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by
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FINAL PROJECT REPORT

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THE AXISYMMETRIC CRACK PROBLEM IN A NONHOMOGENEOUS MEDIUM

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Abstract

In this paper the axisymmetric crack problem for a nonhomogeneous medium is considered. It is assumed that the shear modulus is a function of z approximated by $\mu = \mu_0 e^{\alpha z}$. This is a simple simulation of materials and interfacial zones with intentionally or naturally graded properties. The problem is a mixed mode problem and is formulated in terms of a pair of singular integral equations. With fracture mechanics applications in mind, the main results given are the stress intensity factors as a functions of the nonhomogeneity parameter α for various loading conditions. Also given are some sample results showing the crack opening displacements.

Introduction

In solid mechanics many of the engineering materials such as composites and large variety of bonded materials and structural components are generally modeled as nonhomogeneous continua. The thermomechanical parameters of these materials are usually assumed to be discontinuous functions, very often piece-wise constants, in space variables. However, there are some important applications in which the spatial variation in material properties is continuous. As examples for such materials we may mention the following: (a) Geophysical materials with naturally graded compositions such as shale-sandstone. In studying, for example, hydraulic fracturing of such a medium it would be necessary to take into account the mechanical property gradient of the material. (b) Interfacial regions with graded properties in diffusion bonded materials, in plasma spray coating, and in ion plating (Bataakis and Vogan, 1985, Houck, 1987). (c) Material nonhomogeneity resulting from temperature gradients in the solid in which thermomechanical constants of the medium are significantly dependent on temperature. And finally (d) the tailored materials with predetermined continuously varying volume fractions.

Some of the important applications in the last category of materials may be found in ceramic coating of metal substrates and in metal-ceramic composites with graded properties. It is known that certain strength related properties of ceramic coatings can be improved by layering the interfacial zone going from metal-rich to ceramic-rich compositions. For example, in joining tungsten to zirconia by introducing four intermediate layers that contain 80/20, 60/40, 40/60,

and 20/80 percent W/ZrO₂, respectively, it has been shown that the peak value of the residual stress is reduced by approximately six folds compared to that resulting from direct tungsten/zirconia bonding (Hirano, et al, 1988, Hirano and Yamada, 1988).

The Fatigue and fracture characterization of materials and the related analysis require the solution of certain standard crack problems. With the exception of the problem of "torsion" of an infinite medium containing a penny-shaped crack (Kassir,1972), the existing solutions of the crack problems in nonhomogeneous materials with continuously varying properties have been obtained under the assumption of plane strain, generalized plane stress, or anti-plane shear loading (see, for example, Dhalival and Sing, 1978; Gerasoulis and Srivastav, 1980; Delale and Erdogan, 1983, 1988a, 1988b; Erdogan, 1985; Konda and Erdogan, 1989). One of the practical problems in this area is that of an internal circular crack in a nonhomogeneous solid subjected to "tension." The problem is always a mixed mode problem. In this paper we consider the simplest of these problems, namely that of an axisymmetric crack in an infinite solid with elastic properties varying in the axial direction only (Fig.1). The corresponding plane strain mixed mode problem was recently considered by Konda and Erdogan (1992).

Formulation of the Problem

In the crack problem shown in Fig.1 for a large elastic solid it is assumed that the external loads as well as the geometry are axisymmetric. Therefore, one may easily separate the torsion component of the problem in which $u_\theta=v(r,z)$, $\sigma_{\theta r}$, and $\sigma_{\theta z}$ are the only nonzero displacement and stress components. Similarly,

by solving the remaining problem under the given external loads in the absence of a crack and by using superposition, the problem may be reduced to that of a nonhomogeneous solid containing a penny-shaped crack in which

$$\sigma_{zz}(r,0) = p_1(r), \sigma_{rz}(r,0) = p_2(r), 0 \leq r < a \quad (1a,b)$$

are the only external loads. Furthermore, if the crack radius a is small in comparison with the nearest distance from the crack to the boundary of the solid, then in the perturbation problem the medium can be assumed to be infinite. Needless to say, from the viewpoint of fracture mechanics the perturbation problem would contain all the relevant information such as the stress intensity factors and the crack opening displacements. If we now assume that locally the elastic properties of the medium may be approximated by

$$\mu(y) = \mu_0 e^{\alpha z}, \nu = \text{constant}, \quad (2)$$

for the axisymmetric problem under consideration by using the kinematic relations

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad 2\varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad (3)$$

and the Hooke's law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \quad (i, j = r, \theta, z), \quad (4)$$

the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} &= 0, \end{aligned} \quad (5a,b)$$

may be expressed as follows:

$$\begin{aligned} (\kappa + 1) \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} u + \frac{\partial^2 w}{\partial r \partial z} \right\} + (\kappa - 1) \alpha \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \\ + (\kappa - 1) \left\{ \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z} \right\} = 0, \end{aligned}$$

$$\begin{aligned}
& (\kappa + 1) \left\{ \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} - \frac{\partial^2 w}{\partial z^2} \right\} - (\kappa - 1) \left\{ \frac{\partial^2 u}{\partial r \partial z} - \frac{\partial^2 w}{\partial r^2} \right\} \\
& - \frac{(\kappa - 1)}{r} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) + (3 - \kappa) \alpha \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) + (\kappa + 1) \alpha \frac{\partial w}{\partial z} = 0.
\end{aligned} \tag{6a,b}$$

where u and w are, respectively the r and z components of the displacement vector, μ is the shear modulus, ν the Poisson's ratio, $\lambda/\mu = 2\nu/(1 - 2\nu)$, and $\kappa = 3 - 4\nu$. In addition to the boundary conditions given by (1) and the regularity conditions at infinity (requiring that u and w must vanish as $r^2 + z^2 \rightarrow \infty$), the crack problem must be solved under the following conditions:

$$\sigma_{zz}(r, +0) = \sigma_{zz}(r, -0), \quad \sigma_{rz}(r, +0) = \sigma_{rz}(r, -0), \quad 0 \leq r < \infty, \tag{7a,b}$$

$$u(r, +0) = u(r, -0), \quad w(r, +0) = w(r, -0), \quad a < r < \infty. \tag{8a,b}$$

