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Change analysis is concerned with "fluctuation" of the data (in accordance with probability distributions fitted to a whole sample) from "non-stationarity" (changes in the parameters of probability distributions). To detect change over time in a sequence of observations one forms for various transformations of the data sample change processes on [0,1]; the transformations are called "data score functions". One can choose non-parametric score functions which detect changes of location, scale, skewness, etc. in the probability distribution of the observations. When a parametric model is available for the distribution of each observation one can detect changes in the parameter values by transforming the data by parametric score functions which we call Fisher-score functions. This paper studies the asymptotic distributions (under the null hypothesis of no change) of Fisher-score change processes which are cusums of scored data. They are related to cuscore processes or cumulative score processes.

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LIMIT THEOREMS FOR FISHER-SCORE CHANGE PROCESSES

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0. Introduction

Change analysis is concerned with distinguishing "fluctuation" of the data (in accordance with probability distributions fitted to a whole sample) from "non-stationarity" (changes in the parameters of probability distributions). To detect change over time in a sequence of observations one forms for various transformations of the data sample change processes on [0,1]; the transformations are called "data score functions" (Parzen (1992)). One can choose non-parametric score functions which detect changes of location, scale, skewness, etc. in the probability distribution of the observations. When a parametric model is available for the distribution of each observation one can detect changes in the parameter values by transforming the data by parametric score functions which we call Fisher-score functions.

This paper studies the asymptotic distributions (under the null hypothesis of no change) of Fisher-score change processes which are cusums of scored data. They are related to cuscore processes or cumulative score processes, some of whose applications are described in Box and Ramirez (1992).

1. Fisher-score change processes

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be independent random vectors with distribution functions $F(\mathbf{x}; \mathbf{\Theta}_1), F(\mathbf{x}; \mathbf{\Theta}_2), \ldots, F(\mathbf{x}; \mathbf{\Theta}_n)$, where $\mathbf{\Theta}_1, \mathbf{\Theta}_2, \ldots, \mathbf{\Theta}_n$ are unknown p-dimensional parameter vectors. A basic changepoint problem is the problem of "abrupt change" which tests

$$H_0 : \mathbf{\Theta}_1 = \mathbf{\Theta}_2 = \ldots = \mathbf{\Theta}_n$$

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against the alternative H_A : There is $\tau\epsilon(0,1)$ such that

$$\mathbf{\Theta}_1 = \ldots = \mathbf{\Theta}_{[n\tau]} \neq \mathbf{\Theta}_{[n\tau]+1} = \ldots = \mathbf{\Theta}_n.$$

The "abrupt change" problem motivates the definition of the Fisher-score change processes introduced in (1.3). We digress for a moment to note that test statistics for smooth change models can be formed by inner products of these processes with "change score functions."

We assume that the observations are absolutely continuous or discrete. The density functions (probability mass functions in the discrete case) are denoted by $f(\mathbf{x};\mathbf{\Theta}_1),\ldots,f(\mathbf{x};\mathbf{\Theta}_n)$.

Let $\mathbf{g}_1(\mathbf{x};\mathbf{\Theta}) = (g_{1,1}(\mathbf{x};\mathbf{\Theta}),\ldots,g_{1,p}(\mathbf{x};\mathbf{\Theta}))$, defining Fisher-score functions

$$g_{1,i}(\mathbf{x};\mathbf{\Theta}) = \frac{\partial \log f(\mathbf{x};\mathbf{\Theta})}{\partial \theta_i} \quad , \qquad 1 \leq i \leq p.$$

We estimate the unknown parameter by the usual maximum likelihood method; i.e. $\hat{\mathbf{\Theta}}_n = (\hat{\Theta}_{n,1},\ldots,\hat{\Theta}_{n,p})$ satisfies the estimating equations

$$\sum_{1\leq j\leq n} g_{1,i}\left(\mathbf{X}_j;\hat{\mathbf{\Theta}}_n\right) = 0, \qquad 1 \leq i \leq p. \tag{1.1}$$

A basic statistic in changepoint problems is the process on $0 < t < 1$

$$\mathbf{Z}_n(t) = \left(Z_{n,1}(t),\ldots,Z_{n,p}(t)\right), \tag{1.2}$$

whose components are called Fisher-score change processes defined by

$$Z_{n,i}(t) = \frac{1}{n^{1/2}} \sum_{1\leq j\leq (n+1)t} g_{1,i}\left(\mathbf{X}_j;\hat{\mathbf{\Theta}}_n\right), 0 \leq t < 1, 1 \leq i \leq p \tag{1.3}$$

$(Z_{n,i}(1) = 0, 1 \leq i \leq p)$. They can be considered, for t fixed, to be score test statistics for the hypothesis that the parameter estimators for data up to time $(n+1)t$ are not significantly different from the parameter estimators for all the data, against the alternative hypothesis that there is abrupt change at time $(n+1)t$.

