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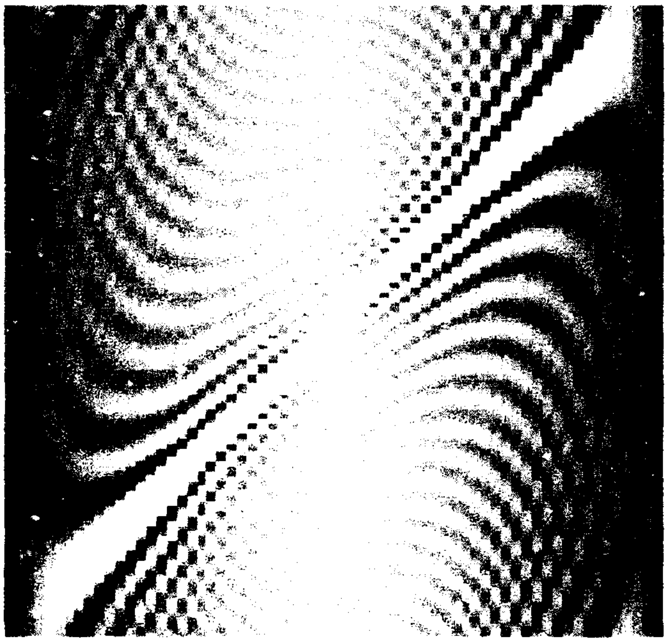
Time-Frequency Localization via the Weyl Correspondence

Jayakumar Ramanathan
Pankaj Topiwala

September 1992



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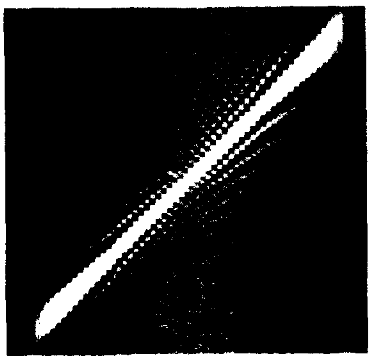

Cover Illustration: The Wigner distribution of a chirp signal in 0 dB noise after filtering the noise by time-frequency localization techniques.

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ABSTRACT

A technique of producing signals whose energy is concentrated in a given region of the time-frequency plane is examined. The degree to which a particular signal is concentrated is measured by integrating the Wigner distribution over the given region. This procedure was put forward by Flandrin, and has been used for time-varying filtering in the recent work of Hlawatsch, Kozek, and Krattenthaler. In this paper, the associated operator is studied. Estimates for the eigenvalue decay and the smoothness and decay of the eigenfunctions are established.

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SECTION 1
INTRODUCTION

It is well known that the time-frequency characteristics of a square integrable signal cannot be arbitrary. For example, no such signal can be both time and band limited. The Heisenberg uncertainty principle provides another quantitative restriction on the joint time-frequency behavior of a square integrable signal. These facts indicate that a signal cannot have all its energy concentrated in a finite region of the time-frequency plane.

Nevertheless, in many applications it is important to use signals whose time-frequency characteristics are highly localized. Among other things, the work of Landau, Pollack, and Slepian [1,2] produced a rigorous development of band limited signals that are as concentrated as possible within a prescribed timespan. More recently, there has been interest in finding signals that are localized in general regions of phase space via methods that keep time and frequency on an equal footing (see the papers of Daubechies and Paul [3,4]).

In this paper, we study a localization technique that uses the Wigner distribution to measure the degree to which a signal is concentrated in a particular region. This leads to a self-adjoint *localization* operator that is easy to study in terms of the Weyl correspondence. Under the Weyl correspondence, a function of two variables—called the *symbol*—is associated with an operator on functions of one variable. The symbol of the localization operator is simply the indicator function of the given region in the time-frequency space. The eigenfunctions of this operator with large eigenvalues span a subspace that can be used to determine the component of a general signal that is concentrated within the given region of the time-frequency plane: one computes the projection (either orthogonal or weighted by the eigenvalues) of a general signal into this subspace. This procedure was put forward by Flandrin [5], who derived a number of useful results, including Lemma 4 below. It has since been developed in the context of time-varying filtering (see for example the papers of Hlawatsch, Kozek, and Krattenthaler [6,7]). This paper is devoted to the further study of the localization operator described above. The asymptotic properties of the eigenvalues are studied and it is established that they are $\mathcal{O}(k^{-3/4})$. In addition, the eigenfunctions with nonzero eigenvalues are shown to have faster than exponential decay in both the time and frequency domains. This, of course, leads to a statement regarding the smoothness of these eigenfunctions. In particular, we show that they are analytic. In the last section, some numerical examples are provided.

SECTION 2
THE WEYL CORRESPONDENCE

The basic properties of the Weyl correspondence that we will need are collected in this section; for a thorough treatment, the reader is referred to the book of Folland [8].

Let $f, g \in L^2(\mathbf{R})$. A general time-frequency shift of f is

$$\rho(\tau,\sigma)f(t) = e^{\pi i\tau\sigma}e^{2\pi i\sigma t}f(t+\tau).$$

The cross-ambiguity function of f and g is

$$\begin{aligned} A(f,g)(\tau,\sigma) &= \langle \rho(\tau,\sigma)f, g\rangle \\ &= \int e^{\pi i\tau\sigma}e^{2\pi i\sigma t}f(t+\tau)\overline{g(t)}\,dt \\ &= \int e^{2\pi i\sigma s}f(s+\tau/2)\overline{g(s-\tau/2)}\,ds. \end{aligned}$$

The value of $A(f,g)$ at a particular point is the cross-correlation between a particular time-frequency shift of f against g. The ambiguity function is therefore a time-frequency cross-correlation between two functions. The Wigner distribution is the two-dimensional Fourier transform of the cross ambiguity function, thus giving it the interpretation of a time-varying spectrum. The Wigner distribution can be written as

$$W(f,g)(\xi,t) = \int e^{-2\pi i\tau\xi}f(t+\tau/2)\overline{g(t-\tau/2)}\,d\tau.$$

Several useful properties of the Wigner distribution are catalogued below.

THEOREM 1. *Let $f, g \in L^2(\mathbf{R})$. Then*

(i) $W(f,g)(\xi,t) \in L^2(\mathbf{R}^2)$ *and* $\|W(f,g)\|^2 = \|f\|^2\|g\|^2$,

(ii) $W(f,g) \in C_0(\mathbf{R}^2)$ *and* $\|W(f,g)\|_\infty \le \|f\|\|g\|$,

(iii) $W(g,f) = \overline{W(f,g)}$,

(iv) $W(\hat{f},\hat{g})(\xi,t) = W(f,g)(t,-\xi)$, *and*

(v) $W(\rho(a,b)f, \rho(a,b)g)(\xi,t) = W(f,g)(\xi-b, t+a)$.

