar{M}_i \bar{\varphi}_i \right]_{s_1}^{s_2} dt - \int_{s_1}^{s_2} \left[ \rho A \bar{v}_i x_i \right]_{t_1}^{t_2} ds - \\ &\int_{s_1}^{s_2} \rho \left[ I_1 \bar{\omega}_1 \varphi_1 + I_2 \bar{\omega}_2 \varphi_2 + J \bar{\omega}_3 \varphi_3 \right]_{t_1}^{t_2} ds \end{aligned} \quad (24)$$

where  $\mathcal{L}$  is the action density function

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \rho A \dot{x}_i \dot{x}_i + \frac{1}{2} \rho (I_1 \omega_1^2 + I_2 \omega_2^2 + J \omega_3^2) - w(e, k_i) + \\ &\lambda_i [l_{i3} (1+e) - x_i'] + f_i^x x_i + m_i^y \varphi_i \end{aligned} \quad (25)$$

and  $\bar{F}_i^1, \bar{M}_i^1$  and  $\bar{F}_i^2, \bar{M}_i^2$  are the applied forces and moments at ends  $s_1$  and  $s_2$ , respectively.

Also  $\bar{v}_i^{1,2}$  and  $\bar{\omega}_i^{1,2}$  are the initial and final linear and angular velocities. The strain energy function  $w$  depends upon the kinematical variables  $e$  and  $k_i$ . The precise nature of this dependence is the constitutive relations that we seek and is a consequence of the stationarity of  $H$ . The functions  $\lambda_i$  are the Lagrange multipliers that allow the constraints (12) to be incorporated within the Hamiltonian. As a result  $x_i$  and  $l_{ij}$  can be regarded as independent variables. Additionally the constraint (12) implies the definition (2)–(3) for the strain  $e$  and hence in (24)  $e$  can also be viewed as an independent variable. To see this we need to note that if each side of (12) is multiplied by itself we obtain  $x_i' x_i' =$

$(1+e)^2 l_{i3} l_{i3} = (1+e)^2$  which is restatement of (2)–(3). The terms  $f_i^x x_i$  and  $m_i^y \varphi_i$  in (25) represent the density of the potential of the applied forces and moments on the rod. As stated in (5) the angles  $\varphi_i$  are the rotations from  $E_i$  to  $e_i$ .

With these preliminaries we note that the Euler equation corresponding variations  $\delta x_i$  is simply  $\lambda'_i + \dot{f}_i^x = \rho A \bar{x}_i$  which when compared with (22) reveals that  $\lambda_i = F_i^x$ . Next considering the variations with respect to  $e$  we obtain

$$\frac{\partial W}{\partial e} = F_i^x l_{i3} = F_3 \quad (26)$$

which is the constitutive relationship determining the axial force  $F_3$  as the derivative of strain energy with respect to axial strain  $e$ .

We now turn to the Euler equation corresponding to the variation  $\delta \varphi_1$ . For this purpose we note that  $l_{ij}$ ,  $\omega_i$  and  $k_i$  depend on  $\varphi_1$ . For orientation angles of the cross section we select the sequence of body rotations first  $\theta_2 e_2$ , second  $\theta_3 e_3$  and the third  $\theta_1 e_1$  with  $\theta_1 \equiv \varphi_1$ . In this sequence the last rotation is through  $\varphi_1$  with respect to which variation is sought. The matrix  $l$  of the direction cosines is given by

$$l = B(\theta_2)C(\theta_3)A(\theta_1) = \begin{bmatrix} C_2 C_3 & -C_1 C_2 S_3 + S_1 S_2 & S_1 C_2 S_3 + C_1 S_2 \\ S_3 & C_1 C_3 & -S_1 C_3 \\ -S_2 C_3 & C_1 S_2 S_3 + S_1 C_2 & -S_1 S_2 S_3 + C_1 C_2 \end{bmatrix} \quad (27)$$

where

$$C_i = \cos \theta_i, \quad S_i = \sin \theta_i \quad (28)$$

$$A(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad B(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$C(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (29)$$