We study the asymptotic properties of $\mathbf{Z}_n(t)$ under the null hypothesis of no change. The true value of the parameter under H_0 is denoted by $\mathbf{\Theta}_0 = (\Theta_{0,1},\ldots,\Theta_{0,p})$. Let $\mathbf{X}$

be a random vector with density function (probability mass function in the discrete case) $f(\mathbf{x};\boldsymbol{\Theta}_0)$. Let

$$g(\mathbf{x};\boldsymbol{\Theta}) = \log f(\mathbf{x};\boldsymbol{\Theta})$$

$$g_{1,i}(\mathbf{x};\boldsymbol{\Theta}) = \frac{\partial}{\partial\Theta_i} g(\mathbf{x};\boldsymbol{\Theta}), \quad 1 \le i \le p$$

$$g_{2,i,j}(\mathbf{x};\boldsymbol{\Theta}) = \frac{\partial^2}{\partial\Theta_i\partial\Theta_j} g(\mathbf{x};\boldsymbol{\Theta}), \quad 1 \le i,j \le p$$

and

$$g_{3,i,j,k}(\mathbf{x};\Theta) = \frac{\partial^3}{\partial\Theta_i\partial\Theta_j\partial\Theta_k} g(\mathbf{x};\boldsymbol{\Theta}), 1 \le i,j,k \le p$$

We assume that there is an open neighborhood Θ_0 of $\boldsymbol{\Theta}_0$ such that the following conditions hold:

C.1 $g(\mathbf{x};\boldsymbol{\Theta})$, $g_{1i}(\mathbf{x};\boldsymbol{\Theta})$, $g_{2,i,j}(\mathbf{x};\boldsymbol{\Theta})$ and $g_{3,i,j,k}(\mathbf{x};\boldsymbol{\Theta})$ $1 \le i,j,k \le p$ exist for all $\mathbf{x}\epsilon R^d$ and $\boldsymbol{\Theta}\epsilon\Theta_0$

C.2 There is a function $M(\mathbf{x})$ such that $EM(\mathbf{X}) < \infty$ and for all $\mathbf{x}\epsilon R^d, \boldsymbol{\Theta}\epsilon\Theta_0$

$$|g_{1,i}(\mathbf{x};\boldsymbol{\Theta})| \le M(\mathbf{x}), \quad 1 \le i \le p$$

$$|g_{2,i,j}(\mathbf{x};\boldsymbol{\Theta})| \le M(\mathbf{x}), \quad 1 \le i \le p$$

$$|g_{3,i,j,k}(\mathbf{x};\boldsymbol{\Theta})| \le M(\mathbf{x}), \quad 1 \le i,j,k \le p$$

C.3 $Eg_{1,i}(\mathbf{X};\boldsymbol{\Theta}_0) = 0, \quad 1 \le i \le p$

C.4 $E|g_{1,i}(\mathbf{X};\boldsymbol{\Theta}_0)|^{2+\delta} < \infty, 1 \le i \le p,$ for some $\delta > 0$

C.5 J^{-1} exists, where $J = \{J_{i,j}, 1 \le i,j \le p\}$ and $J_{i,j} = Eg_{1,i}(\mathbf{X};\boldsymbol{\Theta}_0)g_{1,j}(\mathbf{X};\boldsymbol{\Theta}_0), \quad 1 \le i,\ j \le p$

C.6 $E|g_{2,i,j}(\mathbf{X};\boldsymbol{\Theta}_0)|^2 < \infty$

We show that $\mathbf{Z}_n(t)$ converges weakly to $\boldsymbol{\Gamma}(t) = (\Gamma^{(1)}(t), \ldots, \Gamma^{(p)}(t))$, where $\boldsymbol{\Gamma}(t)$ is a Gaussian process with covariance structure $E\Gamma^{(i)}(t) = 0$ and $E\Gamma^{(i)}(t)\Gamma^{(j)}(s) = J_{i,j}(\min(t,s) - ts)$. This means that $J_{i,i}^{-1/2}\Gamma^{(i)}(t)$ is a Brownian bridge for each $1 \le i \le p$.

To consider the convergence in weighted metrics, we consider the following class of functions:

$Q_{0,1} = \{q : q$ non-decreasing in a neighborhood of zero, non-increasing in a neighborhood of one and $\inf_{\delta \le t \le 1-\delta} q(t) > 0$ for all $0 < \delta < 1/2\}$.

