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13. ABSTRACT *(Maximum 200 words)*

Our research efforts during the grant period were concerned with undetermined coefficient problems in partial differential equations focusing on three distinct areas: problems where the unknown coefficients depend only on the dependent variables; problems giving rise to first order equations where the boundary conditions and the coefficients in the equations have a nonlocal dependence on the variables; and inverse spectral problems, in which unknown coefficients have to be determined from eigenvalue data. These problems originate from situations where one wants to recover unknown material properties, or to determine the nature of an unknown physical law.

A number of theoretical results (existence as well as uniqueness) as well as computational algorithms were derived for model elliptic, parabolic and hyperbolic equations.

During the period of the grant, two major conferences were organized by the principal investigators, 1 Ph.D. thesis was completed, 17 publications appeared, and an additional 5 manuscripts were submitted.

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Final Report to Office of Naval Research for contract N00014-89-J-1108

Michael S. Pilant and William Rundell
Principal Investigators

Introduction.

In our original proposal we agreed to

> ...*investigate problems where the unknown coefficients depend only on the dependent variables; consequently the underlying equations are quasilinear. These problems are related to the determination of unknown physical laws or relationships, and the mathematical formulation is most often in the form of an elliptic or parabolic equation. The nonlinear terms which we recover in our model problems correspond to material properties that have physical significance. The form of the nonlinearities are appropriate for isotropic media. Examples are, the determination of an unknown reaction term* $f(\cdot)$ *in* $u_t - u_{xx} = f(u)$, *or the conductivity* $k(\cdot)$ *in the equation* $\nabla \cdot k(u)\nabla u = 0$. *We seek to determine these functions by giving only overposed boundary data....*

Our research efforts during the contract period were concerned with undetermined coefficient problems in partial differential equations. The work can be broken down into three distinct areas:

- Problems where the unknown coefficients depend *only* on the dependent variables;

- Problems giving rise to first order equations where the boundary conditions and the coefficients in the equations have a nonlocal dependence on the variables;

- Inverse spectral problems, in which unknown coefficients have to be determined from eigenvalue data. [Although not in our original proposal, this third group of problems was suggested at the end of the second year by our program manager, Dr. John Lavery.]

During the period of the contract, two major conferences were organized by the principal investigators [**Arcata, CA. –1989** and **College Station, TX. – 1991**] , 1 Ph.D. thesis was completed, and 17 publications appeared. An additional 5 manuscripts were submitted. A list of conference presentations and invited talks by the principal investigators is given and the end of this final report.

We summarize our research, as well as the publications arising out of this work, in the following three sections.

Recovery of Elliptic and Parabolic Equations from Boundary Data.

The problems modeled by these equations are related to the determination of unknown physical laws or relationships. The nonlinear terms which we seek to recover in our model problems correspond to material properties that have physical significance; temperature dependent specific heats, conductivities, reaction terms, to name just a few. Examples of such problems are – the determination of an unknown reaction term $f(\cdot)$ in $u_t - u_{xx} = f(u)$, or the conductivity $k(\cdot)$ in the equation $\nabla . k(u)\nabla u = 0$. We seek to determine these functions by giving *only* overposed boundary data.

This type of problem is distinct from those that involve media with unknown inhomogeneities; that is, the differential equations contain an unknown coefficient that depends on the independent spatial variable. In a given physical problem both situations may occur, that is, the unknowns have both spatial as well temperature dependence. This is a considerably more difficult problem and has received little attention in its full generality; instead both the limiting cases of dependence on a single one of these variables have received the vast majority of the research efforts. For those cases where the media is isotropic, the assumption that the unknowns depend only on the independent variable may be a very reasonable one. From a mathematical standpoint the two types of problem pose different mathematical difficulties. Both types of inverse problem leads to nonlinear equations, but in the case when the unknown coefficient depends on the spatial variables the direct problem is frequently linear. In contrast, when the coefficient is a function of the dependent variable, the direct problem is nonlinear.