The Weyl correspondence uses the Wigner distribution to define a correspondence between functions of two variables and operators on $L^2(\mathbf{R})$. It is defined, via duality, by

$$\langle L_S f, g\rangle = \iint S(\xi,t)W(f,g)(\xi,t)\,d\xi dt$$

where $f, g \in L^2(\mathbf{R})$ and $S(\xi,t)$ is a function with appropriate decay properties. $S(\xi,t)$ is the symbol of the operator L_S. The following theorem of Pool [9] is useful.

Theorem 2. *A symbol $S(\xi,t) \in L^2(\mathbf{R}^2)$ gives rise to an operator L_S that is Hilbert-Schmidt on $L^2(\mathbf{R})$. Moreover, the mapping $S \mapsto L_S$ is a unitary operator from $L^2(\mathbf{R}^2)$ to the Hilbert-Schmidt operators on $L^2(\mathbf{R})$.*

Lemma 3. *L_S is self-adjoint if $S(\xi,t)$ is real valued.*

The following result was derived by P. Flandrin in [5], based on results of Janssen [10]. We will follow the development in [8].

Lemma 4. *The eigenfunctions of the operator L_S corresponding to a radially symmetric symbol S are:*

$$h_j(t) = \frac{2^{1/4}}{\sqrt{j!}}\left(\frac{-1}{2\sqrt{\pi}}\right)^j e^{\pi t^2}\frac{d^j}{dt^j}e^{-2\pi t^2}.$$

Proof: Theorem 1.105 in [7] shows that

$$W(h_j,h_k)(\xi,t) = \begin{cases} 2(-1)^k\sqrt{\frac{k!}{j!}}e^{-2\pi|z|^2}(2\sqrt{\pi}z)^{j-k}L_k^{(j-k)}(4\pi|z|^2), & \text{for } j \geq k \\ 2(-1)^j\sqrt{\frac{j!}{k!}}e^{-2\pi|z|^2}(2\sqrt{\pi}\bar{z})^{k-j}L_j^{(k-j)}(4\pi|z|^2), & \text{for } k \geq j \end{cases}$$

where $z = t + i\xi$ and $L_k^{(\alpha)}$ is the associated Laguerre polynomial. Set $r = |z|$ and note that

$$\begin{aligned} \langle L_S h_j, h_k\rangle &= \iint S(r)W(h_j,h_k)\,d\xi dt \\ &= \begin{cases} 0, & \text{for } j \neq k \\ (-1)^j 4\pi \int_0^\infty S(r)\,e^{-2\pi r^2}L_j^{(0)}(4\pi r^2)\,rdr, & \text{for } j = k. \end{cases} \end{aligned}$$

Hence

$$\lambda_j = (-1)^j 4\pi \int_0^\infty S(r)\,e^{-2\pi r^2}L_j^{(0)}(4\pi r^2)\,rdr. \tag{1}$$

SECTION 3
LOCALIZATION VIA A CUT-OFF

The localization operator we are concerned with is L_{χ_Ω}, where χ_Ω is the characteristic function of some bounded domain in the $t-\xi$ plane. We will assume that $\Omega \subset [-B,B]\times[-T,T]$. Note that

$$\langle L_{\chi_\Omega} f, g\rangle = \iint_\Omega W(f,g).$$

L_{χ_Ω} is self adjoint since χ_Ω is real valued. Pool's theorem implies that L_{χ_Ω} is Hilbert-Schmidt. Hence there is an orthonormal basis $\phi_1, \phi_2, \ldots$ of $L^2(\mathbf{R})$ and real numbers $\lambda_1, \lambda_2, \ldots$ such that $L_{\chi_\Omega}\phi_k = \lambda_k \phi_k$. The Hilbert-Schmidt norm of L_{χ_Ω} is $\sum |\lambda_k|^2 = |\Omega|$, the measure of Ω. We will assume that the eigenvalues are arranged in order of decreasing absolute value. It is easy to check that the largest positive eigenvalue corresponds to the maximum energy an L^2 function can have within the domain Ω. The corresponding eigenfunction would then be a time function with energy as concentrated as possible within Ω. Our principal aim in this paper is to study the decay of the eigenvalues of L_{χ_Ω} and the smoothness properties of the eigenfunctions.

Several properties of the associated kernel will be of importance.

LEMMA 5. *The kernel of the operator L_{χ_Ω} is given by the equation*

$$K(s,t) = \int \chi_\Omega(\xi, \frac{s+t}{2})\, e^{2\pi i(s-t)\xi}\, d\xi$$

and has the properties

(i) $K(s,t) \equiv 0$ *if* $|s+t| \geq 2T$ *and*

(ii) *if the cross-sections Ω_t of Ω in the ξ direction consist of at most M intervals, then* $|K(s,t)| \leq \frac{CM}{|s|+1}$.

PROOF: The formula for the kernel is well known (see [8]). The kernel can be written as

$$K(s,t) = F(s-t, \frac{s+t}{2})$$

where

$$F(\eta, t) = \int_{\Omega_t} e^{-2\pi i \eta\xi}\, d\xi.$$

Item 1 follows from the observation that $\Omega_t = \emptyset$ if $|t| \geq 2T$. We now verify item 2. By assumption, the cross-section is the union of at most M disjoint intervals: $\Omega_t = \cup_{i=1}^{L}[\alpha_i, \beta_i]$ for some integer $L \leq M$. Therefore one can estimate that

$$\begin{aligned}
|F(\eta,t)| &= \left|\sum e^{\pi i(\alpha_k+\beta_k)\eta} \frac{\sin(\pi(\beta_k - \alpha_k)\eta)}{\pi\eta}\right| \\
&\leq \sum \left|\frac{\sin(\pi(\beta_k - \alpha_k)\eta)}{\pi\eta}\right| \\
&\leq \frac{2L}{|\eta|+1} \leq \frac{2M}{|\eta|+1}
\end{aligned}$$

since $\sin(Ax)/x \leq \frac{\max(2,A)}{|x|+1}$. The estimate only needs to be verified when $|s+t| \leq 2T$. In this case,

$$|s-t| + 2T \geq |s-t| + |s+t| \geq 2|s|.$$

Using this and the estimate for $F(\eta, t)$ yields

$$\begin{aligned}
|K(s,t)| &\leq \frac{CM}{|s-t|+1} \\
&\leq \frac{CM}{2|s|+1-2T}
\end{aligned}$$

for all large s. The estimate in the theorem follows easily by adjusting the constant as necessary.

For domains Ω with piecewise C^1 boundary, we can show that λ_k is $\mathcal{O}(\frac{1}{k^{3/4}})$. The proof is a modification of Weyl's classical work on the asymptotics of eigenvalues of integral equations. The following lemma contains some useful standard facts about the Weyl correspondence. Again we refer the reader to [8] for the proofs.

LEMMA 6.