Subsequently, we find from (7) and (8)

$$\omega_1 = \dot{\theta}_2 S_3 + \dot{\theta}_1, \quad \omega_2 = \dot{\theta}_2 C_1 C_3 + \dot{\theta}_3 S_1, \quad \omega_3 = -\dot{\theta}_2 S_1 C_3 + \dot{\theta}_3 C_1 \quad (30)$$

$$k_1 = \dot{\theta}_2' S_3 + \dot{\theta}_1', \quad k_2 = \dot{\theta}_2' C_1 C_3 + \dot{\theta}_3' S_1, \quad k_3 = -\dot{\theta}_2' S_1 C_3 + \dot{\theta}_3' C_1 \quad (31)$$

From (24) we have

$$\left\{ -\frac{\partial W}{\partial k_i} \frac{\partial k_i}{\partial \varphi_1} + \frac{\partial}{\partial s} \left( \frac{\partial W}{\partial k_i} \frac{\partial k_i}{\partial \varphi_1} \right) + (1+e)F_i^x \left[ \frac{\partial l_{i3}}{\partial \varphi_1} - \frac{\partial}{\partial s} \left( \frac{\partial l_{i3}}{\partial \varphi_1} \right) \right] \right\} + m_i^y$$

$$= \frac{\partial}{\partial t} \left[ \rho I_1 \omega_1 \frac{\partial \omega_1}{\partial \varphi_1} \right] - \rho I_2 \omega_2 \frac{\partial \omega_2}{\partial \varphi_1} - \rho J \omega_3 \frac{\partial \omega_3}{\partial \varphi_1} \quad (32)$$

Noting that  $\theta_1 \equiv \varphi_1$ , we have from (30)  $\partial \omega_1 / \partial \varphi_1 = 1$ ,  $\partial \omega_2 / \partial \varphi_1 = \omega_3$ ,  $\partial \omega_3 / \partial \varphi_1 = -\omega_2$ . Also from (31)  $\partial k_1 / \partial \varphi_1 = 0$ ,  $\partial k_2 / \partial \varphi_1 = k_3$ ,  $\partial k_3 / \partial \varphi_1 = -k_2$  and from (27)  $\partial l_{i3} / \partial \varphi_1 = -l_{i2}$ . Using these results and the inverse of (21a), (32) becomes

$$\left( \frac{\partial W}{\partial k_1} \right)' + k_2 \frac{\partial W}{\partial k_3} - k_3 \frac{\partial W}{\partial k_2} - (1+e)F_2 + m_1^y = \rho I_1 (\dot{\omega}_1 + \omega_2 \omega_3) \quad (33)$$

In exactly the same manner one can proceed to determine the Euler equation for a variation  $\delta \varphi_2$ . Now the consecutive sequence of body rotations  $\theta_3 e_3$ ,  $\theta_1 e_1$  and  $\theta_2 e_2$  is selected with  $\theta_2 \equiv \varphi_2$ . Without going into details one obtains

$$\left( \frac{\partial W}{\partial k_2} \right)' + k_3 \frac{\partial W}{\partial k_1} - k_1 \frac{\partial W}{\partial k_3} - (1+e)F_1 + m_2^y = \rho I_2 (\dot{\omega}_2 + \omega_1 \omega_3) \quad (34)$$

For variation of  $\varphi_3$  we adopt the consecutive sequence of body rotations  $\theta_1 e_1$ ,  $\theta_2 e_2$  with  $\theta_3 = \varphi_3$ . The matrix of direction cosines is

$$l = A(\theta_1)B(\theta_2)C(\theta_3) = \begin{bmatrix} C_2 C_3 & -C_2 S_3 & S_2 \\ S_1 S_2 C_3 + C_1 S_3 & -S_1 S_2 S_3 + C_1 C_3 & -S_1 C_2 \\ -C_1 S_2 C_3 + S_1 S_3 & C_1 S_2 S_3 + S_1 C_3 & C_1 C_2 \end{bmatrix} \quad (35)$$

with angular velocities of the cross section and the curvatures given by

$$\omega_1 = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3, \quad \omega_2 = \dot{\theta}_2 C_3 - \dot{\theta}_1 C_2 S_3, \quad \omega_3 = \dot{\theta}_3 + \dot{\theta}_1 S_2 \quad (36)$$

$$k_1 = \theta_1' C_2 C_3 + \theta_2' S_3, \quad k_2 = \theta_2' C_3 - \theta_1' C_2 S_3, \quad k_3 = \theta_3' + \theta_1' S_2 \quad (37)$$