In earlier work the investigators gave a constructive algorithm, the *FPP method*, for the determination of a state-dependent reaction term f in the reaction diffusion equation $u_t - u_{xx} = f(u)$. We extended this work in several ways: to obtain sharp estimates on the rate of convergence and how this depends on various parameters in the problem [A1]; to the case of systems of equations [A4]; and to the recovery of other coefficients in the equation [A6,A7,A9,A10]. A uniqueness theorem for the case of a steady state recovery of a conductivity (in two space dimensions) was proved in [A2].

For the case of recovery of a nonlinear boundary condition, where the flux depends in an unknown manner on the boundary temperature, we have used the *FPP method* to obtain an efficient numerical scheme for which we were able to give a complete analysis in [A3]. A uniqueness result for the higher spatial dimension case was found in [A5]. A collocation method, which relies on the characteristic of the parabolic equation (the fact that information flows forward in time) was used in [A10] to obtain an alternative formulation of the problem.

In many problems, in particular those with small spatial variation, a reasonable approximation to the case where the coefficients depend on the state variable $u(x,t)$ is to assume instead that the determination is only through the time variable. Even if this model is not sufficiently accurate we may use it to gain a good initial approximation for a scheme such as the *FPP*. A multidimensional recovery of a time-dependent conductivity from boundary measurements was shown in [A8].

Inverse Problems in First Order Equations.

Here we consider the problem of determining unknown coefficients in a first order hyperbolic operator subject to both initial and boundary data. We wish to recover either this data, or coefficients that occur in the equation itself, or in the boundary conditions. To be more specific, we shall consider the single equation $\rho_t(a,t) + \rho_a(a,t) + \lambda\rho(a,t) = 0$. Our choice of independent variables indicates a particular application; a stands for *age*, t for *time* and the density $\rho(a,t)$ denotes the number of individuals of age a at a particular time t. As will be explained in the next section the function λ will correspond to a "death function", and the different types of boundary condition will arise from different assumptions on the birth process. Although we are using the language of *age structured population dynamics*, our " species" could be also be mechanical parts. In this case, $\rho(a,t)$ would denote the number of parts of age a still in service at time t, and λ would be age dependent rate of failures. Our results are by no means be limited to these applications for equations such as this are common throughout the physical sciences.

The initial-boundary conditions are $\rho(a,0) = \phi(a)$ and $\rho(0,t) = \int_0^L \beta(a,\rho)\rho(a,t)\,da$. The functions β and λ are called the "birth" and "death" equations respectively. The usual direct problem is to be given β, λ and ϕ and to use this to recover $\rho(a,t)$.

From a mathematical viewpoint, the inverse problems of recovering the parameters β, λ and possibly ϕ, is complicated by the nonlocal boundary condition at $a = 0$. In addition, the coefficients β and λ can, and in more realistic models do, depend on functionals of the population such as $P(t) = \int \rho(a,t)da$, in this case, the total number of the species present at time t. Such non-local dependence has two effects on the inverse problem: First, it does not allow us to obtain easy-to-use representation formulae; in particular, it essentially disallows any such formula that has pointwise definition. Second, it usually means that the inverse problem is *ill-posed*.

During the last four years we have made considerable progress on some of these inverse problems. For example, we have been able to find conditions under which uniqueness results proven and computational schemes for recovery found for reconstruction of the various parameters. We have, under suitable conditions of overposed data, determined the birth function $\beta(a)$ in [B1], the death function $\lambda(a)$ in [B5,B6], the death function $\lambda(a,P)$ in [B4] and the initial profile ϕ from a later data measurement in [B3]. The paper [B2] is a survey on this subject. Much work remains to be done, in particular the determination of coefficients in systems or the simultaneous determination of multiple coefficients.

Inverse Eigenvalue Problems.