(i) *The operators corresponding to $S(\xi,t)$ and $S(\xi-\xi_0, t-t_0)$ are unitarily equivalent.*

(ii) *Suppose the symbols $S_1(\xi,t)$ and $S_2(\xi,t)$ are related by an orthogonal change of variables. Then the corresponding operators are unitarily equivalent.*

We will first prove that λ_k is $\mathcal{O}(\frac{1}{k^{3/4}})$ for domains Ω of the following form:

$$(*) \qquad \Omega = \{(\xi,t) : \alpha(t) \leq \xi \leq \beta(t), t \in [-T,T]\}$$

where $\alpha(t), \beta(t)$ are C^1 functions with vanishing endpoint values. The kernel associated to the symbol $\chi_\Omega(\xi, t)$ is:

$$K(s,t) = e^{-\pi i(s-t)[\alpha+\beta](\frac{s+t}{2})} \frac{\sin(\pi[\alpha-\beta](\frac{s+t}{2})(s-t))}{\pi(s-t)}.$$

Moreover, it is easy to check that K is a Lipschitz function whose gradient DK exists a.e. and satisfies the inequality

$$|DK| \leq C \qquad \text{a.e.} \tag{2}$$

LEMMA 7. *If Ω is of the form $(*)$ then there are symmetric finite rank approximations*

$$k_N(s,t) = \sum a_Q \chi_Q$$

where Q ranges over squares of the form

$$\{(\xi,t) : i/N \leq \xi \leq (i+1)/N, j/N \leq t \leq (j+1)/N\} \qquad -N^2 \leq i,j \leq N^2 - 1$$

with the property that

$$\iint |K(s,t) - k_N(s,t)|^2\, dsdt \qquad \text{is } \mathcal{O}(1/N). \tag{3}$$

PROOF: Set

$$a_Q = \frac{1}{|Q|} \iint_Q K(s,t)\, dsdt.$$

The symmetry of the kernel K forces symmetry of k_N. Let $R = \cup Q$. Then

$$\begin{aligned}\iint_R |K(s,t) - k_N(s,t)|^2\, dsdt &= \sum \iint_Q |K(s,t) - k_N(s,t)|^2\, dsdt \\ &\leq \sum |Q| \iint_Q |DK|^2 = \frac{1}{N^2} \iint_R |DK|^2\end{aligned}$$

using Poincare's inequality [11]. Since $K(s,t)$ is supported within the strip $|s+t| \leq 2T$, equation (2) yields the estimate

$$\iint_R |K(s,t) - k_N(s,t)|^2\, dsdt \leq \frac{C}{N}. \tag{4}$$

The mean square error over the exterior of R can be handled as follows:

$$\iint_{\mathbf{R}^2-R} |K - k_N|^2 = \iint_{\mathbf{R}^2-R} |K|^2\, ds dt \leq \frac{C}{N} \tag{5}$$

in view of lemma 5. Putting equations (4) and (5) together yields the estimate in equation (3).

In view of Satz III of Weyl's paper [11], one has

$$\lambda^2_{4N^2+1} + \cdots \leq C/N.$$

We will apply this inequality to $N/2$ where $N = [\sqrt{k-1}/2]$. (Here [ ] denotes the greatest integer function.) This yields:

$$\lambda^2_{4(N/2)^2-1} + \cdots + \lambda^2_{4N^2-1} + \cdots + \lambda^2_k \cdots \leq 2C/N.$$

Since the eigenvalues have been arranged in decreasing order, the left-hand side of the above inequality is greater than or equal to $3N^2\lambda^2_k$. This yields an inequality of the form $\lambda^2_k \leq C/N^3$. Clearly $1/N$ is $\mathcal{O}(1/\sqrt{k})$. These remarks imply that λ_k is $\mathcal{O}(\frac{1}{k^{3/4}})$ for domains with property (*). We now use a cut and paste argument to derive the result for general Ω with C^1 boundary.

LEMMA 8. *If $K'(s,t)$ and $K''(s,t)$ are two symmetric real kernels in $L^2(\mathbf{R}^2)$ with*

$$|\lambda'_k| \leq \frac{C'}{k^{3/4}} \quad \text{and} \quad |\lambda''_k| \leq \frac{C''}{k^{3/4}}$$

then the eigenvalues of $K(s,t) = K'(s,t) + K''(s,t)$ must satisfy the same estimate: $|\lambda_k| \leq \frac{C}{k^{3/4}}$ for some constant C.

PROOF: Satz I of Weyl's paper [11] implies that

$$|\lambda_{2k}| \leq |\lambda'_k| + |\lambda''_{k+1}|$$
$$|\lambda_{2k+1}| \leq |\lambda'_{k+1}| + |\lambda''_{k+1}|$$

for all positive integers k. From this it is straightforward to verify that the eigenvalues of $K(s,t)$ have the required decay property.

THEOREM 9. *If Ω is a bounded domain with piecewise C^1 boundary, then λ_k is $\mathcal{O}(\frac{1}{k^{3/4}})$.*

PROOF: Clearly, such an Ω can be decomposed into a finite union of nonoverlapping sub-domains $\Omega = \cup_k \Omega_k$ where each Ω_k can be put into the form (*) after a rigid motion in the plane. By lemma 6, each $L_{\chi_{\Omega_k}}$ has eigenvalues with the sought-after decay property. Lemma 8 then implies that L_{χ_Ω} also has the same property.

This estimate is in fact sharp, at least for annular regions.

PROPOSITION 10. *Let $\Omega = \{(\xi, t) : \epsilon \le (\xi^2 + t^2)^{1/2} \le R\}$ where $0 < \epsilon < R$. Then*

$$0 < \limsup_{k\to\infty} k^{3/4}\lambda_k < \infty.$$

PROOF: According to lemma 4, the k-th eigenvalue is

$$\lambda_k = (-1)^k 4\pi \int_\epsilon^R \exp(-2\pi r^2) L_k(4\pi r^2)\, r dr.$$

The following classical asymptotic expansion for Laguerre polynomials, valid for $x \in [\epsilon', R']$ with $0 < \epsilon' < R'$, will be essential [12, Theorem 8.22.2]:

$$\begin{aligned}\pi^{1/2}x^{1/4}k^{1/4}\exp(-x/2)\, L_k(x) = {} & \cos(2(kx)^{1/2} - \pi/4)\left(1 + A_1(x)k^{-1/2} + \mathcal{O}(k^{-1})\right) \\ & + \sin(2(kx)^{1/2} - \pi/4)\left(B_1(x)k^{-1/2} + \mathcal{O}(k^{-1})\right).\end{aligned}$$