The condition is given in terms of the integral test

$$I(q,c) = \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{cq^2(t)}{t(1-t)}\right) dt.$$

Theorem 1.1. *We assume that* (1.1) *has a unique solution,* C.1–C.6 *hold and* $q_i \epsilon Q_{0,1}$, $1 \leq i \leq p$. *We can define a sequence of Gaussian processes* $\{\mathbf{\Gamma}_n(t) = (\Gamma_{n,1}(t), \ldots, \Gamma_{n,p}(t)),\ 0 \leq t \leq 1\}$ *such that*

$$\{\mathbf{\Gamma}_n(t),\ 0 \leq t \leq 1\} \stackrel{D}{=} \{\mathbf{\Gamma}(t),\ 0 \leq t \leq 1\} \tag{1.4}$$

and

$$\max_{1 \leq i \leq p} \sup_{0<t<1} |Z_{n,i}(t) - \Gamma_{n,i}(t)|/q_i(t) = o_P(1) \tag{1.5}$$

if and only if

$$\max_{1 \leq i \leq p} I(q_i, c) < \infty \text{ for all } c > 0. \tag{1.6}$$

If we are interested in the convergence of the weighted supremum functional, we can establish it under weaker conditions.

Theorem 1.2. *We assume that* (1.1) *has a unique solution,* C.1 – C.6 *hold and* $q_i \epsilon Q_{0,1}$, $1 \leq i \leq p$. *Then, as* $n \to \infty$, *we have*

$$\begin{aligned} &\left\{ \sup_{0<t<1} |Z_{n,1}(t)|/q_1(t), \ldots, \sup_{0<t<1} |Z_{n,p}|/q_p(t) \right\} \\ &\quad \xrightarrow{D} \left\{ \sup_{0<t<1} |\Gamma^{(1)}(t)|/q_1(t), \ldots, \sup_{0<t<1} |\Gamma^{(p)}(t)|/q_p(t) \right\} \end{aligned} \tag{1.7}$$

if and only if

$$\max_{1 \leq i \leq p} I(q_i, c) < \infty \text{ for some } c > 0. \tag{1.8}$$

We can choose $q_i(t) = (t(1-t) \log\log 1(t(1-t)))^{1/2}$ in Theorem 1.2 but this function does not satisfy (1.6). However, the standard deviation $(J_{i,i}t(1-t))^{1/2}$ does not satisfy (1.6) nor (1.8). Let

$$a(x) = (2 \log x)^{1/2}$$

$$b(x) = 2\log x + \frac{1}{2} \log\log x - \frac{1}{2} \log \pi.$$

Theorem 1.3. *We assume that (1.1) has a unique solution and* C.1–C.6 *hold. Then for each* $1 \le i \le p$ *we have*

$$\lim_{n\to\infty} P\left\{a(\log n) \sup_{0<t<1} |Z_{n,i}(t)| / \left(J_{i,i}t(1-t)\right)^{1/2} \le x + b(\log n)\right\} \tag{1.9}$$
$$= \exp(-2e^{-x})$$

for all x.

We note that if $J_{i,j} = 0, i \neq j$, then $a(\log n)\sup_{0<t<1}|Z_{n,i}(t)|/\left(J_{i,i}t(1-t)\right)^{1/2} - b(\log n)$ and $a(\log n)\sup_{0<t<1}|Z_{n,j}(t)|/\left(J_{j,j}t(1-t)\right)^{1/2} - b(\log n)$ are asymptotically independent. This happens, for example, if the observations are normal and the parameters are the mean and the variance.

2. Proofs

We start with a few lemmas. We assume that H_0 holds. Let $||\mathbf{x}|| = \max_{1\le i\le p}|x_i|$, $\mathbf{x} = (x_1, \ldots, x_p)$.

Lemma 2.1. *We assume that (1.1) has a unique solution and* C.1–C.6 *hold. Then, as* $n \to \infty$, *we have for all* $1 \le i \le p$ *that*

$$Z_{n,i}(t) = Z^*_{n,i}(t) + R^{(1)}_{n,i}(t) + R^{(2)}_{n,i}(t),$$

where

$$Z^*_{n,i}(t) = \frac{1}{n^{1/2}}\left\{\sum_{1\le j\le (n+1)t} g_{1,i}\left(\mathbf{X}_j; \mathbf{\Theta}_0\right) - t\sum_{1\le j\le n} g_{1,i}\left(\mathbf{X}_j; \mathbf{\Theta}_0\right)\right\},$$
$$\sup_{0\le t\le 1}|R^{(1)}_{n,i}(t)| = O_p\left(n^{-1/2}\right)$$