The classical inverse Sturm-Liouville problem consists of recovering one of the coefficients $p(x)$, $q(x)$ or $r(x)$ from

$$-\big(p(x)y'\big)' + q(x)y = \lambda r(x)y$$
$$y'(0) - hy(0) = 0$$
$$y'(1) + Hy(1) = 0$$

and a knowledge of spectral data. This data can take various forms and this leads to a family of inverse problems. In each of these, one assumes that a complete spectrum, $\{\lambda_j\}_{j=1}^{\infty}$ is given. It is well known that this is insufficient for recovery of q, and thus some additional information must be provided. In fact questions of existence and uniqueness for the inverse spectral problem have been studied rather thoroughly. Our work concentrated on obtaining fast, robust algorithms for numerical reconstruction from *finite* spectral data. We have obtained results on a variety of methods.

One of these, the so-called method of Rundell and Sacks [C1, C3], uses the Gelfand-Levitan equation to convert the inverse spectral problem to one of an overposed boundary value problem for a certain hyperbolic equation. It can be shown that this problem has a unique solution, and we have formulated several efficient schemes that can be used to effect numerical recovery. This method has an extremely low operational count and its speed of operation is such that it can recover a potential $q(x)$ faster than the best available methods can compute a *single* eigenvalue from a given potential. Essentially, what is obtained from finite data is very close to the finite Fourier series reconstruction of the unknown function. The method is sufficiently flexible to allow most of the standard inverse spectral problems to be handled. There are three papers on the subject [C1,C3,C5] during the grant period.

A second type of scheme seeks to rearrange the total data specified in order to take advantage of a certain shooting algorithm which targets on the *boundary values* at a single endpoint given the spectral data and the boundary data at the other endpoint, that is, we look at the map $F(q, \{\lambda_n\}, h) \to y(1)$. This feature allows for an algorithm that is relatively fast and extremely flexible – it can effect recovery from all types of inverse spectral problems known to the investigators. We have been able to prove convergence of our scheme in a few cases, but numerically it performs exceedingly well [C2,C4].

A third type of numerical scheme relies on a linearization of the Rayleigh-Ritz method for computing eigenvalues from given coefficients. This leads to exceedingly rapid computations. We are able to prove certain convergence results (for small variations of the coefficients from constants). The main advantage of the scheme is in its apparent applicability to inverse spectral methods in higher space dimensions and to problems with discontinuous coefficients. We are actively continuing our efforts in this area and papers are in preparation.

Conference Presentations and Invited Talks During the Contract Period

M. Pilant:

"Survey of Methods for Inverse Coefficient Problems", one hour survey talk, AMS Summer Conference, Arcata 1989.

"Using Domain Decomposition Methods for Computing Singularities of Equations of Mixed Type," SIAM Conference on Domain Decomposition Methods, Houston, March 20-22, 1989. [Chaired session.]

"Parallel algorithms for an Inverse Sturm-Liouville Problem," 20 minute invited address, AMS Meeting, Portland, OR., June 1991.

W. Rundell:

December 1989, Conference on *Inverse problems*, Montpellier, France.

February 1990, Colloquium, University of Houston.

March 1990, *Special Session on Inverse Problems*, American Mathematical Society, Manhatten, Kansas.

April 1990, *Special Session on Invariant Embedding and Inverse Problems*, A.M.S. and S.I.A.M. Albuquerque, New Mexico.

October 1990, Colloquium, Brigham Young University.

November 1990, *Special Session on Differential Equations*, American Mathematical Society, Denton, Texas.

June 1991, *Special Session on Inverse Problems*, American Mathematical Society, Portland, Oregon.

June 1991, *Minisymposium on Inverse Problems*, International Congress of Industrial and Applied Mathematics, Washington D.C.

Publications During the Contract Period

Papers on Elliptic and Parabolic Equations.

[A1] M. S. Pilant and W. Rundell, Fixed point methods for a nonlinear parabolic inverse coefficient problem, *Comm. Partial Differential Equations*, **13**, (4), pp. 469– 493, (1988).