Applying this formula with $x = 4\pi r^2$ yields

$$|\lambda_k| = k^{-1/4}I_0(k) + k^{-3/4}I_1(k) + \mathcal{O}(k^{-5/4}) \tag{6}$$

where

$$I_0(k) = 2^{3/2}\pi^{1/4}\int_\epsilon^R r^{1/2}\cos(4(k\pi)^{1/2}r - \pi/4)\, dr$$

and

$$\begin{aligned}I_1(k) = {} & 2^{3/2}\pi^{1/4}\int_\epsilon^R r^{1/2}A_1(4\pi r^2)\,\cos(4(k\pi)^{1/2}r - \pi/4)\, dr \\ & + 2^{3/2}\pi^{1/4}\int_\epsilon^R r^{1/2}B_1(4\pi r^2)\,\sin(4(k\pi)^{1/2}r - \pi/4)\, dr.\end{aligned}$$

Consider the behavior of $I_0(k)$. Using a double-angle formula, one has

$$I_0(k) = 2^{3/2}\pi^{1/4}\int_\epsilon^R r^{1/2}\cos(4(k\pi)^{1/2}r)\, dr + 2^{3/2}\pi^{1/4}\int_\epsilon^R r^{1/2}\sin(4(k\pi)^{1/2}r)\, dr.$$

These are the Fourier cosine and sine integrals of the smooth function $r^{1/2}$ on the interval $[\epsilon, R]$, evaluated as $4(k\pi)^{1/2}$. As such, both integrals are $\mathcal{O}(k^{-1/2})$. A similar argument shows that the integrals in $I_1(k)$ are $\mathcal{O}(k^{-1/2})$ as well.

Finally, with a little more care one can show that

$$\limsup_{k\to\infty} k^{1/2} I_0(k) > 0.$$

In fact, write the function $r^{1/2} = \ell(r)+(r^{1/2}-\ell(r))$, where $\ell(r)$ is the linear function that interpolates between the endpoint values of $r^{1/2}$ at ϵ and R. The function $r^{1/2}-\ell(r)$ is a Lipschitz function on the real line with support in $[\epsilon, R]$. Consequently, the Fourier sine and cosine integrals of this function evaluated at $4(k\pi)^{1/2}$ are $\mathcal{O}(k^{-1})$. It is therefore enough to show that the integral

$$A_0(k) = \Re\left(e^{i\pi/4}\int_\epsilon^R \ell(y)\exp(4i(k\pi)^{1/2}r)\,dr\right)$$

has the property that

$$\limsup_{k\to\infty} k^{1/2} A_0(k) > 0.$$

This is easy to show directly.

For general domains, we obtain the weaker result that the sequence of eigenvalues is not absolutely summable.

PROPOSITION 11. *The series $\sum \lambda_k$ is not absolutely convergent.*

PROOF: It is well known that $W(\phi_k, \phi_l)$ is an orthonormal basis for $L^2(\mathbf{R}^2)$. The equations $\iint_\Omega W(\phi_k, \phi_l) = \delta_{kl}\lambda_k$ imply that $\chi_\Omega = \sum \lambda_k W(\phi_k, \phi_k)$ in $L^2(\mathbf{R}^2)$. On the other hand, if $\sum |\lambda_k| < \infty$ then $\sum \lambda_k W(\phi_k, \phi_k)$ would have to converge uniformly to an element of $C_0(\mathbf{R}^2)$ (see theorem 1 part 2). This is clearly a contradiction.

We now examine the smoothness and decay of the eigenfunctions. We assume that Ω is an open set contained in a rectangle $[-B, B] \times [-T, T]$ of which all cross-sections in the ξ and t directions consist of at most M intervals. We now examine an equation of the form $L_{\chi_\Omega}\phi = \lambda\phi$, where λ is a nonzero real number and $\phi \in L^2(\mathbf{R})$. Fubini's theorem implies that

$$\lambda\phi(s) = \int K(s,t)\phi(t)\,dt$$

holds for all $s \in \mathbf{R}\backslash Z$, where Z is some set of measure zero. Lemma 5 yields the estimate

$$\begin{aligned}|\lambda\phi(s)| &\le \frac{C}{1+|s|}\int_{-s-2T}^{-s+2T} |\phi(t)|\,dt \\ &\le \frac{C\sqrt{4T}}{|s|+1}\|\phi\|_{L^2} \le \frac{C'}{|s|+1}\end{aligned}$$

for all $s \in \mathbf{R}\backslash Z$. This last estimate implies that $\phi \in L^\infty(\mathbf{R})$. Now define

$$\mathcal{E}(s) = \sup_{|s'|\geq|s|} |\phi(s')|.$$

Note that $\mathcal{E}(s)$ is even and decreasing in $|s|$. The quantity $|\lambda\phi(s)|$ can now be estimated as follows:

$$\begin{aligned} |\lambda\phi(s)| &\leq \frac{C}{|s|+1}\int_{-s-2T}^{-s+2T} |\phi(t)|\,dt \\ &\leq \frac{4CT}{|s|+1}\mathcal{E}(|s|-2T) \end{aligned}$$

for all $s \in (\mathbf{R}\backslash Z) \cap \{s' : |s'| > 2T\}$. Therefore, given any $b > 1$, there is an s_0 such that if $|s| \geq |s_0|$ and $s \in \mathbf{R}\backslash Z$ then

$$\mathcal{E}(|s| + 2T) \leq \frac{1}{b}\mathcal{E}(|s|).$$

Iterating this estimate yields

$$\mathcal{E}(|s| + 2nT) \leq \frac{1}{b^n}\mathcal{E}(|s|).$$

It is straightforward to check that

$$|\phi(s)| \leq Cb^{-|s|/(2T)} \qquad \forall s \in \mathbf{R}\backslash Z. \tag{7}$$

As a consequence, $\hat{\phi}$ is smooth and has all its derivatives in L^2.

Now, by theorem 1 part 4, we have that

$$\iint_{\tilde{\Omega}} W(\hat{\phi}_k, \hat{\phi}_l) = \iint_{\Omega} W(\phi_k, \phi_l) = \lambda_k\delta_{kl}$$

where $\tilde{\Omega} = \{(\xi, t) : (-t, \xi) \in \Omega\}$. Hence $L_{\chi_{\tilde{\Omega}}}\hat{\phi}_k = \lambda_k\hat{\phi}_k$, and by the preceding discussion ϕ is smooth and has all its derivatives in L^2. Now for a given s_1,

$$\left.\frac{d^n\phi}{ds^n}\right|_{s=s_1} = \int \hat{\phi}(\xi)(2\pi i\xi)^n e^{2\pi i s_1\xi} d\xi.$$

Using the estimate analogous to equation (7) for $\hat{\phi}$ with $b >> 1$, one has $|\hat{\phi}(\xi)| \leq Ce^{-4\pi|\xi|}$. Then

$$\begin{aligned} \left|\frac{1}{n!}\frac{d^n\phi}{ds^n}\right|_{s=s_1} &\leq \frac{2C}{n!}\int_0^\infty e^{-4\pi\xi}(2\pi\xi)^n\,d\xi \\ &\leq \frac{C}{n!}\frac{\Gamma(n)}{2^{n+1}} = \frac{C}{2^{n+1}n}. \end{aligned}$$

Hence the power series of ϕ at $s = s_1$ converges in some interval around s_1. Because of the symmetry in the role of ϕ and $\hat{\phi}$, the same observations hold for $\hat{\phi}$. The preceding discussion is summarized in the following theorem.