and

$$\sup_{1/(n+1)\le t\le 1-1/(n+1)} |R^{(2)}_{n,i}(t)|/(t(1-t)) = O_p(1).$$

PROOFS. Conditions C.1 –C.4 imply

$$||\hat{\boldsymbol{\theta}}_n - \mathbf{\Theta}_0|| \overset{a.s.}{=} O(1) \tag{2.1}$$

as $n \to \infty$, and therefore we can assume that $\hat{\boldsymbol{\Theta}}_n \epsilon \Theta_0$. Ibragimov and Hasminskii (1972, 1973a,b) showed that

$$||n\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\Theta}_0\right) - \sum_{1 \le j \le n} \mathbf{g}_1\left(\mathbf{X}_j; \boldsymbol{\Theta}_0\right) J^{-1}|| = o_p(n). \tag{2.2}$$

Let

$$\tilde{\mathbf{g}}_1(\boldsymbol{\Theta}) = E\mathbf{g}_1\left(\mathbf{X}; \boldsymbol{\Theta}_0\right).$$

We write

$$Z_{n,i}(t) = A^{(1)}_{n,i}(t) + A^{(2)}_{n,i}(t), \tag{2.3}$$

where

$$A^{(1)}_{n,i}(t) = \frac{1}{n^{1/2}} \sum_{1 \le j \le (n+1)t} \tau_{1,i}\left(\mathbf{X}_j; \hat{\Theta}_n\right) \tag{2.4}$$

$$A^{(2)}_{n,i}(t) = \frac{(n+1)t}{n^{1/2}}\left(\tilde{g}_{1,i}\left(\hat{\boldsymbol{\Theta}}_n\right) - \tilde{g}_{1,i}\left(\boldsymbol{\Theta}_0\right)\right) \tag{2.5}$$

and

$$\tau_{1,i}\left(\mathbf{x}_j \boldsymbol{\Theta}\right) = g_{1,i}\left(\mathbf{x}; \boldsymbol{\Theta}\right) - \tilde{g}_{1,i}\left(\boldsymbol{\Theta}\right), \quad 1 \le i \le p.$$

Let

$$\tau_{2,i,j}\left(\mathbf{x}; \boldsymbol{\Theta}\right) = \frac{\partial}{\Theta_j} \tau_{1,i}\left(\mathbf{x}; \boldsymbol{\Theta}\right)$$

and

$$\tau_{3,i,j,k}\left(\mathbf{x}; \boldsymbol{\Theta}\right) = \frac{\partial^2}{\Theta_j \Theta_k} \tau_{1,i}\left(\mathbf{x}; \boldsymbol{\Theta}\right).$$

We note that

$$E\tau_{2,i,j}\left(\mathbf{X}; \boldsymbol{\Theta}_0\right) = 0, \quad 1 \le i, \quad j \le p \tag{2.6}$$

and

$$|\tau_{3,i,j,k}\left(\mathbf{x}\right)| \le 2M(\mathbf{x}). \tag{2.7}$$

A two-term Taylor expansion and (2.2) with the central limit theorem yield

$$\tilde{g}_{1,i}\left(\hat{\boldsymbol{\Theta}}_n\right) - \tilde{g}_{1,i}\left(\boldsymbol{\Theta}_0\right) = \sum_{1 \le j \le p} \tilde{g}_{2,i,j}\left(\boldsymbol{\Theta}_0\right)\left(\hat{\Theta}_{n,i} - \Theta_{0,i}\right) \tag{2.8}$$

$$+ O_p\left(\frac{1}{n}\right)$$

Next we use again (2.2) and get

$$n\left(\tilde{g}_{1,i}\left(\hat{\boldsymbol{\Theta}}_n\right)-\tilde{g}_{1,i}\left(\boldsymbol{\Theta}_0\right)\right)=\tilde{\mathbf{g}}_{2,i}\left(\boldsymbol{\Theta}_0\right)\left(\sum_{1\le\ell\le n}\mathbf{g}_1\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)J^{-1}\right)^T \tag{2.9}$$
$$+\,o_p\left(n^{1/2}\right).$$

Observing that $\tilde{g}_{2,i,j}(\boldsymbol{\Theta}_0)=-J_{i,j}$, by (2.9) we have

$$n\left(\tilde{g}_{1,i}\left(\hat{\boldsymbol{\Theta}}_n\right)-\tilde{g}_{1,i}\left(\boldsymbol{\Theta}_0\right)\right)=-\sum_{1\le\ell\le n}g_{1,i}\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)+o_p\left(n^{1/2}\right). \tag{2.10}$$