We now assume the solution of (6) in the form

$$u(r, z) = \int_0^\infty F(z, \rho) \rho J_1(r\rho) d\rho,$$

$$w(r, z) = \int_0^\infty G(z, \rho) \rho J_0(r\rho) d\rho, \tag{9a,b}$$

where J_0 and J_1 are the Bessel functions of the first kind. Substituting from (9) into (6), defining $D = d/dz$, and inverting the related Hankel transforms we find

$$\begin{aligned}
& \{(\kappa - 1)D^2 + \alpha(\kappa - 1)D - (\kappa + 1)\rho^2\}F - \{2\rho D + \alpha(\kappa - 1)\}G = 0, \\
& \{2\rho D + \alpha(3 - \kappa)\rho\}F + \{(\kappa + 1)D^2 + \alpha(\kappa + 1)D - (\kappa - 1)\rho^2\}G = 0,
\end{aligned} \tag{10a,b}$$

where the following relationships have been used:

$$\begin{aligned}
& \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) J_1(r\rho) = -\rho^2 J_1(r\rho), \\
& \left(\frac{d}{dr} + \frac{1}{r} \right) J_1(r\rho) = \rho J_0(r\rho),
\end{aligned}$$

$$\frac{d^2}{dr^2} J_0(r\rho) = \frac{\rho}{r} J_1(r\rho) - \rho^2 J_0(r\rho). \quad (11a-c)$$

The solution of the system of the differential equations(10) is found to be

$$F(z, \rho) = \sum_{k=1}^4 A_k(\rho) e^{m_k z},$$

$$G(z, \rho) = \sum_{k=1}^4 a_k(\rho) A_k(\rho) e^{m_k z}, \quad (12a, b)$$

where the functions A_k , ($k = 1, \dots, 4$) are arbitrary unknowns, $m_1, \dots, m_4$ are the roots of the following characteristic equation

$$m^4 + 2\alpha m^3 + (\alpha^2 - 2\rho^2)m^2 - 2\rho^2\alpha m + \frac{3-\kappa}{\kappa+1}\alpha^2\rho^2 + \rho^4 = 0, \quad (13)$$

and the coefficients $a_1, \dots, a_4$ are given by

$$a_k = - \frac{2\rho m_k + \rho\alpha(3-\kappa)}{(\kappa+1)m_k^2 + \alpha(\kappa+1)m_k - (\kappa-1)\rho}, \quad (k = 1, \dots, 4). \quad (14)$$

The characteristic equation (13) may easily be expressed as

$$(m^2 + \alpha m - \rho^2)^2 + \frac{3-\kappa}{\kappa+1}\alpha^2\rho^2 = 0 \quad (15)$$

from which it follows that

$$m_1 = \overline{m}_3 = -\frac{\alpha}{2} + \frac{1}{2}\left\{\alpha^2 + \rho^2 + i4\sqrt{\frac{3-\kappa}{\kappa+1}}\alpha\rho\right\}^{\frac{1}{2}},$$

$$m_2 = \overline{m}_4 = -\frac{\alpha}{2} - \frac{1}{2}\left\{\alpha^2 + \rho^2 + i4\sqrt{\frac{3-\kappa}{\kappa+1}}\alpha\rho\right\}^{\frac{1}{2}}. \quad (16a, b)$$

After solving for $m_1, \dots, m_4$, the expressions for coefficients a_k given by (14) may be simplified as follows:

$$a_k = - \frac{2m_k \alpha(3-\kappa)}{2\rho + i\alpha\sqrt{(3-\kappa)(1+\kappa)}}, \quad (k = 1, 2)$$

$$a_3 = \overline{a}_1, \quad a_4 = \overline{a}_2. \quad (17a-c)$$

By observing that $\Re(m_1, m_3) > 0$ and $\Re(m_2, m_4) < 0$, to satisfy the

regularity conditions at $z = \pm \infty$ in the solution given by (12) we must delete the terms involving A_1 and A_3 for $z > 0$ and A_2 and A_4 for $z < 0$. Thus, the problem may be considered as that of two half spaces involving two unknowns, A_2 and A_4 , for the upper half space and two, A_1 and A_3 , for the lower half. Of these four unknowns two may be eliminated by using the stress continuity conditions (7). Thus, by eliminating A_1 and A_3 from (12) and (7) it can be shown that

$$F(r, \rho) = \begin{cases} A_2 e^{m_2 z} + A_4 e^{m_4 z}, & z > 0, \\ (\lambda_1 A_2 + \lambda_2 A_4) e^{m_1 z} + (\lambda_3 A_2 + \lambda_4 A_4) e^{m_3 z}, & z < 0 \end{cases} \quad (18)$$

$$G(r, \rho) = \begin{cases} a_2 A_2 e^{m_2 z} + a_4 A_4 e^{m_4 z}, & z > 0, \\ a_1 (\lambda_1 A_2 + \lambda_2 A_4) e^{m_1 z} + a_3 (\lambda_3 A_2 + \lambda_4 A_4) e^{m_3 z}, & z < 0 \end{cases} \quad (19)$$

where the expressions for the coefficients $\lambda_1, \dots, \lambda_4$ are given in Appendix A. The unknowns A_2 and A_4 may now be determined from the mixed boundary conditions (1) and (8).

The Integral Equations

To reduce the mixed boundary conditions (1) and (8) to a system of integral equations we first define the following new unknown functions

$$\begin{aligned} g_1(r) &= \frac{\partial}{\partial r} \{w(r, +0) - w(r, -0)\}, \\ g_2(r) &= \frac{1}{r} \frac{\partial}{\partial r} \{ru(r, +0) - ru(r, -0)\}. \end{aligned} \quad (20a, b)$$

From (9), (18), (19) and (20) it then follows that

$$\begin{aligned} g_1(r) &= \int_0^\infty (-\rho(a_2 + G_2)A_2 - \rho(a_4 + G_4)A_4) \rho J_1(r\rho) d\rho, \\ g_2(r) &= \int_0^\infty (\rho(1 - E_2)A_2 + \rho(1 - E_4)A_4) \rho J_0(r\rho) d\rho, \end{aligned} \quad (21a, b)$$

where the functions E_2, E_4, G_2 and G_4 are defined in Appendix A. Equations (21)

determine A_2 and A_4 in terms of g_1 and g_2 .