THEOREM 12. *Suppose Ω is an open set contained in the interval $[-B, B] \times [-T, T]$ with the property that all its cross-sections in both the ξ and t directions consist of at most M intervals. Then:*

(i) *for any $b > 0$, there is a constant C_b such that*

$$|\phi(s)| \leq C_b e^{-b|s|} \qquad \forall s, \quad \text{and}$$
$$|\hat{\phi}(\xi)| \leq C_b e^{-b|\xi|} \qquad \forall \xi,$$

and

(ii) *ϕ and $\hat{\phi}$ are analytic and have all their derivatives in L^2.*

Note that (1) actually implies (2) in theorem 12 by the Paley-Wiener theorems [13], although we have chosen to give the elementary argument.

It is instructive to consider the case when Ω is a ball centered at the origin. It is well known that, in this case, the ϕ_k are Hermite functions (see lemma 4). The Hermite functions $h_j(t)$ in lemma 4 will satisfy estimates of the form $h_j(t) \leq Ce^{-(\pi-\epsilon)t^2}$ for any $\epsilon > 0$. It is directly evident that they will then satisfy the weaker inequalities in theorem 12. This theorem states that this weaker decay statement holds for general regions in the time-frequency plane. We do not know whether these estimates can be improved.

SECTION 4
NUMERICAL EXAMPLES

For illustration purposes, we provide four numerical examples of time-frequency localization. These examples are obtained by discretizing the kernel in the integral representation of the operator given in lemma 5. We consider localization on domains of the form of a "zigzag," a disk, a rectangle, and a parallelogram, as indicated in figure 1. Note that while the operators here are all Hermitian, so that they have real eigenvalues, they do not generally have real eigenfunctions unless their kernels are real. This is true if the symbol $S(\xi, t)$ satisfies $S(-\xi, t) = S(\xi, t)$. For clarity of eigenfunction representation, projection domains were chosen to allow this symmetry when possible.

Localization on the unusual zigzag domain (figure 1a) produces the unfamiliar eigenfunctions in figure 2 (as these are complex, we have plotted their magnitude). Eight samples per unit length are used, over the time domain [−8,8]. Most of the energy is concentrated in the desired region [−2,2], although there is some leakage. Note that the first three eigenfunctions have one, two, and three peaks, respectively. For the case of the disk (figure 1b), a plot of the eigenvalues is shown in figure 3. As noted in lemma 4, the eigenfunctions in this case are the well-known hermite functions. Next, localization on the rectangle (figure 1c) allows comparison to the prolate spheroidal wavefunctions. The first nine solutions for the prolate spheroidal and Weyl operator cases are depicted in figures 4 and 5. Again, eight samples per unit length are used over the time domain [−8,8]. While there is energy leakage outside the desired domain [−2,2], the amount differs in the two cases. The prolate spheroidal wavefunctions have $1/x$ decay while the Weyl eigenfunctions have exponential decay. This difference is visible when one compares the last three eigenfunctions in each case. A plot of the eigenvalues is also included.

Finally, we provide a simple illustration of these ideas in the context of filtering Gaussian noise from a corrupted linear FM (chirp) signal. Figures 6a and 7a show, respectively, the Wigner distribution intensity plot and the actual plot of the real part of a linear FM chirp centered at 0 frequency. (Although chirp signals used in radar are not centered at 0 frequency, that is irrelevant for our purposes because of the covariance of the Wigner distribution under time and frequency shifts (theorem 1, part 5).) In the discretization, 16 samples per unit length are used over the time domain [−2,2]. No windowing is applied to remove the echo effect in these Wigner plots, although we have found a simple cosine-squared window to be effective. Figures 6b and 7b show this signal after 0 dB Gaussian white noise has been added. To filter the noise, we note that theoretically, the Wigner distribution of a chirp signal is a measure concentrated along a diagonal line corresponding to the slope of the chirp. In particular, a chirp can be localized in any domain containing its time-frequency support.

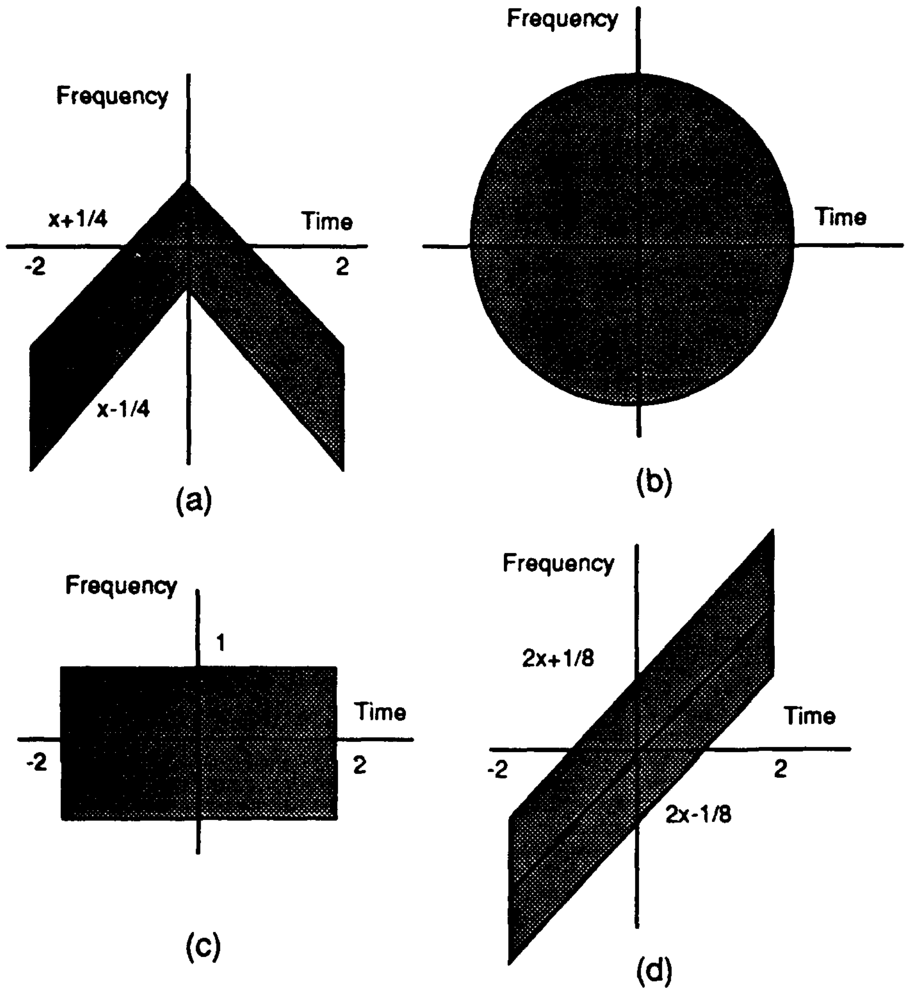

Figure 1. Localization Domains: a) Zigzag Domain, b) Disk of Area 50, c) 4×2 Rectangle, and d) Parallelogram