We use again Taylor expansion and get

$$\Big|\sum_{1\le\ell\le(n+1)t}\left(\tau_{1,i}\left(\mathbf{X}_\ell;\hat{\boldsymbol{\Theta}}_0\right)-\tau_{1,i}\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)\right) \tag{2.11}$$
$$-\sum_{1\le j\le p}\left(\hat{\Theta}_{n,j}-\Theta_{0,j}\right)\sum_{1\le\ell\le(n+1)t}\tau_{2,i,j}\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)\Big|$$
$$\le\frac{p}{2}||\hat{\boldsymbol{\Theta}}_n-\boldsymbol{\Theta}_0||^2\sum_{1\le\ell\le(n+1)t}M\left(\mathbf{X}_\ell\right).$$

Now by (2.6) we can use the invariance principle and by C.2 we can apply the law of large numbers. Thus we obtain

$$\sup_{0\le t\le 1}\Big|\sum_{1\le\ell\le(n+1)t}\left(\tau_{1,i}\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)-\tau_{1,i}\left(\mathbf{X}_\ell;\boldsymbol{\Theta}_0\right)\right)\Big|=O_p(1). \tag{2.11}$$

We showed that

$$\sup_{0\le t\le 1}\Big|A_{n,i}^{(1)}(t)-\frac{1}{n^{1/2}}\sum_{1\le j\le(n+1)t}g_{1,i}\left(\mathbf{X}_j;\boldsymbol{\Theta}_0\right)\Big|=O_p\left(n^{-1/2}\right) \tag{2.12}$$

and

$$\sup_{0\le t\le 1}\Big|\left(A_{n,i}^{(2)}(t)+\frac{t}{n^{1/2}}\sum_{1\le j\le n}q_{1,i}\left(\mathbf{X}_j;\boldsymbol{\Theta}_0\right)\right)/t\Big|=o_p(1). \tag{2.13}$$

By (1.1) we have

$$Z_{n,i}(t)=-\sum_{(n+1)t<j\le n}q_{1,i}\left(\mathbf{X}_j;\hat{\boldsymbol{\Theta}}_n\right),$$

and therefore similarly to (2.3) we have

$$\begin{aligned}
&|Z_{n,i}(t) - \frac{1}{n^{1/2}}\left(\sum_{1\le \ell\le (n+1)t} g_{1,i}(\mathbf{X}_\ell;\mathbf{\Theta}_0) - t\sum_{1\le \ell\le n} g_{1,i}(\mathbf{X}_\ell;\mathbf{\Theta}_0)\right)| \\
&\le |n^{-1/2}\sum_{(n+1)t<j\le n}\left\{\left(g_{1,i}\left(\mathbf{X}_j;\hat{\mathbf{\Theta}}_n\right) - \tilde{g}_{1,i}\left(\hat{\mathbf{\Theta}}_n\right)\right) - g_1\left(\mathbf{X}_j;\mathbf{\Theta}_0\right)\right\}| \\
&+ |\frac{n-(n+1)t}{n^{1/2}}\left(\tilde{g}_{1,i}\left(\hat{\mathbf{\Theta}}_n\right) - \tilde{g}_{1,i}(\mathbf{\Theta}_0)\right) + \frac{1-t}{n^{1/2}}\sum_{1\le j\le n} g_{1,i}\left(\mathbf{X}_j;\Theta_0\right)| \\
&= A^{(3)}_{n,i}(t) + A^{(4)}_{n,i}(t).
\end{aligned}$$

Now similarly to (2.12) and (2.13) one can establish

$$\sup_{0\le t\le 1}|A^{(3)}_{n,i}(t)| = O_p\left(n^{-1/2}\right)$$

and

$$\sup_{1/(n+1)\le t\le 1-1/(n+1)}|A^{(4)}_{n,i}(t)|/(1-t)| = o_p(1),$$

which completes the proof of Lemma 2.1.

Lemma 2.2. *We assume that* C.3 *and* C.4 *hold. We can define a sequence of Gaussian processes* $\{\mathbf{\Gamma}_n(t) = (\Gamma_{n,1}(t),\ldots,\Gamma_{n,p}(t)), 0\le t\le 1\}$ *such that* (1.4) *holds and*

$$n^{\frac{1}{2}-\nu}\max_{1\le i\le p}\sup_{1/(n+1)\le t\le n/(n+1)}|Z^*_{n,i}(t) - \Gamma_{n,1}(t)|/(t(1-t))^\nu = O_p(1) \qquad (2.14)$$

for all $\dfrac{1}{2+\delta}\le \nu\le 1/2$.

