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Solution of a Class of Sturm-Liouville Problems Using the Galerkin Method with Global Basis Functions

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ABSTRACT

Many problems in physics and engineering can be reduced to the Sturm-Liouville (S-L) Problem. For some classes of the S-L problem we develop a method that can rapidly result in computational solutions. Let us assume that we can write the problem as follows:

$$\frac{d^2U_i(r)}{dr^2} + [K(r) - \lambda_i]U_i(r) = 0$$

where L is an eigenvalue and K(r) is some well behaved (i.e. positive definite) function of r. Now let k correspond to some average value of K(r) over the domain of K(r). The boundary conditions are general. Then we easily solve the following problem:

$$\frac{d^2\psi_i(r)}{dr^2} + [k - \lambda_i]\psi_i(r) = 0.$$

This yields global basis functions. We determine an efficient method to obtain solutions of the form:

$$U_i = \sum_{i=1}^{N} a_{ij} \Psi_j$$

in which we determine the best expansion coefficients a_{ij} as well as the eigenvalues L_i's in terms of the l_i's and the a_{ij}'s. We also develop several perturbation methods from the technique. The method is then applied to typical problems found in acoustics and quantum mechanics.

I. INTRODUCTION

One of the most interesting problems of the Sturm-Liouville class pertains to the solution of the vertical component of the normal mode of sound in a wave guide.[1, 2] We shall discuss a general method, based on separation of variables, which is ideally suited for a range-independent wave guide but has been extended to certain range-dependent problems by mode coupling.[3, 4] It is safe to say that this appealing method has been explored by a large number of researchers, and several numerical strategies have been advanced to find solutions. Generally, one is required to solve the vertical component of the problem subject to a variety of environmental constraints. The range part of the solution is analytical. The vertical solutions are, in fact, eigenfunctions that fall into the category of a Sturm-Liouville problem.[5] For the isovelocity case the eigenvalues for a layered bottom and a pressure release surface are simple to find, and the solutions in the water column are simple sine function. For variable velocity profiles the water column can be represented as a sum of isovelocity layers that have discrete changes at the isovelocity interface or the problem can be solved numerically. For certain classes of velocity profiles (with linear behavior) the Airy function, which seems likely to be one of the fastest of the techniques, can be employed.

One may ask the question, "So why another method?" The principal reason is, that for certain applications it is desirable to obtain the normal mode solution in closed analytical form. The motivation behind this research is to formulate the normal mode solution in such a manner that the solution can be written in the form of a spherical representation. This formulation is required for some problems pertaining to scattering from objects in a waveguide. It is quite easy to see that the isovelocity case is just such a representation. This fact suggests that a perturbation about the isovelocity solution be attempted. However, conventional perturbation theory is too limiting and works only with very small departures from the isovelocity case.

The purpose of this work is to develop a method using the isovelocity solution as a set of basic functions that span the solution space that adequately

represents the exact eigenvalue problem. This method uses Sturm-Liouville theory and completeness. A nice outgrowth of the method to be developed is to show why the old perturbation theory is limited and how to extend it by adding a simple term, so that the new perturbation method will work for more severe variations from the isovelocity case.

II. THEORETICAL DEVELOPMENTS

An updated perturbation method

We write the exact solution for unperturbed cases as follows:

$$\frac{d^2\psi_i}{dz^2} + \left[k_0^2 - \lambda_i^0\right]\psi_i = 0 \tag{1}$$

and the desired solution as follows:

$$\frac{d^2U_i}{dz^2} + \left[k^2(z) - \lambda_i\right]U_i = 0\,. \tag{2}$$

For convenience, rewrite Equation 2 as follows:

$$\frac{d^2U_i}{dz^2} + \left[k_0^2 - \lambda_i^0\right]U_i = QU_i\,, \tag{3}$$

where $Q = k_0^2 - k^2(z) + \lambda_i - \lambda_i^0 = q + \Delta\lambda_i$ and where $q(z) = k_0^2 - k^2(z)$ and $\Delta\lambda_i = \lambda_i - \lambda_i^0$.

For the isovelocity case we can rewrite Equation 1 as

$$\frac{d^2\psi_i}{dz^2} + \alpha_i^2\,\psi_i = 0\,, \tag{4}$$

where $\alpha_i^2 = k_0^2 - \lambda_i^0$.