As an elementary example of time-frequency localization, the noisy chirp of figure 7b is projected onto the first two eigenfunctions (weighted according to the eigenvalues) for the domain in figure 1d. This results in the signal in figure 7c, with a Wigner distribution as shown in figure 6c. A plot of the eigenvalues for this localization operator is provided in figure 7d. In fact, the chirp in figure 7a is orthogonal to the second eigenfunction, illustrating an interesting fact. Numerically, the chirp appears to be an exact solution to the problem of localizing onto an infinite diagonal band domain. For more numerical examples of time-frequency localization, the reader is referred to [6,7].

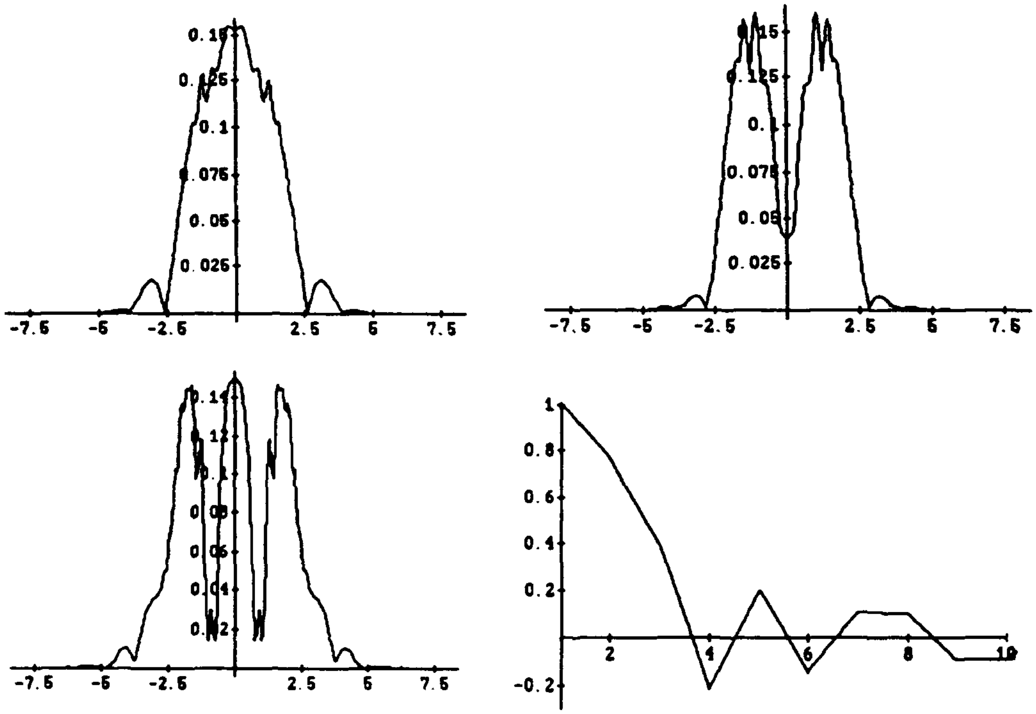

Figure 2. Magnitude of Eigenfunctions 1-3 for Localization onto a Zigzag Domain (in figure 1a), with a Plot of Eigenvalues Sorted According to Decreasing Magnitude

It is interesting at this point to remark on the conjecture of Flandrin [5] that for localization onto convex domains, the top eigenvalue is bounded above by 1. This seems to hold (at least numerically) in our examples, even for figure 1a, which is not convex. However, something like convexity is certainly necessary in general, since we have nonconvex numerical examples where the top eigenvalue exceeds 4.

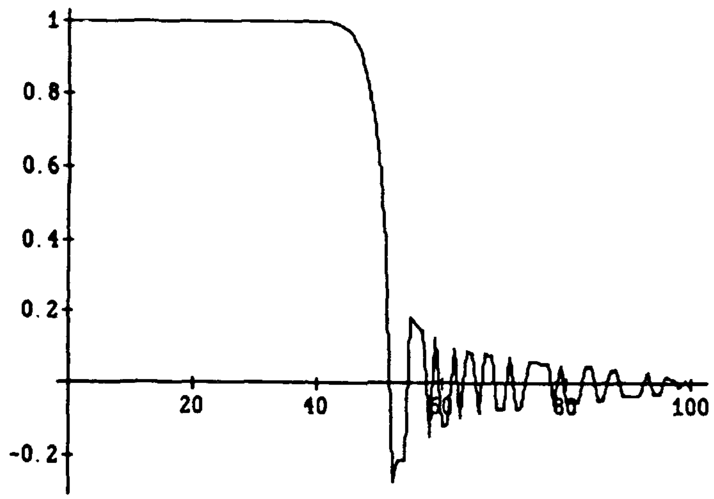

Figure 3. Weyl Eigenvalue Plot for Localization onto a Disk of Area 50 (figure 1b), Sorted According to Decreasing Magnitude