PROOF. Let

$$V_{n,i}(t) = \sum_{1\le j\le (n+1)t} q_{1i}(\mathbf{X}_j;\mathbf{\Theta}_o),$$

and

$$V_{n,i}(1) = \sum_{1\le j\le n} g_{1,i}(\mathbf{X}_j;\mathbf{\Theta}_o).$$

We have

$$n^{1/2}Z^*_{n,i}(t) = \begin{cases} V_{n,i}(t) - t\left(V_{n,i}\left(\frac{1}{2}\right) + \left(V_{n,i}(1) - V_{n,i}\left(\frac{1}{2}\right)\right)\right), 0\le t\le \frac{1}{2} \\ -\left(V_{n,i}(1) - V_{n,i}(t)\right) + (1-t)\left(V_{n,i}\left(\frac{1}{2}\right) + \left(V_{n,i}(1) - V_{n,i}\left(\frac{1}{2}\right)\right)\right), \frac{1}{2}\le t\le 1. \end{cases}$$

By Einmahl (1989) for each n we can define two independent Gaussian processes $\{(G_{n,1}^{(1)}(x), \ldots, G_{n,p}^{(1)}(x)), 0 \leq x \leq n/2\}$ and $\{(G_{n,1}^{(2)}(x), \ldots, G_{n,p}^{(2)}(x)), 0 \leq x \leq n/2\}$ with covariance $EG_{n,i}^{(j)}(x) = 0$, $EG_{n,i}^{(j)}(x)G_{n,k}^{(j)}(y) = J_{i,k}\min(x,y)$, $j = 1,2$, $1 \leq i,k \leq p$ and

$$\max_{1\leq i\leq p} \sup_{1/(n+1)\leq t\leq 1/2} |V_{n,i}(t) - G_{n,i}^{(1)}(nt)|/(nt)^{\frac{1}{2+\delta}} = O_p(1) \tag{2.16}$$

and

$$\max_{1\leq i\leq p} \sup_{1/2\leq t\leq n/(n+1)} |\left(V_{n,i}(1) - V_{n,i}(t)\right) - G_{n,i}^{(2)}(n(1-t))|/(n(1-t))^{\frac{1}{2+\delta}} = O_p(1). \tag{2.17}$$

Now (2.15), (2.16) and (2.17) yield

$$\begin{aligned} n^{\frac{1}{2}-\nu} & \sum_{1/(n+1)\leq t\leq 1/2} |Z_{n,i}^{*}(t) - n^{-1/2}\left(G_{n,i}^{(1)}(nt) - t\left(G_{n,i}^{(1)}\left(\frac{n}{2}\right) + G_{n,i}^{(2)}\left(\frac{n}{2}\right)\right)\right)|/(nt)^{\nu} \\ &= \sup_{1/(n+1)\leq t\leq 1/2} |n^{1/2}Z_{n,i}^{*}(t) - \left(G_{n,i}^{(1)}(nt) - t\left(G_{n,i}^{(1)}\left(\frac{n}{2}\right) + G_{n,i}^{(2)}\left(\frac{n}{2}\right)\right)\right)|/(nt)^{\nu} \qquad (2.18) \\ &= O_p(1) \sup_{1/(n+1)\leq t\leq 1/2} (nt)^{\frac{1}{2+\delta}-\nu} = O_p(1), \end{aligned}$$

and similar arguments give

$$\begin{aligned} n^{\frac{1}{2}-\nu} \sup_{1/2\leq t\leq n/(n+1)} & |Z_{n,i}^{(*)}(t) - n^{-1/2}\Big(-G_{n,i}^{(2)}(n(1-t)) \qquad (2.19) \\ & + (1-t)\left(G_{n,i}^{(1)}\left(\frac{n}{2}\right) + G_{n,i}^{(2)}\left(\frac{n}{2}\right)\right)\Big)|/(1-t)^{\nu} = O_p(1). \end{aligned}$$

We define $\mathbf{\Gamma}_n(t)$ by

$$n^{1/2}\Gamma_{n,i}(t) = \begin{cases} G_{n,i}^{(1)}(nt) - t\left(G_{n,i}^{(1)}\left(\frac{n}{2}\right) + G_{n,i}^{(2)}\left(\frac{n}{2}\right)\right), 0 \leq t \leq 1/2 \\ -G_{n,i}^{(2)}(n(1-t)) + (1-t)\left(G_{n,i}^{(1)}\left(\frac{n}{2}\right) + G_{n,i}^{(2)}\left(\frac{n}{2}\right)\right), 1/2 \leq t \leq 1. \end{cases}$$

It is easy to see that $\mathbf{\Gamma}_n(t)$ satisfies (1.4) and by (2.18), (2.19) we have (2.14).