Impose the following orthonormality conditions:

$$\left(\psi_i, \frac{1}{\rho}\psi_j\right) = \delta_{ij} \text{ and } \left(U_i, \frac{1}{\rho}U_j\right) = \delta_{ij}\,. \tag{5}$$

Assume closure so that we can express:

$$U_i = \sum_{j=1}^{N} a_{ij} \psi_j \,. \tag{6}$$

Insert Equation 6 into Equation 5 and make use of the Galerkin weighted residual method we arrive at

$$\sum_{j=1}^{N} a_{ij} \left(\alpha_i^2 - \alpha_j^2\right) \psi_j = \sum_{j=1}^{N} a_{ij} \left(q(z) + \Delta \lambda_j\right) \psi_j \,, \tag{7}$$

where $i = 1, 2, 3, \ldots N$.

By integrating the overlap of Equation 7 with ψ_i we obtain:

$$\lambda_i = \lambda_i^0 - q_{ii} - \sum_{i \neq j}^{N} \tilde{a}_{ij} q_{ij} \,, \tag{8}$$

where $\tilde{a}_{ij} = \dfrac{a_{ij}}{a_{ii}}$. This yields the Eigenvalue Correction Equation.

Integrate the overlap of Equation 7 with ψ_k to obtain:

$$\tilde{a}_{ik} \left(\alpha_i^2 - \alpha_k^2 - \Delta \lambda_k\right) - \sum_{j \neq i}^{N} q_{jk} \tilde{a}_{ij} = q_{ik} \,, \tag{9}$$

where $k = 1, 2, 3, \ldots N$ and $i \neq k$

This is an eigenvalue equation. We can use the expansion for the eigenvalue above to rewrite the equation as follows:

$$\tilde{a}_{ik} \left(\alpha_i^2 - \alpha_k^2 + H_k\right) - \sum_{j \neq \{i,\, k\}}^{N} q_{jk} \tilde{a}_{ij} = q_{ik} \,, \tag{10}$$

where $k = 1, 2, 3 \ldots N$, $i \neq k$, and the diagonal terms are.

This expression will prove useful later. The H_k's are the higher order terms and are negligible in many cases. The diagonal terms are almost always greater than the off-diagonal terms. As a first iteration via Gauss-Seidel we obtain

$$\tilde{a}_{ik} = \frac{q_{ik}}{\alpha_i^2 - \alpha_k^2 + q_{kk} - \Delta \lambda_k} = \frac{q_{ik}}{\alpha_i^2 - \alpha_k^2 - H_k} \tag{11}$$

where

$$H_i = -\sum_{j \neq i}^{N} a_{ij}\, q_{ij} \,. \tag{12}$$

The term H_k can be a big improvement over the ordinary perturbation term in that often $H_k > q_{ik}$. Thus, we expect that generally we will have convergence if the master matrix (defined by Eq. 10) is diagonally strong. Further, this expression shows that the old theory overestimated the expansion coefficient.

The new complete perturbation expansion for the eigenvalue is now

$$\lambda_i = \lambda_i^0 - q_{ii} - \sum_{i \neq j}^{N} \frac{q_{ij}\, q_{ji}}{\alpha_i^2 - \alpha_j^2 + H_j} \,. \tag{13}$$

This expression indicates that the first-order correction is still the same but that the higher-order corrections are over/underestimated in the old theory.

A solution to the eigenvalue problem and an eigenvalue approach

Reorder the above equations selectively to arrive at

$$Qa = la \tag{14}$$

where $Q_{ij} = -q_{ij}$ $i \neq j$ and $Q_{ii} = \lambda_i^0 - q_{ii}$.

We thus have an eigenvalue problem for the l's. In Equation 14 the matrix Q is real, symmetric, and diagonally strong. A rather good strategy to solve this eigenvalue problem is to use the Householder method to reduce Q to tridiagonal form. For an n × n matrix, this only takes n-2 orthogonal transformations. Since the eigenvalues of this very stable and fast reduction method are unchanged, we need only to find the eigenvalues of a tridiagonal matrix, which can be done in n operations. We have also tried the more general QR algorithm for the above problem; although we arrive at the same set of eigenvalues, the method is understandably somewhat slower. The quantity a in the above expression is rotational (unitary for complex eigenvalues) and should correspond to the expansion coefficients required in reconstructing the exact eigenvalues. Indeed, the fact that matrix a is rotational or unitary under certain conditions guarantees that the newly constructed eigenvalues are orthogonal, as is required for Sturm-Liouville problems.