For this case  $l_{i3}$  does not depend on  $\theta_3$  and hence the Euler variational equation assumes the form

$$\left( \frac{\partial W}{\partial k_3} \right)' + k_1 \frac{\partial W}{\partial k_2} - k_2 \frac{\partial W}{\partial k_1} + m_3^y = \rho [J \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2] \quad (38)$$

The specified boundary conditions at  $s = s_1, s_2$  must be consistent with

$$[(\bar{F}_1 - F_1^x) \delta x_1 + (\bar{M}_1 - M_1) \delta \varphi_1]_{s_1}^{s_2} = 0 \quad (39)$$

for arbitrary and independent variations  $\delta x_1$  and  $\delta \varphi_1$ . Similar restrictions are imposed on initial and final data, i.e.

$$\left\{ \rho A (\dot{x}_1 - \bar{v}_1) \delta x_1 + \rho [I_1 (\omega_1 - \bar{\omega}_1) \delta \varphi_1 + I_2 (\omega_2 - \bar{\omega}_2) \delta \varphi_2 + J (\omega_3 - \bar{\omega}_3) \delta \varphi_3] \right\}_{t_1}^{t_2} = 0 \quad (40)$$

Comparison of equations (33), (34) and (38) with equations (20a,b,c) respectively, establishes the constitutive relations

$$M_i = \frac{\partial W}{\partial k_i} \quad i = 1, 2, 3 \quad (41)$$

### A STRAIN ENERGY FUNCTION

In order to gain an insight into the nature of the strain energy function we consider the strain of the lines and angles in the cross section of the rod. For this purpose we invoke the Kirchhoff hypothesis which assumes that the plane cross sections of rod that are normal to the axial direction in the unstrained state remain normal to the strained axial direction during deformation. Therefore the position vector to a material point in the cross-section before and after deformation can be given by

$$\mathbf{R} = \mathbf{X}(s) + y_1 \mathbf{E}_1(s) + y_2 \mathbf{E}_2(s) \quad (42)$$

and

$$\mathbf{r} = \mathbf{x}(s) + \alpha(s) [y_1 \mathbf{e}_1(s) + y_2 \mathbf{e}_2(s)] \quad (43)$$

respectively. The parameter  $\alpha(s)$  is to be fixed by enforcing traction-free boundary conditions on the lateral surface of the rod.

Using the concept of extensional strains for stretching of line elements and distortion of angles between perpendicular lines as shear strains (Wempner, 1991), we define components of strain by

$$\epsilon_{ij} = \frac{1}{2} (\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) \quad (44)$$

where

$$\begin{aligned} \mathbf{g}_1 &= \frac{\partial \mathbf{r}}{\partial y_1} = \alpha \mathbf{e}_1, \quad \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial y_2} = \alpha \mathbf{e}_2, \\ \mathbf{g}_3 &= \frac{\partial \mathbf{r}}{\partial y_3} = \alpha' y_1 \mathbf{e}_1 + \alpha' y_2 \mathbf{e}_2 + \alpha y_1 \mathbf{e}'_1 + \alpha y_2 \mathbf{e}'_2 + (1 + \epsilon) \mathbf{e}_3 \end{aligned} \quad (45)$$

$$\mathbf{G}_1 = \frac{\partial \mathbf{R}}{\partial y_1} = \mathbf{E}_1, \quad \mathbf{G}_2 = \frac{\partial \mathbf{R}}{\partial y_2} = \mathbf{E}_2,$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial y_3} = y_1 \mathbf{E}'_1 + y_2 \mathbf{E}'_2 + \mathbf{E}_3 \quad (46)$$