By substituting now from (18) and (19) into (9) and by using (3) and (4), the boundary conditions (1) may be expressed as

$$\begin{aligned}\int_0^\infty F_1(\rho)\rho J_0(r\rho)d\rho &= \frac{\kappa-1}{\mu_0} p_1(r), \quad 0 \leq r < a, \\ \int_0^\infty F_2(\rho)\rho J_1(r\rho)d\rho &= \frac{1}{\mu_0} p_2(r), \quad 0 \leq r < a,\end{aligned}\quad (22a,b)$$

$$F_1(\rho) = n_2 A_2 + n_4 A_4,$$

$$F_2(\rho) = v_2 A_2 + v_4 A_4, \quad (23a,b)$$

where the functions n_2 , n_4 , v_2 , and v_4 are again defined in Appendix A. Inverting the transforms given by (21) and from (8) and (20) observing that $g_k(r) = 0$ for $r > a$, ($k = 1,2$), the functions F_1 and F_2 are found to be

$$\begin{aligned}F_1(\rho) &= \int_0^a (-d_{11}J_1(\rho s)g_1(s) + d_{12}J_0(\rho s)g_2(s))sds, \\ F_2(\rho) &= \int_0^a (-d_{21}J_1(\rho s)g_1(s) + d_{22}J_0(\rho s)g_2(s))sds,\end{aligned}\quad (24a,b)$$

where the functions $d_{ij}(\rho)$, ($i, j = 1,2$) are also defined in Appendix A.

In order to avoid working with divergent kernels and to simplify the analysis regarding the asymptotic behavior of the kernels, first both sides of (22) are integrated in r . By using (24) it may then be shown that

$$\begin{aligned}\int_0^\infty J_1(r\rho)d\rho \int_0^a (-d_{11}J_1(\rho s)g_1(s) + d_{12}J_0(\rho s)g_2(s))sds &= \frac{\kappa-1}{\mu_0 r} \left(\int_0^r s p_1(s)ds + C_1 \right) \\ - \int_0^\infty J_0(r\rho)d\rho \int_0^a (-d_{21}J_1(\rho s)g_1(s) + d_{22}J_0(\rho s)g_2(s))sds &= \frac{1}{\mu_0} \left(\int_0^r p_2(s)ds + C_2 \right),\end{aligned}\quad (25a,b)$$

where C_1 and C_2 are arbitrary constants. Equations (25) are essentially the

integral equations that determine g_1 and g_2 . For $\rho \rightarrow \infty$ it can be shown that the functions $d_{ij}(\rho)$ have the following asymptotic behavior:

$$\begin{aligned} d_{11}(\rho) &= -2\frac{\kappa-1}{\kappa+1} + O(\rho^{-2}), & d_{12}(\rho) &= \frac{\kappa-1}{\kappa+1}\frac{\alpha}{\rho} + O(\rho^{-3}), \\ d_{21}(\rho) &= \frac{1}{\kappa+1}\frac{\alpha}{\rho} + O(\rho^{-3}), & d_{22}(\rho) &= -\frac{2}{\kappa+1} + O(\rho^{-2}). \end{aligned} \quad (26a-d)$$

Thus, from (25) and (26) we find

$$\begin{aligned} \int_0^a \{(H_{11}(r,s) + h_{11}(r,s))g_1(s) + h_{12}(r,s)g_2(s)\}sds &= \frac{\kappa+1}{2\mu_0 r} \left(\int^r s p_1(s)ds + C_1 \right), \\ \int_0^a \{h_{21}(r,s)g_1(s) + (H_{22}(r,s) + h_{22}(r,s))g_2(s)\}sds &= \frac{\kappa+1}{2\mu_0} \left(\int^r p_2(s)ds + C_2 \right), \end{aligned} \quad (27a,b)$$

where

$$H_{11}(r,s) = \int_0^\infty J_1(r\rho)J_1(s\rho)d\rho = \frac{2}{\pi} \begin{cases} \frac{1}{s}[K(s/r) - E(s/r)], & s < r, \\ \frac{1}{r}[K(r/s) - E(r/s)], & s > r, \end{cases} \quad (28)$$

$$H_{22}(r,s) = \int_0^\infty J_0(r\rho)J_0(s\rho)d\rho = \frac{2}{\pi} \begin{cases} \frac{1}{r}K(s/r), & s < r, \\ \frac{1}{s}K(r/s), & s > r, \end{cases} \quad (29)$$

$$\begin{aligned} h_{11}(r,s) &= \int_0^\infty D_{11}(\rho)J_1(r\rho)J_1(s\rho)d\rho, \\ h_{12}(r,s) &= -\int_0^\infty D_{12}(\rho)J_0(r\rho)J_1(s\rho)d\rho, \\ h_{21}(r,s) &= -\int_0^\infty D_{21}(\rho)J_1(r\rho)J_0(s\rho)d\rho, \\ h_{22}(r,s) &= \int_0^\infty D_{22}(\rho)J_0(r\rho)J_0(s\rho)d\rho, \end{aligned} \quad (30a-d)$$

$$D_{11}(\rho) = \left(\frac{d_{11}(\rho)}{d_{11}(\infty)} - 1 \right),$$

$$D_{12}(\rho) = \frac{d_{12}(\rho)}{d_{11}(\infty)},$$

$$D_{21}(\rho) = \frac{d_{21}(\rho)}{d_{22}(\infty)},$$

$$D_{22}(\rho) = \left(\frac{d_{22}(\rho)}{d_{22}(\infty)} - 1 \right),$$

$$d_{11}(\infty) = -2\frac{\kappa-1}{\kappa+1}, \quad d_{22}(\infty) = -\frac{2}{\kappa+1}, \quad (31a-f)$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. By differentiating, (27) can further be reduced to the following standard form:

$$\begin{aligned} \frac{1}{\pi} \int_0^a \left(\frac{1}{s-r} + \frac{1}{s+r} \right) g_1(s) ds + \frac{1}{\pi} \int_0^a \sum_{j=1}^2 k_{1j}(r,s) g_j(s) ds &= \frac{\kappa+1}{2\mu_0} p_1(r), \\ \frac{1}{\pi} \int_0^a \left(\frac{1}{s-r} - \frac{1}{s+r} \right) g_2(s) ds + \frac{1}{\pi} \int_0^a \sum_{j=1}^2 k_{2j}(r,s) g_j(s) ds &= \frac{\kappa+1}{2\mu_0} p_2(r), \end{aligned} \quad (32a,b)$$