However, there are some difficulties when one tries to retain a from the process using either the QL or QR algorithms. One problem is that the relative phases between the eigenfunctions are not retained, which leads to

erroneous results when summing the normal mode series. The second problem appears to be the accumulation of numerical errors for large numbers of normal modes. Thus, we use the above method just to obtain the eigenvalues and revert to the master equation that defines a along with H_k to obtain the expansion coefficients. This method seems to work well. Interestingly, the new perturbation method yields rather close agreement with the exact method for fairly strong perturbations, while the old perturbation method performs poorly. The next section presents examples.

III. NUMERICAL EXAMPLES

For the example we chose a pressure release surface with a water column of 300 m over a fluid half-space. The velocity profile (Fig. 1) varies from 1510 m/s at the surface to 1476 m/s at 100 m linearly to 1520 m/s at the bottom. The fluid half-space has a compressional speed of 1600, a density of 1.6, and an attenuation of 0.5 dB/m. Absorption in the bottom is treated by the standard perturbation treatment.

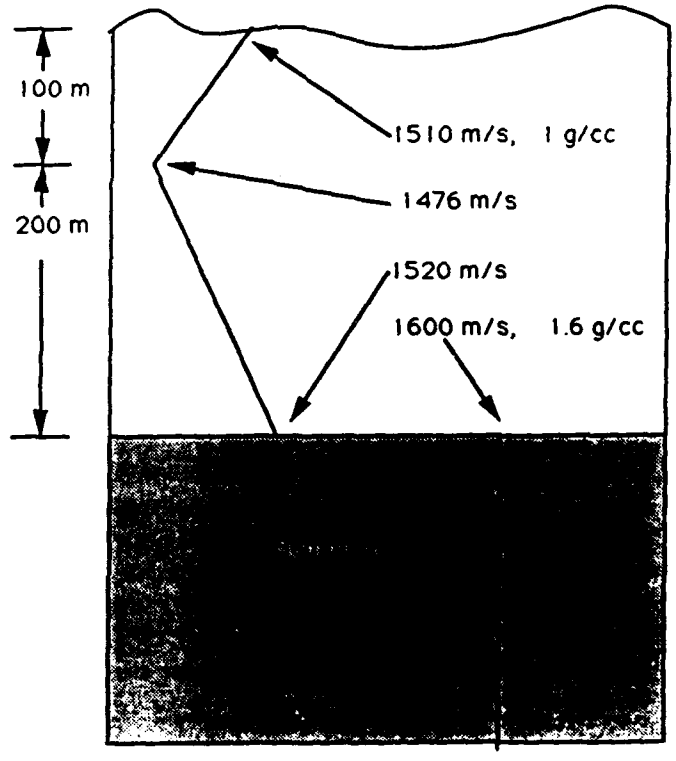

Figure 1. Diagram of the waveguide configuration and parameters used in the example.

We try several strategies for obtaining the isovelocity spanning eigenfunctions. The wisest and safest strategy is to choose the isovelocity to correspond to the smallest value of the velocity profile, in this case, 1476. This value guarantees that we will not miss any modes. It is all right to span the solution space with a larger number of isovelocity eigenfunctions. The boundary condition at the water-ocean bottom interface will exclude imaginary modes if they are encountered. One might also choose the isovelocity as the average value along the water column. The advantage here would be to reduce errors in the predicted eigenvalues, but such a choice obviously might exclude the highest order modes. As a final strategy we choose a small isovelocity value of 1400 m/s to determine the robustness of the method and to see whether or not we could account for some of the continuum which we have ignored. This last strategy did not yield a correct answer.

We also compared our results with Michael D. Collins' finite-element normal mode code. We employed a Taylor's expansion of the inverse of the variable velocity about the isovelocity and dropped all but the linear terms to obtain analytical integrals. This approach introduced an error of a few parts per thousand, so that a comparison with other methods should only be accurate to about three places. Table 1 compares the seven eigenvalues for a 50-Hz signal for the low and the average reference velocities to those predicted by Collins' code. The two reference velocities yield essentially exact agreement. Agreement with the Collins value is within the accuracy tolerance of a few parts per thousand. We also list the eigenvalues for the isovelocity case using the lower reference number. A comparison of the eigenfunction using the two reference wave numbers are in very good agreement. Figure 2 illustrates the seven modes constructed using the lower wave number. It is clear, as expected, that channeling occurs at 100 m. We ran the code for this problem to 400 Hz, and 58 modes and the results were apparently successful.