Using (8) we can establish

$$\mathbf{e}'_i \cdot \mathbf{e}_j = \epsilon_{ijm} \mathbf{k}_m \quad (47)$$

$$\mathbf{E}'_i \cdot \mathbf{E}_j = \epsilon_{ijm} \mathbf{K}_m \quad (48)$$

where  $\mathbf{K}_i$  is the curvatures and twist in the unstrained state. Therefore (9) yields as the strain components

$$\begin{aligned} \epsilon_{11} &= \epsilon_{22} = \frac{1}{2}(\alpha^2 - 1), \quad \epsilon_{12} = 0 \\ \epsilon_{13} &= \frac{1}{2}[\alpha(\alpha' y_1 - \alpha y_2 k_3) + y_2 K_3] \\ \epsilon_{23} &= \frac{1}{2}\alpha(\alpha' y_2 + \alpha y_2 k_3) - y_1 K_3] \\ \epsilon_{33} &= e + \frac{1}{2}e^2 - y_1[(1+e)\alpha k_2 - K_2] + y_2[(1+e)\alpha k_1 - K_1] - y_1 y_2(\alpha^2 k_1 k_2 - K_1 K_2) + \\ &\quad \frac{1}{2}y_1^2[\alpha'^2 + \alpha^2(k_2^2 + k_3^2) - K_2^2 - K_3^2] \\ &\quad + \frac{1}{2}y_2^2[\alpha'^2 + \alpha^2(k_1^2 + k_3^2) - K_1^2 - K_3^2] \end{aligned} \quad (49)$$

For a linear isotropic elastic material the non-zero stress components per unit strained area are

$$\begin{aligned} \sigma^{11} &= (\lambda + G)(\alpha^2 - 1) + \lambda \epsilon^{33}, \quad \sigma^{12} = 0 \\ \sigma^{22} &= (\lambda + G)(\alpha^2 - 1) + \lambda \epsilon^{33}, \quad \sigma^{23} = 2G \epsilon_{23} \\ \sigma^{33} &= \lambda(\alpha^2 - 1) + (\lambda + 2G)\epsilon^{33}, \quad \sigma^{31} = 2G \epsilon_{31} \end{aligned} \quad (50)$$

where  $\lambda$  and  $G$  are Lamé's constant and the shear modulus, respectively. The traction per unit undeformed area is then given by

$$\mathbf{t}^3 = \sigma^{3i} \mathbf{g}_i \quad (51)$$

One can define the axial stress resultant  $F_3$  by

$$\begin{aligned} F_3 &= \int_A \mathbf{t}^3 \cdot \mathbf{e}_3 \, dA = \\ &= A(1+e) \left[ (\lambda + 2G)(e + \frac{1}{2}e^2) + \lambda(\alpha^2 - 1) \right] \\ &\quad + (\lambda + 2G) \left[ I_1[(1+e)\alpha k_1 - K_1] \alpha k_1 + I_2[(1+e)\alpha k_2 - K_2] \alpha k_2 \right. \\ &\quad \left. + \frac{1}{2} I_1(1+e)(\alpha'^2 + \alpha^2 k_1^2 - K_1^2 + \alpha^2 k_3^2 - K_3^2) \right. \\ &\quad \left. + \frac{1}{2} I_2(1+e)(\alpha'^2 + \alpha^2 k_2^2 - K_2^2 + \alpha^2 k_3^2 - K_3^2) \right] \end{aligned} \quad (52)$$

Similarly we have

$$M_1 = \int_A \alpha y_2 t^3 \cdot e_3 \, dA = \alpha(\lambda+2G)I_1 \left[ (1+e)\alpha k_1 - K_1 \right] (1+e) \\ + I_1 \alpha^2 k_1 \left[ \lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{4}e^2) \right] \quad (53)$$

$$M_2 = - \int_A \alpha y_1 t^3 \cdot e_3 \, dA = \alpha(\lambda+2G)I_2 \left[ (1+e)\alpha k_2 - K_2 \right] (1+e) \\ + I_2 \alpha^2 k_2 \left[ \lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{4}e^2) \right] \quad (54)$$

$$M_3 = - \int_A \alpha (y_1 t^3 \cdot e_3 - y_2 t^3 \cdot e_1) \, dA = JG\alpha^2(\alpha^2 k_3 - K_3) \\ + J\alpha^2 k_3 \left[ \lambda(\alpha^2 - 1) + (\lambda + 2G)(e + \frac{1}{4}e^2) \right] \quad (55)$$