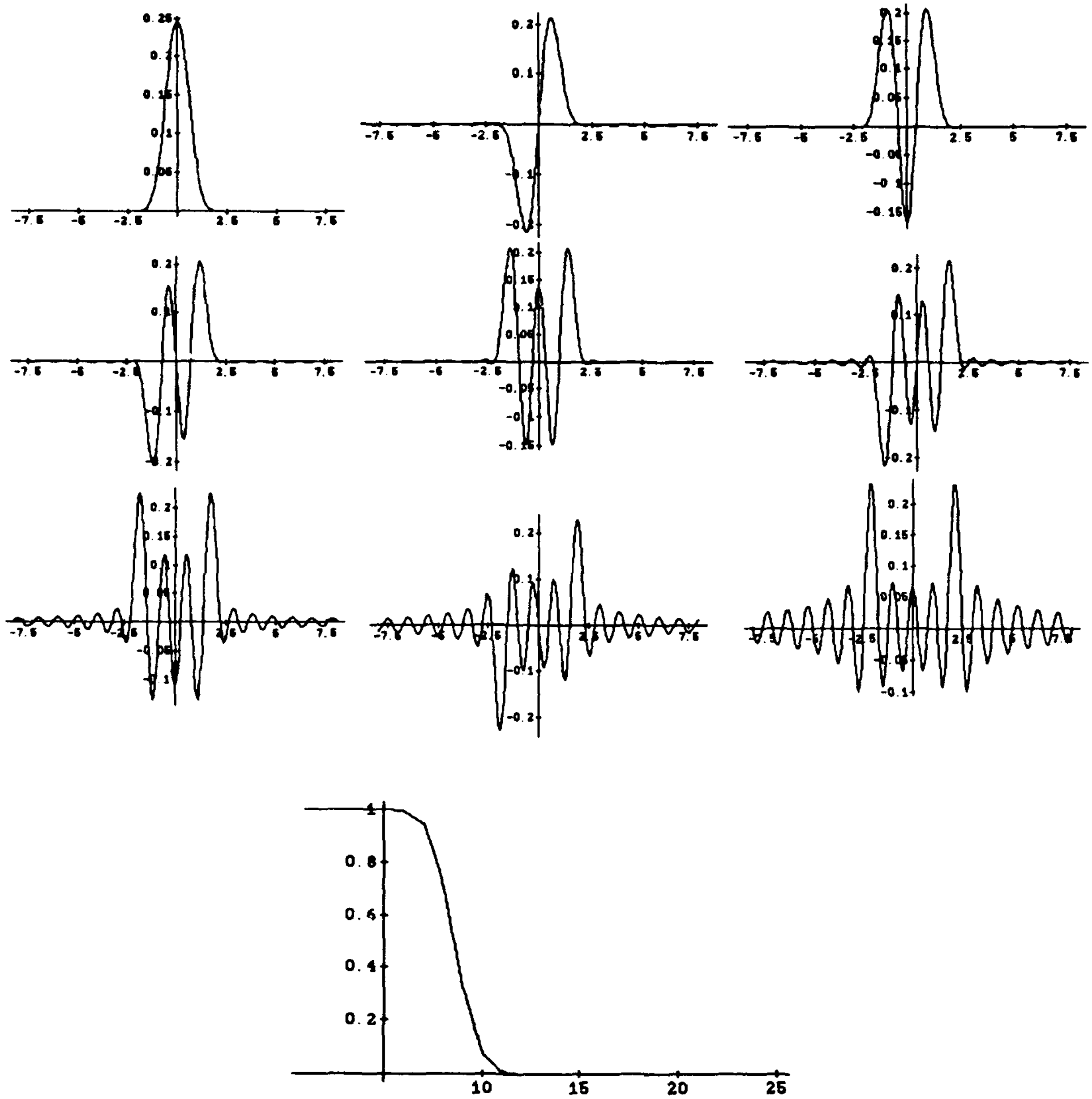

Figure 4. Prolate Spheroidal Wavefunctions 1-9 for Localization on a Rectangle (figure 1c), with Eigenvalue Plot

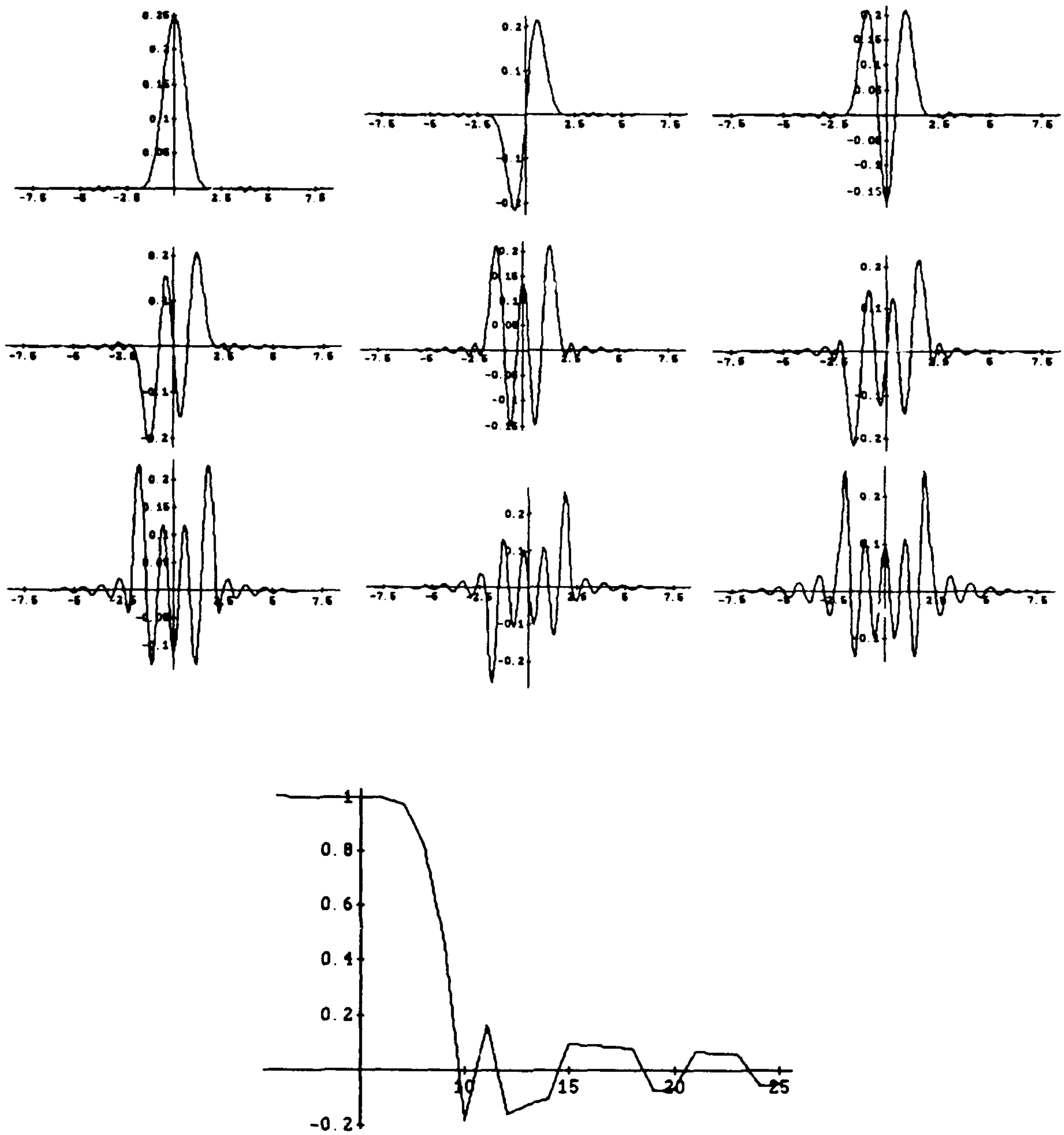

Figure 5. Weyl Eigenfunctions 1-9 for Localization on a Rectangle (figure 1c), and a Plot of Eigenvalues in Decreasing Magnitude