Table 1. Eigenvalues for waveguide problem at 50 Hz.

Mode	FEMODE (M.Collins)	NMEIG (Lower)	NMEIG (Ave)	NMEIG (ISO)
1	0.2113	0.2116	0.2116	0.2126
2	0.2091	0.2086	0.2087	0.2119
3	0.2072	0.2069	0.2070	0.2107
4	0.2054	0.2054	0.2055	0.2092
5	0.2034	0.2032	0.2033	0.2070
6	0.2008	0.2007	0.2006	0.2044
7	0.1978	0.1974	0.1974	0.2012

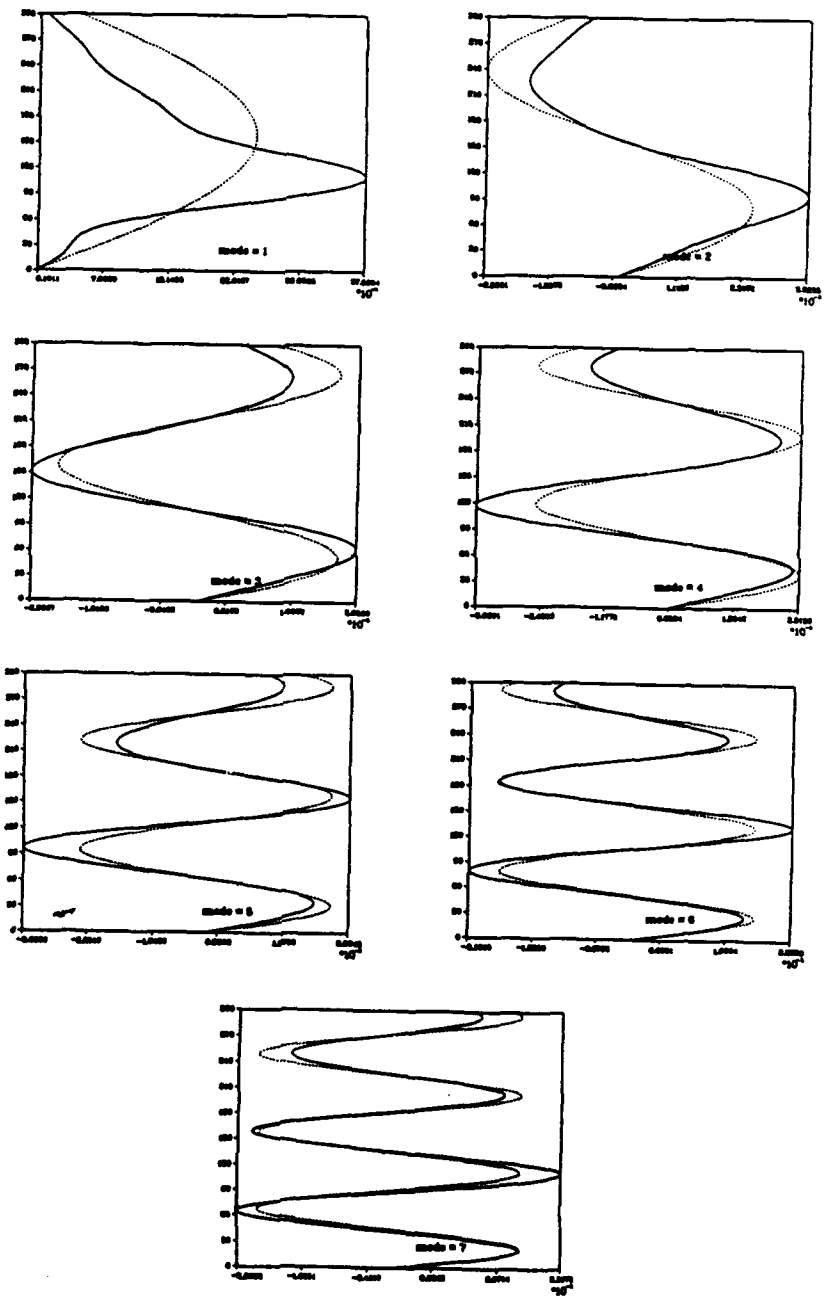

Figure 2. The seven normal modes (solid curves) from the exact method for 50 Hz. The dashed curves are from the isovelocity calculation.