where the Fredholm kernels k_{ij} are given by

$$\begin{aligned} k_{11}(r,s) &= \frac{M_1(r,s)-1}{s-r} + \frac{M_1(r,s)-1}{s+r} + \pi s \int_0^\infty D_{11}(\rho) \rho J_0(r\rho) J_1(s\rho) d\rho, \\ k_{12}(r,s) &= -\pi s \int_0^\infty D_{12}(\rho) \rho J_0(r\rho) J_0(s\rho) d\rho, \\ k_{21}(r,s) &= \pi s \int_0^\infty D_{21}(\rho) \rho J_1(r\rho) J_1(s\rho) d\rho, \\ k_{22}(r,s) &= \frac{M_2(r,s)-1}{s-r} - \frac{M_2(r,s)-1}{s+r} - \pi s \int_0^\infty D_{22}(\rho) \rho J_1(r\rho) J_0(s\rho) d\rho, \end{aligned} \quad (33a-d)$$

$$M_1(r,s) = \begin{cases} \frac{r}{s} E\left(\frac{s}{r}\right) + \frac{s^2-r^2}{r^2} K\left(\frac{s}{r}\right), & s < r, \\ E\left(\frac{r}{s}\right), & s > r, \end{cases} \quad (34)$$

$$M_2(r,s) = \begin{cases} \frac{s}{r} E\left(\frac{s}{r}\right), & s < r, \\ \frac{s^2}{r^2} E\left(\frac{r}{s}\right) - \frac{s^2-r^2}{r^2} K\left(\frac{r}{s}\right), & s > r. \end{cases} \quad (35)$$

Note that the dominant kernels of the system of integral equations (32) are of the generalized Cauchy type (Erdogan, 1978). Thus, expressing the solution by

$$g_k(s) = \frac{h_k(s)}{(a-s)^{\gamma_k} s^{\beta_k}}, \quad k=1,2, \quad (36)$$

and using the function-theoretic method described, for example, by Erdogan (1978), the characteristic equations giving γ_k and β_k may be obtained as follows:

$$\cot \pi \gamma_k = 0, \quad k = 1, 2, \quad (37)$$

$$\cos \pi \beta_1 = -1, \quad \cos \pi \beta_2 = 1. \quad (38a,b)$$

In (36) h_1 and h_2 are unknown bounded functions which are nonzero at the end points $s = a$ and $s = 0$. Equation (37) gives the expected results $\gamma_1 = 1/2$, $\gamma_2 = 1/2$. On the other hand, considering the physical constraints $g_1(0) = 0$ and $u(0, +0) = u(0, -0) = 0$, it may be seen that the admissible roots of (38) are $\beta_1 = -1$, $\beta_2 = 0$ giving

$$g_1(s) = \frac{s}{(a-s)^{1/2}} h_1(s), \quad g_2(s) = \frac{1}{(a-s)^{1/2}} h_2(s). \quad (39)$$

From (20) and (39) it can be shown that for small values of r/a the crack surface displacements would have the following asymptotic form:

$$w^+ - w^- \simeq w_0 + \frac{h_1(0)}{2\sqrt{a}} r^2, \quad u^+ - u^- \simeq \frac{h_2(0)}{2\sqrt{a}} r, \quad r \ll a. \quad (40)$$

Also, from the physical conditions $ru^+ - ru^- = 0$ for $r = 0$ and $r = a$, it is clear that the unknown function g_2 must satisfy

$$\int_0^a r g_2(r) dr = 0. \quad (41)$$

In solving equations such as (32), the accuracy is very highly dependent on the correct evaluation of the kernels k_{ij} , ($i, j = 1, 2$). For this it is necessary that the asymptotic behavior of k_{ij} for $s \rightarrow r$ be examined and the weak singularities, if any, be separated. From (26b,c), (31e,f), and

$$K(\lambda) \rightarrow \log(4/\sqrt{1-\lambda^2}) \quad \text{for } \lambda \rightarrow 1, \quad (42)$$

it can indeed be shown that for $s \rightarrow r$ the kernels k_{ij} have logarithmic singularities

that may be extracted as

$$\begin{aligned}
\frac{M_1(r,s)-1}{s-r} &= -\frac{1}{2r} \log |s-r| - \frac{1}{r} (1 - \log \sqrt{8r}) + m_{11}(r,s), \\
\frac{M_2(r,s)-1}{s-r} &= \frac{1}{2r} \log |s-r| + \frac{1}{r} (2 - \log \sqrt{8r}) + m_{22}, \\
k_{12}(r,s) &= -\frac{\pi}{2} \alpha s H_{22}(r,s) - \pi s \int_0^\infty \left(D_{12}(\rho) \rho + \frac{\alpha}{2} \right) J_0(r\rho) J_0(s\rho) d\rho, \\
k_{21}(r,s) &= \frac{\pi}{2} \alpha s H_{11}(r,s) + \pi s \int_0^\infty \left(D_{21}(\rho) \rho + \frac{\alpha}{2} \right) J_1(r\rho) J_1(s\rho) d\rho,
\end{aligned} \tag{43a-d}$$

where H_{11} and H_{22} are given by (28) and (29) and, by virtue of (42), are also seen to have logarithmic singularities and m_{11} and m_{22} are known functions which are bounded in the closed interval $0 \leq (r,s) \leq a$.

The Solution

The integral equations may be solved by extending the interval and the definition of the unknown functions and the kernels into $(-a,0)$ and by using the properties of the Chebyshev polynomials T_n and U_n (Erdogan,1978). These equations may also be solved by observing from (39) that the orthogonal polynomials corresponding to the weight functions $w_1(y) = (1+y)/\sqrt{1-y}$ and $w_2(y) = 1/\sqrt{1-y}$, $y = 2s/a - 1$, are $P_n^{(-\frac{1}{2},1)}(y)$ and $P_n^{(-\frac{1}{2},0)}(y)$, respectively, and by expressing the unknown functions g_1 and g_2 as follows:

$$\begin{aligned}
g_1(s) = \phi_1(y) &= \frac{1+y}{\sqrt{1-y}} \sum_0^\infty A_{1n} P_n^{(-\frac{1}{2},1)}(y) \\
g_2(s) = \phi_2(y) &= \frac{1}{\sqrt{1-y}} \sum_0^\infty A_{2n} P_n^{(-\frac{1}{2},0)}(y).
\end{aligned} \tag{44a,b}$$