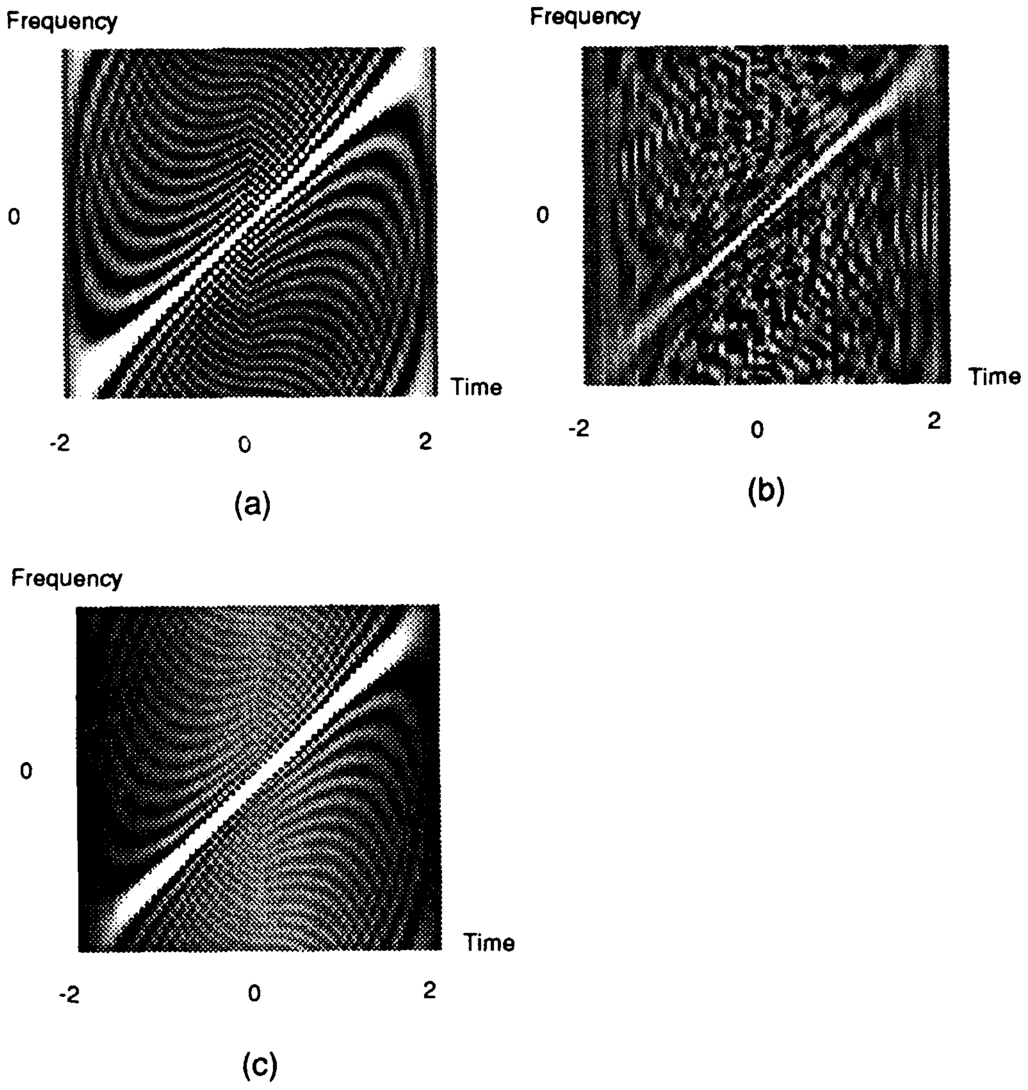

Figure 6. Wigner Distribution Intensity Plot of (a) Chirp Signal, (b) Chirp in 0 dB Gaussian White Noise, and (c) Filtered Noisy Signal

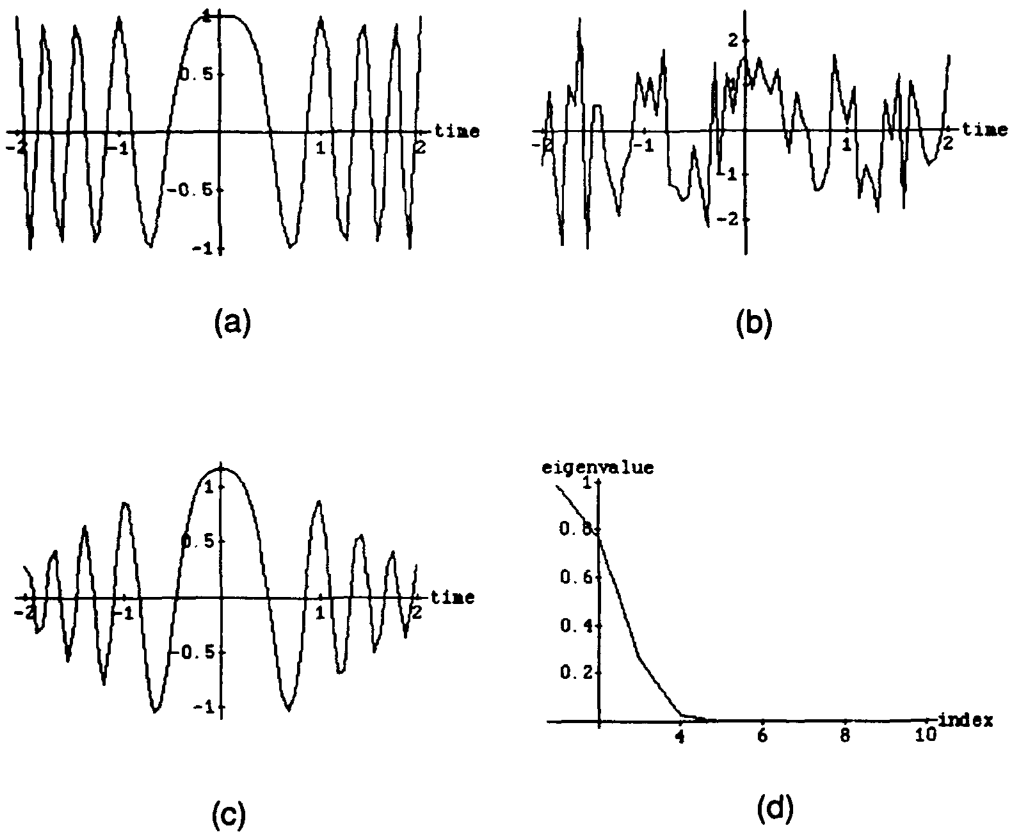

Figure 7. Plots of the Real Part of (a) Chirp Signal, (b) Chirp Signal in 0 dB Gaussian White Noise, and (c) Filtered Noisy Signal; Part (d) Is an Eigenvalue Plot. The First Two Eigenfunctions, Weighted by their Eigenvalues, Are Used in the Filtering.

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