We now want to compare the exact solution with both the old and new perturbation theories. In all cases we use the lower reference isovelocity to generate results. A good test of the method is to examine the expansion coefficients for the three cases (Tables 2, 3, and 4). The exact expansion coefficients (Table 2) and those from the old perturbation theory (Table 3) are widely different, but the coefficients from the new method agree reasonably well with the exact method. A better idea of the agreement can be seen by comparing the exact eigenvalues (Fig. 2) with those produced by the new method illustrated in Figure 3. Visual agreement is also quite good. The old perturbation theory does not even reproduce the correct modal ordering.

Table 2. Normal mode expansion coefficients (exact).

Mode	1	2	3	4	5	6	7	8
1	1.000	0.384	−0.102	−0.241	−0.151	0.014	0.086	0.056
2	0.258	1.000	0.126	−0.168	−0.126	−0.035	−0.031	0.039
3	−0.095	0.129	1.000	0.133	−0.134	−0.095	−0.028	0.006
4	−0.121	−0.162	0.128	1.000	0.149	−0.124	−0.106	−0.032
5	−0.680	−0.116	−0.127	0.148	1.000	0.129	−0.134	−0.119
6	0.001	−0.036	−0.094	0.128	0.134	1.000	0.111	−0.110
7	0.034	0.029	−0.028	−0.111	−0.141	−0.111	1.000	0.119

Table 3. Normal mode expansion coefficients (old perturbation).

Mode	1	2	3	4	5	6	7	8
1	1.000	2.085	−0.402	−0.102	−0.050	0.0005	0.005	0.0006
2	−2.085	1.000	0.928	−0.367	−0.072	−0.029	0.002	0.008
3	0.402	−0.928	1.000	0.652	−0.259	−0.053	−0.018	−0.007
4	0.102	0.367	−0.652	1.000	0.510	−0.198	−0.054	−0.010
5	0.050	0.072	0.259	−0.510	1.000	0.383	−0.158	−0.062
6	−0.0005	0.029	0.053	0.198	−0.383	1.000	0.287	−0.114
7	−0.005	−0.002	0.018	0.054	0.158	−0.287	1.000	0.217

Table 4. Normal mode expansion coefficients (new perturbation).

Mode	1	2	3	4	5	6	7	8
1	1.000	0.338	−0.164	−0.080	−0.063	0.0009	0.013	0.002
2	0.258	1.000	0.163	−0.157	−0.055	−0.034	0.003	0.019
3	−0.124	0.160	1.000	0.154	−0.142	−0.049	−0.052	−0.013
4	−0.056	−0.151	0.151	1.000	0.155	−0.135	−0.062	−0.017
5	−0.046	−0.052	−0.138	0.154	1.000	0.143	−0.131	−0.085
6	0.0006	−0.033	−0.048	−0.135	0.145	1.000	0.131	−0.114
7	0.010	0.003	−0.025	−0.061	−0.131	0.129	1.00	0.117