The integral equations may then be regularized by using the properties of Jacobi polynomials and may be solved by truncating the series in (44) and using an

appropriate collocation technique (Mahajan,1992). However, a somewhat more efficient technique may also be developed by defining

$$g_1(as') = f_1(s') \left(\frac{s'}{1-s'} \right)^{\frac{1}{2}}, \quad g_2(as') = \frac{f_2(s')}{\sqrt{s'(1-s')}}, \quad s' = s/a, \quad (45a,b)$$

and by requiring that $f_1(0) = 0$, $f_2(0) = 0$. The orthogonal polynomials related to the weight functions in (45) are

$$P_n^{(-\frac{1}{2}, \frac{1}{2})}(t) = \frac{\Gamma(n + \frac{1}{2})}{n! \sqrt{\pi}} \frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{\theta}{2}} = \frac{\Gamma(n + \frac{1}{2})}{n! \sqrt{\pi}} t_n(t),$$

$$P_n^{(-\frac{1}{2}, -\frac{1}{2})}(t) = \frac{\Gamma(n + \frac{1}{2})}{n! \sqrt{\pi}} \cos n\theta = \frac{\Gamma(n + \frac{1}{2})}{n! \sqrt{\pi}} T_n(t), \quad (46a,b)$$

$$t = 2s' - 1 = \cos \theta, \quad (1+t)t_n(t) = T_n(t) + T_{n+1}(t), \quad -1 < t < 1. \quad (47)$$

We then express g_1 and g_2 in terms of the following infinite series:

$$\frac{2\mu_0}{\kappa + 1} g_1(as') = \sqrt{\frac{s'}{1-s'}} \sum_0^{\infty} B_{1n} t_n(2s' - 1),$$

$$\frac{2\mu_0}{\kappa + 1} g_2(as') = \frac{1}{\sqrt{s'(1-s')}} \sum_0^{\infty} B_{2n} T_n(2s' - 1). \quad (48a,b)$$

By substituting from (48) into (32) and by observing that

$$\frac{1}{\pi} \int_0^1 \sqrt{\frac{s'}{1-s'}} \frac{t_n(2s' - 1) ds'}{s' - r'} = \frac{1}{\pi} \int_{-1}^1 \frac{T_n(t) + T_{n+1}(t)}{(t-x)\sqrt{1-t^2}} dt$$

$$\frac{1}{\pi} \int_0^1 \sqrt{\frac{s'}{1-s'}} \frac{t_n(2s' - 1) ds'}{s' - r'} = \begin{cases} U_{n-1}(x) + U_n(x), & |x| < 1, \\ G_n(x) + G_{n+1}(x), & x > 1, \end{cases} \quad (49)$$

$$r' = r/a, \quad x = 2r' - 1, \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad |x| < 1, \quad \cos \theta = x, \quad n = 0, 1, \dots, \quad (50)$$

$$G_n(x) = -\frac{(x - \sqrt{x^2 - 1})^n}{\sqrt{x^2 - 1}}, \quad (x > 1), \quad n = 0, 1, \dots \quad (51)$$

the integral equations (32) may be reduced to the following system of equations:

$$\sum_0^{\infty} \{B_{1n}[U_{n-1}(x) + U_n(x) + K_{11}(x)] + B_{2n}K_{12n}(x)\} = P_1(x),$$

$$\sum_0^{\infty} \{B_{1n}K_{21n}(x) + B_{2n}[2U_{n-1}(x) + K_{22n}(x)]\} = P_2(x),$$

$$(-1 < x < 1) \quad (52a,b)$$

where

$$P_j(x) = p_j(r), \quad j = 1, 2, \quad x = \frac{2r}{a} - 1 = 2r' - 1,$$

$$K_{i1n}(x) = \frac{1}{\pi} \int_0^{\infty} k_{i1}(ar', as') \sqrt{\frac{s'}{1-s'}} t_n(2s' - 1) ds', \quad (i = 1, 2),$$

$$K_{i2n}(x) = \frac{1}{\pi} \int_0^{\infty} k_{i2}(ar', as') \frac{T_n(2s' - 1)}{\sqrt{s'(1-s')}} ds', \quad (i = 1, 2), \quad (53a-c)$$

Also, by substituting from (48b) into (41) and by using the orthogonality conditions

$$\frac{1}{\pi} \int_{-1}^1 \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} dt = \begin{cases} 1, & i = j = 0, \\ 1/2, & i = j > 0, \\ 0, & i \neq j, \end{cases} \quad (54)$$

it may easily be shown that

$$B_{20} + \frac{1}{2}B_{21} = 0. \quad (55)$$

Equations (52) and (55) are solved by truncating the series and by using a simple collocation technique. Very fast converging results are obtained if the collocation points x_j are selected as follows:

$$T_N(x_j) = 0, \quad x_j = \cos \theta_j, \quad \theta_j = \frac{\pi}{2N}(2j - 1). \quad (56)$$

From the derivation of the integral equations (32) we observe that the right hand side of (32) represents $\sigma_{zz}(r, 0)$ and $\sigma_{rz}(r, 0)$ for $a < r < \infty$ as well as for $0 \leq r < a$. Thus, defining the modes I and II stress intensity factors by

$$\begin{aligned}
k_1 &= \lim_{r \rightarrow a} \sqrt{2(r-a)} \sigma_{zz}(r, 0), \\
k_2 &= \lim_{r \rightarrow a} \sqrt{2(r-a)} \sigma_{rz}(r, 0),
\end{aligned} \tag{57a,b}$$

and by using (51), from (47), (48) and (32) it can be shown that

$$k_1 = -\sqrt{2a} \sum_0^{\infty} B_{1n}, \quad k_2 = -\sqrt{2a} \sum_0^{\infty} B_{2n}. \tag{58}$$

For a homogeneous medium (i.e., for $\alpha = 0$ in (2)) modes I and II crack problems are uncoupled and the stress intensity factors are given by

$$k_1 = -\frac{2}{\pi\sqrt{a}} \int_0^a \frac{rp_1(r)}{\sqrt{a^2-r^2}} dr, \quad k_2 = -\frac{2}{\pi a^{3/2}} \int_0^a \frac{r^2 p_2(r)}{\sqrt{a^2-r^2}} dr. \tag{59}$$

The Results

The main results of this study are the stress intensity factors calculated for various loading conditions as functions of the nonhomogeneity constant α by defining the shear modulus in $\mu(z) = \mu_0 \exp(\alpha z)$. Table 1 shows the six different loading conditions used in the calculations. The table also shows the corresponding modes I and II stress intensity factors in a homogeneous medium containing a penny-shaped crack of radius a obtained from (59). For the nonhomogeneous medium the normalized stress intensity factors calculated for $\nu = 0.3$ are shown in Table 2. As in the plane strain problem, for all loading conditions both k_1 and k_2 increase with increasing $a\alpha$, the dimensionless nonhomogeneity constant. Comparing the results given in Table 2 with the corresponding plane strain results obtained by Konda and Erdogan (1992), it may be seen that, aside from the factor $2/\pi$ characterizing the difference between penny-shaped and plane strain cracks, the increase in the stress intensity factors k_1 and k_2 with increasing $a\alpha$ in the axisymmetric case is not as severe as in the plane strain crack. For the case of

pressurized cracks (i.e., for $\sigma_{xx}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$ and $\sigma_{yy}(x,0) = -p_0$, $\sigma_{xy}(x,0) = 0$) the comparison of the two sets of results is shown in Table 3.