One may note that the integrability conditions

$$\frac{\partial F_3}{\partial k_1} = \frac{\partial M_1}{\partial e}, \quad \frac{\partial F_3}{\partial k_2} = \frac{\partial M_2}{\partial e}, \quad \frac{\partial F_3}{\partial k_3} = \frac{\partial M_3}{\partial e}, \\ \frac{\partial M_1}{\partial k_2} = \frac{\partial M_2}{\partial k_1}, \quad \frac{\partial M_1}{\partial k_3} = \frac{\partial M_2}{\partial k_1}, \quad \frac{\partial M_2}{\partial k_3} = \frac{\partial M_3}{\partial k_2} \quad (56)$$

are satisfied. Hence existence of a strain energy function is assured and by integration we have

$$W = A \frac{\lambda+2G}{2} e^2 (1+e+\frac{e^2}{4}) - A\lambda(1-\alpha^2)(e+\frac{1}{4}e^2) \\ + \lambda I_1 \alpha^2 (\alpha^2 - 1) \frac{k_1^2}{2} + (\lambda+2G)I_1 \alpha k_1 \left[ \frac{1+2e^2+4e}{4} \alpha k_1 - (1+e)K_1 \right] \\ + \lambda I_2 \alpha^2 (\alpha^2 - 1) \frac{k_2^2}{2} + (\lambda+2G)I_2 \alpha k_2 \left[ \frac{1+2e^2+4e}{4} \alpha k_2 - (1+e)K_2 \right] \\ + \frac{\lambda+2G}{2} (1+e)^2 \left[ I_1 (\alpha'^2 + \alpha^2 k_1^2 - K_1^2) + I_2 (\alpha'^2 + \alpha^2 k_2^2 - K_2^2) \right] \\ + \lambda J \alpha^2 (\alpha^2 - 1) \frac{k_3^2}{2} + \frac{\lambda+2G}{2} J (e+\frac{1}{4}e^2) (\alpha^2 k_3^2 - K_3^2) + \frac{1}{2} G J \alpha (\alpha k_3 - K_3)^2 \quad (57)$$

For an initially straight rod  $K_1$  should be set equal to zero. The above form of  $W$  reflects material isotropy, i.e.  $W(e, k_1, k_2, k_3) = W(e, k_2, k_1, k_3)$ , provided that  $I_1 = I_2$ .

We note that positive curvatures imply positive bending moments and conversely negative curvatures imply negative bending moments provided that  $e > (-1 + 1/\sqrt{3})$  for  $\alpha = 1$ . This shows that equations (53) have a limited range of validity if the sense correspondence between moments and curvatures is to be retained. We also note the second order coupling between the squares of the curvatures and the axial strain  $e$  in (52), which implies that axial force can be generated by bending or twist only.

The parameter  $\alpha(s)$  depends on the boundary conditions applied at the lateral surface of the rod. If the lateral surface is fixed, then  $\alpha(s) = 1$ . For zero tractions on the lateral surface, a condition appropriate for thin flexible rods is adopted according to which the average of  $\sigma^{11}$  and  $\sigma^{22}$  over the cross-section should vanish, i.e.

$$\int_A \sigma^{11} dA = \int_A \sigma^{22} dA = 0 \quad (58)$$

The above condition reduces to

$$\begin{aligned} H(\alpha, k_i, e) &\equiv \alpha' J + \alpha^2 (I_1 k_1^2 + I_2 k_2^2 + J k_3^2 + \frac{A}{\nu}) \\ &- (I_1 K_1^2 + I_2 K_2^2 + J K_3^2) + 2Ae(1 + \frac{1}{2}e) - \frac{A}{\nu} = 0 \end{aligned} \quad (59)$$

This equation should be interpreted as a differential equation for  $\alpha(s)$  when  $k_i$  and  $e$  are known. To achieve this (52) is solved in an iterative procedure in which at every step  $k_i$  and  $e$  are known. We begin by writing (52) as  $\lim_{n \rightarrow \infty} H(\alpha_{n+1}, k_i^n, e_n) = 0$  with  $n = 0, 1, 2, 3, \dots$  For the first iteration ( $n=0$ ),  $e_0 = 0$ ,  $k_i^0 = K_i$ ,  $\alpha_1 = 1$ , and the six equations (15) - (16) after using (22), (33) and (57) contain only six unknown quantities  $F_1, F_2, e_1$  and  $k_i^1$  when the externally applied forces and moments are prescribed. Solution of this set of equations enables one to use  $k_i^1$  in the curvature-orientation angle relations such as (31) to determine the latter i.e.  $\theta_i$ . Now  $l_{ij}(\theta_k)$  are known and one proceeds to determine  $\varphi_i$  from (21) and  $x_i$  from (9) using the appropriate boundary conditions. The solution for the first iteration is complete. One enters (52) with  $e_1$  and  $k_i^1$  and computes  $\alpha_2$  and the iteration proceeds.