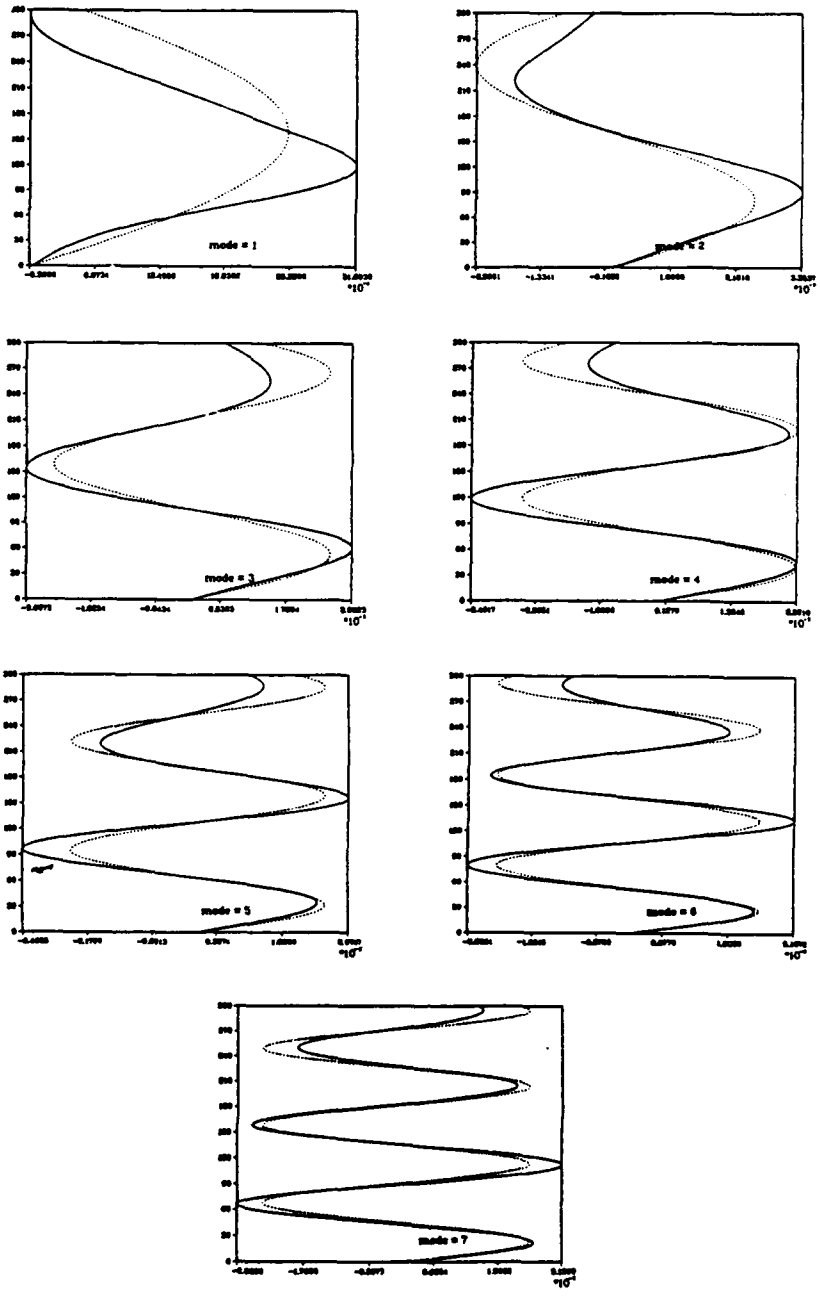

Figure 3. The seven normal modes (solid curves) from the new perturbation method for 50 Hz The dashed curves are from the isovelocity case.

As frequency increases, channeling becomes more pronounced for the lower modes. This seems to create some problem with the new perturbation method because at that point the governing matrix begins to depart from a diagonally strong matrix so that Gauss-Seidel cannot be used in a first iteration. However, a second or third iteration may be adequate for that case, and it could be done rapidly by computer. This problem existed for only the two lowest modes at 400 Hz. The higher modes agreed quite well for the new perturbation method and the exact method.

IV. COMMENTS

The expansion method presented here is useful for formulating a normal-mode solution in a closed mathematical form. An outgrowth of the development was a new perturbation theory that is more powerful than an earlier one commonly used in quantum mechanics. The method is quite fast, however, and calculated 58 modes in about two minutes on a Vax 8650. With further development and testing, it may prove to be a very useful general method. Another advantage of the method would be its use in a range-dependent normal-mode development.

It is also possible to develop a deep-water model by matching the expansion wave functions at some point in the water column (when the velocity profile has a constant slope) to an Airy function.

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VI. REFERENCES

1. Tolstoy, I. and Clay, C. S. *Ocean Acoustics, Theory and Experiment in Underwater Sound* pp. 65–93, American Institute of Physics, New York, 1987.

2. Ahluwalia, D. S. and Keller, J. B. in *Wave propagation and Underwater Acoustics* eds. Keller, J. B. and Papadakis, J. S. pp. 14–85, Springer, Berlin, 1977.

3. Rutherford, S. R. and Hawker, K. E. 'Consistent Coupled Mode Theory of Sound Propagation for a Class of Nonseperable Problems,' *J. Acoust. Soc. Am.* Vol. 70, pp. 554–564, 1981.

4. Evans, R. B. 'A Coupled Mode Solution for Acoustic Propagation in a Waveguide with Stepwise Depth Variations of a Penetrable Bottom' *J. Acoust. Soc. Am.* Vol. 74, pp. 188–195, 1983.

5. Weidmann, J. *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1980.

6. Sakuri, J. J. *Modern Quantum Mechanics*, pp. 289–293, Benjamin/ Cummings, Menlo Park, 1985.

7. Wilkinson, J. H. *The Algebraic Eigenvalue Problem*, Clarendon, Oxford, 1965.