Note that the results given in Table 2 may be used to obtain the stress intensity factors for arbitrary crack surface tractions by superposition to the extent that the tractions may be approximated by the second degree polynomials in r .

After determining g_1 and g_2 or the coefficients B_{1n} and B_{2n} shown in (48), the crack opening displacements may be obtained from (20) as follows:

$$\begin{aligned} w(r, +0) - w(r, -0) &= - \int_r^a g_1(s) ds \\ &= - \frac{a(\kappa + 1)}{4\mu_0} \left\{ B_{10}(\theta' + \frac{1}{2}\sin 2\theta') \right. \\ &\quad \left. + \sum_1^\infty B_{1n} \left(\frac{1}{2(n+1)} \sin(2n+2)\theta' + \frac{1}{2n} \sin 2n\theta' \right) \right\}, \end{aligned} \quad (60)$$

$$\begin{aligned} r[u(r, +0) - u(r, -0)] &= - \int_r^a s g_2(s) ds \\ &= - \frac{a^2(\kappa + 1)}{2\mu_0} \left\{ B_{21} \left(\frac{1}{4} \sin 2\theta' + \frac{1}{8} \sin 4\theta' \right) + \sum_2^\infty B_{2n} \left(\frac{1}{2n} \sin 2n\theta' \right. \right. \\ &\quad \left. \left. + \frac{1}{4n+4} \sin(2n+2)\theta' + \frac{1}{4n-4} \sin(2n-2)\theta' \right) \right\}, \end{aligned} \quad (61)$$

where

$$\cos^2 \theta' = \frac{r}{a}.$$

It should be observed that $u^+ - u^-$ vanishes for $r \rightarrow 0$. This may be seen from (40) and also from (61) and (45) by using the condition (41) as follows:

$$\begin{aligned} \lim_{r \rightarrow 0} [u(r, +0) - u(r, -0)] &= - \lim_{r \rightarrow 0} \frac{1}{r} \int_r^a s g_2(s) ds \\ &= - \lim_{r \rightarrow 0} \frac{d}{dr} \int_r^a s g_2(s) ds \\ &= \lim_{r \rightarrow 0} r \frac{f_2(r)}{\sqrt{r(a-r)}} = 0. \end{aligned} \quad (62)$$

Figures 2 and 3 show some sample results for the crack opening displacements. In Fig.2 the external loads are $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$ and the dimensionless displacement shown is defined by

$$W(r) = \frac{w(r, +0) - w(r, -0)}{w_0}, \quad w_0 = \frac{(1 + \kappa)}{2\mu_0} p_0 a. \quad (63)$$

In Fig.3 it is assumed that $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$ and the displacement is normalized as follows:

$$U(r) = \frac{u(r, +0) - u(r, -0)}{u_0}, \quad u_0 = \frac{(1 + \kappa)}{2\mu_0} q_0 a. \quad (64)$$

From figures 2 and 3 it may be seen that the influence of the nonhomogeneity constant α is much more significant on the axial displacement than on the radial displacement. For reference we note that for $\alpha = 0$ in pressurized plane strain and penny-shaped cracks the crack opening displacements are respectively given by

$$\begin{aligned} u_y(x, +0) - u_y(x, -0) &= w_0 \sqrt{1 - x^2/a^2}, \\ w(r, +0) - w(r, -0) &= \frac{2}{\pi} w_0 \sqrt{1 - r^2/a^2}, \quad w_0 = \frac{(1 + \kappa)}{2\mu_0} p_0 a. \end{aligned} \quad (65a-c)$$

In this study, largely to simplify the analysis, the Poisson's ratio ν is assumed to be constant. In an actual nonhomogeneous medium this, of course, is not possible. The assumption can only be justified if the fracture mechanics parameters of interest, in this case the stress intensity factors, prove to be relatively insensitive to variations in the Poisson's ratio. To give some idea about the influence of the variations in ν on the stress intensity factors, some additional results are given in Table 4. The table shows the normalized stress intensity factors for various values of ν and for a small and a large value of the dimensionless nonhomogeneity constant αa . In these examples the external loads

are assumed to be $\sigma_{zz}(r,0) = p_1(r) = (-p_0, -p_1(r/a), -p_2(r/a)^2)$, $0 \leq r < a$, $\sigma_{rz}(r,0) = p_2(r) = 0$. Particularly considering the fact that the Poisson's ratio of the constituent materials and consequently that of the composite nonhomogeneous medium is likely to vary within a much narrower range than shown in the table, the influence of ν on the stress intensity factors does not seem to be very significant.

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Table 1. Loading conditions used and the corresponding stress intensity factors for $\alpha=0$.

$p_1(r)$	$-p_0$	$-p_1 \frac{r}{a}$	$-p_2 (\frac{r}{a})^2$	0	0	0
$p_2(r)$	0	0	0	$-q_0$	$-q_1 (\frac{r}{a})$	$-q_2 (\frac{r}{a})^2$
k_1	$\frac{2}{\pi} p_0 \sqrt{a}$	$\frac{1}{2} p_1 \sqrt{a}$	$\frac{4}{3\pi} p_2 \sqrt{a}$	0	0	0
k_2	0	0	0	$\frac{1}{2} q_0 \sqrt{a}$	$\frac{4}{3\pi} q_1 \sqrt{a}$	$\frac{3}{8} q_2 \sqrt{a}$

Table 2. The variation of stress intensity factors with α for various loading conditions shown in Table 1, $\nu = 0.3$