PROOF OF THEOREM 1.1. First we assume that

$$I(q_i, c) < \infty \text{ for some } c > 0. \tag{2.20}$$

By Csörgő et al (1986) (2.20) implies

$$\lim_{t \downarrow 0} q_i(t)/\sqrt{t} = \infty \tag{2.21}$$

and

$$\lim_{t \downarrow 1} q_i(t)/(1-t)^{1/2} = \infty. \tag{2.22}$$

Let $\varepsilon > 0$. Lemma 2.1 implies

$$\sup_{\varepsilon \le t \le 1-\varepsilon} |Z_{n,i}(t) - Z^*_{n,i}(t)|/q_i(t) = o_p(1) \tag{2.23}$$

and

$$\sup_{1/(n+1) \le t \le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/(t(1-t))^{1/2} = O_p(1). \tag{2.24}$$

Next we write

$$\begin{aligned} &\sup_{1/(n+1) \le t \le \varepsilon} |Z_{n,i}(t) - Z^*_{n,i}(t)|/q_i(t) \\ &\le \sup_{0<t\le\varepsilon} t^{1/2}/q_i(t) \sup_{1/(n+1) \le t \le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/t^{1/2} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &\sup_{1-\varepsilon \le t \le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/q_i(t) \\ &\le \sup_{1-\varepsilon\le t<1} (1-t)^{1/2}/q_i(t) \sup_{1/(n+1) \le t \le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/(1-t)^{1/2} \end{aligned} \tag{2.26}$$

Putting together (2.21) - (2.26) and choosing ε as small as we wish we get

$$\sup_{1/(n+1) \le t \le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/q_i(t) = o_p(1). \tag{2.27}$$

Using Lemma 2.2 with $\nu = 1/2$ we have

$$\sup_{1/(n+1) \le t \le n/(n+1)} |Z^*_{n,i}(t) - \Gamma_{n,i}(t)|/(t(1-t))^{1/2} = O_p(1). \tag{2.28}$$

Hence by (2.21) and (2.22) similarly to (2.27) we can establish

$$\sup_{1/(n+1) \le t \le n/(n+1)} |Z^*_{n,i}(t) - \Gamma_{n,i}(t)|/q_i(t) = o_p(1). \tag{2.29}$$

The covariance of $\Gamma_{n,i}(t)$ implies that $J_{i,i}^{-1/2}\Gamma_{n,i}(t)$ is a Brownian bridge for each n. By Csörgő et al (1986) condition (1.6) implies

$$\sup_{0\le t\le 1/(n+1)} |\Gamma_{n,i}(t)|/q_i(t) = o_p(1) \tag{2.30}$$

and

$$\sup_{n/(n+1)\le t\le 1} |\Gamma_{n,i}(t)|/q_i(t) = o_p(1). \tag{2.31}$$

Now (1.5) follows form (2.27), (2.29), (2.30) and (2.31).

Next we assume that (1.5) holds. It follows from the definition and (1.1) that $Z_{n,i}(t) = 0$ if $0 \le t < 1/(n+1)$ and $Z_{n,i}(t) = 0$ if $n/(n+1) \le t \le 1$. Thus we have

$$\sup_{0<t<1/(n+1)} |\Gamma_{n,i}(t)|/q_i(t) = o_p(1) \tag{2.32}$$

and

$$\sup_{n/(n+1)\le t<1} |\Gamma_{n,i}(t)|/q_i(t) = o_p(1). \tag{2.33}$$

By definition,

$$\left\{\Gamma_{n,i}(t), 0 \le t \le 1\right\} \stackrel{D}{=} \left\{J_{i,i}^{1/2}B(t),\ 0 \le t \le 1\right\} \tag{2.34}$$

for each n, where $\{B(t), 0 \le t \le 1\}$ is a Brownian bridge. We have (2.32) and (2.33) if and only if

$$\lim_{\varepsilon\downarrow 0} \sup_{0<t\le\varepsilon} |B(t)|/q_i(t) = 0 \qquad a.s. \tag{2.35}$$

and

$$\lim_{\varepsilon\downarrow 0} \sup_{1-\varepsilon\le t\le 1} |B(t)|/q_i(t) = 0 \qquad a.s. \tag{2.36}$$

Using Csörgő et al (1986) we get that (2.35) and (2.36) imply (1.6).

PROOF OF THEOREM 1.2. We showed in the proof of Theorem 1.1 that (1.8) implies

$$\max_{1\le i\le p} \sup_{1/(n+1)\le t\le n/(n+1)} |Z_{n,i}(t) - \Gamma_{n,i}(t)|/q_i(t) = o_p(1). \tag{2.37}$$

Also, (1.8) yields that the limiting random vector is almost surely finite in (1.7) (cf. Csörgő et al (1986)). Since $Z_{n,i}(t) = 0$, if $0 \le t < 1/(n+1)$ and $Z_{n,i}(t) = 0$ if $n/(n+1) \le t \le 1$, the limit theorem in (1.7) follows from (2.37).