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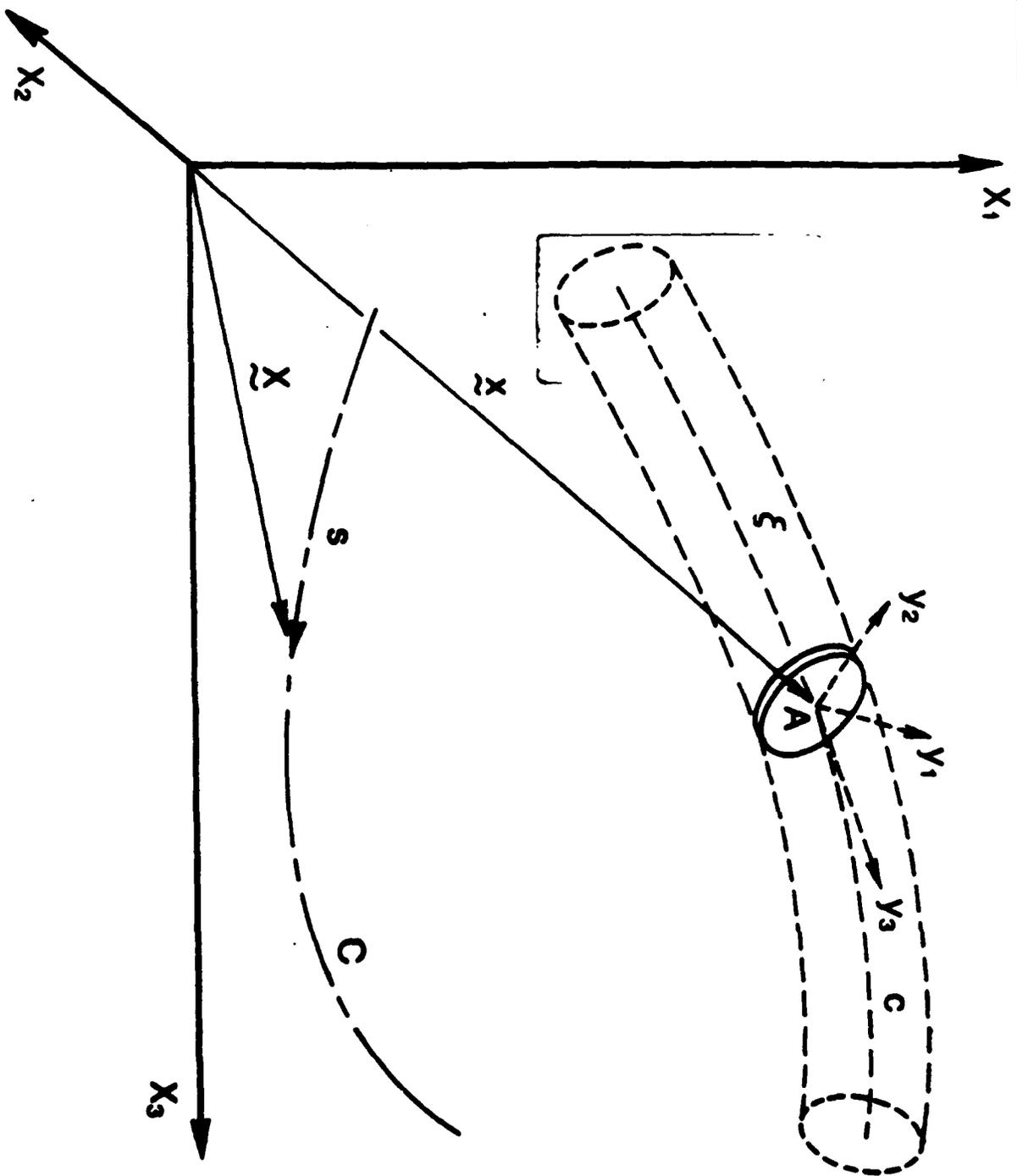


Fig. 1

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