$\sigma_{zz}(r,0) = p_1(r), \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	0.6366	0.0	0.5	0.0	0.4244	0.0
0.1	0.6378	0.0106	0.5008	0.0063	0.4249	0.0042
0.2	0.6413	0.0212	0.5028	0.0125	0.4263	0.0085
0.3	0.6465	0.0319	0.5058	0.0188	0.4284	0.0128
0.4	0.6580	0.0425	0.5097	0.0250	0.4309	0.0169
0.5	0.6608	0.0532	0.5143	0.0313	0.4341	0.0213
0.6	0.6695	0.0639	0.5193	0.0376	0.4376	0.0255
0.75	0.6841	0.0801	0.5279	0.0471	0.4434	0.0319
1.0	0.7115	0.1073	0.5441	0.0631	0.4544	0.0428
1.5	0.7741	0.1628	0.5811	0.0955	0.4795	0.0647
2.0	0.8435	0.2202	0.6219	0.1289	0.5073	0.0872
3.0	0.9943	0.3413	0.7108	0.1988	0.5676	0.1339
4.0	1.1558	0.4714	0.8057	0.2732	0.6319	0.1833
5.0	1.3256	0.6105	0.9053	0.3521	0.6994	0.2355
$\sigma_{zz}(r,0) = 0, \sigma_{rz}(r,0) = p_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	0.0	0.5	0.0	0.4244	0.0	0.3750
0.1	0.000	0.5001	0.000	0.4244	0.000	0.3750
0.2	0.000	0.5003	0.000	0.4246	0.000	0.3751
0.3	0.000	0.5006	0.000	0.4248	0.000	0.3753
0.4	0.000	0.5012	0.000	0.4252	0.000	0.3756
0.5	0.000	0.5018	0.000	0.4256	0.000	0.3759
0.6	0.000	0.5026	0.000	0.4262	0.000	0.3763
0.75	0.000	0.5039	0.000	0.4271	0.000	0.3770
1.0	0.000	0.5068	0.000	0.4290	0.000	0.3784
1.5	0.000	0.5142	0.000	0.4341	0.000	0.3821
2.0	0.000	0.5233	0.000	0.4403	0.000	0.3867
3.0	0.000	0.5447	0.000	0.4550	0.000	0.3976
4.0	-0.0001	0.5682	0.000	0.4713	0.000	0.4097
5.0	-0.0004	0.5926	0.0002	0.4882	-0.0001	0.4224

Table3. Comparison of the stress intensity factors for pressurized plane strain and penny-shaped cracks in a nonhomogeneous medium ($\nu = 0.3$).

αa	0	0.1	0.25	0.5	1.0	2.5	5.0
Plane strain crack							
$\frac{k_1}{p_0\sqrt{a}}$	1	1.008	1.036	1.101	1.258	1.808	2.869
$\frac{k_2}{p_0\sqrt{a}}$	0.0	0.026	0.065	0.129	0.263	0.697	1.567
Penny-shaped crack							
$\frac{k_1}{p_0\sqrt{a}}$	0.637	0.638	0.644	0.661	0.712	0.918	1.326
$\frac{k_2}{p_0\sqrt{a}}$	0.0	0.011	0.026	0.053	0.107	0.280	0.611

Table4. The influence of the Poisson's ratio on the stress intensity factors, loading: $\sigma_{rz}(r,0) = 0$, $\sigma_{zz}(r,0) = (-p_0, -p_1(r/a), p_2(r/a)^2)$, $0 \leq r < a$.

ν	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
$\alpha a = 0.3$						
0.01	0.6428	0.0318	0.5037	0.0188	0.4269	0.0127
0.1	0.6437	0.0318	0.5042	0.0188	0.4273	0.0127
0.2	0.6449	0.0319	0.5049	0.0188	0.4277	0.0127
0.3	0.6465	0.0319	0.5058	0.0188	0.4284	0.0127
0.4	0.6484	0.0319	0.5069	0.0188	0.4291	0.0127
0.45	0.6496	0.0319	0.5077	0.0188	0.4296	0.0127
$\alpha a = 3.0$						
0.01	0.9193	0.3370	0.6659	0.1967	0.5367	0.1327
0.1	0.9389	0.3381	0.6776	0.1973	0.5448	0.1330
0.2	0.9642	0.3395	0.6928	0.1979	0.5552	0.1334
0.3	0.9943	0.3413	0.7108	0.1988	0.5676	0.1339
0.4	1.031	0.3434	0.7328	0.1999	0.5827	0.1346
0.45	1.053	0.3447	0.7458	0.2006	0.5917	0.1349