Now we assume that (1.7) holds. In this case the limiting random vector is almost surely finite. Using (2.34), this can happen only if (1.8) is satisfied.

The proof of Theorem 1.3 is based on the following lemma. Let

$$c(x) = \log \frac{1-x}{x}.$$

Lemma 2.3. *We assume that* C.3 *and* C.4 *hold. If* $1/(n+1) \le \varepsilon_1(n), \varepsilon_2(n) \le n/(n+1)$, $\varepsilon_1(n) < 1 - \varepsilon_2(n)$ *and*

$$\lim_{n\to\infty} \frac{(1-\varepsilon_1(n))(1-\varepsilon_2(n))}{\varepsilon_1(n)\varepsilon_2(n)} = \infty,$$

then we have

$$\lim_{n\to\infty} P\left\{ a\left(\frac{1}{2}(c(\varepsilon_1(n)) + c(\varepsilon_2(n)))\right) \sup_{\varepsilon_1(n)\le t\le 1-\varepsilon_2(n)} |Z^*_{n,i}(t)| / \left(I_{i,i}\, t(1-t)\right)^{1/2} \le x + b\left(\frac{1}{2}(c(\varepsilon_1(n)))\right)\right\} = \exp(-2e^{-x})$$

for all x.

PROOF. It can be found, for example, in Csörgő and Horváth (1990).

PROOF OF THEOREM 1.3. We show that

$$\sup_{1/(n+1)\le t\le n/(n+1)} |Z_{n,i}(t)|/(t(1-t))^{1/2} \text{ and } \sup_{1/(n+1)\le t\le n/(n+1)} |Z^*_{n,i}(t)|/(t(1-t))^{1/2}$$

satisfy the same limit theorem. By Lemma 2.3 we have

$$\sup_{1/(n+1)\le t\le n/(n+1)} |Z_{n,i}(t) - Z^*_{n,i}(t)|/(t(1-t))^{1/2} = O_p(1). \tag{2.38}$$

Now Lemma 2.3 yields

$$(2\log\log\log n)^{-1/2} \sup_{1/(n+1)\le t\le (\log n)/n} |Z^*_{n,i}(t)| / \left(J_{i,i}\, t(1-t)\right)^{1/2} \xrightarrow{P} 1 \tag{2.39}$$

and therefore by (2.38) we have

$$(2\log\log\log n)^{-1/2} \sup_{1/(n+1)\le t\le(\log n)/n} |Z^*_{n,i}(t)| \left(J_{i,i}\, t(1-t)\right)^{1/2} \xrightarrow{P} 1. \tag{2.40}$$

It is easy to see that (2.40) implies

$$a(\log n) \sup_{1/(n+1)\le t\le(\log n)/n} |Z_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} - (x + b(\log n)) \xrightarrow{P} -\infty \tag{2.41}$$

for all x. Similar arguments give

$$a(\log n) \sup_{1-(\log n)/n\le t\le n/(n+1)} |Z_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} - (x + b(\log n)) \xrightarrow{P} -\infty. \tag{2.42}$$

Using again Lemma 2.1 we obtain

$$\sup_{(\log n)/n\le t\le 1/\log n} |Z_{n,i}(t) - Z^*_{n,i}(t)|/(t(1-t))^{1/2} = O_P((\log n)^{-1/2}) \tag{2.43}$$

and

$$\sup_{1-1/\log n\le t\le 1-(\log n)/n} |Z_{n,i}(t) - Z^*_{n,i}(t)|/(t(1-t))^{1/2} = O_P((\log n)^{-1/2}). \tag{2.44}$$

Combining (2.38) with Lemma 2.3 we get

$$a(\log n) \sup_{1\log n\le t\le 1-1/\log n} |Z_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} - (x + b(\log n)) \xrightarrow{P} -\infty \tag{2.45}$$

for all x. Similarly,

$$a(\log n) \sup_{1/\log n\le t\le 1-1/\log n} |Z^*_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} - (x + b(\log n)) \xrightarrow{P} -\infty. \tag{2.46}$$

By (2.41)-(2.46) we have

$$\lim_{n\to\infty} P\left\{a(\log n) \sup_{1/(n+1)\le t\le n/(n+1)} |Z_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} \le x + b(\log n)\right\}$$
$$= \lim_{n\to\infty} P\left\{a(\log n) \sup_{(\log n)/n\le t\le 1-(\log n)/n} |Z^*_{n,i}(t)|/\left(J_{i,i}\, t(1-t)\right)^{1/2} \le x + b(\log n)\right\}$$

and therefore Lemma 2.3 implies the result in Theorem 1.3.

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