[A2] M. S. Pilant and W. Rundell, A uniqueness theorem for determining conductivity from over-specified boundary data, *Journal of Math. Anal. Appl.*, **136**, (1), pp. 20–28, (1988).

[A3] M. S. Pilant and W. Rundell, An iteration method for the determination of an unknown boundary condition in a parabolic initial-boundary value problem, *Proc. Edin. Math. Soc.*, **32**, pp. 59–71, (1989).

[A4] M. S. Pilant and W. Rundell, Multiple undetermined coefficient problems for quasi-linear parabolic equations, *Numerical Methods in P.D.E.*, **5**, pp. 297–311, (1989).

[A5] W. Rundell and H. M. Yin, A Parabolic Inverse Problem with an Unknown Boundary Condition, *Journal of Differential Equations*, **86**, pp. 234–242, (1990).

[A6] M. S. Pilant and W. Rundell, Undetermined coefficient problems for quasilinear parabolic equations, in *Inverse Problems in Partial Differential Equations*, Proceedings of the 1989 Summer Research Conference, Arcata, CA. 1989, William Rundell, David Colton, Richard Ewing editors, SIAM (1990), pp. 165–185.

[A7] M. S. Pilant and W. Rundell, A method for identifying nonlinear terms in parabolic initial-boundary value problems, *Advances in Water Resources*, **14**, (2), pp. 83–88, (1991).

[A8] J. R. Cannon and W. Rundell, Recovering a time dependent coefficient in a parabolic differential equation, *J. Math. Anal. Appl.*, **160**, pp. 572–582, (1991).

[A9] M. S. Pilant and W. Rundell, Recovery of an unknown specific heat by means of overposed data, *Numerical Methods in P.D.E.*, **6**, pp. 1-16, (1990).

[A10] M. S. Pilant and W. Rundell, A collocation method for the identification of nonlinear flux terms in a parabolic initial-boundary value problem, *Krueger Conference Proceedings*, SIAM, Albuquerque, NM, (1992).

Papers on Inverse Problems in First Order Equations.

[B1] W. Rundell, Determining the birth function in an age structured population, *Mathematical Population Studies*, **1**, 4, pp. 377–395, (1989).

[B2] M. S. Pilant and W. Rundell, Age-structured population dynamics, in *Inverse Methods in Action*, P. Sabatier, editor, Springer Verlag, pp. 122–129, (1990)

[B3] M. S. Pilant and W. Rundell, Determining the initial age distribution for an age structured population, *Mathematical Population Studies*, **3**, pp. 3–20, (1991).

[B4] M. S. Pilant and W. Rundell, Determining a coefficient in a first order hyperbolic equation, *SIAM J. Appl. Math.*, **51**, (2), pp. 494-506, (1991).

[B5] W. Rundell, Determining the death rate for an age-structured population, submitted.

[B6] H.W. Engl, W. Rundell and O, Scherzer, A Regularization Scheme for an inverse problem in age-structured populations, submitted.

Papers on Inverse Eigenvalue Problems.

[C1] W. Rundell and P. Sacks, Reconstruction techniques for classical inverse Sturm-Liouville problems, *Mathematics of Computation*, **58**, (197), pp. 161–183, (1992).

[C2] B. Lowe, M. S. Pilant and W. Rundell, The recovery of potentials from finite spectral data, *SIAM J. Math. Anal.*, **23**, pp. 482–504, (1992).

[C3] W. Rundell and P. Sacks, The reconstruction of Sturm-Liouville operators, *Inverse Problems*, **8**, pp. 457–482, (1992)

[C4] B. Lowe and W. Rundell, An inverse problem for a Sturm-Liouville operator, *J. Math. Anal. Appl.*, to appear.

[C5] W. Rundell and P. Sacks, The reconstruction of potentials with bound states, submitted.

[C6] B. Lowe and W. Rundell, The reconstruction of two coefficients in a Sturm-Liouville operator, submitted.