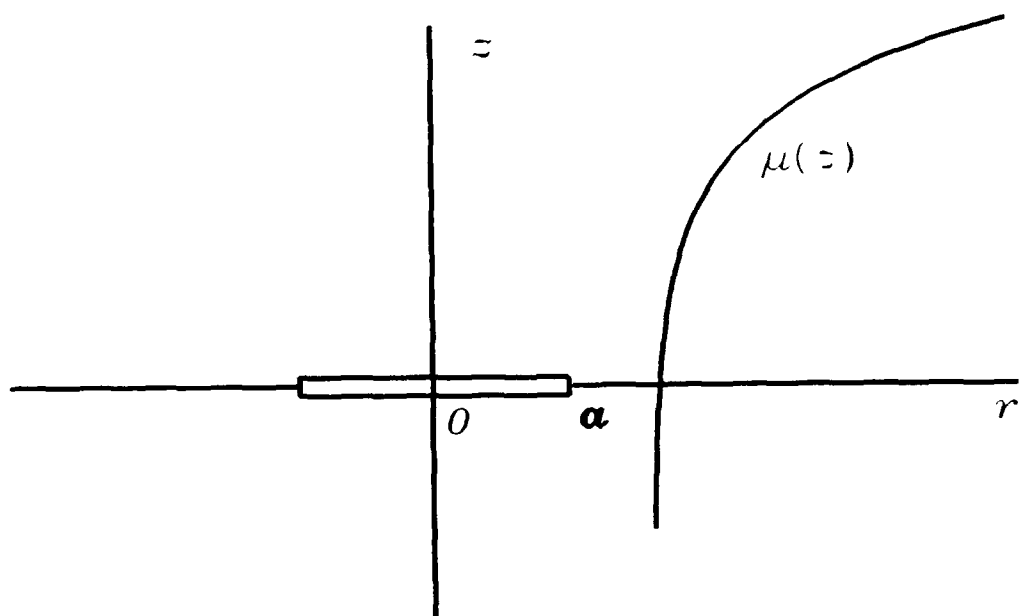


Fig.1 Crack geometry and notation

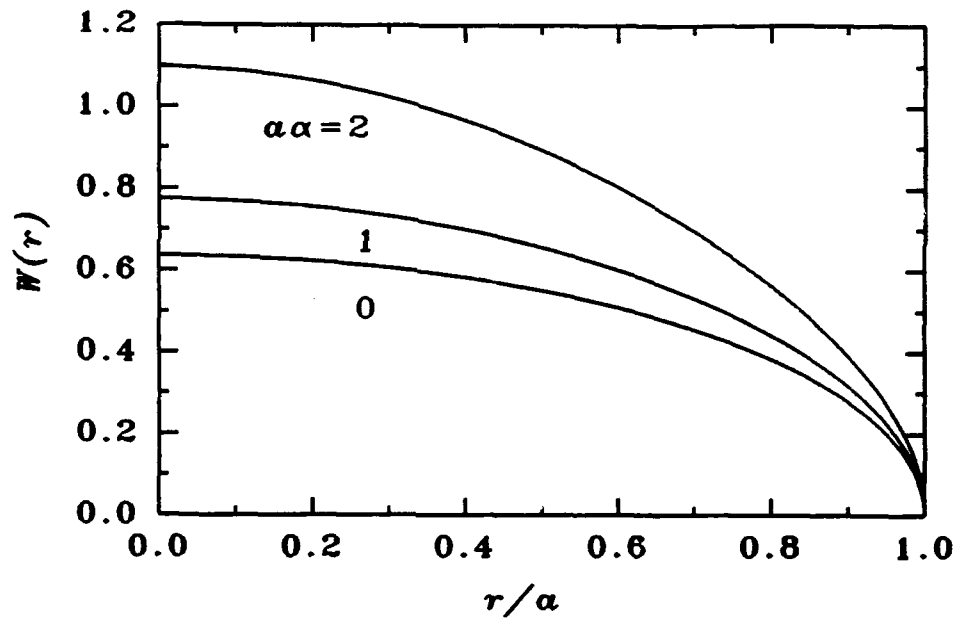


Fig.2 z- component of the normalized crack opening displacement, $W = (w^+ - w^-)/w_0$, $w_0 = (1 + \kappa)p_0a/2\mu_0$ for the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

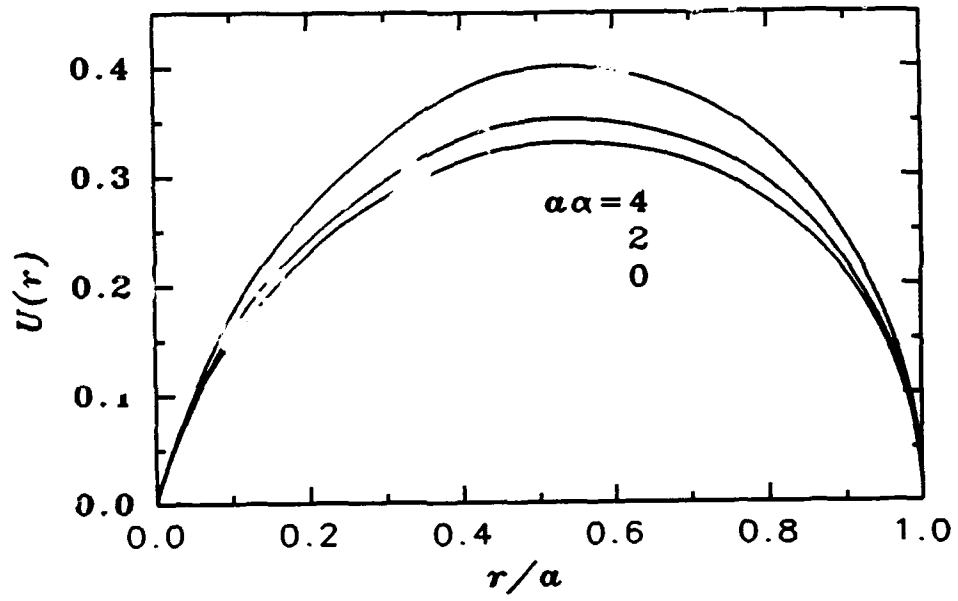


Fig.3. r - component of the normalized crack opening displacement, $U = (u^+ - u^-)/u_0$, $u_0 = (1 + \kappa)q_0a/2\mu_0$ for the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

Appendix

Expressions for various functions that appear in analysis.

$$\begin{aligned}\lambda_1(\rho) &= (n_2 v_3 - n_3 v_2) / \Delta_1, \quad \lambda_2(\rho) = (n_4 v_3 - n_3 v_4) / \Delta_1, \\ \lambda_3(\rho) &= (n_1 v_2 - n_2 v_1) / \Delta_1, \quad \lambda_4(\rho) = (n_1 v_4 - n_4 v_1) / \Delta_1.\end{aligned}\quad (\text{A1-4})$$

$$n_j(\rho) = (3 - \kappa)\rho - (1 + \kappa)m_j a_j, \quad v_j(\rho) = m_j + \rho a_j, \quad j = 1, \dots, 4,$$

$$\Delta_1(\rho) = n_1 v_3 - n_3 v_1. \quad (\text{A5-7})$$

$$E_j(\rho) = \frac{1}{\Delta_1} \{ n_j [m_3 - m_1 + \rho(a_3 - a_1)] + (1 + \kappa) v_j (a_3 m_3 - a_1 m_1) \},$$

$$\begin{aligned}G_j(\rho) &= \frac{1}{\Delta_1} \{ n_j (a_1 m_3 - a_3 m_1) + v_{2j} [(3 - \kappa)(a_3 - a_1)\rho \\ &\quad + (1 + \kappa)(m_3 - m_1)a_1 a_3] \}, \quad (j = 2, 4).\end{aligned}\quad (\text{A8,9})$$

$$d_{11}(\rho) = \frac{n_2}{\Delta_2} (1 - E_4) - \frac{n_4}{\Delta_2} (1 - E_2),$$

$$d_{12}(\rho) = \frac{n_2}{\Delta_2} (a_4 - G_4) - \frac{n_4}{\Delta_2} (a_2 - G_2),$$

$$d_{21}(\rho) = \frac{v_2}{\Delta_2} (1 - E_4) - \frac{v_4}{\Delta_2} (1 - E_2),$$

$$d_{22}(\rho) = \frac{v_2}{\Delta_2} (a_4 - G_4) - \frac{v_4}{\Delta_2} (a_2 - G_2), \quad (\text{A10-13})$$

$$\Delta_2(\rho) = \rho(1 - E_2)(a_4 - G_4) - (1 - E_4)(a_2 - G_2). \quad (\text{A14})$$